Module 0: Introduction[1]

"Pure mathematics is, in its way, the poetry of logical ideas" - Albert Einstein

These notes reflect an attempt to convey, in a conversational tone and at a conceptual level, some of the mathematics used in first-year economics courses. These notes are intended to be *read*, not referenced. Think of them as a narrative providing reasonably rigorous definitions of mathematical concepts together with the intuition associated with their use. There are plenty of references available that are exhaustive in their rigor and completeness, and I encourage you to use them to clear up any confusions you might have, and to fill in the many evident gaps these notes include.

1 About math

It is common for first-year students to find their coursework's technical aspects – often referred to as "math" – both mysterious and challenging.

The mystery reflects a natural misapprehension of the role mathematics plays in economic research, and alleviating this misapprehension requires a deep understanding of what mathematics really is. Unfortunately, I can't tell you what mathematics is – nobody can (not even Einstein, though we can tell from the quote above that he tried). I can only assure you that as your mathematical maturity develops, you will come to appreciate, and indeed be inexorably drawn to the power mathematics lends to rigorous communication.

We were first taught math as it directly relates to the world we experience. Numbers had meaning because we attached them to concepts we were already familiar with – *"What do these groups of apples, oranges, and bananas have in common?"* (This is, in essence, set theory). Mathematical operations also had a similar allegorical pedagogy – you may have been asked *"If I have 4 apples and get 3 more, how many apples do I have?"* The difficulty many people have with math is in struggling to abstract away from these allegories into *"pure"* mathematics.

The challenge you will face in this year's coursework is similarly about developing familiarity and comfort with abstraction. Most introductory math courses eschew abstraction in favor of technique. For example, you learn how to take derivatives of lots of different types of functions, but less time is spent understanding what a derivative really is. I really don't care whether you know how to differentiate inverse trig functions, etc. I need you to know what a derivative is, and why it is important.

[1] These notes (and the notes in all modules and slides) were originally authored by Bruce McGough, then later expanded upon and edited by Owen Jetton in 2024.

More generally, it is abstraction that lends mathematics its power. By abstracting away from all non-essential assumptions, the reasons for the validity of a given statement are clarified. Further, the abstraction itself engenders application across a broad range of environments of scientific inquiry.

2 About mathematics in economics

Economics is an empirical science. Economists observe correlations in the data, form theories to impart causation to the correlations they witness, and confront their theories with the data to test validity. If, after a long process of confrontation and revision, a given theory appears robust then economists may be willing to use this theory to conduct policy analysis and even provide policy prescription.

The above, loose description of the scientific process as it is applied in economics, makes no mention of mathematics; and indeed, with the exception of statistical techniques needed to confront data, no mathematics is required. On the other hand, the key step of *forming theories* may be greatly aided by appealing to the structure of mathematical inquiry. I'll be more specific. A theory is, at its core, a set of assumptions together with the necessary implications of these assumptions. The development and communication of a theory need not be anchored in mathematics: it is perfectly reasonable, and still common among some subfields, to use a narrative approach to lay out a theory intended to explain behavior witnessed in the data. The great advantage of this approach is that the narrative can be developed to allow for realistic modeling assumptions – we can tell stories that appear to well-capture the myriad nuances of a particular economic environment. However, the narrative approach is challenged by the imprecision of language; and, unless great care is taken, there may be confusion about exactly what assumptions are being made, and contest about what implications necessarily follow.

The mathematical approach to developing a theory (usually called a model) is to lay out precise assumptions and use accepted logic to deduce necessary implications. Of course, this is exactly how mathematical inquiry proceeds, which is why much of economic theory looks like, and really is, applied math. The advantage of this approach is that the confusion and contest are eliminated: unless an error has been made, the meaning of the assumptions and validity implications cannot be debated. Instead, any debate about the theory is confined to the *appropriateness* of the assumptions – indeed, if you don't like the implications, the only redress is modification of the assumptions. This advantage cannot be overstated: it provides a highly structured environment for careful debate and measured progress. However, the drawback is also evident: tractable theories often require Herculean assumptions, and the technical progress needed to weaken the assumptions, and thereby surmount their impediment to realistic models, is tedious and time-consuming.

So there's a trade-off between the narrative and mathematical approaches to developing models, which is why both methods, as well as all conceivable hybrids of them, remain prominent in economics.

3 About this course

You will need to understand (or, at least, come to understand) the various concepts identified in the syllabus, as well as the technical details of their application. I need to emphasize four important points:

- New concepts in mathematics are complete mysteries until you understand them, and then they are trivial. Upon reflection, this is not terribly surprising: every result in mathematics is true by definition; understanding why a given result is true requires making numerous non-obvious connections, but once the connections have been made, the result is no longer difficult to understand. It's very easy to solve a maze one someone has marked the path.
- These lectures and lecture notes, as well as the reference text, are intended to aid the development of your conceptual understanding; however, simply reading the notes may not – indeed will very likely not – be enough for you to achieve conceptual competence. You have to create your own conceptual understanding – no one can do it for you – and it takes work.
- The development of your ability to apply the concepts covered in this course to various problems is economics is entirely up to you, and *only* comes through practice. You will be given ample opportunity to practice via the homework problems. Don't become complacent, and if you work in groups, don't trick yourself into believing you understand the problem because the group completed the problem. Working in groups is fine – indeed, encouraged – but you should write up all of the details of all of the problems yourself.
- Many of the important results are contained in exercises. I will work through some of the exercises in class, but my point is this: **don't skip the exercises**.

4 The prerequisites: Logic

When formulating economic theories, as discussed above, all implications of the assumptions are those which *logically follow*, according to basic laws of logic. These need to be understood prior to learning the methods of proofs you will utilize all year long (and hopefully into your career), otherwise you risk misusing those methods. We will discuss how those laws affect math below, but for now, let us focus on what these laws are.

4.1 Basic Laws of Logic

First, some notation: Let P and Q be statements – a statement is "a declarative sentence that is either true or false but not both"[2]. We can use logical operators to create new compound statements that are either true or false.

- **Conjunction:** the conjunction of P and Q is the statement "P *and* Q", denoted $P \wedge Q$, and is only true if **both** P and Q are true.
- **Disjunction**: the disjunction of P and Q is the statement "P *or* Q", denoted $P \vee Q$, and is only true if **either** P or Q are true.
- **Negation**: the negation of a statement P is the statement "*not* P", denoted $\neg P$, and is only true if P is false, and is only false if P is true.
- **Conditional**: the conditional statement is the statement "$If\ P\ then\ Q$, denoted $P \rightarrow Q$, means that Q must be true whenever P is true, and is only false when P is true and Q is false.

The final definition is the one we will consider the most. Theories begin with assumptions and conclude in *theorems*, which are conditional statements. A theorem is *proven* when it logically follows from established assumptions.

Exercise 1 *Show that "$P \rightarrow Q$" is logically equivalent to "$\neg P \vee Q$". Provide an example using two statements.*

Exercise 2 *Show that "$\neg(P \wedge Q)$" (*meaning "not ' P and Q' "*) is logically equivalent to "$\neg P \vee \neg Q$". Provide an example using two statements.*

Exercise 3 *Show that "$P \rightarrow Q$" is logically equivalent to "$\neg Q \rightarrow \neg P$". Provide an example using two statements.* Hint: use the results from Exercises 1 to help.

4.2 Methods of proof

As said above, this course is dedicated to understanding the meaning behind the abstract math we use in Economics rather than learning mathematical techniques. When we discuss "methods of proof" here, we will focus on *why* these methods logically prove a statement.

[2] *Mathematical Reasoning: Writing and Proof*, by Ted Sunstrom, page 1.

Direct. A *direct proof* logically combines axioms, definitions, and other proven theorems to derive a conclusion.

Suppose you want to prove the statement "$P \rightarrow Q$." A direct proof would utilize other conditional statements previously established as true.

Induction. A *proof by induction* proves that a statement is true for a given Natural Number (integers greater than 0) n (call this statement $P(n)$) by establishing $P(1)$ as true, then establishing the statement: "if $P(k)$ is true, then $P(k+1)$ is true," thereby proving that $P(n)$ is true for all Natural Numbers.

The logic here is that if $P(1)$ is true, and if $P(k) \rightarrow P(k+1)$ is true, since the process $k, k+1, (k+1)+1, \ldots$ beginning with $k = 1$ yields the sequence of Natural Numbers $1, 2, 3\ldots$, $P(n)$ is true for all natural numbers.

Contraposition. A *proof by contraposition* proves the *contrapositive* of a statement "$P \rightarrow Q$," which is the statement "$\neg Q \rightarrow \neg P$."

As you established in **Exercise 3**, $P \rightarrow Q$ is logically equivalent to $\neg Q \rightarrow \neg P$. Therefore, proving the latter statement as true, necessitates the former statement as true.

Contradiction. A *proof by contradition* for a given statement P shows that if the negation of P is assumed to be true, then a logical contradiction occurs, hence the statement must be false.

As stated in the definition of negation: $\neg P$ is only true if P is false, and is only false if P is true.

Exhaustion. A *proof by exhaustion* is a method where a conclusion is established by dividing it into a finite number of cases and proving each one separately.

There have to be a finite number of cases possible, and if the statement we are proving is true in each case, then the statement is true. This typically requires sub-proofs within each case.

Construction. A *proof by construction* creates a concrete example with a property to show that something having that property exists, or creates a counterexample to disprove a proposition that all elements have a certain property.

If our statement is about the existence of something with a given property, then constructing an example X with said property and proving X exists, is sufficient for proving the initial statement. Providing a counterexample utilizes the same logic, but now the property is counter to the statement.

5 The prerequisites: Numbers

You are assumed to be familiar with the concepts of *set*, *Cartesian product* and *function*, as well as the particular sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

5.1 $\mathbb{R}$

The important property of $\mathbb{R}$ is that it is *complete*. In fact, $\mathbb{C}$ is also complete, but the concept of *completeness* needs to be generalized to make this connection – we'll get to that. First we need to define the *supremum* of a set. If $A \subset \mathbb{R}$ then the supremum of A, *if it exists*, is defined to be the smallest real number that is greater-than-or-equal-to every element of A. The supremum of A is denoted $\sup A$.

I won't define completeness formally here – I'll wait for the more general definition. By saying $\mathbb{R}$ is complete, we mean that if $A \subset \mathbb{R}$ is bounded above (i.e. there is a real number greater-than-or-equal-to every element of A) then its supremum exists. This can be taken either as a defining feature of the real number field or as an implication of the construction of the real numbers from $\mathbb{Q}$.

Exercise 4 *Show that $\mathbb{Q}$ is not complete.*

Exercise 5 *Show that the supremum is unique.*

Exercise 6 *Define infimum. Show that if $A \subset \mathbb{R}$ is bounded below then the infimum exists.*

5.2 $\mathbb{C}$

The important property of $\mathbb{C}$ is that it is *algebraically closed*. This means that every non-constant polynomial has a root in $\mathbb{C}$ (this is a statement of the fundamental theorem of algebra). Polynomials are among the most important functions in mathematics, and a central reason for their importance is that they are arithmetic in nature: their evaluation only requires addition and multiplication. In particular, a computer can very easily and efficiently evaluate any polynomial. Since $\mathbb{C}$ is algebraically closed, we know that every polynomial equation has a solution. In fact, more can be said, as demonstrated in the next exercise.

Exercise 7 *Show that if p is a complex polynomial of degree n then p has* exactly *n roots in $\mathbb{C}$. (*Hint*: First, remember that a degree n polynomial looks like $p(x) = \sum_{k=0}^{n} a_k x^k$ with $a_n \neq 0$. Next, show that if f and g are polynomials, and* $\deg(f) < \deg(g)$ *then there is are polynomials h and r with* $\deg(h) = \deg(g) - \deg(f)$ *and* $\deg(r) < \deg(f)$ *so that* $g(x) = f(x)h(x) + r(x)$. *To show this, I would use polynomial division and induction on the* *degree of g. Then use this result to show that if z is a root of p then* $p(x) = (x-z)q(x)$ *with* $\deg(q) = \deg(p) - 1$*, and apply induction again.)*

5.3 $\sim$

A final concept is useful to introduce at this stage: *equivalence relation* – you will study this concept in detail in micro theory. Given a set X, a *relation* is a subset R of $X \times X$. If $(x, y) \in R$ we will write $x \sim y$; we refer to $\sim$ as the relation, and say "x is related to y". The relation $\sim$ is an equivalence relation provided it has the following properties:

- Reflexivity: $x \sim x$
- Symmetry: $x \sim y \implies y \sim x$
- Transitivity: $x \sim y$ and $y \sim z$ implies $x \sim z$.

The collection of all elements related to a given element is called an *equivalence class*.

Now, recall that a *partition* of a set X refers to decomposing X into a collection of non-intersecting subsets; more formally it is a collection $\{X_\lambda\}_{\lambda\in\Lambda}$ of subsets of X, where Λ is a possibly infinite index set, such that

$$X = \cup_{\lambda\in\Lambda} X_\lambda \text{ and } \lambda_1 \neq \lambda_2 \implies X_{\lambda_1} \cap X_{\lambda_2} = \varnothing.$$

The relationship between a partition and an equivalence relation is provided in the following exercises.

Exercise 8 *Let* $\{X_\lambda\}_{\lambda\in\Lambda}$ *be a partition of* X*. For* $x, y \in X$ *define* $x \sim y$ *if and only if there exists* $\lambda \in \Lambda$ *such that* $x, y \in X_\lambda$*. Show that* $\sim$ *is an equivalence relation.*

Exercise 9 *Let* $\sim$ *be an equivalence relation on* X*. Show there is a unique partition* $\{X_\lambda\}_{\lambda\in\Lambda}$ *of* X *so that* $x \sim y$ *if and only if there exists* $\lambda \in \Lambda$ *such that* $x, y \in X_\lambda$*.*

Equivalence relations and their associated partitions play important roles in economics: indifference curves and iso-cost curves serve as simple examples. There is, conceptually, a much more general principle here, though: an equivalence relation that results from some type of analysis conveys lost information: from the context of those who can only "observe" the partition, two points that are in the same equivalence class are indistinguishable – whatever information that once identified them has been lost.