# Chapter 5

# Differentation
# Selected Exercise homework

**Exercise 5.1.** Let $f$ be defined for all real $x$, and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real $x$ and $y$. Prove that $f$ is constant.

*Proof.*

**Exercise 5.2.** Suppose $f'(x) > 0$ in $(a, b)$. Prove that $f$ is strictly increasing in $(a, b)$, and let $g$ be its inverse function. Prove that $g$ is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \qquad (a < x < b).$$

*Proof.*

**Exercise 5.3.** Suppose $g$ is a real function on $\mathbb{R}$, with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that $f$ is one-to-one if $\varepsilon$ is small enough.
(A set of admissible values of $\varepsilon$ can be determined which depends only on $M$.)

*Proof.*

**Exercise 5.4.** If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where $C_0, ..., C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

*Proof.*

**Exercise 5.5.** Suppose $f$ is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

*Proof.*

**Exercise 5.6.** Suppose

(a) $f$ is continuous for $x \geq 0$

(b) $f'(x)$ exists for $x > 0$

(c) $f(0) = 0$,

(d) $f'$ is monotonically increasing

Put

$$g(x) = \frac{f(x)}{x} \qquad (x > 0)$$

and prove that $g$ is monotonically increasing.

*Proof.*

**Exercise 5.7.** Suppose $f'(c)$, $g'(c)$ exist, $g'(c) \neq 0$, and $f(c) = g(c) = 0$. Prove that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

(This holds also for complex functions.)

*Proof.*

**Exercise 5.8.** Suppose $f'$ is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon$$

whenever $0 < |t - x| < \delta$, $a \le x \le b$, $a \le t \le b$. (This could be expressed by saying that $f$ is uniformly differentiable on $[a, b]$ if $f'$ is continuous on $[a, b]$.) Does this hold for vector-valued functions too?

*Proof.*

**Exercise 5.9.** Let $f$ be a continuous real function on $\mathbb{R}$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?

*Proof.*

**Exercise 5.10.** Suppose $f$ and $g$ are complex differentiable functions on $(0, 1)$, $f(x) \to 0$, $g(x) \to 0$, $f'(x) \to A$, $g'(x) \to B$ as $x \to 0$, where $A$ and $B$ are complex numbers, $B \neq 0$. Prove that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{A}{B}$$

Compare with **Example 5.18**.
Hint:

$$\frac{f(x)}{g(x)} = \left\{ \frac{f(x)}{x} - A \right\} \cdot \frac{x}{g(x)} + A \cdot \frac{x}{g(x)}.$$

*Proof.*

**Example 5.18.** On the segment $(0,\,1)$, define $f(x) = x$ and $g(x) = x + x^2 e^{i/x^2}$.
Since $|e^{it}| = 1$ for all real $t$, we see that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 1. \tag{36}$$

Next,

$$g'(x) = 1 + \left\{ 2x - \frac{2i}{x} e^{i/x^2} \right\} \qquad (0 < x < 1).$$

so that

$$|g'(x)| \geq \left| 2x - \frac{2i}{x} \right| - 1 \geq \frac{2}{x} - 1. \tag{38}$$

Hence

$$\left| \frac{f'(x)}{g'(x)} \right| = \frac{1}{|g'(x)|} \leq \frac{x}{2 - x}$$

and so

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = 0. \tag{40}$$

By (36) and (40), L'Hospital's rule fails in this case. Note also that $g'(x) \neq 0$ on $(0,\,1)$, by (38).

**Exercise 5.11.** Suppose $f$ is defined in a neighborhood of $x$, and suppose $f''(x)$ exists. Show that

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x)$$

Show by an example that the limit may exist even if $f''(x)$ does not.
Hint: Use **Theorem 5.13**.

*Proof.*

**Exercise 5.18.** Suppose $f$ is a real function on $[a, b]$, $n$ is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a, b]$. Let $\alpha$, $\beta$, and $P$ be as in Taylor's theorm (5.8). Define

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$$

for $t \in [a, b]$, $t \neq \beta$, differentiate

$$f(t) - f(\beta) = (t - \beta)Q(t)$$

$n - 1$ times at $t = \alpha$, and derive the following version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{n-1}(\alpha)}{(n-1)!}(\beta - \alpha)^n.$$

*Proof.*

**Exercise 5.19.** Suppose $f$ is defined in $(-1, 1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1$, $\alpha_n \to 0$, and $\beta_n \to 0$ as $n \to \infty$. Define the difference quotients

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}.$$

Prove the following statements:

(a) If $\alpha_n < 0 < \beta_n$, then $\lim D_n = f'(0)$

(b) If $0 < \alpha_n < \beta_n$ and $\{\beta_n/(\beta_n - \alpha_n)\}$ is bounded, then $\lim D_n = f'(0)$.

(c) If $f'$ is continuous in $(-1, 1)$, then $\lim D_n = f'(0)$.

Give an example in which $f$ is differentiable in $(-1, 1)$ (but $f'$ is not continuous at 0) and in which $\alpha_n$, $\beta_n$ tend to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.

*Proof.*

**Theorem 1.** Let $f$ be continuous mapping of $[a,\, b]$ to $\mathbb{R}^k$, $n \in \mathbb{N}$.
Suppose $f^{(n-1)}$ is continuous and $f^{(n)}$ exists at $[a,\, b]$.
Let $\alpha$, $\beta$ be distinct points on $[a,\, b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$$

Then, there exists $c \in (\alpha,\, \beta)$ such that

$$\|f(\beta) - P(\beta)\| \leq \left\| \frac{f^{(n)}(c)}{n!} \right\| (\beta - \alpha)^n$$

Note that $\|\cdot\|$ is $p$-norm.

**Exercise 5.20.** Formulate and prove an inequality which follows from Taylor's theorem and which remains valid for vector-valued functions.

*Proof.* We will prove **Theorem 1**
Let $u = f(\beta) - P(\beta)$, $g = u \cdot f$.
Then $g$ is continuous mapping of $[a, b]$ to $\mathbb{R}$.
As $u$ is constant vector, $g^{(n)} = u \cdot f^{(n)}$. By **Theorem 5.14**, there exists $c \in (\alpha, \beta)$ such that

$$g(\beta) = P'(\beta) + \frac{g^{(n)}(c)}{n!}(\beta - \alpha)^n \tag{1}$$

where $P'(x) = \sum_{k=0}^{n-1} \frac{u \cdot f^{(k)}(\alpha)}{k!}(x-\alpha)^k$.

$$\begin{aligned}
\therefore\ g(\beta) - P'(\beta) &= u \cdot f(\beta) - \sum_{k=0}^{n-1} \frac{u \cdot f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k \\
&= u \cdot \left\{ f(\beta) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k \right\} \\
&= u \cdot \{f(\beta) - f(\alpha)\} = \|u\|^2 \\
\frac{g^{(n)}(c)}{n!}(\beta - \alpha)^n &= \frac{u \cdot f^{(n)}(c)}{n!}(\beta - \alpha)^n \\
&\leq \|u\| \cdot \|\frac{f^{(n)}(c)}{n!}(\beta - \alpha)^n\| \qquad (\because \text{ Cauchy-Schwarz inequality})
\end{aligned}$$

Using above formulas, (1) can be

$$\|u\|^2 \leq \|u\| \cdot \left\| \frac{f^{(n)}(c)}{n!} \right\| (\beta - \alpha)^n$$

Hence $\|u\| > 0$,

$$\|f(\beta) - P(\beta)\| \leq \left\| \frac{f^{(n)}(c)}{n!} \right\| (\beta - \alpha)^n$$

**Lemma 1.** Let $P_n(x)$ be set of polynomials of degree $3n+1$. Suppose $f(x) = e^{-1/x^2}$ for $x \neq 0$, then

$$\lim_{x\to 0} g\left(\frac{1}{x}\right) f(x) = 0$$

where $g \in P_n(x)$.

*Proof.* Let $x = \dfrac{1}{t}$. It is easy to show that

$$\lim_{x\to 0} g\left(\frac{1}{x}\right) f(x) = 0$$

if and only if

$$\lim_{t\to\infty} g(t) f\left(\frac{1}{t}\right) = \lim_{t\to-\infty} g(t) f\left(\frac{1}{t}\right) = 0$$

using $\varepsilon - \delta$ argument.

So, we will show that $\lim\limits_{t\to\infty} g(t) f\left(\dfrac{1}{t}\right) = L$.

As $\lim\limits_{t\to\infty} \left\{t^{3n+2} - g(t)\right\} = \infty$, there exists $C > 0$ such that $|g(t)| \leq C\, t^{3n+2}$ for all $t \neq 0$.

$$\begin{aligned}
|g(t)| \leq C\, t^{3n+2} &\Leftrightarrow -C\, t^{3n+2} \leq g(t) \leq C\, t^{3n+2} \\
&\Leftrightarrow -C\, t^{3n+2} e^{-t^2} \leq g(t) f\left(\frac{1}{t}\right) \leq C\, t^{3n+2} e^{-t^2} \\
&\Rightarrow \lim_{t\to\infty} -C\, t^{3n+2} e^{-t^2} \leq \lim_{t\to\infty} g(t) f\left(\frac{1}{t}\right) \leq \lim_{t\to\infty} C\, t^{3n+2} e^{-t^2}
\end{aligned}$$

By squeeze theorem, $\lim\limits_{t\to\infty} g(t) f\left(\dfrac{1}{t}\right) = 0$.

The case $t \to -\infty$ is analogous, and proof is completed.

**Lemma 2.** Let $f(x) = e^{-1/x^2}$ for all $x \neq 0$, and $f(0) = 0$.
Then $f(x)$ is infinitely differentiable for $x \in \mathbb{R}$.
Moreover,

$$f^{(n)}(x) = \begin{cases} f(x)Q_n(x) & (x > 0) \\ 0 & (x = 0) \end{cases}$$

where $P_n(x)$ a polynomial function of degree $n$, and $Q_n(x) = P_{3n}\left(\frac{1}{x}\right)$ fulfills the recursive definition

$$\begin{aligned} Q_0(x) &= 1 \\ Q_n(x) &= \frac{2}{x^3}Q_{n-1}(x) + Q'_{n-1}(x) \end{aligned}$$

*Proof.* Let us use mathematical induction.
It is easy to show when $n = 1$.
Assume $n = k$ is true.
Then, $f^{(k+1)}(x)$ is well-defined for $x \neq 0$.
More Specifically,

$$\begin{aligned} f^{(k+1)}(x) &= \left\{e^{-1/x^2}\right\}' Q_k(x) + e^{-1/x^2}\left\{Q_k(x)\right\}' \\ &= \frac{2}{x^3}\, e^{-1/x^2} Q_k(x) + e^{-1/x^2}\, Q'_k(x) \\ &= e^{-1/x^2}\left\{\frac{2}{x^3}Q_k(x) + Q'_k(x)\right\} \\ &= e^{-1/x^2}Q_{n+1}(x) \\ &= f(x)Q_{n+1}(x) \end{aligned}$$

So if we show $f^{(k+1)}(0) = 0$, then we can say that $f^{(k+1)}$ is also differentiable for $x \in \mathbb{R}$.
By definition,

$$\begin{aligned} f^{(k+1)}(0) &= \lim_{x\to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{f^{(k)}(x)}{x} && (\because\ f^{(k)}(0) = 0) \\ &= \lim_{x\to 0} f(x)\frac{Q_k(x)}{x} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{x\to 0} f(x) P_{3k+1}\left(\frac{1}{x}\right) && (\because\ Q_k(x) = P_{3k}\left(\frac{1}{x}\right)) \\
&= 0 && (\because\ \text{By Lemma 1})
\end{aligned}
$$

Hence $f^{(k+1)}(0) = 0$, $f^{(k+1)}$ is also differentiable for $x \in \mathbb{R}$.
By the principle of mathematical induction, $f(x)$ is infinitely differentiable for $x \in \mathbb{R}$.

**Exercise 5.21.** Let $E$ be a closed subset of $\mathbb{R}$. There is a real continuous function $f$ on $\mathbb{R}$ whose zero set is $E$. Is it possible, for each closed set $E$, to find such an $f$ which is differentiable on $\mathbb{R}$, or one which is $n$ times differentiable, or even one which has derivatives of all orders on $\mathbb{R}$?

*Proof.* Let $f(x) = e^{-1/x^2}$ for all $x \neq 0$, and $f(0) = 0$.
By Lemma 2, $f(x)$ is infinitely differentiable, and $f^{(n)}(0)$ for $n \in \mathbb{N}$.
Define function $g$ by

$$g = (f \circ ReLU)(x)$$

with $ReLU(x) = \max(0,\, x)$.
*i.e.*

$$g(x) = \begin{cases} e^{-1/x^2} & (x > 0) \\ 0 & (x \leq 0) \end{cases}$$

$g$ is also infinitely differentiable everywhere on $\mathbb{R}$ and whose zero set is $(-\infty,\ 0]$.
Let us think

$$f_{(a,\,b)}(x) = g(x-a)g(b-x)$$

where $-\infty \leq a < b \leq \infty$.
Note that $f_{(-\infty,\,b)}(x) = e^{-\frac{1}{(x-b)^2}}$ for $x > b$, and similarly when $b = \infty$.
As set $E$ is closed, complement of $E$ is open set consisting of a union of disjoint open intervals, so $E^c = \bigcup_i (a_i,\ b_i)$
Let $l$ be collection of disjoint open intervals, and define function $h$ by

$$h(x) = \sum_{(a,\,b)\,\in\, l} f_{(a,\,b)}(x)$$

$h$ is well-defined since at least one of the terms in the sum isn't 0 for any $x \notin E$, infinitely differentiable on $\mathbb{R}$, and has zero set $E$.

**Exercise 5.22.** Suppose $f$ is a real function on $(-\infty,\ \infty)$. Call $x$ a *fixed point* of $f$ if $f(x) = x$.

(a) If $f$ is differentiable and $f\,'(t) \neq 1$ for every real $t$, prove that $f$ has at most one fixed point.

(b) Show that the function $f$ defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f\,'(t) < 1$ for all real $t$.

(c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real $t$, prove that a fixed point $x$ of $f$ exists, and that $x = \lim x_n$, where $x_1$ is an arbitary real number and

$$x_{n+1} = f(x_n)$$

for $n = 1,\ 2,\ 3, ....$

(d) Show that the process described in (c) can be visualized by the zig-zag path

$$(x_1,\ x_2) \to (x_2,\ x_2) \to (x_2,\ x_3) \to (x_3,\ x_3) \to (x_3,\ x_4) \to \cdots$$

*Proof.*