MAU22C00 - TUTORIAL 2 SOLUTIONS

1) Prove $A \setminus (A \setminus B) \subseteq B$.

Solution: This is done by examining where the elements lie in the set to the left of $\subseteq$ and proving they also lie in B. To this end, take $x \in A \setminus (A \setminus B)$. By the definition of $X \setminus Y = X \cap Y^c$, we have

$$x \in A \setminus (A \setminus B) \Rightarrow x \in A \cap (A \setminus B)^c \Rightarrow x \in A \text{ AND } x \in (A \setminus B)^c$$

Applying the definition of $\setminus$ again. we conclude $x \in A$ AND $x \in (A \cap B^c)^c$. Using De Morgan's laws for the later, we get $x \in A$ AND $x \in A^c \cup (B^c)^c$. Let's focus more on the later, with the knowledge that $x \in A$.

$$x \in A^c \cup (B^c)^c \Rightarrow x \in A^c \text{ OR } x \in (B^c)^c \Rightarrow x \in A^c \text{ OR } x \in B$$

Since x cannot be in both A and A^c at the same time, we conclude $x \in B$ (now ignoring A). What we have shown:

$$\forall x (x \in A \setminus (A \setminus B) \Rightarrow x \in B)$$

So $A \setminus (A \setminus B) \subseteq B$ as required. $\square$

Remark. Veitch diagrams and/or Venn diagrams will **NOT** be accepted as a form of proof in set theory. Please bear in mind that only a solution of this kind is acceptable as a proof to an assertion in set theory.

2) In the country of Tannu Tuva, a valid license plate consists of any digit except 0, followed by any two letters of the English alphabet, followed by any two digits.

(a) Let D be the set of all digits and L the set of all letters. With this notation, write the set of all possible license plates as a Cartesian product.

(b) How many possible license plates are there?

Solution: (a) The first character on the license plate belongs to the set $D \setminus \{0\}$ consisting of all digits but zero. The second character belongs to L, the third also to L, the fourth to D, and the fifth to D. The set of all possible license plates is the Cartesian product of all these sets in order, namely $(D \setminus \{0\}) \times L \times L \times D \times D$.

(b) The number of possible license plates is exactly the number of elements in the Cartesian product $(D \setminus \{0\}) \times L \times L \times D \times D$ from part (a). The number of elements of a **finite** Cartesian product is thus the product of the number of elements in each of the finite sets composing the product. We'll see in Hilary term what happens when we look at Cartesian products of infinite sets. Since there are 10 digits, $\#(D) = 10$, which means $\#(D \setminus \{0\}) = 9$. There are 26 letters in the English alphabet, so $\#(L) = 26$. We conclude that

$$\#((D \setminus \{0\}) \times L \times L \times D \times D = 9 \times 26 \times 26 \times 10 \times 10 = 608,400.$$

3) (From the 2016-2017 Annual Exam) Let Q denote the relation on the set $\mathbb{Z}$ of integers, where integers x and y satisfy xQy if and only if

$$x - y = (x - y)(x + 2y).$$

Determine the following:

(i) Whether or not the relation Q is *reflexive*;

(ii) Whether or not the relation Q is *symmetric*;

(iii) Whether or not the relation Q is *transitive*;

(iv) Whether or not the relation Q is an *equivalence relation*;

Justify your answers.

Solution: $x, y \in \mathbb{Z}$ satisfy xQy iff $x - y = (x - y)(x + 2y)$, which is equivalent to $(x - y)(x + 2y - 1) = 0$, i.e., $x = y$ or $x + 2y - 1 = 0$.

(i) **Reflexivity:** The relation Q is reflexive because xQx holds for all $x \in \mathbb{Z}$ as $x - x = (x - x)(x + 2x) = 0$.

(ii) **Symmetry:** The relation Q is not symmetric because if $x \neq y$, then xQy holds if $x + 2y = 1$, thus for yQx we would need $y + 2x = 1$, which only holds at the same time with $x + 2y = 1$ when $x = y = \dfrac{1}{3} \notin \mathbb{Z}$.

(iii) **Transitivity:** The relation Q is not transitive. Assume xQy and yQz hold for $x, y, z \in \mathbb{Z}$. There are 4 cases to consider:

Case 1: $x = y$ and $y = z$, then $x = z$, so xQz as needed.

Case 2: $x = y$ and $y + 2z = 1$, then $x + 2z = 1$, so xQz as needed.

Case 3: $x + 2y = 1$ and $y = z$, then $x + 2z = 1$, so xQz as needed.

Case 4: $x + 2y = 1$ and $y + 2z = 1$, then $x + 2(1 - 2z) = 1$, so $x + 2 - 4z = 1$, i.e., $x - 4z = -1$. This last equation is satisfied for example for $x = 3$, $z = 1$. Take $y = -1$ in order to satisfy $x + 2y = 1$. We see that $x + 2z = 3 + 2 = 5 \neq 1$, so xQz fails. We have constructed a counterexample.

(iv) **Equivalence relation:** The relation Q is not an equivalence relation because while reflexive, it fails to be symmetric and transitive.