Student Online Teaching Advice Notice

The materials and content presented within this session are intended solely for use in a context of teaching and learning at Trinity.

Any session recorded for subsequent review is made available solely for the purpose of enhancing student learning.

Students should not edit or modify the recording in any way, nor disseminate it for use outside of a context of teaching and learning at Trinity.

Please be mindful of your physical environment and conscious of what may be captured by the device camera and microphone during videoconferencing calls.

Recorded materials will be handled in compliance with Trinity's statutory duties under the Universities Act, 1997 and in accordance with the University's policies and procedures.

Further information on data protection and best practice when using videoconferencing software is available at https://www.tcd.ie/info_compliance/data-protection/.

3.2.2 Properties of Set Operations

Correspondence between Logic and Set Theory

Logical Connective	Set operation/property
$\wedge$	intersection $\cap$
$\vee$	union $\cup$
$\neg$	complement $(\ )^C$
$\rightarrow$	subset $\subseteq$
$\leftrightarrow$	equality of sets $=$

Recall:

Definition: Let A, B be two sets. The intersection $A \cap B = \{x \mid x \in A \wedge x \in B\}$

Definition: Let A, B be two sets. The union $A \cup B = \{x \mid x \in A \vee x \in B\}$

Definition: Let A, U be sets s.t. $A \subseteq U$. The complement of A in $U = U \backslash A = A^C = \{x \mid x \in U \wedge x \notin A\}$

Definition: Let A, B be sets. A is a subset of B if all elements of A are elements of B, **i.e.** $\forall x(x \in A \rightarrow x \in B)$.

Definition: Let A, B be sets. A=B if and only if all elements of A are elements of B and all elements of B are elements of A,
i.e. $A = B \leftrightarrow [\forall x(x \in A \rightarrow x \in B)] \wedge [\forall y(y \in B \rightarrow y \in A)]$

As a result, various properties of set operations become obvious:

- Commutativity
  - $A \cap B = B \cap A$ comes from the tautology $(P \wedge Q) \leftrightarrow (Q \wedge P)$ (#31 on the list of tautologies posted in Course Documents)
  - $A \cup B = B \cup A$ comes from the tautology $(P \vee Q) \leftrightarrow (Q \vee P)$ (# 32 on the list of tautologies)
- Associativity
  - $(A \cup B) \cup C = A \cup (B \cup C)$ comes from the tautology $[(P \vee (Q \vee R)] \leftrightarrow [(P \vee Q) \vee R]$ (# 33 on the list of tautologies)
  - $(A \cap B) \cap C = A \cap (B \cap C)$ comes from the tautology $[(P \wedge (Q \wedge R)] \leftrightarrow [(P \wedge Q) \wedge R]$ (# 34 on the list of tautologies)
- Distributivity
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ comes from the tautology $[(P \wedge (Q \vee R)] \leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$ (# 29 on the list of tautologies)

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ comes from the tautology $[(P \vee (Q \wedge R)] \leftrightarrow [(P \vee Q) \wedge (P \vee R)]$ (# 30 on the list of tautologies)

- De Morgan Laws in Set Theory
  - $(A \cap B)^C = A^C \cup B^C$ comes from the tautology $\neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$ (# 18 on the list of tautologies)
  - $(A \cup B)^C = A^C \cap B^C$ comes from the tautology $\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$ (# 19 on the list of tautologies)
- Involutivity of the Complement
  - $(A^C)^C = A$ comes from the tautology $\neg(\neg P) \leftrightarrow P$ (# 3 on the list of tautologies)

NB: An involution is a map such that applying it twice gives the identity. Familiar examples: reflecting across the x-axis, the y-axis, or the origin in the plane.

- Transitivity of Inclusion
  - $A \subseteq B \wedge B \subseteq C \to A \subseteq C$ comes from the tautology

$$[(P \to Q) \wedge (Q \to R)] \to (P \to R)$$

(# 14 on the list of tautologies)

- Criterion for proving equality of sets, which comes from the tautology $(P \leftrightarrow Q) \leftrightarrow [(P \to Q) \wedge (Q \to P)]$ (#22 on the list of tautologies)
  - $A = B \leftrightarrow A \subseteq B \wedge B \subseteq A$
- Criterion for proving non-equality of sets
  - $A \neq B \leftrightarrow (A \backslash B) \cup (B \backslash A) \neq \emptyset$

3.3 Example Proof in Set Theory

Proposition: $\forall A, B$ sets. $(A \cap B) \cup (A \backslash B) = A$

Proof: Use the criterion for proving equality of sets from above, **i.e.** inclusion in both directions.

<u>Show $(A \cap B) \cup (A \backslash B) \subseteq A$</u>: $\forall x \in (A \cap B) \cup (A \backslash B), x \in (A \cap B)$ or $x \in A \backslash B$. If $x \in (A \cap B)$, then clearly $x \in A$ as $A \cap B \subseteq A$ by definition. If $x \in A \backslash B$, then by definition $x \in A$ and $x \notin B$, so definitely $x \in A$. In both cases, $x \in A$ as needed.

<u>Show $A \subseteq (A \cap B) \cup (A \backslash B)$</u>: $\forall x \in A$, we have two possibilities, namely $x \in B$ or $x \notin B$. If $x \in B$, then $x \in A$ and $x \in B$, so $x \in A \cap B$. If $x \notin B$, then $x \in A$ and $x \notin B$, so $x \in A \backslash B$. In both cases, $x \in (A \cap B)$ or $x \in (A \backslash B)$, so $x \in (A \cap B) \cup (A \backslash B)$ as needed.

qed