MAU22C00 - TUTORIAL 1 SOLUTIONS

1) Give an example of sentences P and Q such that the implication $P \to Q$ is true, but its converse $Q \to P$ is false. Give another example of sentences P and Q such that the implication $P \to Q$ is true and its converse $Q \to P$ is also true. Given an implication $P \to Q$, what can you deduce about the truth value of its converse?

Solution: Let P be: 'The set A is finite.' Let Q be: 'The set A has a finite subset B.' Clearly, $P \to Q$ is true as every finite set A has finite subsets. For example, we can set $B = A$ as every set is its own subset. $Q \to P$ is false, however. There are infinite sets that have finite subsets. An example is $A = \mathbb{N}$, the set of natural numbers, and $B = \{2, 4, 6\}$. The fact that a set has a finite subset does **not** imply the set itself is finite.

Now, let P be: "0=1." Let Q be: "Every real number is either even or odd." Both P and Q are false. As we know from the truth table of the implication, if both P and Q are false, both implications $P \to Q$ and $Q \to P$ are true.

Moral of the story: Given a true implication $P \to Q$, there is **NOTHING** that can be said about the truth value of the converse $Q \to P$.

2) Suppose we are told the following:
If turtles can sing then artichokes can fly. Artichokes can fly implies turtles can sing and dogs can't play chess. Dogs can play chess if and only if turtles can sing.
Deduce that turtles can't fly.

Proof.
We convert these statments into a logical format;
P = "turtles can sing".
Q = "artichokes can fly".
R = "dogs can't play chess".

We can assume the following hypotheses from the above paragraph;

(a) $P \to Q$

(b) $Q \to (P \wedge R)$

(c) $\neg R \leftrightarrow P$

We wish to prove $\neg P$. We do so as follows:

(1) $Q \rightarrow (\neg R \wedge R)$ - substitution of (c) into (b).

(2) $\neg(\neg R \wedge R) \rightarrow \neg Q$ - contrapositive of (1) (tautology #24 on the list of tautologies posted in Course Documents)

(3) $R \vee \neg R$ - law of the excluded middle (tautology #1 on the list of tautologies)

(4) $\neg(\neg R \wedge R)$ - De Morgan's law applied to (3) (tautology #18) and substitution of $\neg(\neg R)$ by R by the law of double negation (tautology #3)

(5) $\neg Q$ - modus ponens (2, 4) (tautology #10)

(6) $\neg Q \rightarrow \neg P$ - contrapositive of (a) (tautology #24)

(7) $\neg P$ - modus ponens (5,6) (tautology #10)

$\square$

3) Prove that the order of quantifiers cannot be reversed without potentially modifying the truth value by providing a counterexample. The easiest counterexample can be given for two quantifiers. Find a sentence in predicate logic $P(x, y)$ such that $\forall x\, \exists y\, P(x, y)$ is true, but $\exists y\, \forall x\, P(x, y)$ is false. Do not forget to specify to which set x and y belong!

Solution: Let the domain of both x and y be $\mathbb{R}$, the set of real numbers. Let $P(x, y)$ be the statement $x + y = 0$. $\forall x\, \exists y\, P(x, y)$ is true because we can let $y = -x$. The statement $\exists y\, \forall x\, P(x, y)$ is false because there is no $y \in \mathbb{R}$ such that for every $x \in \mathbb{R}$, $x + y = 0$.

This phenomenon is true in general. Any existentially bound variable in a statement that is true can be expressed as a function of all universally bound variable to its left. In this case, in the statement $\forall x\, \exists y\, P(x, y)$, the variable x is universally bound (bound by the universal quantifier $\forall$), whereas the variable y is existentially bound (bound by the existential quantifier $\exists$). The statement is true because y can be expressed as a function of all universally bound variables to its left. Here we have only one universally bound variable to y's left, variable x, and the formula defining

y as a function of x is $y = -x$. For $\exists y \, \forall x \, P(x, y)$, variable y is existentially bound and has no universally bound quantifiers to its left, which means $\exists y \, \forall x \, P(x, y)$ would have to be true for y a constant, since y cannot be given as a formula involving any variables. Clearly, the statement $x + y = 0$ is not true for a constant over the real numbers $\mathbb{R}$, so $\exists y \, \forall x \, P(x, y)$ is false.