An ol' method

Gaussian elimination is a simple method to solve a linear equation. We can also find the inverse and determinant of a matrix.

Suppose you have a non-singular matrix M. To find the inverse of M, we use elementary row operations to reduce it into an identity matrix. If we apply the same operations (in the order) to an identity matrix, the identity matrix turns into M^{-1}.

Consider the matrix,

$$\begin{bmatrix} 2 & 5 \\ 7 & 1 \end{bmatrix}$$

The following operations,

$$R_1 \to \frac{1}{2} R_1$$
$$R_2 \to R_2 - 7R_1$$
$$R_2 \to \frac{-2}{33} R_2$$
$$R_1 \to R_1 - \frac{5}{2} R_2$$

reduce M into a 2×2 identity matrix. The same operations on the identity matrix result in the inverse of $M^{-}1$,

$$\begin{bmatrix} -0.03030303 & 0.15151515 \\ 0.21212121 & -0.06060606 \end{bmatrix}$$

Finding the inverse using Gaussian elimination...

```python
import numpy as np

def inverse(M: np.array) -> np.array:
    n = len(M)
    I = np.eye(n)
    for i in range(n):
        I[i] /= M[i][i]
        M[i] /= M[i][i]
        for j in range(n):
            if i != j:
                I[j] -= M[j][i] * I[i]
                M[j] -= M[j][i] * M[i]
    return I

print(inverse(np.array([[2., 5.], [7., 1.]])))
```

But why do elementary operations on I lead to M^{-1}?

tl;dr - The three elementary row operations - scale a row, swap two rows, and subtract multiple of a row from another row are equivalent to left matrix multiplication.

Swapping

To swap rows i and j of M, multiply M by I', where $I'_{kk} = 1$, $I'_{ij} = 1$, $I'_{ji} = 1$, and $k \neq i \neq j$.

$$I' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M' = I'M$$

where M' is M but row 1 and 2 swapped,

$$M' = \begin{bmatrix} M_2 \\ M_1 \\ M_3 \\ M_4 \end{bmatrix}$$

Scaling

To scale row i of M by k, multiply M by I', where $I'_{jj} = 1$, $i \neq j$, and $I'_{ii} = k$.

$$I' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M' = I'M$$

where M' is M but row 2 scaled by 3,

$$M' = \begin{bmatrix} M_1 \\ 3M_2 \\ M_3 \\ M_4 \end{bmatrix}$$

Subtracting

To subtract multiple of row j from row i, multiply M by I', where $I'_{ii} = 1$, $i \neq j$, and $I'ij = -k$ where k is a +ve integer ($-k$ because subtraction).

$$I' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M' = I'M$$

where M\ is M but 2 times row 1 is subtracted from row 3,

$$M' = \begin{bmatrix} M_1 \\ M_2 \\ M_3 - 2M_1 \\ M_4 \end{bmatrix}$$

Together

Let E be a matrix that denotes an elementary row operation. 4 operations are 4 left matrix multiplications,

$$E_1E_2E_3E_4M = I$$

Multiply by M^{-1} on both sides,

$$E_1E_2E_3E_4I = M^{-1}$$

and so we see that the same 4 operations on I result in M^{-1}. The row operations in the previous section are equivalent to matrices,

$$E_1 = \begin{bmatrix} 1 & \frac{-5}{2} \\ 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{-2}{33} \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 \\ -7 & 1 \end{bmatrix} \quad E_4 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

i.e.,

$$\begin{bmatrix} 1 & \frac{-5}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{-2}{33} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -7 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.03030303 & 0.15151515 \\ 0.21212121 & -0.06060606 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 7 & 1 \end{bmatrix}^{-1}$$