ST2004 Applied Probability I

Extra Exercises

1. Suppose 50% of emails are spam emails. A software can detect 99% of spam emails, and the probability that a non-spam email is detected as spam is 5%. If an email is detected as spam, what is the probability that it is in fact a non-span email?

2. A fair coin is tossed 3 times. Let A be the event that there are two heads in total, and B be the event that the first toss is heads. Are A and B independent events?

3. Let $X_1, X_2, \ldots, X_n$ be n independent discrete random variables with $\mathbb{P}(X_i = c) = 1$ for $i = 1, \ldots, n$. Let $X = \sum_{i=1}^{n} X_i$. Compute $\mathbb{E}(X)$ and $Var(X)$.

4. A service provider has a weekly demand Y following a probability density function given by

$$f(y) = \begin{cases} y, & 0 \leq y < 1, \\ 1, & 1 \leq y < 1.5, \\ 0, & \text{otherwise} \end{cases}$$

   The profit of the service provider is given by $U = 10Y - 4$.

   (a) Find the probability density function of U.

   (b) Find $\mathbb{E}(U)$.

5. Let $Y_1, Y_2, \ldots, Y_n$ be independent and identically distributed random variables. Let $V = \min(Y_1, \ldots, Y_n)$.

   (a) Show that the cdf (cumulative distirbution function) of V is given by

$$F_V(v) = 1 - (1 - F_Y(v))^n$$

   where $F_Y(\cdot)$ is the cdf of Y_i, $i = 1, \ldots, n$.

   (b) A system contains 10 components, each of which operate independently for an exponential distributed time with mean 5 years before failure. The system will fail when any of the components fails. What is the distribution of the time until failure of the system?

6. For each of two construction jobs, the contract is independently assigned at random to one of three firms A, B, C where each of the nine outcomes $(A, A), (A, B), \cdots, (C, C)$ are equally likely. Let Y_1, Y_2 denote the numbers of contracts assigned to firms A and B respectively. Each firm can receive 0, 1, or 2 contracts.

   (a) Find the joint probability mass function for Y_1 and Y_2.

   (b) Find the marginal probability mass functions of Y_1 and Y_2.

   (c) Confirm that Y_1 and Y_2 each follows a binomial distribution with $n = 3$ and $p = 1/3$. Explain why.

   (d) Are Y_1 and Y_2 independent?