Chapter 5

Differentation

The derivative of a real function

Definition 5.1. Let f be defined (and real-valued) on $[a, b]$. For any $x \in [a, b]$ form the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \qquad (a < t < b,\ t \neq x) \tag{1}$$

and define

$$f'(x) = \lim_{t \to x} \phi(t), \tag{2}$$

Theorem 5.2. Let f be defined on $[a, b]$. If f is differentiable at a point $x \in [a, b]$, then f is continuous at x.

Proof. ■

Theorem 5.3. Suppose f and g are defined on $[a, b]$ and are differentiable at a point $x \in [a, b]$. Then $f + g$, and fg, and f/g are differentable at x, and

(a) $(f + g)'(x) = f'(x) + g'(x)$;

(b) $(fg)'(x) = f'(x) + g'(x)$;

(c) $\left(\dfrac{f}{g}\right)'(x) = \dfrac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$

In (c), we assume of course that $g(x) \neq 0$

Proof. ■

Theorem 5.4. Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If

$$h(t) = g(f(t)) \qquad (a \leq t \leq b),$$

then h is differentiable at x, and

$$h'(x) = g'(f(x))\, f'(x) \tag{3}$$

Proof. ■

Mean value theorems

Definition 5.5. Let f be a real function defined on a metric space X. We say that f has a *local maximum* at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$

Local minima are defined likewise.

Theorem 5.6. Let f be defined on $[a, b]$; if f has a local maximum at a point $x \in (a,\ b)$, and if $f'(x)$ exists, then $f'(x) = 0$.

Proof. ■

Theorem 5.7. If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b), then there is a point $x \in (a, b)$ at which

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$$

Note that differentiability is not required at the endpoints.

Proof. ■

Theorem 5.8. If f is a real continuous function on $[a, b]$ which is differentiable in (a, b), then there is a point $x \in (a, b)$ at which

$$f(b) - f(a) = (b - a)f'(x)$$

Proof. ■

Theorem 5.9. Suppose f is differentiable in $(a,\, b)$

(a) If $f\,'(x) \geq 0$ for all $x \in (a,\, b)$, then f is monotonically increasing.

(b) If $f\,'(x) = 0$ for all $x \in (a,\, b)$, then f is constant.

(c) If $f\,'(x) \leq 0$ for all $x \in (a,\, b)$, then f is monotonically decreasing.

Proof. ■

The continuity of derivatives

Theorem 5.10. Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = \lambda$

Corollary 5.11. If f is differentiable on $[a, b]$, then f' cannot have any simple dis-continuities on $[a, b]$

Proof. ■

L'Hospitial's rule

Theorem 5.12. Suppose f and g are real and differentiable in (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq +\infty$. Suppose

$$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a. \tag{4}$$

If

$$f(x) \to 0 \text{ and } g(x) \to 0 \text{ as } x \to a, \tag{5}$$

or if

$$g(x) \to +\infty \text{ as } x \to a, \tag{6}$$

then

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to a \tag{7}$$

The analogous statement is of course aslo true if $x \to b$, or if $g(x) \to -\infty$ in (5).

Proof. ■

Definition 5.13. If f has a derivative f' on an interval, and if f' is itself differentiable, we denote the derivative of f' by f'' and call f'' the second derivative of f. Continuing in this manner, we obtain functions

$$f,\ f',\ f'',\ f^{(3)},\ ...,\ f^{(n)},$$

each of which is the derivative of the preceding one. $f^{(n)}$ is called the nth derivative, or the derivative of order n, of f.

Taylor's theorem

Theorem 5.14. Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let $\alpha, \ \beta$ be distinct points of $[a, b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k \tag{8}$$

Then there exists a point x between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n \tag{9}$$

For $n = 1$, this is just the mean value theorem. In general, the theorem shows that f can be approximated by a polynomial of degree $n-1$, and that (9) allows us to estimate the error, if we know bounds on $|f^{(n)}(x)|$.

Proof. ■

Differentiation of vector-valued functions

Remarks 5.15. Definition 5.1. applies without any change to complex functions f defined on $[a, b]$, and Theorems 5.2 and 5.3, as well as their proofs, remain valid. If f_1 and f_2 are the real and imaginary parts of f, that is if

$$f(t) = f_1(t) + i f_2(t)$$

for $a \leq t \leq b$, where $f_1(t)$ and $f_2(t)$ are real, then we clearly have

$$f\,'(x) = f_1{}'(x) + i f_2{}'(x); \tag{10}$$

also, f is differentiable at x if and only if both f_1 and f_2 are differentiable at x.

Theorem 5.16. Suppose $\mathbf{f}$ is a continuous mapping of $[a, b]$ into $\mathbb{R}^k$ and $\mathbf{f}$ is differentiable in (a, b). Then there exists $x \in (a, b)$ such that

$$|\mathbf{f}(b) - \mathbf{f}(a)| \leq (b - a)|\mathbf{f}'(x)|.$$

Proof. ■