Mathematics for Computer Science

MIT 6.042J/18.062J

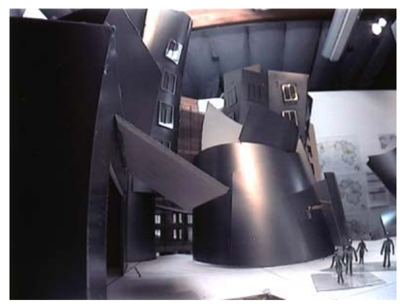
Induction II

The MIT Stata Center

http://web.mit.edu/buildings/statacenter

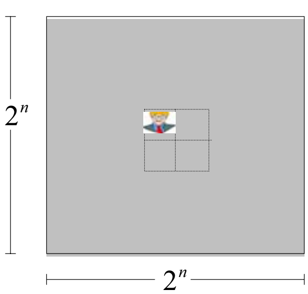
The Stata Center Today

The Stata Center Plaza



The Gehry/Gates Plaza

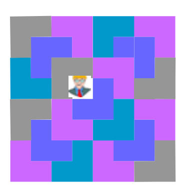
Goal: tile the squares, except one in the middle for Bill.



The Gehry/Gates Plaza

Gehry specifies L-shaped tiles covering three squares: 

For example, for 8 x 8 plaza might tile for Bill this way:



The Gehry/Gates Plaza

Theorem: For any $2^n \times 2^n$ plaza, we can make Bill happy.

Proof: (by induction on n)

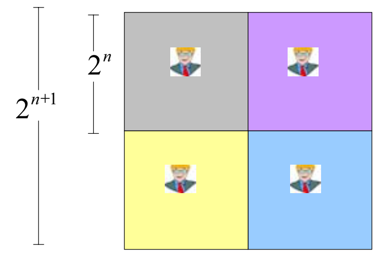
$P(n)$ ::= can tile $2^n \times 2^n$ with Bill in middle.

Base case: (n=0)

(no tiles needed)

The Gehry/Gates Plaza

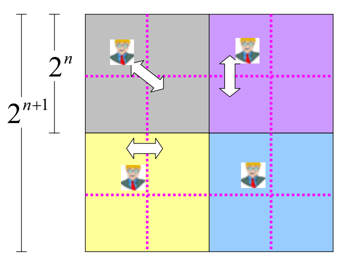
Induction step: assume can tile $2^n \times 2^n$, prove can handle $2^{n+1} \times 2^{n+1}$.



The Gehry/Gates Plaza

Method: Divide into subsquares

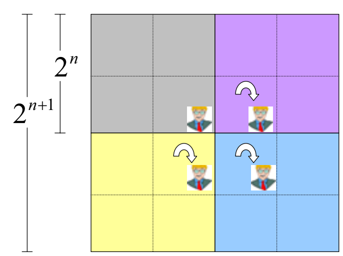
Relocate the subsquares as indicated.



The Gehry/Gates Plaza

Method: after relocation have:

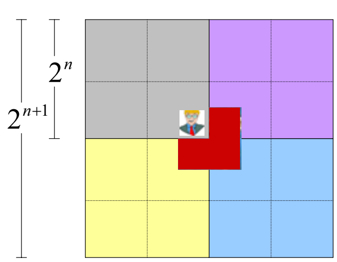
Now rotate the squares as indicated.



The Gehry/Gates Plaza

Method: after rotation have:

Now put tile in center



Done!

The Gehry/Gates Plaza

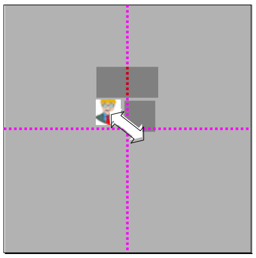
Did I fool you?

The Gehry/Gates Plaza

Bug:

Division into subsquares may

2^n



cut a tile!

The Gehry/Gates Plaza

The fix:

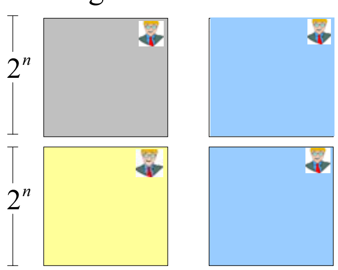
Prove that we can always find a tiling with Bill in the corner.

The Gehry/Gates Plaza

Induction step:

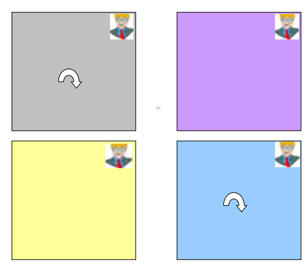
Assume we can get Bill in corner of $2^n \times 2^n$.

Prove we can get Bill in corner of $2^{n+1} \times 2^{n+1}$.



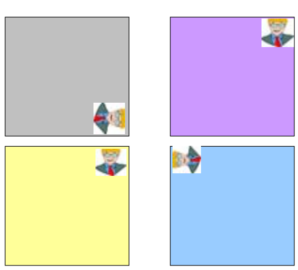
The Gehry/Gates Plaza

Method: Rotate the squares as indicated.



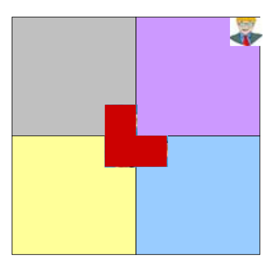
The Gehry/Gates Plaza

Method: Rotate the squares as indicated.
after rotation have:



The Gehry/Gates Plaza

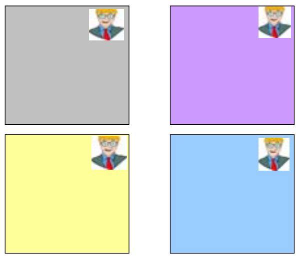
Method: Now group the squares together, and fill the center with a tile.



Done!
For real!

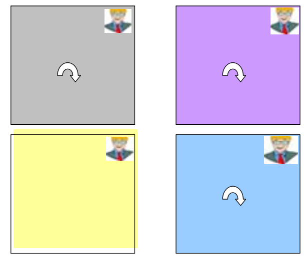
The Gehry/Gates Plaza

Note: Once have Bill in corner,
can get Bill in middle:



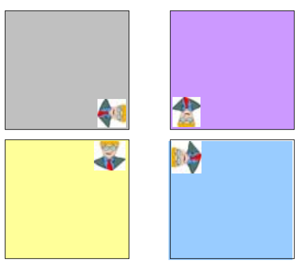
The Gehry/Gates Plaza

Method:
Rotate the squares as indicated.



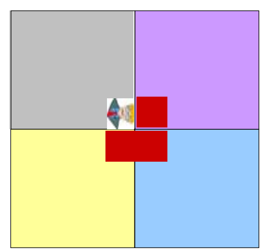
The Gehry/Gates Plaza

Method: after rotation have:



The Gehry/Gates Plaza

Method: Now group the 4 squares together,
and insert a tile.



Done!
Bill in
middle

Ingenious Induction Hypotheses

Note 1: To prove
"Bill in middle,"
we proved *something else:* "Bill in corner."

Inductive (Recursive) Procedures

Note 2: The induction proof of "Bill in corner" implicitly defines a recursive procedure for constructing a $2^{n+1} \times 2^{n+1}$ corner tiling from a $2^n \times 2^n$ tiling.

Ingenious Induction Hypotheses

Note 3: Other times it helps to choose a *stronger hypothesis* than the desired result. Result at n+1 becomes harder to prove -- but we have a stronger hypothesis at n to prove it with!

Problems

Class Problem 1

A False Proof

Theorem: All horses are the same color.

Proof: (by induction on n)
Induction hypothesis:
$P(n)$ ::= any set of n horses have the same color
Base case (n=0):
No horses so *vacuously* true!

A False Proof

(Inductive case)
Assume any n horses have the same color.
Prove that any n+1 horses have the same color.

n+1

A False Proof

(Inductive case)
Assume any n horses have the same color.
Prove that any n+1 horses have the same color.

Second set of n horses have the same color

First set of n horses have the same color

A False Proof

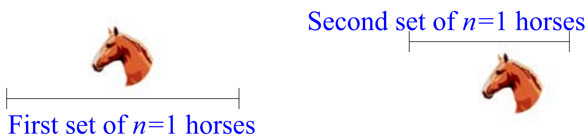
(Inductive case)
Assume any n horses have the same color.
Prove that any n+1 horses have the same color.

Therefore the set of n+1 have the same color!

A False Proof

What is wrong? $n = 1$

Proof that $P(n) \rightarrow P(n+1)$ is false if $n = 1$, because the two horse groups *do not overlap*.



A False Proof

Proof that $P(n) \rightarrow P(n+1)$ is false if $n = 1$, because the two horse groups *do not overlap*.

(But proof works for all $n \neq 1$)

Problems

Class Problems
2 & 3