Math Camp Discussion 8

1 Eigendecomposition of a Matrix

Consider the following matrix, $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. Find the matrices Q and Λ such that Λ is a diagonal matrix whose diagonal elements are the eigenvalues of A, and such that: $A = Q\Lambda Q^{-1}$

$$\underbrace{\det(A - \lambda I)}_{=0} = \det\left(\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix} \right)$$

$$= (1-\lambda)(4-\lambda) + 2 = 0$$

$$= \lambda^2 - 5\lambda + 6 = 0$$

$$\underline{(\lambda - 3)(\lambda - 2) = 0}$$

$\lambda_1 = 2$

$A\vec{v}_1 = 2\vec{v}_1$

$A\vec{v}_1 = 2I\vec{v}_1$

$(A - 2I)\vec{v}_1 = 0$, $\vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$

$\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-a + 2b = 0$

$a = 2b$

$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\lambda_1 = 2$

$\lambda_2 = 3$, $\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$A\vec{v}_2 = 3\vec{v}_2$, $Q = [\vec{v}_1 \ \vec{v}_2]$

$(A - 3I)\vec{v}_2 = 0$

$\begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-a + b = 0$

$a = b$

$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda_2 = 3$

$Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $Q^{-1} = \frac{1}{dc + Q} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

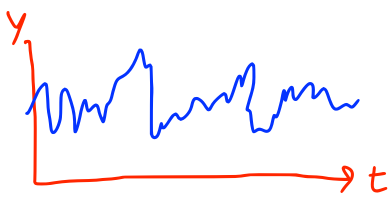
$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $\boxed{Q\Lambda Q^{-1} = A}$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

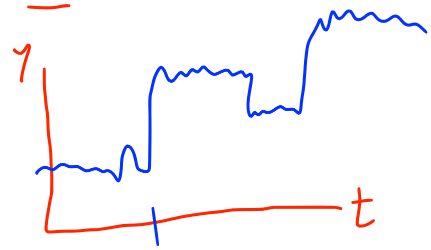
2 Vector Autoregression Eigenvalues

Prove that a vector autoregression (VAR) is stationary when the absolute value of each eigenvalue of the coefficient matrix is less than 1.

VAR: $y_t = \begin{bmatrix} y_{1t} \\ y_{2t} \\ \vdots \\ y_{nt} \end{bmatrix}$ $\quad y_t = \underset{=0}{c} + A\, y_{t-1} + \varepsilon_t$ white noise $\quad E\,\varepsilon_t = 0$

$$y_1 = A\, y_0 + \varepsilon_1$$
$$y_2 = A\, y_1 + \varepsilon_2 = A\,(A\, y_0 + \varepsilon_1) + \varepsilon_2 = A^2 y_0 + A\varepsilon_1 + \varepsilon_2$$
$$\vdots$$
$$y_s = A^s y_0 + A^{s-1}\underline{\varepsilon_1} + A^{s-2}\underline{\varepsilon_2} + \dots + \underline{\varepsilon_s}$$





$$A = Q \Lambda Q^{-1} \qquad A^s = Q \Lambda^s Q^{-1}$$

$$s \to \infty, \quad \Lambda^s \to \begin{bmatrix} \lambda_1^s & 0 & & 0 \\ 0 & \lambda_2^s & & 0 \\ & & \ddots & \\ 0 & 0 & & \lambda_n^s \end{bmatrix} \to 0_{n \times n} \qquad |\lambda_j| < 1 \quad \forall j = 1 \dots n$$