ST2004 Applied Probability I

Tutorial 2

1. Let X be a discrete random variable with probability mass function $p(x)$, and let $g(X)$ be a function of X, and let c be a constant. Prove that

$$\mathbb{E}(cg(X)) = c\mathbb{E}(g(X)).$$

Solution:

$$\begin{aligned}\mathbb{E}(cg(X)) &= \sum_x cg(x)p(x) \\ &= c\sum_x g(x)p(x) \\ &= c\mathbb{E}(g(X)).\end{aligned}$$

2. Consider a random variable X with the following probability distribution.

x	$p(x)$
0	1/8
1	1/4
2	3/8
3	1/4

Compute the expected value and variance of X.

Solution:
The expected value of X is given by

$$\begin{aligned}\mu = \mathbb{E}(X) &= \sum_{x=0}^{3} xp(x) \\ &= 0\left(\frac{1}{8}\right) + 1\left(\frac{1}{4}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{4}\right) \\ &= \frac{7}{4}\end{aligned}$$

The variance can be computed as follows

$$\begin{aligned}Var(X) &= \mathbb{E}((X-\mu)^2) = \sum_{x=0}^{3}(x-\mu)^2 p(x) \\ &= (0-1.75)^2\left(\frac{1}{8}\right) + (1-1.75)^2\left(\frac{1}{4}\right) + (2-1.75)^2\left(\frac{3}{8}\right) + (3-1.75)^2\left(\frac{1}{4}\right) \\ &= 0.9375\end{aligned}$$

3. Let X be a random variblae giving the number of heads minus the number of tails in three tosses of a coin. Assume the coin is biased so that a head is twice likely to occur as a tail. Compute the probability mass function of X.

Solution:
The sample space is $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$. The random variable X is given by

$$X(HHH) = 3, X(HHT) = X(HTH) = X(THH) = 1,$$

$$X(HTT) = X(THT) = X(TTH) = -1, X(TTT) = -3.$$

We have $\mathbb{P}(X = 3) = \mathbb{P}(\{HHH\}) = (2/3)^3$, since the probabiliy of having a head is $2/3$. For $X = 1$, we hve

$$\mathbb{P}(X = 1) = \mathbb{P}(\{HHT, HTH, THH\}) = 3(2/3)^2(1/3).$$

Similarly,

$$\mathbb{P}(X = -1) = 3(2/3)(1/3)^2, \mathbb{P}(W = -3) = (1/3)^3.$$

Therefore, the pmf of X is given by

x	$p(x)$
-3	0.037
-1	0.222
1	0.444
3	0.296

You should verify that $p(x)$ sum to 1.

4. When a basketball player takes his first shot he succeeds with probability $1/2$. If he misses his first shot, his second shot will go in with probability $1/3$. If he misses his first 2 shorts then his third shot will go in with probability $1/4$. If he misses his first 3 shots his next shot will go in with probability $1/5$. If we misses his first 4 shots then the coach will remove him from the game. Assume that the player keeps shooting until he succeeds or he is removed from the game. Let X denote the number of shots he misses until his first success or until he is removed from the game. Calculate the probability mass function of X.

Solution:
We note that X can take value $\{0, 1, 2, 3, 4\}$. We have $\mathbb{P}(X = 0) = 1/2$.

$$\begin{aligned}
\mathbb{P}(X = 1) &= \mathbb{P}(\text{misses 1st shot And makes 2nd shot}) \\
&= \mathbb{P}(\text{misses 1st shot})\mathbb{P}(\text{makes 2nd shot}|\text{misses 1st shot}) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}
\end{aligned}$$

$$
\begin{aligned}
\mathbb{P}(X=2) &= \mathbb{P}(\text{misses 1st shot And misses 2nd shot And makes 3rd shot}) \\
&= \mathbb{P}(\text{misses 1st shot And 2nd shot})\mathbb{P}(\text{makes 3rd shot}|\text{misses 1st and 2nd shots}) \\
&= \mathbb{P}(\text{misses 1st shot})\mathbb{P}(\text{misses 2nd shot}|\text{misses 1st shot}) \\
&\quad \mathbb{P}(\text{makes 3rd shot}|\text{misses 1st and 2nd shots}) \\
&= \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4}
\end{aligned}
$$

Using this pattern we get

$$\mathbb{P}(X=3) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{1}{5},$$

and

$$\mathbb{P}9X=4) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}.$$

Therefore, the pmf of X is given by

x	$p(x)$
0	0.5
1	0.167
2	0.083
3	0.05
4	0.20

You should verify that $p(x)$ sum to 1.

5. An examination consists of 10 multiple choice questions, in each of which a candidate has to deduce which one of give suggested answers is correct. A completely unprepared student guesses each answer completely random. What is the probability that this student gets 8 or more questions correct?

Solution:
Let X be the random variable denoting the number of answers guessed correctly. For each question the probability of success is $1/5$. Thus, X follows a binomial distribution with $n=10, p=0.2$. Thus,

$$\mathbb{P}(X \geq 8) = \frac{10}{8}0.2^8 0.8^2 + \frac{10}{9}0.2^9 0.8 + \frac{10}{10}0.2^{10}0.8^0 = 0.000078 \tag{1}$$