MAU22C00: TUTORIAL 17 PROBLEMS
COUNTABILITY OF SETS

For each of the following sets, determine whether it is finite, countably infinite, or uncountably infinite. Justify your answer.

1) $\bigcup_{q\in\mathbb{Q}} L_q$ where $L_q = \{(x,y) \in \mathbb{R}^2 \mid x = q\}$.

2) $\{a^p \mid p \in \mathbb{N} \text{ and } a = e^{q\pi i} \text{ for } q \in \mathbb{Q}\}$

3) $\{a^p \mid p \in \mathbb{N} \text{ and } a = e^{q\pi i} \text{ for } q \in \mathbb{R} \setminus \mathbb{Q}\}$

4) $\{(x,y,z) \in \mathbb{N}^3 \mid x^2 + y^2 = z^2 \text{ and } x,y,z \in \mathbb{N}^*\}$, the Pythagorean triplets that give the lengths of the legs and the hypothenuse of a right triangle.

5) $\{(x,y) \in \mathbb{R}^2 \mid y = x^2 + 1\}$

6) $\mathcal{P}(J_n) \times \mathcal{P}(\mathbb{N})$, where $J_n = \{1,\ldots,n\}$ and $\mathcal{P}(A)$ is the power set of a set A.

7) $\mathbb{R}^n$ for $n \geq 1$.

Solution: 1) $L_q = \{(x,y) \in \mathbb{R}^2 \mid x = q\} = \{q\} \times \mathbb{R} \sim \mathbb{R}$. Therefore, $\bigcup_{q\in\mathbb{Q}} L_q$ is a countably infinite union of disjoint uncountably infinite sets, so it must itself be uncountably infinite as it contains $\{0\} \times \mathbb{R} \sim \mathbb{R}$, which is uncountably infinite.

2) $\{a^p \mid p \in \mathbb{N} \text{ and } a = e^{q\pi i} \text{ for } q \in \mathbb{Q}\}$ is a finite set. Let $q = \dfrac{r}{s}$ for $r,s \in \mathbb{Z}$, $s \neq 0$, $(r,s) = 1$. Therefore, $a^p = e^{\frac{pr\pi i}{s}}$, which assumes one of s values $e^{\frac{\pi i}{s}}, e^{\frac{2\pi i}{s}}, \ldots, e^{\frac{(s-1)\pi i}{s}}, e^{\frac{s\pi i}{s}}$ depending upon the value of p. We conclude that our set is finite

$$\{a^p \mid p \in \mathbb{N} \text{ and } a = e^{q\pi i} \text{ for } q \in \mathbb{Q}\} = \left\{e^{\frac{\pi i}{s}}, e^{\frac{2\pi i}{s}}, \ldots, e^{\frac{(s-1)\pi i}{s}}, e^{\frac{s\pi i}{s}}\right\}.$$

3) $A = \{a^p \mid p \in \mathbb{N} \text{ and } a = e^{q\pi i} \text{ for } q \in \mathbb{R} \setminus \mathbb{Q}\}$ is countably infinite. Since $q \in \mathbb{R} \setminus \mathbb{Q}$, $a^{p_1} \neq a^{p_2}$ if $p_1 \neq p_2$, so the map $f : \mathbb{N} \to A$ given by $f(p) = a^p$ is a bijection. Therefore,

$$A = \{a^p \mid p \in \mathbb{N} \text{ and } a = e^{q\pi i} \text{ for } q \in \mathbb{R} \setminus \mathbb{Q}\} \sim \mathbb{N}.$$

4) $\{(x,y,z) \in \mathbb{N}^3 \mid x^2 + y^2 = z^2 \text{ and } x,y,z \in \mathbb{N}^*\} \subset \mathbb{N}^3$, and we know from class that $\mathbb{N}^3$ is countably infinite. Therefore, our set can be

finite or countably infinite. We will prove that it is countably infinite by showing that it has a countably infinite subset. We remark that

$$(3,4,5) \in \{(x,y,z) \in \mathbb{N}^3 \mid x^2+y^2=z^2 \text{ and } x,y,z \in \mathbb{N}^*\}$$

as $3^2+4^2=9+16=25=5^2$. Furthermore,

$$(3p,4p,5p) \in \{(x,y,z) \in \mathbb{N}^3 \mid x^2+y^2=z^2 \text{ and } x,y,z \in \mathbb{N}^*\}$$

for every $p \in \mathbb{N}^*$ as $3^2p^2+4^2p^2=9p^2+16p^2=5^2p^2$. Since $\mathbb{N}^* \sim \mathbb{N}$ is countably infinite, the subset $\{(3p,4p,5p) \mid p \in \mathbb{N}^*\}$ is countably infinite, hence our set must likewise be countably infinite.

5) Consider the subset A of $\{(x,y) \in \mathbb{R}^2 \mid y=x^2+1\}$ given by

$$A = \{(x,y) \in \mathbb{R}^2 \mid y=x^2+1\} \cap [(0,1)\times\mathbb{R}].$$

The function $f(x)=x^2+1=y$ is a bijection on $(0,1)$ (easy to check). Therefore, $\{(x,y) \in \mathbb{R}^2 \mid y=x^2+1\} \cap [(0,1)\times\mathbb{R}] \sim (0,1)$, so the set A is uncountably infinite as we proved in class that $(0,1)$ was uncountably infinite. Since $A \subset \{(x,y) \in \mathbb{R}^2 \mid y=x^2+1\}$, the set $\{(x,y) \in \mathbb{R}^2 \mid y=x^2+1\}$ must itself be uncountably infinite. Note that we have employed here a very standard technique for showing a set is uncountably infinite. It suffices to show it has an uncountably infinite subset.

6) As you saw during Michaelmas term, the number of elements of a set with n elements is 2^n, so $\mathcal{P}(J_n)$ is a finite set with 2^n elements, where $n \geq 1$ by the definition of J_n. By contrast, we proved in class that $\mathcal{P}(\mathbb{N})$ is uncountably infinite. Thus, our set is a Cartesian product of a finite set with an uncountably infinite set. Since $J_n=\{1,\ldots,n\}$ for $n \geq 1$, the subset containing just the element 1 is always in $\mathcal{P}(J_n)$ for every $n \geq 1$, $\{1\} \in \mathcal{P}(J_n)$. Therefore, $\{1\}\times\mathcal{P}(\mathbb{N}) \subset \mathcal{P}(J_n)\times\mathcal{P}(\mathbb{N})$, but $\{1\}\times\mathcal{P}(\mathbb{N}) \sim \mathcal{P}(\mathbb{N})$. We conclude that $\mathcal{P}(J_n)\times\mathcal{P}(\mathbb{N})$ has an uncountably infinite subset, so it itself must be uncountably infinite.

7) For $n=1$, we have already shown in class that $\mathbb{R}^1=\mathbb{R}$ was uncountably infinite. Now for $n \geq 2$ consider

$$\mathbb{R}^n = \{(x_1,\ldots,x_n) \mid x_i \in \mathbb{R}\ \forall i\}.$$

The set

$$\mathbb{R}\times\{0\}\cdots\{0\} = \{(x_1,0\ldots,0) \mid x_1 \in \mathbb{R}\} \subset \mathbb{R}^n,$$

but $\mathbb{R}\times\{0\}\cdots\{0\} \sim \mathbb{R}$, which is uncountably infinite. Therefore, $\mathbb{R}^n$ has an uncountably infinite subset, which means it must itself be uncountably infinite.