MAU22C00: TUTORIAL 18 SOLUTIONS

1) Is $\{x \in \mathbb{R}^+ \mid \log x \in \mathbb{R} \setminus \mathbb{Q}\}$ finite, countably infinite, or uncountably infinite? Justify your answer. The set $\mathbb{R}^+$ is the set of all positive real numbers.

2) Is $\displaystyle\bigcup_{n=1}^{10} \left\{ \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = n^2\} \cap \{(x,y) \in \mathbb{R}^2 \mid y^2 - x^4 = 0\} \right\}$ finite, countably infinite, or uncountably infinite? Justify your answer.

3) Let $A = \{0,1\}$. Is $(0^* \circ 1^*) \cap \{A^* \circ 11 \circ A^*\}$ finite, countably infinite, or uncountably infinite? Justify your answer.

4) Prove that the language generated by a regular expression is countable. Give an example of a regular expression that generates a finite language and another example of a regular expression that generates a countably infinite language. Justify your answers.

5) Consider the language over the binary alphabet $A = \{0,1\}$ given by $L = \{0^m 1^{2m} \mid m \in \mathbb{N}\}$.

(a) Use the Pumping Lemma to show L is not a regular language.

(b) Is the language L finite, countably infinite, or uncountably infinite? Justify your answer.

(c) A language L' over the same alphabet $A = \{0,1\}$ is called a *sublanguage* of L if $L' \subset L$. Let $\mathcal{C}$ be the set of sublanguages of L. Is $\mathcal{C}$ finite, countably infinite, or uncountably infinite? Justify your answer.

Solution: 1) The log function $log : \mathbb{R}^+ \to \mathbb{R}$ is bijective as you learned before coming to university, which means $\{x \in \mathbb{R}^+ \mid \log x \in \mathbb{R} \setminus \mathbb{Q}\}$ is in bijective correspondence with $\mathbb{R} \setminus \mathbb{Q}$. We showed in lecture that $\mathbb{R}$ is uncountably infinite, while $\mathbb{Q}$ is countably infinite. We also showed in lecture that taking out a countably infinite set from an uncountably infinite one leaves an uncountably infinite set. Therefore, $\{x \in \mathbb{R}^+ \mid \log x \in \mathbb{R} \setminus \mathbb{Q}\}$ is uncountably infinite.

2) $y^2 - x^4 = (y - x^2)(y + x^2) = 0$ so for each n, we are intersecting the circle centered at the origin of radius n with the two parabolae $y = x^2$ and $y = -x^2$. This gives us four intersection points, and we have ten values for n. For different values of n, we get different intersection points, so we

have a total of 40 points in the set $\bigcup_{n=1}^{10} \left\{ \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=n^2\} \cap \{(x,y) \in \mathbb{R}^2 \mid y^2-x^4=0\} \right\}$, which is thus finite.

3) $(0^* \circ 1^*) \cap \{A^* \circ 11 \circ A^*\} = 0^* \circ 11 \circ 1^*$. Each of 0^* and 1^* is countably infinite, so the given set is countably infinite.

4) By definition, a set is countable, if it is finite or countably infinite. A regular expression is built up from $\emptyset$, ϵ, and the letters of the alphabet A via the Kleene star $*$, concatenation, and union. The Kleene star makes a countably infinite set out of a finite one. Concatenation gives a set whose size matches the size of the biggest set in the concatenation. In other words, the concatenation of strings from two finite sets will yield a finite set. The concatenation of strings from a finite set with a countably infinite set will yield a countably infinite set, whereas the concatenation of strings from two countably infinite sets yields a countably infinite set. Union behaves just like concatenation. Therefore, from a finite set via the Kleene star, union, and concatenation, we can only obtain a finite set or a countably infinite set. This concludes our proof. To give the required examples, let us consider the binary alphabet $A = \{0,1\}$. The regular expression $\{01\} \cup \{11\}$ yields a regular language with two elements, whereas the regular expression $0^* \cup 1^*$ gives the regular language consisting of all strings of just 0's and all strings of just 1's, which is countably infinite as the sequence of strings ϵ, 0, 00, 000, etc. is inside this language.

5) (a) If L is a regular language, then it has a pumping length p. In order to consider just one case, we work with $w = 0^p 1^{2p} \in L$. According to the Pumping Lemma, w is to be decomposed as xuy, where $|u| \geq 1$ and $|xu| \leq p$. Since $|xu| \leq p$, u can only consist of zeroes. Let $u = 0^{n_1}$, for some $n_1 \geq 1$. Clearly, $xu^2y \notin L$ as $xu^2y = 0^{p+n_1}1^{2p}$, so the length of the first sequence of zeroes is not one half that of the second sequence of zeroes violating the pattern of the language.

(b) The language L is countably infinite. Consider the function $f : \mathbb{N} \to L$ given by $f(m) = 0^m 1^{2m}$. It is easy to see that f is both injective and surjective hence bijective. Therefore, L is in one-to-one correspondence with $\mathbb{N}$, hence L is countably infinite.

(c) Since a language L' is a sublanguage of L if $L' \subset L$, the set of sublanguages of L, $\mathcal{C}$, is exactly the power set of L, which is denoted by $\mathcal{P}(L)$. We proved in part (b) that L is countably infinite, so $L \sim \mathbb{N}$. Therefore, $\mathcal{P}(L) \sim \mathcal{P}(\mathbb{N})$. As we proved in lecture, $\mathcal{P}(\mathbb{N})$ is uncountably infinite, so $\mathcal{C} = \mathcal{P}(L)$ must likewise be uncountably infinite.