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7.7 Homomorphisms and Isomorphisms

Task: Understand the most natural functions between objects in abstract algebra such as semigroups, monoids or groups.

Definition: Let $(A, *)$ and $(B, *)$ both be semigroups, monoids or groups. A function $f : A \to B$ is called a homomorphism if

$$f(x * y) = f(x) * f(y) \; \forall x, y \in A.$$

In other words, if f is a function that respects (behaves well with respect to) the binary operation.

Examples:

1. Consider $(\mathbb{Z}, +, 0)$ and $(\mathbb{R}^*, \times, 1)$.
   Pick $a \in \mathbb{R}^*$, then $f(n) = a^n$ is a homomorphism between $(\mathbb{Z}, +, 0)$ and $(\mathbb{R}^*, \times, 1)$ because $(\mathbb{R}^*, \times, 1)$ is a group, and we proved for groups that $a^{m+n} = f(m+n) = a^m * a^n = f(m) * f(n)\ \forall m, n \in \mathbb{Z}$.
2. More generally, $\forall a \in A$ invertible, where $(A, *)$ is a monoid with identity element e, $f(m) = a^m$ gives a homomorphism between $(\mathbb{Z}, +, 0)$ and $(A', *, e)$, where as before $A' = \{a^m \mid m \in \mathbb{Z}\} \subset A$.
   We get even better behaviour if we require $f : A \to B$ to be bijective.

Definition: Let $(A, *)$ and $(B, *)$ both be semigroups, monoids or groups. A function $f : A \to B$ is called an isomorphism if $f : A \to B$ is both bijective <u>AND</u> a homomorphism.

Examples:

1. Let $A' = \{2^m \mid m \in \mathbb{Z}\} = \{1, 2, \frac{1}{2}, 4, \frac{1}{4}, ...\}$
   $f(m) = 2^m$ from $(\mathbb{Z}, +, 0)$ to $(A', \times, 1)$ is an isomorphism since $2^m \neq 2^n$ if $m \neq n$.
2. Let $A' = \{(-1)^m \mid m \in \mathbb{Z}\} = \{-1, 1\}$
   $f(m) = (-1)^m$ from $(\mathbb{Z}, +, 0)$ to $(A', \times, 1)$ is <u>NOT</u> an isomorphism since it's not injective $(-1)^2 = (-1)^4 = 1$.

Theorem: Let $(A, *)$ and $(B, *)$ both be semigroups, monoids or groups. The inverse $f^{-1} : B \to A$ of any isomorphism $f : A \to B$ from A to B is itself an isomorphism.

Proof: If $f : A \to B$ is an isomorphism $\Rightarrow f : A \to B$ is bijective $\Rightarrow f^{-1} : B \to A$ is bijective (proven when we discussed functions).

To show $f^{-1} : B \to A$ is a homomorphism, let $u, v \in B$. $\exists x, y \in A$ s.t. $x = f^{-1}(u)$ and $y = f^{-1}(v)$, but then $u = f(x)$ and $v = f(y)$.

Since $f : A \to B$ is a homomorphism, $f(x * y) = f(x) * f(y) = u * v$. Then $f^{-1}(u * v) = f^{-1}(f(x * y)) = x * y = f^{-1}(u) * f^{-1}(v)$ as needed.

qed

Definition: Let $(A, *)$ and $(B, *)$ both be semigroups, monoids or groups. If $\exists f : A \to B$ an isomorphism betwen A and B, then $(A, *)$ and $(B, *)$ are said to be isomorphic.

Remark: "Isomorphic" comes from "iso" same and "morph$\overline{e}$ " form: the same abstract algebra structure on both $(A, *)$ and $(B, *)$ given to you in two different guises. As the French would say: "Même Marie, autre chapeau" same Mary, different hat.

8 Formal Languages

Task: Use what we learned about structures in abstract algebra in order to make sense of formal languages and grammars.
Let A be a finite set. When studying formal languages, we call A an alphabet and the elements of A letters.

Examples:

1. $A = \{0, 1\}$ binary digits
2. $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ decimal digits
3. $A =$ letters of the English alphabet

Definition: $\forall n \in \mathbb{N}^*$, we define a word of length n in the alphabet A as being any string of the form $a_1 a_2 \cdots a_n$ $s.t.$ $a_i \in A \quad \forall i, 1 \leq i \leq n$. Let A^n be the set of all words of length n over the alphabet A.

Remark: There is a one-to-one correspondence between the string $a_1 a_2 \cdots a_n$ and the ordered n-tuple $(a_1, a_2, ..., a_n) \in A^n = \underbrace{A \times ... \times A}_{n\ times}$, the Cartesian product of n copies of A.

Definition: Let $A^+ = \bigcup\limits_{n=1}^{\infty} A^n = A^1 \cup A^2 \cup A^3 \cup ....$ A^+ is the set of all words of positive length over the alphabet A.

Examples:

1. $A = \{0, 1\}$, A^+ is the set of all binary strings of finite length that is at least one, **i.e.** $0, 1, 01, 10, 00, 11$, etc.
2. If $A =$ letters of the English alphabet, then A^+ consists of all non-empty strings of finite length of letters from the English alphabet.

It is useful to also have the empty word ε in our set of strings. ε has length 0. Define $A^0 = \{\varepsilon\}$ and then adjoin the empty word ε to A^+. We get $A^* = \{\varepsilon\} \cup A^+ = A^0 \cup \bigcup\limits_{n=1}^{\infty} A^n = \bigcup\limits_{n=0}^{\infty} A^n$.

Notation: We denote the length of a word w by $\mid w \mid$.