Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science
School of Computer Science and Statistics

Integrated Computer Science **Semester 2**
Year 1
BA (Mod) Computer Science and Language
Year 1
BA (Mod) Computer Science and Business
Year 2

Mathematics II

Tuesday, 18 May 2021 **Online** **12:00 – 14:30**

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Instructions to Candidates:

- You must download and save a local copy of this test paper before you begin work on your solution.
- This is an individual assignment.
- Atempt ALL PARTS OF QUESTION 1 in SECTION A. This question counts for 40% of this exam.
- Attempt ALL QUESTIONS from SECTION B. Each question counts for 20% of this exam.
- You must show all of your workings to be eligible for full marks.
- Answers without any workings will not get any marks.
- Your clearly legible, handwritten work should be submitted as a single pdf file that also includes a signed CSU12002 Assessment Declaration form as the first page of your submission.
- Only one file submission is permitted.
- It is your responsibility to confirm that the file you have uploaded as your solution is the correct file before you hit the submit button.
- It is your responsibility to verify that your solution file is correctly uploaded and suitable for grading.

- All submissions must be through Blackboard. The submission time is recorded in Blackboard.

SECTION A

Answer all parts of Question 1.

Question 1

(a) Prove that the following statement holds for all integers a.

$5a + 4$ is even only if $a + 7$ is odd.

Identify the method of proof that you used.

[5 marks]

(b) Express each of the following statements using predicates, quantifiers, and logical connectives where necessary.

(i) If neither Mary nor Ronan is available, then Bob and Mary study together.

(ii) If Alice is not available, then Bob asks Mary and Mary asks someone.

[5 marks]

(c) Determine whether the following argument is correct or incorrect. Give a fully justified reason for your answer by defining suitable propositions, stating the argument you are going to check using those propositions and then determining whether the argument is valid or not. You may not use truth tables to check the validity of the argument.

Either the program does not terminate or p eventually becomes 0.

If p becomes 0, q will eventually be 0.

The program terminates. Therefore, q will eventually be 0.

[5 marks]

(d) Given y is an integer, use proof by contradiction to prove that

If $5y + 2$ is even, then y is even.

[5 marks]

(e) Show the following two sets are equal:

$$
\begin{aligned}
A &= \left\{ \left(t - 1, \frac{1}{t} \right) \mid t \in \mathbb{R}, t \neq 0 \right\} \\
B &= \left\{ (x, y) \in \mathbb{R}^2 \mid y = \frac{1}{x+1}, x \neq -1 \right\}
\end{aligned}
$$

[5 marks]

(f) C and D are subsets of $\mathbb{R}^2$ given by:

$$\begin{aligned} C &= \{(x,y) \in \mathbb{R}^2 \mid y \geq 1\} \\ D &= \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\} \end{aligned}$$

Sketch each of the following sets on separate diagrams: C, D, $C \cup D$ and $D \setminus C$.

[5 marks]

(g) Find the multiplicative inverse of 426 modulo 743, showing all of your calculations.

[5 marks]

(h) Determine whether $5^7 \equiv 7^5 \pmod{11}$ using methods that do not require you to have access to a calculator. Justify your answer fully using the methods learnt in CSU12002 and show all of your workings.

[5 marks]

[40 marks]

SECTION B

Answer ALL questions from this Section.

Question 2

(a) Determine whether the following argument is correct or incorrect. Give a fully justified reason for your answer by defining suitable predicates, stating the argument you are going to check using these and then determining whether the argument is valid or not using the methods we have met in CSU12002.

If I take a day off to go sailing it either rains or is not windy.

I took Monday off or I took Wednesday off.

Monday was a windy day when it did not rain.

It was windy on Wednesday.

Therefore, it rained on Wednesday.

[10 marks]

(b) With reference to the proof techniques and theorems you met in CSU12002, discuss:

(i) the role that definitions and axioms play in the creation of proofs.

(ii) the essential role that proof plays in the foundation of mathematical knowledge.

[10 marks]

[20 marks]

Question 3

(a) Let the set operator, $\bigtriangledown$, be defined as follows

$$A \bigtriangledown B = \overline{A \cup B}$$

where $\overline{S}$ is the complement of S.
Determine by Veitch diagram, or otherwise, whether:

i) $A \bigtriangledown B = \overline{A} \cap \overline{B}$

ii) $A \cap B = (A \bigtriangledown A) \bigtriangledown (B \bigtriangledown B)$

iii) $A \cup B = (A \bigtriangledown B) \bigtriangledown (A \bigtriangledown B)$

iv) $(A \bigtriangledown B) \cup C = (A \cap C) \bigtriangledown (B \cap C)$

[12 marks]

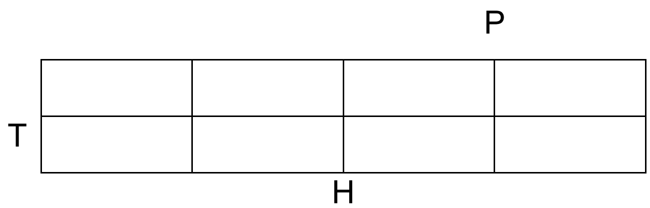
(b) 160 people were surveyed concerning which classical music composers they like.

- 60 like Tchaikovsky, 85 like Chopin, 82 like Handel
- 20 like Tchaikovsky and Handel
- 43 like Tchaikovsky and Chopin
- 25 like Handel and Chopin
- 13 people like all three composers

i) How many people like none of the three composers

ii) How many people like just Tchaikovsky and neither of the other two.

The following is a suggested labelling of a Veitch diagram, where:
T: **T**chaikovsky P: Cho**P**in, H: **H**andel.



[8 marks]

[20 marks]

Question 4

(a) If possible, solve the following linear congruence. Provide a full justification for your answer.

$$21x \equiv 6 \pmod{70}$$

[5 marks]

(b) Find the value of the unique integer x satisfying $0 \leq x < 13$ for which

$$3^{3002} \equiv x \pmod{13}.$$

[5 marks]

(c) Find an integer x such that $x \equiv 2 \pmod{5}$, $x \equiv 1 \pmod{13}$ and $x \equiv 5 \pmod{17}$. Show all of your workings and state any results you make use of in your calculations.

[10 marks]

[20 marks]

CSU12002 Tautologies

1. $P \vee \neg P$ Law of the excluded middle
2. $\neg(P \wedge \neg P)$ Law of non-contradiction
3. $\neg\neg P \leftrightarrow P$ Law of double negation
4. $(P \wedge Q) \rightarrow P$ Basis for simplification
5. $(P \wedge Q) \rightarrow Q$ Basis for simplification
6. $P \rightarrow P \vee Q$ Basis for addition
7. $Q \rightarrow P \vee Q$ Basis for addition
8. $Q \rightarrow (P \rightarrow Q)$
9. $\neg P \rightarrow (P \rightarrow Q)$
10. $[P \wedge (P \rightarrow Q)] \rightarrow Q$ Modus ponens
11. $[\neg Q \wedge (P \rightarrow Q)] \rightarrow \neg P$ Modus tollens
12. $\neg P \wedge (P \vee Q) \rightarrow Q$
13. $P \rightarrow [Q \rightarrow (P \wedge Q)]$
14. $[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$ Transitivity of implications
15. $(P \rightarrow Q) \rightarrow [(P \vee R) \rightarrow (Q \vee R)]$
16. $(P \rightarrow Q) \rightarrow [(P \wedge R) \rightarrow (Q \wedge R)]$
17. $[(P \leftrightarrow Q) \wedge (Q \leftrightarrow R)] \rightarrow (P \leftrightarrow R)$ Transitivity of equivalences
18. $\neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$ De Morgan's law
19. $\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$ De Morgan's law
20. $\neg(P \rightarrow Q) \leftrightarrow P \wedge \neg Q$
21. $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$
22. $(P \leftrightarrow Q) \leftrightarrow [(P \rightarrow Q) \wedge (Q \rightarrow P)]$
23. $(P \leftrightarrow Q) \leftrightarrow [(P \wedge Q) \vee (\neg Q \wedge \neg P)]$
24. $(P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P)$ Law of contraposition (The contrapositive)
25. $[(P \rightarrow Q) \wedge (P \rightarrow R)] \leftrightarrow [P \rightarrow (Q \wedge R)]$
26. $[(P \rightarrow R) \wedge (Q \rightarrow R)] \leftrightarrow [(P \vee Q) \rightarrow R]$ Basis for proof by cases
27. $[P \rightarrow (Q \rightarrow R)] \leftrightarrow [(P \wedge Q) \rightarrow R]$
28. $(P \rightarrow Q \wedge \neg Q) \leftrightarrow \neg P$ Basis for indirect proofs
29. $[(P \wedge (Q \vee R)] \leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$ Law of distributivity
30. $[(P \vee (Q \wedge R)] \leftrightarrow [(P \vee Q) \wedge (P \vee R)]$ Law of distributivity
31. $(P \wedge Q) \leftrightarrow (Q \wedge P)$ Law of commutativity
32. $(P \vee Q) \leftrightarrow (Q \vee P)$ Law of commutativity
33. $[(P \vee (Q \vee R)] \leftrightarrow [(P \vee Q) \vee R]$ Law of associativity
34. $[(P \wedge (Q \wedge R)] \leftrightarrow [(P \wedge Q) \wedge R]$ Law of associativity

CSU12002 Inference Rules

1 Rules of inference involving no quantifiers

1) **Propositional Consequence (PC):** In a proof, any statement that is a propositional consequence of previous steps in the proof can be asserted.

2) **Modus Ponens:** In a proof containing both P and $P \to Q$, the statement Q can be asserted.

$$\begin{array}{l} P \\ P \to Q \\ \hline \therefore Q \end{array}$$

3) **Conditional Proof/Direct Proof:** Assume P. If Q can be proven from the assumption P, then the implication $P \to Q$ can be asserted.

$$\begin{array}{l} \text{Assume } P \\ \text{Intermediate steps} \\ Q \\ \hline \therefore P \to Q \end{array}$$

4) **Indirect Proof/Proof by Contradiction/Reductio ad Absurdum:** Assume $\neg P$ and prove ANY contradiction, then P holds.

$$\begin{array}{l} \text{Assume } \neg P \\ \text{Intermediate steps} \\ \text{Any contradiction} \\ \hline \therefore P \end{array}$$

5) **Proof by Cases:** $Q \vee R$ is one of the steps of the proof. $Q \to P$ and $R \to P$ are also steps in the proof. Then P can be asserted.

$$\begin{array}{l} Q \vee R \\ \text{Assume } Q \\ \text{Intermediate steps} \\ P \text{ (end of case 1)} \\ \text{Assume } R \\ \text{Intermediate steps} \\ P \text{ (end of case 2)} \\ \hline \therefore P \end{array}$$

6) **Biconditional Rule:** If the implications $P \to Q$ and $Q \to P$ appear in the course of the proof, then $P \leftrightarrow Q$ can be asserted.

$$\begin{array}{l} P \to Q \\ Q \to P \\ \hline \therefore P \leftrightarrow Q \end{array}$$

7) **Substitution:** Let $S(P)$ be a statement containing P as a sub-statement. Let $S(P/Q)$ denote a statement that results from $S(P)$ by replacing one or more occurrences of the statement P by the statement Q. From $P \to Q$ and $S(P)$, $S(P/Q)$ can be asserted provided no free variables of P or Q become quantified in $S(P)$ or $S(P/Q)$.

$$\begin{array}{l} P \leftrightarrow Q \\ S(P) \\ \hline \therefore S(P/Q) \end{array}$$

8) **Conjunction:** If statements P and Q appear as steps in the proof, the compound statement $P \land Q$ can be asserted.

$$\begin{array}{l} P \\ Q \\ \hline \therefore P \land Q \end{array}$$

9) **Modus Tollens:** If $P \to Q$ and $\neg Q$ are both steps in the proof, then $\neg P$ can be asserted.

$$\begin{array}{l} P \to Q \\ \neg Q \\ \hline \therefore \neg P \end{array}$$

10) **Contrapositive Conditional Proof:** If the assumption $\neg Q$ leads to the conclusion $\neg P$, then the implication $P \to Q$ can be asserted.

$$\begin{array}{l} \text{Assume} \neg Q \\ \text{Intermediate steps} \\ \neg P \\ \hline \therefore P \to Q \end{array}$$

2 Rules of inference involving quantifiers

11) **De Morgan's Laws for Quantifiers (Axioms):**

    (a) $\neg(\forall x\, P(x)) \leftrightarrow \exists x\, \neg P(x)$;

    (b) $\neg(\exists x\, P(x)) \leftrightarrow \forall x\, \neg P(x)$;

12) **Universal Specification Axiom (US):** If the domain of the variable is not empty, $\forall x\, P(x) \to P(t)$ can be asserted, where t is an object in the domain and P is a statement depending upon one free variable.

13) **Universal Generalization Rule of Inference (UG):** If $P(x)$ can be proven, where x is a free variable representing an arbitrary element of a certain domain, $\forall x\, P(x)$ can be asserted.

14) **Existential Specification Rule of Inference (ES):** If a step of the form $\exists x\, P(x)$ appears in the proof, $P(c)$ can be asserted, where c is a constant symbol.

15) **Existential Generalization Axiom (EG):** $P(t) \to \exists x\, P(x)$ can be asserted, if t is an object in the domain of x (assumed not empty).