Math 215 Exam #2 Practice Problems

1. For each of the following statements, say whether it is true or false. If the statement is true, prove it. If false, give a counterexample.

   (a) If Q is an orthogonal matrix, then $\det Q = 1$.

   (b) Every invertible matrix can be diagonalized.

   (c) Every diagonalizable matrix is invertible.

   (d) If the matrix A is not invertible, then 0 is an eigenvalue of A.

   (e) If $\vec{v}$ and $\vec{w}$ are orthogonal and P is a projection matrix, then $P\vec{v}$ and $P\vec{w}$ are also orthogonal.

   (f) Suppose A is an $n \times n$ matrix and that there exists some k such that $A^k = 0$ (such matrices are called *nilpotent* matrices). Then A is not invertible.

2. Let Q be an $n \times n$ orthogonal matrix. Show that if $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$, then so is $\{Q\vec{v}_1, \ldots, Q\vec{v}_n\}$.

3. Let
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

   (a) Let R be the region in the plane enclosed by the unit circle. If T is the linear transformation of the plane whose matrix is A, what is the area of $T(R)$?

   (b) Find the matrix for the transformation T^{-1} *without* doing elimination.

4. Let ℓ be the line in $\mathbb{R}^3$ through the vector $\vec{a} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

   (a) Find a basis for the orthogonal complement of ℓ.

   (b) If $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, write $\vec{v}$ as a sum
$$\vec{v} = \vec{v}_1 + \vec{v}_2,$$
   where $\vec{v}_1 \in \ell$ and $\vec{v}_2 \in \ell^\perp$.

5. Find the line $C + Dt$ that best fits the data
$$(-1, 1),\ (0, 1),\ (1, 2).$$

6. Let ℓ be the line through a vector $\vec{a} \in \mathbb{R}^n$ and let P be the matrix which projects everything in $\mathbb{R}^n$ to ℓ.

   (a) Show that the trace of P equals 1.

   (b) What can you say about the eigenvalues of P?

7. Suppose A is a 2×2 matrix with eigenvalues λ_1 and λ_2 corresponding to non-zero eigenvectors $\vec{v}_1$ and $\vec{v}_2$, respectively. If $\lambda_1 \neq \lambda_2$, show that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.