Math Camp

Module #4: Functions on $\mathbb{R}$

Recall continuity

Intuition: continuity preserves proximity

Usefulness: existence of equilibria, existence of optima, allowance for approximation

Formalities: A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous at* $x \in \mathbb{R}$ provided that whenever $x_n \to x$ it follows that $f(x_n) \to f(x)$.

- Makes perfect sense in a metric space
- Local notion

Preservation of continuity

Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous. Then the following functions are continuous:

- $f + g$
- $f \cdot g$
- $f \circ g$

Question: Are polynomials continuous?

Intermediate value theorem

Theorem (Intermediate value theorem)

Let $f : [a,b] \to \mathbb{R}$ *be continuous. If* $f(a) > 0 > f(b)$ *or if* $f(a) < 0 < f(b)$ *then there exists* $c \in (a,b)$ *such that* $f(c) = 0$*.*

Remark: This is the univariate version of “the continuous image of a path-connected set is path-connected.”

Differentiability

A function $f : \mathbb{R} \to \mathbb{R}$ is *differentiable at* x, provided the following limit, denoted $f'(x)$, exists:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

- Local notion
- $f(x + \Delta x) - f(x) \approx f'(x)\Delta x$
- $df = f'(x)dx$
- A differentiable function is continuous

The calculus

Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable. Then

- $h = f + g \implies h' = f' + g'$
- $h = f \cdot g \implies h' = f' \cdot g + g' \cdot f$
- $h = f \circ g \implies h' = (f' \circ g) \cdot g'$

Differentiating polynomials

- What is the derivative of a constant?
- Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=x$. What is $f'(x)$?
- Finish the job
- What is the derivative of $1/x$?

Mean value theorem

Theorem (Mean value theorem)

Let U be an open set containing the interval $[a,b]$, and let $f : U \to \mathbb{R}$ be differentiable. Then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Integration

Intuition: Integration measures average global behavior

- If $f : [a,b] \to \mathbb{R}$ the *average* value of f is

$$\text{avg}(f) = \frac{1}{b-a} \sum_{x \in [a,b]} f(x)dx$$

- Suppose $F : [a,b] \to \mathbb{R}$ such that $F'(x) = f(x)$. Then

$$F(b) - F(a) = \sum_{x \in [a,b]} dF(x) = \sum_{x \in [a,b]} F'(x)dx = \sum_{x \in [a,b]} f(x)dx$$

- Thus $F'(x) = f(x)$ implies $\text{avg}(f) = \dfrac{F(b) - F(a)}{b-a}$

Fundamental Theorem of calculus

Theorem (Fundamental theorem of calculus)

Let $g:[a,b]\to\mathbb{R}$ *be continuous. Define* $G:[a,b]\to\mathbb{R}$ *by*

$$G(x)=\int_a^x g(s)ds.$$

Then $G'(x)=g(x)$*.*

Corollary (Second fundamental theorem of calculus)

If $f:[a,b]\to\mathbb{R}$ *is continuous and* $F:(a,b)\to\mathbb{R}$ *is differentiable with* $F'(x)=f(x)$ *then*

$$F(b)-F(a)=\int_a^b f(x)dx$$

Exponents and logs

- For $x \geq 1$ define $\log(x) \equiv \int_1^x t^{-1} dt$
- For $x \in (0,1)$, define $\log(x) \equiv -\log(1/x)$
- For $x \in \mathbb{R}$, define $\exp(x) = \log^{-1}(x)$

$$\begin{aligned}
\log(\exp(x)) = x \quad & \implies \quad \frac{d}{dx}\log(\exp(x)) = \frac{d}{dx}x \\
& \implies \quad \frac{1}{\exp(x)}\frac{d}{dx}\exp(x) = 1 \\
& \implies \quad \frac{d}{dx}\exp(x) = \exp(x)
\end{aligned}$$

- $x^\alpha \equiv \exp(\alpha \log(x))$ for $x > 0, \alpha \in \mathbb{R}$