# A Theorem on Inverses of Tridiagonal Matrices

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ABSTRACT

Tridiagonal or Jacobi matrices arise in many diverse branches of mathematics and have been studied extensively. However, there is little written about the inverses of such matrices. In this paper we characterize those matrices with nonzero diagonal elements whose inverses are tridiagonal. The arguments given are elementary and show that matrices with tridiagonal inverses have an interesting structure.

## 1. INTRODUCTION

In [1] we gave a probabilistic proof of a theorem which characterized the inverses of positive definite symmetric tridiagonal matrices. The proof appeals to the properties of covariance matrices which are necessarily positive definite and symmetric, but the restriction to positive definiteness seems artificial in the statement of the theorem. Furthermore, the theorem also applies to nonsymmetric matrices if we suitably extend a definition in [1]. In this paper we give an elementary algebraic proof of the theorem which eliminates the need for these restrictions. However, we feel that the original proof is still important not only because it illustrates a number of important probabilistic ideas, but also because it was the natural avenue for discovery of the theorem.

There is a theorem along similar lines in Gantmacher and Krein [2]. See also Karlin [3, pp. 110–114].

## 2. MAIN RESULT

DEFINITION. A matrix $A$ is tridiagonal if $A_{ij}=0$ for $|i-j|>1$.



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DEFINITION. Suppose $R$ is an $n\times n$ matrix whose diagonal elements $R_{22}, R_{33}, \dots, R_{n-1n-1}$ are nonzero. We say that $R$ has the triangle property if

$$R_{ij}=\frac{R_{ik}R_{kj}}{R_{kk}}\qquad\text{for all}\quad i<k<j\quad\text{and all}\quad i>k>j. \tag{2.1}$$

For symmetric matrices Eq. (2.1) reduces to the definition given in [1]:

$$R_{ij}=\frac{R_{ik}R_{kj}}{R_{kk}}\qquad\text{for all}\quad i<k<j; \tag{2.1S}$$

or equivalently

$$R_{ij}=\frac{R_{ii+1}R_{i+1i+2}\cdots R_{j-1j}}{R_{i+1i+1}\cdots R_{j-1j-1}}\qquad\text{for all}\quad i<j. \tag{2.2}$$

Equation (2.2) shows that in a symmetric matrix with the triangle property all elements $R_{ij}$ are determined by the elements on the main diagonal and superdiagonal. Figure 1 illustrates this formula. To determine the element $R_{ij}$ in the matrix $R$, draw horizontal and vertical lines back to the main diagonal and then multiply along the superdiagonal to find the numerator and along the main diagonal to find the denominator as indicated in the figure. Hence the name triangle property.

For a nonsymmetric matrix, Eq. (2.2) shows all elements in $R$ above the main diagonal are determined by the elements on the main diagonal and superdiagonal, while a similar formula determines all elements below the main diagonal in terms of the elements on the main diagonal and subdiagonal.

Our main theorem is:

THEOREM 1. *Assume $R$ is a nonsingular $n\times n$ matrix whose diagonal elements $R_{22},\dots,R_{n-1n-1}$ are nonzero. The matrix $R$ has the triangle property if and only if its inverse $A$ is tridiagonal.*

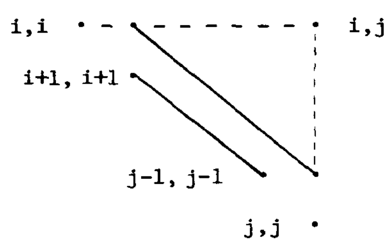


FIG. 1.

Before giving the proof of the theorem we wish to compare it with the theorem of Gantmacher and Krein [2].

DEFINITION. Following Karlin [3, p. 110], we call $R$ a Green's matrix if there exist numbers $a_1,\dots,a_n,b_1,\dots,b_n$ such that

$$R_{ij}=a_{\min(i,j)}b_{\max(i,j)}=\begin{cases} a_ib_j, & i\leqslant j,\\ a_jb_i, & i\geqslant j.\end{cases}$$

Gantmacher and Krein use the term "einpaarige matrix," and in [1] we used the terminology that $R$ "factors."

The theorem in Gantmacher and Krein [2], remark (g) on p. 95, states that nonsingular symmetric tridiagonal matrices whose superdiagonal elements are nonzero and nonsingular Green's matrices composed from nonzero $a_i, b_i$ $(i=1,\dots,n)$ are inverses of each other. This statement is incorrect, as the following counterexample shows. The matrix

$$\begin{bmatrix} 1 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}$$

is a symmetric tridiagonal matrix with nonzero superdiagonal elements, and yet its inverse

$$\begin{bmatrix} 0 & 1 & -1\\ 1 & -1 & 1\\ -1 & 1 & 0\end{bmatrix}$$

has zero entries. However, this inverse is a Green's matrix with $a_1=1$, $a_2=-1$, $a_3=0$ and $b_1=0$, $b_2=1$, $b_3=-1$. There is no error in the arguments preceding Remark (g) on p. 95. This remark was merely an inaccurate summary of those arguments. The theorem is true if we remove the restriction on the $a_i$, $b_i$ and allow them to assume the value zero. (*Note*: If $R$ is a nonsingular Green's matrix, $a_1$ and $b_n$ are necessarily nonzero.)

The theorem of Gantmacher and Krein can then be stated as follows:

THEOREM 2 (Gantmacher, Krein). *The matrix $R$ is a nonsingular Green's matrix if and only if its inverse $A$ is a symmetric tridiagonal matrix with nonzero superdiagonal elements.*

Neither Theorem 1 nor Theorem 2 is a special case of the other. In Theorem 2 there is a restriction on the superdiagonal elements of $A$ but no restriction on the diagonal elements of $R$. In our theorem there is a restriction on the diagonal elements of $R$, but no restriction on the superdiagonal or subdiagonal elements of $A$, and we allow $R$ to be nonsymmetric.

A nonsingular Green's matrix composed from nonzero $a_i$, $b_i$ has the triangle property [Eq. (2.1S)], since

$$\frac{R_{ik}R_{kj}}{R_{kk}}=\frac{a_ib_ka_kb_j}{a_kb_k}=R_{ij}.$$

Hence by Theorem 1 its inverse $A$ is tridiagonal, and it is easily seen that no superdiagonal element of $A$ can be zero. Conversely, if $A$ is a nonsingular symmetric tridiagonal matrix with nonzero superdiagonal elements whose inverse $R$ has nonzero diagonal elements, then by Theorem 1, $R$ has the triangle property. If a superdiagonal element of $R$ were zero, one could show the corresponding superdiagonal element of $A$ was zero. Therefore all superdiagonal elements of $R$ are nonzero, and setting

$$a_i=\frac{R_{11}R_{22}\cdots R_{ii}}{R_{12}R_{23}\cdots R_{i-1i}},\qquad b_j=\frac{R_{12}\cdots R_{j-1j}}{R_{11}\cdots R_{j-1j-1}},$$

we find $R$ is a Green's matrix composed from nonzero $a_i$, $b_i$. Thus if one wishes to restate the original theorem of Gantmacher and Krein in terms of Green's matrices composed from nonzero $a_i$, $b_i$, the theorem is a corollary of Theorem 1.

## 3. PROOF OF THEOREM 1

We first define

$$d_{ij}\equiv R_{ii}R_{jj}-R_{ij}R_{ji}\qquad \text{for all}\quad 1\leqslant i,j\leqslant n$$

and begin with the lemma,

LEMMA 1. *If $R$ has the triangle property and if $d_{kk+1}=0$ for some $k, k=1,\dots,n-1$, then $R$ is singular.*

*Proof.* Suppose $d_{kk+1}=R_{kk}R_{k+1k+1}-R_{kk+1}R_{k+1k}=0$.

*Case 1.* If $R_{kk}R_{k+1k+1}=0$, then $k=1$ or $n-1$. Suppose $k=1$. Then $R_{11}R_{22}=0$. $R_{22}\neq 0$, so $R_{11}=0$. Since $R_{11}R_{22}-R_{12}R_{21}=0$, $R_{12}=0$ or $R_{21}=0$.

If $R_{12}=0$, then

$$R_{1k}=\frac{R_{12}R_{2k}}{R_{22}}=0 \qquad \text{for} \quad k>2.$$

Thus $R_{1k}=0$ for $k=1,\dots,n$, which implies $R$ is singular.

If $R_{21}=0$, then

$$R_{k1}=\frac{R_{k2}R_{21}}{R_{22}}=0 \qquad \text{for} \quad k>2.$$

Then $R_{k1}=0$ for $k=1,\dots,n$, and $R$ is singular.

The case $k=n-1$ is similar.

*Case* 2. $R_{kk}R_{k+1k+1}\neq 0$. Then $R_{kk+1}R_{k+1k}\neq 0$ also. Let

$$\alpha=\frac{R_{kk}}{R_{k+1k}}=\frac{R_{kk+1}}{R_{k+1k+1}}.$$

Then for $j\geqslant k+1$

$$R_{kj}=\frac{R_{kk+1}R_{k+1j}}{R_{k+1k+1}}=\alpha R_{k+1j},$$

while for $j\leqslant k$

$$R_{k+1j}=\frac{R_{k+1k}R_{kj}}{R_{kk}}=\frac{R_{kj}}{\alpha}.$$

Thus $R_{kj}=\alpha R_{k+1j}$ for $j=1,\dots,n$. Since the $k$th row is a multiple of the $k+1$st row, $R$ is singular. ∎

Remark. Actually the determinant of a matrix $R$ with the triangle property is given by

$$\det R=\frac{d_{12}d_{23}\cdots d_{n-1n}}{R_{22}\cdots R_{n-1n-1}}, \tag{3.1}$$

from which the above lemma follows immediately. See [4] for a proof of this formula.

Given a nonsingular matrix $R$ with the triangle property, it now follows that the inverse is tridiagonal because it can be written down explicitly:

$$A_{ij}=\begin{cases}\dfrac{-R_{ij}}{d_{ij}} & \text{for } |i-j|=1,\\[2mm] R_{ii}\dfrac{d_{i-1i+1}}{d_{i-1i}d_{ii+1}} & \text{for } i=j\neq 1,n,\\[2mm] \dfrac{R_{22}}{d_{12}} & \text{for } i=j=1,\\[2mm] \dfrac{R_{n-1n-1}}{d_{n-1n}} & \text{for } i=j=n,\\[2mm] 0 & \text{for } |i-j|>1.\end{cases} \tag{3.2}$$

One verifies that $RA=I$ by a direct calculation, making use of the triangle property. The lemma guarantees all the denominators are nonzero. The discovery of this formula for the $A_{ij}$ came from investigation of Markovian normal densities [5]. This completes the proof of the theorem in one direction.

Now assume $A$ is a tridiagonal, nonsingular matrix. We wish to show that $R=A^{-1}$ has the triangle property provided $R_{22},\dots,R_{n-1n-1}$ are not zero.

We use the standard symbol $R\begin{pmatrix} i_1,\dots,i_k\\ j_1,\dots,j_k\end{pmatrix}$ to denote the minor formed from the $i_1,\dots,i_k$ rows and $j_1,\dots,j_k$ columns of $R$. We likewise use the corresponding notation for the matrix $A$.

Since $R_{22},\dots,R_{n-1n-1}$ are not zero, the triangle property is equivalent to proving that

$$R\begin{pmatrix} i & k\\ k & j\end{pmatrix}=0 \quad \text{for all} \quad i<k<j,$$

$$R\begin{pmatrix} k & i\\ j & k\end{pmatrix}=0 \quad \text{for all} \quad j<k<i.$$

Since $A^{-1}=R$, by the well-known formula for inverse determinants, (see e.g. [3, p. 5]),

$$R\begin{pmatrix} i & k\\ k & j\end{pmatrix}=(-1)^{i+j}\frac{A\begin{pmatrix} 1,\dots,k-1,k+1,\dots,j-1,j+1,\dots,n\\ 1,\dots,i-1,i+1,\dots,k-1,k+1,\dots,n\end{pmatrix}}{\det A}.$$

Thus we must show that

$$A\begin{pmatrix} 1,\ldots,k-1,k+1,\ldots,j-1,j+1,\ldots,n \\ 1,\ldots,i-1,i+1,\ldots,k-1,k+1,\ldots,n \end{pmatrix}=0$$

for $i<k<j$. (The other case is totally analogous.)

We claim that since $A$ is tridiagonal, the above determinant is obviously zero. Consider the first $k-1$ rows of the associated matrix. The last $n-k$ elements of each of these row vectors are all zero, i.e., we have $k-1$ vectors which have nonzero components in at most $k-2$ places (the same $k-2$ components for all $k-1$ vectors). Thus these $k-1$ vectors are linearly dependent, and the above determinant is zero. This completes the proof of Theorem 1. ∎

*We wish to thank the referee for the above proof of the second part of Theorem 1; i.e., if A is tridiagonal, then R has the triangle property. His argument was much shorter than the original one.*

## REFERENCES

1 W. Barrett and P. Feinsilver, Gaussian families and a theorem on patterned matrices, *J. Appl. Probability* 15:514–522 (1978).

2 F. R. Gantmacher and M. G. Krein, *Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme*, Akademie-Verlag, Berlin, 1960, p. 95.

3 S. Karlin, *Total Positivity*, Vol. 1, Stanford U. P., Calif., 1968.

4 W. Barrett and P. Feinsilver, Inverses of banded matrices, *Linear Algebra and Appl.*, to appear.

5 W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. 2, 2nd ed., Wiley, New York, 1971, pp. 94–99.

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