Module 4: Functions on $\mathbb{R}$

This module concerns real-valued functions of one variable, i.e. functions $f : \mathbb{R}^n \to \mathbb{R}$. As emphasized in a previous chapter, economics is inherently a multivariate affair; however, much of the analysis is conducted either by reducing a multivariate problem into a decoupled collection of univariate problems, or by extending univariate results to multivariate settings. The point is this: most of the main ideas and techniques are already present in the univariate world – a world which is attractive for many reasons, including its admission of graphical approaches.

1 Continuity

A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous at* $x \in \mathbb{R}$ provided that whenever $x_n \to x$ it follows that $f(x_n) \to f(x)$. The domain $\mathbb{R}$ in this definition could be replaced by any subset of $\mathbb{R}$, though usually the domain is restricted to be open or closed. Observe that this definition makes perfect sense also for functions between two metric spaces, and indeed this is what it means for a function between two metric spaces to be continuous at a point. It's a good exercise to show that this definition comports with the one you perhaps remember:

Exercise 1 *Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x \in \mathbb{R}$ provided that for any $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$|y - x| < \delta \implies |f(y) - f(x)| < \varepsilon.$$

Importantly, continuity is a *local* notion: the definition speaks of continuity at a *point*. We say that the function is *continuous* on its domain (in the above case the domain is $\mathbb{R}$) provided that it is continuous at every point in the domain.

Continuity preserves some notion of nearness. Intuitively, if y is near x then $f(y)$ is near $f(x)$. However, notice, from exercise 1, that a more precise intuition is as follows: you can make $f(y)$ near $f(x)$ by choosing y sufficiently near x.[1] Continuity is useful in economic analysis for many reasons – principal among them the intermediate value theorem guaranteeing solutions to equations – but, at perhaps the broadest level, the benefit is this: If f is continuous at x then the behavior of f near x can't be too crazy.

The following proposition shows that continuity is preserved by many common point-wise transformations.

Proposition 1 *Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous. Then the following functions are continuous:*

[1] How near you need to choose y may depend on the point x; the stronger notion of *uniform continuity* eliminates this dependence.

1. *$f+g$*
2. *$f \cdot g$*
3. *$f \circ g$*

Exercise 2 *Are polynomials continuous? Answer using the following steps.*

1. *Let $\alpha \in \mathbb{R}$ and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \alpha$. Show that f is continuous.*
2. *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x$. Show that f is continuous.*
3. *Use parts 1 and 2 of this problem, together with the above proposition to show that polynomials are continuous.*

The most important result on continuous functions from $\mathbb{R}$ to $\mathbb{R}$ is the *intermediate value theorem*.

Theorem 1 (Intermediate value theorem) *Let $f : [a,b] \to \mathbb{R}$ be continuous. If $f(a) > 0 > f(b)$ or if $f(a) < 0 < f(b)$ then there exists $c \in (a,b)$ such that $f(c) = 0$.*

The proof of this theorem was sketched in a previous chapter. The importance of the theorem concerns solutions to equations. It is an existence theorem. It provides precise conditions under which an equation is guaranteed to have a solution.

Existence theorems are wonderfully powerful as they often require only weak assumptions; however there is a trade off: most important existence theorems do not guarantee uniqueness – there may be many zeros of f in (a,b) – and, more significantly, they are not typically *constructive*: they provide no method for constructing the object whose existence is establishedby the result.

Exercise 3 *Consider a competitive market for one good. Let $q = S(p)$ and $q = D(p)$ be supply and demand curves respectively. Assume S and D are continuous functions. Under what conditions can you guarantee the existence of a market equilibrium?*

2 Differentiability

A function $f : \mathbb{R} \to \mathbb{R}$ is *differentiable at x*, provided the following limit, denoted $f'(x)$, exists:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

The domain $\mathbb{R}$ may be replaced by an open set. Observe that, like continuity, differentiability is a local notion: it is defined at a given point in the domain. A function is *differentiable* if it is differentiable at every point in the domain.

Whereas we may think of continuous functions as reasonably well-behaved – things don't get too crazy near a point of continuity – differentiable functions are *very* well-behaved; so well-behaved, in fact, that, locally, they act just like linear functions. This is the reason differentiability is so important and calculus is so useful!

To make this point more clear, notice that the definition of the derivative above means that for small values of Δx, the following approximation is pretty good:

$$f(x+\Delta x) - f(x) \approx f'(x)\Delta x. \tag{1}$$

Thus, letting $z = x + \Delta x$, we conclude that if z is near x (i.e. Δx is small) then

$$f(z) \approx f(x) + f'(x)(z-x),$$

i.e. the function f looks like the line $y = mz + b$ where $m = f'(x)$ and $b = f(x) - f'(x)x$. In particular, $f'(x)$ is the slope of the line tangent to the graph of f at x.

Another important way to intuit the local linearity of differentiable functions is to rewrite (1) as $\Delta f \approx f'(x)\Delta x$. Now recall that a linear function from $\mathbb{R}$ to $\mathbb{R}$ is exactly and only multiplication by a real number. Thus we may interpret the real number $f'(x)$ as defining a linear function $\mathbb{R} \to \mathbb{R}$. Which linear function? The linear function taking small changes in x to approximations of the corresponding small changes in f. Viewed this way – as a linear map – the derivative is often referred to as the *differential*, and written $df = f'(x)dx$, where here the dx and df are called *infinitesimals*, and loosely thought of as infinitely small: they should not be viewed as having meaning in isolation, but only when written in relation to other differentials. We think of df as the small change in f induced by the small change dx in x, and this small change in f is measured by their derivatives.

Exercise 4 *Show that a differentiable function is continuous.*

Like continuity, differentiability is preserved under many common point-wise transformations.

Proposition 2 *Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable. Then the following functions are differentiable:*

1. $f + g$
2. $f \cdot g$
3. $f \circ g$

While the definition of differentiability is somewhat daunting, it turns out that calculating derivatives is often mechanical, hence the name: the calculus.

Proposition 3 *Suppose $f,g:\mathbb{R}\to\mathbb{R}$ are differentiable. Then*

1. $h = f+g \implies h' = f'+g'$
2. $h = f\cdot g \implies h' = f'\cdot g + g'\cdot f$
3. $h = f\circ g \implies h' = (f'\circ g)\cdot g'$

Exercise 5 *How to differentiate polynomial? Answer using the following steps.*

1. *Let $\alpha\in\mathbb{R}$ and define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=\alpha$. Show that f is differentiable and $f'(x)=0$.*
2. *Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=x$. Show that f is differentiable and $f'(x)=1$.*
3. *Use parts 1 and 2 of this problem, together with the above propositions to show that polynomials are differentiable, and to compute the derivative.*

Exercise 6 *Let $f:(0,\infty)\to\mathbb{R}$ be defined by $f(x)={}^1\!/_x$. Show that $f'(x)=-{}^1\!/_{x^2}$.*

Exercise 7 *Show that if $f:\mathbb{R}\to\mathbb{R}$ is differentiable and $f'>0$ everywhere then f is invertible.*

The most important result on differentiable functions is the *mean value theorem*.

Theorem 2 (Mean value theorem) *Let U be an open set containing the interval $[a,b]$, and let $f:U\to\mathbb{R}$ be differentiable. Then there exists $c\in(a,b)$ such that*

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Graphically, this theorem is straightforward: the RHS is the slope of the line connecting the endpoints of the graph of f, i.e. the secant line. The theorem says this slope is equal to the slope of the tangent line at some point inside the interval (a,b). A more important (and more generalizable) intuition is as follows: think of f as the position of something at a given time $t\in[a,b]$, so that f' is the instantaneous velocity. Then the RHS is the average velocity over the time period a to b. The theorem says that at some point in the interval of time (a,b) the instantaneous velocity must equal the average velocity.

The principal importance for economics of the mean value theorem is that it is a special case of Taylor's theorem, so I will put off discussion of applications until the next chapter.

Exercise 8 *Suppose $f, g : [a,b] \to \mathbb{R}$. Show that if $f' = g'$ then there is a constant α such that $f(x) = g(x) + \alpha$.*

A nod to higher-order derivatives is worth making here. Note that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable then f' may be viewed as a function $\mathbb{R} \to \mathbb{R}$, and it may be differentiable. If it is, then f'' is called the *second-derivative* of f. This process may be repeated so long as the derivative exists. The notation $f^{(k)}$ is used to denote the kth derivative of f. In general we say that f is C^k (pronounced "see kay") if its derivative may be computed k-times *and* as a function, $f^{(k)}$ is continuous.

Exercise 9 *What is the geometric interpretation of f''?*

3 Integration

Whereas differentiation allows for the local analysis of a given function, integration speaks to a measure of its global behavior. Intuitively, given a function $F : [a,b] \to \mathbb{R}$, we may use differential calculus to think about small changes in f given small changes in $x \in [a,b]$, i.e. $dF = f(x)dx$, where here, for rhetorical reasons, I am using f to suggest the derivative of F. Conversely, suppose that, instead of knowing the function F, we know how it changes at each point in an interval. What can we say about the total change $F(b) - F(a)$? It is perhaps not unreasonable to expect that we may compute this total change by adding up all the little changes:

$$F(b) - F(a) = \sum dF = \sum f(x)dx. \tag{2}$$

Riemann integration provides the machinery needed to make these intuitive, if meaningless statements meaningful.

We need some terminology. A *partition* of an interval $[a,b]$ is

$$\mathscr{P}_n = \{x_0 = 1, x_1, \dots, x_n = b\}$$

with $x_i < x_{i+1}$. Note that a partition as defined here partitions the interval $[a,b]$ into a collection of intervals with endpoints x_i, x_{i+1} (these intervals may be open or closed depending on how you choose to construct the partition). Given $\mathscr{P}_n$, we say that $\hat{x} = \{\hat{x}_1, \dots, \hat{x}_n\}$ is *admissible* if $\hat{x}_i \in [x_{i-1}, x_i]$. The partition/admissible-point pair $(\mathscr{Q}_m, \hat{y})$ is a *refinement* of the pair $(\mathscr{P}_n, \hat{x})$ provided that three properties hold:

1. $m \geq n$
2. as sets $\mathscr{P}_n \subset \mathscr{Q}_m$
3. for each $i \in \{1, \dots, n\}$ there is a $j \in \{1, \dots, m\}$ so that $\hat{x}_i = \hat{y}_j$.

Intuitively, $(\mathscr{Q}_m, \hat{y})$ is a refinement if it dissects the interval more finely that $\mathscr{P}_n$, while at the same time respecting the pair $(\mathscr{P}_n, \hat{x})$'s cutoffs and admissible points.

Given f, $\mathscr{P}_n$, and admissible $\hat{x}$ we may define the associated *Riemann sum* as

$$\mathscr{R}(f, \mathscr{P}_n, \hat{x}) = \sum_{i=1}^{n} f(\hat{x}_i)\,\Delta x_i, \tag{3}$$

where $\Delta x_i = x_i - x_{i-1}$. Think of (3) as providing a formal approximation to the right-most expression in (2), i.e.

$$\sum f(x)dx \approx \sum_{i=1}^{n} f(\hat{x}_i)\,\Delta x_i.$$

This, like with derivatives, has a geometric interpretation. Given an interval $[a,b]$, and continuous function f over the interval, the Riemann Sum approximates the area between f and 0 by summing the geometric area of n partitioned rectangles – the rectangles have height $f(\hat{x}_t)$ and width Δx_i. Note that the area of individual rectangles can be negative if there exists $c \in [a,b]$ such that $f(c) < 0$. Thereby, the Riemann sum can be thought of as the area of the partitioned rectangles with height $f(\hat{x}_i) > 0$ (area above 0) *minus* the area of the partitioned rectangles with height $f(\hat{x}_i) < 0$ (area below 0).

To make this approximation precise, i.e. take a limit.

Informally, the *Riemann integral of f over $[a,b]$*, written $\int_a^b f(x)dx$, is the limit, when it is exists, of the Riemann sums (3), where the limit is taken over refinements. More formally, f is *Riemann integrable* on $[a,b]$ provided there a real number $\int_a^b f(x)dx$ such that for any $\varepsilon > 0$ there is a pair $(\mathscr{P}_n, \hat{x})$ with the property that whenever $(\mathscr{Q}_m, \hat{y})$ is a refinement of $(\mathscr{P}_n, \hat{x})$ it follows that

$$\left| \mathscr{R}(f, \mathscr{Q}_m, \hat{y}) - \int_a^b f(x)dx \right| < \varepsilon.$$

Importantly, *continuous functions are Riemann integrable.*

We conclude the following: if $F'(x) = f(x)$ and f is Riemann integrable then $F(b) - F(a) = \int_a^b f(x)dx$. This, of course, is not a proof: we have relied on the intuitive connection $dF = f'(x)dx$. The careful justification of this step is at the heart of the following theorem:

Theorem 3 (Fundamental theorem of calculus) *Let $g : [a,b] \to \mathbb{R}$ be continuous. Define $G : [a,b] \to \mathbb{R}$ by*

$$G(x) = \int_a^x g(s)ds.$$

Then $G'(x) = g(x)$.

Exercise 10 *Let $f : [a,b] \to \mathbb{R}$ be continuous. Is there a function $F : [a,b] \to \mathbb{R}$ whose derivative is given by f?*

To see how this theorem provides the answer to our original quest of finding the total change in F given all the little changes, we need the following result:

Exercise 11 *Show that if $F, G : [a,b] \to \mathbb{R}$ are continuous and differentiable on (a,b), and if $F'(x) = G'(x)$ then $F(b) - F(a) = G(b) - G(a)$.*

Now assume that we know $F' \equiv f$ is continuous on $[a,b]$. Suppose we can find $G : [a,b] \to \mathbb{R}$ such that $G'(x) = f(x)$. Then, by the FTC, together with exercise 8 we know that $G(x) = \int_a^x f(x)dx + \alpha$. Observing that

$$G(a) = \int_a^a f(x)dx + \alpha = \alpha,$$

it follows that

$$G(b) - G(a) = \int_a^b f(x)dx.$$

Finally, since $F' = f = G'$, it follows from exercise 11 that $F(b) - F(a) = G(b) - G(a)$, whence

$$F(b) - F(a) = \int_a^b f(x)dx.$$

as desired. Thus, if we are given the little changes and want to compute the total change, instead of adding up all the little changes which is a difficult limit to even contemplate, all we need to do is to find a function whose derivative gives us the little changes, and then evaluate this function at the endpoints.

The fundamental theorem is used in many ways in economics. A common application is to solve differential equations, the idea being that you may know how a variable changes over time and you want to compute its value at each point in time. The next exercise provides an example.

Exercise 12 *Suppose that you know the change in the price level at time t is given by βt, where $\beta > 0$. What is the price level in time $t = 10$? Here you should assume that the price level is capture by a differentiable function $p : \mathbb{R}_+ \to \mathbb{R}_+$.*

More generally, it is useful to think of an integral as capturing an aggregrate or an average associated to a given function. For example, if a model has a collection of consumers indexed by the unit interval then aggregate consumption is the integral of individual consumption levels over the interval.

4 Exponents and logs

The derivative of $1/n \cdot x^n$ is x^{n-1} whenever $n \neq 0$. So what about $n = 0$? Or, perhaps said differently, what function is it whose derivative is x^{-1}? That's easy to answer: use the fundamental theorem. For $x \geq 1$ define

$$\log(x) = \int_1^x \frac{1}{t} dt, \tag{4}$$

and for $x \in (0,1)$ let $\log(x) \equiv -\log(1/x)$. This is also often referred to as the *natural log* and denoted $ln(x)$, however since it is the main logarithm you will be dealing with in economics, we will refer to it as "the logarithm" and $log(x)$.

Exercise 13 *Show that* $\frac{d}{dx} \log x = 1/x$.

Note that (4) is the definition of the logarithm: it's a limit of Riemann sums, and you know what its derivative is because you know the FTC.

It can be shown that the range of log is $\mathbb{R}$. By the fundamental theorem, we know that log is a differentiable function; also, since its derivative is positive, we know that it is invertible. Let $\exp : \mathbb{R} \to (0, \infty)$ be its inverse. Let's compute the derivative of exp. We find

$$\begin{aligned} \log(\exp(x)) = x \quad &\Longrightarrow \quad \frac{d}{dx}\log(\exp(x)) = \frac{d}{dx}x \\ &\Longrightarrow \quad \frac{1}{\exp(x)} \frac{d}{dx}\exp(x) = 1 \\ &\Longrightarrow \quad \frac{d}{dx}\exp(x) = \exp(x), \end{aligned}$$

where the second implication uses the chain rule.

Exercise 14 *Show that if* $f : \mathbb{R} \to \mathbb{R}$ *is such that* $f'(x) = f(x)$ *then* $f(x) = \alpha \exp(x)$ *for some* $\alpha \in \mathbb{R}$.

Exercise 15 *Let* $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. *Here* $0! \equiv 1$. *Assume this sum converges for all x and that you are free to differentiate across an infinite sum. Compute* $f'(x)$.

The exponential function, and its inverse, the logarithm, are the most important functions in mathematics. They are very likely the only functions you know of that are not polynomials (trig functions are built via exponentials). They have the following dual properties:

1. $\log(xy) = \log(x) + \log(y)$ and $\exp(x)\exp(y) = \exp(x+y)$.
2. $\log\left(x^{-1}\right) = -\log(x)$ and $\exp(-x) = \exp(x)^{-1}$.

These properties allow us to define the more general exponential of a positive real number:

$$x^\alpha \equiv \exp(\alpha \log(x)) \text{ for } x > 0, \alpha \in \mathbb{R}.$$

The following exercise has you verify that this function behaves like you think it should.

Exercise 16 *Show the following:*

1. $x^{\alpha}x^{\beta} = x^{\alpha+\beta}$
2. $x^{-\alpha} = \frac{1}{x^{\alpha}}$
3. $\frac{d}{dx}x^{\alpha} = \alpha x^{\alpha-1}$
4. $(x^{\alpha})^{\beta} = x^{\alpha\beta}$
5. $\log(x^r) = r\log(x)$

The uses of logs and exponents in economics are myriad. Here are two examples.

Exercise 17 *Suppose gdp y grows at a constant rate $r > 0$, i.e. y satisfies $\dot{y} = ry$. Solve for y as a function of t. More generally, the exponential function can be defined as the solution to this differential equation.*

Exercise 18 *If $f : \mathbb{R} \to \mathbb{R}$ and $y = f(x)$ define the* elasticity *of y with respect to x as $\varepsilon_{yx} = \frac{dy}{dx}\frac{x}{y}$. Intuitively the elasticity is the percent change in y given a percent change in x; the advantage elasticity is that it's a unit-free measure of change. Let $f(x) = x^{\alpha}$ for some $\alpha \in \mathbb{R}$.*

1. *Compute ε_{yx}.*
2. *Let $\hat{y} = \log y$ and $\hat{x} = \log x$. Compute $\frac{d\hat{y}}{d\hat{x}}$.*

Thus by taking log transforms of the data and then performing a regression, the estimated coefficients are the elasticities.