Chapter 2

Linear Transformations and Matrices

Review

Definition 1. Let $\beta = \{u_1,\ u_2, ...,\ u_n\}$ be an ordered basis for a finite dimensional vector space V. For $x \in \mathsf{V}$, let $a_1,\ a_2, ...,\ a_n$ be the unique scalars such that

$$x = \sum_{i=1}^{n} a_i u_i$$

We define the **coordinate vector** x **relative to** β, denoted $[x]_\beta$, by

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

.

Definition 2. Using the notation above, we call the $m \times n$ matrix A defined by $A_{ij} = a_{ij}$ the **matrix representation of T in the ordered basis β and** γ and write $A = [\mathsf{T}]_\beta^\gamma$. If $\mathsf{V} = \mathsf{W}$ and $\beta = \gamma$, then we write $A = [\mathsf{T}]_\beta$. Notice that the jth column of A is simply $[T(v_j)]_\gamma$.

Theorem 2.1. Let V, W, and Z be finite-dimensional vector spaces with ordered bases α, β and γ, respectively. Let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ and $\mathsf{U} : \mathsf{W} \to \mathsf{Z}$ be linear transformations. Then

$$[\mathsf{UT}]_\alpha^\gamma = [\mathsf{U}]_\beta^\gamma [\mathsf{T}]_\alpha^\beta$$

Theorem 2.2. Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. For each j $(1 \leq j \leq p)$ let u_j and v_j denote the jth columns of AB and B, respectively. Then

(a) $u_j = Av_j$

(b) $v_j = Be_j$, where e_j is the jth standard vector of F^p

i.e.

$$AB = \begin{pmatrix} Av_1 \\ Av_2 \\ \vdots \\ Av_p \end{pmatrix}$$

Theorem 2.3. Let V and W be finite-dimensional vector spaces having ordered basis β and γ, respectively, and let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be linear. Then, for each $u \in \mathsf{V}$, we have

$$[\mathsf{T}(u)]_\gamma = [\mathsf{T}]_\beta^\gamma [u]_\beta$$

Theorem 2.4. Let β and β' be two ordered basis for a finite-dimensional vector space V, and let $Q = [\mathsf{I_V}]_\beta^{\beta'}$, $\mathsf{T} : \mathsf{V} \to \mathsf{V}$. Then

(a) Q is invertible

(b) For any $v \in \mathsf{V}$, $[v]_\beta = Q[v]_{\beta'}$

(c) $\mathsf{T}_{\beta'} = Q^{-1}[\mathsf{T}]_\beta Q$

The matirx $Q = [\mathsf{I_V}]_\beta^{\beta'}$ is called **change of coordinate matrix**. Also, we say that Q **changes β'-coordinates into β-coordinates**.
That is, jth column of Q is $[x'_j]_\beta$, where $\beta' = \{x'_1,\ x'_2, ...,\ x'_n\}$.

Chapter 5

Diagonalization

Eigenvalues and Eigenvectors

Definition 3. A linear operator T on a finite-dimensional vector space V is called diagonalizable if there is an ordered basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix. A square matrix A is called **diagonalizable** if L_A is diagonalizable.

Definition 4. Let T be a linear operator on a vector space $|mathsfV$ A nonzero vector $v \in V$ is called an **eigenvector** of T if there exists a scalar λ such that $\mathsf{T}(v) = \lambda v$. The scalar λ is called the **eigenvalue** corresponding to the eigenvector v.

Let A be in $\mathsf{M}_{n\times n}(F)$. A nonzero vector $v \in \mathsf{F}^n$ is called an **eigenvector** of A if v is an eigenvector of L_A; that is, if $Av = \lambda v$ for some scalar λ. The scalar λ is called the **eigenvalue** of A corresponding to the eigenvector v

Theorem 5.1. A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if there exists an ordered basis β for V consisting of eigenvectors of T.

Furthermore, if T is diagonalizable, $\beta = \{v_1,\ v_2, ...,\ v_n\}$ is an ordered basis of eigenvectors of T, and $D = [\mathsf{T}]_\beta$, then D is a diagonal matrix and D_{jj} is the eigenvalue corresponding to v_j for $1 \leq j \leq n$.

Proof. ∎

Theorem 5.2. Let $A \in \mathsf{M}_{n\times n}(F)$. THen a scalar λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.

Proof. ■

Definition 5. Let $A \in \mathsf{M}_{n\times n}(F)$. The polynomial $f(t) = \det(A - tI_n)$ is called the **characteristic polynomial** of A.

Theorem 5.3. Let $A \in M_{n\times n}(F)$.

(a) The characteristic polynomial of A is a polynomial of degree n with leading coefficient $(-1)^n$

(b) A has at most n distinct eigenvalues.

Proof. ■

Theorem 5.4. Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. A vector $v \in \mathsf{V}$ is an eigenvalue of T corresponding to λ if and only if $v \neq \mathit{0}$ and $v \in \mathsf{N}(\mathsf{T} - \lambda \mathsf{I})$.

Proof. ■

Problem 5.1. Let T be a linear operator on a finite-dimensional vector space V, and let β be an ordered basis for V. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of $[\mathsf{T}]_\beta$.

Proof. ■

Problem 5.2. Prove following theorems.

(a) Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T

(b) Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

(c) State and prove results analogous to (a) and (b) for matrices.

Proof. ∎

Problem 5.3. Prove that the eigenvalues of an upper triangular matrix M are the diagonal entries of M.

Proof. ∎

Problem 5.4. Let V be a finite-dimensional vector space, and let λ be any scalar.

(a) For any ordered basis β for V, prove that $[\lambda \mathsf{I}_{\mathsf{V}}]_\beta = \lambda I$.

(b) Compute the characteristic polynomial of $\lambda \mathsf{I}_{\mathsf{V}}$.

(c) Show that $\lambda \mathsf{I}_{\mathsf{V}}$ is diagonalizable and has only one eigenvalue.

Proof. ∎

Problem 5.5. A **scalar matrix** is a square matrix of the form λI for some scalar λ; that is, a scalar matrix is a diagonal matrix in which all the diagonal entries are equal.

(a) Prove that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.

(b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.

(c) Prove that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

Proof. ■

Problem 5.6.

(a) Prove that similar matrices have the same characteristic polynomial.

(b) Show that the definition of the characteristic polynomial of a linear operator on a finite-dimensional vector space V is independent of the choice of basis for V.

Proof. ■

Problem 5.7. Let T be a linear operator on a finite-dimensional vector space V over a field F, let β be an ordered basis for V, and let $A = [\mathsf{T}]_\beta$. In reference to Figure 5.1, prove the following.

$$\begin{tikzcd} \mathsf{V} \arrow[r, "\mathsf{T}"] \arrow[d, "\phi_\beta"'] & \mathsf{V} \arrow[d, "\phi_\beta"] \\ \mathsf{F}^n \arrow[r, "\mathsf{L}_A"] & \mathsf{F}^n \end{tikzcd}$$

Figure 5.1

(a) If $v \in \mathsf{V}$ and $\Phi_\beta(v)$ is an eigenvector of A corresponding to the eigenvalue λ, then v is an eigenvector T corresponding to λ.

(b) If λ is an eigenvalue of A (and hence of T), then a vector $y \in \mathsf{F}^n$ is an eigenvector of A corresponding to λ if and only if $\Phi_\beta^{-1}(y)$ is an eigenvector of T corresponding to λ.

Proof. ■

Problem 5.8. For any square matrix A, prove that A and A^T have the same characteristic polynomial (and hence the same eigenvalues).

Proof. ∎

Problem 5.9. Solve following problems.

(a) Prove that similar matrices have the same trace.

(b) How would you define the trace of a linear operator on a finite-dimensional vector space? Justify that your definition is well-defined.

Proof. ∎

Problem 5.10. Let T be the linear operator on $\mathsf{M}_{n\times n}(R)$ defined by $\mathsf{T}(A) = A^T$.

(a) Show that ± 1 are the only eigenvalues of T.

(b) Describe the eigenvectors corresponding to each eigenvalue of T.

(c) Find an ordered basis β for $\mathsf{M}_{2\times 2}(R)$ such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

(d) Find an ordered basis β for $\mathsf{M}_{n\times n}(R)$ such that $[\mathsf{T}]_\beta$ is a diagonal matrix for $n > 2$.

Proof. ■

Problem 5.11. Let $A,\ B \in \mathsf{M}_{n\times n}(C)$.

(a) Prove that if B is invertible, then there exists a scalar $c \in C$ such that $A + cB$ is not invertible.

(b) Find nonzero 2×2matrices A and B such that both A and $A + cB$ are invertible for all $c \in C$.

Proof. ■

Problem 5.12. Let A and B be similar $n \times n$ natrices. Prove that there exists an n-dimensional vector space V, a linear operator T on V, and ordered basis β and γ for V such that $A = [\mathsf{T}]_\beta$ and $B = [\mathsf{T}]_\gamma$.

Proof. ■

Problem 5.13. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$$

Prove that $f(0) = a_0 = det(A)$. Deduce that A is invertible if and only if $a_0 \neq 0$.

Proof. ■

Problem 5.14. Solve following problems.

(a) Let T be a linear operator on a vector space V over the field F, and let $g(t)$ be a polynomial with coefficients from F. Prove that if x is an eigenvector of T with corresponding eigenvalue λ, then $g(\mathsf{T}))(x) = g(\lambda)(x)$. That is, x is an eigenvector of $g(\mathsf{T})$ with corresponding eigenvalue $g(\lambda)$.

(b) State and prove a comparable result for matrices.

(c) Verify (b) for the matrix A in Exercise 3(a) with polynomial $g(t) = 2t^2 - t + 1$, eigenvector $x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, and corresponding eigenvalue $\lambda = 4$.

Note that matrix A in Exercise 3(a) is $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ for $F = R$.

Proof. ∎