

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Mathematics for Computer Science

MIT 6.042J/18.062J

Induction II

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L2-2.1

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The MIT Stata Center



<http://web.mit.edu/buildings/statacenter>

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L2-2.2

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Stata Center Today

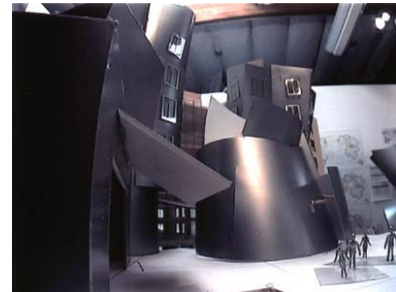


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L2-2.3

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Stata Center Plaza



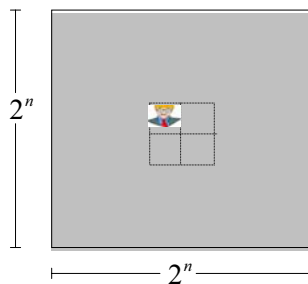
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L2-2.4

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Gehry/Gates Plaza

Goal: tile the squares, except one in the middle for Bill.



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L2-2.5

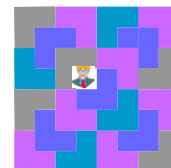
6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Gehry/Gates Plaza

Gehry specifies L-shaped tiles covering three squares:



For example, for 8 x 8 plaza might tile for Bill this way:



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L2-2.6

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Gehry/Gates Plaza

Theorem: For any $2^n \times 2^n$ plaza, we can make Bill happy.

Proof: (by induction on n)

$P(n) ::=$ can tile $2^n \times 2^n$ with Bill in middle.

Base case: ($n=0$)



(no tiles needed)

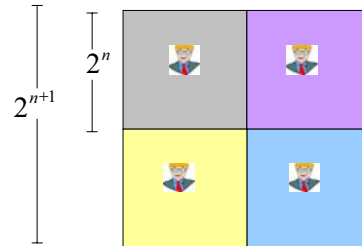
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L2-2.7

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Gehry/Gates Plaza

Induction step: assume can tile $2^n \times 2^n$,
prove can handle $2^{n+1} \times 2^{n+1}$.



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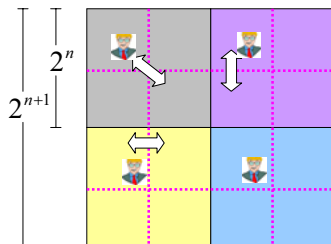
L2-2.8

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Gehry/Gates Plaza

Method: Divide into subsquares

Relocate the subsquares as indicated.



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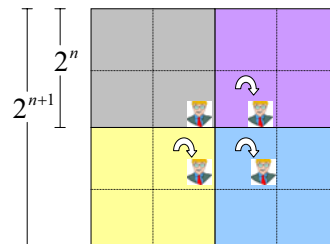
L2-2.9

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Gehry/Gates Plaza

Method: after relocation have:

Now rotate the squares as indicated.



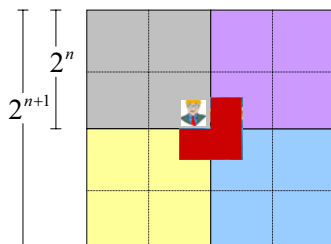
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6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Gehry/Gates Plaza

Method: after rotation have:

Now put tile in center



Done!

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L2-2.11

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Gehry/Gates Plaza

Did I fool you?

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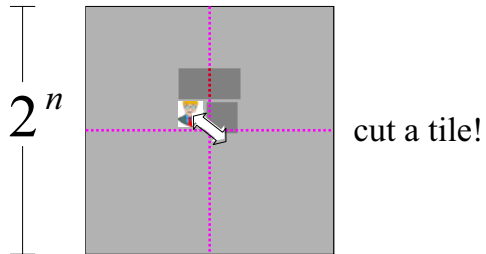
L2-2.12

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Gehry/Gates Plaza

Bug:

Division into subsquares may



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L2-2.13

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Gehry/Gates Plaza

The fix:

Prove that we can always find a tiling with Bill **in the corner**.

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L2-2.14

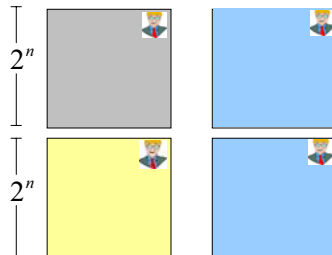
6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Gehry/Gates Plaza

Induction step:

Assume we can get Bill **in corner** of $2^n \times 2^n$.

Prove we can get Bill in corner of $2^{n+1} \times 2^{n+1}$.



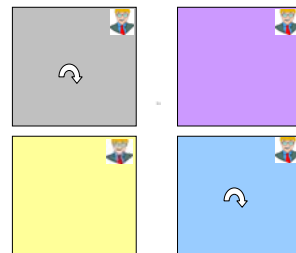
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L2-2.15

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Gehry/Gates Plaza

Method: Rotate the squares as indicated.



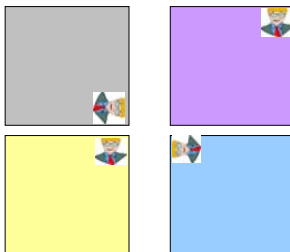
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L2-2.16

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Gehry/Gates Plaza

Method: Rotate the squares as indicated.
after rotation have:



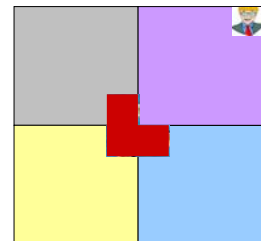
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L2-2.17

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Gehry/Gates Plaza

Method: Now group the squares together,
and fill the center with a tile.



Done!
For real!

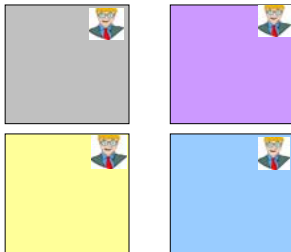
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L2-2.18

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Gehry/Gates Plaza

Note: Once have Bill in corner,
can get Bill in middle:



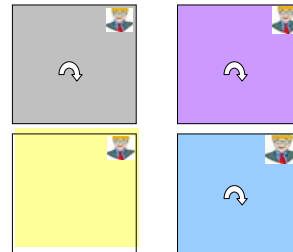
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L2-2.19

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Gehry/Gates Plaza

Method:
Rotate the squares as indicated.



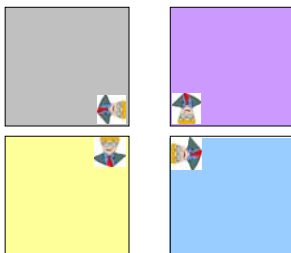
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L2-2.20

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Gehry/Gates Plaza

Method: after rotation have:



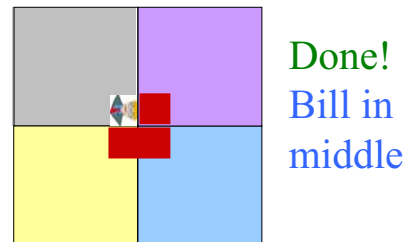
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L2-2.21

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

The Gehry/Gates Plaza

Method: Now group the 4 squares together,
and insert a tile.



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L2-2.22

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Ingenious Induction Hypotheses

Note 1: To prove
“Bill in middle,”
we *proved something else*: “Bill in corner.”

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L2-2.23

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Inductive (Recursive) Procedures

Note 2: The induction proof of
“Bill in corner” implicitly defines
a *recursive procedure* for
constructing a $2^{n+1} \times 2^{n+1}$ corner
tiling from a $2^n \times 2^n$ tiling.

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L2-2.24

6	9	10	7
12	10	5	
3	5	14	
15	8	11	4

Ingenious Induction Hypotheses

Note 3: Other times it helps to choose a *stronger hypothesis* than the desired result. Result at $n+1$ becomes harder to prove -- but we have a stronger hypothesis at n to prove it with!

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L2-2.25

6	9	10	7
12	10	5	
3	5	14	
15	8	11	4

Problems

Class Problem 1

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L2-2.26

6	9	10	7
12	10	5	
3	5	14	
15	8	11	4

A False Proof

Theorem: All horses are the same color.

Proof: (by induction on n)

Induction hypothesis:

$P(n) ::=$ any set of n horses have the same color

Base case ($n=0$):

No horses so *vacuously* true!



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L2-2.27

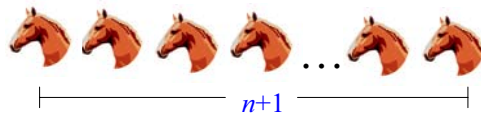
6	9	10	7
12	10	5	
3	5	14	
15	8	11	4

A False Proof

(Inductive case)

Assume any n horses have the same color.

Prove that any $n+1$ horses have the same color.



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L2-2.28

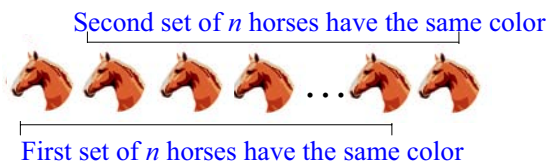
6	9	10	7
12	10	5	
3	5	14	
15	8	11	4

A False Proof

(Inductive case)

Assume any n horses have the same color.

Prove that any $n+1$ horses have the same color.



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L2-2.29

6	9	10	7
12	10	5	
3	5	14	
15	8	11	4

A False Proof

(Inductive case)

Assume any n horses have the same color.

Prove that any $n+1$ horses have the same color.



Therefore the set of $n+1$ have the same color!

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L2-2.30

6	9	13	7
12	10	5	
3	4	14	
15	8	11	2

A False Proof

What is wrong? $n=1$

Proof that $P(n) \rightarrow P(n+1)$
is **false** if $n=1$, because the two
horse groups *do not overlap*.



First set of $n=1$ horses

Second set of $n=1$ horses



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L2-2.31

6	9	13	7
12	10	5	
3	4	14	
15	8	11	2

A False Proof

Proof that $P(n) \rightarrow P(n+1)$
is **false** if $n=1$, because the two
horse groups *do not overlap*.

(But proof works for all $n \neq 1$)

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L2-2.32

6	9	13	7
12	10	5	
3	4	14	
15	8	11	2

Problems

Class Problems 2 & 3

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L2-2.33