Midterm 2

Midterm 1 Review: Midterm 1 Study Guide

Section 2.2: The Inverse of a Matrix

Definitions

- Invertible Matrix
  - An n x n matrix A where $AA^{-1} = I$
- Inverse of a Matrix
  - A^{-1} where $AA^{-1} = I$
- Singular
  - Not invertible
- Determinant of a 2x2 matrix
  - ad - bc
- Elementary Matrix (E)
  - Matrix obtained by performing a **single** row operation on an **identity matrix**
  - Are invertible: inverse of an elementary matrix, E, is another elementary matrix of the same type that **transforms E back to I**
  - **All** elementary matrices are invertible
- Row Equivalent Matrices
  - Matrices that can **transform into each other** through a sequence of elementary row operations

Key Notes

- Invertible = **non**singular
- **Not** invertible = singular
- Inverse of a 2x2 Matrix
  - $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
    $$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
    - If (ad - bc) = 0, then A is **not invertible**
- Ax = b can be rewritten using inverses **only if A is invertible**
  - $$\begin{aligned} Ax &= b \\ A^{-1}Ax &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$
  - Of course, you can still use the **row reduction** method to solve Ax = b
  - For all b in R^n, x = A^{-1}b is a **unique solution**
    - Invertible matrices have **no free variables**
    - Unique solution
- Product of n x n invertible matrices is invertible
  - Inverse of product is the product of the inverses in **reverse order**

- When an elementary row operations is performed on an $m \ x \ n$ matrix A, the resulting matrix can be written as EA
    - What if we had **multiple elementary row operations** on A?
        - $E_k ... E_2 E_1 A$
- Method to find the inverse
    - Row reduce A to the identity matrix while performing the **same row operations on the identity matrix** at the same time
    - [A | I] => [I | A^{-1}]
- A matrix is invertible if and only if it is **row equivalent** to the identity
    - Pivots in every row and column **(onto & one-to-one)**

Section 2.3: Characterizations of Invertible Matrices

Definitions

- Linear Transformation
    - Mapping between two vector spaces (R^n's) that preserves all vector addition & scalar properties
- Invertible Linear Transformation
    - Linear transformation T: R^n -> R^n is invertible if there is **another linear transformation S**: R^n -> R^n such that:
        - S(T(x)) = x for all x in R^n
        - T(S(x)) = x for all x in R^n
    - Equivalent to saying:
        - $A^{-1}Ax = (I)x$

Key Notes

- The IMT

- **The Invertible Matrix Theorem**

  Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

  a. A is an invertible matrix.
  b. A is row equivalent to the $n \times n$ identity matrix.
  c. A has n pivot positions.
  d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  e. The columns of A form a linearly independent set.
  f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
  g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
  h. The columns of A span $\mathbb{R}^n$.
  i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
  j. There is an $n \times n$ matrix C such that $CA = I$.
  k. There is an $n \times n$ matrix D such that $AD = I$.
  l. A^T is an invertible matrix.

- Let A and B be square matrices:
  - If AB = I, then A and B are **both invertible**
  - $B = A^{-1}$ & $A = B^{-1}$
- How to determine if a linear transformation is **invertible**?
  - Let a matrix A represent the linear transformation
  - If A is invertible, then the linear transformation is invertible
    - $$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} : reflection\, through\, y - axis\, (invertible)$$
      $$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} : projection\, onto\ x - axis\, (not\, invertible)$$

Section 2.4: Partitioned Matrices

Definitions

- Partitioned Matrix
  - Matrix divided up into separate blocks
- Block Diagonal Matrix
  - A partitioned matrix where all blocks except the main diagonal are 0's
  - Is invertible if the main diagonal blocks are invertible

Key Notes

- Adding 2 partitioned matrices A and B
  - A and B must be the **same size** and partitioned in the **exact same way**
    - Add **block by block**
- Scaling partitioned matrices

        - Scale block by block
- Multiplying 2 partitioned matrices A and B (A*B)
    - Column partition of A **must equal** row partition of B
    - Number of columns in partition A = number of rows in partition B
        - Just like multiplying regular matrices
        - (2 x 2) * (2 x 1) => (2 x 1)
        - (3 x 4) * (4 x 1) => (3 x 1)
- Inverses of Partitioned Matrices

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} : \ (A,\ B,\ and\, C\, are\, invertible)$$

$$AX\ +\ BZ\ =\ I_n$$
$$AY\ +\ BW\ =\ 0$$
$$0X\ +\ CZ\ =\ 0$$
$$0Y\ +\ CW\ =\ I_n$$

    - $Solve\ for\ X, Y, Z, W\ in\ terms\ of\ A, B, C$

Section 2.5: Matrix Factorizations

Definitions

- Factorization of a matrix
    - Expression of a matrix as the product of two or more matrices
- Row interchanges
    - Swapping rows when row reducing
- Lower triangular matrix
    - Entries above the main diagonal are all 0's
- Upper triangular matrix
    - Entries below the main diagonal are all 0's
- Algorithm for an LU Factorization
    - 1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
    - 2. Place entries in L such that the *same sequence of row operations* reduces L to I.

Key Notes

- LU Factorization
    - Why do we use it?
        - More efficient to solve a sequence of equations with the same coefficient matrix (Ax = b_1, Ax = b_2, ... , Ax = b_n) by LU factorization than row reducing the equations every single time
- Let A be an $m\ x\ n$ matrix that can be row reduced to echelon form **without row exchanges**, then:

- $$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
  $$\qquad L \qquad\qquad\qquad\qquad U$$
    - **L**: m x m lower triangular matrix with one's on the main diagonal
    - **U**: m x n echelon form of A
- Rewriting Ax = b using A = LU
  - $$\boxed{\begin{aligned} L\mathbf{y} &= \mathbf{b} \\ U\mathbf{x} &= \mathbf{y} \end{aligned}}$$
  - Ax = b -> L(Ux) = b
- The LU Factorization Algorithm
  - How do we get U?
    - Row reduce A to echelon form using only row replacements that add a multiple of one row to another <u>below</u> it
  - How do we get L?
    - Take the row replacement operations you did on A when getting echelon form
      - Basically: find the **elementary matrices** that transform A into U
    - Then, **reverse** the signs and input them in their respective spots in the m x m identity matrix
      - Replace the 0's below the main diagonal with the row replacement "coefficients"
      - Basically: after finding all the elementary matrices, take their **inverses**

    $$E_p \cdots E_1 A = U$$

  $$A = (E_p \cdots E_1)^{-1} U = LU$$

  - $$L = (E_p \cdots E_1)^{-1}$$
- Using the LU Decomposition
  - After constructing A = LU, solve Ax = LUx = b by:
    - 1. Forward solve for y in Ly = b
      - R1(x) + R2 -> R2
      - Modify rows **below** using **above rows**

            - 2. Backwards solve for x in Ux = y
                - R2(x) + R1 -> R1
                - Modify rows **above** using **below rows**

Section 2.6: The Leontief Input-Output Model

Definitions

- Production vector in R^n (x)
    - Lists the output of each sector for one year
- Final demand vector (d)
    - Lists the value of goods and services produced for the consumers (nonproductive part of the economy)
- Intermediate demand (Cx)
    - The demand for goods and services that the producers (sectors) need as inputs for their own production
        - Ex: electricity sector needs inputs from the water sector and vice versa
- Consumption matrix (C)
    - How much each sector consumes from other sectors in terms of percentages
- Column sum
    - The sum of the entries in a column

Key Notes

- The Leontief Input-Output Model (Production Equation)
    - THE LEONTIEF INPUT–OUTPUT MODEL, OR PRODUCTION EQUATION

      $$\underbrace{\mathbf{x}}_{\text{Amount produced}} = \underbrace{C\mathbf{x}}_{\text{Intermediate demand}} + \underbrace{\mathbf{d}}_{\text{Final demand}} \tag{4}$$
    - Can be rewritten as:
        - (I - C)x = d
            - Solve for x (amount produced) by row reduction
        - x = $(I - C)^{-1}$ * d
            - Solve for x (amount produced) by multiplying
- For a good economy, the column sum of each sector should be **less than 1**
    - A sector should in general require less than one unit's worth of inputs to produce one unit of output
- Output vector (x)
    - x_i: entry i of vector x
        - Number of units produced by sector i

- Internal consumption (C)
    - 2 equivalent ways of defining entries of C where an entry is $c_{i,j}$:
        - Sector *i* <u>sends</u> a proportion of its units to sector *j*
        - Sector *j* <u>requires</u> a proportion of the units created by sector *i*
- Consumption matrix (Cx)
    - Total output for each sector (per one unit) is the sum of the columns for each sector
- A Formula for $(I - C)^{-1}$
    - As an economy is introduced to a demand vector, the equation starts off as:
        - $x = d$
    - However, production will require intermediate demand from other sectors, and then that intermediate demand will require more inputs from even more sectors
        - $x = d + Cd + C^2d + C^3d + \ldots$
            - $\Rightarrow (I + C + C^2 + C^3)d$
    - $$(I - C)^{-1} \approx I + C + C^2 + C^3 + \cdots + C^m$$ when the column sums of C are less than 1.
        - We can approximate $(I - C)^{-1}$ by making m as large as possible
            - Add as many intermediate demands as we can
- Economic Importance of Entries in $(I - C)^{-1}$
    - Entries used to predict how the production x will have to change when the final demand d changes
        - Remember: $x = (I - C)^{-1} * d$
    - The entries in each column of $(I - C)^{-1}$ are the *increased* amounts each sector has to produce in order to satisfy *an increase of 1 unit* in the final demand

Section 2.7: Applications to Computer Graphics

Definitions

- Homogeneous coordinates
    - Each point (x, y) in R^2 can be identified with the point (x, y, 1) on the plane in R^3 that lies one unit above the xy - plane
- Composite transformations
    - Multiplication of 2 or more basic transformations

Key Notes

- Why do we use homogeneous coordinates?
    - Translations are **not** linear transformations

- Homogeneous coordinates are allowed to be **scalars**
  - (3, 5, 1) = (6, 10, 2)
- (x, y) -> (x + h, y + k)
  - Translation cannot be represented by an R^2 matrix multiplication
  - (x, y, 1) -> (x + h, y + k, 1)
  - $$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+h \\ y+k \\ 1 \end{bmatrix}$$
    - Translation not possible if we used a 2x2 identity matrix
- Linear transformations in R^2 represented with homogeneous coordinates are written as partitioned matrices:
  - $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$ where A is a 2x2 matrix
  - Examples
    - $$\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} s & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

      Counterclockwise rotation about the origin, angle φ | Reflection through $y = x$ | Scale x by s and y by t
- Composite Transformations
  - "Add" on more transformation matrices **to the left** of the other transformations
    - First transformation is always the **rightmost** (modifies the x vector first)
- Homogeneous 3D Coordinates
  - (X, Y, Z, H) are homogeneous coordinates for (x, y, z) if H ≠ 0 and
    - $$x = \frac{X}{H}, \qquad y = \frac{Y}{H}, \quad \text{and} \quad z = \frac{Z}{H}$$

Section 2.8: Subspaces of R^n

Definitions

- Subset of R^n
  - Any collection of vectors that are in R^n
- Subspace of R^n
  - A subset H in R^n that has 3 properties:
    - The zero vector is in H
    - $\vec{u} + \vec{v} \in H$ (closed under addition)

        - $c\vec{u} \in H$ (closed under scalar multiplication)
    - Subspace can be written as the **Span{}** of some amount of linearly independent vectors
- Column Space of a Matrix A **(m x n)**
    - Col A: the subspace of $\mathbf{R}^m$ spanned by $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}$
    - Essentially all the **pivot columns**
- Null Space of a Matrix A **(m x n)**
    - Null A: the subspace of $\mathbf{R}^n$ spanned by the set of all vectors $\mathbf{x}$ that solve $\mathbf{Ax} = \mathbf{0}$
- Basis for a Subspace $\boldsymbol{H}$ of $\mathbf{R}^n$
    - A linearly independent set in $\boldsymbol{H}$ that spans $\boldsymbol{H}$
        - **DOES NOT CONTAIN THE ZERO VECTOR (BECAUSE IT IS LINEARLY INDEPENDENT) UNLIKE THE SPAN**
- Standard Basis for $\mathbf{R}^n$
    - $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$

Key Notes

- If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are in $\mathbf{R}^n$ and $\boldsymbol{H} = Span\{v_1, v_2\}$, then $\boldsymbol{H}$ is a subspace of $\mathbf{R}^n$
    - $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ must be in $\mathbf{R}^n$ for this relation to work
- For $\boldsymbol{v}_1, \dots, \boldsymbol{v}_p$ in $\mathbf{R}^n$, the set of all linear combinations of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_p$ is a subspace of $\mathbf{R}^n$
    - $\boldsymbol{Span\{v_1, \dots, v_p\}}$ = subspace spanned by $\boldsymbol{v}_1, \dots, \boldsymbol{v}_p$
- Is **b** in the column space of **A**?
    - Same as : Is **b** a linear combination of **A**?
    - Same as : Is **b** in the Span of **A**?
- Is **H** a subspace of $\mathbf{R}^n$**?**
    - Basically asking if H has **n linearly independent** columns
    - Does H have **no free variables?**
- Subspaces vs. Bases
    - Subspaces => **Span**$\{v_1, \dots, v_n\}$
        - **Includes the 0 vector**
    - Bases => $\{v_1, \dots, v_n\}$
- Defining a basis for column space A
    - Number of entries for each vector = number of rows in matrix A
    - Number of vectors in the basis = number of pivot columns
    - What vectors can you include in the basis?
        - Scalar multiples
        - The identity matrix columns **only if** every column is pivotal in A
- Finding the Column Space
    - Row reduce the matrix
        - **Row operations do not affect linear dependence relations**

    - Determine the pivot columns
    - Create a basis/subspace using the pivot columns in the **original matrix**
        - Not the row reduced one
    - If **every** column is **linearly independent**, then the elementary vectors are included in the column space
        - Linear combinations of elementary vectors can get you **any column** of the original matrix
- Finding the Null Space
    - Determine all the free variables
    - Rewrite system in **parametric vector form**
    - Vectors created in parametric vector form generate the null space

Section 2.9: Dimension and Rank

Definitions

- Coordinates
    - Weights that map our vectors to get to some point in the span of the vectors
- Coordinate Vector
    - Suppose the set $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ is a basis for a subspace H. For each $\mathbf{x}$ in H, the **coordinates of x relative to the basis** $\mathcal{B}$ are the weights $c_1, \dots, c_p$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p$, and the vector in $\mathbb{R}^p$

      $$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

      is called the **coordinate vector of x (relative to** $\mathcal{B}$**)** or the $\mathcal{B}$**-coordinate vector of x.**[1]
- Dimension of a Subspace
    - dim H: the number of vectors in a basis of H
    - dim{0} = 0
- Rank of a Matrix A
    - Dimension of the column space of A
    - Number of pivots in A

Key Notes

- Why we choose to write bases:
    - Each vector in **H** can be written in **only one way** as a linear combination of the **basis** vectors
- A plane through 0 in R^3 is two-dimensional
    - 3x3 matrix A has 2 pivots
- A line through 0 in R^2 is one-dimensional

    - 2x2 matrix A has one pivot
- Any two choices of bases of a non-zero subspace H have the **same dimension**

    - dim R^n = n
    - dim(Col A) = number of pivots
    - dim(Null A) = number of free variables
- dim(Col A) = rank A
- Rank Theorem
    - If A has n columns, then:
        - rank A + dim(Null A) = n
    - Number of pivots + number of free variables = number of columns
- Basis Theorem
    - Any two bases for a subspace have the same dimension (cardinality)
    - Many choices for the basis of a subspace
- Continuation of the Invertible Matrix Theorem with Rank
    - **The Invertible Matrix Theorem (continued)**

      Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

      m. The columns of A form a basis of $\mathbb{R}^n$.
      n. Col $A = \mathbb{R}^n$
      o. dim Col $A = n$
      p. rank $A = n$
      q. Nul $A = \{\mathbf{0}\}$
      r. dim Nul $A = 0$

Section 3.1: Introduction to Determinants

Definitions

- A_{ij} submatrix
    - Delete the ith row and jth column of matrix A
    - Remaining elements will form the new submatrix
- Determinant for a 2x2
    - A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ -> det A = ad - bc
- Cofactor expansion
    - A way to solve determinants for square matrices that are 3x3 and greater

Key Notes

- Signs of cofactor expansions
    - Depends on position of element a_{ij} in the matrix

- $$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$
- Shortcut for finding the determinant
  - Row reduce to REF
    - Effects of row operations on determinant covered in 3.2
  - Multiply all the numbers on the main diagonal

Section 3.2: Properties of Determinants

Definitions

- Column Operations
  - Same effect on determinants as row operations
  - This is true because the determinant of A = determinant of A^T (transpose)

Key Notes

- Row operations on determinants
  - Row replacement: nothing
  - Row swap: multiply determinant by negative one
  - Row scale: multiply determinant by scale
- Summary of elementary matrices' determinants
  - *If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then*

    $$\det EA = (\det E)(\det A)$$

    where

    $$\det E = \begin{cases} 1 & \textit{if E is a row replacement} \\ -1 & \textit{if E is an interchange} \\ r & \textit{if E is a scale by r} \end{cases}$$
- More specific example of row scaling on determinants
  - $$\begin{bmatrix} \cdot & \cdot & \cdot \\ 5k & -2k & 3k \\ \cdot & \cdot & \cdot \end{bmatrix} = k \begin{bmatrix} \cdot & \cdot & \cdot \\ 5 & -2 & 3 \\ \cdot & \cdot & \cdot \end{bmatrix}$$
  - Row divided by k
    - Determinant is multiplied by 1/k
- If A is invertible (every column is pivotal)
  - det A ≠ 0
- If A is not invertible
  - det A = 0
  - At least one entry on the main diagonal of REF is 0

- $$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$
- When A is not invertible, the rows are linearly dependent
  - If A is square, then the columns are also linearly dependent
- det A = det A^T
- det AB = (det A)(det B)
- det A^{-1} = 1 / (det A)

Section 3.3: Volume and Linear Transformations

Definitions

- Parallelepiped: a parallelogram in R^n where n > 2

Key Notes

- If A is a 2x2 matrix:
  - Area of the parallelogram determined by the columns of A is **| det A |**
- If A is a 3x3 matrix:
  - Area of the **parallelepiped** determined by the columns of A is **| det A |**
- Row/column swaps and replacements do not affect the **absolute value** of the determinant
- Linear transformations on a parallelepiped
  - Area of T(S) = | det A | * { area of S }
    - T: linear transformation determined by matrix A
    - S: parallelogram

Section 4.9: Applications to Markov Chains

Definitions

- Probability vector
  - A vector with **nonnegative** entries that **sum to 1**
- Stochastic matrix
  - A **square** matrix whose columns are **probability vectors**
- Markov Chain
  - A sequence of probability vectors {x_0, x_1, x_2, ...} together with a stochastic matrix {P} such that:
    - $x_1 = Px_0\,,\quad x_2 = Px_1,\quad x_{k+1} = Px_k$
- Steady State Vector
  - A probability q such that $Pq = q$

    - Every stochastic matrix has a steady state vector
- Regular stochastic matrix
    - Stochastic matrix where some power of it will contain only **strictly positive entries**
        - P^k where all entries > 0

Key Notes

- How to find the next outcome of a Markov Chain?
    - Simply multiply P by x_k to find x_{k+1}
- How to find a steady state vector?
    - $Pq = q$
      $Pq - q = 0$
      $(P - I)q = 0$
    - After finding a basis for the null space of (P - I) q = 0, remember to make sure that the **column sum is 1**
        - Steady state vector is a **probability vector**
- The initial state has **no effect** on the long term behavior of the Markov Chain

Section 5.1: Eigenvectors and Eigenvalues

Definitions

- Eigenvector of an n x n matrix A:
    - **Nonzero** vector x such that Ax = λx for some scalar λ
- Eigenvalue of A:
    - A scalar λ where there is a **nontrivial solution** x of Ax = λx
- Eigenspace of an eigenvalue
    - Contains the zero vector and **all eigenvectors corresponding to λ**

Key Notes

- Determine if a vector x is an eigenvector
    - A*x => see if product is a scalar multiple of x
- Finding the eigenvector from an eigenvalue (7)
    - Solve (A - 7I)x = 0
    - Then, do the parametric vector form of what you have left
- Finding the eigenvalue λ
    - Solve (A - λI)x = 0 for a **nontrivial solution**
    - Find the set of all solutions to the **null space** of (A - λI)
- Eigenvalues of a **triangular matrix** are the entries on the **main diagonal**
- 0 is an eigenvalue of A if and only if A is **not invertible**
    - Ax = 0x

- $Ax = 0$: x is a nontrivial solution if A is not invertible
- Eigenvectors that correspond to distinct eigenvalues are **linearly independent**
    - Opposite is not always true
        - $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$: eigenvectors are linearly independent but have the **same eigenvalue**

Section 5.2: The Characteristic Equation

Definitions

- The Characteristic Polynomial:
    - det(A - λI)
- The Characteristic Equation
    - det(A - λI) = 0
- Trace
    - Sum of the diagonal entries in a matrix
- Algebraic Multiplicity of an Eigenvalue
    - The number of times the eigenvalue shows up as roots of the characteristic polynomial
- Geometric Multiplicity of an Eigenvalue
    - The dimension of Null (A - λI) for a given eigenvalue λ

Key Notes

- How to find eigenvalues?
    - Solve (A - λI)x = 0 for a **nontrivial solution**
    - Find the set of all solutions to the **null space** of (A - λI)
- Continuation of IMT
    - For A: $n\ x\ n$ matrix, A is invertible if and only if:
        - The number 0 **is not** an eigenvalue of A
        - The determinant of A **is not 0**
- Finding the characteristic polynomial using **trace** and **determinant** for a characteristic polynomial of **2**
    - λ^2 - λ(trace) + det A
- **Warnings:**
    - Cannot determine the eigenvalues of a matrix from its reduced from
    - Row operations **change** the eigenvalues

Theorems

Chapter 2

Theorem 4: Finding the Inverse of a 2x2 Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

Theorem 5: Alternate Method of Finding the Solution Set

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Theorem 6: Properties of Invertible Matrices

a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T$$

Theorem 7: Finding the Inverse of a Matrix

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Theorem 8: The Invertible Matrix Theorem

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

Theorem 9: Invertible Linear Transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying equations (1) and (2).

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \tag{1}$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \tag{2}$$

Theorem 10: Column-Row Expansion of AB

Column–Row Expansion of *AB*

If A is $m \times n$ and B is $n \times p$, then

$$AB = [\,\text{col}_1(A) \quad \text{col}_2(A) \quad \cdots \quad \text{col}_n(A)\,] \begin{bmatrix} \text{row}_1(B) \\ \text{row}_2(B) \\ \vdots \\ \text{row}_n(B) \end{bmatrix} \tag{1}$$

$$= \text{col}_1(A)\,\text{row}_1(B) + \cdots + \text{col}_n(A)\,\text{row}_n(B)$$

Theorem 11: Solving the Output Vector (x)

Let C be the consumption matrix for an economy, and let $\mathbf{d}$ be the final demand. If C and $\mathbf{d}$ have nonnegative entries and if each column sum of C is less than 1, then $(I - C)^{-1}$ exists and the production vector

$$\mathbf{x} = (I - C)^{-1}\mathbf{d}$$

has nonnegative entries and is the unique solution of

$$\mathbf{x} = C\mathbf{x} + \mathbf{d}$$

Theorem 12: Finding the Null Space of Matrix A

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions of a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Theorem 13: Determining the Column Space of Matrix A

The pivot columns of a matrix A form a basis for the column space of A.

Theorem 14: The Rank Theorem

The Rank Theorem

If a matrix A has n columns, then rank A + dim Nul $A = n$.

Theorem 15: The Basis Theorem

The Basis Theorem

Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

Chapter 3

Theorem 1: Cofactor Expansion to find Determinants

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors in (4) is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Theorem 2: Shortcut to Computing Determinant

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

Theorem 3: Row Operations on Determinants

Row Operations

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.
b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.
c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

Theorem 4: IMT DLC: Determinant

A square matrix A is invertible if and only if $\det A \neq 0$.

Theorem 5: Transpose Equivalence for Determinants

If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Theorem 6: Multiplicative Property of Determinants

Multiplicative Property

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

Theorem 9: Determinants as Area and Volume

If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Theorem 10: Linear Transformations on Area/Volume

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\} \tag{5}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\} \tag{6}$$

Chapter 4

Theorem 18: Long-term Behavior of a Markov Chain

If P is an $n \times n$ regular stochastic matrix, then P has a unique steady-state vector $\mathbf{q}$. Further, if $\mathbf{x}_0$ is any initial state and $\mathbf{x}_{k+1} = P\mathbf{x}_k$ for $k = 0, 1, 2, \ldots$, then the Markov chain $\{\mathbf{x}_k\}$ converges to $\mathbf{q}$ as $k \to \infty$.

Chapter 5

Theorem 1: Eigenvalues of a Triangular Matrix

The eigenvalues of a triangular matrix are the entries on its main diagonal.

Theorem 2: Eigenvectors for Distinct Eigenvalues

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.