Math Camp

Module #5: Functions on $\mathbb{R}^n$

$\mathbb{R}^n$

A *Hilbert space* is a complete, normed linear space (i.e. Banach space) in which the norm is induced by an inner product.

$\mathbb{R}^n$ is a Hilbert space, with inner product given by the usual dot product:

$$\|x\|_2 = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Continuous functions

- Same as before: continuous functions preserve convergent sequences.
- On $\mathbb{R}$ the main result was the IVT
- Recall that $K \subset \mathbb{R}^n$ is *convex* if $x, y \in K \implies \alpha x + (1-\alpha)y \in K$

Theorem (Brouwer)

Let $K \subset \mathbb{R}^n$ be compact and convex, and let $f : K \to K$ be continuous. Then there is a point $x \in K$ such that $f(x) = x$.

Remark: this is an existence theorem

Continuous functions

$f : \mathbb{R}^n \to \mathbb{R}^n$ is a *contraction* if there is a $\beta \in (0,1)$ such that for any $x, y \in \mathbb{R}^n$ we have that

$$\|f(x) - f(y)\| \leq \beta \|x - y\|.$$

Theorem (Contraction mapping theorem)

Let $X \subset \mathbb{R}^n$ be closed. Suppose $f : X \to X$ is a contraction. Then there exists a unique $x^* \in X$ such that $f(x^*) = x^*$. Furthermore, if x_0 is any point in X then

$$f^n(x_0) \equiv f(f^{n-1}(x_0)) \to x^*.$$

Remark: this theorem is constructive!

Continuous functions

Recall:

Theorem (Attainment of extrema)

Let $K \subset \mathbb{R}^n$ be compact and $f : K \to \mathbb{R}$ be continuous. Then there exists $x \in K$ such that

$$f(x^*) = \sup_{x \in K} f(x) < \infty.$$

Partial derivatives

Let $f : \mathbb{R}^n \to \mathbb{R}$. Then the *partial derivative of* f *with respect to* x_i at $x \in \mathbb{R}^n$, when it it exists, is computed as

$$f_i(x) = \lim_{\Delta x_i \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + \Delta x_i, x_{i+1}, \ldots, x_n) - f(x)}{\Delta x_i}.$$

Remark: f_{ij} is the partial derivative of f_i with respect to x_j. If f_{ij} and f_{ji} are continuous then they are equal.

Total differentials

if $f : \mathbb{R}^n \to \mathbb{R}$ then the *total differential* of f is

$$df = \sum_{i=1}^{n} f_i \, dx_i = f_i \, dx_i$$

Example: Use the Keynesian cross to determine the impact on output of a rise in taxes.

Derivatives

- Let $f : \mathbb{R}^n \to \mathbb{R}$ and $dx = (dx_1, \ldots, dx_n)^T$
- Let Df be the *row* vector $(f_1, \ldots, f_n)$ (linear functional!)
- $df = \sum_{i=1}^{n} f_i \, dx_i = Df \cdot dx$
- More generally, if $f : \mathbb{R}^n \to \mathbb{R}^m$ then $Df(x) : \mathbb{R}^m \to \mathbb{R}^m$ is the unique linear map satisfying

$$\lim_{\|\Delta x\| \to 0} \frac{\|f(x+\Delta x) - f(x) - Df(x)(\Delta x)\|}{\|\Delta x\|} = 0$$

Derivatives

If $f : \mathbb{R}^n \to \mathbb{R}^m$ write $f = \begin{pmatrix} f^1 \\ \vdots \\ f^m \end{pmatrix}$

Theorem (The Jacobian)

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable and f^i_j is continuous then

$$Df(x) = \begin{pmatrix} f^1_1 & \cdots & f^1_n \\ \vdots & \ddots & \vdots \\ f^m_1 & \cdots & f^m_n \end{pmatrix}$$

Derivatives

Notice that $f : \mathbb{R}^n \to \mathbb{R}$ implies

$$df = \begin{pmatrix} df^1 \\ \vdots \\ df^m \end{pmatrix} = \begin{pmatrix} Df^1 dx \\ \vdots \\ Df^m dx \end{pmatrix} = \begin{pmatrix} Df^1 \\ \vdots \\ Df^m \end{pmatrix} dx = Df \cdot dx$$

Taylor's theorem: univariate

If $f : \mathbb{R} \to \mathbb{R}$ is C^{N+1} then

$$f(x) = f(x^*) + \sum_{n=1}^{N} \frac{f^{(n)}(x^*)}{n!}(x - x^*)^n + \mathcal{O}\left(|x - x^*|^{N+1}\right).$$

Taylor's theorem: multivariate

Let $f : \mathbb{R}^n \to \mathbb{R}$.

- $Df(x) \in (\mathbb{R}^n)^* \cong \mathbb{R}^n$.
- Thus $Df : \mathbb{R}^n \to \mathbb{R}^n$.
- So $D^2f(x) : \mathbb{R}^n \to \mathbb{R}^n$ and $D^2f : \mathbb{R}^n \to \mathbb{R}^{n \times n}$.
- The *Hessian* is D^2f, which is the matrix of second partials.
- Taylor's theorem

$$f(x) = f(x^*) + Df(x^*)(x - x^*) + \frac{1}{2}(x - x^*)^T D^2f(x^*)(x - x^*) + \mathscr{O}\left(\|x - x^*\|^3\right)$$

Comparative statics

Comparative statics: let $f : \mathbb{R}^m \oplus \mathbb{R}^n \to \mathbb{R}^n$ and consider the equation

$$f(x,y) = 0 \in \mathbb{R}^n$$

If x moves by dx, how must y move (dy) so that

$$f(x+dx, y+dy) = 0?$$

Implicit function theorem

Let $f : \mathbb{R}^m \oplus \mathbb{R}^n \to \mathbb{R}^n$ and define the matrix partials

$$D_x f(x,y) = \begin{pmatrix} f^1_{x_1}(x,y) & \cdots & f^1_{x_m}(x,y) \\ \vdots & \ddots & \vdots \\ f^n_{x_1}(x,y) & \cdots & f^n_{x_m}(x,y) \end{pmatrix}$$

$$D_y f(x,y) = \begin{pmatrix} f^1_{y_1}(x,y) & \cdots & f^1_{y_n}(x,y) \\ \vdots & \ddots & \vdots \\ f^n_{y_1}(x,y) & \cdots & f^n_{y_n}(x,y) \end{pmatrix}$$

Theorem (Implicit function theorem)

Suppose $f : \mathbb{R}^m \oplus \mathbb{R}^n \to \mathbb{R}^n$ *has continuous first partials. If* $f(x^*, y^*) = 0$ *and* $\det D_y f(x^*, y^*) \neq 0$ *then there exists open set* $U \subset \mathbb{R}^m$, *with* $x^* \in U$, *and a continuously differentiable function* $g : U \to \mathbb{R}^n$ *so that*

- $y^* = g(x^*)$
- $x \in U \implies f(x, g(x)) = 0$
- $Dg(x^*) = -D_y f(x^*, y^*)^{-1} \circ D_x f(x^*, y^*)$