Math 215 Exam #1 Practice Problems

1. For each of the following statements, say whether it is true or false. If the statement is true, prove it. If false, give a counterexample.

   (a) If A is a 2×2 matrix such that $A(Ax) = 0$ for all $x \in \mathbb{R}^2$, then A is the zero matrix.

   (b) A system of 3 equations in 4 unknowns can never have a unique solution.

   (c) If V is a vector space and S is a finite set of vectors in V, then some subset of S forms a basis for V.

   (d) Suppose A is an $m \times n$ matrix such that $A\mathbf{x} = \mathbf{b}$ can be solved for any choice of $\mathbf{b} \in \mathbb{R}^m$. Then the columns of A form a basis for $\mathbb{R}^m$.

   (e) Given 3 equations in 4 unknowns, each describes a hyperplane in $\mathbb{R}^4$. If the system of those 3 equations is consistent, then the intersection of the hyperplanes contains a line.

   (f) If A is a symmetric matrix (i.e. $A = A^T$), then A is invertible.

   (g) If $m < n$ and A is an $m \times n$ matrix such that $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$, then there exists $\mathbf{z} \in \mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{z}$ has infinitely many solutions.

   (h) The set of polynomials of degree ≤ 5 forms a vector space.

2. For each of the following, determine whether the given subset is a subspace of the given vector space. Explain your answer.

   (a) **Vector Space:** $\mathbb{R}^4$.
   Subset: The vectors of the form
   $$\begin{bmatrix} a \\ b \\ 0 \\ d \end{bmatrix}.$$

   (b) **Vector Space:** $\mathbb{R}^2$.
   Subset: The solutions to the equation $2x - 5y = 11$.

   (c) **Vector Space:** $\mathbb{R}^n$.
   Subset: All $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = 2x$ where A is a given $n \times n$ matrix.

   (d) **Vector Space:** $\mathbb{R}^3$.
   Subset: The intersection of P_1 and P_2, where P_1 and P_2 are planes through the origin.

   (e) **Vector Space:** All polynomials.
   Subset: The quadratic (i.e. degree 2) polynomials.

   (f) **Vector Space:** All real-valued functions.
   Subset: Functions of the form $f(t) = a\cos t + b\sin t + c$ for $a, b, c \in \mathbb{R}$.

3. Consider the matrix
   $$A = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}.$$

   (a) Under what conditions on a is A invertible?

   (b) Choose a non-zero value of a that makes A invertible and determine A^{-1}.

   (c) For each value of a that makes A non-invertible, determine the dimension of the nullspace of A.

4. Consider the system of equations

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & x_3 & - & 3x_4 & = & b_1 \\ x_1 & + & 2x_2 & + & 2x_3 & - & 5x_4 & = & b_2 \\ 2x_1 & + & 4x_2 & + & 3x_3 & - & 8x_4 & = & b_3 \end{array}.$$

(a) Find all solutions when the above system is homogeneous (i.e. $b_1 = b_2 = b_3 = 0$). Find a basis for the space of solutions to the homogeneous system.

(b) Let S be the set of vectors $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that the system can be solved. What is the dimension of S?

(c) It's easy to check that the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ is a solution to the system that arises when $b_1 = 3$, $b_2 = 5$, and $b_3 = 8$. Find *all* the solutions to this system.