Chapter 6

The Riemann-Stieltjes Integral

List of Theorems

Definition and existence of the integral

Definition 6.1. Let $[a,\ b]$ be a given interval, By a partition P of $[a,\ b]$ we mean a finite set of points $x_0,\ x_1,\ \ldots,\ x_n$, where

$$a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$$

We write

$$\Delta x_i = x_i - x_{i-1} \qquad (i = 1, 2, ..., n).$$

Now suppose f is a bounded real function defined on $[a,\ b]$. Corresponding to each partition P of $[a,\ b]$ we put

$$\begin{aligned} M_i &= \sup\ f(x) \qquad (x_{i-1} \leq x \leq x_i), \\ m_i &= \inf\ f(x) \qquad (x_{i-1} \leq x \leq x_i), \\ U(P, f) &= \sum_{i=1}^{n} M_i \Delta x_i, \\ L(P, f) &= \sum_{i=1}^{n} m_i \Delta x_i \end{aligned}$$

and finally

$$\overline{\int_a^b} f dx = \inf U(P,\ f) \tag{1}$$

$$\underline{\int_a^b} f dx = \sup L(P,\ f) \tag{2}$$

where the inf and the sup are taken over all partitions P of $[a,\ b]$. The left members of (1) and (2) are called **the upper** and **lower Riemann integrals** of f over $[a,\ b]$, respectively.

If the upper and lower integrals are equal, we say that f is **Riemann integrable** on $[a,\ b]$, we write $f \in \mathscr{R}$ (that is, $\mathscr{R}$ denotes the set of Riemann integrable functions), and we denote the common value (1) of (2) by

$$\int_a^b f dx \tag{3}$$

or by

$$\int_a^b f(x) dx \tag{4}$$

Definition 6.2. Let α be a monotonically increasing function on $[a,\ b]$ (since $\alpha(a)$ and $\alpha(b)$ are finite, it follows that α is bounded on $[a,\ b]$). Corresponding to each partition P of $[a,\ b]$, we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

It is clear that $\Delta\alpha_i \geq 0$. For any real function f which is bounded on $[a,\ b]$ We put

$$U(P,\ f,\ \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i L(P,\ f,\ \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

where M_i, m_i have the same meaning as in **Definition 6.1**, and we define

$$\overline{\int_a^b} f\, d\alpha = \inf U(P,\ f,\ \alpha) \tag{5}$$

$$\underline{\int_a^b} f\, d\alpha = \sup L(P,\ f,\ \alpha) \tag{6}$$

the inf and sup again being taken over all partitions.

If the left members of (5) and (6) are equal, we denote their common value by

$$\int_a^b f\, d\alpha \tag{7}$$

or sometimes by

$$\int_a^b f(x)\, d\alpha(x) \tag{8}$$

This is the **Riemann-Stieltjes integral** (or simply the **Stieltjes integral**) of f w.r.t α, over $[a,\ b]$. If (7) exists, *i.e.*, if (5) and (6) are equal, we say that f is integrable w.r.t α, in the Riemann sense, and write $f \in \mathscr{R}(\alpha)$.

Definition 6.3. We say that the partition P^* is a **refinement** of P if $P^* \supset P$ (that is, if every point of P is a point of P^*). Given two partitions, P_1 and P_2, we say that P^* is their **common refinement** if $P^* = P_1 \cup P_2$.

Theorem 6.4. If P^* is a refinement of P, then

$$L(P,\ f,\ \alpha) \leq L(P^*,\ f,\ \alpha) \tag{9}$$

and

$$U(P^*,\ f,\ \alpha) \leq U(P,\ f,\ \alpha). \tag{10}$$

Proof.

Theorem 6.5. $\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$

Proof.

Theorem 6.6. $f \in \mathscr{R}(\alpha)$ on $[a,\ b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that

$$U(P,\ f,\ \alpha) - L(P,\ f,\ \alpha) < \varepsilon \tag{11}$$

Proof.

Theorem 6.7.

(a) If (11) holds for some P and some ε, then (11) holds (with the same ε) for every refinement of P.

(b) If (11) holds for $P = \{x_0, \ldots, x_n\}$ and if s_i, t_i are arbitary points in $[x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta\alpha_i < \varepsilon$$

(c) If $f \in \mathscr{R}(\alpha)$ and the hypotheses of (b) holds, then

$$\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\, d\alpha\right| < \varepsilon$$

Proof.

Theorem 6.8. If f is continuous on $[a, b]$ then $f \in \mathscr{R}(\alpha)$ on $[a, b]$

Proof.

Theorem 6.9. If f is monotonic on $[a, b]$, and if α is continuous on $[a, b]$, then $f \in \mathscr{R}(\alpha)$. (We still assume, of course, that α is monotonic.)

Proof.

Theorem 6.10. Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathscr{R}(\alpha)$.

Proof.

Theorem 6.11. Suppose $f \in \mathscr{R}(\alpha)$ on $[a, b]$, $m \leq f \leq M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in \mathscr{R}(\alpha)$ on $[a, b]$.

Proof.

Properties of the integral

Theorem 6.12.

(a) If $f_1 \in \mathscr{R}(\alpha)$ and $f_2 \in \mathscr{R}(\alpha)$ on $[a, b]$, then

$$f_1 + f_2 \in \mathscr{R}(\alpha)$$

$cf \in \mathscr{R}(\alpha)$ for every constant c, and

$$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha,$$
$$\int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha$$

(b) If $f_1(x) \leq f_2(x)$ on $[a, b]$, then

$$\int_a^b f_1\, d\alpha \leq \int_a^b f_2\, d\alpha.$$

(c) If $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and if $a < c < b$, then $f \in \mathscr{R}(\alpha)$ on $[a, c]$ and on $[c, b]$, and

$$\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha$$

(d) If $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and if $|f(x)| \leq M$ on $[a, b]$, then

$$\left|\int_a^b f\, d\alpha\right| \leq M[\alpha(b) - \alpha(a)].$$

(e) If $f \in \mathscr{R}(\alpha_1)$ and $f \in \mathscr{R}(\alpha_2)$, then $f \in \mathscr{R}(\alpha_1 + \alpha_2)$ and

$$\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2;$$

If $f \in \mathscr{R}(\alpha)$ and c is a positive constant, then $f \in \mathscr{R}(c\alpha)$ and

$$\int_a^b f\, d(c\alpha) = c\int_a^b f\, d\alpha.$$

Proof.

Theorem 6.13. If $f \in \mathscr{R}(\alpha)$ and $g \in \mathscr{R}(\alpha)$ on $[a, b]$, then

(a) $fg \in \mathscr{R}(\alpha)$;

(b) $|f| \in \mathscr{R}(\alpha)$ and $\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$

Proof.

Definition 6.14. The unit step function I is defined by

$$I(x) = \begin{cases} 0 & (x \le 0), \\ 1 & (x > 0). \end{cases}$$

Theorem 6.15. If $a < s < b$, f is bounded on $[a, b]$, f is continuous at s, and $\alpha(x) = I(x - s)$, then

$$\int_a^b f\, d\alpha = f(s)$$

Proof.

Theorem 6.16. Suppose $c_n \geq 0$ for 1, 2, 3, ..., $\sum c_n$ converges, $\{s_n\}$ is a sequence of distinct points in (a, b) and

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n). \tag{12}$$

Let f be continuous on $[a, b]$. Then

$$\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} c_n f(s_n). \tag{13}$$

Proof.

Theorem 6.17. Assume α increases monotonically and $\alpha' \in \mathscr{R}$ on $[a, b]$. Let f be a bounded real function on $[a, b]$.
Then $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$. In that case

$$\int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx. \tag{14}$$

Proof.

Remarks 6.18. The two preceding theorems illustrate the generality and flexibility which are inherent in the Stieltjes process of integration. If α is a pure step function [this is the name often given to functions of the form (12)], the integral reduces to a finite or infinite series. If α has an integrable derivative, the integral reduces to an ordinary Riemann integral. This makes it possible in many cases to study series and integrals simultaneously, rather than separately.

Theorem 6.19 (change of variable)**.** Suppose φ is a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$ and $f \in \mathscr{R}(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by

$$\beta(y) = \alpha(\varphi(y)), \qquad g(y) = f(\varphi(y)). \tag{15}$$

Then $g \in \mathscr{R}(\beta)$ and

$$\int_A^B g\, d\beta = \int_a^b f\, d\alpha \tag{16}$$

Proof.

Integration and differentiation

Theorem 6.20. Let $f \in \mathscr{R}$ on $[a, b]$. For $a \le x \le b$, put

$$F(x) = \int_a^x f(t)\,dt.$$

Then F is continuous on $[a, b]$; furthermore, if f is continuous at a point x_0 of $[a, b]$, then F is differentiable at x_0, and

$$F'(x_0) = f(x_0).$$

Proof.

Theorem 6.21 (**The fundamental theorem of calculus**)**.** If $f \in \mathscr{R}$ on $[a, b]$ and if there is a differentiable function F on $[a, b]$ such that $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof.

Theorem 6.22 (**Integration by parts**)**.** Suppose F and G are differentiable functions on $[a, b]$, $F' = f \in \mathscr{R}$, and $G' = g \in \mathscr{R}$. Then

$$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx.$$

Proof.

Integration of vector-valued functions

Definition 6.23. Let $f_1, \ldots, f_k$ be real functions on $[a, b]$ and let $\mathbf{f} = (f_1, \ldots, f_k)$ be the corresponding mapping of $[a, b]$ into $\mathbb{R}^k$. If α increases monotonically on $[a, b]$, to say that $\mathbf{f} \in \mathscr{R}(\alpha)$ means that $f_j \in \mathscr{R}(\alpha)$ for $j = 1, \ldots, k$. If this is the case, we define

$$\int_a^b \mathbf{f}\, d\alpha = \left(\int_a^b f_1\, d\alpha, \ldots, \int_a^b f_k\, d\alpha \right).$$

In other words, $\int \mathbf{f}\, d\alpha$ is the point in $\mathbb{R}^k$ whose j-th coordinate is $\int f_j\, d\alpha$.

Theorem 6.24. If $\mathbf{f}$ and $\mathbf{F}$ map $[a, b]$ into $\mathbb{R}^k$, if $\mathbf{f} \in \mathscr{R}$ on $[a, b]$, and if $\mathbf{F}' = \mathbf{f}$, then

$$\int_a^b \mathbf{f}(t)\,dt = \mathbf{F}(b) - \mathbf{F}(a)$$

Proof.

Theorem 6.25. If $\mathbf{f}$ maps $[a, b]$ into $\mathbb{R}^k$ and if $\mathbf{f} \in \mathscr{R}(\alpha)$ for some monotonically increasing function α on $[a, b]$, then $|\mathbf{f}| \in \mathscr{R}(\alpha)$, and

$$\left| \int_a^b \mathbf{f} \, d\alpha \right| \leq \int_a^b |\mathbf{f}| \, d\alpha. \tag{17}$$

Definition 6.26. A continuous mapping γ of an interval $[a, b]$ into $\mathbb{R}^k$ is called a **curve** in $\mathbb{R}^k$. To emphasize the parameter interval $[a, b]$, we may also say that γ is a curve on $[a, b]$.

If γ is one-to-one, γ is called an **arc**

If $\gamma(\alpha) = \gamma(\beta)$, γ is said to be a **closed curve**.

Theorem 6.27. If γ' is continuous on $[a, b]$, then γ is **rectifiable**, and

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)|\, dt < \infty.$$

Proof.