Math Camp

Module #6: Optimization

Optimization

Static problem

Assume that $K(\eta) \subset \mathbb{R}^n$ is a collection of compact sets parameterized by $\eta \in \mathbb{R}^m$.

$$V(\eta) = \max_{x \in K(\eta)} f(x, \eta)$$

$$x^*(\eta) = \underset{x \in K(\eta)}{\operatorname{argmax}} f(x, \eta).$$

Optimization

Univariate, unconstrained case

Let $f : \mathbb{R} \to \mathbb{R}$ be C^3.

- Suppose x^* solves $\max_{x \in \mathbb{R}} f(x)$. Write

$$f(x) = f(x^*) + f'(x^*)(x - x^*) + \mathcal{O}\left(|x - x^*|^2\right).$$

  Then $f'(x^*) = 0$.

- Suppose $f'(x^*) = 0$ and $f''(x^*) < 0$. Write

$$\begin{aligned} f(x) &= f(x^*) + f'(x^*)(x - x^*) + f''(x^*)(x - x^*)^2 + \mathcal{O}\left(|x - x^*|^3\right) \\ &= f(x^*) + f''(x^*)(x - x^*)^2 + \mathcal{O}\left(|x - x^*|^3\right) \end{aligned}$$

  Then x^* solves $\max_{x \in \mathbb{R}} f(x)$.

Optimization

Equality constraints

Consider the problem

$$\max_{x \in K(c)} f(x)$$

where $K(c) = \{x \in \mathbb{R}^n : g(x) = c\}$, with $g : \mathbb{R}^n \to \mathbb{R}^m$ and $c \in \mathbb{R}^m$.

Strategy: look for necessary conditions: $df \leq 0$ and $dg = 0$.

Optimization

Theorem (Method of Lagrange: necessity)

Let $f : \mathbb{R}^n \to \mathbb{R}$ *and* $g : \mathbb{R}^n \to \mathbb{R}^m$ *be* C^2 *and set*

$$L(x, \lambda) = f(x) + \lambda^T (c - g(x)).$$

If

$$x^* \in \arg\max f(x)$$
$$g(x) = c$$

and if $\text{rank}(Dg(x^*)) = m$ *then there exists* $\lambda^* \in \mathbb{R}^m$ *such that*

- $L_{x_i}(x^*, \lambda^*) = 0$ *for* $i = 1, \ldots, n$
- $L_{\lambda_j}(x^*, \lambda^*) = 0$ *for* $j = 1, \ldots, m$

Optimization

Concavity is measured with the *bordered Hessian*

$$D^2L(x,\lambda) = \begin{pmatrix} L_{x_1x_1} & \cdots & L_{x_1x_n} & L_{x_1\lambda_1} & \cdots & L_{x_1\lambda_m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ L_{x_nx_1} & \cdots & L_{x_nx_n} & L_{x_n\lambda_1} & \cdots & L_{x_n\lambda_m} \\ L_{\lambda_1x_1} & \cdots & L_{\lambda_1x_n} & L_{\lambda_1\lambda_1} & \cdots & L_{\lambda_1\lambda_m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ L_{\lambda_mx_1} & \cdots & L_{\lambda_mx_n} & L_{\lambda_m\lambda_1} & \cdots & L_{\lambda_m\lambda_m} \end{pmatrix}$$

$$= \begin{pmatrix} D_x^2L & -Dg^T \\ -Dg & 0 \end{pmatrix}$$

Theorem (Method of Lagrange: sufficiency)

If (x^*,λ^*) satisfies Lagrange's FOC and if $v^TD_x^2L(x^*,\lambda^*)v < 0$ for all $v \notin \mathrm{ker}Dg(x^*)$, then x^* is a local maximum.

Optimization

Theorem (Envelope theorem)

Consider the maximization problem

$$V(\eta) = \max_{x \in K(\eta)} f(x, \eta), \text{ where } \eta \in \mathbb{R}^k \text{ and } K(\eta) = \{x \in \mathbb{R}^n : g(x, \eta) = 0\}$$

If (x^*, λ^*) *satisfy the Lagrange conditions and constitutes a local maximum then*

$$V_{\eta_i}(\eta) = L_{\eta_i}(x^*, \lambda^*).$$

Example: re-consider the problem

$$V(I) = \max \; u(x, y)$$
$$p_x x + p_y y = I$$

What is V_I?

Optimization

The Problem:

$$\max_{x\in\mathbb{R}^n} f(x) \quad \text{s.t} \quad \begin{aligned} & g^i(x) = c_i,\ i\in\{1,\ldots,m\} \\ & h^i(x) \le 0,\ i\in\{1,\ldots,k\} \end{aligned} \tag{1}$$

Let $L(x,\lambda,\mu) = f(x) + \lambda^T(c - g(x)) - \mu^T h(x)$

Theorem (Kuhn-Tucker)

If x^* solves (1) then there exist $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^k$ so that

- $L_{x_i}(x^*,\lambda^*,\mu^*) = 0,\ i\in\{1,\ldots,n\}$
- $L_{\lambda_i}(x^*,\lambda^*,\mu^*) = 0,\ i\in\{1,\ldots,m\}$
- $L_{\mu_i}(x^*,\lambda^*,\mu^*) \ge 0$ *and* $\mu_i^* \ge 0$, *w/cs* $i\in\{1,\ldots,k\}$