Math 215 HW #3 Solutions

1. Problem 1.6.6. Use the Gauss–Jordan method to invert

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Solution: Start with the augmented matrix $[A_1 \; I]$:

$$\left[\begin{array}{ccc ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Then the only row on the left that doesn't already look like the identity matrix is the second row; we just need subtract rows 1 and 3 from row 2, which gives:

$$\left[\begin{array}{ccc ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Hence,

$$A_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

To find A_2^{-1}, start with the augmented matrix $[A_2 \; I]$:

$$\left[\begin{array}{ccc ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right].$$

Replace the first row by half of itself and add half of the first row to the second:

$$\left[\begin{array}{ccc ccc} 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right].$$

Next, add a third of the second row to the first, add 2/3 the second row to the third, and multiply the second row by 2/3:

$$\left[\begin{array}{ccc ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right].$$

Finally, multiply the third row by 3/4, then add 1/3 of the result to row 1 and add 2/3 of the result to row 2:

$$\left[\begin{array}{ccc ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right].$$

Thus,

$$A_2^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}.$$

To find A_3^{-1}, start with the augmented matrix $[A_3 \ I]$:

$$\left[\begin{array}{ccc ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

First, switch rows 1 and 3:

$$\left[\begin{array}{ccc ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right].$$

Now, subtract row 2 from row 1 and subtract row 3 from row 2:

$$\left[\begin{array}{ccc ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right].$$

Thus,

$$A_3^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

2. Problem 1.6.8. Show that $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ has no inverse by solving $Ax = 0$, and by failing to solve

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solution: Note that (as discussed in class on Friday),

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a solution of $Ax = 0$. The fact that this equation has such a solution implies that A is not invertible. To see this, note that if A were invertible, we could multiply both sides of the above equation by A^{-1}, yielding $x = A^{-1}0 = 0$. Since the given solution x is not zero, this is clearly impossible.

Another proof that A is not invertible is as follows. If A were invertible, then there would exist A^{-1} such that $AA^{-1} = I$. Assuming $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, this means

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This implies that

$$\begin{aligned} a + c &= 1 \\ b + d &= 0 \\ 3a + 3c &= 0 \\ 3b + 3d &= 1. \end{aligned}$$

The third equation can be re-written as $3(a+c) = 0$ or, dividing both sides by 3, as $a+c = 0$. But this directly contradicts the first equation, meaning that there is no solution to this system; equivalently, A is not invertible.

3. Problem 1.6.14. If B is square, show that $A = B + B^T$ is always symmetric and $K = B - B^T$ is always *skew-symmetric*—which means that $K^T = -K$. Find these matrices A and K when $B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$, and write B as the sum of a symmetric matrix and a skew-symmetric matrix.

   Solution: Suppose B is an $n \times n$ matrix with entries as indicated:

$$B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix}.$$

   Then

$$B^T = \begin{bmatrix} b_{11} & b_{21} & \dots & b_{n1} \\ b_{12} & b_{22} & \dots & b_{n2} \\ \vdots & \vdots & & \vdots \\ b_{1n} & b_{2n} & \dots & b_{nn} \end{bmatrix},$$

   so

$$A = B + B^T = \begin{bmatrix} 2b_{11} & b_{12} + b_{21} & \dots & b_{1n} + b_{n1} \\ b_{21} + b_{12} & 2b_{22} & \dots & b_{2n} + b_{n2} \\ \vdots & \vdots & & \vdots \\ b_{n1} + b_{1n} & b_{n2} + b_{2n} & \dots & 2b_{nn} \end{bmatrix},$$

   which is clearly a symmetric matrix.

   Likewise,

$$K = B - B^T = \begin{bmatrix} 0 & b_{12} - b_{21} & \dots & b_{1n} - b_{n1} \\ b_{21} - b_{12} & 0 & \dots & b_{2n} - b_{n2} \\ \vdots & \vdots & & \vdots \\ b_{n1} - b_{1n} & b_{n2} - b_{2n} & \dots & 0 \end{bmatrix},$$

   which is certainly skew-symmetric.

   Now, when $B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$, we get

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}, \qquad K = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

Now, notice that

$$A + K = (B + B^T) + (B - B^T) = 2B;$$

this suggests that we try adding $\frac{1}{2}A$ and $\frac{1}{2}K$:

$$\frac{1}{2}A + \frac{1}{2}K = \frac{1}{2}(B + B^T) + \frac{1}{2}(B - B^T) = B.$$

Therefore, using the computed A and K from above,

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is indeed the sum of a symmetric matrix and a skew-symmetric matrix.

4. Problem 1.6.18. Under what conditions on their entries are A and B invertible?

$$A = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$$

Solution: Since square a matrix is invertible if and only if elimination yields the same number of pivots as rows, we just need to do elimination on A and B and see what conditions on their entries ensure that we get a pivot in every row.

First, we do elimination on A. Notice that, if $f = 0$, then the third row is all zeros and there can never be a third pivot. So it must be the case that $f \neq 0$ if A is invertible. This then ensures there is a pivot in the first column; to make the pivot actually occur at f, switch rows 1 and 3:

$$\begin{bmatrix} f & 0 & 0 \\ d & e & 0 \\ a & b & c \end{bmatrix}.$$

Now, subtract $\frac{d}{f}$ times row 1 from row 2 and subtract $\frac{a}{f}$ times row 1 from row 3 (note that these fractions are well-defined because $f \neq 0$):

$$\begin{bmatrix} f & 0 & 0 \\ 0 & e & 0 \\ 0 & b & c \end{bmatrix}.$$

If $e = 0$ then the second row is all zeros, meaning that there can never be a pivot in that row. Thus, if A is invertible, it must be the case that $e \neq 0$. This then implies that there *is* a pivot in the second column, and we can eliminate the entry below it by subtracting $\frac{b}{e}$ times row 2 from row 3 (note that this fraction is well-defined because $e \neq 0$):

$$\begin{bmatrix} f & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & c \end{bmatrix}.$$

We already know there are pivots in the first two rows; there will be a pivot in the third row only if $c \neq 0$. Hence, if A is to be invertible, it must be the case that $c \neq 0$. Therefore, the conditions which ensure that A is invertible are:

$$c \neq 0, \quad e \neq 0, \quad f \neq 0.$$

Turning to B, note that the third row will be all zeros (and, thus, never have a pivot) unless $e \neq 0$. Hence, if B is to be invertible, it must be the case that $e \neq 0$. Also, if there is to be a pivot in the first column, then either a or c must be nonzero. If a is nonzero, then we can eliminate c by subtracting $\frac{c}{a}$ times row 1 from $row2$ (which is well-defined since $a \neq 0$):

$$\begin{bmatrix} a & b & 0 \\ 0 & d - \frac{cb}{a} & 0 \\ 0 & 0 & e \end{bmatrix}.$$

Then, in order to have a pivot in the second row, it must be the case that

$$d - \frac{cb}{a} \neq 0$$

or, equivalently,

$$ad - bc \neq 0.$$

On the other hand, if $c \neq 0$, so we can switch rows 1 and 2 to get

$$\begin{bmatrix} c & d & 0 \\ a & b & 0 \\ 0 & 0 & e \end{bmatrix}.$$

Then we can eliminate a by subtracting $\frac{a}{c}$ times row 1 from row 2 (which is well-defined since $c \neq 0$):

$$\begin{bmatrix} c & d & 0 \\ 0 & b - \frac{ad}{c} & 0 \\ 0 & 0 & e \end{bmatrix}.$$

Again, if we are to have a pivot in the second row, it must be the case that

$$b - \frac{ad}{c} \neq 0$$

or, equivalently,

$$bc - ad \neq 0.$$

Therefore, either $a \neq 0$ and $ad - bc \neq 0$, or $c \neq 0$ and $bc - ad \neq 0$. However, $ad - bc \neq 0$ is equivalent to $bc - ad \neq 0$, and this inequality requires that either a or c is nonzero (if both were zero then the left hand side would be zero). Hence, the simplified conditions under which A is invertible are:

$$ad - bc \neq 0 \quad \text{and} \quad e \neq 0.$$

5. Problem 1.6.26. If A has column 1 + column 2 = column 3, show that A is not invertible:

   (a) Find a nonzero solution x to $Ax = 0$. The matrix is 3 by 3.

   (b) Elimination keeps column 1 + column 2 = column 3. Explain why there is no third pivot.

Solution: For part (a), suppose

$$A = \begin{bmatrix} a & b & a+b \\ c & d & c+d \\ e & f & e+f \end{bmatrix}.$$

Then, if $x = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, we have that

$$Ax = \begin{bmatrix} a & b & a+b \\ c & d & c+d \\ e & f & e+f \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a+b-(a+b) \\ c+d-(c+d) \\ e+f-(e+f) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, $Ax = 0$, so A is not invertible.

For part (b), since elimination keeps column 1 + column 2 = column 3 and since, after elimination, the first and second entries in the third row will be zero, we have that the third entry must equal $0+0=0$. Thus, the whole third row is zero, so there is no third pivot.

6. Problem 1.6.38. Invert these matrices A by the Gauss–Jordan method starting with $[A\ I]$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Solution: For the first choice of A, we write the augmented matrix $[A\ I]$:

$$\begin{bmatrix} 1 & 0 & 0 & & 1 & 0 & 0 \\ 2 & 1 & 3 & & 0 & 1 & 0 \\ 0 & 0 & 1 & & 0 & 0 & 1 \end{bmatrix}.$$

Then subtracting two times row 1 from row 2 and subtracting three times row 3 from row 2 yields

$$\begin{bmatrix} 1 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 1 & 0 & & -2 & 1 & -3 \\ 0 & 0 & 1 & & 0 & 0 & 1 \end{bmatrix}.$$

Hence,

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

For the second choice of A, write the augmented matrix $[A\ I]$:

$$\begin{bmatrix} 1 & 1 & 1 & & 1 & 0 & 0 \\ 1 & 2 & 2 & & 0 & 1 & 0 \\ 1 & 2 & 3 & & 0 & 0 & 1 \end{bmatrix}.$$

Subtracting row 1 from rows 2 and 3 yields:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right].$$

In turn, subtracting row 2 from rows 1 and 3 yields:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right].$$

Finally, subtracting row 3 from row 2 yields:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right].$$

Hence,

$$A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

7. Problem 1.6.40. True or false (with a counterexample if false and a reason if true):

(a) A 4 by 4 matrix with a row of zeros is not invertible.

Solution: True. There can never be a pivot in a row of all zeros, so the matrix can have at most 3 pivots and hence cannot be invertible.

(b) A matrix with 1s down the main diagonal is invertible.

Solution: False. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. This matrix has 1s down the main diagonal, but it's clear that the first elimination step will yield all zeros in the second row, so this A is not invertible.

(c) If A is invertible then A^{-1} is invertible.

Solution: True. Since $AA^{-1} = I$ and $A^{-1}A = I$, we see that A is the inverse of A^{-1} (i.e. $(A^{-1})^{-1} = A$), and so A^{-1} is invertible.

(d) If A^T is invertible then A is invertible.

Solution: True. From Equation 1M in the textbook,

$$(A^T)^{-1} = (A^{-1})^T,$$

or, in other words,

$$\left[(A^T)^{-1}\right]^T = A^{-1},$$

so A is invertible.

8. Problem 1.6.52. Show that $A^2 = 0$ is possible but $A^T A = 0$ is not possible (unless $A =$ zero matrix).

   Solution: First, note that if $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, then $A \neq 0$, but

$$A^2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

   So we see that $A^2 = 0$ is possible even if $A \neq 0$.

   On the other hand, to show that $A^T A \neq 0$ whenever $A \neq 0$, suppose A is an $m \times n$ matrix which is not equal to zero. Writing it out, we have that

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

   where $a_{jk} \neq 0$ for at least one choice of j and k. Hence,

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}.$$

   Therefore,

$$\begin{aligned} A^T A &= \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^m a_{i1}^2 & \sum_{i=1}^m a_{i1}a_{i2} & \dots & \sum_{i=1}^m a_{i1}a_{in} \\ \sum_{i=1}^m a_{i2}a_{i1} & \sum_{i=1}^m a_{i2}^2 & \dots & \sum_{i=1}^m a_{i2}a_{in} \\ \vdots & \vdots & & \vdots \\ \sum_{i=1}^m a_{in}a_{i1} & \sum_{i=1}^m a_{in}a_{i2} & \dots & \sum_{i=1}^m a_{in}^2 \end{bmatrix}. \end{aligned}$$

   Notice that the diagonal entries are all sums of squares and so are all non-negative. Moreover, since $a_{jk} \neq 0$, we have that

$$\sum_{i=1}^m a_{ik}^2 = a_{1k}^2 + a_{2k}^2 + \ldots + a_{jk}^2 + \ldots + a_{mk^2} > 0.$$

   Since this is exactly the diagonal entry in the kth row of $A^T A$, we see that, indeed, $A^T A \neq 0$.

9. Problem 1.7.4. Write down the 3 by 3 finite-difference matrix equation $\left(h = \frac{1}{4}\right)$ for

$$-\frac{d^2 u}{dx^2} + u = x, \qquad u(0) = u(1) = 0.$$

Solution: First, note that we approximate the second derivative by

$$\frac{\Delta^2 u}{\Delta x^2} = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}.$$

Also, since we can only measure u at the mesh points $x = jh$, we substitute jh for x and use the approximation for the second derivative to get

$$-\frac{u(jh+h) - 2u(jh) + u(jh-h)}{h^2} + u(jh) = jh.$$

Multiplying both sides by h^2 and using the notation $u_k = u(kh)$ yields the finite-difference equation

$$-u_{j+1} + 2u_j - u_{j-1} + h^2 u_j = h^2 jh.$$

Combining terms and using the fact that $h = 1/4$ yields

$$-u_{j+1} + \left(2 + \frac{1}{16}\right) u_j - u_{j-1} = \frac{j}{64}.$$

When $j = 1$, this yields the equation

$$-u_2 + \frac{33}{16} u_1 - u_0 = \frac{1}{64}.$$

Since $u_0 = u(0) = 0$, this simplifies to

$$-u_2 + \frac{33}{16} u_1 = \frac{1}{64}. \tag{1}$$

When $j = 2$, the difference equation becomes

$$-u_3 + \frac{33}{16} u_2 - u_1 = \frac{2}{64}. \tag{2}$$

When $j = 3$, the difference equation becomes

$$-u_4 + \frac{33}{16} u_3 - u_2 = \frac{3}{64}.$$

Since $u_4 = u(1) = 0$, this simplifies as

$$\frac{33}{16} u_3 - u_2 = \frac{3}{64}. \tag{3}$$

We can combine equations (1), (2), and (3) into the single matrix equation

$$\begin{bmatrix} \frac{33}{16} & -1 & 0 \\ -1 & \frac{33}{16} & -1 \\ 0 & -1 & \frac{33}{16} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{64} \\ \frac{2}{64} \\ \frac{3}{64} \end{bmatrix}.$$