Math Camp

Module #1 Vector spaces and linear maps

Part I: enacting peace

“The introduction of numbers as coordinates is an act of violence.” *H. Weyl*

Why is there math in economics?

There's not.

Definition. A *theory* is a set of assumptions together with the necessary implications.

Two approaches to forming theories

1. Construct a narrative to explain behavior witnessed in data

   - Pro: language allows for nuance and therefore realistic modeling assumptions

   - Con: language is imprecise so assumptions and their implications can be unclear

About math in economics (continued)

2. Use math to lay out precise assumptions and deduce necessary implications

- Pro: meaning of assumptions and what implications follow cannot be debated; debate is limited to appropriateness of assumptions

- Con: tractable assumptions are necessarily restrictive, limiting reach of theory

About this course

Four important points

- New concepts in mathematics are completely mysterious until they are trivial

- Simply reading the lecture slides/text may not be enough to understand the concepts

- Your ability to apply course concepts only comes through self practice

- Don’t skip the exercises

Prereqs

- What about $\mathbb{R}$?
  - $\mathbb{R}$ is the completion of the real numbers
  - Let $A \subset \mathbb{R}$. The *supremum* of A, denoted $\sup A$ is the least upper bound of A (may be infinite).
  - If $A \subset \mathbb{R}$ is bounded above then $\sup A \in \mathbb{R}$. (!!!!!)

- What about $\mathbb{C}$?
  - $\mathbb{C}$ is algebraically closed: every non-constant polynomial has a root in $\mathbb{C}$
  - Every polynomial equation has a solution in $\mathbb{C}$

Prereqs

What is it about $\mathbb{R}$?

- Why don't we use complex numbers as our go-to?
- I don't know.

$\sim$

Definition. A *relation* on a set X is a subset R of $X \times X$

- if $(x, y) \in R$ then we write $x \sim y$
- R is an equivalence relation if it has the following properties
  - reflexivity $x \sim x$
  - symmetry $x \sim y \Rightarrow y \sim x$
  - transitivity $x \sim y$ and $y \sim z$ implies $x \sim z$
- the collection of all elements related to a given element is called an equivalence class

The Prerequisites

Definition. A *partition* of a set X is a collection of non-intersecting subsets whose union is X.

More formally, $\{X_\lambda\}_{\lambda \in \Lambda}$ is a partition of X provided

- $X_\lambda \subset X$ for all $\lambda \in \Lambda$
- $X = \cup_{\lambda \in \Lambda} X_\lambda$
- $\lambda_1 \neq \lambda_2 \Rightarrow X_{\lambda_1} \cap X_{\lambda_2} = \emptyset$

Linear spaces and linear maps

Why vector spaces?

Economics is a multivariate affair...

- trade offs only occur when there are two or more outcomes
- linear (or vector) spaces are the simplest environment capable of handling multi-dimensional problems
- start abstract: no reference yet to bases, dimension or coordinates

Linear spaces and linear maps

Vector spaces and direct sums

Definition A *vector space* is a set V that is closed under addition, and under scalar multiplication by elements of $\mathbb{R}$

- they are related by the distributed property:

$$\alpha \in \mathbb{R}, v, w \in V \Rightarrow \alpha(v+w) = \alpha v + \alpha w$$

- V contains a special element:

$$v \in V \implies v + 0 = v$$

- More generally, could be any field.

Linear spaces and linear maps

Vector spaces and direct sums

- If V and W are vector spaces then $V \oplus W$ is the Cartesian product of V and W such that
  - $x \in V \oplus W$ implies $x = (v_x, w_x)$, with $v_x \in V, w_x \in W$
  - $x + y = (v_x + w_x, v_y + w_y)$
- Q: what is $\mathbb{R} \oplus \mathbb{R}$?

Linear spaces and linear maps

Spans and subsets

Definition. A *subspace* W of V is a subset of V that is, itself, a vector space.
Definition. If $A \subset V$ then the *span* of A is the set of all finite linear combinations of elements of A:

$$\text{span}(A) = \left\{ \sum_{k=1}^{m} \alpha_k a_k \text{ such that } m \in \mathbb{N}, \alpha_k \in \mathbb{R}, a_k \in A \right\}$$

- span(A) is the smallest subspace of V containing A.

Linear spaces and linear maps

Linear Independence and bases

Definition. $A \subset V$ is *linearly independent* if no element of A can be written as a finite linear combination of other elements of A.

More formally, A is linearly independent if whenever

$$\{a_1, ..., a_n\} \subset A \text{ and } \sum_{k=1}^{n} \alpha_k a_k = 0$$

it follows that $\alpha_k = 0$ for $k = 1, \ldots n$.

Definition. A *basis* of V is a linearly independent set $B \subset V$ that spans V.

Linear spaces and linear maps

Linear Independence and bases

Theorem (Fundamental theorem of linear algebra)

if $A = \{a_1, ..., a_n\} \subset V$ *is a basis for* V *and* $B = \{b_1, ..., b_m\} \subset V$ *is linearly independent then* $m \leq n$

Linear spaces and linear maps

Dimension

If $B \subset V$ is a basis then $\dim V = |B|$.

Let $\dim V = n$. Then

- If $B \subset V$ is linearly independent then B is a basis for *span*(B) and $\dim span(B) = |B|$.

- If $B \subset V$ is linearly independent and $|B| = m < n$ then there exists $C \subset V$ so that $B \cup C$ is a basis for V.

- If W is a subspace of V then $\dim(W) \leq \dim(V)$.

Linear spaces and linear maps

Linear functions

Category Theory: everything is objects and arrows

In the category of linear spaces,

- objects are vector spaces
- arrows are linear maps

Definition. If V and W are vector spaces then a map $f : V \to W$ is *linear* provided

$$f(\alpha v_1 + \beta v_2) = \alpha f(v_1) + \beta f(v_2)$$

Linear spaces and linear maps

Linear functions

Definition. A linear map is an *isomorphism* if it is invertible.

If there is an isomorphism between the vector spaces V and W then we say that V and W are *isomorphic*, written $V \cong W$.

Linear functionals

Definition. A *linear functional* α on V is a linear map $\alpha : V \to \mathbb{R}$.

- The set of linear functionals, V^*, called the *dual space*
- V^* is a vector space
- If $\dim V < \infty$ the $V^* \cong V$.

Linear spaces and linear maps

Linear functions

Let $f : V \to W$ be a linear map.

Definition. The *kernel (or nullspace)* of f is the collection of vectors that f sends to zero:

$$ker(f) = \{v \in V : f(v) = 0\} \subset W$$

- $ker(f)$ is a subspace of V
- The *nullity* of f is the dimension of its kernel

Definition. The *range* of f is the image of V in W under f:

$$f(V) = \{w \in W : \exists v \in V \text{ with } f(v) = w\} \subset W$$

- The range of f is a subspace of W
- The *rank* of f is the dimension of the range

Linear spaces and linear maps

Linear Functions

Rank Nullity Theorem: if V is finite dimensional and $f : V \to W$ is a linear map then the rank of f plus the nullity of f equals the dimension of V

$$\dim ker(f) + \dim f(V) = \dim V$$

Linear spaces and linear maps

Information and nullity

Let $f : V \to W$ b a linear map. Define the following relation on V:

$$v_1 \sim v_2 \iff f(v_1) = f(v_2) \text{ i.e. } v_1 - v_2 \in ker(f)$$

- $\sim$ is an equivalence relation
- $\sim$ measures the information lost by f

Linear spaces and linear maps

Linear extensions

Let V and W be vector spaces and $B \subset V$ a basis. Let $\phi : B \to W$ be *any* function. Then there exists unique linear map $\Phi : V \to W$ such that the following diagram commutes:

$$\begin{tikzcd}
V \arrow[r, "\Phi"] & W \\
B \arrow[u, hook] \arrow[ur, "\phi"']
\end{tikzcd}$$

Let $B = \{b_1, \ldots, b_n\}$ be a basis for V and $\{w_1, \ldots, w_n\} \subset W$. Define a linear map Φ from V to W by sending b_i to $\phi(b_i) = w_i$, and *extending linearly:*

$$v = \sum_{k=1}^{n} \beta_k b_k \to \sum_{k=1}^{n} \beta_k \phi(b_k) = \sum_{k=1}^{n} \beta_k w_k \equiv \Phi(v).$$

Linear spaces and linear maps

Linear extensions

Let V and W be vector spaces and $B \subset V$ a basis. Let $\phi : B \to W$ be *any* function. Then there exists unique linear map $\Phi : V \to W$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\Phi"] & W \\ B \arrow[u, hook] \arrow[ur, "\phi"'] & \end{tikzcd}$$

Note that the behavior of linear map is entirely characterized by its behavior on a basis

Linear spaces and linear maps

Characterization of finite dimensions vector spaces

Theorem. $\dim(V) = n \implies V \cong \mathbb{R}^n$

Linear spaces and linear maps

Dynamics and Decompositions

Let $\dim V = n$ and $f : V \to V$ be linear. Given $v_0 \in V$, define $v_{t+1} = f(v_t)$.

- The *dynamic* f traces a path/orbit in the vector space V.
- The orbits of f partition V.
- The subspace $W \subset V$ is *invariant* (under the action of f) provide $f(W) \subset W$.

Theorem (Schur Decomposition) Let $\dim V = n$ and $f : V \to V$ be linear. Then there is a collection of invariant subspaces $\{V_k\}_{k=1}^{n}$ such that

$$\dim(V_k) = k \text{ and } V_k \subset V_{k+1}$$

Linear spaces and linear maps

Dynamics and decompositions

Definition. The scalar $\lambda \in \mathbb{R}$ of a linear map f is an *eigenvalue* provided there exists $v \in V$ such that $f(v) = \lambda v$

- v is called an associated *eigenvector*.
- if v is an eigenvector associated to λ then f scales v by λ.
- the collection of all eigenvectors associated with an eigenvalue is called the *eigenspace*.

Linear spaces and linear maps

Eigenspace decomposition (greatest thing ever!)

The set up:

- $\dim V = m$ and $f : V \to V$ linear
- $V(\lambda)$is the eigenspace associated with λ
- Assume the eigenvalues are distinct.

Then there is an isomorphism $\phi : V \to \oplus_{i=1}^m V(\lambda_i)$ such that

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "\phi"'] & V \\
\bigoplus_{i=1}^m V(\lambda_i) \arrow[r, "\hat{f}"'] & \bigoplus_{i=1}^m V(\lambda_i) \arrow[u, "\phi^{-1}"']
\end{tikzcd}$$