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3 Set Theory

Task: Understand enough set theory to make sense of other mathematical objects in abstract algebra, graph theory, etc.

Set theory started around 1870's $\rightarrow$ late development in mathematics but now taught early in one's maths education due to the Bourbaki school.

Definition: A set is a collection of objects. $x \in A$ means the element x is in the set A (**i.e.** belongs to A).

Examples:

1. All students in a class.
2. $\mathbb{N}$ the set of natural numbers starting at 0.

   $\mathbb{N}$ is defined via the following two axioms:

   (a) $0 \in \mathbb{N}$
   (b) if $x \in \mathbb{N}$, then $x + 1 \in \mathbb{N}$ ($x \in \mathbb{N} \rightarrow x + 1 \in \mathbb{N}$)
3. $\mathbb{R}$ set of real numbers also introduced axiomatically. The hardest axiom is the last one: completeness. $\mathbb{R}$ is constructed from $\mathbb{Q}$ in one of two ways: via Dedekind cuts or Cauchy sequences.

   $\mathbb{R}$ is the set of real numbers. The axioms governing $\mathbb{R}$ are:

   (a) Additive closure: $\forall x, y\, \exists z (x + y = z)$
   (b) Multiplicative closure: $\forall x, y, \exists z (x \times y = z)$
   (c) Additive associativity: $\forall x, y, z \quad x + (y + z) = (x + y) + z$
   (d) Multiplicative associativity: $\forall x, y, z \quad x \times (y \times z) = (x \times y) \times z$
   (e) Additive commutativity: $\forall x, y \quad x + y = y + x$
   (f) Multiplicative commutativity: $\forall x, y \quad x \times y = y \times x$
   (g) Distributivity: $\forall x, y, z \quad x \times (y + z) = (x \times y) + (x \times z)$ and $(y + z) \times x = (y \times x) + (z \times x)$
   (h) Additive identity: There is a number, denoted 0, such that for all $x, x + 0 = x$
   (i) Multiplicative identity: There is a number, denoted 1, such that for all $x, x \times 1 = 1 \times x = x$

(j) Additive inverses: For every x there is a number, denoted $-x$, such that $x + (-x) = 0$

(k) Multiplicative inverses: For every nonzero x there is a number, denoted x^{-1}, such that $x \times x^{-1} = x^{-1} \times x = 1$

(l) $0 \neq 1$

(m) Irreflexivity of $<$: $\sim (x < x)$

(n) Transitivity of $<$: If $x < y$ and $y < z$, then $x < z$

(o) Trichotomy: Either $x < y, y < x$, or $x = y$

(p) If $x < y$, then $x + z < y + z$

(q) If $x < y$ and $0 < z$, then $x \times z < y \times z$ and $z \times x < z \times y$

(r) Completeness: If a nonempty set of real numbers has an upper bound, then it has a *least* upper bound.

4. $\emptyset$ is the empty set (The set with no elements).

Definition: Let A, B be sets. A=B if and only if all elements of A are elements of B and all elements of B are elements of A,
i.e. $A = B \leftrightarrow [\forall x(x \in A \rightarrow x \in B)] \wedge [\forall y(y \in B \rightarrow y \in A)]$

3.1 Two Ways to Describe Sets

1. The enumeration/roster method: list all elements of the set.
   NB: order is irrelevant.
   $A = \{0, 1, 2, 3, 4, 5\} = \{5, 0, 2, 3, 1, 4\}$

2. The formulaic/set builder method: give a formula that generates all elements of the set.
   $A = \{x \in \mathbb{N} \mid 0 \leq x \wedge x \leq 5\} = \{0, 1, 2, 3, 4, 5\} = \{x \in \mathbb{N} : 0 \leq x \wedge x \leq 5\}$

Using $\mathbb{N}$ and the set-builder method, we can define:

$\mathbb{Z} = \{m - n \mid \forall m, n \in \mathbb{N}\}$

$n = 0$ and m any natural number $\Rightarrow$ we generate all of $\mathbb{N}$

$m = 0$ and n any natural number $\Rightarrow$ we generate all negative integers

$0 - 1 = -1$

$0 - 2 = -2$

etc.

$\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z} \wedge q \neq 0\}$

Definition: A set A is called finite if it has a finite number of elements; otherwise, it is called infinite.

3.2 Set Operations

Task: Understand how to represent sets by Venn diagrams. Understand set union, intersection, complement, and difference.

Definition: Let A, B be sets. A is a subset of B if all elements of A are elements of B, **i.e.** $\forall x(x \in A \to x \in B)$. We denote that A is a subset of B by $A \subseteq B$

Example: $\mathbb{N} \subseteq \mathbb{Z}$

Definition: Let A, B be sets. A is a proper subset of B if $A \subseteq B \wedge A \neq B$, **i.e.** $A \subseteq B \wedge \exists x \in B \ s.t. \ x \notin A$.

Notation: $A \subset B$

Example: $\mathbb{N} \subset \mathbb{Z}$ since $\exists(-1) \in \mathbb{Z}$ such that $-1 \notin \mathbb{N}$.

NB: $\forall A$ a set, $\emptyset \subseteq A$

Recall: $B \subseteq C$ means $\forall x(x \in B \to x \in C)$, but $\emptyset$ has no elements, so in $\emptyset \subseteq A$ the quantifier $\forall$ operates on a domain with no elements. Clearly, we need to give meaning to $\exists$ and $\forall$ on empty sets.

Boolean Convention

$$\left.\begin{array}{l}\forall \text{ is true on the empty set} \\ \exists \text{ is false on the empty set}\end{array}\right\} \text{Consistent with common sense}$$

Definition: Let A, B be two sets. The union $A \cup B = \{x \mid x \in A \ \vee \ x \in B\}$

Definition: Let A, B be two sets. The intersection $A \cap B = \{x \mid x \in A \wedge x \in B\}$

Definition: Let A, B be sets. A and B are called disjoint if $A \cap B = \emptyset$

Definition Let A, B be two sets. $A - B = A \backslash B = \{x \mid x \in A \ \wedge \ x \notin B\}$

Examples:

$$\begin{array}{ll} A = \{1,2,5\} & B = \{1,3,6\} \\ A \cup B = \{1,2,3,5,6\} & A \cap B = \{1\} \\ A \backslash B = \{2,5\} & B \backslash A = \{3,6\} \end{array}$$

Definition: Let A, U be sets s.t. $A \subseteq U$. The complement of A in $U = U \backslash A = A^C = \{x \mid x \in U \wedge x \notin A\}$

Remark: The notation A^C is unambiguous only if the universe U is clearly defined or understood.

3.2.1 Venn Diagrams

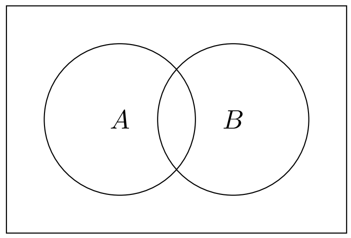
Schematic representation of set operations.

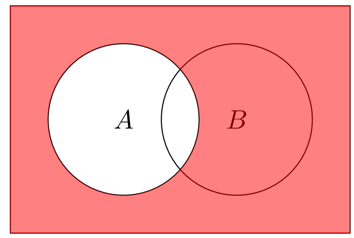
Pros of Venn diagrams:

Very easy to visualize

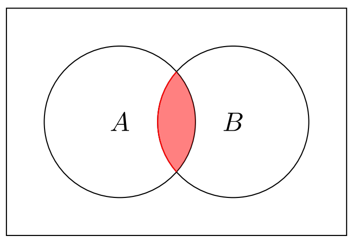
Cons of Venn diagrams:

1. Misleading if for example $A \subset B$ or sets are in some other non standard configuration;

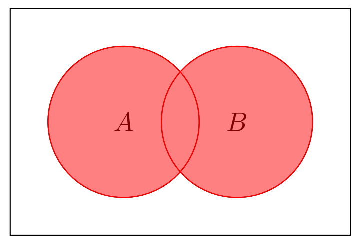




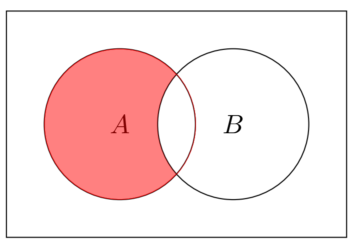
A^C



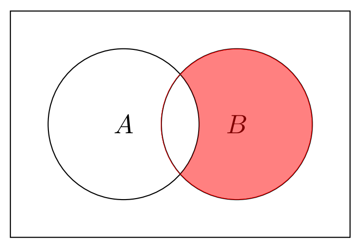
$A \cap B$



$A \cup B$



$A \backslash B$



$B \backslash A$

2. Not helpful if a lot of sets are involved;

3. Not helpful if sets are infinite or have some peculiar structure.

Moral of the story: Venn diagrams will **NOT** be accepted as proof of any statement in set theory. Instead, we will introduce rigorous ways of proving assertions in set theory.