Université de Pierre et Marie Curie
Course Project

Recherche rapide d'un triangle contenant un point dans un maillage

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Submission Date: 13.01.2018

Contents

1 Project description and aim

The aim of this project is to implement an algorithm which searches for a triangle in a convexe mesh covering a point (x, y). A linear complexity (in number of vertices of the mesh) is desired.

2 The template class T3 and the class Triangle

2.1 The template class T3

Objects of the class `T3` represent elements of a three dimensional space in `T`. Hereby, the data type `T` is defined using a template. Within this project, either `T3<int>` for the definition of a triangle (see 2.2) or `T3<double>` for the definition of the coordinates of a vertex (see 3) is used. The private members x, y and z of type `T` store the entries of the `T3` vector. The class has a default constructor, a constructor by copy and a constructor creating a vector given the three vector entries. Moreover, the class has operators to access and modify the entries, add to elements of `T3`, multiply an element of `T3` by a scalar of the type `T` and calculate the scalar product of two `T3` vectors. The operator $<$ compares two elements of `T3`, $v_1 = \begin{pmatrix} x_1 & y_1 & z_1 \end{pmatrix}^T$ and $v_2 = \begin{pmatrix} x_2 & y_2 & z_2 \end{pmatrix}^T$, in the following way:

$$v_1 < v_2 \quad \Leftrightarrow \quad \begin{cases} x_1 < x_2 \text{ or} \\ x_1 = x_2 \text{ and } y_1 < y_2 \text{ or} \\ x_1 = x_2 \text{ and } y_1 = y_2 \text{ and } z_1 < z_2 \end{cases} .$$

This allows a lexicographic ordering of a list of `T3` elements (see 3.2.2, `T3<int>`). For three `T3` vectors, which define the vertices of a triangle, the method *oriented_vol* computes the corresponding signed volume. Let a, b and c denote `T3` vectors. The oriented volume A_{abc} is defined as

$$A_{abc} = \left(\overrightarrow{ab} \times \overrightarrow{bc}\right)^T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (b_1 - a_1)(c_2 - b_2) - (b_2 - a_2)(c_1 - b_1)$$

The sign is positive for triangles oriented in trigonometric sense. This method is used in the method `Maillage::Promenade` 3.3 to determine a suitable 'walking' direction.

2.2 The class Triangle

The class Triangle inherits form the class T3. Its derived members i, j, k are specialized as integers representing the position of its defining vertices in the array of the points of the given mesh. This array is a member of the class mesh in 3. In addition, it has three extra integer members `Triangle::int neighbor1, neighbor2, neighbor3` representing the position of the adjacent triangles in the array of triangles which is also a member of the class mesh. Notice that theses index lies between zero and the number of triangles minus one.
An adjacent triangle $t^{'}$ is said to be `neighbor1` of a triangle t if, saying i represents the first index of a vertex of t but i does not represent a vertex of $t^{'}$. The same holds for `neighbor1` and `neighbor2`. Note that this relation is not symmetric, that is to say that $t^{'}$ is *neighbor1* does not imply that t is `neighbor1` of $t^{'}$. This assignment facilitates the access to the following adjacent triangle in the algorithm `Mesh::Promenade` (see 3.3).
At the creation of a new triangle the latter three members are initialized by -1 which means that a triangle has no neighbors when it is created. The neighbors are set via the functions `Mesh::setAdjacencyViaMultimap` 3.2.1, respectively `Mesh::setAdjacencyViaList` 3.2.2. After the execution of one of these two functions `neighbor1` $= 1$ describes the case that there is no adjacent triangle on the opposite side of the first vertex. Since the neighbors are private members there are getter and setter in order to read and manipulate them.

3 The class Mesh

The class Mesh contains all necessary information of the mesh and the search of a vertex in (or outside) the mesh can be realized by its member functions. Its private members are pointers to the arrays of

vertices and triangles of the mesh as integers who store the size of these two lists. In order to create an object of type Mesh the name of the desired .msh file has to be transmitted. The file is read by the functions `Mesh::LoadVertices` and `Mesh::LoadTriangles` 3.1 who then initialize the members `Mesh:: Triangle* triangles, T3<double> vertices, int numbTriangles, int numbVertices`. Since the members are private there are getter and setter for `numbTriangles` and `numbVertices` as well as getter for `triangles` and `vertices`. The functions `LoadTriangles` and `LoadVertices` are their setters.

3.1 LoadVertices and LoadTriangles

Both functions are called in the constructor of the class `Mesh` setting the members `vertices` and `triangles` They work basically the same way with the small difference that they create arrays of different data types (`T3<double>` and `T3<int>`) and searching for different key words in the .msh file. At first, the .msh file is opened according to its name. Then the functions search for the line indicating the number of vertices respectively the number of triangles. This is implemented by using an `std::fstream` and the function `std::getline` who stores the information of a line and skips to the next line.
To make the code work the next line must include the according number which is then stored in `numbVertices`, respectively `numbTri`. The data is read as a string which is transformed to an integer by the command `astoi`. These numbers also define the size of the arrays of type `T3<double>` and `Triangle` which are created by `new` in order to allocate the necessary memory. The following lines must include the coordinates of all vertices, respectively the positions of the triangles in the table of vertices. The lines are read as strings who are casted in to `doubles` respectively `integers`.
Each line defines a vertex or a triangle which is then stores in the correspondent array. Both functions finally return the arrays filled with the vertices or triangles.

3.2 Find the adjacent triangle

3.2.1 setAdjacencyViaMultimap

The goal of this function is the initialization of the members `neighbor1, neighbor2, neighbor3` of the class Triangle. As a reminder, they are defined by their position in the array of triangles. Their initialization is realized by the container `std::multimap`. Each triangle $t = (v_{i_1}, v_{i_2}, v_{i_3})$ is represented by the sequence (i_1, i_2, i_3), where these indices indicate the position of the vertices v_{i_1}, v_{i_2} and v_{i_3} in the array of vertices. For each t three pairs are added to the multimap, where the edges (v_{i_k}, v_{i_l}), $k, l \in \{1, 2, 3\}$, $k \neq l$, stored by the data type $pair < int, int >$ (more precisely $pair < i_k, i_l >$), represent the keys, whereas the mapped value is an integer representing the position of the triangle in the array of triangles. Hence, the initialization if the multimap is realized in $3n_T$ loops, so in $O(n_T)$.
Essentially for the functioning of the method is the ordering of the of the indices i_1, i_2, i_3 in t. As in `setAdjacencyViaList`, an edge $\{v_{i_k}, v_{i_l}\}$ has to be uniquely identified by the pair (i_k, i_l) which should not be mixed up with (i_l, i_k). This is achieved by demanding that for each triangle $t = (v_{i_1}, v_{i_2}, v_{i_3})$ the vertices are ordered in a manner such that $i_1 < i_2 < i_3$. Thus, an edge (v_{i_k}, v_{i_l}) can uniquely be identified by the pair (i_k, i_l), $i_k < i_l$.
We find the adjacent triangles by the following three steps:

1. We range over the multimap and in each step we save the current key (i_k, i_l), its position in the list (this is the current iterator) and the index i of the triangle $t = (v_{i_1}, v_{i_2}, v_{i_3})$. Then we move the iterator to the next element of the list and erase the pair $((i_k, i_l), i)$ of the multimap which is realized in constant time since the iterator belonging to $((i_k, i_l), i)$ was saved.

2. Now searching by the function $find$ for the stored key (i_k, i_l) allows us to determine if there is another index j associated to (i_k, i_l). If this is the case (the iterator returned by find does not point to the last element of the multimap) the triangles represented by i and j are adjacent. The function $find$ needs $O(\log(n_T))$ in order to return an iterator. Note that the pair $((i_k, i_l), j)$ is not erased. This would yield an error because the actual iterator would try to access that element.

3. The indices i and j are set as the neighbors of each other where j is set as neighbor $k \in \{1, 2, 3\}$ if the index k does not represent a vertex of the triangle represented by j. The same rule is applied to i as a neighbor of j. This assignment is realized in constant time.

The application of all three steps needs a time of $O(n_T)O(\log(n_T)) = O(n\log(n_T))$.

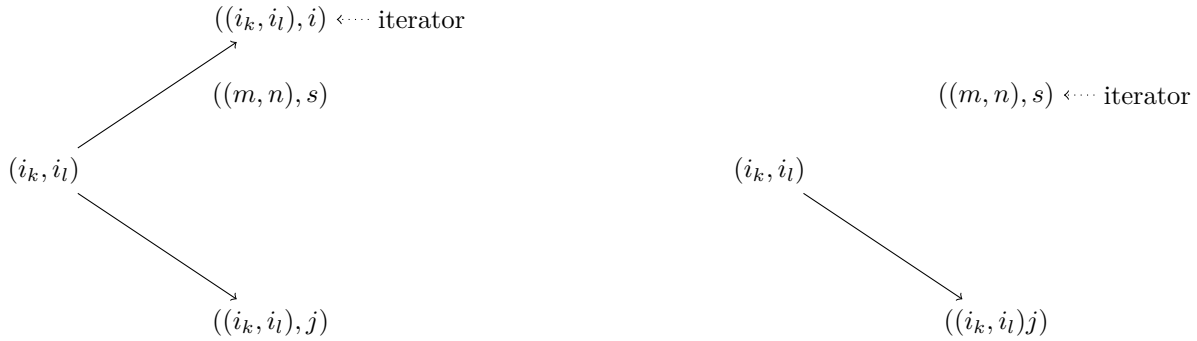


Figure 1: The initial situation for determining the neighbors (left). The element$((i_k, i_l), i)$ is erased from the map in the iterator points to another element, not necessarily $((i_k, i_l), j)$. This essentially depends on the internal ordering of the multimap.

3.2.2 setAdjacencyViaList

As in `setAdjacencyViaMultimap`, this method sets the members, describing the neighbors, for each triangle in the considered mesh. In the same manner as before, the members are described using their position in the triangle array. The idea of this method is to create a list in each adjacent triangles appear in consecutive order. After this list is created, the adjacencies of the triangles can be set in linear complexity in the number of triangles.
The list with the desired property is created in the following way: For each triangle three new triangles are created and appended to the list. For the triangle $t = (v_{i_1}, v_{i_2}, v_{i_3})$ with the index i, a new triangle for every edge is defined. More precisely, the triangles $(i_1, i_2, i), (i_1, i_3, i)$ and (i_2, i_3, i) are pushed on the list. This can be done in $O(n_T)$. As in `setAdjacencyViaMultimap`, it is necessary that an edge is with increasing vertex indexing, so $i_1 < i_2 < i_3$ has to hold. In order to archive that two neighboring triangle are positioned one after another, the list is sorted in lexicographic order. First, edges of triangles including vertex 1 are listed. These are ordered with respect to the second vertex defining the list triangle's edge. For two list entries with coinciding edge, the list is ordered with respect to the triangle index stored in the last list triangle's vertex. In this case, two neighboring triangles are considered. Then, list triangles with leading entry 2 followed by higher indices that are not yet considered are appended. In Figure 3.2.2 the structure of the sorted list is depicted.

$$
\begin{array}{ccc}
1 & * & * \\
1 & * & * \\
 & \vdots & \\
i_1 & i_2 & i \\
i_1 & i_2 & j \\
i_1 & i_3 & i \\
 & \vdots & \\
i_1 & * & * \\
i_2 & i_3 & i \\
i_2 & i_3 & k \\
 & \vdots & \\
n_T & * & *
\end{array}
$$

Figure 2: Illustration of the list structure

The increasing order for indices of an edge is crucial since otherwise neighboring triangles would not appear in consecutive order. The complexity for sorting this list of size $3n_T$ is $O(n_T log n_T)$.
After the creation of this list, the members of the mesh triangles can be set. Due to the lexicographical ordering, the list allows now to define adjacencies between triangles with a number of comparison which is linear in the number of triangles. Two consecutive triangles of the list are considered. The first two entries of the triangles are compared. If they are equal, the triangles share an edge and, hence, can be defined as adjacent. Particularly, if the triangles with the indices i and j have the vertices v_k and v_l in common, the triangles (k, l, i) and (k, l, j) appear consecutively in the list. The head element of the list is popped from the front and the comparison is repeated for new top triangles. If the first two vertices of successive triangles do not coincide, the neighbor attached to this edge was either already set in the step before, or the edge is located on the border of the considered mesh. In this case, the corresponding mesh triangle's member is not modified and will still be defined as -1. The comparison of every pair of triangles has constant complexity. As a list of size $3n_T$ is considered, the complexity of setting the neighbors given this list is $O(n_T)$.
In summary, the method `setAdjacencyViaList` has complexity $O(n_T log n_T) + O(n_T) = O(n_T log n_T)$.

3.2.3 Vergleich

Laufzeitvergleich, komplexität

list average execution time: 0.0750262
multimap average execution time: 0.0887504

best / worst case

3.3 L'algorithme promenade

`promenade` is a method of the class Mesh which is given a triangle `T` and a point p of the type `T3<double>`. As output a triangle in the mesh covering p is returned.
Let `T` $= (c_1, c_2, c_3)$, where the vertices c_1, c_2 and c_3 are of type `T3<double>`. Hereby, the vertices are considered in trigonometric sense. Thus, the oriented volume of the triangle `T` is positive. Recall that `T3::oriented_volume` is a method of the class `T3` which takes two additional `T3` vectors as input (see 2.1).
For each edge of the triangle `T`, a new triangle containing the edge and the point p is considered. We are interested in the oriented volumes of the triangles $(c_1, c_2, p), (c_2, c_3, p)$ and (c_3, c_1, p). If the oriented volume of one of those new triangles is positive, then p and the triangle `T` are contained in the half plane defined by the currently considered edge of `T`. Consequently, p is contained in `T` if the oriented volumes of the triangles $(c_1, c_2, p), (c_2, c_3, p)$ and (c_3, c_1, p) are all positive. In this case, the method `Mesh::promenade` returns the input triangle `T`. On the other hand, if there is one triangle with negative oriented volume, `T` is not covering p. `T` and p are separated by the regarded edge. We choose one edge who leads to a negative oriented volume for the triangle formed with p and consider the adjacent triangle `N` sharing this edge with `T`. Thereafter, the method `Mesh::promenade` is called recursively with starting triangle `N`. With this choice of neighboring triangles, the path generated by the algorithm heads in the direction of p. In addition, it should be paid attention to the existence of a neighboring triangle. Triangles on the border of the mesh just have two neighbors, the remaining member is initialized by -1.
In order to make sure that a triangle in treated in trigonometric sense, the oriented volume of the starting triangle `T` $= (c_1, c_2, c_3)$ is consulted. If `T3::oriented_volume` returns a negative value for the oriented volume, the results for oriented volumes of the triangles $(c_1, c_2, p), (c_2, c_3, p)$ and (c_3, c_1, p) are negated. Crossing an edge from one triangle to an adjacent one, the sense in which this edge is viewed has to be reversed to guarantee a consideration of the triangles in trigonometric sense. Using the above approach, this can be treated efficiently without need to know the traversed edge and its indices in the neighbor triangle.
Optionally, the method `Mesh::promenade` takes also a vector with entries of the class Triangle, called path, as input variable. The current triangle is pushed back on the vector and, due to recursively calling of the method, the vector stores the consecutively passed triangles when the algorithm terminates. The path is used to visualize the results of the method `Mesh::promenade` in 3.4.

The method terminates when the desired covering triangle, if it exists, is found. Otherwise, if the point p is located outside of the mesh, the method returns a triangle on the border. The only edge forming a triangle with negative oriented volume with p will be part of the mesh border. Thus, no adjacent neighbor exists. The returned triangle is the closest triangle to p.

terminates because? XXX

Following the algorithm, there are three cases for the number of possible walking directions:

1. Exactly one edge forms a triangle with negative oriented volume with p,
2. two neighboring triangles are valid choices or
3. the current triangle already covers p.

If there are two neighboring triangles to chose from, different approaches can be pursued. We implemented a random neighbor selection (see 3.3.1) and a method choosing the edge forming the triangle with minimal oriented volume with p (see 3.3.2).

3.3.1 Random neighbor selection

The method `random_neg` is given three numbers of the type `double` and returns an integer. If at least one input variable is strictly negative, the output takes a value between one and three indicating the handover position of a negative input variable. Otherwise, if non of the input variables is negative, -1 is returned. In the case of several negative input variables, the variable is chosen randomly. Hereby, the time XXX is used to generate random numbers.
Using this approach for the choice of the neighbor triangle in `Mesh::promenade`, the algorithms returns different paths when executing it repeatedly with the same starting triangle and point p.
#include <time.h>
srand(time(NULL));
details zu return wert?? XXX

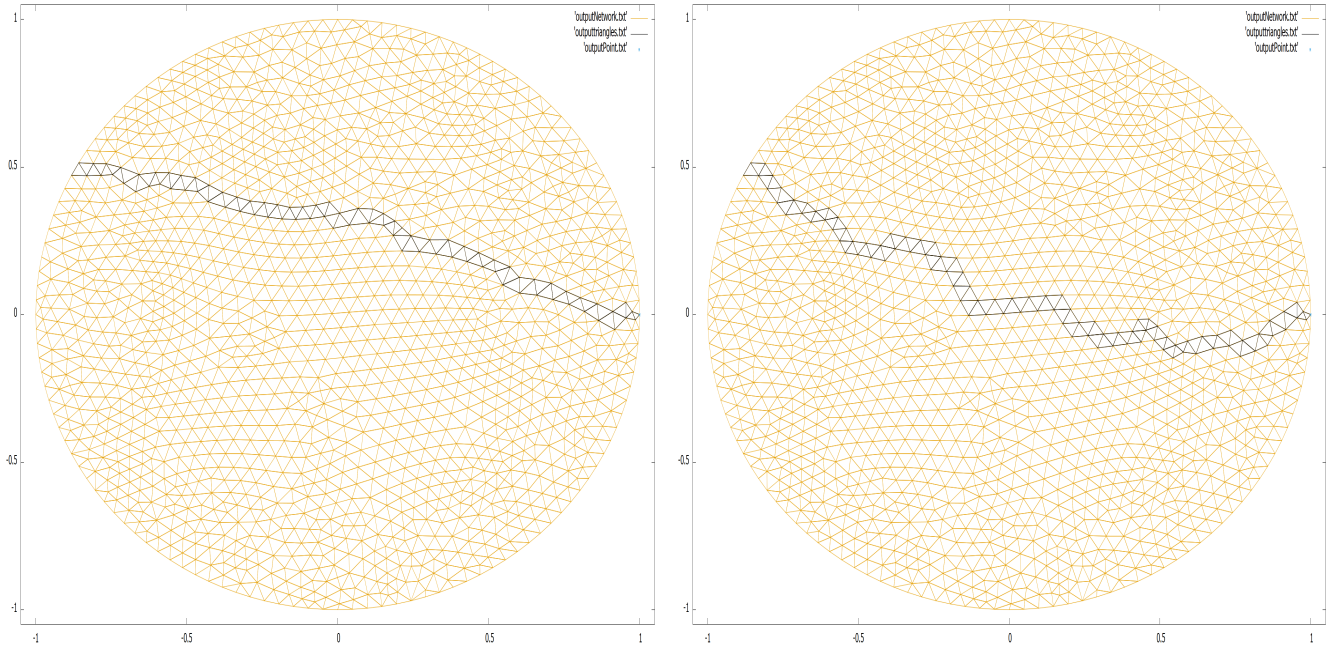
3.3.2 Selection of neighbor with minimal oriented volume

Another option to choose the consecutive triangle is to select the edge forming the triangle with minimal oriented volume with the point to cover. The method `min_neg` returns the smallest of the three given numbers or -1 if none of them is negative.
Unlike `random_neg`, the neighbor choice with `min_neg` XXX

same path length for multiply execution with same starting triangle and point p in most cases
only case in which neighbor selection is not uniquely determined: p on half plane

3.3.3 Can an arbitrary choice of the consecutive triangle be a better choice?

This example shows that a random choice of the consecutive triangle can be better. Assume that the algorithm `promenade` currently considers the triangle A of Figure 3.3.3 and searches for a covering triangle of the point p. Hence, the algorithm will proceed with triangle B or C.
If we use the size of the oriented volume (the oriented volume of the convex hull of p and the lower right edge of A, respectively the upper right edge of A) we will clearly choose C, as illustrated in the second figure (the green area is clearly bigger than the blue one). This is obviously the worse choice since B is a covering triangle of the point p.
An important observation is that the regularity of the triangles plays an important role. In this case, regularity means that all triangles of the mesh have a 'similar' area, 'similar' angles and its vertices are adjacent to a 'similar' number of triangles.
XXX bsp erklären



3.3.4 Empirical demonstration that the deterministic choice of the consecutive triangle is better in regular meshs

If we take a look at the figures 3.3.4 and 3.3.4 SELBE NUMMER IN PDF, it becomes empirically clear that the minimum choice of the consecutive triangle speeds up the algorithm. Figure 3.3.4 even states that the path created by the minimum choice is always shorter or as long as the path created by the random choice. Looking at the runtime in 3.3.4, we see that the runtime is about as twice as high for the random choice when the starting triangle and the given point have a big distance XXXX (are remote / distant / far away (from each other / in the mesh)).
Nevertheless, it has to be kept in mind that the underlying mesh plays an important role. In this case, the mesh 'maillage5.msh', USED FOR THE CREATION OF FIGURE X, is regular in the sense explained in the previous section (see 3.3.3).

3.4 The visualization via gnuplot

The visualization is done by the function `Mesh::exportGnuplot` which is able to visualize two different inputs:

1. The result of the algorithm promenade, that is a sequence of adjacent triangles from a starting triangle to a triangle covering the searched point. In this case, the input variable `vector<triangle> triangles_path` contains the sequence of triangles and the input variable `const T3<double>* points` contains just one point, hence, the input `int numbpoint`, denoting the size of the array `points`, is 1.

2. Given a mesh and the vertices of another mesh, it can visualize the set of the covering triangles (a subset of triangles of the first mesh) of the points of the second mesh. In this case, the input variable `vector<triangle> triangles_path` contains the covering triangles and `points` stores the vertices of the second mesh.

The function writes four text files, one for all triangles of the mesh, one for the triangles in `triangles_path`, one for the points in `points` and one for the commands which shall be executed by gnuplot. This is all realized by a simple `std::ofstream` variable, allowing to create and manipulate a text file.
The script for gnuplot contains a line which keeps the plot open until some key is hit in the terminal. The actual execution in the terminal is achieved by the command `system` which executes its input string

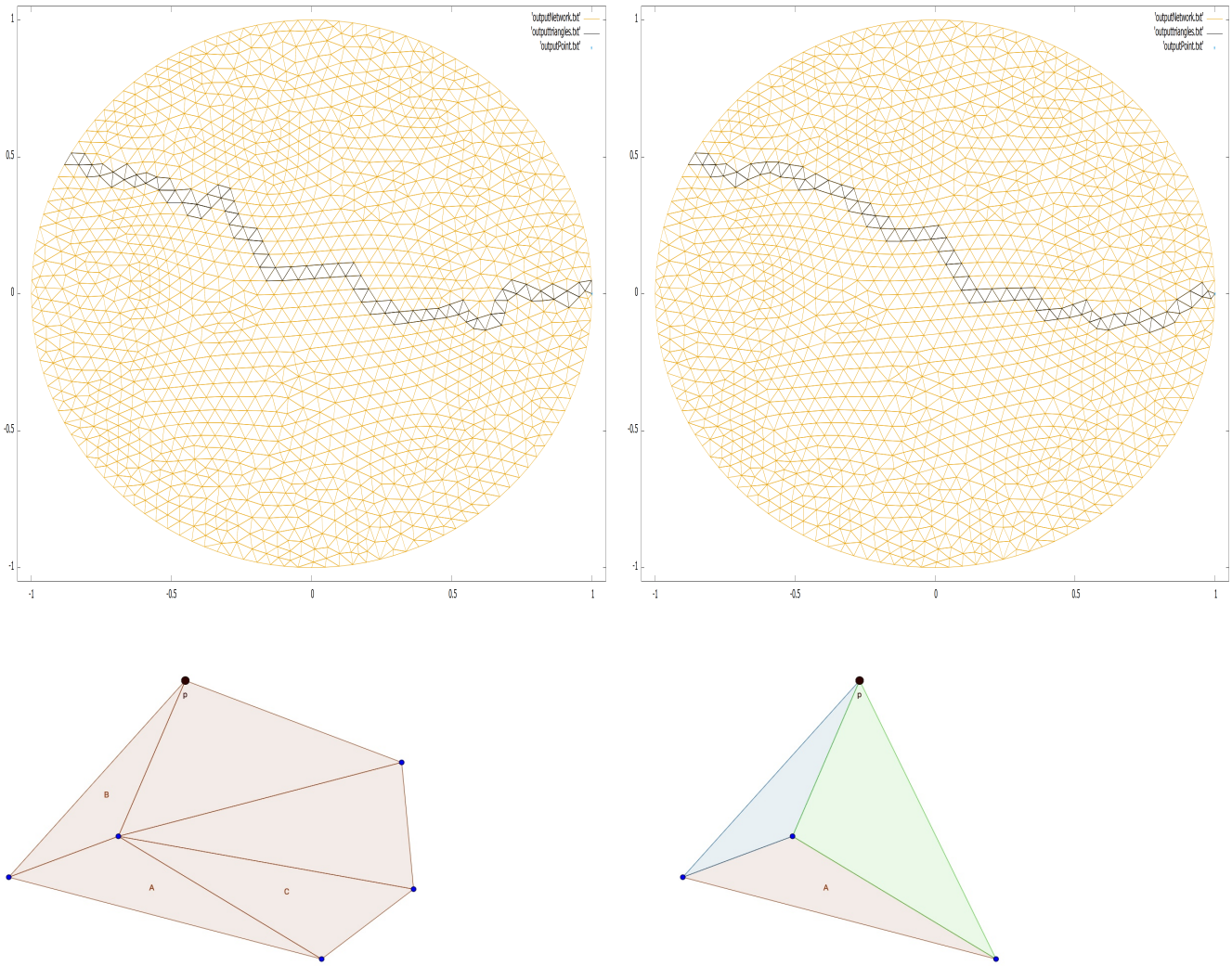


Figure 3: The illustration of 3.3.3.

in the terminal. XXXX The actual execution is achieved by typing the command `system` in the terminal. ???

4 Cover vertices of a mesh by another mesh

The aim of this section is to find the covering triangle for every vertex of another mesh in the same domain. This can be useful when a mesh is refined or coarsened. For this purpose, the promenade algorithm is executed many times, once for every vertex. In order nevertheless to obtain a good running time, the starting triangle for `promenade` should be chosen in an efficient way. The idea is to use starting triangles for which we already know that they are located close to the vertex to cover.

The function `findVertices` is given two meshes, `Mesh m` and `Mesh M`, and a reference of a vector of triangles `coveringTriangles` and returns `coveringTriangle` after modification. The function's goal is to search for covering triangles in `M` for the vertices of `m`. At the position i of `coveringTriangles` the triangle in `M` covering the i-th vertex of the mesh `m` is stored. All covering triangles are initialized by 0. Note that the indexing of the vertices in the mesh files starts at 1.

First, two random triangles, one in `m` and one in `M`, are chosen. In `findVertices` covering triangles for the vertices of the randomly chosen triangle `firstTriangle_m = (firstVertex, secondVertex,`

thirdVertex) in m are computed. In order to find a covering triangle for firstVertex, Mesh::promenade is executed given the random triangle firstStartTri chosen in the M. After calculation of the covering triangle for firstVertex indexed by firstTriangle_M, we can use that triangle as starting triangle for the search of a covering triangle for secondVertex. As firstVertex and secondVertex belong to one edge in m, we expect the covering triangles of those vertices to be close in M. In the same manner, we use the covering triangle of secondVertex as starting triangle for covering thirdVertex. Hereby, the version of Mesh::promenade not storing the taken path is executed.

why start with random triangle of mesh m (JUST START WITH FIRST TRI?)

Now, we computed covering triangles for all the vertices of firstTriangle_m. Originating from this triangle, we start a recurrence to find the remaining covering triangles. Neighboring triangles, neighbors of the neighboring triangles and so on will be considered.
In findVertices the function Triangle_Recurrence is called. The two meshes, the vector coveringTriangles and the already covered triangle firstTriangle_m are passed. Triangle_Recurrence modifies the vector coveringTriangles and, thereafter, findVertices returns the completed vector.
Triangle_Recurrence searches for covering triangles for the vertices of neighboring triangles of the already covered triangle. Again, 'close' covering triangle that are already found are used to accelerate the termination of the algorithm promenade.
We start with the member neighbor1 of firstTriangle_m. If this neighbor triangle exists and its vertices are not yet covered, we do so using the same procedure as in findVertices. After the triangle is covered, we consider its member neighbor1. At the moment when either a neighbor does not exist or is already covered, we go on to the member neighbor2 and afterwards to the member neighbor3. If all members are covered, we return to the calling triangle and considered its next neighbor. This terminates when a cover triangle for all vertices of the mesh m is found and the vector coveringTriangles is completed.
searching structure is similar to approach of depth-first search
selectAdjacentPoint to select Triangle to start *promenade* (if the vertex is not already covered), then cover neighbors of this triangle

5 Test environment- the main function

The cpp file main offers the possibility to test the properties of the program:

1. Find the neighbors of a triangle. If a valid triangle is index chosen, the triangles and its neighbors are marked in the chosen mesh. This case offers the extra option to choose if the adjacency should be set via list or mutlimap. Essentially the functions setAdjacencyViaList or setAdjacencyViaMultimap and exportGnuplot are called.

2. Find the triangle for a point (the point can be chosen). As a result, the path to the covering triangle of the point is displayed. If the point lies outside of the mesh, the path to the nearest triangle is marked in the mesh. In this case, the functions Promenade and exportGnuplot are called.

3. Find the covering triangles for the vertices of another mesh. These option offers to display all covering triangles of a the vertices of a second mesh. Note that the mesh whose vertices is supposed to be broader (a finer mesh can be embedded in a broader mesh as well but then each triangle serves as a covering triangle).

All three options contain the option to choose the mesh where the mesh becomes finer from maillage1.msh increasingly to maillage5.msh. Of course, it is also possible to add own meshs. In this case yet, it has to be respected that the triangle's indices ordering must be increasing, as mentioned in the sections before. Moreover, the mesh's interval should be at maximum $[1.5, 1.5]$ for a convenient display since the axis of the plot for gnuplot are predefined as the interval $[2, 2]$.
The test environment is not robust face to wrong inputs. That means that, for example, if an integer is demanded but a letter is typed, the program has to be started again. The same happens it an index is demanded but a the typed input lies outside of the range (an assert exception is thrown then).

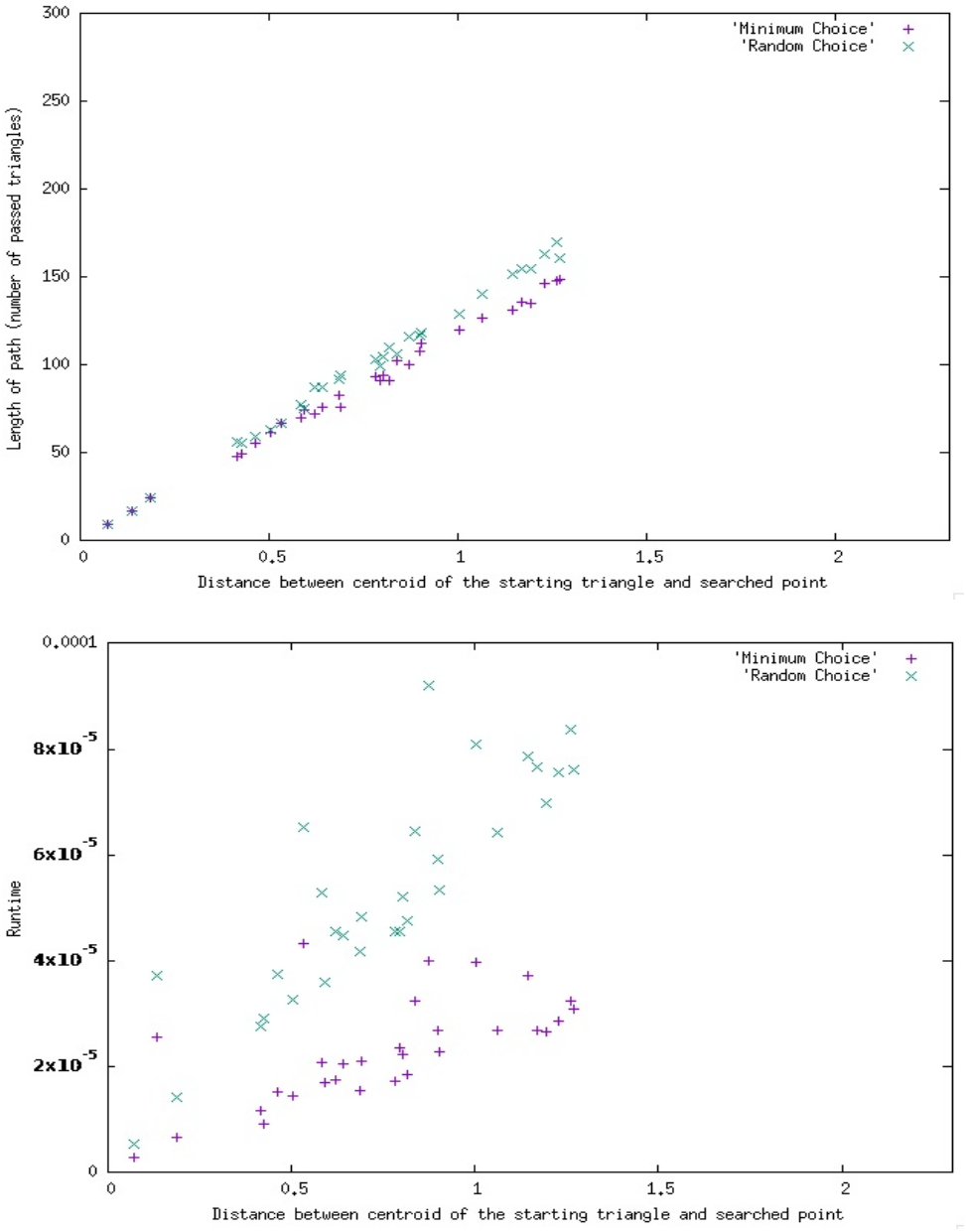


Figure 4: The pathlength and the runtime compared using min_neg and $random_neg$. In this example, 30 data points were randomly created in the file 'TimeMeasurements.cpp'. The underlying mesh is 'maillage5.msh'.

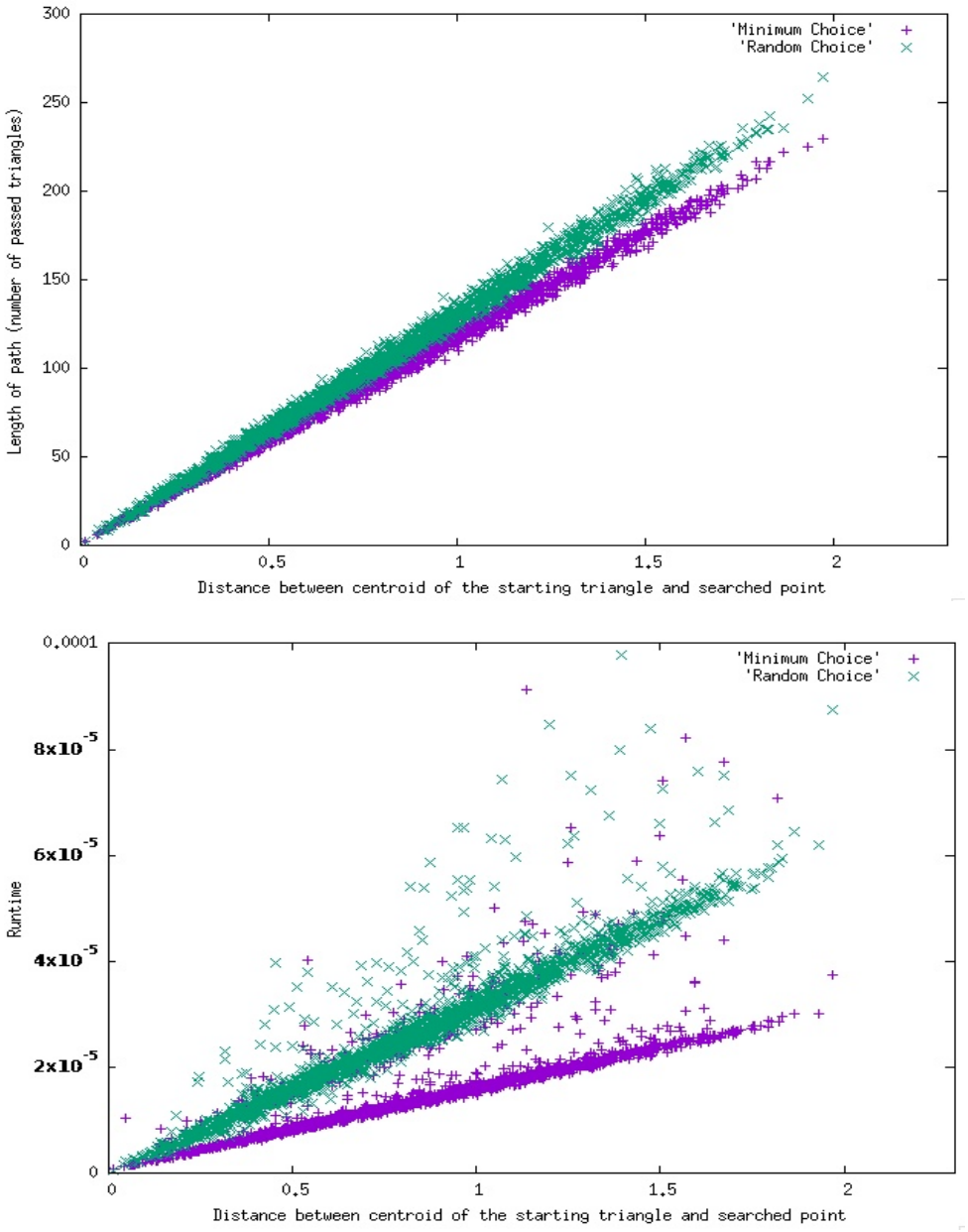


Figure 5: Here, 2000 data points were randomly created.