

# 18.06 (Spring 2014) Problem Set 2 Solutions

These 8 problems are worth 80 points. MITx problems are worth 20 points.

- Imagine that the 2nd difference matrix  $S$  (with 1, -2, 1 down three central diagonals) is INFINITE. Multiply  $S$  with these infinite vectors (infinite in both directions):

(a) all-ones  $x = (\dots, 1, 1, 1, 1, \dots)'$

$$Sx = (x_{i-1} - 2x_i + x_{i+1})_{i=-\infty}^{\infty} = (1 - 2 * 1 + 1)_{i=-\infty}^{\infty} = (\dots, 0, 0, 0, 0, \dots)'$$

(b) linear  $(\dots, -2, -1, 0, 1, 2, 3, \dots)'$

$$Sx = (x_{i-1} - 2x_i + x_{i+1})_{i=-\infty}^{\infty} = ((i-1) - 2 * i + (i+1))_{i=-\infty}^{\infty} = (\dots, 0, 0, 0, 0, 0, \dots)'$$

(c) squares  $(\dots, 4, 1, 0, 1, 4, 9, \dots)'$   $Sx = (x_{i-1} - 2x_i + x_{i+1})_{i=-\infty}^{\infty} = ((i-1)^2 - 2 * i^2 + (i+1)^2)_{i=-\infty}^{\infty} = 2 * (\dots, 1, 1, 1, 1, 1, \dots)'$

(d) cubes  $(\dots, -8, -1, 0, 1, 8, 27, \dots)'$   $Sx = (x_{i-1} - 2x_i + x_{i+1})_{i=-\infty}^{\infty} = ((i-1)^3 - 2 * i^3 + (i+1)^3)_{i=-\infty}^{\infty} = (6i)_{i=-\infty}^{\infty} = 6 * (\dots, -2, -1, 0, 1, 2, 3, \dots)'$

How do the answers match up with 2nd derivatives of  $1, x, x^2, x^3$  ?

The term in all four cases is equal to the second derivative, evaluated at integer values of  $x$ . Specifically, (a) - (d) match the 2nd derivative of  $1, x, x^2, x^3$ , which are  $0, 0, 2, 6x$ .

- Find the inverse of the 4 by 4 backward difference matrix  $B$  (main diagonal of 1's and subdiagonal of -1's).

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Interpret as the fundamental theorem of calculus.

When the matrix  $B^{-1}$  is applied to a vector  $f$ , the  $s^{th}$  entry of the resulting vector is the sum  $\sum_{t=1}^s f_t$ . Hence,  $B^{-1}$  is just the discretization of the integral operator  $\int_0^s f(t)dt$ . Since  $Bf$  takes the difference of consecutive entries in  $f$ ,  $B$  is just the discretization of the derivative operator. The FTC says that  $\frac{d}{ds} \int_0^s f(t)dt = \int_0^s \frac{d}{dt} f(t)dt = f(s)$ . Hence, in discretized form, this is the same as saying  $BB^{-1}f = B^{-1}Bf = f$ .

The inverse of the derivative is the integral.

- If the permutation  $P$  has 1's on the antidiagonal (from the  $(1, n)$  entry down to the  $(n, 1)$  entry) is this an even or odd permutation (depending on  $n$ ) ?

$P(x_1, x_2, \dots, x_{n-1}, x_n)' = (x_n, x_{n-1}, \dots, x_2, x_1)'$ , so  $P$  vertically reflects the entries of  $x$ . This is the same as transposing the  $i^{th}$  and  $(n-i)^{th}$  entry of  $x$  for each  $i \leq \frac{n}{2}$ . We will do this  $\lfloor \frac{n}{2} \rfloor$  times, with a single entry left in the middle and not transposed if  $n$  is odd (here " $\lfloor \frac{n}{2} \rfloor$ " just means round  $\frac{n}{2}$  down to the nearest integer). Hence, if  $n$  is a multiple of 4 or one more than a multiple of 4, then the permutation will consist of an even number of transpositions and hence will be an even permutation. Otherwise, it will be an odd permutation.

4. The LU decomposition of A is

$$\begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix} = A = LU = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & -\frac{3}{8} & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 \\ 0 & \frac{8}{3} & -1 \\ 0 & 0 & \frac{21}{8} \end{pmatrix}$$

The pivots are 8,  $\frac{8}{3}$ , and  $\frac{21}{8}$  (the diagonal entries of U). The multipliers are  $-\frac{1}{3}$ , 0, and  $-\frac{3}{8}$  (the entries below the diagonal of L). The determinant is  $\det(A) = 21$ .

5. Now let

$$A = \begin{pmatrix} 4 & 10 & 0 \\ 8 & b & 4 \\ 4 & 0 & 1 \end{pmatrix}.$$

What value of  $b$  interferes with normal elimination ? What should you do in this case? Which  $b$  makes the matrix singular ?

$$A = \begin{pmatrix} 4 & 10 & 0 \\ 8 & b & 4 \\ 4 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 10 & 0 \\ 0 & b-20 & 4 \\ 0 & -10 & 1 \end{pmatrix} \xrightarrow{b \neq 20} \begin{pmatrix} 4 & 10 & 0 \\ 0 & b-20 & 4 \\ 0 & 0 & 1 + \frac{40}{b-20} \end{pmatrix}$$

A value of  $b = 20$  interferes with the second step of the above elimination, so instead we should exchange rows 2 and 3 to get

$$\begin{pmatrix} 4 & 10 & 0 \\ 0 & -10 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

The determinant of this matrix is  $4(b+20)$ . Therefore when  $b = -20$ , the matrix is singular.

6. Problem 30 page 91 (Section 2.5)

$$\det(A) = -c^3 + 9c^2 - 14c = -c(c-7)(c-2)$$

A is not invertible if and only if  $\det(A) = 0$ , i.e., when  $c = 0, 7$  or  $2$ .

7. Problem 40 page 92 (Section 2.5)

$$\begin{pmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{pmatrix} = I \rightarrow A^{-1} = \begin{pmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

8.

(a) Suppose every row of  $A$  adds up to zero. Why is  $A$  singular ?

In this case the sum of all the columns of  $A$  must be the zero column-vector, so  $A$  must be singular.

(b) Suppose every column of  $A$  adds to zero. Why is  $A$  singular ?

In this case the sum of all the rows of  $A$  must be the zero row-vector, so  $A$  must be singular.