MAU22C00: TUTORIAL 15 PROBLEMS MINIMAL SPANNING TREES AND DIRECTED GRAPHS

1) Prove that any subgraph (V', E') of a connected graph (V, E) is contained in some spanning tree of $(V, E) \iff (V', E')$ contains no circuits.

2) (Annual Exam Trinity Term 2018) Consider the connected undirected graph with vertices A, B, C, D, E, F, G, H, I, J, K, and L, and with edges listed with associated costs in the following table:

CF	JK	IJ	AD	CH	EI	BL	CE	HG	BH
2	2	3	3	3	4	5	6	6	7
EF	FJ	GK	CD	DE	HL	AC	FH	EJ	AB
8	8	9	9	10	10	10	11	12	14

Determine the minimum spanning tree generated by Prim's Algorithm, starting from the vertex F, where that algorithm is applied with the queue specified in the table above. For each step of the algorithm, write down the edge that is added.

3) (Annual Exam Trinity Term 2018)

(a) How many distinct directed graphs with three vertices $V = \{a, b, c\}$ are there? Justify your answer.

(b) Using the one-to-one correspondence between directed graphs and relations, draw a directed graph that corresponds to a relation on $V = \{a, b, c\}$ that is reflexive **but neither** symmetric **nor** transitive. Justify your answer.

(c) Using the one-to-one correspondence between directed graphs and relations, draw a directed graph that corresponds to a relation on $V = \{a, b, c\}$ that is symmetric **but neither** reflexive **nor** transitive. Justify your answer.

(d) Using the one-to-one correspondence between directed graphs and relations, draw a directed graph that corresponds to a relation on $V = \{a, b, c\}$ that is transitive **but not** reflexive **nor** symmetric. Justify your answer.

Solution: 1) We have to prove an equivalence, which we will do by proving each implication in turn:

" $\Longrightarrow$ " (V', E') is contained in a spanning tree (V, E'') of (V, E). Therefore, $V' \subseteq V$, $E' \subseteq E$, and (V', E') is a subgraph of (V, E'').

Since (V, E'') is a spanning tree of (V, E), it is a tree, so by definition it contains no circuits. Therefore, its subgraph (V', E') cannot contain any circuits either.

" $\Longleftarrow$ " (V', E') contains no circuits. If (V', E') is itself a spanning tree of (V, E), it is clearly contained in a spanning tree, and there is nothing to prove; otherwise, if $V' \subsetneq V$ or (V', E') is not connected, then let $\tilde{E} = E \setminus E'$. Since (V, E) is connected, if $V' \subsetneq V$ or (V', E') is not connected, $\tilde{E} \neq \emptyset$. We seek to add edges from $\tilde{E}$ and their endpoints not already in V' to (V', E') in order to construct a spanning tree. Since $\tilde{E}$ is a finite set, we can write it as $\tilde{E} = \{e_1, e_2, \ldots, e_m\}$. We now examine each edge in $\tilde{E}$ in turn. Consider e_1. If adding e_1 to (V', E') produces a circuit, then discard e_1; otherwise, add e_1 to (V', E') along with any of its endpoints not already in (V', E') and denote by (V_1, E_1) the resulting graph. If e_1 is discarded, then let $(V_1, E_1) = (V', E')$. We continue this process. Consider e_2. If adding e_2 to (V_1, E_1) produces a circuit, then discard e_2; otherwise, add e_2 to (V_1, E_1) along with any of its endpoints not already in (V_1, E_1) and denote by (V_2, E_2) the resulting graph. If e_2 is discarded, then let $(V_2, E_2) = (V_1, E_1)$. At step j of this process, consider e_j. If adding e_j to (V_{j-1}, E_{j-1}) produces a circuit, then discard e_j; otherwise, add e_j to (V_{j-1}, E_{j-1}) along with any of its endpoints not already in (V_{j-1}, E_{j-1}) and denote by (V_j, E_j) the resulting graph. If e_j is discarded, then let $(V_j, E_j) = (V_{j-1}, E_{j-1})$. The process stops after e_m is considered, hence after m steps. Since the starting subgraph (V', E') had no circuits and since all edges not already in E' were considered and only added if no circuit was created, the resulting graph (V_m, E_m) cannot contain any circuits. Note that we added all vertices not already in V' that were endpoints of edges added. Assume $\exists\, v \in V \setminus V_m$. Since all edges in E are either in E' or were considered by the algorithm, v cannot be an endpoint of either an edge in E' or in $\tilde{E}$, but $E = E' \cup \tilde{E}$. Therefore, v is not incident to any edge in E. We conclude that $\deg v = 0$ in (V, E), which means (V, E) contains an isolated vertex. $v \notin V'$, $\deg v = 0$, and $V' \neq \emptyset$ together imply that (V, E) has at least two components $\Rightarrow\!\Leftarrow$ as (V, E) is connected. Therefore, $V \setminus V_m = \emptyset$, so $V_m = V$. We have obtained a subgraph (V_m, E_m) of (V, E) that has no circuits and satisfies $V_m = V$. To show, (V_m, E_m) is a spanning tree of (V, E), we must show that it is connected. Assume not, then (V_m, E_m) contains at least two components. Assume vertices v and w belong to different components of (V_m, E_m). (V, E) is connected, so there exists a path from v to w via edges in (V, E). Some of those edges in this path then do not belong to E_m $\Rightarrow\!\Leftarrow$ as these edges were in $\tilde{E}$ and should have been added at some step i of the algorithm since their addition could not have created a circuit. Therefore, (V_m, E_m) is connected, so it is a spanning tree of (V, E) containing (V', E'). $\square$

2) The edges are added in the following order: CF, CH, CE, EI, IJ, JK, HG, BH, BL, CD, and AD.

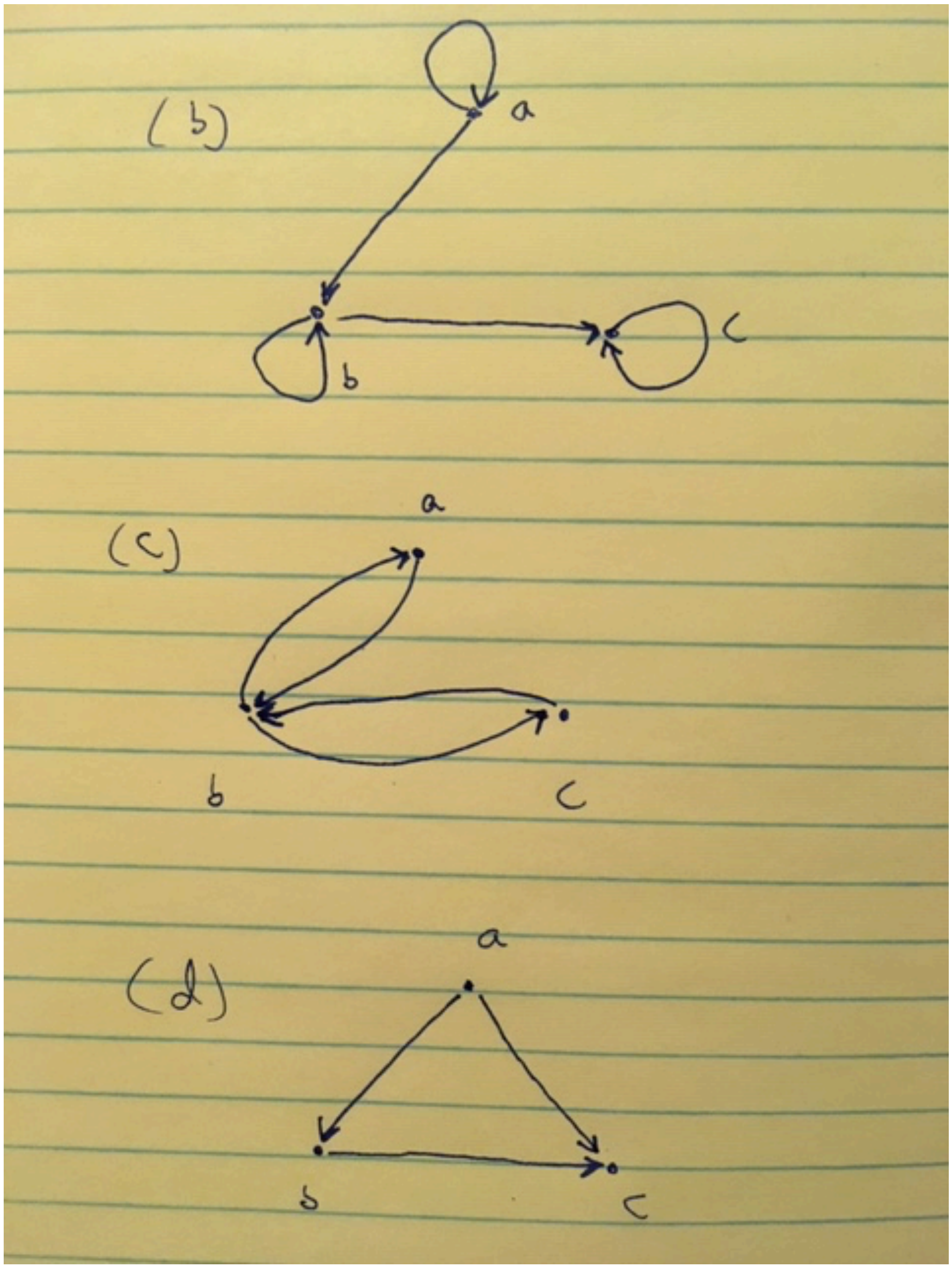
3) (a) An edge of a directed graph is any pair in $V \times V$. Since V has 3 elements, there are $3 \times 3 = 9$ different such pairs hence possible edges. A directed graph on three vertices $V = \{a, b, c\}$ will have as its set of edge a subset of this 9-element set. As the power set of a set of nine elements has $2^9 = 512$ elements, there are 512 distinct directed graphs on 3 vertices $\{a, b, c\}$.

For (b), (c), (d), note that reflexivity is universally quantified, whereas symmetry and transitivity are given by implications, which are vacuously true if the antecedents (the 'if" part of the statements) fail to be true. The graphs are drawn at the end of the solutions.

(b) $E = \{(a, a), (b, b), (c, c), (a, b), (b, c)\}$. Since (b, a) and (c, b) are missing, the relation isn't symmetric. Since (a, c) is missing but (a, b) and (b, c) are present, the relation isn't transitive. The presence of (a, a), (b, b), and (c, c) makes the relation reflexive.

(c) $E = \{(a, b), (b, a), (b, c), (c, b)\}$. No (a, a) in E means the relation isn't reflexive. No (a, c) and (c, a) in E makes the relation non-transitive. The pairs (a, b) with (b, a) and (b, c) with (c, a) make the relation symmetric.

(d) $E = \{(a, b), (b, c), (a, c)\}$. Those three pairs make the relation transitive. Since (b, a) is not present, the relation isn't symmetric. Since (a, a) isn't present, the relation isn't reflexive.



(b)
a
b
c
(c)
a
b
c
(d)
a
b
c