Math Camp

Module #3: Metric spaces and continuous functions

Metric space

Intuition: a *metric space* is a set on which is a natural notion of distance.

Usefulness: Economics is about tradeoffs, which must be measured

Formally: let X be a non-empty set and let $d : X \times X \to \mathbb{R}_+$ have the following properties:

- $d(x,y) = 0 \Leftrightarrow x = y$
- (symmetry) $d(x,y) = d(y,x)$
- (triangle inequality) $d(x,z) \leq d(x,y) + d(y,z)$

The pair (X,d) is a metric space, with d a metric on X.

Comments on metric spaces

- The “arrows” in the category of metric spaces are *continuous functions*
- Intuitively, continuous functions preserve proximity
- *sequential approximation* is the name of the game
- Examples:
  - $\mathbb{R}$ and $\mathbb{C}$
  - $\mathbb{R}^n$
  - $C([a,b])$

Normed linear spaces

Intuition: a *norm* measures the “length” of a vector and induces a metric on a vector space.

Usefulness: A norm induces a metric: $d(v,w) = \|v-w\|$

Formally, a norm on V is a map $\|\cdot\| : V \to \mathbb{R}_+$ satisfying

- $\|v\| = 0 \Leftrightarrow v = 0$
- $\|\alpha v\| = |\alpha|\|v\|$ where, here, $\alpha \in \mathbb{C}$ or $\mathbb{R}$ and $v \in V$
- (triangle inequality) $\|v+w\| \leq \|v\| + \|w\|$

Examples of normed vector spaces

- $(\mathbb{R}^n, \|\cdot\|_2)$ where $\|v\|_2 = \left(\sum_{i=1}^n v_i^2\right)^{1/2}$
- $(\mathbb{R}^n, \|\cdot\|_1)$ where $\|v\|_1 = \sum_{i=1}^n |v_i|$
- $(\mathbb{R}^n, \|\cdot\|_\infty)$ where $\|v\|_\infty = \max_{i=1,\dots,n} |v_i|$

Examples of normed vector spaces

- $\left(\ell^2(\mathbb{N}), \|\cdot\|_2\right)$ where

$$\begin{aligned}\ell^2(\mathbb{N}) &= \left\{ f : \mathbb{N} \to \mathbb{R} \text{ such that } \sum_n f(n)^2 < \infty \right\} \\ \|f\|_2 &= \left(\sum_n f(n)^2 \right)^{1/2}\end{aligned}$$

- $\left(\ell^1(\mathbb{N}), \|\cdot\|_1\right)$ where

$$\begin{aligned}\ell^1(\mathbb{N}) &= \left\{ f : \mathbb{N} \to \mathbb{R} \text{ such that } \sum_n |f(n)| < \infty \right\} \\ \|f\|_1 &= \sum_n |f(n)|\end{aligned}$$

Examples of normed vector spaces

- $(\ell^\infty(\mathbb{N}), \|\cdot\|_\infty)$ where

$$\begin{aligned} \ell^\infty(\mathbb{N}) &= \left\{ f : \mathbb{N} \to \mathbb{R} \text{ such that } \sup_n |f(n)| < \infty \right\} \\ \|f\|_\infty &= \sup_n |f(n)| \end{aligned}$$

- $(C(X), \|\cdot\|_\infty)$ where X is a compact metric space,

$$\begin{aligned} C(X) &= \text{set of continuous functions from } X \text{ to } \mathbb{R} \\ \|f\|_\infty &= \sup_{x \in X} |f(x)| \end{aligned}$$

Examples of normed vector spaces

- $\left(L^2(\Omega), \|\cdot\|_2\right)$
  - (Ω, μ) is a probability space
  - $L^2(\Omega)$ = set of random variables on Ω with finite second moments, i.e. $Ex^2 < \infty$
  - $\|x\|_2 = \left(Ex^2\right)^{1/2}$

Completeness

Intuition: A metric space is complete if every sequence that should converge does converge.

Usefulness: existence of equilibria

Formalities:

- $x_n \to x$ provided $d(x_n, x) \to 0$
- A sequence is *Cauchy* provided that for any $\varepsilon > 0$ there is some $N > 0$ so that $m, n > N$ imply $d(x_n, x_m) < \varepsilon$.
- A metric space is *complete* provided Cauchy sequences converge

Openness and closedness

Intuition: a set is open if the ”neighbors” of every point in the set are also in the set.

Usefulness: show something is true for a point, conclude that it’s true for a lot of points.

Formalities:

- The ball of radius ε around x is $B(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$.
- $A \subset X$ is *open* provided that $x \in A$ implies there is ball around x that is completely contained in A.
- $A \subset X$ is is *closed* if its complement is open.

Openness and closedness

- Openness is closed under finite intersections and arbitrary unions
- Closedness is closed under finite unions and arbitrary intersections.
- The *interior* of $A \subset X$ is the union of all open sets contained in A
- The *closure* of $A \subset X$ is the intersection of all closed sets containing A.

Compactness

Intuitively, a compact set is not too big and contains it boundary. It's a bit like finiteness.

Usefulness: continuous functions on compact sets attain extrema.

Formally, $A \subset X$ is *compact* provided every sequence has a convergent subsequence.

Theorem (Heine-Borel)

The set $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$.

- Intuition: continuous functions respect “nearness.”
- Definition: f is continuous if it maps convergent sequences to convergent sequences.

Connectedness

Let $A \subset X$.

- Intuition: Any two points in a path-connected set can be “connected” by a continuous curve that stays in the set.
- A is path connected if for any $x_1, x_2 \in A$ there is a continuous function $f : [0,1] \to A$ such that $f(0) = x_1$ and $f(1) = x_2$.

Important results

- The continuous image of a compact set is compact.
- The continuous image of a path-connected set is path-connected.
- A real-valued function achieves its extrema on a compact set.

Convexity

Let $(V, \|\cdot\|)$ be a normed vector space and $K \subset V$.

- Intuition: Any two points in a convex set can be "connected" by a line that stays in the set.
- Definition: K is convex if for any $v_1, v_2 \in K$ and $\alpha \in (0,1)$ it follows that $\alpha v_1 + (1-\alpha) v_2 \in K$.

Theorem (Brouwer, Schauder)

Let $K \subset V$ be compact and convex, and let $f : K \to K$ be continuous. Then there is an $x \in K$ such that $f(x) = x$.