Homework sheet 5: ORTHOGONALITY AND LEAST-SQUARES PROBLEM
Year 2011-2012

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1. Let $u = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$, $w = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$, $x = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$. Compute the following quantities: $w \cdot w$, $x \cdot w$, $\dfrac{x \cdot w}{w \cdot w}$, $\dfrac{1}{u \cdot u}u$, $\dfrac{x \cdot w}{x \cdot x}x$, $||x||$.

   Solution: Respectively: 35; 5; $\dfrac{1}{7}$, $\begin{pmatrix} -1/5 \\ 2/5 \end{pmatrix}$, $\begin{pmatrix} 30/49 \\ -10/49 \\ 15/49 \end{pmatrix}$.

2. Let $v = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$, $w = \begin{bmatrix} 8/3 \\ 2 \end{bmatrix}$. Find a unit vector in the direction of each given vector.

   Solution: Respectively: $\dfrac{1}{\sqrt{61}}\begin{pmatrix} -6 \\ 4 \\ -3 \end{pmatrix} = \dfrac{\sqrt{61}}{61}\begin{pmatrix} -6 \\ 4 \\ -3 \end{pmatrix}$, $\begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}$.

3. Find the distance between $x = \begin{pmatrix} 0 \\ -5 \\ 2 \end{pmatrix}$ and $z = \begin{pmatrix} -4 \\ -1 \\ 8 \end{pmatrix}$.

   Solution:$\text{dist}(u, z) = \|u - z\| = 2\sqrt{17}$.

4. Let $u = [5, -6, 7]^T$ and W be the subset of all vectors x in $\mathbb{R}^3$ such that $u \cdot x = 0$. Give a geometric interpretation of W and show that W is a subspace.

   Solution: W is a plane in $\mathbb{R}^3$ that contains the origin with normal vector u. Hence it is a subspace. Another way to prove it is to check that $0 \in W$, $x + y \in W$ for $x, y \in W$, $\lambda x \in W$ for $\lambda \in \mathbb{R}$ and $x \in W$.

5. Let $u = (u_1, u_2, u_3)$. Explain why $u \cdot u \geq 0$. When is $u \cdot u = 0$?

   Solution:$u \cdot u = u_1^2 + u_2^2 + u_3^2$. As $u \cdot u$ is equal to the sum of positive numbers, $u \cdot u \geq 0$. The inner product $u \cdot u = 0$ if and only if the three positive numbers in the addition vanish, that is, $u_1^2 = u_2^2 = u_3^2 = 0$. Equivalently, $u_1 = u_2 = u_3 = 0$. Thus $u \cdot u = 0$ if and only is u is the zero vector.

6. Let $v = \begin{pmatrix} a \\ b \end{pmatrix}$. Describe the set H of vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ that are orthogonal to v.

Solution:$H = \text{span}\left\{\begin{pmatrix} -b \\ a \end{pmatrix}\right\}$.

7. Determine which set of vectors are orthogonal:

$$S_1 = \left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}\right\}, \quad S_2 = \left\{\begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix}\right\}.$$

Solution: S_1 is orthogonal, S_2 is not orthogonal.

8. Show that $u_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ is an orthogonal basis for $\mathbb{R}^3$. Then express the vector $x = [5, -3, 1]^T$ as a linear combination of the $u's$.

   Solution: It needs to be proved that $u_1 \cdot u_2 = u_1 \cdot u_3 = u_2 \cdot u_3 = 0$. As they are three orthogonal vectors two by two in $\mathbb{R}^3$, which are different from the zero vector, they are linearly independent.

$$x = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix} = \frac{x \cdot u_1}{u_1 \cdot u_1}u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}u_2 + \frac{x \cdot u_3}{u_3 \cdot u_3}u_3 = \frac{4}{3}u_1 + \frac{1}{3}u_2 + \frac{1}{3}u_3.$$

9. Compute the orthogonal projection of $y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ onto the line through $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and the origin.

   Solution:$\text{proj}_v y = \frac{y \cdot v}{v \cdot v}v = \begin{pmatrix} 2/5 \\ -6/5 \end{pmatrix}$.

10. Let $y = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ and $u = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$. Write y as the sum of a vector in span$\{u\}$ and a vector orthogonal to u.

    Solution: An orthogonal vector to u is $v = \begin{pmatrix} -1 \\ 7 \end{pmatrix}$. Then

$$y = \frac{y \cdot u}{u \cdot u}u + \frac{y \cdot v}{v \cdot v}v = \frac{2}{5}u + \frac{4}{5}v.$$

11. Let $y = \begin{pmatrix} -3 \\ 9 \end{pmatrix}$, $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Compute the distance from y to the line through u and the origin.

    Solution: The distance is given by the norm/length of the following vector$y - \text{proj}_u y = \begin{pmatrix} -6 \\ 3 \end{pmatrix}$. Hence, the distance is $3\sqrt{5}$.

12. Determine which of the following sets of vectors are orthonormal. If a set is only orthogonal, normalize the vectors to produce an orthonormal set.

$$S_1 = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right\}, \; S_2 = \left\{ \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix} \right\},$$

$$S_3 = \left\{ \begin{pmatrix} \frac{1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix} \right\}.$$

Solution: S_1 is not orthogonal, then it is not orthonormal. S_2 is orthogonal, but the second vector is not a unit vector. After normalizing, we obtain the following orthonormal set: $\left\{ \begin{pmatrix} -2/3 \\ 1/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} \sqrt{5}/5 \\ 2\sqrt{5}/5 \\ 0 \end{pmatrix} \right\}$. S_3 is orthonormal.

13. Let $\{v_1, v_2\}$ be an orthogonal set of nonzero vectors and let c_1, c_2 be any nonzero scalars. Show that $\{c_1\, v_1, c_2\, v_2\}$ is also an orthogonal set.

Solution: Using the properties of the inner product, we have $(c_1 u_1) \cdot (c_2 u_2) = (c_1 c_2)(u_1 \cdot u_2) = 0$.

14. Let W be the subspace spanned by the vectors $u's$. Write y as the sum of a vector in W and a vector orthogonal to W.

$$a)\;\; y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \quad , \quad u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad , \quad u_2 = \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}$$

$$b)\;\; y = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \quad , \quad u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad , \quad u_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad , \quad u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

Solution: We first have to compute $\text{proj}_W y$, then $y - \text{proj}_W y$ to obtain a vector orthogonal to W.

$$a)\; y = \begin{pmatrix} 3/2 \\ 7/2 \\ 1 \end{pmatrix} + \begin{pmatrix} -5/2 \\ 1/2 \\ 2 \end{pmatrix};$$

$$b)\; y = \begin{pmatrix} 5 \\ 2 \\ 3 \\ 6 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \\ 2 \\ 0 \end{pmatrix}.$$

15. Let W be the subspace spanned by the vectors u_1 and u_2. Find the closest point to y in the subspace W.

$$y = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix} \quad , \quad u_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} \quad , \quad u_2 = \begin{bmatrix} -3 \\ -1 \\ -1 \\ 5 \end{bmatrix}$$

Solution:$\text{proj}_W y = \begin{pmatrix} -1 \\ -5 \\ -3 \\ 9 \end{pmatrix}$.

16. Let $y = [7, 9]^T$, $u_1 = [1/\sqrt{10}, -3/\sqrt{10}]^T$, and $W = \text{Gen}\{u_1\}$.

   a) Let U be the 2×1 matrix whose only column is u_1. Compute $U^T U$ and UU^T.

   b) Compute $\text{proy}_W y$ and $(UU^T)y$.

Solution: $U^T U = 1$, $UU^T = \begin{pmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{pmatrix}$, $\text{proj}_W y = (UU^T)y = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$.

17. Let $u_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $u_2 = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$, $u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Show that u_1 and u_2 are orthogonal but that u_3 is not orthogonal to u_1 or u_2. Show that u_3 is not in the subspace W spanned by u_1 and u_2. Use this fact to construct a nonzero vector v in $\mathbb{R}^3$ that is orthogonal to u_1 and u_2.

Solution: It must be proved that $u_1 \cdot u_2 = 0$, $u_3 \cdot u_1 = -2 \neq 0$, $u_3 \cdot u_2 = 2 \neq 0$. To check that u_3 is not in the vector space spanned by u_1 and u_2 we compute the determinant of the matrix given by the those three vectors. As the determinant is different from zero, the three vectors are linearly independent. Thus, u_3 is not in the subspace W spanned by u_1 and u_2. A nonzero orthogonal vector to u_1 and u_2 is obtained as follows:

$$u_3 - \text{proj}_W u_3 = \begin{pmatrix} 0 \\ 2/5 \\ 1/5 \end{pmatrix}.$$

18. Use the Gram-Schmidt process to produce an orthonormal basis for the subspace W spanned by the following vectors:

$$\text{(a)} \quad \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ -7 \end{pmatrix} \quad \text{(b)} \quad \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ 14 \\ -7 \end{pmatrix} \quad \text{(c)} \quad \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix}.$$

Solution:

$$\text{(a)}\quad \begin{pmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix},\ \frac{1}{\sqrt{105}}\begin{pmatrix} 5 \\ 4 \\ -8 \end{pmatrix}, \quad \text{(b)}\quad \begin{pmatrix} 3/(5\sqrt{2}) \\ -4/(5\sqrt{2}) \\ 5/\sqrt{2} \end{pmatrix},\ \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

$$\text{(c)}\quad \frac{1}{\sqrt{15}}\begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix},\ \frac{1}{\sqrt{61}}\begin{pmatrix} 4 \\ 6 \\ -3 \\ 0 \end{pmatrix}.$$

19. Use the Gram-Schmidt process to find an orthogonal basis for the column space of each of the following matrices:

$$\text{(a)}\quad \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix} \quad \text{(b)}\quad \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}.$$

Solution:(a) $\left\{ \frac{1}{2\sqrt{3}}\begin{pmatrix} -1 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2\sqrt{3}}\begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{2\sqrt{3}}\begin{pmatrix} -1 \\ -1 \\ 3 \\ -1 \end{pmatrix} \right\}.$

(b) $\left\{ \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$

20. Find a least-squares solution of $Ax = b$ by constructing the normal equations for $\hat{x}$ and solving for $\hat{x}$. In the first two exercises compute the least-squares error associated with the least-squares solution found.

$$a)\ A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$$

$$b)\ A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$$

$$c)\ A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$$

Solution: (a) $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$, zero error. (b) $\begin{pmatrix} 4/3 \\ -1/3 \end{pmatrix}$, the error is $2\sqrt{5}$. (c) $\begin{pmatrix} 5-z \\ -3+z \\ z \end{pmatrix}$.

21. Find the orthogonal projection of b onto the column space of A and find a least-squares solution of $Ax = b$.

   a) $A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix} \quad , \quad b = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$

   b) $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} \quad , \quad b = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$

   c) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \quad , \quad b = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$

   Solution:(a) $\text{proj}_{\text{ColA}} b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\widehat{x} = \begin{pmatrix} 2/7 \\ 1/7 \end{pmatrix}$. (b) $\text{proj}_{\text{ColA}} b = \begin{pmatrix} 4 \\ -1 \\ 4 \end{pmatrix}$, $\widehat{x} = \begin{pmatrix} 3 \\ 1/2 \end{pmatrix}$.
   (c) $\text{proj}_{\text{ColA}} b = \begin{pmatrix} 5 \\ 2 \\ 3 \\ 6 \end{pmatrix}$, $\widehat{x} = \begin{pmatrix} 1/3 \\ 14/3 \\ -5/3 \end{pmatrix}$.

22. Describe all least-squares solutions of the system: $\left\{ \begin{array}{l} x + y = 2 \\ x + y = 4 \end{array} \right.$. **Solution:** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3-y \\ y \end{pmatrix}$.

23. Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fits the given data points:

   a) $(0,1), (1,1), (2,2), (3,2)$.

   b) $(2,3), (3,2), (5,1), (6,0)$.

   Solution:(a) $y = \dfrac{9}{10} + \dfrac{2}{5}x$. (b) $y = \dfrac{43}{10} - \dfrac{7}{10}x$.

24. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be the data that must be fit by a least-squares line. Show that the normal equations have a unique solution if and only if the data include at least two data points with different x-coordinates.

   Solution: Find the system of linear equations imposing that the straight line $y = \beta_0 + \beta_1 x$ must contain the following points three points: $\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} =$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

The augmented matrix of the system has the following echelon form:

$$\begin{pmatrix} 1 & x_1 & y_1 \\ 0 & x_2 - x_1 & y_2 - y_1 \\ 0 & x_3 - x_1 & y_3 - y_1 \end{pmatrix}.$$

The system has a unique solution if the last column is not a pivot column. That is, $x_2 - x_1 \neq 0$ or $x_3 - x_1 \neq 0$, these two equations could be satisfied simultaneously.

25. A certain experiment produces the data (1,1.8), (2,2.7), (3,3.4), (4,3.8), (5,3.9). Describe the model that produces a least-squares fit of these points by a function of the form $y = \beta_1 x + \beta_2 x^2$.

Solution:

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1{,}8 \\ 2{,}7 \\ 3{,}4 \\ 3{,}8 \\ 3{,}9 \end{pmatrix}$$

Additional exercises: D. C. Lay "Linear algebra and its applications", 2012.

- **Sections 6.1-6.6.**