ST2004 Applied Probability I

Tutorial 1

1. An urn contains 6 red balls and 3 blue balls. One ball is selected at random and is replaced by a ball of the other color. A second ball is then chosen. What is the conditional probability that the first ball selected is red, given that the second ball was red?

Solution:
The sample space is given by $\Omega = \{rr, rb, br, bb\}$ where rr denotes that the both balls selected are red, etc. Now, we can compute the probability of each outcome in the sample space:

$$\begin{aligned}
\mathbb{P}(rr) &= \frac{6}{9}\frac{5}{9} = \frac{30}{81} \\
\mathbb{P}(rb) &= \frac{6}{9}\frac{4}{9} = \frac{24}{81} \\
\mathbb{P}(br) &= \frac{3}{9}\frac{7}{9} = \frac{21}{81} \\
\mathbb{P}(bb) &= \frac{3}{9}\frac{2}{9} = \frac{6}{81}
\end{aligned}$$

Let A be the event that the second ball is red, and B be the event that the first ball is red. Then $A = \{rr, br\}$ and $B = \{rr, rb\}$. We want to compute

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{30/81}{30/81 + 21/81} = \frac{10}{17}.$$

2. Sixty percent of new drivers have had driver education. During their first year, new drivers without driver education have probability 0.08 of having an accident, but new drivers with driver education have only a 0.05 probability of an accident. What is the probability a new driver has had driver education, given that the driver has had no accident the first year?

Solution:
Let E denote the event that the new driver has had driver education, and A represent the event that the new driver who has had an accident in the first year. We have $\mathbb{P}(E) = 0.6$, $\mathbb{P}(E^c) = 1 - \mathbb{P}(E) = 0.4$. We are also given that

$$\mathbb{P}(A^c|E) = 1 - \mathbb{P}(A|E) = 1 - 0.05 = 0.95,$$

and

$$\mathbb{P}(A^c|E^c) = 1 - \mathbb{P}(A|E^c) = 1 - 0.08 = 0.92.$$

By Bayes Theorem,

$$\begin{aligned}\mathbb{P}(E|A^c) &= \frac{\mathbb{P}(A^c|E)\mathbb{P}(E)}{\mathbb{P}(A^c|E)\mathbb{P}(E)+\mathbb{P}(A^c|E^c)\mathbb{P}(E^c)}\\ &= \frac{0.95\times 0.6}{0.95\times 0.6+0.92\times 0.4}\\ &= 0.6077.\end{aligned}$$

3. A study of the post-treatment behaviour of a large number of drug abusers suggests that the likelihood of conviction within a two-year period after treatment may depend on the offender's education. The proportions of the total number of cases falling in four education-conviction categories follow:

Status within two years after treatment

Education	*Convicted*	*Not Convicted*	*Totals*
10 *Years or more*	0.10	0.30	0.40
9 *Years or less*	0.27	0.33	0.60
Totals	0.37	0.63	1.00

Suppose that a single offender is selected from the treatment program. Define the events:

A: The offender has 10 or more years of education,
B: The offender is convicted within two years after completion of treatment.

Find the probabilities for the events
$(a)A$ $(b)B$ $(c)A\cap B$ $(d)A\cup B$
$(e)\bar{A}$ $(f)\overline{A\cup B}$ $(g)\overline{A\cap B}$
(h) Event A given that event B has occurred
(i) Event B given that event A has occurred.

Solution:

(a) $\mathbb{P}(A) = 0.40$,
(b) $\mathbb{P}(B) = 0.37$,
(c) $\mathbb{P}(A\cap B) = 0.10$.
(d) $\mathbb{P}(A\cup B) = 0.10+0.30+0.27 = 0.67$
(e) $\mathbb{P}(\bar{A}) = 1-\mathbb{P}(A) = 1-0.40 = 0.60$
(f) $\mathbb{P}(\overline{A\cup B}) = 1-\mathbb{P}(A\cup B) = 0.33$
(g) $\mathbb{P}(\overline{A\cap B}) = 1-\mathbb{P}(A\cap B) = 0.90$
(h) $\mathbb{P}(A|B) = \mathbb{P}(A\cap B)/\mathbb{P}(B) = 0.10/0.37 = 0.27$
(i) $\mathbb{P}(B|A) = \mathbb{P}(A\cap B)/\mathbb{P}(A) = 0.10/0.40 = 0.25$

4. An advertising agency notes that approximately 1 in 50 potential buyers of a product sees a given magazine ad and 1 in 5 sees a corresponding ad on television. One in 100 sees both. One in 3 actually purchases the product after seeing the ad, 1 in 10 without seeing it. What is the probability that a randomly selected potential customer will purchase the product?

Solution:
Let M and T denote the events that a randomly selected person has seen the magazine or television ad respectively, and B the event that a randomly selected person buys the product.

$$\mathbb{P}(M) = 0.02, \quad \mathbb{P}(T) = 0.2, \quad \mathbb{P}(M \cap T) = 0.01$$

$$\mathbb{P}(B|(M \cup T)) = 1/3, \quad \mathbb{P}(B|\overline{M \cup T}) = 0.1$$

$$\mathbb{P}(M \cup T) = \mathbb{P}(M) + \mathbb{P}(T) - \mathbb{P}(M \cap T) = 0.02 + 0.2 - 0.01 = 0.21$$

$$\mathbb{P}(\overline{M \cup T}) = 1 - \mathbb{P}(M \cup T) = 1 - 0.21 = 0.79$$

$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(M \cup T)\mathbb{P}(B|(M \cup T)) + \mathbb{P}(\overline{M \cup T})\mathbb{P}(B|(\overline{M \cup T}) \\ &= (0.21)(1/3) + (0.79)(0.1) = 0.149\end{aligned}$$

5. The computing system at a large university is currently undergoing shutdown for repairs. Previous shutdowns have been due to one of hardware failure, software failure or power (electronic) failure. The system is forced to shutdown 73% of the time when it experiences hardware problems, 12% when it experiences software problems, and 88% of the time when it experiences electronic problems. Maintenance engineers have determined that the probabilities of hardware, software and power problems are 0.2, 0.5 and 0.3 respectively. Given that the system is undergoing shutdown, what is the probability that the problem is due to hardware failure? Software failure? Power failure?

Solution:
Let A_1, A_2, A_3 denote hardware, software and power failure, and S denote shutdown.

$$\begin{aligned}P(A_1|S) &= \frac{P(S|A_1)P(A_1)}{P(S|A_1)P(A_1) + P(S|A_2)P(A_2) + P(S|A_3)P(A_3)} \\ &= \frac{0.73 \times 0.2}{0.73 \times 0.2 + 0.12 \times 0.5 + 0.88 \times 0.3} = 0.3106\end{aligned}$$

We applied the Bayes' formula $P(A_j|B) = P(A_j)P(B|A_j)/P(B)$ and for the denominator

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B|A_i)$$

Similarly, $P(A_2|S) = 0.1277$, $P(A_3|S) = 0.5617$.