Worksheet: Eigenvalues and Eigenvectors

1. (10 points) Consider the matrix $A = \begin{bmatrix} 7 & 5 \\ 3 & -7 \end{bmatrix}$

   (a) Find the eigenvalues and eigenvectors of A.

   (b) Find matrices P and D such that $A = PDP^{-1}$ where P is invertible and D diagonal.

   (c) Compute A^3.

7. Let A be a 3×3 matrix with eigenvalues 2, -1 and 3.

   (a) (2 Pts.) Find the eigenvalues of A^{-1}.

   (b) (2 Pts.) Find the determinant of A.

   (c) (2 Pts.) Find the determinant of A^{-1}.

   (d) (2 Pts.) Find the eigenvalues of $A^2 - A$.

5. (a) (4 Pts.) Find all the eigenvalues of the matrix,

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix}.$$

   (b) (1 Pts.) Can you decide how many linearly independent eigenvectors has this matrix without actually computing them? Justify your answer.

6. Let k be any number, and consider the matrix A given by

$$A = \begin{bmatrix} 2 & -2 & 4 & -1 \\ 0 & 3 & k & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   (a) (3 Pts.) Find the eigenvalues of A, and their corresponding multiplicity. Show your work.

   (b) (7 Pts) Find the number k such that there exists an eigenspace $E_A(\lambda)$ that is two dimensional, and find a basis for this $E_A(\lambda)$. The notation $E_A(\lambda)$ means the eigenspace corresponding to the eigenvalue λ of matrix A. Show your work.

7. (a) (5 Pts.) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Show your work.

   (b) (3 Pts.) Find matrices P and D such that $A = PDP^{-1}$, where P is invertible an D diagonal. Show your work.

(3i) (20 points) Find the eigenvalues and associated eigenvectors of the following matrix

$$A = \begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}.$$

(ii) (10 points) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(iii)(5 points) Compute A^{10}. You may express your answer as the product of not more than 3 matrices.

1. (12 pts) (a) Find the eigenvalues of the matrix $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

   (b) Find an eigenvector for each eigenvalue.

7. For each of the matrices below, either diagonalize it, i.e. find a matrix X such that $X^{-1}AX$ etc. is diagonal, or explain why its not possible: (10p each)

$$(a)\quad A = \begin{pmatrix} 4 & 0 \\ 2 & 5 \end{pmatrix}, \qquad (b)\quad B = \begin{pmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \qquad (c)\quad C = \begin{pmatrix} 16 & -12 & 0 \\ -12 & 9 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

4. In a population consisting of rabbits and foxes let r_n and f_n denote the number of rabbits and foxes at the beginning of the year n, respectively. The number of rabbits and foxes at the beginning of the next year are given by the formulas

$$r_{n+1} = 1.2r_n - 0.3f_n \quad \text{and} \quad f_{n+1} = 0.4r_n + 0.4f_n.$$

These two equations can be combined to yield the recurrence equation

$$\underbrace{\begin{bmatrix} r_{n+1} \\ f_{n+1} \end{bmatrix}}_{P_{n+1}} = \underbrace{\begin{bmatrix} 1.2 & -0.3 \\ 0.4 & 0,4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} r_n \\ f_n \end{bmatrix}}_{P_n}.$$

(a) (2 points) Find the eigenvalues and the associated eigenspaces of the matrix A.

(b) (2 points) Is the matrix A diagonalizable? If it is not, justify your answer. If it is diagonalizable, write down an eigenvalue decomposition for A?

6. Consider the eigenvalue decompostion of the matrix A given below.

$$A = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}}_{V^{-1}}$$

(a) (2 points) Write down the eigenvalues and an eigenvector associated with each eigenvalue of A.

(b) (2 points) Calculate A^5.

(c) (2 points) Calculate the determinant of A.

Problem 4. Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

a) Find all of the eigenvalues of A.

b) Find a basis for the eigenspace corresponding to each of the eigenvalues you found in part (a).

c) Is A diagonalizable? Why or why not?