Chapter 6

The Riemann-Stieltjes Integral Selected Exercise

List of Exercise

Exercise 1. Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is countinuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f d\alpha = 0$.

Proof.

Exercise 2. Suppose $f \geq 0$, f is countinuous on $[a, b]$, and $\displaystyle\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare this with **Exercise 1**.)

Proof.

Exercise 3. Define three functions β_1, β_2, β_3 as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j(x) = 1$ if $x > 0$ for $j = 1, 2, 3$; and $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_3(0) = \frac{1}{2}$. Let f be a bounded function on $[-1, 1]$.

(a) Prove that $f \in \mathscr{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then

$$\int f \, d\beta_1 = f(0).$$

(b) State and prove a similiar result for β_2.

(c) Prove that $f \in \mathscr{R}(\beta_3)$ if and only if f is continuous at 0.

(d) If f is continuous at 0, prove that

$$\int f \, d\beta_1 = \int f \, d\beta_2 = \int f \, d\beta_3 = f(0).$$

Proof.

Exercise 4. If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that $f \notin \mathscr{R}$ on $[a, b]$ for any $a < b$.

Proof.

Exercise 5. Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathscr{R}$ on $[a, b]$. Does it follow that $f \in \mathscr{R}$? Does the answer change if we assume that $f^3 \in \mathscr{R}$?

Proof.

Exercise 6. Let P be the Cantor set constructed in **Sec. 2.44**. Let f be a bounded real function on $[0, 1]$ which is continuous at every point outside P. Prove that $f \in \mathscr{R}$ on $[0, 1]$.
Hint : P can be covered be finitely many segments whose total length can be made as small as desired. Proceed as in **Theroem 6.10**.

Sec 2.44 (The Cantor set)**.** The set which we are now going to construct shows that there exist perfect sets in R^1 which contain no segment.
Let E_0 be the intrval $[0, 1]$. Remove the segment $(\frac{1}{3}, \frac{2}{3})$, and let E_1 be the union of the intervals

$$\left[0, \frac{1}{3}\right] \quad \left[\frac{2}{3}, 1\right]$$

Remove the middle thirds of these intervals, and let E_2 be the union of the intervals

$$\left[0, \frac{1}{9}\right] \quad \left[\frac{2}{9}, \frac{3}{9}\right] \quad \left[\frac{6}{9}, \frac{7}{9}\right] \quad \left[\frac{8}{9}, 1\right]$$

Continuing in this way, we obtain a sequence of compact sets E_n, such that

(a) $E_1 \supset E_2 \supset E_3 \supset \cdots$;

(b) E_n is the union of 2^n intervals, each of length 3^{-n}.

The set

$$P = \bigcap_{n=1}^{\infty} E_n$$

is called the *Cantor set*. P is clearly compact, and P is not empty.

Proof.

Proof. (Continued...)

Exercise 7. Suppose f is a real function on $(0, 1]$ and $f \in \mathscr{R}$ on $[c, 1]$ for every $c > 0$. Define

$$\int_0^1 f(x)dx = \lim_{c \to 0} \int_c^1 f(x)dx$$

if this limit exists (and is finite).

(a) If $f \in \mathscr{R}$ on $[0, 1]$, show that this definition of the integral agrees with the old one.

(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

Proof.

Exercise 8. Suppose $f \in \mathscr{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

if the limit exists (and is finite). In that case, we say that the integral on the left *converges*. If it also converges after f has been replaced by $|f|$, it is said to converge *absolutely*.
Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that

$$\int_1^\infty f(x)dx$$

converges if and only if

$$\sum_{n=1}^\infty f(n)$$

converges. (This is the so-called "integral test" for convergence of series.)

Proof.

Exercise 9. Show that integration by parts can sometimes be applied to the "improper" integrals defined in **Exercise 7** and **Exercise 8**. (State appropriate hypotheses, formulate a theorem, and prove it!) For instance show that

$$\int_0^\infty \frac{\cos x}{1+x} dx = \int_0^\infty \frac{\sin x}{(1+x)^2} dx.$$

Show that one of these integrals converges absolutely, but that the other does not.

Proof.

Exercise 10. Let p and q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

(a) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

(b) If $f \in \mathscr{R}(\alpha)$, $g \in \mathscr{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p d\alpha = 1 = \int_a^b g^q d\alpha,$$

then

$$\int_a^b fg \, d\alpha \leq 1.$$

(c) If f and g are complex functions in $\mathscr{R}(\alpha)$, then

$$\left| \int_a^b fg \, d\alpha \right| \leq \left\{ \int_a^b |f|^p \, d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q \, d\alpha \right\}^{1/q}.$$

This is *Hölder's inequality*. When $p = q = 2$ it is usually called the *Schwarz inequality*.

(d) Show that *Hölder's inequality* is also true for the "improper" integrals described in **Exercise 7** and **Exercise 8**.

Proof.

Exercise 11. Let α be a fixed increasing function on $[a, b]$. For $u \in \mathscr{R}(\alpha)$, define

$$\|u\|_2 = \left\{ \int_a^b |u|^2 \, d\alpha \right\}^{1/2}.$$

Suppose f, g, $h \in \mathscr{R}(\alpha)$, and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the *Schwarz inequality*, as in the proof of **Theorem 1.37.**

Proof.

Exercise 17. Suppose α increases monotonically on $[a, b]$, g is continuous, and $g(x) = G'(x)$ for $a \le x \le b$. Prove that

$$\int_a^b \alpha(x)g(x)\,dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G\,d\alpha.$$

Hint : Take g real, without loss of generality. Given $P = \{x_0, x_1, \ldots, x_n\}$, choose $t_i \in (x_{i-1}, x_i)$ so that $g(t_i)\Delta x_i = G(x_i) - G(x_{i-1})$. Show that

$$\sum_{i=1}^{n} \alpha(x_i)g(t_i)\Delta x_i = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^{n} G(x_{i-1})\Delta\alpha_i.$$