MAU22C00 - TUTORIAL 3 SOLUTIONS

1) (From the 2016-2017 Annual Exam) Let Q denote the relation on the set $\mathbb{Z}$ of integers, where integers x and y satisfy xQy if and only if

$$x - y = (x - y)(x + 2y).$$

Determine the following:

(i) Whether or not the relation Q is *anti-symmetric*;
(ii) Whether or not the relation Q is a *partial order*.

Justify your answers.

Solution: $x, y \in \mathbb{Z}$ satisfy xQy iff $x - y = (x - y)(x + 2y)$, which is equivalent to $(x - y)(x + 2y - 1) = 0$, i.e., $x = y$ or $x + 2y - 1 = 0$.

(i) **Anti-symmetry:** The relation Q is anti-symmetric. Having xQy and yQx when $x \neq y$ would imply $x + 2y = 1$ and $y + 2x = 1$ hold simultaneously, which gives $x = y = \frac{1}{3} \notin \mathbb{Z}$. Therefore, xQy and yQx can both be true only if $x = y$.

(ii) **Partial order:** The relation Q is not a partial order because while reflexive and anti-symmetric, it fails to be transitive as seen in tutorial 2.

2) Let A be a set, and let $\mathcal{A} = \{A_\alpha \mid \alpha \in I\}$, where I is an indexing set, be any partition of the set A. Define a relation R on A as follows: $x, y \in A$ satisfy xRy iff $x, y \in A_\alpha$ for some $\alpha \in I$. In other words, xRy iff x and y belong to the same set of the partition. Prove that R is an equivalence relation and that the partition R defines on A is precisely the given partition $\mathcal{A}$.

(Hint: Recall we discussed in lecture the one-to-one correspondence between partitions and equivalence relations, and this is the proof direction I sketched in lecture without providing the details.)

Solution: First, let us prove R is an equivalence relation:

Reflexivity: For any $x \in A$, since $\mathcal{A} = \{A_\alpha \mid \alpha \in I\}$ is a partition of A, there exists $\alpha \in I$ such that $x \in A_\alpha$. The element x is in the same set A_α as itself, so xRx.

Symmetry: If xRy, then by definition $x, y \in A_\alpha$ for some $\alpha \in I$, i.e. x and y belong to the same set of the partition. Therefore, yRx holds as well.

Transitivity: If xRy, then by definition $x, y \in A_\alpha$ for some $\alpha \in I$. If yRz, then z belongs to the same set of the partition as y, namely $z \in A_\alpha$ for the same α. Thus, $x, y, z \in A_\alpha$, which means xRz holds as well.

The partition determined by R is exactly $\mathcal{A}$: If $x \in A_\alpha$, then the equivalence class of x given by $[x]_R = A_\alpha$ by the very definition of R. Since $\mathcal{A}$ is a partition of A and it consists of the set of equivalence classes determined by the relation R, we conclude that R determines the partition $\mathcal{A}$ as needed.