ST2004 Applied Probability I

Tutorial 4

1. The lifetime of tablet computers are well modelled by an exponential distribution with a mean of 3 years.

   (a) What is the probability that this tablet is functional for more than 4 years?

   (b) A sales executive owns two of these tablets. What is the probability that at least 1 of the tablets is functional for more than 4 years?

   (c) What is the variance of the lifetime of the tablet?

   (d) Given that a tablet is 1 year old and is functional, what is the probability it will function for 4 more years?

   Solution:
   Let T be the lifetime of tablet. $T \sim \text{Exp}(\lambda)$.
   We have $\mathbb{E}(T) = 3 = \frac{1}{\lambda}$. Thus, $\lambda = \frac{1}{3}$.

   (a)

   $$\begin{aligned} \mathbb{P}(T > 4) &= 1 - \mathbb{P}(T \leq 4) \\ &= 1 - F(4) \\ &= 1 - (1 - e^{-4/3}) \\ &= 0.2636 \end{aligned}$$

   (b) $\mathbb{P}(T \leq 4) = 1 - 0.2636 = 0.7364$.

   $$\begin{aligned} \mathbb{P}(\text{at least 1 lasts longer than 4 years}) &= 1 - \mathbb{P}(\text{both last less than 4 years}) \\ &= 1 - (\mathbb{P}(T \leq 4))^2 \\ &= 1 - 0.7364^2 \\ &= 0.4577 \end{aligned}$$

   (c) $Var(T) = \frac{1}{\lambda^2} = 9$.

   (d) By memoryless property of exponential distribution,

   $$\mathbb{P}(T > 5 | T > 1) = \mathbb{P}(T > 4) = 0.2636.$$

2. The density function of a random variable is given below:

   $$f(x) = \begin{cases} \frac{1}{6} & 2 \leq x \leq d \\ 0 & \text{otherwise} \end{cases}$$

   (a) What value of d will make this a valid probability density?

   (b) What distribution does X follow?

   (c) Calculate $\mathbb{P}(2.5 \leq X \leq 5)$.

   (d) What is the variance of X?

Solution:

(a) Since $\int_2^d f(x)dx = \int_2^d \frac{1}{6}dx = 1$, we have $d = 8$.

(b) $X \sim Unif(2, 8)$.

(c)

$$\begin{aligned} \mathbb{P}(2.5 \le X \le 5) &= \int_{2.5}^{5} \frac{1}{6}dx \\ &= 0.4167 \end{aligned}$$

(d) $Var(X) = \frac{(8-2)^2}{12} = 3$.

3. Let X have the density function

$$f(x) = \begin{cases} \frac{2x}{k^2}, & 0 \le x \le k \\ 0\ , & \text{elsewhere.} \end{cases}$$

For what value of k is the variance of X equal to 2?
Solution:

The expected value of X is

$$\begin{aligned} \mathbb{E}(X) &= \int_0^k x\frac{2x}{k^2}dx \\ &= \frac{2}{3}k. \end{aligned}$$

$$\begin{aligned} \mathbb{E}(X^2) &= \int_0^k x^2 f(x)dx \\ &= \int_0^k x^2\frac{2x}{k^2}dx \\ &= \frac{1}{2}k^2. \end{aligned}$$

Hence, the variance is given by

$$\begin{aligned} Var(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\ &= \frac{1}{2}k^2 - \frac{4}{9}k^2 \\ &= \frac{1}{18}k^2. \end{aligned}$$

Since we require the variance to be 2, $k = 6$ or -6. But $k > 0$, hence $k = 6$.

4. A supplier of kerosene has a 150-gallon tank that is filled at the beginning of each week. His weekly demand shows a relative frequency behavior that increases steadily up to 100 gallons and then levels off between 100 and 150 gallons. If Y denotes weekly demand in hundreds of gallons, the relative frequency of demand can be modeled by

$$f(y) = \begin{cases} y & , 0 \le y \le 1 \\ 1 & , 1 < y \le 1.5 \\ 0 & , \text{elsewhere} \end{cases}$$

(a) Find $F(y)$.

(b) Find $\mathbb{P}(0 \le Y \le 0.5)$.

(c) Find $\mathbb{P}(0.5 \le Y \le 1.2)$.

(d) Find $\mathbb{P}(Y > 1.2 | Y > 0.2)$.

Solution:

(a) Recall $F(y) = \int_{-\infty}^{y} f(t)dt$

For $y < 0$, there is nothing to compute, because $f(y) = 0$ for $y < 0$, therefore $F(y) = 0$. Similary for $y > 1.5$, $F(y) = 1$ (otherwise it wouldn't be a distribution, because it needs to integrate to 1).

To show that formally, for $y < 0$:

$$F(y) = \int_{-\infty}^{y} f(t)dt = \int_{-\infty}^{y} 0dt = 0$$

For $0 \le y \le 1$,

$$F(y) = \int_{-\infty}^{y} f(t)dt = \int_{-\infty}^{0} f(t)dt + \int_{0}^{y} f(t)dt = 0 + \int_{0}^{y} tdt = y^2/2$$

For $1 < y \le 1.5$,

$$F(y) = \int_{-\infty}^{0} f(t)dt + \int_{0}^{1} f(t)dt + \int_{1}^{y} f(t)dt = 0 + \int_{0}^{1} tdt + \int_{1}^{y} 1dt = 1/2 + y - 1 = y - 1/2$$

For $y > 1.5$ (similar to $y < 0$ case, there is nothing really to show, o.w. it's not a valid p.d.f.)

$$F(y) = \int_{-\infty}^{0} f(t)dt + \int_{0}^{1} f(t)dt + \int_{1}^{1.5} f(t)dt + \int_{1.5}^{y} f(t)dt = 0 + (1/2) + (1.5 - 1) + 0 = 1$$

Hence,

$$F(y) = \begin{cases} 0 & , y < 0 \\ y^2/2 & , 0 \le y \le 1 \\ y - 1/2 & , 1 < y \le 1.5 \\ 1 & , y > 1.5 \end{cases}$$

(b) $\mathbb{P}(0 \le Y \le 0.5) = F(0.5) - F(0) = 1/8 - 0 = 1/8$.

(c) $\mathbb{P}(0.5 \le Y \le 1.2) = F(1.2) - F(0.5) = 1.2 - 1/2 - 1/8 = 0.575$.

(d) $\mathbb{P}(Y > 1.2 | Y > 0.2) = \frac{\mathbb{P}(\{Y>1.2\} \cap \{Y>0.2\})}{\mathbb{P}(Y>0.2)} = \frac{\mathbb{P}(Y>1.2)}{\mathbb{P}(Y>0.2)} = \frac{1-F(1.2)}{1-F(0.2)} = \frac{1-(1.2-0.5)}{1-0.2^2/2} = 0.306$

5. Accidents occur in a factory according to a Poisson process at the rate of 2 per week.

(a) What is the probability of 2 accidents occurring in each of the next 3 weeks?

(b) What is the probability that the 3rd accident (starting from now) occurs during the second week?

Solution:

(a) The probability of 2 accidents in a week is given by

$$\mathbb{P}(X = 2) = \frac{1}{2!} 2^2 e^{-2} = 0.2707$$

By independence, the probability of 2 accidents occurring in each of three consecutive weeks is $0.2707^3 = 0.0198$.

(b) There are three possibilities

i. None in week 1, at least 3 in week 2

ii. 1 in week 1, at least 2 in week 2

iii. 2 in week 1, at least in week 2

Therefore the probability we need to compute is

$$\mathbb{P}(X = 0)\mathbb{P}(X \geq 3) + \mathbb{P}(X = 1)\mathbb{P}(X \geq 2) + \mathbb{P}(X = 2)\mathbb{P}(X \geq 1)$$

We have that

$$\begin{aligned} \mathbb{P}(X = 0) &= \frac{1}{0!} 2^0 e^{-2} = 0.1353 \\ \mathbb{P}(X = 1) &= \frac{1}{1!} 2^1 e^{-2} = 0.2707 \\ \mathbb{P}(X = 2) &= \frac{1}{2!} 2^2 e^{-2} = 0.2707 \end{aligned}$$

Then

$$\begin{aligned} \mathbb{P}(X = 0)\mathbb{P}(X \geq 3) &= 0.1353 \times (1 - 0.1353 - 0.2707 - 0.2707) = 0.0437 \\ \mathbb{P}(X = 1)\mathbb{P}(X \geq 2) &= 0.2707 \times (1 - 0.1353 - 0.2707) = 0.1608 \\ \mathbb{P}(X = 2)\mathbb{P}(X \geq 1) &= 0.2707 \times (1 - 0.1353) = 0.2341 \end{aligned}$$

Therefore the probability is 0.4386.