MAU22C00 - TUTORIAL 4 SOLUTIONS

1) (From the 2016-2017 Annual Exam) Let $f : [-2, 2] \to [-15, 1]$ be the function defined by $f(x) = x^2 + 3x - 10$ for all $x \in [-2, 2]$. Determine whether or not this function is injective and whether or not it is surjective. Justify your answers.

Injectivity: $f(x) = x^2 + 3x - 10 = (x - 2)(x + 5)$ This function is not injective on the interval $[-2, 2]$. Acceptable justifications: drawing the graph, providing two values $x_1, x_2 \in [-2, 2]$, $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$ (for example, looking at the sign of its derivative to figure out when the function is increasing or decreasing in order to find those values), etc. Note that $f(-1) = f(-2) = -12$ and $-1 \neq -2$ for $-2, -1 \in [-2, 2]$.

Surjectivity: $f(x) = x^2 + 3x - 10$ is not surjective on the interval $[-2, 2]$. Acceptable justifications: drawing the graph, providing a value in $[-15, 1]$ that $f(x)$ does not assume, showing the minimum value occurs at $-\frac{3}{2}$, where $f\left(-\frac{3}{2}\right) = -12.25 > -15$, etc.

2) Use mathematical induction to prove the geometric series formula, which states that for any $a, r \in \mathbb{R}$ with $r \neq 1$ and any $n \in \mathbb{N}^*$,

$$a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{(1 - r^n)}{(1 - r)}.$$

Solution: Fix $a, r \in \mathbb{R}$, $r \neq 1$.
Base case: $n = 1$**.**
Then

$$a\frac{(1 - r^1)}{(1 - r)} = a(1) = a$$

as required.
Induction step: Assume true for $n = k$**.**
Prove true for $n = k + 1$**.**

$$a + ar + ar^2 + \cdots + ar^{k-1} + ar^k = a\frac{(1 - r^k)}{(1 - r)} + ar^k$$

$$= a\left(\frac{(1 - r^k)}{(1 - r)} + \frac{(1 - r)r^k}{(1 - r)}\right) = a\left(\frac{(1 - r^k) + (r^k - r^{k+1})}{(1 - r)}\right)$$

$$= a\frac{(1 - r^{k+1})}{(1 - r)}$$

as required.

3) Where is the fallacy in the following argument by induction?

Statement: If p is an even number and $p \geq 2$, then p is a power of 2.

"Proof:" We give a proof using strong induction on the even number p. Denote by $P(n)$ the statement "if n is an even number and $n \geq 2$, then $n = 2^j$, where $j \in \mathbb{N}$."

Base case: Show $P(2)$. $2 = 2^1$, so 2 is indeed a power of 2.

Inductive step: Assume $p > 2$ and that $P(n)$ is true for every n such that $2 \leq n < p$ (the strong induction hypothesis). We have to show that $P(p)$ also holds. We consider two cases:

Case 1: p is odd, then there is nothing to show.

Case 2: p is even. Since $p \geq 4$ and p is an even number, we can write $p = 2n$ with $2 \leq n < p$. By the inductive hypothesis, $P(n)$ holds, so we conclude that $n = 2^j$ for some $j \in \mathbb{N}$. Since $p = 2n = 2 \times 2^j = 2^{j+1}$, we conclude that $P(p)$ also holds.

Solution: The argument fails at the inductive step as it is possible that $p = 2n$ and n is not even. For example, if $p = 6 = 2 \times 3$, the argument in the inductive step fails.