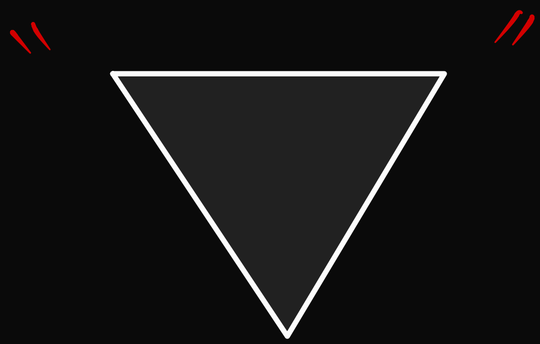
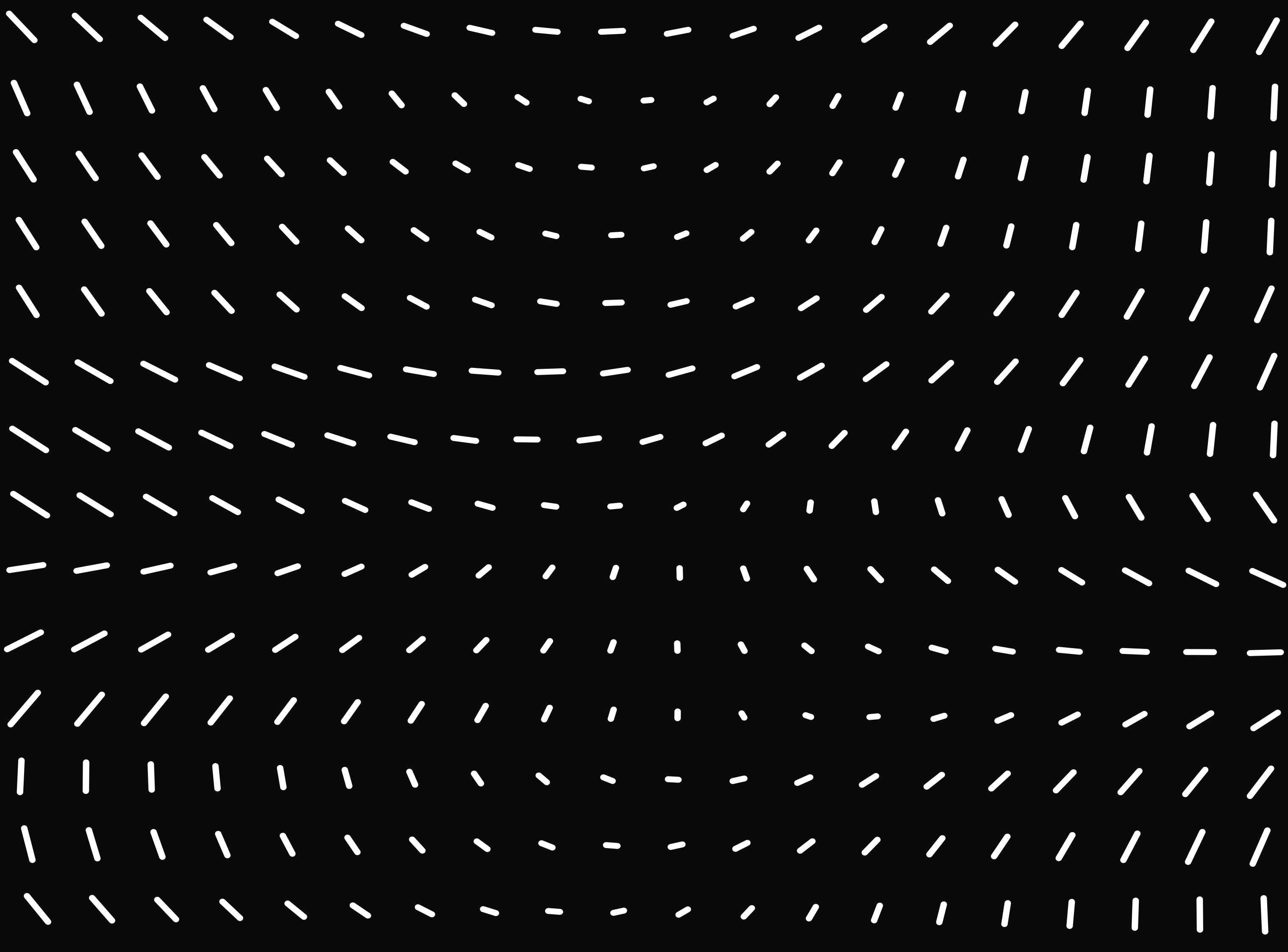
Div, Grad & Curl





PDE's & Vector Calculus

Conservation Laws
Mass, Momentum & Energy
$\Downarrow$

Vector calc
∇ Grad
$\nabla \cdot$ Div
$\nabla \times$ Curl

$\left.\begin{array}{l} \nabla \text{ Grad} \\ \nabla\cdot \text{ Div} \\ \nabla\times \text{ Curl} \end{array}\right\}$ is language for PDE

Grad

'Del' 'Nabla'

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

or

$$\nabla = \begin{array}{cc} \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} & \begin{array}{c} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \end{array}$$

Scalar $f \longrightarrow$ vector field

Grad $f(x,y,z)$

$$\nabla f = \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{bmatrix}$$

It gives a vector direction at every pt of how that temperature is increasing the fastest along what direction.

Temp distribution

it shows directions where it max go from cold to hot

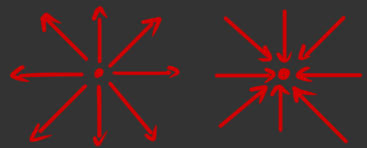
Div

vector $\vec{f}$ $\longrightarrow$ Scalar field $\qquad \nabla \cdot \vec{f}$

$(\vec{f} = \hat{i} f_1 + \hat{j} f_2 + \hat{k} f_3)$

$$\nabla \cdot \vec{f} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

It computes how much vector field is kind of instantaneously or locally expanding outward or contracting inward.



If its going to be greater than zero if vector field is kind of sourcing out blowing stuff away from one point and

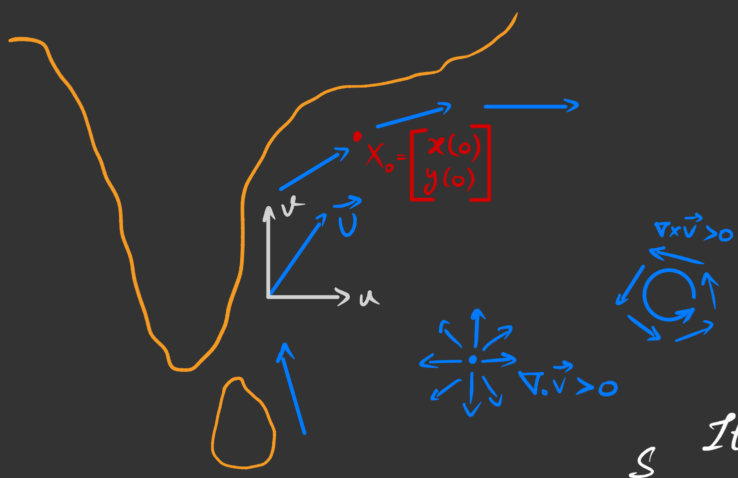
- Its negative if its pulling stuff in or kind of attracting stuff in into the vector field.
- If $\nabla \cdot \vec{f} = 0$, its is called incompressible.

Curl

vector f $\longrightarrow$ vector field $\qquad \nabla \times \vec{f}$

$$\nabla \times \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \hat{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

- Curl measure how much stuff is kind of swirling around in a circle, it measures kind of the circulation around a point



$$\vec{U}(x,t) = \begin{bmatrix} u(x,y,t) \\ v(x,y,t) \end{bmatrix}$$ vector field

s It is solution of PDE (i.e Navier stokes)

one PDE

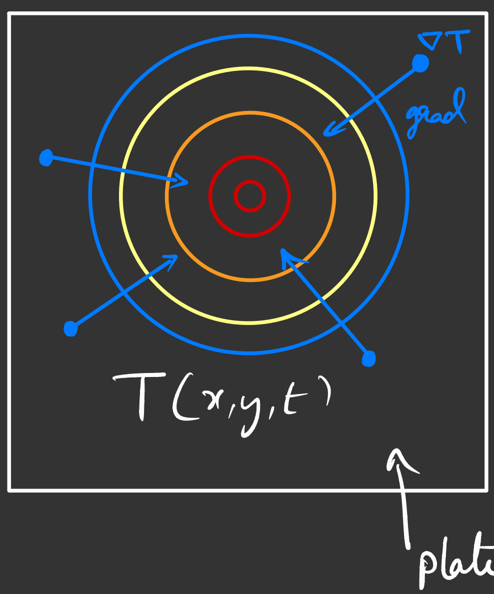
$$\frac{\partial \vec{U}}{\partial t} = N(\vec{U}, t)$$

Non-linear func

we can also find the path the lost person in the ocean, since the person takes the dynamics of the flow.

$$\frac{d\vec{X}_0}{dt} = \vec{U}(\vec{x}, t)$$

ex:



it gives a vector field where the Temp increasing the fastest.

$$\frac{\partial T}{\partial t} = \alpha^2 \nabla^2 T$$
$$= \alpha^2 \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

plate heated with Blow torch at centre

Grad

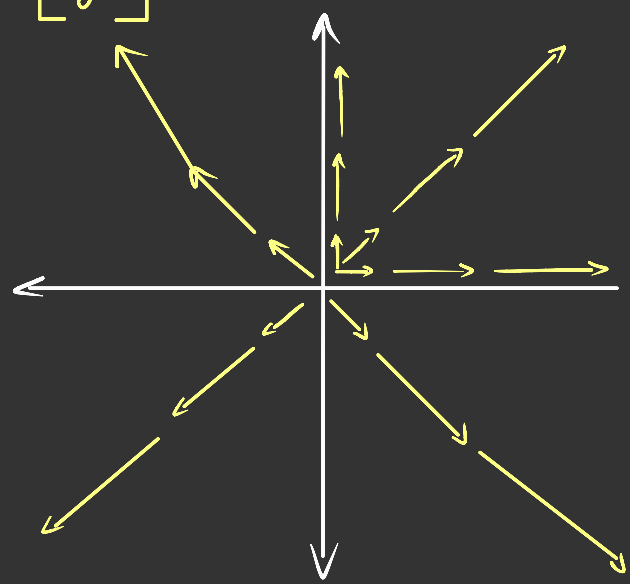
$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} f$$

Scalar $f(x,y)$

$$\nabla f = \text{grad}(f) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

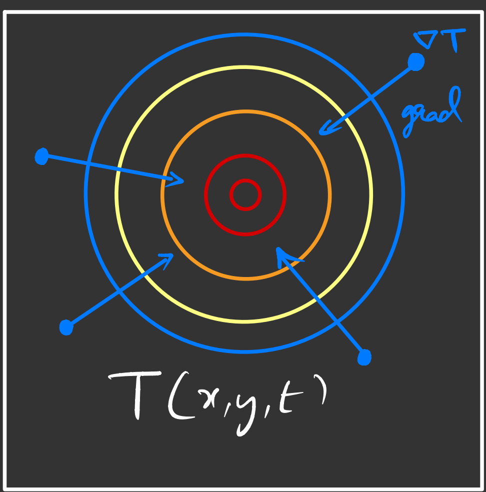
Let $f = x^2 + y^2$

$$\nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$



Grad is Linear : $\nabla(f_1 + f_2) = \nabla f_1 + \nabla f_2$

$$\nabla(\alpha f) = \alpha \nabla f$$



$$\nabla T = \begin{bmatrix} \partial T/\partial x \\ \partial T/\partial y \end{bmatrix}$$

Plate with T hot at centre

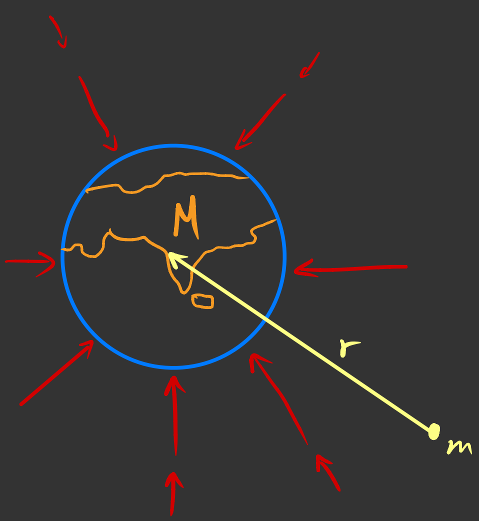
- Gradient provide fastest way to approach the highest T, so it provides optimum value.

- Directional Derivative : Derivative of f in a direction $\vec{v}$

$$D_{\vec{v}} f = \frac{1}{\|\vec{v}\|} \vec{v} \cdot (\nabla f)$$

Normalize $\vec{v}$

<u>Object in Gravitational field :</u>



Newtons Law

$$\vec{F} = -\frac{mMG}{r^3}\vec{r} = -\nabla V$$

Gravitational Energy

$$V = -\frac{mMG}{r}$$

Divergence $\nabla.$

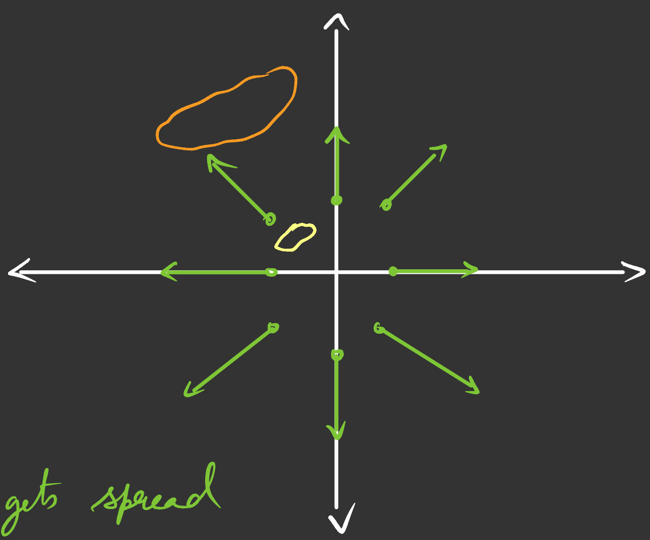
$$f = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} \qquad \nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

$$\mathrm{div}(f) = \nabla \cdot \vec{f}$$
$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

$\nabla\cdot$ is "linear" $\longrightarrow$ $\nabla\cdot\left(\vec{f}_A+\vec{f}_B\right) = \nabla\cdot\vec{f}_A + \nabla\cdot\vec{f}_B$

$$\nabla\cdot\left(\alpha\vec{f}_c\right) = \alpha\,\nabla\cdot\vec{f}_c$$

Example of +ve div.

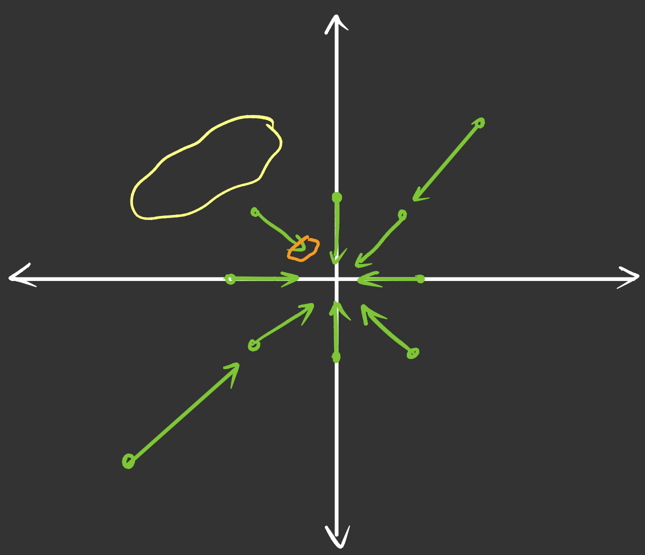


$$\vec{f}(x,y) = x\hat{i}+y\hat{j} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\nabla\cdot f = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1+1 = 2$$

example that let assume an oil spill, it gets spread after some time with spatial.

Example of −ve div.:

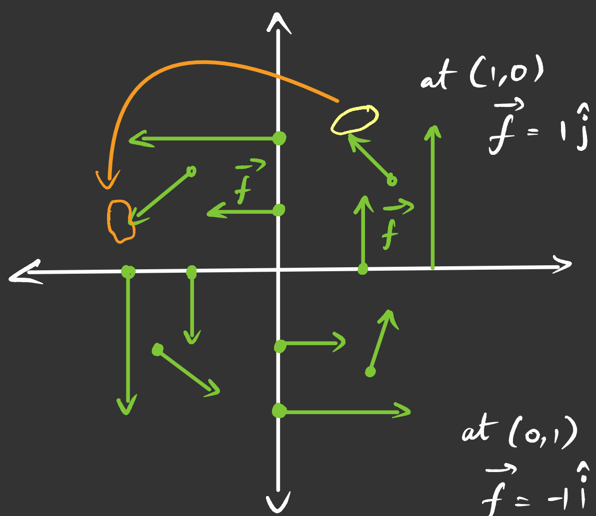


$$\vec{f}(x,y) = -x\hat{i}-y\hat{j} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

$$\nabla\cdot\vec{f} = \frac{\partial(-x)}{\partial x} + \frac{\partial(-y)}{\partial y} = -1-1 = -2$$

Flow is converging

Example of Zero divergence



$$\vec{f}(x,y) = -y\hat{i}+x\hat{j} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

$$\nabla\cdot\vec{f} = \frac{\partial(-y)}{\partial x} + \frac{\partial(x)}{\partial y} = 0$$

Divergence free vector field

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{f}(x,y) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{vector field}}$$

if we integrate we can find the oil spread

$$\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} e^t x(0) \\ e^t y(0) \end{bmatrix}$$

$$\nabla \cdot \nabla f = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \nabla^2 f$$

(Laplacian)

Curl

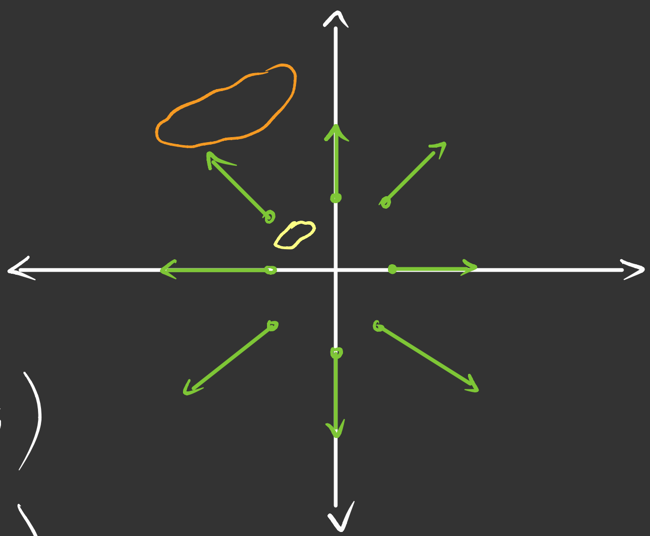
$$\nabla = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \qquad \vec{f} = \begin{bmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{bmatrix} = \begin{bmatrix} xy \\ -\sin z \\ 1 \end{bmatrix} = xy\,\hat{i} - \sin z\,\hat{j} + 1\hat{k}$$

$$\operatorname{curl}(\vec{f}) = \nabla \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -\sin z & 1 \end{vmatrix}$$

$$\nabla \times \vec{f} = \hat{i}(\cos z) - \hat{j}(0) + \hat{k}(-x) = \cos z\,\hat{i} - x\,\hat{k}$$

$$\nabla \times \vec{f} = \begin{bmatrix} \cos z \\ 0 \\ -x \end{bmatrix}$$

Ex1:



$$\vec{f}(x,y) = x\hat{i} + y\hat{j} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & 0 \end{vmatrix} = \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \hat{k}(0-0) = 0$$

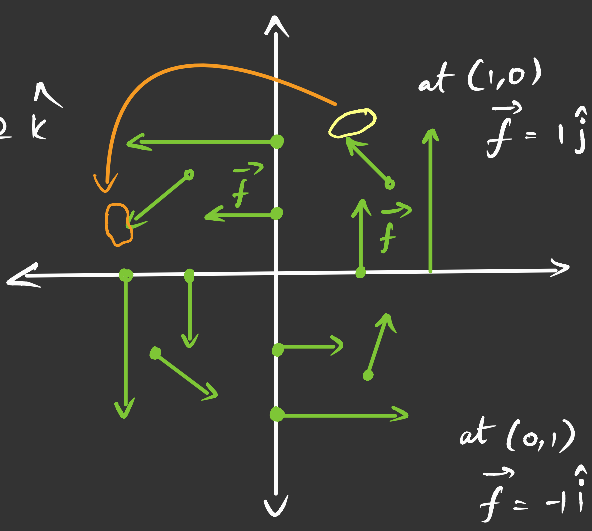
curl free $\longrightarrow$ Irrotational

Ex2:

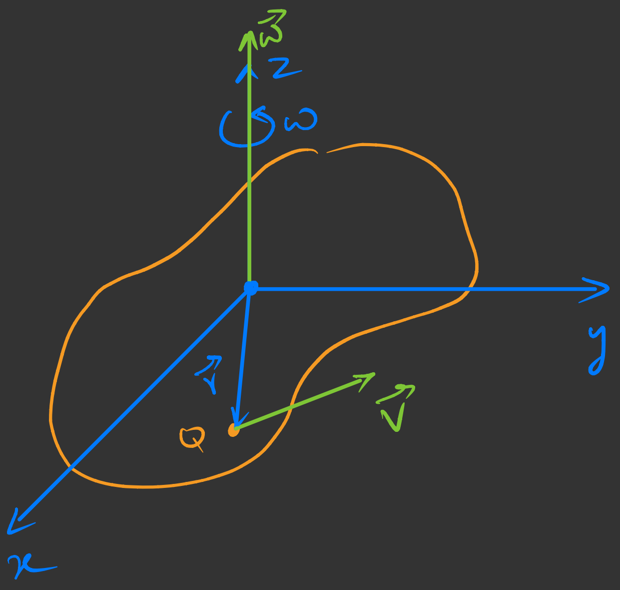
$$\vec{f}(x,y) = -y\hat{i} + x\hat{j} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

$$\nabla \times \vec{f} = \hat{k}\left(\frac{\partial x}{\partial x} - \frac{\partial -y}{\partial y}\right) = \hat{k}(1+1) = 2\hat{k}$$

" Irrotational "



Curl as solid body rotation



$$\vec{v} = \vec{\omega} \times \vec{r}$$

angular velocity vector

if $\vec{\omega} = \omega \hat{k}$ & $r = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{v} = \begin{vmatrix} i & j & k \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \hat{i}(-\omega y) + \hat{j}(\omega x)$$

- some pt along z axis does not move
- pt near axis of rotation, moves slower
- pt far away from axis of rotation, moves faster

$$\text{curl}(\vec{v}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = \hat{k}\left(\frac{\partial(\omega x)}{\partial x} - \frac{\partial(-\omega y)}{\partial y}\right)$$

$$= \hat{k}(\omega + \omega) = 2\omega\hat{k}$$

- Curl of grad $= 0$

Any potential flow vector field is irrotational

$$\nabla \times \underbrace{\nabla f}_{\text{potential flow solution}} = 0 \text{ for all } f$$

- Div of curl $= 0$

$$\nabla \cdot (\nabla \times \vec{f}) = 0 \text{ for all } \vec{f}$$

if a vector field had divergence & curl component (swirling & expanding).

Curl will only pull out the swirling part of the vector and always have divergence free.