Chapter 7

Sequence and series of functions

List of Theorems

Discussion of main problem

Definition 7.1. Suppose $\{f_n\}$, $n = 1,2,3,\ldots$, is a sequence of functions defined on a set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. We can then define a function f by

$$f(x) = \lim_{n\to\infty} f_n(x) \quad (x \in E). \tag{1}$$

Under these circumstances we say that $\{f_n\}$ converges on E and that f is the limit, or the limit function, of $\{f_n\}$. Sometimes we shall use a more descriptive terminology and shall say that "$\{f_n\}$ converges to f **pointwise** on E" if (1) holds. Similarly, if $\sum f_n(x)$ converges for every $x \in E$, and if we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (x \in E),$$

the function f is called the sum of the series $\sum f_n$.
To say that f is continuous at x means

$$\lim_{t\to x} f(t) = f(x).$$

Hence, to ask whether the limit of a sequence of continuous functions is continuous is the same as to ask whether

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t), \tag{2}$$

i.e., whether the order in which limit processes are carried out is **immaterial**. On the left side of (2), we first let $n \to \infty$, then $t \to x$; on the right side, $t \to x$ first, then $n \to \infty$.

Example 7.2. For $m = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots$, let

$$s_{m,n} = \frac{m}{m+n}.$$

Then, for every fixed n,

$$\lim_{m\to\infty} s_{m,n} = 1,$$

so that

$$\lim_{n\to\infty}\lim_{m\to\infty} s_{mn} = 1.$$

On the other hand, for every fixed m,

$$\lim_{n\to\infty} s_{mn} = 0,$$

so that

$$\lim_{m\to\infty}\lim_{n\to\infty} s_{m,n} = 0.$$

Example 7.3. Let

$$f_n(x) = \frac{x^2}{(1+x^2)^n} \quad (x \text{ is real; } n = 0,\ 1,\ 2,\ \ldots),$$

and consider

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}.$$

Prove that $f(x)$ is convergences, and may have a discontinuous sum.

Uniform convergence

Definition 7.4. We say that a sequence of function $\{f_n\}$, $n = 1, 2, 3, \ldots$, converges **uniformly** on E to a function f if for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \varepsilon$$

for all $x \in E$.
It is clear that every uniformly convergence sequence is pointwise convergent.(*why*?)
We say that the series $\sum f_n(x)$ converges uniformly on E if the sequence $\{s_n\}$ of **partial sums** defined by

$$\sum_{i=1}^{n} f_i(x) = s_n(x)$$

converges uniformly on E.

Theorem 7.5. The sequence of functions $\{f_n\}$ defined on E, converges uniformly on E if and only if for every $\varepsilon > 0$ there exists an integer N such that $m \geq N$, $n \geq N$, $x \in E$ implies

$$|f_n(x) - f_m(x)| \leq \varepsilon.$$

Note that suppose the cauchy condition holds, by Theorem 3.11, the sequence $\{f_n(x)\}$ converges.

Proof.

Theorem 7.6. Suppose

$$\lim_{n\to\infty} f_n(x) = f(x) \quad (x \in E).$$

Put

$$M_n = \sup_{x\in E} |f_n(x) - f(x)|.$$

Then $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$.

Proof.

Theorem 7.7. Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose

$$|f_n(x)| \leq M_n \quad (x \in E,\ n = 1,\ 2,\ 3,\ \ldots).$$

Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Note that the converse is not asserted (and is, in fact, not true).

Proof.

Theorem 7.8. Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that

$$\lim_{t \to x} f_n(t) = A_n \quad (n = 1,\ 2,\ 3,\ \ldots).$$

Then $\{A_n\}$ converges, and

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n.$$

i.e., the conclusion is that

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t).$$

Proof.

Theorem 7.9. If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E.

This is very important.

Proof.

Theorem 7.10. Suppose K is compact, and

(a) $\{f_n\}$ is a sequence of continuous functions on K,

(b) $\{f_n\}$ converges pointwise to a continuous function f on K,

(c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1,\ 2,\ 3,\ \ldots,$.

Then $f_n \to f$ uniformly on K.

Proof.

Definition 7.11. If X is a metric space, $\mathscr{C}(X)$ will denote the set of all complex valued, continuous, bounded functions with domain X. We associate with each $f \in \mathscr{C}(X)$ its supremum norm

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Since f is assumed to be bounded, $\|f\| < \infty$. It is obvious that $\|f\| = 0$ only if $f(x) = 0$ for every $x \in X$, that is, only if $f = 0$. If $h = f + g$, then

$$|h(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\|$$

for all $x \in X$; hence

$$\|f + g\| \leq \|f\| + \|g\|.$$

If we define the distance between $f \in \mathscr{C}(X)$ and $g \in \mathscr{C}(X)$ to be $\|f - g\|$, it follows that **Axioms 2.15** for a metric are satisfied.

Theorem 7.12. The above metric makes $\mathscr{C}(X)$ into a complete metric space.

Proof.

Uniform convergence and integration

Theorem 7.13. Let α be monotonically increasing on $[a, b]$. Suppose $f \in \mathscr{R}(\alpha)$ on $[a, b]$, for $n = 1, 2, 3, \ldots$, and suppose $f_n \to f$ uniformly on $[a, b]$. Then $f \in \mathscr{R}(\alpha)$ on $[a, b]$, and

$$\int_a^b f\, d\alpha = \lim_{n\to\infty} \int_a^b f_n\, d\alpha.$$

(The existence of the limit is part of the conclusion.)

Proof.

Corollary 7.14. If $f_n \in \mathscr{R}(\alpha)$ on $[a, b]$ and if

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (a \le x \le b),$$

the series converging uniformly on $[a, b]$, then

$$\int_a^b f\, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n\, d\alpha.$$

i.e., the series may be integrated term by term.

Proof.

Uniform convergence and differentiation

Theorem 7.15. Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and

$$f'(x) = \lim_{n\to\infty} f_n'(x) \quad (a \le x \le b).$$

Proof.

Theorem 7.16. There exists a real continuous function on the real line which is nowhere differentiable.

Proof.

Equicontinuous familites of functions

Definition 7.17. Let $\{f_n\}$ be a sequence of functions defined on a set E. We say that $\{f_n\}$ is **pointwise bounded** on E if the sequence $\{f_n(x)\}$ is bounded for every $x \in E$, that is, if there exists a finite-valued function ϕ defined on E such that

$$|f_n(x)| < \phi(x) \quad (x \in E, n = 1,\ 2,\ 3,\ \ldots).$$

We say that $\{f_n\}$ is uniformly bounded on E if there exists a number M such that

$$|f_n(x)| < M \quad (x \in E,\ n = 1,\ 2,\ 3,\ \ldots).$$

Definition 7.18. A family $\mathscr{F}$ of complex functions f defined on a set E in a metric space X is said to be equicontinuous on E if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon$$

whenever $d(x, y) < \delta$, $x \in E$, $y \in E$, and $f \in \mathscr{F}$. Here d denotes the metric of X.

Theorem 7.19. If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.

Proof.

Theorem 7.20. If K is a compact metric space, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \ldots$, and if $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous on K.

Proof.

Theorem 7.21. If K is compact, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \ldots$, and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then

(a) $\{f_n\}$ is uniformly bounded on K,

(b) $\{f_n\}$ contains a uniformly convergent subsequence.

Proof.

The stone-weierstrass theorem

Theorem 7.22. If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomials P_n such that

$$\lim_{n\to\infty} P_n(x) = f(x)$$

uniformly on $[a, b]$. If f is real, the P_n may be taken real.

This is the form in which the theorem was originally discovered by weierstrass.

Proof.

Continued....

Corollary 7.23. For every interval $[-a, a]$ there is a sequence of real polynomials P_n such that $P_n(0) = 0$ and such that

$$\lim_{n\to\infty} P_n(x) = |x|$$

uniformly on $[-a, a]$.

Proof.

Definition 7.24. A family $\mathscr{A}$ of complex functions defined on a set E is said to be an **algebra** if (i) $f + g \in \mathscr{A}$. (ii) $fg \in \mathscr{A}$. (iii) $cf \in \mathscr{A}$ for all $f \in \mathscr{A}$, $g \in \mathscr{A}$, and for all complex constants c, that is, if $\mathscr{A}$ is closed under addition, multiplication, and scalar multiplication. We shall also have to consider algebras of real functions; in this case, (iii) is of course only required to hold for all real c.
If $\mathscr{A}$ has the property that $f \in \mathscr{A}$ whenever $f_n \in \mathscr{A} \quad (n = 1,\ 2,\ 3,\ \ldots)$ and $f_n \to f$ uniformly on E, then $\mathscr{A}$ is said to be uniformly closed.
Let $\mathscr{B}$ be the set of all functions which are limits of uniformly convergent sequences of members of $\mathscr{A}$. Then $\mathscr{B}$ is called the **uniform closure** of $\mathscr{A}$.

Theorem 7.25. Let $\mathscr{B}$ be the uniform closure of an algebra $\mathscr{A}$ of bounded functions. Then $\mathscr{B}$ is a uniformly closed algebra.

Proof.

Definition 7.26. Let $\mathscr{A}$ be a family of functions on a set E. Then $\mathscr{A}$ is said to **separate points** on E if to every pair of distincts point x_1, $x_2 \in E$ there corresponds a function $f \in \mathscr{A}$ such that $f(x_1) \neq f(x_2)$.
If to each $x \in E$ there corresponds a function $g \in \mathscr{A}$ such that $g(x) \neq 0$, we say that $\mathscr{A}$ vanishes at no point of E.
The algebra of all polynomials in one variable clearly has these properties on $\mathbb{R}$. An example of an algebra which does not separate points is the set of all even polynomials, say on $[-1, 1]$, since $f(-x) = f(x)$ for every even function f.

Theorem 7.27. Suppose $\mathscr{A}$ is an algebra of functions on a set E, $\mathscr{A}$ separates points on E, and $\mathscr{A}$ vanishes at no point of E. Suppose x_1, x_2 are distinct points of E, and c_1, c_2 are constants (real if $\mathscr{A}$ is a real algebra). Then $\mathscr{A}$ contains a function f such that

$$f(x_1) = c_1, \quad f(x_2) = c_2.$$

Proof.

Lemma 7.28. If $f \in \mathscr{R}$, then $|f| \in \mathscr{R}$.

Proof.

Lemma 7.29. If $f \in \mathscr{R}$ and $g \in \mathscr{R}$, then $\max(f,\, g) \in \mathscr{R}$ and $\min(f,\, g) \in \mathscr{R}$.

Proof.

Lemma 7.30. Given a real function f, continuous on K, a point $x \in K$, and $\varepsilon > 0$, there exists a function $g_x \in \mathscr{B}$ such that $g_x(x) = f(x)$ and

$$g_x(t) > f(t) - \varepsilon \quad (t \in K).$$

Proof.

Lemma 7.31. Given a real function f, continuous on K, and $\varepsilon > 0$, there exists a function $h \in \mathscr{B}$ such that

$$|h(x) - f(x)| < \varepsilon \quad (x \in K).$$

Since $\mathscr{B}$ is uniformly closed, this statement is equivalent to the conclusion of the theorem.

Proof.

Theorem 7.32. Let $\mathscr{A}$ be an algebra of real continuous functions on a compact set K. If $\mathscr{A}$ separates points on K and if $\mathscr{A}$ vanishes at no point of K, then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K.

Proof.

Theorem 7.33. Suppose $\mathscr{A}$ is a self-adjoint algebra of complex continuous functions on a compact set K, $\mathscr{A}$ separates points on K, and $\mathscr{A}$ vanishes at no point of K. Then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all complex continuous functions on K. *i.e.*, $\mathscr{A}$ is dense $\mathscr{C}(K)$.

Proof.