MAU22C00: TUTORIAL 13 SOLUTIONS
GRAPH THEORY

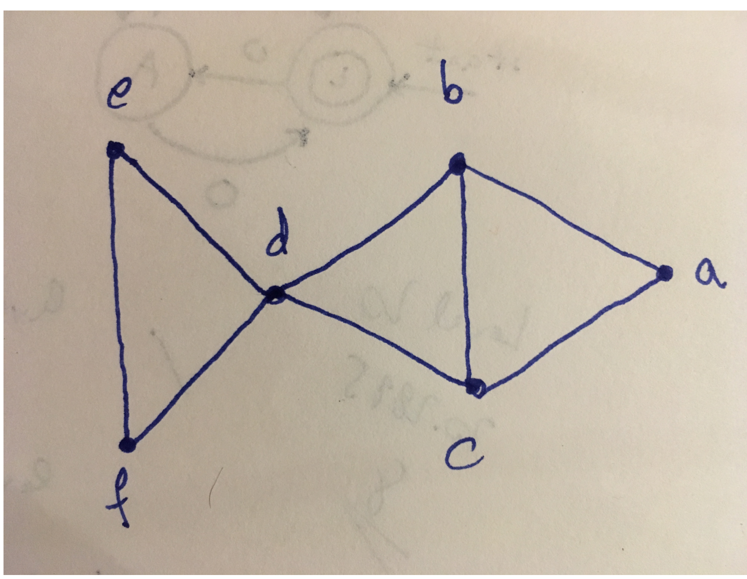
1) For what type of p and q does the complete bipartite graph $K_{p,q}$ have a Hamiltonian circuit? Justify your answer.

Solution: Recall that a bipartite graph satisfies that its vertices are partitioned into two sets V_1 and V_2 such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$, the set of all vertices. In the case of the complete bipartite graph $K_{p,q}$, the number of elements in V_1 is p, and the number of elements in V_2 is q. We must have $p = q \geq 2$ for a Hamiltonian circuit to exist as we hop from a vertex in V_1 to a vertex in V_2 and back.

2) Let (V, E) be the graph with vertices a, b, c, d, e, and f and edges ab, ac, bc, bd, cd, de, df, and ef.

(a) Does this graph have a Hamiltonian circuit? Justify your answer.
(b) Is this graph a tree? Justify your answer.
(c) If it is not a tree, how many distinct spanning trees does it have?

Solution: Let (V, E) be the graph with vertices a, b, c, d, e, and f and edges ab, ac, bc, bd, cd, de, df, and ef. Here is the graph:



(a) No, as we would have to pass through vertex d twice.
(b) It is not a tree as it contains circuits defd, abca, and bcdb.

(c) We have to break up each of the three circuits by deleting one edge per circuit. The complication is that circuits abca, and bcdb share edge bc. To break up circuit defd, we delete one of de, df, or ef (3 possibilities). To break up circuits abca and bcdb, we could
- either delete one of db and dc (2 possibilities) and one of bc, ba, and ac (3 possibilities) for a total of $2 \cdot 3 = 6$ possibilities
- or keep both db and dc, in which case we must delete bc to break up circuit dcdb and delete either ab or ac for a total of $1 \cdot 2 = 2$ additional possibilities.

Altogether, we have 8 possibilities to break up circuits abca and bcdb and 3 independent possibilities to break up circuit defd for a total of $8 \cdot 3 = 24$ distinct spanning trees.

3) Consider the statement "A graph (V, E) is a tree $\iff \#(E) = \#(V) - 1$. " What hypothesis is needed for this equivalence to be true? Give an example to show why this hypothesis is necessary.

Solution: The missing hypothesis is "connected." If the graph (V, E) is not connected we could have something like the graph with vertices a, b, c, d, and e and edges ab, bc, cd, and da, where the vertex e is isolated. This graph has 5 vertices and 4 edges, but it contains the circuit abcda, so it is not acyclical, and it has two connected components, so it is not connected. Therefore, it cannot be a tree.

Recall that
$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$
read as " n choose k" gives the number of distinct combinations of k objects taken out of a possible n objects for $n \geq k \geq 0$ with the convention $0! = 1$.

4) Consider the complete graph K_n for $n = 2, 3, 4$. In each of the three cases

(a) Is this graph a tree? Justify your answer.
(b) If it is not a tree, how many distinct spanning trees does it have?

(Hint: How many edges does K_n have?)

Solution: In a complete graph K_n every vertex is connected to every other vertex, so the degree of every vertex is $n-1$. We have n vertices, so the number of edges in K_n is $\dfrac{n(n-1)}{2}$ as each edge is counted twice. Out of $\dfrac{n(n-1)}{2}$ edges, we are supposed to choose $n-1$ to construct a spanning tree as we have n vertices, so a tree connecting them has

$n-1$ edges. Therefore, we first check whether our K_n has any circuits. If it does not, it is a tree. If it does, then the count

$$\binom{\frac{n(n-1)}{2}}{n-1}$$

gives the number of ways $n-1$ edges can be chosen, but in certain configurations depending on n, we can get graphs (V, E) satisfying $\#(E) = \#(V) - 1$ that are not connected (as we saw in the previous problem). We have to count those and subtract them from

$$\binom{\frac{n(n-1)}{2}}{n-1}$$

in order to get the number of distinct spanning trees.

$n = 2$ We have 2 vertices and 1 edge, so K_2 is a tree and hence its own spanning tree (1 choice of spanning tree).

$n = 3$ We have 3 vertices and 3 edges, K_3 contains a circuit, so it is not a tree. The number of distinct spanning trees is

$$\binom{3}{2} = \frac{3!}{1!\,2!} = 3$$

as it is not possible in this case to construct subgraphs of K_3 with 3 vertices and 2 edges that are disconnected.

$n = 4$ We have 4 vertices and $\dfrac{4 \cdot 3}{2} = 6$ edges, K_4 contains a number of circuits, so it is not a tree. The number of ways we can choose 3 edges out of 6 is

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20,$$

but there are

$$4 = \binom{4}{1}$$

different disconnected subgraphs of K_4 consisting of a triangle plus an isolated point. Those are not spanning trees of K_4, so the number of distinct spanning trees is

$$\binom{6}{3} - \binom{4}{1} = \frac{6!}{3!\,3!} - 4 = 20 - 4 = 16.$$