Math Camp

Module #1 Vector spaces and linear maps

Part I: enacting peace

"The introduction of numbers as coordinates is an act of violence." *H. Weyl*

$a_1 \subset X$

$\exists\ x_1 \in X$

$x_1 \notin a_1$

$a_2 \subseteq X$

Linear spaces and linear maps

Why vector spaces?

Economics is a multivariate affair...

- trade offs only occur when there are two or more outcomes
- linear (or vector) spaces are the simplest environment capable of handling multi-dimensional problems
- start abstract: no reference yet to bases, dimension or coordinates

Linear spaces and linear maps

Vector spaces and direct sums

$v, w \in V$. $v + w \in V$

Definition A *vector space* is a set V that is closed under addition, and under scalar multiplication by elements of $\mathbb{R}$

$v \in V$, $\alpha \in \mathbb{R}$

$\alpha \cdot v \in V$

- they are related by the distributed property:

$$\alpha \in \mathbb{R}, v, w \in V \Rightarrow \alpha(v+w) = \alpha v + \alpha w$$

- V contains a special element:

$\alpha \in \mathbb{R}$ $\alpha \cdot v = v$

$$v \in V \implies v + 0 = v$$

$\alpha = 1$

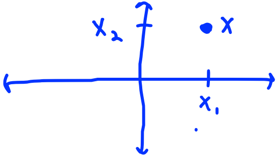
- More generally, could be any field.

Linear spaces and linear maps

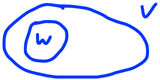
Vector spaces and direct sums

- If V and W are vector spaces then $V \oplus W$ is the Cartesian product of V and W such that
  - $x \in V \oplus W$ implies $x = (v_x, w_x)$, with $v_x \in V, w_x \in W$
  - $x + y = \cancel{(v_x + w_x, v_y + w_y)}$ $= (v_x + v_y, w_x + w_y)$
- Q: what is $\mathbb{R} \oplus \mathbb{R}$?

$x = (x_1, x_2)$



Linear spaces and linear maps



Spans and subsets

Definition. A *subspace* W of V is a subset of V that is, itself, a vector space.

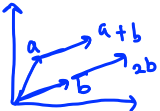
Definition. If $A \subset V$ then the *span* of A is the set of all finite linear combinations of elements of A:

$$\text{span}(A) = \left\{ \sum_{k=1}^{m} \alpha_k a_k \text{ such that } m \in \mathbb{N}, \alpha_k \in \mathbb{R}, a_k \in A \right\}$$

- span(A) is the smallest subspace of V containing A.

$$A \subset V \qquad \text{span}(A) = \alpha \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \beta \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$A = \left\{ \underbrace{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{a}, \underbrace{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}_{b} \right\}$$



Linear spaces and linear maps

$A = \{a_1, a_2, a_3\}$

Linear Independence and bases

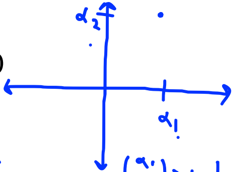
$\alpha_1 a_1 + \alpha_2 a_2 = a_3$

Definition. $A \subset V$ is *linearly independent* if no element of A can be written as a finite linear combination of other elements of A.

More formally, A is linearly independent if whenever

$\alpha \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\{a_1, ..., a_n\} \subset A \text{ and } \sum_{k=1}^{n} \alpha_k a_k = 0$$



it follows that $\alpha_k = 0$ for $k = 1, \ldots n$.

$B = \left\{ \underset{b_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \underset{b_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right\}$

$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \alpha_1 \cdot b_1 + \alpha_2 \cdot b_2$

Definition. A *basis* of V is a linearly independent set $B \subset V$ that spans V.

$B = \{b_1, \ldots b_n\}$ $\quad \exists \alpha_1, \ldots, \alpha_n \in \mathbb{R}$ $\quad \sum_{k=1}^{n} \alpha_k \cdot b_k = w$

$w \in V$

Linear spaces and linear maps

Linear Independence and bases

Theorem (Fundamental theorem of linear algebra)

if $A = \{a_1, \ldots, a_n\} \subset V$ *is a basis for* V *and* $B = \{b_1, \ldots, b_m\} \subset V$ *is linearly independent then* $m \le n$

$V = \mathbb{R}^2$ $\quad A = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ $\quad n = 2$

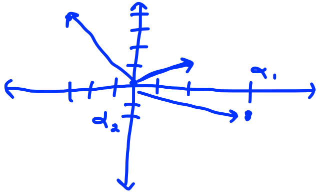
$B = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$ $\quad m = 1$,

$B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix}, \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \right\}$ $\quad \beta \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$

$B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix} \right\}$ $\quad m = 2$

$2\beta_1 - 3\gamma_1 = \alpha_1$

$\beta_2 + 4\gamma_2 = \alpha_2$



Linear spaces and linear maps

Dimension

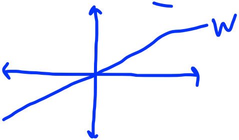
If $B \subset V$ is a basis then $\dim V = |B|$.

Let $\dim V = n$. Then

- If $B \subset V$ is linearly independent then B is a basis for *span*(B) and $\dim span(B) = |B|$.
- If $B \subset V$ is linearly independent and $|B| = m < n$ then there exists $C \subset V$ so that $B \cup C$ is a basis for V.
- If W is a subspace of V then $\dim(W) \leq \dim(V)$.

$B = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$ $\quad |B| = 1$

$v = \mathbb{R}^2$



$C = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

$B \cup C = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

Linear spaces and linear maps

Linear functions

Category Theory: everything is objects and arrows

In the category of linear spaces,

$F : V \to W$

- objects are vector spaces
- arrows are linear maps

Definition. If V and W are vector spaces then a map $f : V \to W$ is *linear* provided

$$f(\alpha v_1 + \beta v_2) = \alpha f(v_1) + \beta f(v_2)$$

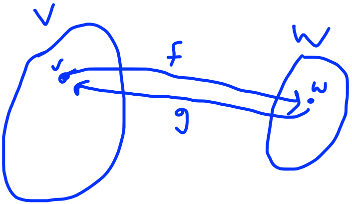
Linear spaces and linear maps

$f : V \to W$ $g(f(v)) = v$
$g : W \to V$

Linear functions

Definition. A linear map is an *isomorphism* if it is invertible.

If there is an isomorphism between the vector spaces V and W then we say that V and W are *isomorphic*, written $\boxed{V \cong W}$.



Linear spaces and linear maps

Linear functionals

Definition. A *linear functional* α on V is a linear map $\alpha : V \to \mathbb{R}$.

- The set of linear functionals, V^*, called the *dual space*
- V^* is a vector space
- If $\dim V < \infty$ the $V^* \cong V$.

$$p_1 x_1 + p_2 x_2 = I$$

$$(p_1 \; p_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = I$$

$\mathbb{R}^2$

$$V^* = \{(\alpha_1, \alpha_2),\ \alpha_1, \alpha_2 \in \mathbb{R}\}$$

$$f : V \to V^*$$

$$g : V^* \to V$$

Linear spaces and linear maps

Linear functions

Let $f : V \to W$ be a linear map.

Definition. The *kernel (or nullspace)* of f is the collection of vectors that f sends to zero:

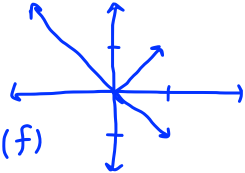
$$ker(f) = \{v \in V : f(v) = 0\} \subset \cancel{W}\; V$$

- $ker(f)$ is a subspace of V
- The *nullity* of f is the dimension of its kernel

$$(1, 1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$x_1 = -x_2$$

$$\left\{\alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \alpha \in \mathbb{R}\right\} = ker(f)$$



$$f = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

Linear spaces and linear maps

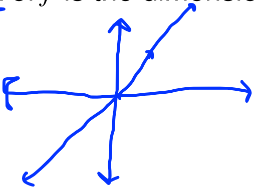
Linear functions

Definition. The *range* of f is the image of V in W under f:

$$f(V) = \{w \in W : \exists v \in V \text{ with } f(v) = w\} \subset W$$

- The range of f is a subspace of W
- The *rank* of f is the dimension of the range

$f = \{\begin{pmatrix}1\\1\end{pmatrix}\}$



$f(V) = \{\alpha \begin{pmatrix}1\\1\end{pmatrix}, \alpha \in \mathbb{R}\}$

$f: \mathbb{R}^2 \to \mathbb{R}$

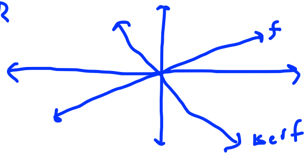
Linear spaces and linear maps

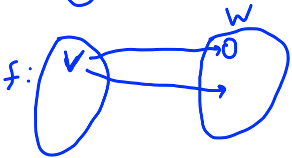
Linear Functions

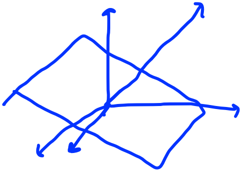
Rank Nullity Theorem: if V is finite dimensional and $f : V \to W$ is a linear map then the rank of f plus the nullity of f equals the dimension of V

$$\dim ker(f) + \dim f(V) = \dim V$$

$f : \mathbb{R}^2 \to \mathbb{R}$







Linear spaces and linear maps

Information and nullity

Let $f : V \to W$ be a linear map. Define the following relation on V:

$$v_1 \sim v_2 \iff f(v_1) = f(v_2) \text{ i.e. } v_1 - v_2 \in ker(f)$$

- $\sim$ is an equivalence relation
- $\sim$ measures the information lost by f

Linear spaces and linear maps

Linear extensions

Let V and W be vector spaces and $B \subset V$ a basis. Let $\phi : B \to W$ be *any* function. Then there exists unique linear map $\Phi : V \to W$ such that the following diagram commutes:

$$
\begin{tikzcd}
V \arrow[r, "\Phi"] & W \\
B \arrow[u, hook] \arrow[ur, "\phi"'] &
\end{tikzcd}
$$

Let $B = \{b_1, \ldots, b_n\}$ be a basis for V and $\{w_1, \ldots, w_n\} \subset W$. Define a linear map Φ from V to W by sending b_i to $\phi(b_i) = w_i$, and *extending linearly:*

$$v = \sum_{k=1}^{n} \beta_k b_k \to \sum_{k=1}^{n} \beta_k \phi(b_k) = \sum_{k=1}^{n} \beta_k w_k \equiv \Phi(v).$$

Linear spaces and linear maps

Linear extensions

Let V and W be vector spaces and $B \subset V$ a basis. Let $\phi : B \to W$ be *any* function. Then there exists unique linear map $\Phi : V \to W$ such that the following diagram commutes:

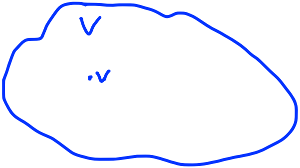
$$\begin{tikzcd} V \arrow[r, "\Phi"] & W \\ B \arrow[u, hook] \arrow[ur, "\phi"'] & \end{tikzcd}$$

Note that the behavior of linear map is entirely characterized by its behavior on a basis

Linear spaces and linear maps

Characterization of finite dimensions vector spaces

Theorem. $\dim(V) = n \implies \underline{V \cong \mathbb{R}^n}$



$\mathbb{R}^n: \{e_1, \dots, e_n\}$

$$e_i = \begin{array}{c} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow i\text{th} \\ \\ \\ \end{array}$$

$\mathcal{B} = \{b_1, \dots, b_2\}$

$$b_1 = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = a_1 e_1 + \dots + a_n e_n$$

Linear spaces and linear maps

Dynamics and Decompositions

Let $\dim V = n$ and $f : V \to V$ be linear. Given $v_0 \in V$, define $v_{t+1} = f(v_t)$.

- The *dynamic* f traces a path/orbit in the vector space V.
- The orbits of f partition V.
- The subspace $W \subset V$ is *invariant* (under the action of f) provide $f(W) \subset W$.

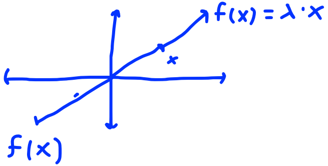
Linear spaces and linear maps

Dynamics and decompositions

Theorem (Schur Decomposition) Let $\dim V = n$ and $f : V \to V$ be linear. Then there is a collection of invariant subspaces $\{V_k\}_{k=1}^{n}$ such that

$$\dim(V_k) = k \text{ and } V_k \subset V_{k+1}$$

Linear spaces and linear maps



Dynamics and decompositions

Definition. The scalar $\lambda \in \mathbb{R}$ of a linear map f is an *eigenvalue* provided there exists $v \in V$ such that $f(v) = \lambda v$

- v is called an associated *eigenvector*.
- if v is an eigenvector associated to λ then f scales v by λ.
- the collection of all eigenvectors associated with an eigenvalue is called the *eigenspace*.

Linear spaces and linear maps

Eigenspace decomposition (greatest thing ever!)

The set up:

- $\dim V = m$ and $f : V \to V$ linear
- $V(\lambda)$is the eigenspace associated with λ
- Assume the eigenvalues are distinct.

Then there is an isomorphism $\phi : V \to \oplus_{i=1}^{m} V(\lambda_i)$ such that

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "\phi"] & V \\
\bigoplus_{i=1}^{m} V(\lambda_i) \arrow[r, "\hat{f}"] & \bigoplus_{i=1}^{m} V(\lambda_i) \arrow[u, "\phi^{-1}"]
\end{tikzcd}$$