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## 5.2 Inverting Functions

**Task:** Figure out which properties a function has to satisfy so that its action can be undone, **i.e. when** we can define an inverse to the original function.

Given $f : A \to B$, want $f^{-1} : B \to A$ s.t. $f^{-1} \circ f : A \to A$ is the identity
$f^{-1} \circ f(x) = f^{-1}(f(x)) = x$
$A \xrightarrow{f} B \xrightarrow{f^{-1}} A$
It turns out $f$ has to satisfy two properties for $f^{-1}$ to exist:

1. Injective
2. Surjective

**Definition:** A function $f : A \to B$ is called injective or an injection (sometimes called one-to-one) if $f(x) = f(y) \Rightarrow x = y$

**Examples:**
$\sin x : [0, \frac{\pi}{2}] \to \mathbb{R}$ is injective
$\sin x : \mathbb{R} \to \mathbb{R}$ is not injective because $\sin 0 = \sin \pi = 0$

**Definition:** A function $f : A \to B$ is called surjective or a surjection (sometimes called onto) if $\forall z \in B\ \exists x \in A$ s.t. $f(x) = z$.

**Remark:** $f$ assigns a value to each element of $A$ by its definition as a function, but it is not required to cover all of $B$. $f$ is surjective if its range is all of $B$.

**Examples:**
$\sin x : \mathbb{R} \to [-1, 1]$ is surjective
$\sin x : \mathbb{R} \to \mathbb{R}$ is not surjective since $\nexists x \in \mathbb{R}$ s.t. $\sin x = 2$. We know $|\sin x| \leq 1\ \forall x \in \mathbb{R}$

**Definition:** A function $f : A \to B$ is called bijective or a bijection if $f$ is both injective and surjective.

**Example:** $f : \mathbb{R} \to \mathbb{R} \qquad f(x) = 2x + 1$ is bijective.

- Check injectivity: $f(x_1) = f(x_2) \Rightarrow 2x_1 + 1 = 2x_2 + 1 \Leftrightarrow 2x_1 = 2x_2 \Leftrightarrow x_1 = x_2$ as needed.
- Check surjectivity: $\forall z \in \mathbb{R}\ f(x) = z$ means $2x + 1 = z$.
  Solve for $x$: $2x = z - 1 \Rightarrow x = \frac{z-1}{2} \in \mathbb{R} \Rightarrow f$ is surjective.

**Remark:** All bijective functions have inverses because we can define the inverse of a bijection and it will be a function:

- Surjectivity ensures $f^{-1}$ assigns an element to every element of $B$ (its domain).
- Injectivity ensures $f^{-1}$ assigns to each element of $B$ one and only one element of $A$.

**Conclusion:** $f : A \to B$ bijective $\Rightarrow f^{-1}$ exists, **i.e.** $f^{-1}$ is a function. It turns out (reverse the arguments above) that $f^{-1}$ exists $\Rightarrow f : A \to B$ is bijective.
Altogether we get the following theorem:

**Theorem:** Let $f : A \to B$ be a function. $f^{-1}$ exists $\Leftrightarrow f : A \to B$ is bijective.

**Q:** How do we find the inverse function $f^{-1}$ given $f : A \to B$?

**A:** If $f(x) = y$, solve for $x$ as a function of $y$ since $f^{-1}(f(x)) = f^{-1}(y) = x$ as $f^{-1} \circ f$ is the identity.

**Example:** $f(x) = 2x + 1 = y$. Solve for $x$ in terms of $y$.
$f : \mathbb{R} \to \mathbb{R}$
$2x = y - 1 \qquad x = \frac{y-1}{2}$

## 5.3 Functions Defined on Finite Sets

**Task:** Derive conclusions about a function given the number of elements of the domain and codomain, if finite; understand the pigeonhole principle.

**Proposition:** Let $A, B$ be sets and let $f : A \to B$ be a function. Assume $A$ is finite. Then $f$ is injective $\Leftrightarrow f(A)$ has the same number of elements as $A$.

**Proof:**

$A$ is finite so we can write it as $A = \{a_1, a_2, ..., a_p\}$ for some $p$. Then $f(A) = \{f(a_1), f(a_2), ..., f(a_p)\} \subseteq B$. A priori, some $f(a_i)$ might be the same as some $f(a_j)$. However, $f$ injective $\Leftrightarrow f(a_i) \neq f(a_j)$ whenever $i \neq j \Leftrightarrow f(A)$ has exactly $p$ elements just like $A$.

**qed**

**Corollary 1** Let $A, B$ be finite sets such that $\#(A) = \#(B)$. Let $f : A \to B$ be a function. $f$ is injective $\Leftrightarrow f$ is bijective.

**Proof:**

"$\Rightarrow$" Suppose $f : A \to B$ is injective. Since $A$ is finite, by the previous proposition, $f(A)$ has the same number of elements as $A$, but $f(A) \subseteq B$ and $B$ has the same number of elements as $A \Rightarrow \#(A) = \#(f(A)) = \#(B)$, which means $f(A) = B$, **i.e.** $f$ is also surjective $\Rightarrow f$ is bijective.

"$\Leftarrow$" $f$ is bijective $\Rightarrow f$ is injective.

**qed**

**Corollary 2 (The Pigeonhole Principle)** Let $A, B$ be finite sets, and let $f : A \to B$ be a function. If $\#(B) < \#(A)$, $\exists\, a, a' \in A$ with $a \neq a'$ such that $f(a) = f(a')$.

**Remark:** The name pigeonhole principle is due to Paul Erdös and Richard Rado. Before it was known as the principle of the drawers of Dirichlet. It has a simple statement, but it's a very powerful result in both mathematics and computer science.

**Proof:** Since $f(A) \subseteq B$ and $\#(B) < \#(A)$, $f(A)$ cannot have as many elements as $A$, so by the proposition, $f$ cannot be injective, namely $\exists\, a, a' \in A$ with $a \neq a'$ (**i.e.** distinct elements) s.t. $f(a) = f(a')$.

**qed**

**Examples:**

1. You have 8 friends. At least two of them were born the same day of the week. #(days of the week) $= 7 < 8$.
2. A family of five gives each other presents for Christmas. There are 12 presents under the tree. We conclude at least one person got three presents or more.
3. In a list of 30 words in English, at least two will begin with the same letter. #(Letters in the English alphabet) $= 26 < 30$.