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Theorem: For any equivalence relation R on a set A, its equivalence classes form a partition of A, **i.e.**

1. $\forall x \in A, \exists y \in A$ s.t. $x \in [y]$ (every element of A sits somewhere)
2. $xRy \Leftrightarrow [x] = [y]$ (all elements related by R belong to the same equivalence class)
3. $\neg(xRy) \Leftrightarrow [x] \cap [y] = \emptyset$ (if two elements are not related by R, the they belong to disjoint equivalence classes)

Proof:

1. Trivial. Let $y = x$. $x \in [x]$ because R is an equivalence relation, hence reflexive, so xRx holds.
2. We will prove $xRy \Leftrightarrow [x] \subseteq [y]$ and $[y] \subseteq [x]$
   "$\Rightarrow$" Fix $x \in A, [x] = \{z \in A \mid xRz\} \Rightarrow \forall y \in A$ s.t. $xRy, y \in [x]$. Furthermore, $[y] = \{w \in A \mid yRw\}$
   $\Rightarrow \forall w \in [y], yRw$ but $xRy \Rightarrow xRw$ by transitivity. Therefore, $w \in [x]$. We have shown $[y] \subseteq [x]$.
   Since R is an equivalence relation, it is also symmetric. **i.e.** $xRy \Leftrightarrow yRx$. So by the same argument with x and y swapped $yRx \Rightarrow [x] \subseteq [y]$. Thus $xRy \Rightarrow [x] = [y]$.
   "$\Leftarrow$" $[x] = [y] \Rightarrow y \in [x]$ but $[x] = \{y \in A \mid xRy\}$
3. "$\Rightarrow$" We will prove the contrapositive. Assume $[x] \cap [y] \neq \emptyset \Rightarrow \exists z \in [x] \cap [y]$. $z \in [x]$ means xRz, whereas $z \in [y]$ means $yRz \Leftrightarrow zRy$ because R is symmetric. We thus have xRz and $zRy \Rightarrow xRy$ by the transitivity of R. xRy contradicts $\neg(xRy)$ so indeed $\neg(xRy) \Rightarrow [x] \cap [y] = \emptyset$
   "$\Leftarrow$" Once again we use the contrapositive:

Assume $\neg(\neg(xRy)) \Leftrightarrow xRy$. By part (2), $xRy \Rightarrow [x] = [y] \Rightarrow [x] \cap [y] \neq \emptyset$ since $x \in [x]$ and $y \in [y]$, **i.e.** these equivalence classes are non-empty. We have obtained the needed contradiction.

qed

Q: What partition does "=" impose on $\mathbb{R}$?

A: $[x] = \{x\}$ since $E = \{(x, x) \mid x \in \mathbb{R}\}$ the diagonal.
The one-element equivalence class is the smallest equivalence class possible (by definition, an equivalence class cannot be empty as it contains x itself). We call such a partition the <u>finest</u> possible partition.

Remark: The theorem above shows how every equivalence relation partitions a set. It turns out every partition of a set can be used to define an equivalence relation: xRy if x and y belong to the same subset of the partition (check this is indeed an equivalence relation!). Therefore, there is a 1-1 correspondence between partitions and equivalence relations: to each equivalence relation there corresponds a partition and vice versa.

4.3 Partial Orders

Task: Understand another type of relation with special properties.

Definition: Let A be a set. A relation R on A is called anti-symmetric if $\forall x, y \in A$ s.t. $xRy \wedge yRx$, then $x = y$.

Definition: A <u>partial order</u> is a relation on a set A that is reflexive, anti-symmetric, and transitive.

Examples:

1. $A = \mathbb{R}$ $\quad\quad\leq$ "less than or equal to" is a partial order
   (a) $\forall x \in \mathbb{R}$, $x \leq x \to$ reflexive
   (b) $\forall x, y \in \mathbb{R}$ s.t. $x \leq y \wedge y \leq x \implies x = y \to$ anti-symmetric
   (c) $\forall x, y, z \in \mathbb{R}$ s.t. $x \leq y \wedge y \leq z \implies x \leq z \to$ transitive
   Same conclusion if $A = \mathbb{Z}$ or $A = \mathbb{N}$
2. A is a set. Consider $P(A)$, the power set of A. The relation $\subseteq$ "being a subset of" is a partial order.
   (a) $\forall B \in P(A), B \subseteq B \to$ reflexive.
   (b) $\forall B, C \in P(A), B \subseteq C \wedge C \subseteq B \implies B = C$ (recall the criterion for proving equality of sets) $\to$ anti-symmetric
   (c) $\forall B, C, D \in P(A)$ s.t. $B \subseteq C \wedge C \subseteq D \implies B \subseteq D \to$ transitive

The most important example of a partial order is example (2) "being a subset of".

Q: Why is "being a subset of" a partial order as opposed to a total order?

A: There might exist subsets B, C of A s.t. neither $B \subseteq C$ nor $C \subseteq B$ holds, **i.e.** where B and C are not related via inclusion.