Q1. **(Eigenvalues and eigenvectors)** Determine whether the vectors $\mathbf{v}_1 = (1,2),\ \mathbf{v}_2 = (3,4),\ \mathbf{v}_3 = (5,6)$ are eigenvectors of the given matrix $\mathbf{A}$. If so, what are the eigenvalues?

$$\mathbf{A} = \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}.$$

Solution Note that

$$\begin{aligned}
\mathbf{A}\mathbf{v}_1 &= \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \neq \lambda\mathbf{v}_1 \quad \text{for any } \lambda, \\
\mathbf{A}\mathbf{v}_2 &= \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} = 2\mathbf{v}_2, \\
\mathbf{A}\mathbf{v}_3 &= \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix} \neq \lambda\mathbf{v}_3 \quad \text{for any } \lambda.
\end{aligned}$$

Therefore, $\mathbf{v}_1$ is not an eigenvector of $\mathbf{A}$, $\mathbf{v}_2$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2, $\mathbf{v}_3$ is not an eigenvector of $\mathbf{A}$.

Q2. **(Diagonalization)** Verify the vectors $\mathbf{v}_1 = (1,-1,1),\ \mathbf{v}_2 = (1,1,0),\ \mathbf{v}_3 = (1,0,-1)$ are eigenvectors of the given matrix $\mathbf{A}$. Then find a diagonalization of the matrix.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Solution Note that

$$\begin{aligned}
\mathbf{A}\mathbf{v}_1 &= \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = -2\mathbf{v}_1, \\
\mathbf{A}\mathbf{v}_2 &= \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \mathbf{v}_2, \\
\mathbf{A}\mathbf{v}_3 &= \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \mathbf{v}_3.
\end{aligned}$$

Therefore, $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$ are eigenvectors of $\mathbf{A}$ corresponding to eigenvalues $\lambda_1 = -2,\ \lambda_2 = 1,\ \lambda_3 = 1$, respectively. Since $\mathbf{v}_2$ and $\mathbf{v}_3$ (corresponding to the same eigenvalue) are not parallel, we have three linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of $\mathbf{A}$. Thus $\mathbf{A}$ is diagonalizable. A diagonalization of $\mathbf{A}$ is

$$\boxed{\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}^{-1}.}$$

Q3. **(Diagonalizability)** Show that if $\mathbf{A}$ is diagonalizable, then $\mathbf{A}^2$ is also diagonalizable.

Solution If $\mathbf{A}$ is diagonalizable, then $\mathbf{A} = \mathbf{PDP}^{-1}$ for some invertible matrix $\mathbf{P}$ and diagonal matrix $\mathbf{D}$. Thus,

$$\mathbf{A}^2 = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) = \mathbf{PD}^2\mathbf{P}^{-1}.$$

Since $\mathbf{D}$ is diagonal, we can write

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Then

$$\mathbf{D}^2 = \begin{bmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^2 \end{bmatrix}$$

is also a diagonal matrix. By definition, $\mathbf{A}^2 = \mathbf{PD}^2\mathbf{P}^{-1}$ is a diagonalization of $\mathbf{A}^2$ and hence $\mathbf{A}^2$ is also diagonalizable.

Q4. **(Eigenvalues and eigenvectors)**

(a) Find an example of 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$, such that $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvectors but distinct eigenvalues.

Solution Let a, b, c be real, distinct. Consider the diagonal matrix

$$\operatorname{diag}(a, b, c) \overset{\text{def.}}{=} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Then one can always choose $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ be the eigenvectors of the matrix, with distinct eigenvalues a, b, c. In particular, we may take

$$\boxed{\mathbf{A} = \operatorname{diag}(1, 2, 3), \qquad \mathbf{B} = \operatorname{diag}(4, 5, 6)}$$

for an example.

(b) Find an example of 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$, such that $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues but distinct eigenvectors.

Solution Let $\mathbf{A}$ be an 3×3 matrix having three distinct real eigenvalues λ_1, λ_2, λ_3. Also let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be the corresponding (linearly independent) eigenvectors of $\mathbf{A}$. Then $\mathbf{A}$ is diagonalizable and $\mathbf{A} = \mathbf{PBP}^{-1}$, where $\mathbf{P} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ is invertible and $\mathbf{B} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$ is diagonal. Thus, $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues (i.e., λ_1, λ_2, λ_3) but distinct eigenvectors (since one can always choose $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be the eigenvectors of $\mathbf{B}$). In particular, we may take

$$\boxed{\mathbf{A} = \begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix}, \qquad \mathbf{B} = \operatorname{diag}(-1, 0, 3).}$$

Q5. **(Diagonalizable matrix)** Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 4 & 0 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}, \qquad \mathbf{P} = \begin{bmatrix} 1 & 0.1 & 0.01 & 0.001 & 0.0001 \\ 0 & 1 & 10 & 100 & 1000 \\ 0 & 0 & e & e^2 & e^3 \\ 0 & 0 & 0 & \pi & \sqrt{\pi} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Is $\mathbf{A}$ diagonalizable? Find the eigenvalues of $\mathbf{P}^{-1}\mathbf{A}^{2006}\mathbf{P}$.

<u>**Solution**</u> $\mathbf{A}$ is a lower-triangular matrix of which the eigenvalues are given by the diagonal entries (i.e., 1, 2, 3, 4, 5 being the eigenvalues of $\mathbf{A}$). Since $\mathbf{A}$ has five distinct eigenvalues, $\mathbf{A}$ is diagonalizable. That is, there exist an invertible $\mathbf{Q}$ and a diagonal $\mathbf{D}$ such that $\mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^{-1}$, where $\mathbf{D} = \mathrm{diag}\,(1,2,3,4,5)$. It follows that $\mathbf{A}^{2006} = \mathbf{Q}\mathbf{D}^{2006}\mathbf{Q}^{-1}$. Remark that the given matrix $\mathbf{P}$ is invertible. Multiplying $\mathbf{P}^{-1}$ to the left and $\mathbf{P}$ to the right, the matrix $\mathbf{P}^{-1}\mathbf{A}^{2006}\mathbf{P}$ is diagonalizable because $\mathbf{P}^{-1}\mathbf{A}^{2006}\mathbf{P} = \bar{\mathbf{Q}}\mathbf{D}^{2006}\bar{\mathbf{Q}}^{-1}$, where $\bar{\mathbf{Q}} = \mathbf{P}^{-1}\mathbf{Q}$ is invertible and $\mathbf{D}^{2006}$ is diagonal. Therefore, $\mathbf{P}^{-1}\mathbf{A}^{2006}\mathbf{P}$ has the same eigenvalues of $\mathbf{D}^{2006}$, which are

$$\boxed{1, \qquad 2^{2006}, \qquad 3^{2006}, \qquad 4^{2006}, \qquad 5^{2006}.}$$

Q6. **(Diagonalizable matrix)** Find eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Write down the diagonalization of $\mathbf{A}$ if it is diagonalizable.

<u>**Solution**</u> The characteristic equation of $\mathbf{A}$ is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, or

$$\det \begin{bmatrix} -\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} = (-\lambda)^2(1-\lambda) + (1-\lambda) = (1-\lambda)(\lambda^2+1) = 0$$

which gives the eigenvalues

$$\boxed{\lambda_1 = 1, \qquad \lambda_2 = i, \qquad \lambda_3 = -i.}$$

· For $\lambda_1 = 1$,

$$\mathbf{A} - 1\cdot\mathbf{I} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

By solving $(\mathbf{A} - 1\cdot\mathbf{I})\,\mathbf{x} = \mathbf{0}$, $\mathbf{x} = (x_1,\, x_2,\, x_3)$, we have $x_1 = x_3 = 0$. Thus,

$$\mathbf{x} = (0,\, x_2,\, 0) = x_2\,(0,\, 1,\, 0),$$

we get the eigenvector

$$\boxed{\mathbf{v}_1 = (0,\, 1,\, 0).}$$

· For $\lambda_2 = i$,

$$\mathbf{A} - i\cdot\mathbf{I} = \begin{bmatrix} -i & 0 & -1 \\ 0 & 1-i & 0 \\ 1 & 0 & -i \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By solving $(\mathbf{A} - i\cdot\mathbf{I})\,\mathbf{x} = \mathbf{0}$, $\mathbf{x} = (x_1,\, x_2,\, x_3)$, we have $x_1 = i\,x_3$ and $x_2 = 0$. Thus,

$$\mathbf{x} = (i\,x_3,\, 0,\, x_3) = x_3\,(i,\, 0,\, 1),$$

we get the eigenvector

$$\boxed{\mathbf{v}_2 = (i,\, 0,\, 1).}$$

Similarly, for $\lambda_3 = -i$, we get the eigenvector

$$\boxed{\mathbf{v}_3 = (-i,\ 0,\ 1).}$$

Since $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ form a basis of the complex Euclidean space $\mathbb{C}^3$, we get the diagonalization of $\mathbf{A}$

$$\boxed{\mathbf{A} = \begin{bmatrix} 0 & i & -i \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix} \begin{bmatrix} 0 & i & -i \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1}.}$$

Q7. **(Diagonalizable matrix)** Find eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix}$. Write down the diagonalization of $\mathbf{A}$ if it is diagonalizable.

<u>Solution</u> The characteristic equation of $\mathbf{A}$ is $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$, or

$$\begin{aligned} \det \begin{bmatrix} 2-\lambda & 2 & -2 \\ -5 & 1-\lambda & 2 \\ -2 & 4 & -1-\lambda \end{bmatrix} &\overset{C_2+C_3}{=} \det \begin{bmatrix} 2-\lambda & 2 & 0 \\ -5 & 1-\lambda & 3-\lambda \\ -2 & 4 & 3-\lambda \end{bmatrix} \\ &\overset{-R_3+R_2}{=} \det \begin{bmatrix} 2-\lambda & 2 & 0 \\ -3 & -3-\lambda & 0 \\ -2 & 4 & 3-\lambda \end{bmatrix} \\ &= (3-\lambda)\,[(2-\lambda)(-3-\lambda)+6] \\ &= (3-\lambda)\,\lambda\,(1+\lambda) \;=\; 0 \end{aligned}$$

which gives the distinct real eigenvalues

$$\boxed{\lambda_1 = -1, \qquad \lambda_2 = 0, \qquad \lambda_3 = 3.}$$

· For $\lambda_1 = -1$,

$$\mathbf{A} - (-1)\cdot \mathbf{I} = \begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}.$$

By solving $(\mathbf{A} - (-1)\cdot \mathbf{I})\,\mathbf{x} = \mathbf{0}$, $\mathbf{x} = (x_1,\ x_2,\ x_3)$, we have $x_1 = x_3/2$ and $x_2 = x_3/4$. Thus,

$$\mathbf{x} = (\frac{x_3}{2},\ \frac{x_3}{4},\ x_3) = \frac{x_3}{4}\,(2,\ 1,\ 4),$$

we get the eigenvector

$$\boxed{\mathbf{v}_1 = (2,\ 1,\ 4).}$$

· For $\lambda_2 = 0$,

$$\mathbf{A} - 0\cdot \mathbf{I} = \begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

By solving $(\mathbf{A} - 0\cdot \mathbf{I})\,\mathbf{x} = \mathbf{0}$, $\mathbf{x} = (x_1,\ x_2,\ x_3)$, we have $x_1 = x_3/2$ and $x_2 = x_3/2$. Thus,

$$\mathbf{x} = (\frac{x_3}{2},\ \frac{x_3}{2},\ x_3) = \frac{x_3}{2}\,(1,\ 1,\ 2),$$

we get the eigenvector

$$\boxed{\mathbf{v}_2 = (1,\ 1,\ 2).}$$

· For $\lambda_3 = 3$,

$$\mathbf{A} - 3\cdot \mathbf{I} = \begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

By solving $(\mathbf{A} - 3 \cdot \mathbf{I})\,\mathbf{x} = \mathbf{0}$, $\mathbf{x} = (x_1, x_2, x_3)$, we have $x_1 = 0$ and $x_2 = x_3$. Thus,

$$\mathbf{x} = (0,\, x_3,\, x_3) = x_3\,(0,\, 1,\, 1),$$

we get the eigenvector

$$\boxed{\mathbf{v}_3 = (0,\, 1,\, 1).}$$

Since $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ form a basis of the Euclidean space $\mathbb{R}^3$, we get the diagonalization of $\mathbf{A}$

$$\boxed{\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}^{-1}.}$$

Q8. **(Non-diagonalizable matrix)** Find eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Write down the diagonalization of $\mathbf{A}$ if it is diagonalizable.

Solution The characteristic equation of $\mathbf{A}$ is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, or

$$\det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^3 = 0$$

which gives the one and only one eigenvalue

$$\boxed{\lambda = 1 \quad (\lambda \text{ being a repeated eigenvalue}).}$$

· For $\lambda = 1$,

$$\mathbf{A} - 1 \cdot \mathbf{I} = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

By solving $(\mathbf{A} - 1 \cdot \mathbf{I})\,\mathbf{x} = \mathbf{0}$, $\mathbf{x} = (x_1, x_2, x_3)$, we have $x_2 = x_3 = 0$. Thus,

$$\mathbf{x} = (x_1,\, 0,\, 0) = x_1\,(1,\, 0,\, 0),$$

we get the eigenvector

$$\boxed{\mathbf{v} = (1,\, 0,\, 0).}$$

Note that the eigenvector $\mathbf{v}$ forms a basis for $\mathrm{Nul}\,(\mathbf{A} - 1 \cdot \mathbf{I})$, but it does not form a basis for the Euclidean space $\mathbb{R}^3$. We have not enough linearly independent eigenvectors for spanning $\mathbb{R}^3$. There is no way to construct an invertible $\mathbf{P}$ that diagonalizes $\mathbf{A}$ and hence $\mathbf{A}$ is not diagonalizable.