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1 Review of Propositional Logic

Task: Recall enough propositional logic to see how it matches up with set theory.

Definition: A proposition is any declarative sentence that is either true or false.

1.1 Connectives

		Connectives	Notation in Maths
and	$\wedge$		
or	$\vee$	"Inclusive or"	
not	$\neg$	Sometimes denoted $\sim$	
implies	$\rightarrow$	if/then; called implication	$\Rightarrow$
if and only if	$\leftrightarrow$	Called equivalence	$\Leftrightarrow$

1.1.1 Truth Table of the Connectives

Let P, Q be propositions:

P	Q	P$\wedge$Q
F	F	F
F	T	F
T	F	F
T	T	T

P	Q	P$\vee$Q
F	F	F
F	T	T
T	F	T
T	T	T

P	$\neg$P
F	T
T	F

NB In some textbooks, T is denoted by 1, and F is denoted by 0.

P	Q	P$\rightarrow$Q
F	F	T
F	T	T
T	F	F
T	T	T

NB Note that the only instance when an implication (if/then statement) denoted by $P \rightarrow Q$ is false is when the hypothesis (P) is true, but the conclusion (Q) is false.

P	Q	P$\leftrightarrow$Q
F	F	T
F	T	F
T	F	F
T	T	T

NB The truth table for the equivalence says that both P and Q must have the same truth value, i.e. both be true or both be false for the equivalence to be true.

Priority of the Connectives

Highest to Lowest: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

1.2 Important Tautologies

$$\begin{array}{rcl} (P \rightarrow Q) & \leftrightarrow & (\neg P \vee Q) \\ (P \leftrightarrow Q) & \leftrightarrow & [(P \rightarrow Q) \wedge (Q \rightarrow P)] \\ \neg(P \wedge Q) & \leftrightarrow & (\neg P \vee \neg Q) \\ \neg(P \vee Q) & \leftrightarrow & (\neg P \wedge \neg Q) \end{array} \left.\vphantom{\begin{array}{c}a\\b\end{array}}\right\} \text{De Morgan Laws}$$

(these have parallels in in set theory)

As a result, $\neg$ and $\vee$ together can be used to represent all of $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

Less obvious: One connective called the Sheffer stroke P|Q (which stands for "not both P and Q" or "P nand Q") can be used to represent all of $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ since $\neg$P $\leftrightarrow$ P|P and P $\vee$ Q $\leftrightarrow$ (P|P) | (Q|Q).

Recall that if P$\rightarrow$Q is a given implication, then Q$\rightarrow$P is called the <u>converse</u> of P$\rightarrow$Q, while $\neg$Q $\rightarrow$ $\neg$P is called the <u>contrapositive</u> of P $\rightarrow$ Q.

1.3 Indirect Arguments/Proofs by Contradiction/Reductio ad absurdum

Based on the tautology (P$\rightarrow$Q) $\leftrightarrow$ ($\neg$Q $\rightarrow$ $\neg$P)

Example: Famous argument that $\sqrt{2}$ is irrational.

Proof:

Suppose $\sqrt{2}$ is rational, then it can be expressed in fraction form as $\frac{a}{b}$ with a and b integers, $b \neq 0$. Let us **assume** that our fraction is reduced, **i.e.** the only common divisor of the numerator a and denominator b is 1.

Then,

$$\sqrt{2} = \frac{a}{b}$$

Squaring both sides, we have

$$2 = \frac{a^2}{b^2}$$

Multiplying both sides by b^2 yields

$$2b^2 = a^2$$

Therefore, 2 divides a^2, i.e. a^2 is even. If a^2 is even, then a is also even, namely $a = 2k$ for some integer k.

Substituting the value of $2k$ for a, we have $2b^2 = (2k)^2$ which means that $2b^2 = 4k^2$. Dividing both sides by 2, we have $b^2 = 2k^2$. That means 2 divides b^2, so b is even.

This implies that both a and b are even, which means that both the numerator and the denominator of our fraction are divisible by 2. This contradicts our **assumption** that the numerator a and the denominator b have no common divisor except 1. Since we found a contradiction, our assumption that $\sqrt{2}$ is rational must be false. Hence the theorem is true.

qed

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2 Predicate logic and Quantifiers

Task: Understand enough predicate logic to make sense of quantified statements.

In predicate logic, propositions depend on variables x, y, z, so their truth value may change depending on which values these variables assume: $P(x), Q(x,y), R(x,y,z)$

2.1 Introduce quantifiers

2.1.1 $\exists$ existential quantifier

Syntax: $\exists x P(x)$

Definition: $\exists x P(x)$ is true if $P(x)$ is true for some value of x. It is false otherwise.

2.1.2 $\forall$ universal quantifier

Syntax: $\forall x P(x)$

Definition: $\forall x P(x)$ is true if $P(x)$ is true for all allowable values of x. It is false otherwise.

2.1.3 $\exists!$ for one and only one (additional quantifier standard in maths)

Syntax: $\exists! x P(x)$

Definition: $\exists! x P(x)$ is true if $P(x)$ is true for exactly one value of x and false for all other values of x; otherwise, $\exists! x P(x)$ is false.

Example: $P(x) : x$ is/was the pope and x is Argentine.
(Compound statement; two sentences with connector $\wedge$ between them)
$\exists! x P(x)$ is true with x being Pope Francis.
Now, set $Q(x) : x$ is/was the pope and x is Brazilian.
$\exists! x Q(x)$ is false as there has not been a Brazilian pope so far.
In fact, $\exists x Q(x)$ is also false.

2.2 Alternation of Quantifiers

$\forall x \exists y \forall z \quad P(x, y, z)$

NB: The order <u>cannot</u> be exchanged as it might modify the truth value of the statement (think of examples with two quantifiers).

2.3 Negation of Quantifiers

$$\begin{aligned} \neg(\exists x P(x)) &\quad \leftrightarrow \quad \forall x \neg P(x) \\ \neg(\forall x P(x)) &\quad \leftrightarrow \quad \exists x \neg P(x) \end{aligned}$$

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3 Set Theory

Task: Understand enough set theory to make sense of other mathematical objects in abstract algebra, graph theory, etc.

Set theory started around 1870's $\to$ late development in mathematics but now taught early in one's maths education due to the Bourbaki school.

Definition: A set is a collection of objects. $x \in A$ means the element x is in the set A (**i.e.** belongs to A).

Examples:

1. All students in a class.
2. $\mathbb{N}$ the set of natural numbers starting at 0.

   $\mathbb{N}$ is defined via the following two axioms:

   (a) $0 \in \mathbb{N}$

   (b) if $x \in \mathbb{N}$, then $x+1 \in \mathbb{N}$ ($x \in \mathbb{N} \to x+1 \in \mathbb{N}$)
3. $\mathbb{R}$ set of real numbers also introduced axiomatically. The hardest axiom is the last one: completeness. $\mathbb{R}$ is constructed from $\mathbb{Q}$ in one of two ways: via Dedekind cuts or Cauchy sequences.

   $\mathbb{R}$ is the set of real numbers. The axioms governing $\mathbb{R}$ are:

   (a) Additive closure: $\forall x, y \, \exists z (x + y = z)$

   (b) Multiplicative closure: $\forall x, y, \exists z (x \times y = z)$

   (c) Additive associativity: $\forall x, y, z \quad x + (y + z) = (x + y) + z$

   (d) Multiplicative associativity: $\forall x, y, z \quad x \times (y \times z) = (x \times y) \times z$

   (e) Additive commutativity: $\forall x, y \quad x + y = y + x$

   (f) Multiplicative commutativity: $\forall x, y \quad x \times y = y \times x$

   (g) Distributivity: $\forall x, y, z \quad x \times (y + z) = (x \times y) + (x \times z)$ and $(y + z) \times x = (y \times x) + (z \times x)$

   (h) Additive identity: There is a number, denoted 0, such that for all $x, x + 0 = x$

   (i) Multiplicative identity: There is a number, denoted 1, such that for all $x, x \times 1 = 1 \times x = x$

(j) Additive inverses: For every x there is a number, denoted $-x$, such that $x + (-x) = 0$

(k) Multiplicative inverses: For every nonzero x there is a number, denoted x^{-1}, such that $x \times x^{-1} = x^{-1} \times x = 1$

(l) $0 \neq 1$

(m) Irreflexivity of $<$: $\sim (x < x)$

(n) Transitivity of $<$: If $x < y$ and $y < z$, then $x < z$

(o) Trichotomy: Either $x < y, y < x$, or $x = y$

(p) If $x < y$, then $x + z < y + z$

(q) If $x < y$ and $0 < z$, then $x \times z < y \times z$ and $z \times x < z \times y$

(r) Completeness: If a nonempty set of real numbers has an upper bound, then it has a *least* upper bound.

4. $\emptyset$ is the empty set (The set with no elements).

Definition: Let A, B be sets. A=B if and only if all elements of A are elements of B and all elements of B are elements of A,
i.e. $A = B \leftrightarrow [\forall x(x \in A \to x \in B)] \wedge [\forall y(y \in B \to y \in A)]$

3.1 Two Ways to Describe Sets

1. The enumeration/roster method: list all elements of the set.
NB: order is irrelevant.
$A = \{0, 1, 2, 3, 4, 5\} = \{5, 0, 2, 3, 1, 4\}$

2. The formulaic/set builder method: give a formula that generates all elements of the set.
$A = \{x \in \mathbb{N} \mid 0 \leq x \wedge x \leq 5\} = \{0, 1, 2, 3, 4, 5\} = \{x \in \mathbb{N} : 0 \leq x \wedge x \leq 5\}$

Using $\mathbb{N}$ and the set-builder method, we can define:
$\mathbb{Z} = \{m - n \mid \forall m, n \in \mathbb{N}\}$
$n = 0$ and m any natural number $\Rightarrow$ we generate all of $\mathbb{N}$
$m = 0$ and n any natural number $\Rightarrow$ we generate all negative integers
$0 - 1 = -1$
$0 - 2 = -2$
etc.
$\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z} \wedge q \neq 0\}$

Definition: A set A is called finite if it has a finite number of elements; otherwise, it is called infinite.

3.2 Set Operations

Task: Understand how to represent sets by Venn diagrams. Understand set union, intersection, complement, and difference.

Definition: Let A, B be sets. A is a subset of B if all elements of A are elements of B, **i.e.** $\forall x(x \in A \to x \in B)$. We denote that A is a subset of B by $A \subseteq B$

Example: $\mathbb{N} \subseteq \mathbb{Z}$

Definition: Let A, B be sets. A is a proper subset of B if $A \subseteq B \wedge A \neq B$, **i.e.** $A \subseteq B \wedge \exists x \in B \ s.t. \ x \notin A$.

Notation: $A \subset B$

Example: $\mathbb{N} \subset \mathbb{Z}$ since $\exists(-1) \in \mathbb{Z}$ such that $-1 \notin \mathbb{N}$.

NB: $\forall A$ a set, $\emptyset \subseteq A$

Recall: $B \subseteq C$ means $\forall x(x \in B \to x \in C)$, but $\emptyset$ has no elements, so in $\emptyset \subseteq A$ the quantifier $\forall$ operates on a domain with no elements. Clearly, we need to give meaning to $\exists$ and $\forall$ on empty sets.

Boolean Convention

$$\left.\begin{array}{l}\forall \text{ is true on the empty set} \\ \exists \text{ is false on the empty set}\end{array}\right\} \text{Consistent with common sense}$$

Definition: Let A, B be two sets. The union $A \cup B = \{x \mid x \in A \ \vee \ x \in B\}$

Definition: Let A, B be two sets. The intersection $A \cap B = \{x \mid x \in A \wedge x \in B\}$

Definition: Let A, B be sets. A and B are called disjoint if $A \cap B = \emptyset$

Definition Let A, B be two sets. $A - B = A \backslash B = \{x \mid x \in A \ \wedge \ x \notin B\}$

Examples:

$$\begin{array}{ll} A = \{1,2,5\} & B = \{1,3,6\} \\ A \cup B = \{1,2,3,5,6\} & A \cap B = \{1\} \\ A \backslash B = \{2,5\} & B \backslash A = \{3,6\} \end{array}$$

Definition: Let A, U be sets s.t. $A \subseteq U$. The complement of A in U = $U \backslash A = A^C = \{x \mid x \in U \wedge x \notin A\}$

Remark: The notation A^C is unambiguous only if the universe U is clearly defined or understood.

3.2.1 Venn Diagrams

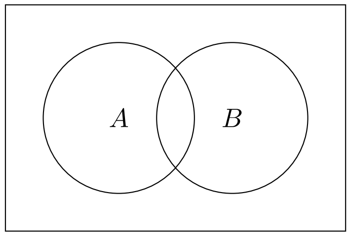
Schematic representation of set operations.

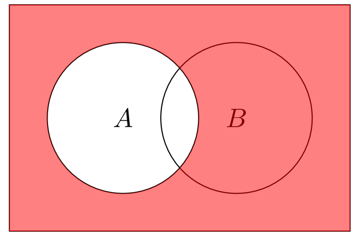
Pros of Venn diagrams:

Very easy to visualize

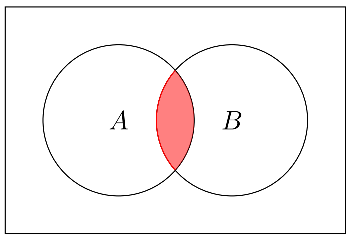
Cons of Venn diagrams:

1. Misleading if for example $A \subset B$ or sets are in some other non standard configuration;

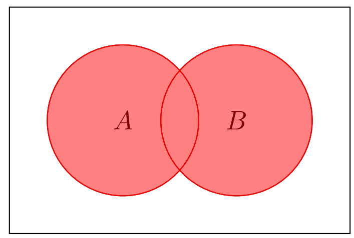




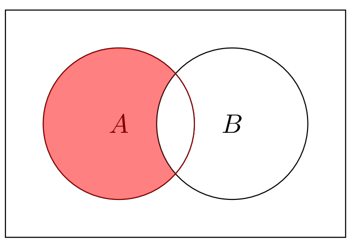
A^C



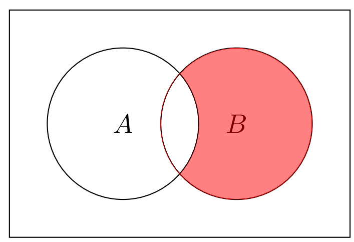
$A \cap B$



$A \cup B$



$A \backslash B$



$B \backslash A$

2. Not helpful if a lot of sets are involved;

3. Not helpful if sets are infinite or have some peculiar structure.

Moral of the story: Venn diagrams will **NOT** be accepted as proof of any statement in set theory. Instead, we will introduce rigorous ways of proving assertions in set theory.

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3.2.2 Properties of Set Operations

Correspondence between Logic and Set Theory

Logical Connective	Set operation/property
$\wedge$	intersection $\cap$
$\vee$	union $\cup$
$\neg$	complement $(\ )^C$
$\to$	subset $\subseteq$
$\leftrightarrow$	equality of sets $=$

Recall:

Definition: Let A, B be two sets. The intersection $A \cap B = \{x \mid x \in A \wedge x \in B\}$

Definition: Let A, B be two sets. The union $A \cup B = \{x \mid x \in A \ \vee \ x \in B\}$

Definition: Let A, U be sets s.t. $A \subseteq U$. The complement of A in $U = U \backslash A = A^C = \{x \mid x \in U \wedge x \notin A\}$

Definition: Let A, B be sets. A is a subset of B if all elements of A are elements of B, **i.e.** $\forall x(x \in A \to x \in B)$.

Definition: Let A, B be sets. A=B if and only if all elements of A are elements of B and all elements of B are elements of A,
i.e. $A = B \leftrightarrow [\forall x(x \in A \to x \in B)] \wedge [\forall y(y \in B \to y \in A)]$

As a result, various properties of set operations become obvious:

- Commutativity
  - $A \cap B = B \cap A$ comes from the tautology $(P \wedge Q) \leftrightarrow (Q \wedge P)$ (#31 on the list of tautologies posted in Course Documents)
  - $A \cup B = B \cup A$ comes from the tautology $(P \vee Q) \leftrightarrow (Q \vee P)$ (# 32 on the list of tautologies)
- Associativity
  - $(A \cup B) \cup C = A \cup (B \cup C)$ comes from the tautology $[(P \vee (Q \vee R)] \leftrightarrow [(P \vee Q) \vee R]$ (# 33 on the list of tautologies)
  - $(A \cap B) \cap C = A \cap (B \cap C)$ comes from the tautology $[(P \wedge (Q \wedge R)] \leftrightarrow [(P \wedge Q) \wedge R]$ (# 34 on the list of tautologies)
- Distributivity
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ comes from the tautology $[(P \wedge (Q \vee R)] \leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$ (# 29 on the list of tautologies)

  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ comes from the tautology $[(P \vee (Q \wedge R)] \leftrightarrow [(P \vee Q) \wedge (P \vee R)]$ (# 30 on the list of tautologies)

- De Morgan Laws in Set Theory
  - $(A \cap B)^C = A^C \cup B^C$ comes from the tautology $\neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$ (# 18 on the list of tautologies)
  - $(A \cup B)^C = A^C \cap B^C$ comes from the tautology $\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$ (# 19 on the list of tautologies)
- Involutivity of the Complement
  - $(A^C)^C = A$ comes from the tautology $\neg(\neg P) \leftrightarrow P$ (# 3 on the list of tautologies)

NB: An involution is a map such that applying it twice gives the identity. Familiar examples: reflecting across the x-axis, the y-axis, or the origin in the plane.

- Transitivity of Inclusion
  - $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$ comes from the tautology

$$[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$$

    (# 14 on the list of tautologies)

- Criterion for proving equality of sets, which comes from the tautology $(P \leftrightarrow Q) \leftrightarrow [(P \rightarrow Q) \wedge (Q \rightarrow P)]$ (#22 on the list of tautologies)
  - $A = B \leftrightarrow A \subseteq B \wedge B \subseteq A$
- Criterion for proving non-equality of sets
  - $A \neq B \leftrightarrow (A \backslash B) \cup (B \backslash A) \neq \emptyset$

3.3 Example Proof in Set Theory

Proposition: $\forall A, B$ sets. $(A \cap B) \cup (A \backslash B) = A$

Proof: Use the criterion for proving equality of sets from above, **i.e.** inclusion in both directions.

<u>Show $(A \cap B) \cup (A \backslash B) \subseteq A$</u>: $\forall x \in (A \cap B) \cup (A \backslash B), x \in (A \cap B)$ or $x \in A \backslash B$. If $x \in (A \cap B)$, then clearly $x \in A$ as $A \cap B \subseteq A$ by definition. If $x \in A \backslash B$, then by definition $x \in A$ and $x \notin B$, so definitely $x \in A$. In both cases, $x \in A$ as needed.

<u>Show $A \subseteq (A \cap B) \cup (A \backslash B)$</u>: $\forall x \in A$, we have two possibilities, namely $x \in B$ or $x \notin B$. If $x \in B$, then $x \in A$ and $x \in B$, so $x \in A \cap B$. If $x \notin B$, then $x \in A$ and $x \notin B$, so $x \in A \backslash B$. In both cases, $x \in (A \cap B)$ or $x \in (A \backslash B)$, so $x \in (A \cap B) \cup (A \backslash B)$ as needed.

qed

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3.4 The Power Set

Task: Understand what the power set of a set A is.

Definition: Let A be a set. The power set of A denoted $P(A)$ is the collection of all subsets of A.

Recall: $\emptyset \subseteq A$. It is also clear from the definition of a subset that $A \subseteq A$.

Examples:

1. $A = \{0, 1\}$
   $P(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$
2. $A = \{a, b, c\}$
   $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
3. $A = \emptyset$
   $P(A) = \{\emptyset\}$
   $P(P(A)) = \{\emptyset, \{\emptyset\}\}$

NB: $\emptyset$ and $\{\emptyset\}$ are different objects. $\emptyset$ has no elements, whereas $\{\emptyset\}$ has one element.

Remark: $P(A)$ and A are viewed as living in separate worlds to avoid phenomena like Russell's paradox.

Q: If A has n elements, how many elements does $P(A)$ have?

A: 2^n

Theorem: Let A be a set with n elements, then $P(A)$ contains 2^n elements.

Proof: Based on the on/off switch idea.

$\forall x \in A$, we have two choices: either we include x in the subset or we don't (on vs off switch). A has n elements $\Rightarrow$ we have 2^n subsets of A.

qed

Alternate Proof: Using mathematical induction.

NB: It is an axiom of set theory (in the ZFC standard system) that every set has a power set, which implies no set consisting of all possible sets could exist, else what would its power set be?

3.5 Cartesian Products

Task: Understand sets like $\mathbb{R}^1$ in a more theoretical way.

Recall from Calculus:

$\mathbb{R} = \mathbb{R}^1 \ni x$

$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \ni (x_1, x_2)$

$\vdots$

$\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{\text{n times}} = \mathbb{R}^n \ni (x_1, x_2, ..., x_n)$

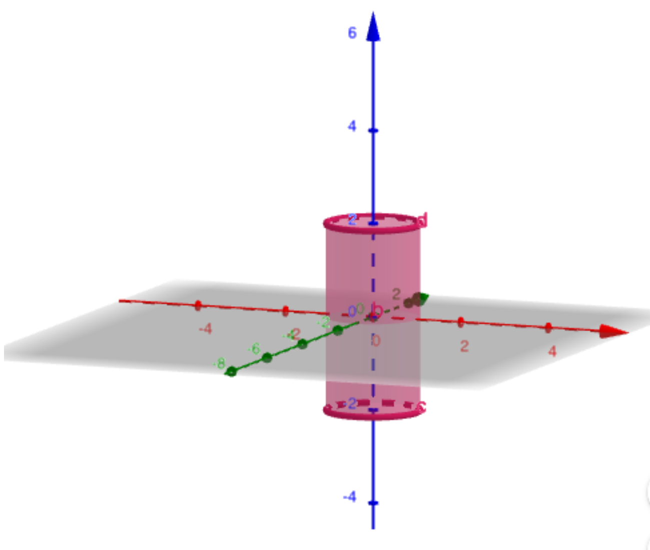
These are examples of Cartesian products.

Definition: Let A, B be sets. The Cartesian product denoted by $A \times B$ consists of all ordered pairs (x, y) $s.t.$ $x \in A \land y \in B$, **i.e.**

$$A \times B = \{(x, y) \mid x \in A \land y \in B\}$$

Further Examples:

1. $A = \{1, 3, 7\}$
   $B = \{1, 5\}$
   $A \times B = \{(1, 1), (1, 5), (3, 1), (3, 5), (7, 1), (7, 5)\}$
   NB: The order in which elements in a pair matters: $(7, 1)$ is different from $(1, 7)$. This is why we call (x, y) an <u>ordered</u> pair.
2. $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \leftarrow$ circle of radius 1
   $B = \{z \in \mathbb{R} \mid -2 \leq z \leq 2\} = [-2, 2] \leftarrow$ closed interval
   $A \times B \leftarrow$ cylinder of radius 1 and height 4



3.5.1 Cardinality (number of elements) in a Cartesian product

If A has m elements and B has p elements, $A \times B$ has mp elements.

Examples:

1. $\#(A) = 3 \qquad A = \{1, 3, 7\}$
   $\#(B) = 2 \qquad B = \{1, 5\}$
   $\#(A \times B) = 3 \times 2 = 6$

2. Both A and B are infinite sets, so $A \times B$ is infinite as well.

Remark: We can define Cartesian products of any length, **e.g.** $A \times A \times B \times A$, $B \times A \times B \times A \times B$, etc. If all sets are finite, the number of elements is the product of the numbers of elements of each factor. If $\#(A) = 3$ and $\#(B) = 2$ as above, $\#(A \times B \times A) = 3 \times 2 \times 3 = 18$ and $\#(B \times A \times B) = 2 \times 3 \times 2 = 12$.

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4 Relations

Task: Define subsets of Cartesian products with certain properties. Understand the predicates " = " (equality) and other predicates in predicate logic in a more abstract light.

Start with $x = y$. The element x is some relation R to y (equality in this case). We can also denote it as xRy or $(x, y) \in E$

Let x, y in $\mathbb{R}$, then $E = \{(x, x) \mid x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}$.

The "diagonal" in $\mathbb{R} \times \mathbb{R}$ gives exactly the elements equal to each other.

More generally:

Definition: Let A, B be sets. A subset of the Cartesian product $A \times B$ is called a relation between A and B. A subset of the Cartesian product $A \times A$ is called a relation on A.

Remark: Note how general this definition is. To make it useful for understanding predicates, we will need to introduce key properties relations can satisfy.

Example: $A = \{1, 3, 7\}$ $\qquad B = \{1, 2, 5\}$

We can define a relation S on $A \times B$ by $S = \{(1, 1), (1, 5), (3, 2)\}$. This means $1S1$, $1S5$ and $3S2$ and no other ordered pairs in $A \times B$ satisfy S.

Remark: The relations we defined involve 2 elements, so they are often called <u>binary relations</u> in the literature.

4.1 Equivalence Relations

Task: Define the most useful kind of relation.

Definition: A relation R on a set A is called

1. <u>reflexive</u> iff (if and only if) $\forall x \in A, xRx$

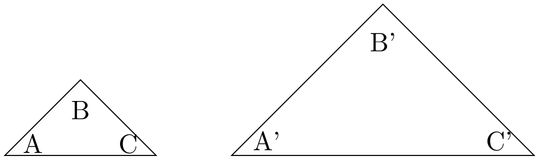
2. <u>symmetric</u> iff $\forall x, y \in A, xRy \rightarrow yRx$
3. <u>transitive</u> iff $\forall x, y, z \in A, xRy \wedge yRz \rightarrow xRz$

An equivalence relation on A is a relation that is reflexive, symmetric, and transitive.

Notation: Instead of xRy, an equivalence relation is often denoted by $x \equiv y$ or $x \sim y$.

Examples:

1. "=" equality is an equivalence relation.
   (a) $x = x$ reflexive
   (b) $x = y \Rightarrow y = x$ symmetric
   (c) $x = y \wedge y = z \Rightarrow x = z$ transitive
2. $A = \mathbb{N}$
   $x \equiv y \bmod 3$ is an equivalence relation. $x \equiv y \bmod 3$ means $x - y = 3m$ for some $m \in \mathbb{Z}$, **i.e.** x and y have the same remainder when divided by 3. The set of all possible remainders is $\{0, 1, 2\}$
   NB: In correct logic notation, $x \equiv y \bmod 3$ if $\exists m \in \mathbb{Z}\ s.t.\ x - y = 3m$
   (a) $x \equiv x \bmod 3$ since $x - x = 0 = 3 \times 0 \rightarrow$ reflexive
   (b) $x \equiv y \bmod 3 \Rightarrow y \equiv x \bmod 3$ because $x \equiv y \bmod 3$ means $x - y = 3m$ for some $m \in \mathbb{Z} \Rightarrow y - x = -3m = 3 \times (-m) \Rightarrow y \equiv x \bmod 3 \rightarrow$ symmetric
   (c) Assume $x \equiv y \bmod 3$ and $y \equiv z \bmod 3$
   $x \equiv y \bmod 3 \Rightarrow \exists m \in \mathbb{Z}$ s.t. $x - y = 3m \Rightarrow y = x - 3m$
   $y \equiv z \bmod 3 \Rightarrow \exists p \in \mathbb{Z}$ s.t. $y - z = 3p \Rightarrow y = z + 3p$
   Therefore, $x - 3m = z + 3p \Leftrightarrow x - z = 3p + 3m = 3(p + m)$
   Since $p, m \in \mathbb{Z}, p + m \in \mathbb{Z} \Rightarrow x \equiv z \bmod 3 \rightarrow$ transitive.
3. Let $f : A \rightarrow A$ be any function on a non-empty set A. We define the relation $R = \{(x, y) \mid f(x) = f(y)\}$
   (a) $\forall x \in A, f(x) = f(x) \Rightarrow (x, x) \in R \rightarrow$ reflexive
   (b) If $(x, y) \in R$, then $f(x) = f(y) \Rightarrow f(y) = f(x)$, **i.e.** $(y, x) \in R \rightarrow$ symmetric
   (c) If $(x, y) \in R$ and $(y, z) \in R$, then $f(x) = f(y)$ and $f(y) = f(z)$, which by the transitivity of equality implies $f(x) = f(z)$, **i.e.** $(x, z) \in R$ as needed, so R is transitive as well.
   $f(x)$ can be $e^x, \sin x, |x|$, etc.



4. Let Γ be the set of all triangles in the plane. $ABC \sim A'B'C'$ if ABC and $A'B'C'$ are similar triangles, **i.e.** have equal angles.

   (a) $\forall ABC \in \Gamma, ABC \sim ABC$ so $\sim$ is reflexive

   (b) $ABC \sim A'B'C' \Rightarrow A'B'C' \sim ABC$ so $\sim$ is symmetric

   (c) $ABC \sim A'B'C'$ and $A'B'C' \sim A"B"C" \Rightarrow ABC \sim A"B"C"$, so $\sim$ is transitive

   Clearly (a), (b), (c) use the fact that equality of angles is an equivalence relation.

Exercise: For various predicates you've encountered, check whether reflexive, symmetric or transitive. Examples of predicates include $\neq$, $<$, $>$, $\leq$, $\geq$, $\subseteq$, $\rightarrow$, $\leftrightarrow$