MAU22C00: TUTORIAL 21 SOLUTIONS
TURING MACHINES

1) Recall the Turing machine constructed in lecture that decides the language $L = \{0^m 1^m \mid m \in \mathbb{N}, m \geq 1\}$. For input string 0011, write down each configuration in turn starting with the initial configuration of the Turing machine and ending with the accepting configuration.

2) (Annual Exam 2020) Let L_1 and L_2 be two Turing-recognisable languages over the same finite alphabet A. Construct an enumerator that outputs $L_1 \cap L_2$.

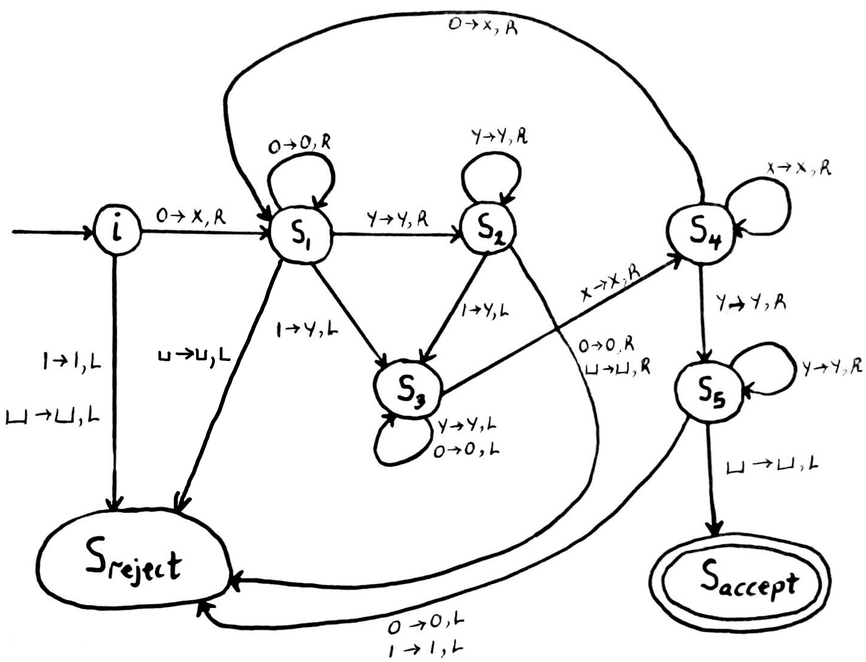
3) Recall Hilbert's 10th Problem from lecture. What is the size of D_1? Finite, countably infinite or uncountably infinite? Remember that each polynomial in D_1 has integer coefficients.

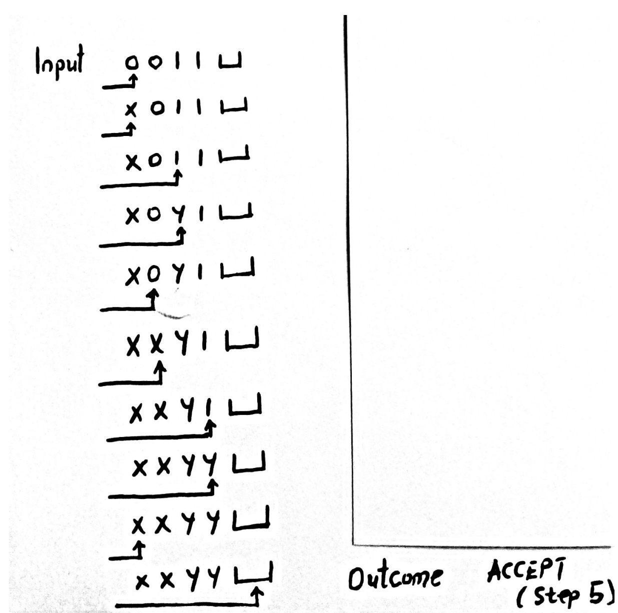
4) (Annual Exam 2020) In lecture, we defined the language

$$E_{DFA} = \{\langle B \rangle \;\big|\; B \text{ is a DFA and } L(B) = \emptyset\}$$

when we examined whether the emptiness testing problem for deterministic finite state acceptors was a Turing-decidable language. Is E_{DFA} finite, countably infinite, or uncountably infinite? Justify your answer.

Solution: 1) Recall from lecture the transition diagram of the Turing machine that decides the language $L = \{0^m 1^m \mid m \in \mathbb{N}, m \geq 1\}$:





Here is how input string 0011 is processed:
Correspondingly, we have the following configurations: $\epsilon i0011$, xS_1011, $x0S_111$, xS_30y1, ϵS_3x0y1, xS_40y1, xxS_1y1, $xxyS_21$, xxS_3yy, xS_3xyy, xxS_4yy, $xxyS_5y$, $xxyyS_5\epsilon$, and finally $xxyS_{accept}y$. Note that the diagram showing how 0011 is processed skips some steps showing how we

move to get to the next character we wish to process, whereas the list of configurations exactly traces what the Turing machine does including which states it enters. Each configuration in the list yields the next one, and going from one to the other is given by a transition that is marked on the transition diagram of the Turing machine.

2) Let M be a Turing machine that recognises L_1, and let N be a Turing machine that recognises L_2. A^* has an enumeration as a sequence

$$A^* = \{w_1, w_2, \dots\}.$$

We construct our enumerator that outputs $L_1 \cap L_2$ as follows:

$E =$ Ignore the input

1. Repeat the following for $i = 1, 2, 3, \dots$
2. Run M for i steps on each input $w_1, w_2, \dots, w_i$.
3. If any computations accept, run N for i steps on the corresponding w_j.
3. Print out every w_j that N accepts.

Note that we need to run M and N for only i steps because either of these Turing machines could loop if we do not specify a number of steps. Each w_j that is printed out has been accepted by both M and N, which means it belongs to $L_1 \cap L_2$.

3) Recall that D_1 is given by

$$D_1 = \{p(x) \mid \exists x \in \mathbb{Z} \text{ such that } p(x) = 0\},$$

where each $p(x)$ has integer coefficients. Note that $p(x)$ could have any degree as long as that degree is at least 1 (polynomials of degree zero are by definition constant polynomials, so they cannot have a root, hence we exclude them). Before we worry about how many of these polynomials have roots in $\mathbb{Z}$, let us first figure out what is the size of the set of polynomials $p(x)$ of degree at least 1 with integer coefficients. Let us call this set B. Let B_d be the set of polynomials $p(x)$ of degree d with integer coefficients. Since the degree of the polynomials in the set B is at least 1, we can represent B as the union

$$B = \bigcup_{d=1}^{\infty} B_d.$$

Note that $B_d \cap B_{d'} = \emptyset$ if $d \neq d'$ because if a polynomial has degree d, it cannot have degree d' if $d \neq d'$. Therefore, B is the union of the disjoint sets B_d. Let us now figure out the size of each B_d. Let $p(x) \in B_d$. Since $p(x)$ has degree d, it can be written as $p(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$, where $a_d \neq 0$. Note that we can put each element $p(x)$ of B_d in bijective correspondence with the $(d+1)$-tuple $(a_d, a_{d-1}, \dots, a_1, a_0)$ of its coefficients. Therefore, $B_d \sim \{(a_d, a_{d-1}, \dots, a_1, a_0) \in \mathbb{Z}^{n+1} \mid a_d \neq 0\}$ since all coefficients of $p(x)$ must be integers, hence in $\mathbb{Z}$.

$$\{(a_d, a_{d-1}, \dots, a_1, a_0) \in \mathbb{Z}^{n+1} \mid a_d \neq 0\} = (\mathbb{Z} \setminus \{0\}) \times \mathbb{Z}^n$$

because $a_d \neq 0$, so $a_d \in \mathbb{Z} \setminus \{0\}$. Therefore, $B_d \sim (\mathbb{Z} \setminus \{0\}) \times \mathbb{Z}^n$. We proved in lecture that $\mathbb{Z}$ is countably infinite. When we take away one element, namely 0, it stays countably infinite, so $\mathbb{Z} \setminus \{0\}$ is countably infinite. Thus, $(\mathbb{Z} \setminus \{0\}) \times \mathbb{Z}^n$ is a Cartesian product of finitely many countably infinite sets. We conclude that $(\mathbb{Z} \setminus \{0\}) \times \mathbb{Z}^n$ must be countably infinite, hence B_d must be countably infinite as it is in bijective correspondence with it. We conclude that B as the countably infinite disjoint union

$$B = \bigcup_{d=1}^{\infty} B_d$$

of countably infinite sets, B_d must itself be countably infinite by a theorem proven in the unit on countability of sets. Note that $D_1 \subset B$, so the set we are interested in, D_1, is a subset of a countably infinite set. Therefore, D_1 could be finite or countably infinite. We will prove that D_1 is countably infinite by showing it contains a sequence, namely a countably infinite subset. D_1 consists of all polynomials $p(x)$ of degree at least 1 with integer coefficients that have at least one integer root. Define $p_i(x) = x - i$. Therefore $p_1(x) = x - 1$, $p_2(x) = x - 2$, etc. Clearly, each p_i has root $i \in \mathbb{N} \subset \mathbb{Z}$, so $p_i(x) \in D_1$. The polynomials $p_i(x)$ for $i \geq 1$ form a sequence $\{p_1(x), p_2(x), p_3(x), \dots\}$, and $\{p_1(x), p_2(x), p_3(x), \dots\} \subset D_1$.Therefore, D_1 is countably infinite. Note this argument was a sandwich argument as we showed

$$\{p_1(x), p_2(x), p_3(x), \dots\} \subset D_1 \subset B,$$

with $\{p_1(x), p_2(x), p_3(x), \dots\}$ and B both countably infinite. Note also that D_n is likewise countably infinite. The argument is fundamentally the same as the one given for D_1, but we have to account for the fact that $p(x_1, \dots, x_n)$ has monomials in terms of the n variables $x_1, \dots, x_n$, so writing out its coefficients requires the use of the multi-index notation.

4) The language

$$E_{DFA} = \{\langle B \rangle \mid B \text{ is a DFA and } L(B) = \emptyset\}$$

consists of two conditions:

- B is a DFA.
- $L(B) = \emptyset$.

Let C be the set of all DFA's that accept languages over a given finite alphabet A. We first need to figure out the size of C. Recall from the unit on formal languages and grammars that a language L is regular $\Leftrightarrow$ L is accepted by a DFA. As proven in lecture, the set of regular languages is countably infinite, so C must also be countably infinite. Clearly, $E_{DFA} \subsetneq C$ as the condition $L(B) = \emptyset$ shrinks the size of the set. E_{DFA} is thus a subset of a countably infinite set, so it could be finite or countably infinite. Note that for any $m \in \mathbb{N}^*$, we can construct a DFA B with m non-accepting states ensuring that $L(B) = \emptyset$. Since $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ is a countably infinite set with an element taken out hence still countably infinite, we have shown E_{DFA} has a countably infinite subset. We conclude that E_{DFA} must be countably infinite. Once again, this was a sandwich argument.