Simulations of Astrophysical Convection

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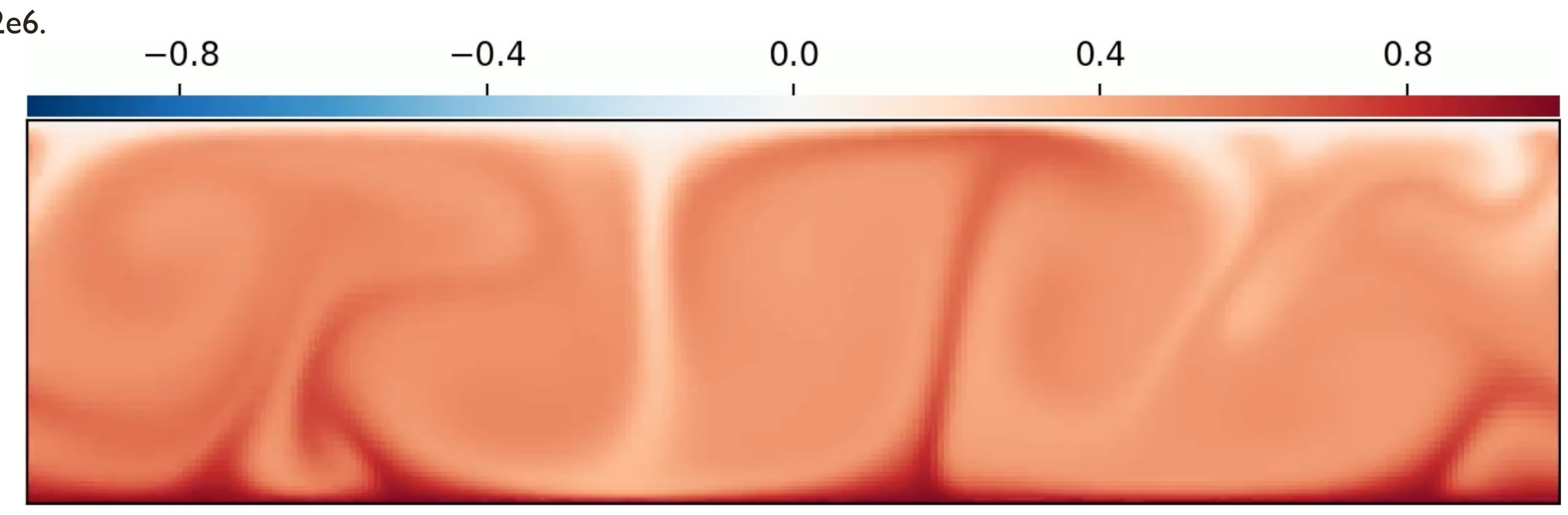
This study investigates Rayleigh-Bénard convection using numerical simulations with the Dedalus framework. The critical Rayleigh number for the onset of convection was estimated to be 1708, consistent with theoretical expectations. The relationship between the Nusselt number and the Rayleigh number was explored, revealing two distinct regimes: an exponential-like increase for Ra < 1×10^5, and a more linear relationship for Ra > 1×10^5. The experiment was extended to spherical geometry, investigating the effect of rotation on convection in a spherical shell. Results showed that rotation enhances convective heat transport efficiency within the studied parameter range. Future work could focus on incorporating compressibility or magnetic fields for more realistic stellar convection simulations, or solving problems related to computing power.

1. Introduction

Rayleigh-Bénard convection is a type of natural convection that occurs when a fluid is heated from below and cooled from above [1]. This process is driven by buoyancy forces arising from temperature-induced density variations in the fluid. Rayleigh-Bénard convection is characterised by the Rayleigh number (Ra), a dimensionless parameter that quantifies the relative strength of buoyancy forces to viscous and thermal dissipation. When Ra exceeds a critical value the fluid transitions from a purely conductive state to a convective state, marked by the formation of organized convection cells. In this investigation we set out to understand the onset and characteristics of Rayleigh-Bénard convection using numerical simulations.

An example of Rayleigh-Bénard convection is seen below, clearly visualising the formation of these convective cells. This fluid was modelled using the equations described in section 3, with the Rayleigh number set to Ra = 2e6.



2. Experimental Aims

- Determine the critical Rayleigh number for the onset of convection within a two-dimensional fluid layer
- Explore the relationship between the Nusselt number (a measure of convective heat transfer efficiency) and the Rayleigh number for a range of Ra values
- Extend the study into spherical geometry and to investigate the effect of rotation on heat transport within a spherical shell

3. Background and Theory

3.1 - Governing Equations and the Boussinesq Approximation

The Boussinesq approximation is a simplification used in fluid dynamics where density variations are ignored except when they directly affect the buoyancy force in the momentum equation [2]. The following three equations, along with the appropriate boundary conditions, form a complete mathematical description of Rayleigh-Bénard convection under the Boussinesq approximation:

1. $\nabla \cdot u = 0$
2. $\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla P + b\,\hat{z} + \nu\nabla^2 u$
3. $\frac{\partial b}{\partial t} - \kappa\nabla^2 = -u \cdot \nabla \mathrm{b}$

- u represents the velocity vector field
- P is the pressure gradient
- b is the temperature-driven buoyancy of the system
- ν represents the kinematic viscosity of the fluid
- κ is the thermal diffusivity

Where equation 1 represents the mass continuity equation under the assumption that our fluid is incompressible, equation 2 describes how the momentum of the system evolves over time, and equation 3 describes the evolution of the buoyancy within the fluid as a result of the temperature gradient.

3.2 - Non-dimensional Parameters

Non-dimensional parameters characterize the behaviour and dynamics of our convection systems. Their importance lies in their ability to simplify complex problems - by expressing the governing equations and boundary conditions in terms of non-dimensional parameters, we can identify the key factors that control the system's behaviour, regardless of the specific values of the dimensional variables.

Three of these parameters are of particular significance to this experiment, seen below.

The Rayleigh Number	The Nusselt Number	The Prandtl Number
The Rayleigh number characterizes the driving forces of convection, quantifying the relative importance of buoyancy forces, which promote convection, and viscous and thermal dissipation, which tend to suppress convection.	The Nusselt number quantifies the efficiency of convective heat transfer in a fluid system. It represents the ratio of the total heat transfer to the heat transfer that would occur by conduction alone.	The Prandlt number represents the ratio of viscous diffusion to thermal diffusion within a fluid, or in other words the ratio of the fluid's ability to transport momentum (viscosity) to its ability to conduct heat (thermal conductivity).

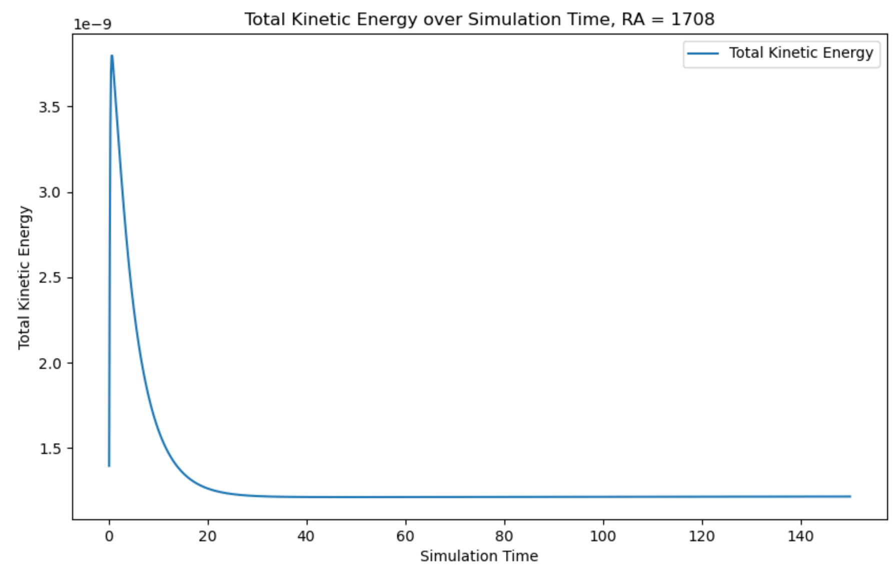
3.2 - Dedalus

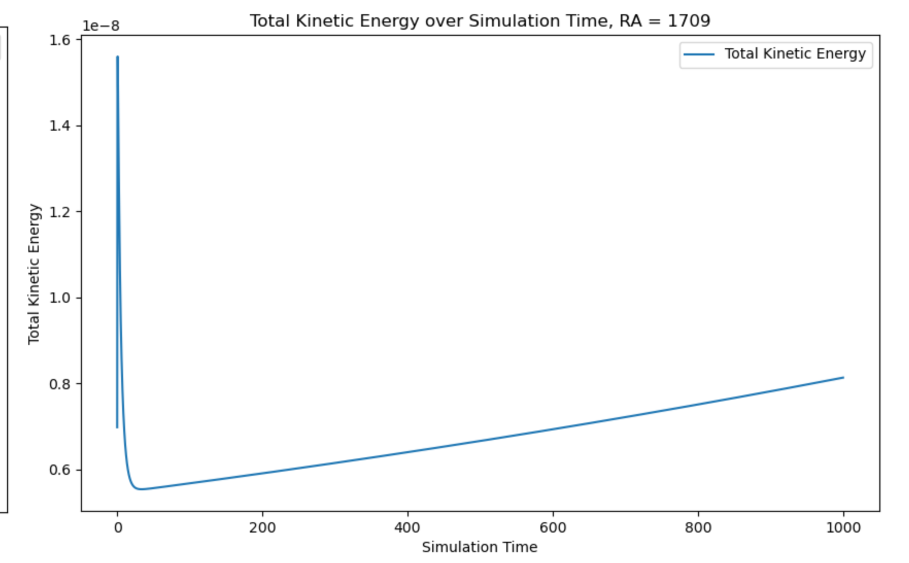
Tying this all together: the Dedalus framework. Dedalus is an open-source software written in Python used to solve complex PDEs of the sort we handle in this experiment. It is also used to visualise the solutions to these PDEs. All media created throughout this experiment was made using Dedalus, and the code written to do so was built off the free-to-use examples within the Dedalus documentation [3].

4. Results and Discussion

4.1- Investigating the Critical Rayleigh Number

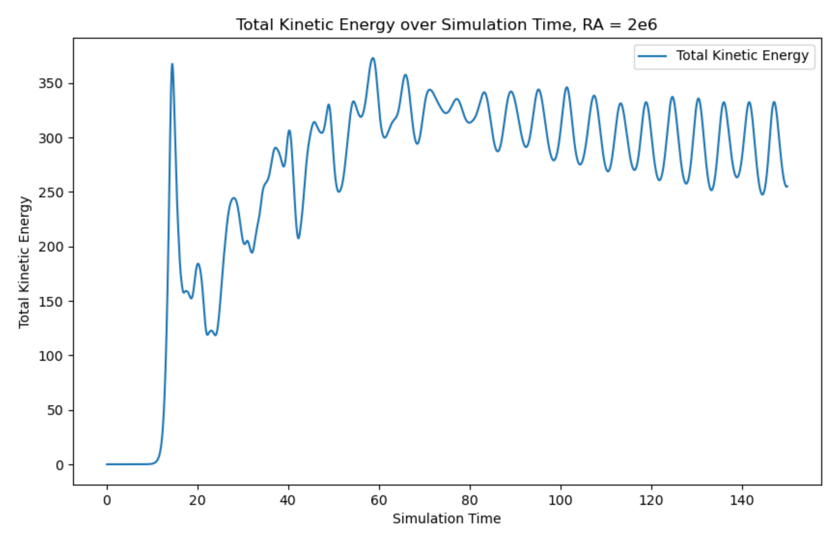
It is established that when the Rayleigh number of a system surpasses the critical threshold, buoyant forces begin to dominate over viscous damping forces, resulting in convection becoming the primary mechanism for heat transfer through the fluid [4]. The hypothesis proposed in this study is that by examining the total kinetic energy of the fluid over the simulation run time, it may be possible to estimate the critical Rayleigh number by observing how the relationship between the total kinetic energy and time evolves as the Rayleigh number is varied. Because the fluid under inspection has no mass property we inspect the kinetic energy per unit mass.

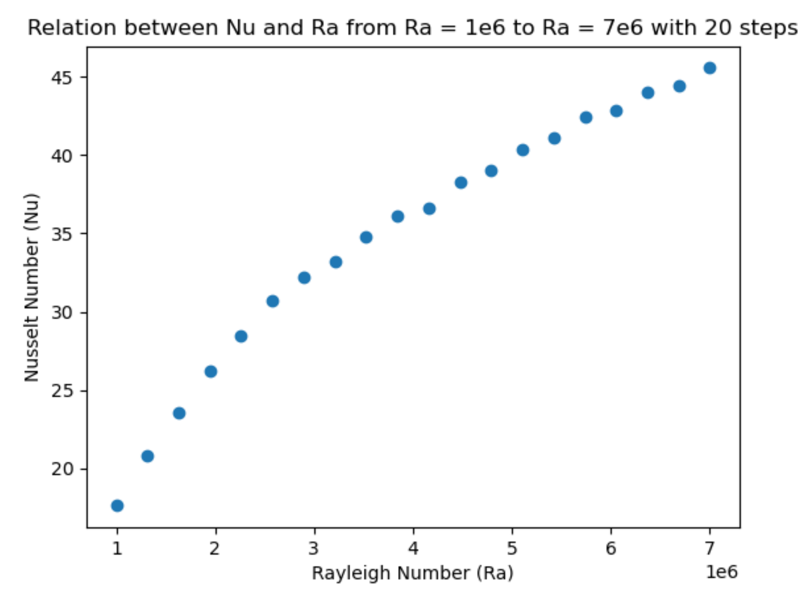




The figures above depict the kinetic energy density of a fluid over the run time of the simulation for 2-D Rayleigh-Bénard convection as seen in Section 1. We can see that when Ra is set to 1708, the total kinetic energy has a spike due to initial perturbations caused by the setup, which then swiftly drops to a constant low value. When Ra is set to Ra = 1709, the kinetic energy density begins to rise rather than remain at a constant low value. This shows us that the buoyant forces are beginning to overcome the viscous, and the system is transitioning toward a state of convection rather than conduction. We can therefore estimate the critical value of the Rayleigh number to be R_{crit} = 1708. In this section of the experiment the Prandtl number is kept at Pr = 1.

4.2 – The Relation between the Nusselt number and the Rayleigh number

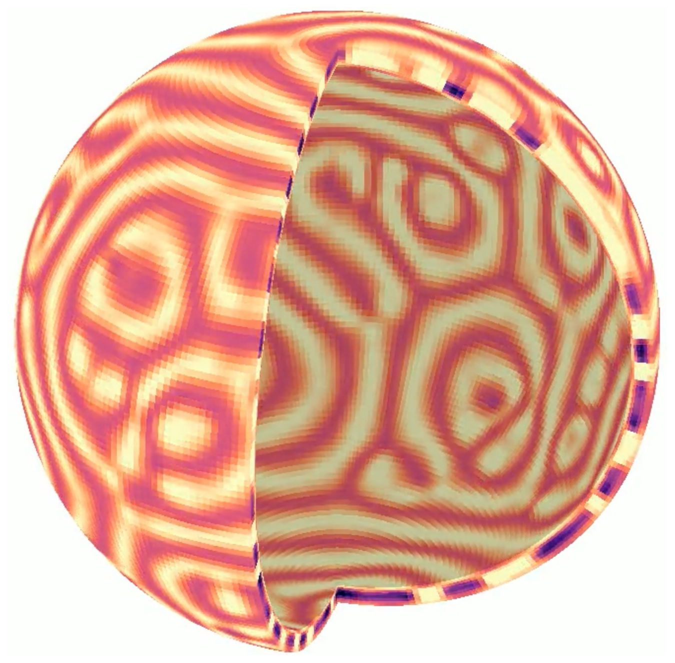


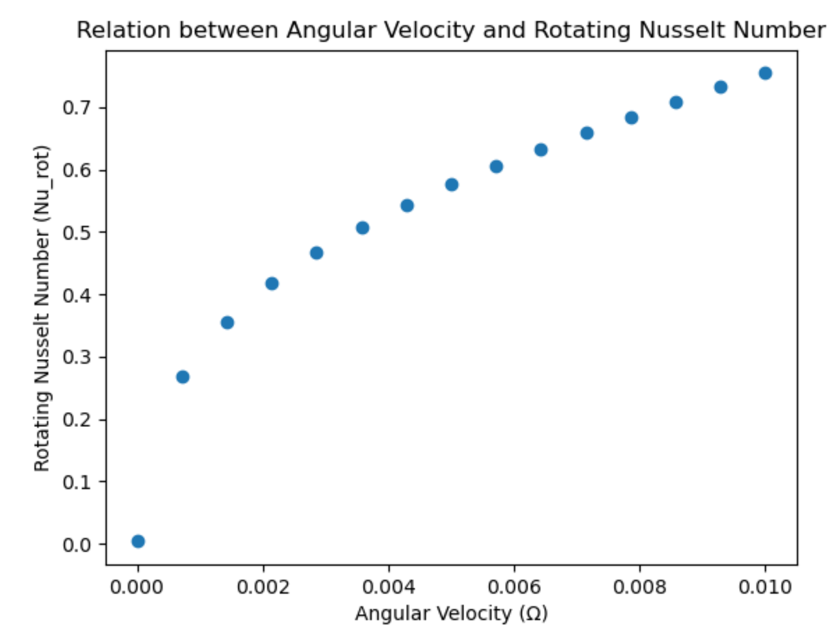


At high values of Ra we start to see more turbulent action compared to the smooth increase we saw previously when looking at the kinetic energy per unit mass over time. It is interesting to note that at around the same value of Ra is when we begin to see a linear-appearing relation between the Nusselt number and the Rayleigh number for the same system. This suggests that as the Rayleigh number increases, the 'efficiency' of the convective heat transfer also increases.

4.3 – Modelling the Convection Zone in Stars: Investigating the Relationship between the Nusselt Number and Rotation Rate for a Spherical Convective Shell

After changing coordinate basis to 3-dimensional spherical to represent a fluid between two radial boundaries and heated from the inner boundary, rotation can be added to the simulation by adding the Coriolis term to the RHS of the momentum equation (eq. 2), $-2\Omega\times u$. The code was manipulated to simulate 500 seconds of convection, calculate the average Nusselt number across the system and then loop over a range of angular velocities, Ω, in radians per second. A plot was then produced to visualise the relation between the rotational Nusselt number and the rotation rate.





Above: On the left, an example of the spherical shell system under inspection, with Ra = 3500 and no rotation after 500 simulation seconds. On the right, the plot produced by looping the simulation over a range of rotation rates and calculating the average Nusselt number for each iteration. Video examples of the simulation outputs can be seen on the QR code at the top of the poster.

5. Conclusions and Further Work

In this study, we successfully investigated the onset and characteristics of Rayleigh-Bénard convection using the Dedalus framework. By inspecting the total kinetic energy evolution over time for different Rayleigh numbers, we estimated the critical Rayleigh number to be Ra = 1708, aligning with the expected value CITE. We also explored the relationship between the Nusselt number (Nu) and Rayleigh number (Ra), identifying two distinct regimes: an exponential-like increase in Nu for small changes in Ra below Ra ~1×10^5, and a more linear relationship above this threshold. This indicates that the system undergoes a transition in its heat transfer behaviour around a critical Rayleigh number as we start to see turbulent behaviour in the kinetic energy profile of the system. The experiment was successfully extended into spherical geometry, investigating the effect of rotation on convection for a spherical convective shell, visualising the relationship between the Nusselt number and the rotation rate. Inspecting the Nu-Ω plot we can see that for the range of values of Ω we selected, the rotation increases the efficiency of convective heat transport within the system.

During our investigation into the effect of rotation on convection within spherical geometry, we were restrained by computational power. This was because as the rotation rate increased, the computational burden increased, meaning that we had to increase the numerical resolution of our simulations. This made simulations with a rotation rate above our highest value, $\Omega = 1\times10^{-2}$ unfeasible as it would simply have taken too long to investigate. Future work could focus on finding a solution to this computational limitation to investigate the effects of higher rotation rates on convection. Additionally, incorporating compressibility or magnetic fields into our simulations could provide a more accurate representation of the convection zone within stars, offering potential avenues for further research.

References

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