1. (a) Describe the formal language over the alphabet $\{a, b, c\}$ generated by the context-free grammar whose non-terminals are $\langle S\rangle$ and $\langle A\rangle$, whose start symbol is $\langle S\rangle$, and whose production rules are the following:

   i. $\langle S\rangle \to a\langle S\rangle a$
   ii. $\langle S\rangle \to b\langle A\rangle$
   iii. $\langle A\rangle \to b\langle A\rangle b$
   iv. $\langle A\rangle \to c\langle A\rangle$
   v. $\langle A\rangle \to c$

   (b) Use the Pumping Lemma to prove that the language from part (a) is not regular.

   **Answer:**

   (a) Let $f : \{b, c\}^* \to M$ be a function that takes in a binary string and counts how many $b$ elements appear. The amount of elements that appear is stored into the variable $M$. There may be a $b$ element before the substring given by $f$. The number of $b$ elements that appear within the substring must be equal to the amount that appear after the string. There must be the same amount of $a$ elements beginning and ending the entire string. There may also be one $c$ element in the centre of the string.

   (b)

2. Let $L$ be the language consisting of all binary numbers divisible by 8. Note that any binary number starting with 0 and containing more than one symbol is considered improper and should be rejected.

   (a) Draw a deterministic finite state acceptor that accepts the language $L$. Carefully label all the states including the starting state and the finishing states as well as all the transitions. Make sure you justify it accepts all strings in the language $L$ and no others.

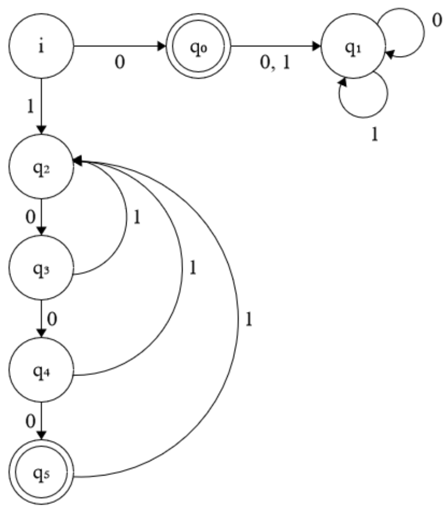
   (b) Devise a regular grammar in normal form that generates the language $L$. Be sure to specify the start symbol, the non-terminals, and all the production rules. Make sure you justify it generates all strings in the language $L$ and no others.

   (c) Prove by applying the definition of a regular language that the language $L$ is regular.

   (d) Write down a regular expression that gives $L$ and justify your answer.

   **Answer:**

   (a) Deterministic finite acceptor:




where $i$ is the initial state, $q_0$ and $q_5$ are accepting states, and $q_1, q_2, q_3$, and $q_4$ are non-accepting states. If the initial number is 0, then it mustn't have any other symbols, and so $q_0$ is the first accepting state. If it does, the DFA gets stuck in a non-accepting loop. If the initial value is 1, then if the next three values are 0, it reaches $q_5$, the other accepting state. This is because for a binary number to be divisible by 8, it must end in "000".

(b) The grammar in normal form can be represented as

i. $\langle S\rangle \rightarrow 1\langle A\rangle$
ii. $\langle A\rangle \rightarrow 0\langle A\rangle$
iii. $\langle A\rangle \rightarrow 1\langle A\rangle$
iv. $\langle A\rangle \rightarrow 0\langle B\rangle$
v. $\langle B\rangle \rightarrow 0\langle B\rangle$
vi. $\langle B\rangle \rightarrow \varepsilon$

The start symbol is $\langle S\rangle$, non-terminals are $\langle A\rangle$ and $\langle B\rangle$, and the terminals are $0, 1$, and the empty word $\varepsilon$. This system ensures only binary numbers divisible by 8 as it begins with 1, which is as specified by the question. It can then go to $\langle A\rangle$, which is preceded by either 1 or 0, followed by another non-terminal $\langle A\rangle$. $\langle A\rangle$ can also lead to another non-terminal $\langle B\rangle$, which is prepended by 0. $\langle B\rangle$ can then lead to either 0 or the empty word $\varepsilon$, which ends the automaton. In this way, once the DFA reaches $\langle B\rangle$, it is stuck in a loop whereby it must either add another 0, or add the empty word, ending the loop. This is all also in normal form, which is also specified by the question.

(c) A regular language can be defined as any language where every word in the language can be recognised by a finite state acceptor. Given that we have already created a DFA for the language $L$, the language is clearly a regular language.

(d) Let $A = \{0, 1\}$. Then the regular expression is $1{*}A{*}000$. This begins with 1, which the question states the strings must begin with, and ends with 000, which any binary string divisible by 8 must contain. The middle can be any combination of 0 and 1.

3. Let A be a finite alphabet.

   (a) Let L be a regular language over the alphabet A. Prove that $A^*\backslash L$, the complement of $L$ in $A^*$, is also a regular language. (Hint: Think about the equivalent conditions characterising a regular language and figure out which one is easiest to check here.)

   (b) Let $L_1$ and $L_2$ be regular languages over the alphabet $A$. Prove that their intersection $L_1 \cap L2$ is a regular language. (Hint: Use part (a) and de Morgan's.)

**Answer:**

(a) Let $D = \{S, A, i, t, F\}$ be a deterministic finite state acceptor. If $L$ is a regular language, then it has a DFA such that every word in $L$ can be recognised by it. If we invert $D$, we get $\bar{D} = \{S, A, i, t, S\backslash F\}$, where the accepting and non-accepting states are switched. This DFA $\overline{D}$ now accepts $\overline{L}$, which is the complement of $L$, which is also denoted as $A^*\backslash L$.

(b) In (a), we proved that for any regular language $L$, $\overline{L}$ is also regular. If $L_1$ and $L_2$ are regular, then $\overline{L_1}$ and $\overline{L_2}$ are also regular. $\overline{L_1} \cup \overline{L_2}$ is also regular, since regular languages are closed under union. Then, by de Morgan's law, $\overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2$, and so the intersection is also a regular language.

4. Let $(V, E)$ be the graph with vertices $a, b, c, d, e, f, g, h, i$, and $j$, and edges $ab, bc, af, bf$, $cf, ef, fg, fi, fj, ad, ch, de, gh, ei, ij$, and $gj$.

   (a) Draw this graph.

   (b) Write down this graph's incidence table and its incidence matrix.

   (c) Write down this graph's adjacency table and its adjacency matrix.

   (d) Is this graph complete? Justify your answer.

   (e) Is this graph bipartite? Justify your answer.

   (f) Is this graph regular? Justify your answer.

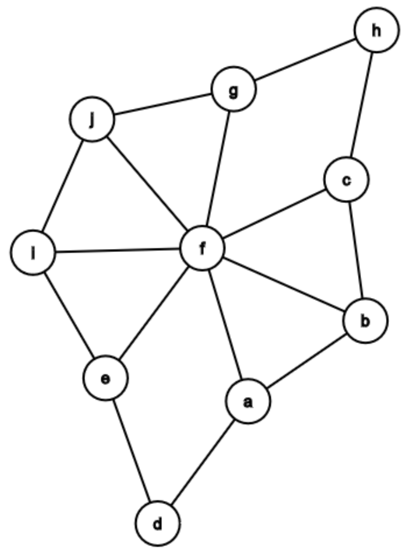
   (g) Does this graph have any regular subgraph? Justify your answer.

   (h) Give an example of an isomorphism $\varphi$ from the graph $(V, E)$ to itself satisfying that $\varphi(b) = b$.

   (i) Is the isomorphism from part (h) unique or can you find another isomorphism $\psi$ that is distinct from $\varphi$ but also satisfies that $\psi(b) = b$? Justify your answer.

**Answer:**

(a) Graph:




(b) Incidence table:

| | ab | bc | af | bf | cf | ef | fg | fi | fj | ad | ch | de | gh | ei | ij | gj |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| b | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| c | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| d | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| e | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| f | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| g | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| h | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| i | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| j | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |

Incidence matrix:

$$\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}$$

(c) Adjecency table:

| | a | b | c | d | e | f | g | h | i | j |
|---|---|---|---|---|---|---|---|---|---|---|
| a | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| b | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| c | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| d | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| e | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 |
| f | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| g | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| h | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| i | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 |
| j | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 |

Adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

(d) The graph is not complete. In order for it to be complete, $\forall a, b \in V$ s.t. $a \neq b, ab \in E$ is an edge. In other words, any two vertices must have an edge between them. But the edge $bj$ does not exist within the graph, so the graph is not complete.

(e) The graph is not bipartite.

(f) No, because for a graph to be regular, all vertices must have the same degree. But vertex $a$ has a degree of 3, whereas the vertex $f$ has a degree of 7.

(g) Yes. The vertices $\{a, b, f\}$ have edges $\{ab, bf, af\}$, which forms a 2-regular subgraph.

(h) The rotation of the graph about $b$ is an isomorphism that satisfies $\varphi(b) = b$.

(i) It is unique, because there are no points that can be swapped or changed to create another isomorphism.