# Module 3: Metric Spaces and Continuity

Loosely speaking, a *space* is a set with some additional structure. For example, a vector space is a set together with a linear structure (i.e. addition and scalar multiplication), and a probability space is a set together with a collection of *events* (i.e. measurable subsets) and a *probability measure* providing the likelihood that a given event occurs. In this chapter we study *metric spaces*, which are comprised of sets together with some associated notion of distance, i.e. a *metric*.

Let $X$ be a non-empty set and let $d : X \times X \to \mathbb{R}_+$ have the following properties:

1. $d(x,y) = 0 \Leftrightarrow x = y$
2. (symmetry) $d(x,y) = d(y,x)$
3. (triangle inequality) $d(x,z) \leq d(x,y) + d(y,z)$

The pair $(X,d)$ is a metric space, with $d$ a metric on $X$.

Intuitively, the metric $d$ measures the distance between points in $X$. The presence of a metric structure on a given set of interest allows the practitioner to appeal to *function continuity* and *sequential approximation.* As we will discuss in more detail in the next chapter, continuous functions preserve nearness: intuitively, if $f$ is a continuous function between metric spaces then $x$ near $y$ implies $f(x)$ near $f(y)$.[1]

Sequential approximation exploits ability to define convergence on a metric space, as we will do below. The idea is as follows: perhaps you would like to say some property holds for a given $x \in X$; and, while you cannot show it holds for $x$ directly, perhaps you can show the property holds for each element in a sequence $\{x_n\}$. If you can also show that $x_n \to x$ and that this property respects limits, you're golden.[2]

**Exercise 1** *Show that $\mathbb{R}^2_+$ is a metric space – be sure to define the metric.*

[1] More rigorously, it's the local, dual version of this statement that defines continuity: $f$ is continuous at $x$ provided that you can make $f(y)$ as close as you like to $f(x)$ by choosing $y$ sufficiently close to $x$.

[2] Continuity can be defined on sets with a more general structure known as a *topology*. In economics, almost all topological spaces are, themselves, metric spaces (a metric space is a special case of a topological space) or, at least, they are naturally metrizable, which means there is a definable metric which reproduces the topology. Sequential approximation can also be generalized, but here it is more subtle: without a metric, the sequences must be replaced with *nets* (or, equivalently, filters), which may be viewed as uncountable analogous to sequences.

# 1 Normed linear spaces

Most metric spaces encountered in economics are (subsets of) vector spaces coupled with *norms* that measure the *lengths* of their vectors, i.e. *normed linear (or vector) spaces*. If $V$ is a vector space then a norm on $V$ is a map $\|\cdot\| : V \to \mathbb{R}_+$ satisfying the following properties:

1. $\|v\| = 0 \Leftrightarrow v = 0$
2. $\|\alpha v\| = |\alpha|\|v\|$ where, here, $\alpha \in \mathbb{C}$ or $\mathbb{R}$ and $v \in V$
3. (triangle inequality) $\|v + w\| \leq \|v\| + \|w\|$

The relationship between a norm and a distance measure is given by the following exercise:

**Exercise 2** *Show that if $(V, \|\cdot\|)$ is a normed vector space and $d : V \times V \to \mathbb{R}_+$ is defined by*

$$d(v,w) = \|v - w\|$$

*then $(V,d)$ is a metric space.*

The following are examples of normed vector spaces commonly found in economic analysis. Some of them may not be familiar, and/or include terms not yet defined in these notes, but we will get to the details.

1. $(\mathbb{R}^n, \|\cdot\|_2)$ where $\|v\|_2 = \left(\sum_{i=1}^n v_i^2\right)^{1/2}$
2. $(\mathbb{R}^n, \|\cdot\|_1)$ where $\|v\|_1 = \sum_{i=1}^n |v_i|$
3. $(\mathbb{R}^n, \|\cdot\|_\infty)$ where $\|v\|_\infty = \max\limits_{i=1,\ldots,n} |v_i|$
4. $\left(\ell^2(\mathbb{N}), \|\cdot\|_2\right)$ where

$$\begin{aligned} \ell^2(\mathbb{N}) &= \left\{ f : \mathbb{N} \to \mathbb{R} \text{ such that } \sum_n f(n)^2 < \infty \right\} \\ \|f\|_2 &= \left( \sum_n f(n)^2 \right)^{1/2} \end{aligned}$$

5. $\left(\ell^1(\mathbb{N}), \|\cdot\|_1\right)$ where

$$\begin{aligned} \ell^1(\mathbb{N}) &= \left\{ f : \mathbb{N} \to \mathbb{R} \text{ such that } \sum_n |f(n)| < \infty \right\} \\ \|f\|_1 &= \sum_n |f(n)| \end{aligned}$$

6. $(\ell^\infty(\mathbb{N}), \|\cdot\|_\infty)$ where

$$\begin{aligned} \ell^\infty(\mathbb{N}) &= \left\{ f : \mathbb{N} \to \mathbb{R} \text{ such that } \sup_n |f(n)| < \infty \right\} \\ \|f\|_\infty &= \sup_n |f(n)| \end{aligned}$$

7. $(C(X), \|\cdot\|_\infty)$ where $X$ is a compact metric space,

$$\begin{aligned} C(X) &= \text{set of continuous functions from } X \text{ to } \mathbb{R} \\ \|f\|_\infty &= \sup_{x \in X} |f(x)| \end{aligned}$$

8. $\left(L^2(\Omega), \|\cdot\|_2\right)$ where $(\Omega, \mu)$ is a probability space,

$$\begin{aligned} L^2(\Omega) &= \text{set of random variables on } \Omega \text{ with finite second moments, i.e. } Ex^2 < \infty \\ \|x\|_2 &= \left(Ex^2\right)^{1/2} \end{aligned}$$

Hopefully you recognize a broad pattern here.

**Exercise 3** *Determine the relationships among $\ell^1(\mathbb{N}), \ell^2(\mathbb{N})$ and $\ell^\infty(\mathbb{N})$, i.e. which, if any, are included in which?*

# 2 Completeness

A sequence in a metric space $(X,d)$ is just what you think: an infinite list of elements of $X$, often, but not always (see exercises below) denoted $\{x_n\}$. We say that the sequence converges to a point $x$, written $x_n \to x$, provided that $d(x_n, x) \to 0$ in $\mathbb{R}$. Formally, $x_n \to x$ provided that for any $\varepsilon > 0$ there is some $N > 0$ so that $n \geq N$ implies $d(x_n, x) < \varepsilon$.

**Exercise 4** *Show that if $\{x_n\}$ is a sequence in a metric space $(X,d)$ and if $x_n \to x$ and $x_n \to y$ then $x = y$.*

**Exercise 5** *Let $x^n \in \mathbb{R}^\mathbb{N}$ be given by $x^n_m = 1/m$ if $m \leq n$ and $0$ otherwise.*[3] *Let $x \in \mathbb{R}^\mathbb{N}$ be given by $x_m = 1/m$.*

1. *Show that $x^n \in \ell^1, \ell^2$ and $\ell^\infty$.*
2. *In which, if any, of these spaces does $x^n \to x$?*

[3] Remember, $\mathbb{R}^\mathbb{N}$ is the set of real sequences, with no other restrictions. It is a vector space, but not a natural normed vector space.

**Exercise 6** *Let $x^n \in \mathbb{R}^{\mathbb{N}}$ be given by $x^n_m = \delta_{mn}$. Does $x^n$ converge to zero in any of $\ell^1, \ell^2$ or $\ell^\infty$?*

Let $\{x_n\}$ be a sequence in a metric space $(X,d)$. The sequence is *Cauchy* provided that for any $\varepsilon > 0$ there is some $N > 0$ so that $m,n > N$ imply $d(x_n,x_m) < \varepsilon$. Intuitively, the terms in a Cauchy sequence get, and stay close together. A metric space is *complete* provided that Cauchy sequences converge, i.e. if $\{x_n\}$ is a Cauchy sequence then there is a point $x \in X$ such that $x_n \to x$.

**Exercise 7** *Show this point is unique.*

The intermediate value theorem (IVT), which applies to continuous functions from $\mathbb{R}$ to $\mathbb{R}$, is the fundamental tool for establishing the existence of a solution to an equation. Indeed, to show there is a solution to an equation of the form $f(x) = 0$ for some continuous function $f$, it is enough, by the IVT, to find some $a < b$ so that $f(a) < 0 < f(b)$ (or vice-versa). The essential step in the proof of this theorem is to form the set $\mathscr{A}$ of all points $z \in [a,b]$ such that $f(z) \geq 0$. Since $\mathbb{R}$ is complete, so too is $[a,b]$, and therefore the infimum of this set $\mathscr{A}$ exists. It is a nice exercise, then, to show that $f(\inf(\mathscr{A})) = 0$.

The concept of completeness generalized to metric spaces allows us to extend the intermediate value theorem in a variety of natural ways, which will provide for solutions to equations of interest. Somewhat more specifically, the intermediate value theorem can be viewed as a fixed point theorem. To see this, recall that if $f : [a,b] \to [a,b]$ then a *fixed point* of $f$ is a real number $x \in [a,b]$ such that $f(x) = x$. By the IVT, if $f$ is continuous then it has a fixed point. Indeed, suppose $f(a) > a$ and $f(b) < b$ (if either of these conditions doesn't hold then $f$ has a fixed point!) and let $g(x) = f(x) - x$. Then $g$ is continuous, $g(a) > 0$ and $g(b) < 0$. Extensions of the IVT to more general metric spaces come in the form of fixed point theorems.

One more, somewhat conceptually harder notion is needed. Let $\{x_n\}$ be a sequence of real numbers. Two definitions:

1. $\limsup x_n = \lim\limits_{n\to\infty} \sup\{x_k : k \geq n\}$
2. $\liminf x_n = \lim\limits_{n\to\infty} \inf\{x_k : k \geq n\}$

To make sense of this definition, let

$$\hat{x}^n = \sup\{x_k : k \geq n\} \text{ and } \hat{x}_n = \inf\{x_k : k \geq n\}.$$

Notice that $\{\hat{x}^n\}$ is a *decreasing sequence* and $\{\hat{x}_n\}$ is an *increasing sequence*. Now suppose that the original sequence $\{x_n\}$ is bounded, i.e. there is some interval $[-M,M] \subset \mathbb{R}$ such that $x_n \in [-M,M]$ for all $n$. Then $\{\hat{x}^n\}$ is a decreasing sequence that is bounded below and $\{\hat{x}_n\}$ is an *increasing sequence* that is bounded above. By the completeness of the real numbers, it follows that both of these sequences converge, i.e. both $\limsup x_n$ and $\liminf x_n$ exist.

Intuitively, $\liminf$s and $\limsup$s concern subsequences. A subsequence is any infinite list of numbers that comes from a sequence, with the ordering preserved. Thus $\{x_2, x_4, x_6, \ldots\}$ is a subsequence of $\{x_n\}$. Of course a given $\{x_n\}$ cannot be expected to converge to anything. For example, $y_n = n$ diverges to $\infty$ and $x_n = (-1)^n$ bounces between $\pm 1$ forever, never settling down. On the other hand, if the sequence is bounded, as in the latter case, then it will have convergent subsequences, and $\liminf$ and $\limsup$ identify the smallest and largest limits of these convergent subsequences. In particular, if $\{x_n\}$ is a bounded sequence of real numbers then the following hold:

1. $\limsup x_n$ and $\liminf x_n$ exist and $\liminf x_n \leq \limsup x_n$
2. There is a subsequence of $\{x_n\}$ converging to $\limsup x_n$
3. There is a subsequence of $\{x_n\}$ converging to $\liminf x_n$
4. If $x \in \mathbb{R}$ is the limit of a subsequence of $\{x_n\}$ then $\liminf x_n \leq x \leq \limsup x_n$.

**Exercise 8** *Show that if $\{x_n\}$ is a sequence in $(X, d)$ which converges to $x \in X$ then every subsequence of $\{x_n\}$ also converges to $x$.*

The following exercise shows the power of this concept:

**Exercise 9** *Let $\{x_n\}$ be a sequence of real numbers.*

1. *Show that $\{x_n\}$ converges to some $x \in \mathbb{R}$ if and only if $\limsup x_n = \liminf x_n$.*
2. *Using our earlier definition of completeness on $\mathbb{R}$ (and the previous part of this exercise) show that Cauchy sequences converge.*

# 3 Openness, closedness, and compactness

First a little notation. If $x \in X$ then the ball of radius $\varepsilon$ around $x$ is defined as

$$B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}.$$

A subset $A$ of a metric space $(X, d)$ is *open* provided that for each $x \in A$ there is some ball around $x$ that is completely contained in $A$. A subset $A$ of a metric space $(X, d)$ is *closed* provided that its complement in $X$ is open. Think of open and closed sets as generalizing the concepts of open and closed intervals.

**Exercise 10** *Some exercises.*

1. *Show that* $\bar{B}(x,\varepsilon) = \{y \in X : d(x,y) \leq \varepsilon\}$ *is closed.*
2. *Show that* $(1,2) \cup (3,\infty)$ *is open in* $\mathbb{R}$.
3. *Show that the intersection of two open sets is open and of two closed sets is closed.*
4. *Are there any subsets of* $\mathbb{R}$ *that are both open and closed?*

Our final metric space notion is compactness. A subset $A$ of a metric space $(X,d)$ is *compact* provided that any sequence in $A$ has a subsequence that converges to some point in $A$. Intuitively, a compact set is not too big (otherwise it could have sequences with points that are always far from each other) and contains it boundary (otherwise a sequence could converge to some point on the boundary, but this would mean that every subsequences converges to a point on the boundary).

Compact sets constrain the behavior of continuous functions defined on them. Most importantly for us, and as will be examined in detail in later chapters, if $f$ is a real-valued continuous function on a compact set $A \subset X$ then $f$ realizes it maximum and minimum at some points in $A$, i.e. there exists $a_{\max}$ and $a_{\min}$ in $A$ so that

$$a_{\max} = \arg\max_{x \in A} f(x) \text{ and } a_{\min} = \arg\min_{x \in A} f(x).$$

Thus if we restrict attention to compact choice sets, we will know that our agents' optimization problems have solutions.

It turns out that compact sets in $\mathbb{R}^n$ are nicely characterized. A set is bounded if it can be contained in some open ball. We have the following:

**Theorem 1 (Heine-Borel)** *The set* $A \subset \mathbb{R}^n$ *is compact if and only if it is closed and bounded.*

**Exercise 11** *Observe that the "closed unit ball"* $D$ *in* $\mathbb{R}^n$*, as given by*

$$D(\mathbb{R}^n) = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$$

*is compact. Show that* $D\left(\ell^2(\mathbb{N})\right)$ *is not compact.*