Module 0: Introduction

These notes reflect my attempt to convey, in a conversational tone and at a conceptual level, some of the mathematics used in first-year economics courses. These notes are intended to be *read*, not referenced. Think of them as a narrative providing reasonably rigorous definitions of mathematical concepts together with the intuition associated with their use. There are plenty of references available that are exhaustive in their rigor and completeness, and I encourage you to use them to clear up any confusions you might have, and to fill in the many evident gaps these notes include.

1 About math

It is common for first-year students to find their coursework's technical aspects – often referred to as "math" – both mysterious and challenging.

The mystery reflects a natural misapprehension of the role mathematics plays in economic research, and alleviating this misapprehension requires a deep understanding of what mathematics really is. Unfortunately, I can't tell you what mathematics is – nobody can. I can only assure you that as your mathematical maturity develops, you will come to appreciate, and indeed be inexorably drawn to the power mathematics lends to rigorous communication.

The coursework's challenge is more about familiarity and comfort with abstraction. Most introductory math courses eschew abstraction in favor of technique. For example, you learn how to take derivatives of lots of different types of functions, but less time is spent understanding what a derivative really is. I really don't care whether you know how to differentiate inverse trig functions, etc. I need you to know what a derivative is, and why it is important.

More generally, it is abstraction that lends mathematics its power. By abstracting away from all non-essential assumptions, the reasons for the validity of a given statement are clarified. Further, the abstraction itself engenders application across a broad range of environments of scientific inquiry.

2 About mathematics in economics

Economics is an empirical science. Economists observe correlations in the data, form theories to impart causation to the correlations they witness, and confront their theories with the data to test validity. If, after a long process of confrontation and revision, a given theory appears robust then economists may be willing to use this theory to conduct policy analysis and even provide policy prescription.

The above, loose description of the scientific process as it is applied in economics, makes no mention of mathematics; and indeed, with the exception of statistical techniques needed to confront data, no mathematics is required. On the other hand, the key step of *forming theories* may be greatly aided by appealing to the structure of mathematical inquiry. I'll be more specific. A theory is, at its core, a set of assumptions together with the necessary implications of these assumptions. The development and communication of a theory need not be anchored in mathematics: it is perfectly reasonable, and still common among some subfields, to use a narrative approach to lay out a theory intended to explain behavior witnessed in the data. The great advantage of this approach is that the narrative can be developed to allow for realistic modeling assumptions – we can tell stories that appear to well-capture the myriad nuances of a particular economic environment. However, the narrative approach is challenged by the imprecision of language; and, unless great care is taken, there may be confusion about exactly what assumptions are being made, and contest about what implications necessarily follow.

The mathematical approach to developing a theory (usually called a model) is to lay out precise assumptions and use accepted logic to deduce necessary implications. Of course, this is exactly how mathematical inquiry proceeds, which is why much of economic theory looks like, and really is, applied math. The advantage of this approach is that the confusion and contest are eliminated: unless an error has been made, the meaning of the assumptions and validity implications cannot be debated. Instead, any debate about the theory is confined to the *appropriateness* of the assumptions – indeed, if you don't like the implications, the only redress is modification of the assumptions. This advantage cannot be overstated: it provides a highly structured environment for careful debate and measured progress. However, the drawback is also evident: tractable theories often require Herculean assumptions, and the technical progress needed to weaken the assumptions, and thereby surmount their impediment to realistic models, is slow in coming.

So there's a trade-off between the narrative and mathematical approaches to developing models, which is why both methods, as well as all conceivable hybrids of them, remain prominent in economics.

3 About this course

You will need to understand (or, at least, come to understand) the various concepts identified in the syllabus, as well as the technical details of their application. I need to emphasize four important points:

- New concepts in mathematics are complete mysteries until you understand them, and then they are trivial. Upon reflection, this is not terribly surprising: every result in mathematics is true by definition; understanding why a given result is true requires making numerous non-obvious connections, but once the connections have been made, the result is no longer difficult to understand. It's very easy to solve a maze one someone has marked the path.

- My lectures and lecture notes, as well as the reference text, are intended to aid the development of your conceptual understanding; however, simply reading the notes may not – indeed will very likely not – be enough for you to achieve conceptual competence. You have to create your own conceptual understanding – no one can do it for you – and it takes work. It's a lot like exercise: if you aren't sweating and uncomfortable you're not doing it right.
- The development of your ability to apply the concepts covered in this course to various problems is economics is entirely up to you, and *only* comes through practice. You will be given ample opportunity to practice via the homework problems. Don't become complacent, and if you work in groups, don't trick yourself into believing you understand the problem because the group completed the problem. Working in groups is fine – indeed, encouraged. But you should write up all of the details of all of the problems yourself.
- Many of the important results are contained in exercises. I will work through some of the exercises in class, but my point is this: don't skip the exercises.

4 The prerequisites

You are assumed to be familiar with the concepts of *set*, *Cartesian product* and *function*, as well as the particular sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

4.1 $\mathbb{R}$

The important property of $\mathbb{R}$ is that it is *complete*. In fact, $\mathbb{C}$ is also complete, but the concept of *completeness* needs to be generalized to make this connection – we'll get to that. First we need to define the *supremum* of a set. If $A \subset \mathbb{R}$ then the supremum of A, *if it exists*, is defined to be the smallest real number that is greater-than-or-equal-to every element of A. The supremum of A is denoted $\sup A$.

I won't define completeness formally here – I'll wait for the more general definition. By saying $\mathbb{R}$ is complete, we mean that if $A \subset \mathbb{R}$ is bounded above (i.e. there is a real number greater-than-or-equal-to every element of A) then its supremum exists. This can be taken either as a defining feature of the real number field or as an implication of the construction of the real numbers from $\mathbb{Q}$.

Exercise 1 *Show that $\mathbb{Q}$ is not complete.*

Exercise 2 *Show that the supremum is unique.*

Exercise 3 *Define infimum. Show that if $A \subset \mathbb{R}$ is bounded below then the infimum exists.*

4.2 $\mathbb{C}$

The important property of $\mathbb{C}$ is that it is *algebraically closed*. This means that every non-constant polynomial has a root in $\mathbb{C}$ (this is a statement of the fundamental theorem of algebra). Polynomials are among the most important functions in mathematics, and a central reason for their importance is that they are arithmetic in nature: their evaluation only requires addition and multiplication. In particular, a computer can very easily and efficiently evaluate any polynomial. Since $\mathbb{C}$ is algebraically closed, we know that every polynomial equation has a solution. In fact, more can be said, as demonstrated in the next exercise.

Exercise 4 *Show that if p is a complex polynomial of degree n then p has* exactly *n roots in $\mathbb{C}$. (Ok, you may need a hint for this one. First, remember that a degree n polynomial looks like $p(x) = \sum_{k=0}^{n} a_k x^k$ with $a_n \neq 0$. Next, show that if f and g are polynomials, and $\deg(f) < \deg(g)$ then there is are polynomials h and r with $\deg(h) = \deg(g) - \deg(f)$ and $\deg(r) < \deg(f)$ so that $g(x) = f(x)h(x) + r(x)$. To show this, I would use polynomial division and induction on the degree of g. Then use this result to show that if z is a root of p then $p(x) = (x - z)q(x)$ with $\deg(q) = \deg(p) - 1$, and apply induction again.)*

4.3 $\sim$

A final concept is useful to introduce at this stage: *equivalence relation* – you will study this concept in detail in micro theory. Given a set X, a *relation* is a subset R of $X \times X$. If $(x, y) \in R$ we will write $x \sim y$; we refer to $\sim$ as the relation, and say "x is related to y". The relation $\sim$ is an equivalence relation provided it has the following properties:

- Reflexivity: $x \sim x$
- Symmetry: $x \sim y \implies y \sim x$
- Transitivity: $x \sim y$ and $y \sim z$ implies $x \sim z$.

The collection of all elements related to a given element is called an *equivalence class*.

Now, recall that a *partition* of a set X refers to decomposing X into a collection of non-intersecting subsets; more formally it is a collection $\{X_\lambda\}_{\lambda\in\Lambda}$ of subsets of X, where Λ is a possibly infinite index set, such that

$$X = \cup_{\lambda\in\Lambda} X_\lambda \text{ and } \lambda_1 \neq \lambda_2 \implies X_{\lambda_1} \cap X_{\lambda_2} = \varnothing.$$

The relationship between a partition and an equivalence relation is provided in the following exercises.

Exercise 5 *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a partition of X. For $x, y \in X$ define $x \sim y$ if and only if there exists $\lambda \in \Lambda$ such that $x, y \in X_\lambda$. Show that $\sim$ is an equivalence relation.*

Exercise 6 *Let $\sim$ be an equivalence relation on X. Show there is a unique partition $\{X_\lambda\}_{\lambda\in\Lambda}$ of X so that $x \sim y$ if and only if there exists $\lambda \in \Lambda$ such that $x, y \in X_\lambda$.*

Equivalence relations and their associated partitions play important roles in economics: indifference curves and iso-cost curves serve as simple examples. There is, conceptually, a much more general principle here, though: an equivalence relation that results from some type of analysis conveys lost information: from the context of those who can only "observe" the partition, two points that are in the same equivalence class are indistinguishable – whatever information that once identified them has been lost.