

MAU22C00 - TUTORIAL 5 SOLUTIONS

- 1) Let $A = \{3^p \mid p \in \mathbb{N}\}$ with the operation of multiplication.
- (a) Is (A, \cdot) a semigroup? Justify your answer.
 - (b) Is (A, \cdot) a monoid? Justify your answer.
 - (c) Does (A, \cdot) have invertible elements? If so, which of its elements are invertible? Justify your answer.

Solution: (a) Yes, (A, \cdot) is a semi-group. $A \subset \mathbb{Q}^*$, and $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ is a monoid under the operation of multiplication. We proved in lecture that if $a \in M$ for M a monoid with operation $*$ and $m, n \in \mathbb{N}$, then $a^m * a^n = a^{m+n}$. Here $a = 3$ and since addition is a binary operation on \mathbb{N} as we showed in class, multiplication is a binary operation on A . The associativity of multiplication on A follows from the associativity of addition on \mathbb{N} and the theorem that if $a \in M$ for M a monoid with operation $*$ and $m, n \in \mathbb{N}$, then $a^m * a^n = a^{m+n}$.

(b) Yes, (A, \cdot) is a monoid. $3^0 = 1$ is the identity element on A because any $b \in A$ is of the form 3^p , so $b \cdot 1 = a^p \cdot a^0 = a^{p+0} = a^{0+p} = 1 \cdot b = a^p = b$.

(c) By the theorem on powers proved in lecture that was quoted above, $3^m \cdot 3^n = 3^{m+n}$. The condition for invertibility $3^{m+n} = 3^0$ for $m \geq 0$ and $n \geq 0$ is only satisfied if $m = n = 0$. Therefore, $3^0 = 1$ is the only invertible element of A .

- 2) (Slightly modified question from the annual exam 2017-2018) Let $A = \{(x, y) \in \mathbb{R}^2 \mid x + 2y = 0\}$ with the operation of addition given by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

- (a) Is $(A, +)$ a semigroup? Justify your answer.
- (b) Is $(A, +)$ a monoid? Justify your answer.
- (c) Does $(A, +)$ have invertible elements? If so, which of its elements are invertible? Justify your answer.
- (d) What geometric object is the set A in \mathbb{R}^2 ?

Solution: (a) Yes, $(A, +)$ is a semi-group. If $x_1 = -2y_1$ and $x_2 = -2y_2$, then $x_1 + x_2 = -2y_1 - 2y_2 = -2(y_1 + y_2)$, so $+$ is a binary operation on A . We proved in lecture that addition is an associative binary operation on \mathbb{R} , so $+$ is associative on A as associativity will function component by component in the vector (x, y) .

(b) Yes, $(A, +)$ is a monoid. $(0, 0)$ is the identity element on A because for any $(x, y) \in A$,

$$(x, y) + (0, 0) = (x + 0, y + 0) = (0 + x, 0 + y) = (0, 0) + (x, y) = (x, y).$$

(c) For any $(x, y) \in A$, $(-x, -y)$ is its inverse because $(x, y) + (-x, -y) = (-x, -y) + (x, y) = (0, 0)$. Therefore, all elements of A are invertible.

(d) A is the line passing through the origin $(0, 0)$ and the point $(2, -1)$ as $2 + 2(-1) = 0$.