18.06 (Spring 14) Problem Set 5

This problem set is due Thursday, March 20, 2014 by 4pm in E17-131. The problems are out of the 4th edition of the textbook. This homework has 8 questions worth 100 points in total. Please WRITE NEATLY. You may discuss with others (and your TA), but you must turn in your own writing.

1. Least squares approximation: if $b = (5, 13, 17)$ at $t =$ (-1,1,2) find the best line $C + Dt$, and the vectors p and e. Check that $e^T p = 0$.

   Solution:

   $$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, x = \begin{bmatrix} C \\ D \end{bmatrix}, b = \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix}$$

   The system is solvable and has solution $x = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$. Hence the best line is $9 + 4t$, which fits perfectly giving zero error $e = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and $p = b$.

2. Least squares approximation: If b takes values 0, 1, 3, respectively 4 at the corners $(x, y) = \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$ of a square, find the best least squares fit to b by a plane $C + Dx + Ey$.

   Solution:

   $$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, x = \begin{bmatrix} C \\ D \\ E \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

   The system is **not** solvable. We solve $A^T A\widehat{x} = A^T b$, where

   $$A^T A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, A^T b = \begin{bmatrix} 8 \\ -3 \\ -3 \end{bmatrix}$$

   which gives

   $$\widehat{x} = \begin{bmatrix} 2 \\ -3/2 \\ -3/2 \end{bmatrix}$$

   So the best fitting plane is $2 - 3/2x - 3/2y$. (You can also check by runnnig $lsqlin(A, b)$ in MATLAB.)

3. Consider the reflection matrix $R = I - 2P$, where P is a projection.

(a) Show that $R^2 = I$.

(b) For vectors v in $C(P)$, what is Rv?

(c) For vectors w orthogonal to $C(P)$, what is Rw?

Solution:

(a) $R^2 = (I-2P)^2 = I^2 - 2PI - 2IP + 4P^2 = I - 4P + 4P = I$. (We used the fact that $P^2 = P$, since P is a projection).

(b) Let $v = Px \in C(P)$. Then $Rv = (I-2P)v = v - 2Pv = v - 2P(Px) = v - 2P^2x = v - 2Px = v - 2v = -v$.

(c) Let $w = (I-P)x \in C(P)^\perp$. Then $Rw = (I-2P)w = w - 2Pw = w - 2P(I-P)x = w - 2Px + 2P^2x = w - 2Px + 2Px = w$.

4. If $q_1, q_2, \ldots, q_n$ are orthonormal vectors in $\mathbf{R}^n$, what combination of the q's produces a given vector v ? How do you know that the q's are a basis for $\mathbf{R}^n$?

Solution:

$$v = \sum_{i=1}^{n} (q_i^T v) \cdot q_i$$

The vectors are orthonormal, hence linearly independent, so they form a basis. Another way to verify is to look at the projection matrix on the span of the q_i's. Consider the matrix with the q_i's as column vectors $A = \left[q_1 | \ldots | q_n\right]$. Then $A^TA = I$. Hence the projection matrix onto their span is $P = A(A^TA)^{-1}A^T = AI^{-1}A^T = AA^T = (A^TA)^T = I$.

5. (a) In class, you may have seen that in the projection matrix formula $P = A(A^TA)^{-1}A^T$, we always want A to have linearly independent columns. Why?

(b) Given an $m \times n$ matrix A $(m > n)$, what is the projection matrix P_{null} onto the nullspace of A^T?

Solution:

(a) If the columns are not linearly independent, A does not have full column rank. So A^TA is not full rank, and hence not invertible.

(b) Recall that $N(A^T) \perp C(A)$. Let P be the projection on $C(A)$. Hence $P = A(A^TA)^{-1}A^T$. The projection onto the null space of A^T is perpendicular to that on the column space of A, hence $P_{null} = I - P = I - A(A^TA)^{-1}A^T$.