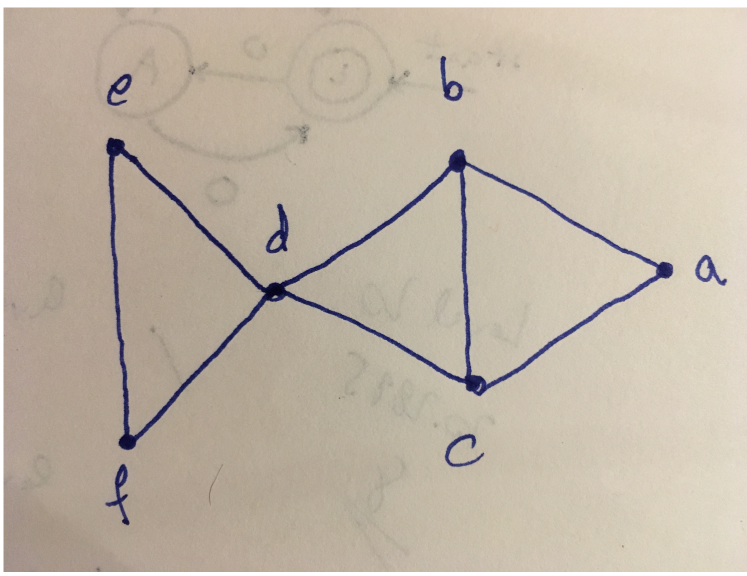
MAU22C00: TUTORIAL 12 SOLUTIONS
GRAPH THEORY

1) Let (V, E) be the graph with vertices a, b, c, d, e, and f and edges ab, ac, bc, bd, cd, de, df, and ef.

(a) Does this graph have an Eulerian trail? Justify your answer.

(b) Does this graph have an Eulerian circuit? Justify your answer.

Solution: Let (V, E) be the graph with vertices a, b, c, d, e, and f and edges ab, ac, bc, bd, cd, de, df, and ef. Here is the graph:



(a) $\deg b = \deg c = 3$, so we have two vertices of odd degree, whereas the rest of the vertices have even degrees $\deg a = \deg e = \deg f = 2$ and $\deg d = 4$. By the corollary in lecture 37, this graph must have an Eulerian trail.

(b) Since not all vertices have even degrees, which is a necessary condition for the existence of an Eulerian circuit (Corollary 2 in lecture 35), this graph does not have an Eulerian circuit.

2) For what type of n does the complete graph K_n have an Eulerian circuit? Justify your answer.

Solution: In a complete graph K_n every vertex is connected to every other vertex, so the degree of every vertex is $n-1$. By Euler's theorem we proved in lecture 37, we need $n-1$ to be even, so n must be odd. Note that we need $n \geq 3$ to have a circuit in the first place, so for $n \geq 3$, n odd K_n has an Eulerian circuit.

3) For what type of n does the complete graph K_n have an Eulerian trail that is not a circuit? Justify your answer.

Solution: Since all vertices have the same degree in the complete graph K_n, we cannot be in the case where all but two of the vertices have odd degree and the rest have even degree unless $n = 2$. Therefore, K_n has an Eulerian trail only for $n = 2$.

4) For what type of p and q does the complete bipartite graph $K_{p,q}$ have an Eulerian circuit? Justify your answer.

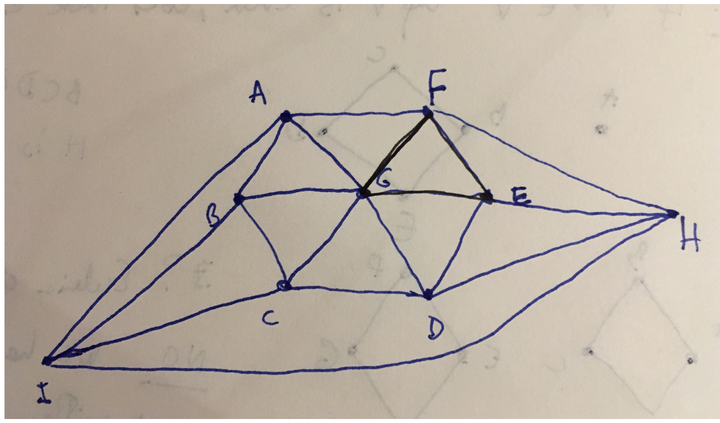
Solution: Recall that a bipartite graph satisfies that its vertices are partitioned into two sets V_1 and V_2 such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$, the set of all vertices. In the case of the complete bipartite graph $K_{p,q}$, the number of elements in V_1 is p, and the number of elements in V_2 is q. Therefore, $\forall v \in V_1$, deg v $= q$, and $\forall v \in V_2$, deg v $= p$ as the graph is a complete bipartite graph. For the degrees of all vertices to be even, we must have that both p and q are even to guarantee the existence of an Eulerian circuit. Furthermore, the total number of vertices should be at least 3 for a circuit to exist, so $p \geq 2$, $q \geq 2$ and both are even.

5) For what type of p and q does the complete bipartite graph $K_{p,q}$ have an Eulerian trail that is not a circuit? Justify your answer.

Solution: Either $p \geq 1$ is odd and $q = 2$ or vice versa $p = 2$ and $q \geq 1$ is odd as we need two vertices to have odd degree and the rest to have even degrees and the degrees of vertices in the same set of the partition, V_1 and V_2, is the same.

6) Illustrate Lemma B in lecture 36 by finding the longest circuit starting and ending at vertex G, which has no edges in common with circuit $EFGE$ in the graph with vertices A, B, C, D, E, F, G, H, and I, and edges AI, BI, CI, HI, AB, AG, AF, BC, BG, CD, CG, DG, DE, DH, EF, EG, EH, FG, and FH.

Solution: Here is the graph:



An example of the longest circuit we could find starting and ending at G, which has no edges in common with circuit $EFGE$ is $GAFHEDHICDGCBIABG$.