Lie theory cheat sheet

$\text{Lie group } \mathcal{M}, \circ$		size	dim	$\big \qquad \mathcal{X} \in \mathcal{M}$	Constraint	$m{ au}^\wedge \in \mathfrak{m}$	$oldsymbol{ au} \in \mathbb{R}^m$
Vector n-D	$ \mathbb{R}^n, +$	n	n	$oldsymbol{\mathbf{v}} \in \mathbb{R}^n$	$ \mathbf{v} - \mathbf{v} = 0$	$\mathbf{v} \in \mathbb{R}^n$	$\mathbf{v} \in \mathbb{R}^n$
Complex number	$ S^1, \cdot $	2	1	$ \mathbf{z} \in \mathbb{C}$	$\mathbf{z}^*\mathbf{z} = 1$	$i\theta \in i\mathbb{R}$	$\theta \in \mathbb{R}$
2D Rotation	$\mid SO(2), \cdot \mid$	4	1	R	$\mathbf{R}^{ op}\mathbf{R} = \mathbf{I}$	$\begin{bmatrix} \theta \end{bmatrix}_{\times} = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} \in \mathfrak{so}(2)$	$\theta \in \mathbb{R}$
2D Rigid Motion	\mid SE(2), ·	9	3	$ \mid \mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix} $	$\mathbf{R}^{ op}\mathbf{R} = \mathbf{I}$	$\begin{bmatrix} [\theta]_{\times} & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(2)$	$\left \begin{array}{c} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{array} \right \in \mathbb{R}^3 \ \left \begin{array}{c} \boldsymbol{\rho} \end{array} \right $
Quaternion	S^3, \cdot	4	3	$\mathbf{q}\in\mathbb{H}$	$\mathbf{q}^*\mathbf{q} = 1$	$\theta/2 \in \mathbb{H}_p$	$oldsymbol{ heta} \in \mathbb{R}^3$
3D Rotation	$\mid SO(3), \cdot \mid$	9	3	R	$\mathbf{R}^{ op}\mathbf{R} = \mathbf{I}$	$ \left \begin{bmatrix} \boldsymbol{\theta} \end{bmatrix}_{\times} = \begin{bmatrix} 0 & -\theta_z & \theta_y \\ \theta_z & 0 & -\theta_x \\ -\theta_y & \theta_x & 0 \end{bmatrix} \in \mathfrak{so}(3) $	$oldsymbol{ heta} oldsymbol{ heta} \in \mathbb{R}^3$
3D Rigid Motion	\mid SE(3), ·	16	6		$\mathbf{R}^{ op}\mathbf{R} = \mathbf{I}$	$\begin{bmatrix} [\boldsymbol{\theta}]_{\times} & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3)$	$igg egin{bmatrix} oldsymbol{ ho} \ oldsymbol{ heta} \end{bmatrix} \in \mathbb{R}^6 \ igg $

Operation	Inverse	Compose	Exp	Log	Right- \oplus	Right- ⊖
Right Jacobians	$oxed{\mathbf{J}_{oldsymbol{\chi}}^{oldsymbol{\chi}^{-1}} = -\mathbf{A}\mathbf{d}_{oldsymbol{\chi}}}$	$ \begin{vmatrix} \mathbf{J}_{\boldsymbol{\chi}}^{\boldsymbol{\chi} \circ \boldsymbol{\gamma}} = \mathbf{A} \mathbf{d}_{\boldsymbol{\gamma}^{-1}} \\ \mathbf{J}_{\boldsymbol{\gamma}}^{\boldsymbol{\chi} \circ \boldsymbol{\gamma}} = \mathbf{I} \end{vmatrix} $	$oxed{\mathbf{J}^{\mathrm{Exp}(oldsymbol{ au})}_{oldsymbol{ au}} = \mathbf{J}_r(oldsymbol{ au})}$	$\mathbf{J}^{\mathrm{Log}(oldsymbol{\chi})}_{oldsymbol{\chi}} = \mathbf{J}^{-1}_r(oldsymbol{ au})$	$ \begin{vmatrix} \mathbf{J}_{\boldsymbol{\chi}}^{\boldsymbol{\chi} \oplus \boldsymbol{\tau}} = (\mathbf{A} \mathbf{d}_{\mathrm{Exp}(\boldsymbol{\tau})})^{-1} \\ \mathbf{J}_{\boldsymbol{\tau}}^{\boldsymbol{\chi} \oplus \boldsymbol{\tau}} = \mathbf{J}_r(\boldsymbol{\tau}) \end{vmatrix} $	$egin{aligned} \mathbf{J}_{oldsymbol{\chi}}^{oldsymbol{\gamma}\ominusoldsymbol{\chi}} &= -\mathbf{J}_{l}^{-1}(oldsymbol{ au}) \ \mathbf{J}_{oldsymbol{ au}}^{oldsymbol{\gamma}\ominusoldsymbol{\chi}} &= & \mathbf{J}_{r}^{-1}(oldsymbol{ au}) \end{aligned}$

Note: In accordance to manif implementation, all Jacobians in this document are **right Jacobians**, whose definition reads: $\frac{\delta f(X)}{\delta X} = \lim_{\varphi \to 0} \frac{f(X \oplus \varphi) \ominus f(X)}{\varphi}$. However, notice that one can relate the left- and right- Jacobians with the Adjoint, $\frac{\varepsilon_{\partial f(\mathcal{X})}}{\partial \mathcal{X}} \mathbf{Ad}_{\mathcal{X}} = \mathbf{Ad}_{f(\mathcal{X})} \frac{\kappa_{\partial f(\mathcal{X})}}{\partial \mathcal{X}}$, see [1] Eq. (46).

[1] J. Solà, J. Deray, and D. Atchuthan, "A micro Lie theory for state estimation in robotics," Tech. Rep. IRI-TR-18-01, Institut de Robòtica i Informàtica Industrial, Barcelona, 2018. Available at arxiv.org/abs/1812.01537.

\mathcal{M}, \circ Op	Identity	Inverse	Compose	Act	Exp	Log
\mathbb{R}^n , +	$\mathbf{v} = [0]$	$-\mathbf{v}$	$ \mathbf{v}_1 + \mathbf{v}_2 $	$ \mathbf{v} + \mathbf{p} $	v	v
S^1, \cdot	z = 1 + i 0	z^*	$ \hspace{.05cm} z_1 \hspace{.05cm} z_2 \hspace{.05cm}$	z v	$z = \cos\theta + i\sin\theta$	$\theta = \arctan2(\operatorname{Im}(z), \operatorname{Re}(z))$
$SO(2), \cdot$	$\mathbf{R} = \mathbf{I}$	$\mathbf{R}^{-1} = \mathbf{R}^{\top}$	${f R}_1 \ {f R}_2$	$\mathbf{R} \cdot \mathbf{v}$	$\mathbf{R} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$	$\theta = \arctan2(r_{21}, r_{11})$
$\mathrm{SE}(2), \cdot$	$\mathbf{M} = egin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix} = \mathbf{I}$	$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{R}^{\top} & -\mathbf{R}^{\top} \mathbf{t} \\ 0 & 1 \end{bmatrix}$	$egin{aligned} \mathbf{M}_1 \mathbf{M}_2 = egin{bmatrix} \mathbf{R}_1 \mathbf{R}_2 & \mathbf{t}_1 + \mathbf{R}_1 \mathbf{t}_2 \ 0 & 1 \end{bmatrix}$	$\mathbf{M} \cdot \mathbf{p} = \mathbf{t} + \mathbf{R}\mathbf{p}$	$\mathbf{M} = \begin{bmatrix} \operatorname{Exp}(\theta) & \mathbf{V}(\theta) \boldsymbol{\rho} \\ 0 & 1 \end{bmatrix} {}^{(1)}$	$egin{aligned} oldsymbol{ au} = egin{bmatrix} oldsymbol{ ho} \ heta \end{bmatrix} = egin{bmatrix} \mathbf{V}^{-1}(heta) \mathbf{p} \ heta \end{bmatrix} ^{(1)} \end{aligned}$
S^3 , ·	$\mathbf{q} = 1 + i0 + j0 + k0$	$\mathbf{q}^* = w - ix - jy - jz$	$\mathbf{q}_1 \; \mathbf{q}_2$	$\mathbf{q} \mathbf{v} \mathbf{q}^*$	$\mathbf{q} = \cos\frac{\theta}{2} + \mathbf{u}\sin\frac{\theta}{2}$	$oldsymbol{ heta} = 2 \mathbf{v} rac{rctan2(\ \mathbf{v}\ , w)}{\ \mathbf{v}\ }$
$SO(3), \cdot$	$\mathbf{R} = \mathbf{I}$	$\mathbf{R}^{-1} = \mathbf{R}^{\top}$	$\mathbf{R}_1 \; \mathbf{R}_2$	$\mathbf{R} \cdot \mathbf{v}$	$\mathbf{R} = \mathbf{I} + \sin \theta \left[\mathbf{u} \right]_{\times} + (1 - \cos \theta) \left[\mathbf{u} \right]_{\times}^{2}$	$oldsymbol{ heta} = rac{ heta(\mathbf{R} - \mathbf{R}^T)^\wedge}{2\sin heta}$
$SE(3), \cdot$	$\mathbf{M} = egin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix} = \mathbf{I}$	$ M^{-1} = \begin{bmatrix} \mathbf{R}^{\top} & -\mathbf{R}^{\top} \mathbf{t} \\ 0 & 1 \end{bmatrix} $	$egin{aligned} \mathbf{M}_1 \mathbf{M}_2 = egin{bmatrix} \mathbf{R}_1 \mathbf{R}_2 & \mathbf{t}_1 + \mathbf{R}_1 \mathbf{t}_2 \ 0 & 1 \end{bmatrix}$	$\mathbf{M} \cdot \mathbf{p} = \mathbf{t} + \mathbf{R} \mathbf{p}$	$\mathbf{M} = \begin{bmatrix} \operatorname{Exp}(\boldsymbol{\theta}) & \mathbf{V}(\boldsymbol{\theta}) \boldsymbol{\rho} \\ 0 & 1 \end{bmatrix} {}^{(2)}$	$egin{aligned} oldsymbol{ au} = egin{bmatrix} oldsymbol{ ho} \ oldsymbol{ heta} \end{bmatrix} = egin{bmatrix} \mathbf{V}^{-1}(oldsymbol{ heta}) \ \mathrm{Log}(\mathbf{R}) \end{bmatrix} \ \end{pmatrix}$

\mathcal{M}, \circ Ad/Jac	Ad	\mathbf{J}_r	\mathbf{J}_l	$\begin{array}{c c} \mathbf{J}_{\boldsymbol{\chi}}^{\boldsymbol{\chi}\cdot\mathbf{p}} & \mathbf{J}_{\mathbf{p}}^{\boldsymbol{\chi}\cdot\mathbf{p}} \\ & (\mathrm{Act}) \end{array}$	
\mathbb{R}^n , +	$\mathbf{I} \in \mathbb{R}^{n \times n}$	I	I	I	I
S^1 , ·	1 1	1	1	$\left \mathbf{R} \left[1 \right]_{\times} \mathbf{v} \right $	\mathbf{R}
SO(2), ·	1	1	1	$\mathbf{R}\left[1\right]_{\times}\mathbf{v}$	\mathbf{R}
$\mathrm{SE}(2), \cdot$	$\begin{bmatrix} \mathbf{R} & -[1]_{\times} \mathbf{t} \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \sin \theta/\theta & (1-\cos \theta)/\theta & (\theta\rho_1-\rho_2+\rho_2\cos \theta-\rho_1\sin \theta)/\theta^2\\ (\cos \theta-1)/\theta & \sin \theta/\theta & (\rho_1+\theta\rho_2-\rho_1\cos \theta-\rho_2\sin \theta)/\theta^2\\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \sin \theta/\theta & (\cos \theta - 1)/\theta & (\theta \rho_1 + \rho_2 - \rho_2 \cos \theta - \rho_1 \sin \theta)/\theta^2 \\ (1 - \cos \theta)/\theta & \sin \theta/\theta & (-\rho_1 + \theta \rho_2 + \rho_1 \cos \theta - \rho_2 \sin \theta)/\theta^2 \\ 0 & 0 & 1 \end{bmatrix}$	$\left \begin{array}{cc} \left[\mathbf{R} & \mathbf{R} \left[1 \right]_{\times} \mathbf{p} \right] \end{array} \right $	R
S^3 , ·	$\mathbf{R}(\mathbf{q})$	$\mathbf{I} - rac{1-\cos heta}{ heta^2}\left[oldsymbol{ heta} ight]_ imes + rac{ heta-\sin heta}{ heta^3}\left[oldsymbol{ heta} ight]_ imes$	$\mathbf{I} + rac{1-\cos heta}{ heta^2}\left[oldsymbol{ heta} ight]_ imes + rac{ heta-\sin heta}{ heta^3}\left[oldsymbol{ heta} ight]_ imes$	$-\mathbf{R}(\mathbf{q}) \left[\mathbf{v}\right]_{\times} $ (3)	$\mathbf{R}(\mathbf{q})^{(3)}$
SO(3), ·	R	$\mathbf{I} - rac{1-\cos heta}{ heta^2}\left[oldsymbol{ heta} ight]_ imes + rac{ heta-\sin heta}{ heta^3}\left[oldsymbol{ heta} ight]_ imes$	$\mathbf{I} + rac{1-\cos heta}{ heta^2} \left[oldsymbol{ heta} ight]_ imes + rac{ heta-\sin heta}{ heta^3} \left[oldsymbol{ heta} ight]_ imes$	$\left -\mathbf{R}\left[\mathbf{v} ight]_{ imes} ight $	\mathbf{R}
SE(3), ·	$\left egin{array}{cc} \left[egin{array}{cc} \egin{array}{cc} egin{array}{cc} egin{array}{cc} \eq \egin{array}{cc} \egin{array} $	$\begin{bmatrix} \mathbf{J}_r(\boldsymbol{\theta}) & \mathbf{Q}(-\boldsymbol{\rho}, -\boldsymbol{\theta}) \\ 0 & \mathbf{J}_r(\boldsymbol{\theta}) \end{bmatrix}^{(4)}$	$\begin{bmatrix} \mathbf{J}_l(\boldsymbol{\theta}) & \mathbf{Q}(\boldsymbol{\rho},\boldsymbol{\theta}) \\ 0 & \mathbf{J}_l(\boldsymbol{\theta}) \end{bmatrix} {}^{(4)}$	$\left \begin{array}{cc} \left[\mathbf{R} & -\mathbf{R} \left[\mathbf{p} \right]_{\times} \right] \end{array} \right $	R

$$^{(1)}\mathbf{V}(\theta) = \frac{\sin\theta}{\theta}\mathbf{I} + \frac{1-\cos\theta}{\theta}\left[1\right]_{\times}$$

$$^{(2)}\mathbf{V}(\theta) = \mathbf{I} + \frac{1-\cos\theta}{\theta} \left[\mathbf{u}\right]_{\times} + \frac{\theta-\sin\theta}{\theta} \left[\mathbf{u}\right]_{\times}^{2}$$

$${}^{(2)}\mathbf{V}(\theta) = \mathbf{I} + \frac{1-\cos\theta}{\theta} \left[\mathbf{u} \right]_{\times} + \frac{\theta-\sin\theta}{\theta} \left[\mathbf{u} \right]_{\times}^{2}$$

$${}^{(3)}\mathbf{R}(\mathbf{q}) = \begin{bmatrix} w^{2}+x^{2}-y^{2}-z^{2} & 2(xy-wz) & 2(xz+wy) \\ 2(xy+wz) & w^{2}-x^{2}+y^{2}-z^{2} & 2(yz-wx) \\ 2(xz-wy) & 2(yz+wz) & w^{2}-x^{2}-y^{2}+z^{2} \end{bmatrix}$$

$$^{(4)}\mathbf{Q}(\boldsymbol{\rho},\boldsymbol{\theta}) = 1/2\left[\boldsymbol{\rho}\right]_{\times} + \frac{\theta - \sin\theta}{\theta^{3}}(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\rho}\right]_{\times} + \left[\boldsymbol{\rho}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times} + \left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right) - \frac{1 - \frac{\theta^{2}}{2} - \cos\theta}{\theta^{4}}(\left[\boldsymbol{\theta}\right]_{\times}^{2}\left[\boldsymbol{\rho}\right]_{\times} + \left[\boldsymbol{\rho}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right) - \frac{1}{2}(\frac{1 - \frac{\theta^{2}}{2} - \cos\theta}{\theta^{4}} - 3\frac{\theta - \sin\theta - \frac{\theta^{3}}{6}}{\theta^{5}})(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\rho}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \theta^{2} - \cos\theta}{\theta^{4}} - \frac{\theta^{2} - \cos\theta}{\theta^{4}}) + \frac{\theta^{2} - \sin\theta}{\theta^{4}}(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \theta^{2} - \cos\theta}{\theta^{4}} - \frac{\theta^{2} - \sin\theta}{\theta^{4}}) + \frac{\theta^{2} - \sin\theta}{\theta^{4}}(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \theta^{2} - \cos\theta}{\theta^{4}} - \frac{\theta^{2} - \sin\theta}{\theta^{4}}) + \frac{\theta^{2} - \sin\theta}{\theta^{4}}(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \theta^{2} - \cos\theta}{\theta^{4}} - \frac{\theta^{2} - \sin\theta}{\theta^{4}}) + \frac{\theta^{2} - \sin\theta}{\theta^{4}}(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \theta^{2} - \cos\theta}{\theta^{4}} - \frac{\theta^{2} - \sin\theta}{\theta^{4}}) + \frac{\theta^{2} - \sin\theta}{\theta^{4}}(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \theta^{2} - \cos\theta}{\theta^{4}} - \frac{\theta^{2} - \sin\theta}{\theta^{4}}) + \frac{\theta^{2} - \sin\theta}{\theta^{4}}(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol$$