ST2004 Applied Probability I

Tutorial 3

1. The probability that an electronic component fails in the first day of operation is 0.005. If 400 items are tested independently and whether they fail or not after a day will be recorded.

   (a) What is the distribution of the number of items that fail?

   (b) What is the probability that at least two items fail?

   (c) Give the Poission approximation to (a) and compute the approximate answer to (b) based on the Poisson approximation? Comment on the accuracy of the approximation.

   Solution:

   (a) An appropriate model is the binomial. The number that fail is bin$(400, 0.005)$.

   (b)

$$\begin{aligned}\mathbb{P}(X \geq 2) &= 1 - \mathbb{P}(X < 2) \\ &= 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) \\ &= 1 - \binom{400}{0} 0.005^0 0.995^{400} - \binom{400}{1} 0.005^1 0.995^{399} \\ &= 0.595\end{aligned}$$

   (c) Using Poisson approximation to binomial distribution,

$$\mathbb{P}(X \geq 2) \approx 1 - \frac{2^0 e^{-2}}{0!} - \frac{2^1 e^{-2}}{1!} = 0.594$$

2. The number of cars that pass through a junction per minute can be modelled by a Poisson(3) distribution.

   (a) What is the probability that at least two cars pass through the junction in a minute?

   (b) What is the probability that at least thirty cars pass through the junction in ten minutes?

   (c) What is the expected value and standard deviation of the number of cars that pass through the junction per hour?

   Solution:

   (a) Let X be the number of cars per miniute. Then $X \sim \text{Poi}(3)$.

$$\begin{aligned}\mathbb{P}(X \geq 2) &= 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) \\ &= 1 - \frac{3^0 e^{-3}}{0!} - \frac{3^1 e^{-3}}{1!} = 0.801\end{aligned}$$

(b) Let X be the number of cars in 10 minutes. We have $X \sim \text{Poi}(30)$. We have

$$\mathbb{P}(X \geq 30) = 1 - \sum_{k=0}^{29} \frac{30^k e^{-30}}{k!} = 0.524$$

(c) Let X be the number of cars per hour. $Z \sim \text{Poi}(180)$.

$$\mathbb{E}(X) = Var(X) = 180$$

$$SD(Z) = \sqrt{Var(X)} = 13.4$$

3. Suppose the probability mass function of X is given by the

x	$p(x)$
0	0.41
1	0.37
2	0.16
3	0.05
4	0.01

Construct the cumulative distribution function of X.

Solution:

We have $F(x) = 0$ for $x < 0$, $F(x) = \mathbb{P}(X = 0)$ for $0 \leq x < 1$, $F(x) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1)$ for $1 \leq x < 2$ and so on. Thus the CDF is

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.41 & 0 \leq x < 1 \\ 0.78 & 1 \leq x < 2 \\ 0.94 & 2 \leq x < 3 \\ 0.99 & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

4. Let X be a continuous random variable with the following probability density function:

$$f(x) = \begin{cases} \sin(x) & 0 \leq x \leq A \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of the constant A?

(b) What is the corresponding cumulative distribution function $F(x)$?

Solution:

(a) $f(x)$ must integrate to 1 and $f(x) \geq 0$ for all x. We have

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^A \sin(x)dx = 1 - \cos(A) = 1.$$

Thus, $A = \frac{\pi}{2}$.

(b)

$$F(x) = \int_{-\infty}^{x} f(s)ds = \begin{cases} 0 & x < 0 \\ 1 - \cos(x) & 0 \leq x \leq \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x \end{cases}$$

5. It is known that in a school 30% of the students own an Iphone. A researcher asks students in the school at random whether they own an Iphone. Let X be the random variable representing the number of students asked up to and including the first person who owns an Iphone. Determine

   (a) $\mathbb{P}(X = 4)$

   (b) $\mathbb{P}(X > 4)$

   (c) $\mathbb{P}(X < 6)$

   Solution:
   $X \sim \text{Geo}(0.3)$.

   (a) $\mathbb{P}(X = 4) = 0.7^3 0.3 = 0.1029$.

   (b) $\mathbb{P}(X \geq 5) = 1 - \mathbb{P}(X \leq 4) = 1 - (1 - 0.7^4) = 0.7^4 = 0.2401$.

   (c)

$$\begin{aligned} \mathbb{P}(X < 6) &= \mathbb{P}(X \leq 5) \\ &= 0.3 + 0.3 \times 0.7 + 0.3 \times 0.7^2 + 0.3 \times 0.7^3 + 0.3 \times 0.7^4 \\ &= 0.8319 \end{aligned}$$

6. Jason and Rebecca are rolling a fair 6-sided die and winner of the game will be the first person to get a 6. Jason rolls the die first. Determine the probability that

   (a) Jason wins on his second throw.

   (b) Jason wins on his third throw.

   (c) Jason wins the game.

   Solution:

   (a) This is the event that both players do not get a 6 on their first throw, and Jason gets a 6 on his second their throw. The probability is

$$\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{25}{216}.$$

   (b) This is the event that both players do not get a 6 on their first two throws, and Jason gets a 6 on his third their throw. The probability is

$$\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{625}{7776}.$$

(c) This is the event that Jason wins on his first throw or Jason wins on his second throw or ... The probability is

$$
\begin{aligned}
& \frac{1}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} + \cdots \\
= \; & \frac{1}{6} + \frac{1}{6} \times \frac{25}{36} + \frac{1}{6} \times \left(\frac{25}{36}\right)^2 + \cdots \\
= \; & \frac{1/6}{1 - 25/36} \\
= \; & \frac{6}{11}
\end{aligned}
$$