MAU22C00: TUTORIAL 16 PROBLEMS
COUNTABILITY OF SETS

For each of the following sets, determine whether it is finite, countably infinite, or uncountably infinite. Justify your answer.

1) The set of integers divisible by 7.

2) $\{11^p \mid p \in \mathbb{Z}\}$

3) $\left\{\left(\frac{m}{3}, \frac{n}{5}\right) \in \mathbb{R}^2 \;\middle|\; m, n \in \mathbb{Z}\right\}$

4) $\{x \in \mathbb{C} \mid x^4 - 2x - 1 = 0\}$

5) $\{(x, y) \in \mathbb{R}^2 \mid y = x^6\} \cap \mathbb{Z}^2$

6) $\{x \in \mathbb{R} \mid \sin x = 1\}$

7) $\bigcup_{q \in \mathbb{Q}} L_q$ where $L_q = \{(x, y) \in \mathbb{R}^2 \mid x = q\} \cap (\mathbb{Q} \times \mathbb{N})$.

Solution: 1) An integer $m \in \mathbb{Z}$ is divisible by 7 if there exists some integer $p \in \mathbb{Z}$ such that $m = 7p$. Therefore, the set A of integers divisible by 7 is given by $A = \{7p \mid p \in \mathbb{Z}\}$. The function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(p) = 7p$ is a bijection (check!). Therefore, $A \sim \mathbb{Z}$, and we know from lecture that $\mathbb{Z}$ is countably infinite. Therefore, the set A of integers divisible by 7 is countably infinite.

2) $\{11^p \mid p \in \mathbb{Z}\} \sim \mathbb{Z}$ via the bijection $f(p) = 11^p$ (check it is a bijection). Therefore, the set is countably infinite.

3) $\mathbb{Z} \times \mathbb{Z} \subset \left\{\left(\frac{m}{3}, \frac{n}{5}\right) \in \mathbb{R}^2 \;\middle|\; m, n \in \mathbb{Z}\right\} \subset \mathbb{Q} \times \mathbb{Q}$. Since both $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$ and $\mathbb{Q} \times \mathbb{Q} = \mathbb{Q}^2$ are countably infinite as proven in class, the set itself is sandwiched between two countably infinite sets, so it must be countably infinite.

4) $\{x \in \mathbb{C} \mid x^4 - 2x - 1 = 0\}$ consists of all roots of the polynomial $x^4 - 2x - 1 = 0$, which has degree 4. Therefore, there are at most 4 roots over $\mathbb{R}$ and exactly 4 roots over $\mathbb{C}$ counted with multiplicity by the Fundamental Theorem of Algebra. It means our set must be finite.

5) $\{(x, y) \in \mathbb{R}^2 \mid y = x^6\} \cap \mathbb{Z}^2$ is a subset of $\mathbb{Z}^2$ by definition, and $\mathbb{Z}^2$ is countably infinite as proven in class. A subset of a countably infinite set could be either finite or countably infinite. It remains to figure out which one of the two is true for our set. We note that the set of all pairs (x, x^6) for $x \in \mathbb{Z}$ is a subset of our set. The set of all such pairs is countably infinite because $\{(x, x^6) \mid x \in \mathbb{Z}\} \sim \mathbb{Z}$ (it is in one-to-one

correspondence with $\mathbb{Z}$.) Therefore, $\{(x,y) \in \mathbb{R}^2 \mid y = x^6\} \cap \mathbb{Z}^2$ is countably infinite.

6)

$$\{x \in \mathbb{R} \mid \sin x = 1\} = \left\{\frac{\pi}{2} + 2\pi n \mid n \in \mathbb{Z}\right\} \sim \mathbb{Z}$$

because the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = 2\pi n$ is a bijection, so the set must be countably infinite.

7) $L_q = \{(x,y) \in \mathbb{R}^2 \mid x = q\} \cap (\mathbb{Q} \times \mathbb{N}) = \{q\} \times \mathbb{N} \sim \mathbb{N}$. Therefore, $\bigcup_{q\in\mathbb{Q}} L_q$ is a countably infinite union of disjoint countably infinite sets and thus countably infinite by the theorem proven in class.