Logic & the search for truth

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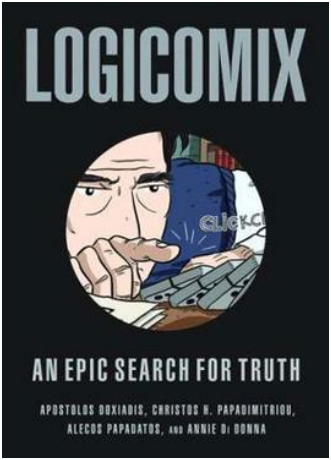

Logic & the search for truth

Challenges to

- truth

  Liar's Paradox: 'I am lying'

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- sets (membership $\in$)
  Russell set $R = \{x \mid \text{not } x \in x\}$

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  $R \in R \iff \text{not } R \in R$

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- search (one by one)

  Cantor: cannot count subsets of $\{0, 1, 2, \ldots\}$

$$\begin{array}{ll}
s_1 & = 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \ldots \\
s_2 & = 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1 \ldots \\
s_3 & = 0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0 \ldots \\
s_4 & = 1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1 \ldots \\
s_5 & = 1\,1\,0\,1\,0\,1\,1\,0\,1\,0\,1 \ldots \\
s_6 & = 0\,0\,1\,1\,0\,1\,1\,0\,1\,1\,0 \ldots \\
s_7 & = 1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0 \ldots \\
s_8 & = 0\,0\,1\,1\,0\,0\,1\,1\,0\,0\,1 \ldots \\
s_9 & = 1\,1\,0\,0\,1\,1\,0\,0\,1\,1\,0 \ldots \\
s_{10} & = 1\,1\,0\,1\,1\,1\,0\,0\,1\,0\,1 \ldots \\
s_{11} & = 1\,1\,0\,1\,0\,1\,0\,0\,1\,0\,0 \ldots \\
\vdots & \quad \vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots \ddots
\end{array}$$

$$s = 1\,0\,1\,1\,1\,0\,1\,0\,0\,1\,1 \ldots$$

https://upload.wikimedia.org/wikipedia/commons/e/ed/Diagonal_argument_01.pdf

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- computability
  Turing: Halting Problem

The Halting Problem

Given a program P and data D, return either 0 or 1 (as output), with 1 indicating that P halts on input D

$$\mathrm{HP}(P, D) \;:=\; \begin{cases} 1 & \text{if } P \text{ halts on } D \\ 0 & \text{otherwise} \end{cases}$$

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$$\mathrm{HP}(P, D) \quad := \quad \begin{cases} 1 & \text{if } P \text{ halts on } D \\ 0 & \text{otherwise} \end{cases}$$

Theorem (Turing) *No TM computes HP.*

The proof is similar to the Liar's Paradox distributed as follows

H: 'L speaks the truth'
L: 'H lies'

with a spoiler L (exposing H as a fraud).

Proof of uncomputability

Given a TM P that takes two arguments, we show P does not compute HP by defining a TM $\overline{P}$ such that

$$P(\overline{P}, \overline{P}) \quad \neq \quad \mathrm{HP}(\overline{P}, \overline{P}) \ .$$

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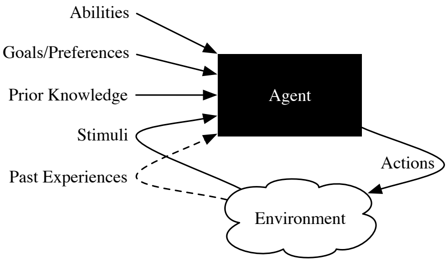
Let

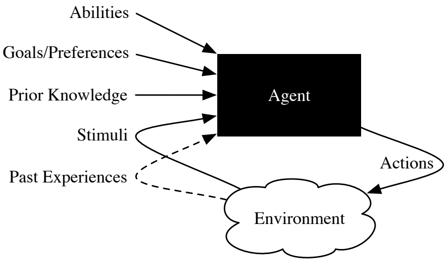
$$\overline{P}(D) \quad :\simeq \quad \begin{cases} 1 & \text{if } P(D, D) = 0 \\ \text{loop} & \text{otherwise.} \end{cases}$$

and notice

$$\begin{aligned} \mathrm{HP}(\overline{P}, \overline{P}) &= \begin{cases} 1 & \text{if } \overline{P} \text{ halts on } \overline{P} \\ 0 & \text{otherwise} \end{cases} && \text{(def of HP)} \\ &= \begin{cases} 1 & \text{if } P(\overline{P}, \overline{P}) = 0 \\ 0 & \text{otherwise} \end{cases} && \text{(def of } \overline{P}\text{)} \end{aligned}$$

$\square$


Abilities
Goals/Preferences
Prior Knowledge
Stimuli
Past Experiences
Agent
Actions
Environment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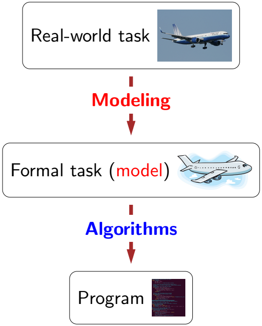
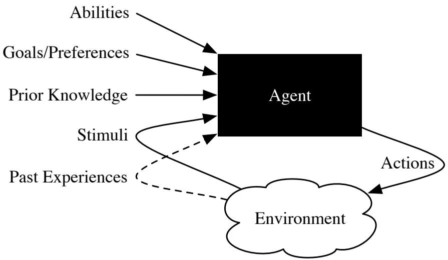
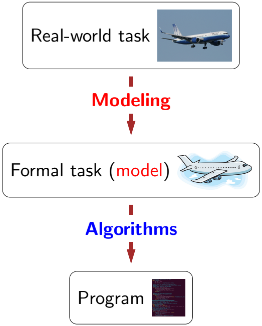
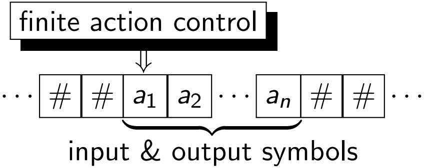

Real-world task
Modeling
Formal task (model)
Algorithms
Program

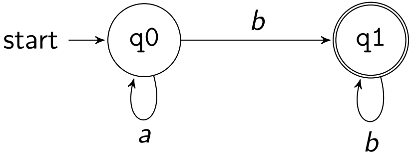




Church-Turing thesis: Program $\approx$ Turing machine



Finite state machine (fsm)



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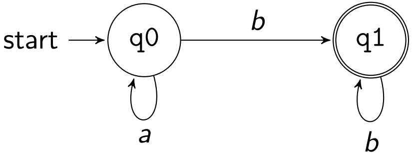

A **fsm** `M` is a triple `[Trans, Final, Q0]` where

- `Trans` is a list of triples `[Q,X,Qn]` such that `M` may, at state `Q` seeing symbol `X`, change state to `Qn`
- `Final` is a list of `M`'s final (i.e. accepting) states
- `Q0` is `M`'s initial state.

E.g.
```
Trans = [[q0,a,q0],[q0,b,q1],[q1,b,q1]]
Final = [q1]
Q0 = q0
```

From strings to fsm's

Encode strings as lists; e.g. 102 as `[1,0,2]`.

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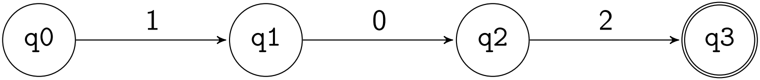

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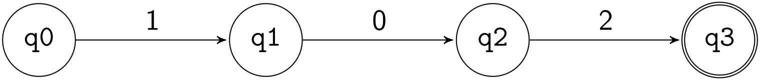


```
% string2fsm(+String, ?TransitionSet, ?FinalStates)
string2fsm([], [], [q0]).
string2fsm([H|T], Trans, [Last]) :-
```

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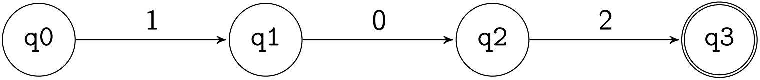


```
% string2fsm(+String, ?TransitionSet, ?FinalStates)
string2fsm([], [], [q0]).
string2fsm([H|T], Trans, [Last]) :-
           mkTL(T, [H], [[q0, H, [H]]], Trans, Last).
```

```
% mkTL(+More, +LastSoFar, +TransSoFar, ?Trans, ?Last)
mkTL([], L, Trans, Trans, L).
mkTL([H|T], L, TransSoFar, Trans, Last) :-
     mkTL(T, [H|L], [[L,H,[H|L]]|TransSoFar],
          Trans, Last).
```

States as histories (in reverse)

More on *states-as-histories*

Encoding q0 as [] leads to the simplification

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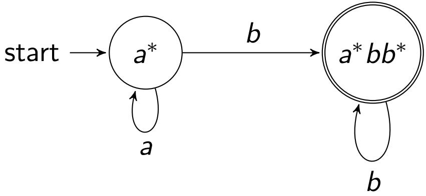

for *state-as-set-of-equivalent-histories*,
where equivalence has to do with acceptable futures . . .

Exercise

Define a 4-ary predicate

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accept(+Trans,+Final,+Q0,?String)
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that is true exactly when `[Trans,Final,Q0]` is a fsm that accepts `String` (encoded as a list).

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That is, write a Prolog program to answer queries such as

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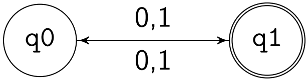
true
```



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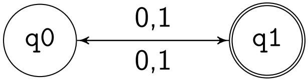
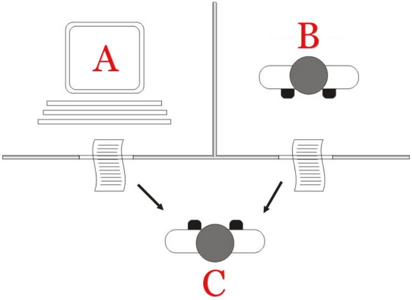
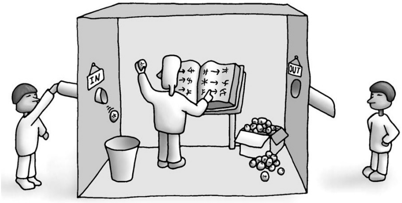
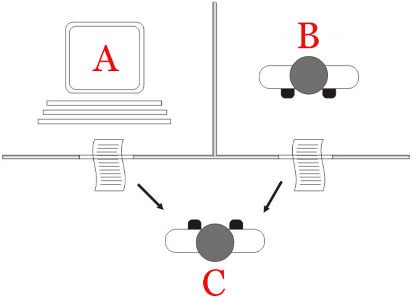


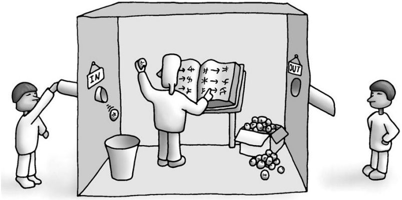

```
test(String) :- string2fsm(String, Trans, Final),
                accept(Trans, Final, q0, String).
```

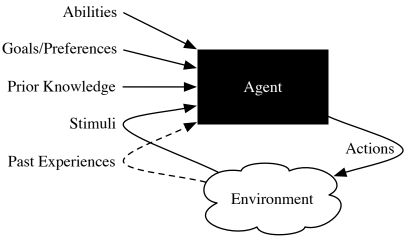

A
B
C


IN
OUT

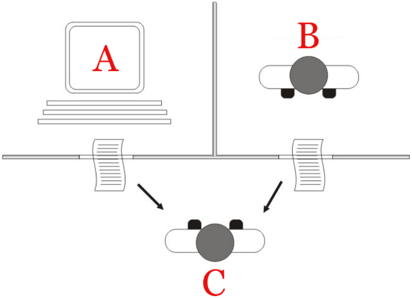
the black box

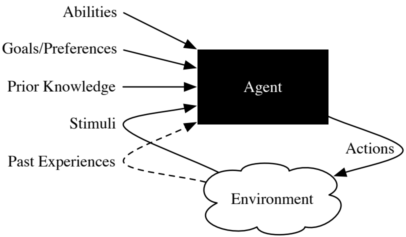


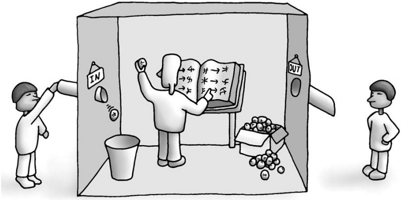




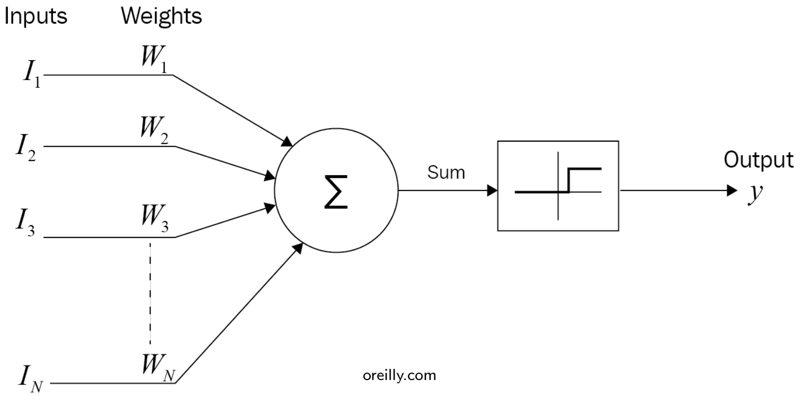
Peering inside the black box







Agents from neurons (perceptrons)



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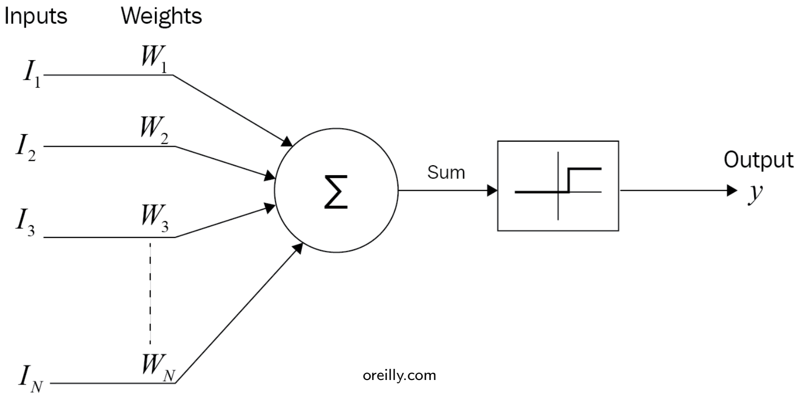
Agents from neurons (perceptrons)

$$\text{Sum} = W_1 I_1 + W_2 I_2 + \cdots + W_N I_N$$



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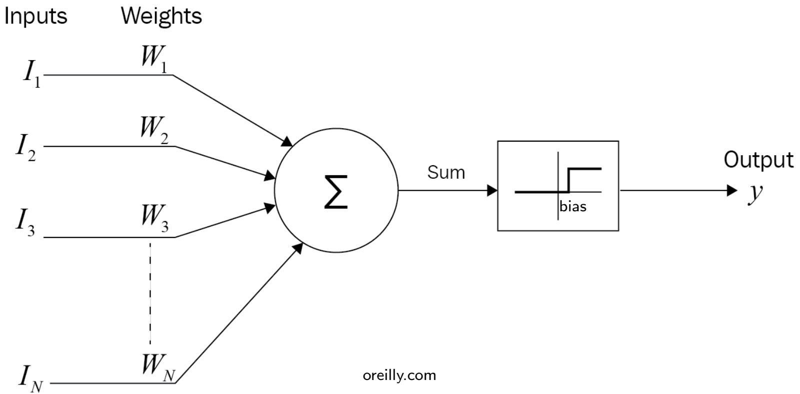
Agents from neurons (perceptrons)

$$\text{Sum} = W_1 I_1 + W_2 I_2 + \cdots + W_N I_N$$

$$y = \begin{cases} 1 & \text{if Sum} \geq \text{bias} \\ 0 & \text{if Sum} < \text{bias} \end{cases}$$



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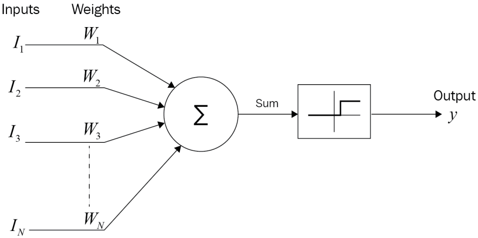
Finite state aggregations



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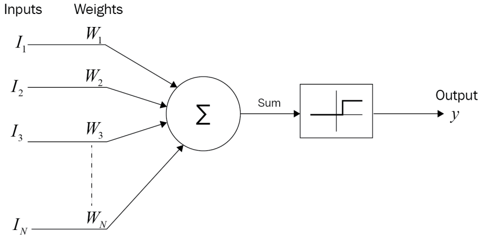

Kleene 1956

$$I_1, I_2 \cdots I_N \;=\; q, a$$

$q \overset{a}{\rightsquigarrow} y \iff$ output y on input a at q

input a given by k *input cells* $\mathrm{N}_1 \ldots \mathrm{N}_k$
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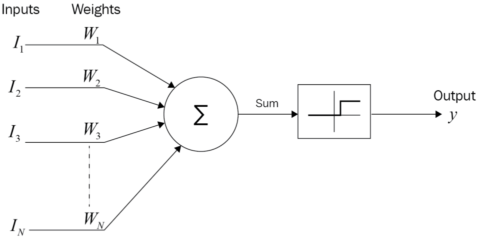

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McCulloch & Pitts 1943: neural activity is *all-or-none*

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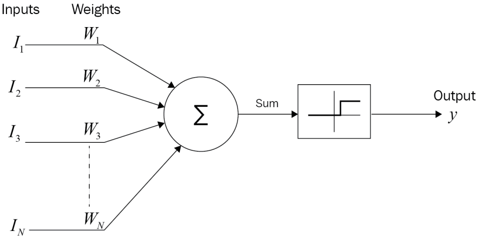

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Rabin & Scott 1959: work with any finite sets A, Q and $\rightarrow$

Logical abstractions away from physics and biology

1976 Turing Award to Rabin & Scott for their joint paper

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