Math 215 Final Exam Practice Problems

1. For each of the following statements, say whether it is true or false. If the statement is true, prove it. If false, give a counterexample.
   (a) If eigenvectors $\vec{x}$ and $\vec{y}$ correspond to distinct eigenvalues, then $\vec{x}^H\vec{y} = 0$.
   (b) Let A be an $m \times n$ matrix and let $\vec{b}$ be a vector in $\mathbb{R}^m$. If $m < n$, then $A\vec{x} = \vec{b}$ has infinitely many solutions.
   (c) If A is an $m \times n$ real matrix, then the nullspace of A^T is the orthogonal complement of the column space of A.
   (d) If S and T are subspaces of $\mathbb{R}^2$, then their union (i.e., the set of vectors which are in either S or T) is also a subspace of $\mathbb{R}^2$.
   (e) Let S be a plane through the origin in $\mathbb{R}^3$ and let P be the matrix which projects onto the plane S. Then for any $\vec{v} \in \mathbb{R}^3$,
   $$\|\vec{v}\|^2 = \|P\vec{v}\|^2 + \|\vec{v} - P\vec{v}\|^2.$$

2. Is the set of all orthogonal $n \times n$ real matrices a vector space?

3. Suppose the first row of A is $2, 3$ and its eigenvalues are $i, -i$. Find A.

4. If $\vec{x}_1, \vec{x}_2$ are the columns of S, what are the eigenvalues and eigenvectors of
$$A = S\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}S^{-1} \quad \text{and} \quad B = S\begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix}S^{-1}?$$

5. If A is an $n \times (n-1)$ matrix with rank $n-2$ and $\vec{b} \in \mathbb{R}^n$ such that $A\vec{v} = \vec{b}$ for some $\vec{v} \in \mathbb{R}^{n-1}$, what is the dimension of the space of *all* solutions to the equation $A\vec{x} = \vec{b}$?

6. (a) Prove that the eigenvalues of A are the eigenvalues of A^T.
   (b) If A and B are $n \times n$ real symmetric matrices, then AB and BA have the same eigenvalues. [HINT: Use part (a)]

7. Is there a real 2×2 matrix $A \neq I$ such that $A^3 = I$?

8. Suppose A is diagonalizable and that
$$p(\lambda) = \det(A - \lambda I) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0.$$
Show that
$$p(A) = c_nA^n + c_{n-1}A^{n-1} + \ldots + c_1A + c_0I$$
is the zero matrix.

   NOTE: The fact that this is true for *any* matrix (regardless of whether it's diagonalizable) is called the *Cayley-Hamilton Theorem.*

9. Suppose $A = \vec{u}\vec{v}^T$ where $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $\vec{v} \neq \vec{0}$.
   (a) What is the rank of A?
   (b) Show that $\vec{u}$ is an eigenvector of A. What is the corresponding eigenvalue?
   (c) What are the other eigenvalues of A?
   (d) Compute the trace of A in two different ways.

10. Let V be an n-dimensional vector space and suppose $T : V \to V$ is a linear transformation such that the range of T is equal to the set of vectors that T sends to $\vec{0}$. In other words,

$$\{T(\vec{v}) : \vec{v} \in V\} = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}.$$

(a) Show that n must be even.

(b) Give an example of such a T.

11. Find the LU decomposition of the matrix

$$\begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix}.$$

12. Let A be the matrix

$$A = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

and suppose $\vec{u}(t)$ solves the differential equation

$$\frac{d\vec{u}}{dt} = A\vec{u}(t)$$

subject to the initial condition $\vec{u}(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. What happens to $\vec{u}(t)$ as $t \to \infty$?