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5 Functions

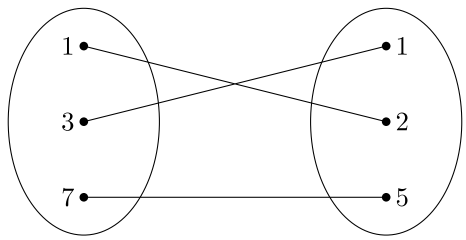
Task: Define a function rigorously and make sense of terminology associated to functions.

Definition: Let A, B be sets. A function $f : A \to B$ is a rule that assigns to every element of A one and only one element of B, **i.e.** $\forall x \in A\ \exists! y \in B$ s.t. $f(x) = y$. A is called the domain of f and B is called the codomain.

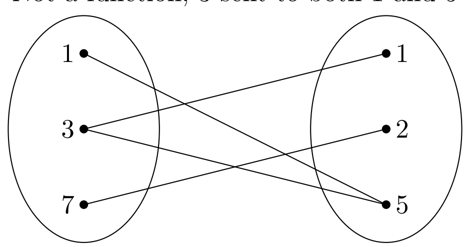
Examples:

1. $A = \{1, 3, 7\}$
   $B = \{1, 2, 5\}$

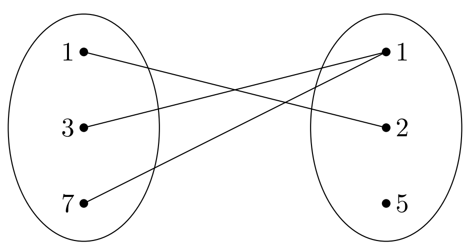
Is a function.



Not a function; 3 sent to both 1 and 5



Is a function.



2. $A = B = \mathbb{R}$ $\qquad F : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ is called the identity function.

Definition: Let A, B be sets, and let $f : A \to B$ be a function. The range of f denoted by $f(A)$ is the subset of B defined by $f(A) = \{y \in B \mid \exists x \in A$ s.t. $f(x) = y\}$.

Definition: Let A be a set. A Boolean function on A is a function $f : A \to \{T, F\}$, which has A as its domain and the set of truth values $\{T, F\}$ as is codomain. $f : A \to \{T, F\}$ thus assigns truth values to the elements of A.

Function are often represented by graphs. If $f : A \to B$ is a function, the graph of f denoted $\Gamma(f)$ is the subset of the Cartesian product of the domain with the codomain $A \times B$ given by $\{(x, f(x)) \mid x \in A\}$.

Q: Is it possible to obtain every subset of $A \times B$ as the graph of some function?

A: No! For $f : A \to B$ to be a function $\forall x \in A \quad \exists! y \in B$ s.t. $f(x) = y$, so for $\Gamma \subseteq A \times B$ to be the graph of some function, Γ must satisfy that $\forall x \in A \quad \exists! y \in B$ s.t. $(x, y) \in \Gamma$. Then we can define f by letting $y = f(x)$.

NB For the usual set-up of a function $f : \mathbb{R} \to \mathbb{R}$, this observation amounts to the "vertical line test," which you have seen before coming to university.

5.1 Composition of Functions

Task: Understand the natural operation that allows us to combine functions.

$$\begin{tikzcd}
A \arrow[r, "f(x)"] \arrow[rr, bend right, "g \circ f : A \to C"'] & B \arrow[r, "g(x)"] & C
\end{tikzcd}$$

Example:

$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = 2x$

$g : \mathbb{R} \to \mathbb{R} \qquad g(x) = \cos x$

$g \circ f(x) = g(f(x)) = g(2x) = \cos(2x)$

$f \circ g(x) = f(g(x)) = f(\cos x) = 2\,(\cos x) = 2\,\cos x$