MAU22C00 - TUTORIAL 4

1) (From the 2016-2017 Annual Exam) Let $f : [-2, 2] \to [-15, 1]$ be the function defined by $f(x) = x^2 + 3x - 10$ for all $x \in [-2, 2]$. Determine whether or not this function is injective and whether or not it is surjective. Justify your answers.

2) Use mathematical induction to prove the geometric series formula, which states that for any $a, r \in \mathbb{R}$ with $r \neq 1$ and any $n \in \mathbb{N}^*$,

$$a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{(1 - r^n)}{(1 - r)}.$$

3) Where is the fallacy in the following argument by induction?

Statement: If p is an even number and $p \geq 2$, then p is a power of 2.

"Proof:" We give a proof using strong induction on the even number p. Denote by $P(n)$ the statement "if n is an even number and $n \geq 2$, then $n = 2^j$, where $j \in \mathbb{N}$."

Base case: Show $P(2)$. $2 = 2^1$, so 2 is indeed a power of 2.

Inductive step: Assume $p > 2$ and that $P(n)$ is true for every n such that $2 \leq n < p$ (the strong induction hypothesis). We have to show that $P(p)$ also holds. We consider two cases:

Case 1: p is odd, then there is nothing to show.

Case 2: p is even. Since $p \geq 4$ and p is an even number, we can write $p = 2n$ with $2 \leq n < p$. By the inductive hypothesis, $P(n)$ holds, so we conclude that $n = 2^j$ for some $j \in \mathbb{N}$. Since $p = 2n = 2 \times 2^j = 2^{j+1}$, we conclude that $P(p)$ also holds.