Contents

5.2 Diagonalizability

Problem 1. For each of following matrices $A \in \mathsf{M}_{n\times n}(R)$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(a) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

(g) $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$

Proof.

Problem 2. For each of the following linear operator T on a vector space V, test T for diagonalizability, and if T is diagonalizable, find a basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

(a) $\mathsf{V} = \mathsf{P}_3(R)$ and T is defined by $\mathsf{T}(f(x)) = f\,'(x) + f\,''(x)$, respectively.

(b) $\mathsf{V} = \mathsf{P}_2(R)$ and T is defined by $\mathsf{T}(ax^2 + bx + c) = cx^2 + bx + a$.

(c) $\mathsf{V} = \mathsf{R}^3$ and T is defined by

$$\mathsf{T}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ -a_1 \\ 2a_3 \end{pmatrix}.$$

(d) $\mathsf{V} = \mathsf{P}_2(R)$ and T is defined by $\mathsf{T}(f(x)) = f(0) + f(1)(x + x^2)$.

(e) $\mathsf{V} = \mathsf{C}^2$ and T is defined by $\mathsf{T}(z,\ w) = (z + iw,\ iz + w)$.

(f) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ and T is defined by $\mathsf{T}(A) = A^T$.

Proof.

Problem 3. Prove the matrix version of corollary to **Theorem 5.5**: If $A \in \mathsf{M}_{n\times n}(F)$ has n distinct eigenvalues, then A is diagonalizable.

Proof.

Problem 4. State and prove the matrix version of **Theorem 5.6**

Theorem 5.6. The characteristic polynomial of any diagonalizable linear operator splits.

Proof.

Problem 5.

(a) Justify the test for diagonalizability and the method for diagonalization stated in this section.

(b) Formulate the results in (a) for matrices.

Proof.

Problem 6. For
$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in \mathsf{M}_{2\times 2}(R),$$
find an expression for A^n, where n is an arbitrary positive integer.

Proof.

Problem 7. Suppose that $A \in \mathsf{M}_{n\times n}(F)$ has two distinct eigenvalues, λ_1 and λ_2, and that $\dim(\mathsf{E}_{\lambda_1}) = n-1$. Prove that A is diagonalizable.

Proof.

Problem 8. Let T be a linear operator on a finite-dimensional vector space V, and suppose there exists an ordered basis β for V such that $[\mathsf{T}]_\beta$ is an upper triangular matrix

(a) Prove that the characteristic polynomial for T splits.

(b) State and prove an analogous result ofr matrices.

The converse of (a) is treated in **Problem 32** of **Section 5.4.**

Proof.

Problem 9. Let T be a linear operator on a finite-dimensional vector space V with the distinct eigenvalues λ_1, λ_2, ..., λ_k and corresponding multiplicities m_1, m_2, ..., m_k. Suppose that β is a basis for V such that $[\mathsf{T}]_\beta$ is an upper triangular matrix. Prove that the diagonal entries of $[\mathsf{T}]_\beta$ are λ_1, λ_2, ..., λ_k and that each λ_i occurs m_i times $(1 \leq i \leq k)$.

Proof.

Problem 10. Let A be an $n \times n$ matrix that is similar to an upper triangular matrix and has the distinct eigenvalues λ_1, λ_2, ..., λ_k with corresponding multiplicities m_1, m_2, ..., m_k. Prove the following statements.

(a) $\mathrm{tr}(A) = \displaystyle\sum_{i=1}^{k} m_i \lambda_i$

(b) $\det(A) = (\lambda_1)^{m_1}(\lambda_2)^{m_2} \cdots (\lambda_k)^{m_k}$.

Proof.

Problem 11. Let T be an invertible linear operator on a finite-dimensional vector space V.

(a) Recall that for any eigenvalue λ of T, λ^{-1} is an eigenvalue of T^{-1}(**Exericse 8** of **Section 5.1**). Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

(b) Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

Proof.

Problem 12. Let $A \in \mathsf{T}_{n\times n}(F)$. Recall form **Problem 14** of **Section 5.1** that A and A^T have the same characteristic polynomial and hence share the same eigenvalues with the same multiplicities. For any eigenvalue λ of A and A^T, let E_λ and $\mathsf{E'}_\lambda$ denote the corresponding eigenspaces for A and A^T, respectively.

(a) Show by way of example that for a given common eigenvalue, these two eigenspaces need not be the same.

(b) Prove that for any eigenvalue λ, $\dim(\mathsf{E}_\lambda) = \dim(\mathsf{E'}_\lambda)$.

(c) Prove that if A is diagonalizable, then A^T is also diagonalizble.

Proof.

Problem 20. Let $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be subspaces of a finite-dimensional vector space V such that

$$\sum_{i=1}^{k} \mathsf{W}_i = \mathsf{V}.$$

Prove that V is the direct sum of $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ if and only if

$$\dim(\mathsf{V}) = \sum_{i=1}^{k} \dim(\mathsf{W}_i).$$

Proof.

Problem 21. Let V be a finite-dimensional vector space with a basis β, anad let β_1, β_2, …, β_k be a partition of β (*i.e.*, β_1, β_2, …, β_k are subsets of β such that $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ and $\beta_i \cap \beta_j = \varnothing$ if $i \neq j$). Prove that $\mathsf{V} = \operatorname{span}(\beta_1) \oplus \operatorname{span}(\beta_2) \oplus \cdots \oplus \operatorname{span}(\beta_k)$.

Proof.

Problem 22. Let T be a linear operator on a finite-dimensional vector space V, and suppose that the distinct eigenvalues of T are $\lambda_1, \lambda_2, \ldots, \lambda_k$. Prove that

$$\operatorname{span}\left(\{x \in \mathsf{V} : x \text{ is an eigenvector of } \mathsf{T}\}\right) = \mathsf{E}_{\lambda_1} \oplus \mathsf{E}_{\lambda_2} \oplus \cdots \oplus \mathsf{E}_{\lambda_k}.$$

Proof.

Problem 23. Let W_1, W_2, K_1, $|mathsfK_2$, …, K_p, M_1, M_2, …, M_q be subspaces of a vector space V such that $\mathsf{W}_1 = \mathsf{K}_1 \oplus \mathsf{K}_2 \oplus \cdots \oplus \mathsf{K}_p$ and $\mathsf{W}_2 = \mathsf{M}_1 \oplus \mathsf{M}_2 \oplus \cdots \oplus \mathsf{M}_q$. Prove that if $\mathsf{W}_1 \cap \mathsf{W}_2 = \{\mathit{0}\}$, then

$$\mathsf{W}_1 + \mathsf{W}_2 = \mathsf{W}_1 \oplus \mathsf{W}_2 = \mathsf{K}_1 \oplus \mathsf{K}_2 \cdots \oplus \mathsf{K}_p \oplus \mathsf{M}_1 \oplus \mathsf{M}_2 \oplus \cdots \oplus \mathsf{M}_q.$$

Proof.

5.6 Invariant subspaces and the Cayley–Hamilton Theorem

Problem 2. For each of the following linear operatos T on the vector space V, determine whether the given subspace W ia a T-invariant subspace of V.

(a) $\mathsf{V} = \mathsf{P}_3(R)$, $\mathsf{T}(f(x)) = f'(x)$, and $\mathsf{W} = \mathsf{P}_2(R)$

(b) $\mathsf{V} = \mathsf{P}(R)$, $\mathsf{T}(f(x)) = xf(x)$, and $\mathsf{W} = \mathsf{P}_2(R)$

(c) $\mathsf{V} = \mathsf{R}^3$, $\mathsf{T}(a,\ b,\ c) = (a+b+c,\ a+b+c,\ a+b+c)$, and $\mathsf{W} = \{(t,\ t,\ t) : t \in R\}$

(d) $\mathsf{V} = \mathsf{C}([0,\ 1])$, $\mathsf{T}(f(t)) = \left[\int_0^1 f(x)dx\right] t$, and $\mathsf{W} = \{f \in \mathsf{V} : f(t) = at + b \text{ for some } a \text{ and } b\}$

(e) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$, $\mathsf{T}(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A$, and $\mathsf{W} = \left\{A \in \mathsf{V} : A^T = A\right\}$

Proof.

Problem 3. Let T be a linear operator on a finite-dimensional vector space V, Prove that the following subspaces are T-invarinat.

(a) $\{\mathit{0}\}$ and V

(b) $\mathsf{N}(\mathsf{T})$ and $\mathsf{R}(\mathsf{T})$.

(c) E_λ, for any eigenvalue λ of T.

Proof.

Problem 4. Let T be a linear operator on a vector space V, and let W be a T-invariant subspace of V. Prove that W is $g(\mathsf{T})$-invariant for any polynomail $g(t)$.

Proof.

Problem 5. Let T be a linear operator on a vector space V. Prove that the intersection of any collection of T-invariant subspaces of V is a T-invariant subspace of V.

Proof.

Problem 6. For each linear operator T on the vector space V, find an ordered basis for the T-cyclic subspace generated by the vector z.

(a) $\mathsf{V} = \mathsf{R}^4$, $\mathsf{T}(a,\ b,\ c,\ d) = (a+b,\ b-c,\ a+c,\ a+d)$, and $z = e_1$.

(b) $\mathsf{V} = \mathsf{P}_3(R)$, $\mathsf{T}(f(x)) = f''(x)$, and $z = x^3$.

(c) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$, $\mathsf{T}(A) = A^T$, and $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(d) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$, $\mathsf{T}(A) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} A$, and $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Proof.

Problem 7. Prove that the restriction of a linear operator T to a T-invariant subspace is a linear operator on that subspace.

Proof.

Problem 8. Let T be alinear operator on a vector space with a T-invariant subspace W. Prove that if v is an eigenvector of T_W with corresponding eigenvalue λ, then the same is true for T.

Proof.

Problem 9. For each linear operator T and cyclic subspace W in **Problem 6**, compute the characteristic polynomial of T_{W} in two ways, as in **Example 6**.

Proof.

Problem 10. For each linear operator in **Problem 6**, find the characteristic polynomial $f(t)$ of T, and verify that the characteristic polynomial of T_{W} (computed in **Problem 9**) divides $f(t)$.

Proof.

Problem 11. Let T be a linear operator on a vector space V, let v be a nonzero vector in V, and let W be the T-cyclic subspace of V generated by v. Prove that

(a) W is T-invariant.

(b) Any T-invazriant subspace of V containing v also contains W.

Proof.

Problem 12. Prove that $A = \begin{pmatrix} B_1 & B_2 \\ O & B_3 \end{pmatrix}$ in the proof of **Theorem 5.21**.

Proof.

Problem 13. Let T be a linear operator on a vector space V, let v be a nonzero vector in V, and let W be the T-cyclic subspace of V generated by v. For any $w \in \mathsf{V}$, prove that $w \in \mathsf{W}$ if and only if there exists a polynomial $g(t)$ such that $w = g(\mathsf{T})(v)$.

Proof.

Problem 14. Prove that the polynomial $g(t)$ of **problem 13** can always be chosen so that its degree is less than or equal to $\dim(\mathsf{W})$.

Proof.

Problem 15. Use the Cayley-Hamilton theorem to prove its corollary for matrices. $Warning$!: If $f(t) = \det(A - tI)$ is the characteristic polynomial of A, it is tempting to prove that $f(A) = O$ by saying $f(A) = \det(A - AI) = det(O) = 0$. But this argument is *non-sense*, Why?

Proof.

Problem 16. Let T be a linear operator on a finite-dimensional vector space V.

(a) Prove that if the characteristic polynomial of T spilits, then so does the characteristic polynomial of the restriction of T to any T-invariant subspace of V.

(b) Deduce that if the characteristic polynomial of T splits, then any non-trivial T-invariant subspace of V contains an eigenvector of T.

Proof.

Problem 17. Let A be an $n \times n$ matrix. Prove that

$$\dim\left(\operatorname{span}\left(I_n,\ A,\ A^2, \ldots\right)\right) \leq n.$$

Proof.

Problem 18. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0.$$

(a) Prove that A is invertible if and only if $a_0 \neq 0$.

(b) Prove that if A is invertible, then

$$A^{-1} = (-1/a_0)\left[(-1)^n A^{n-1} + a_{n-1} A^{n-2} + \cdots + a_1 I_n\right].$$

(c) Use (b) to compute A^{-1} for

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}.$$

Proof.

Problem 19. Let A denote $K \times k$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix},$$

where a_0, a_1, $\ldots$, a_{k-1} are arbitrary scalars. Prove that the characteristic polynomial of A is

$$(-1)^k(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k).$$

Proof.

Problem 20. Let T be a linear operator on a vector space V, and suppose that V is a T-cyclic subspace of itself. Prove that if U is a linear operator on V, then $\mathsf{UT} = \mathsf{TU}$ if and only if $\mathsf{U} = g(\mathsf{T})$ for soem polynomial $g(t)$.
Hint: Suppose that V is generated by v. Choose $g(t)$ according to **Problem 13** so that $g(\mathsf{T})(v) = \mathsf{U}(v)$.

Proof.

Problem 21. Let T be a linear operator on a two-dimensional vector space V. Prove that either V is a T-cyclic subspace of itself or $\mathsf{T} = c\mathsf{I}$ for some scalar c.

Proof.

Problem 22. Let T be a linear operator on a two-dimensional vector space V and suppose that $\mathsf{T} \neq c\mathsf{I}$ for any scalar c. Show that if U is any linear operator on V such that $\mathsf{UT} = \mathsf{TU}$, then $\mathsf{U} = g(\mathsf{T})$ for some polynomial $g(t)$.

Proof.

Problem 23. Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V. Suppose that v_1, v_2, $\ldots$, v_k are eigenvectors of T corresponding to distinct eigenvalues. Prove that if $v_1 + v_2 + \cdots + v_k$ is in W, then $v_i \in \mathsf{W}$ for all i.

Proof.

Problem 24. Prove that the restriction of a diagonalizable linear operator T to any nontrivial T-invariant subspace is also diagonalizable.

Proof.

Problem 33. Let T be alinear operator on a vector space V, and let W_1, W_2, …, W_k be T-invariant subspaces of V. Prove that $\mathsf{W}_1 + \mathsf{W}_2 + \cdots + \mathsf{W}_k$ is also a T-invariant subspace of V.

Proof.

Problem 34. Give a direct proof of **Theorem 5.25** for the case $k = 2$. (This result is used in the proof of **Theroem 5.24**.)

Proof.

Problem 35. Prove **Theorem 5.25**.

Proof.

Problem 36. Let T be a linear operator on a finite-dimensional vector space V. Prove that T is diagonalizable if and only if V is the direct sum of one-dimensional T-invariant subspaces.

Proof.

Problem 37. Let T be alinear operator on a finite-dimensional vector space V, and let W_1, W_2, …, W_k be T-invariant subspaces of V such that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$. Prove that

$$\det(\mathsf{T}) = \det\left(\mathsf{T}_{\mathsf{W}_1}\right)\det\left(\mathsf{T}_{\mathsf{W}_2}\right)\cdots\det\left(\mathsf{T}_{\mathsf{W}_k}\right).$$

Proof.

Problem 38. Let T be alinear operator on a finite-dimensional vector space V, and let W_1, W_2, $\ldots$, W_k be T-invariant subspaces of V such that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$. Prove that T is diagonalizable if and only if $\mathsf{T}_{\mathsf{W}_i}$ is diagonalizable for all i.

Problem 39. Let $\mathcal{C}$ be a collection of diagonalizable linear operatos on a finite-dimensional vector space V. Prove that there is an ordered basis β such that $[\mathsf{T}]_\beta$ is a diagonal matrix for all $\mathsf{T} \in \mathcal{C}$ if and only if the operatos of $\mathcal{C}$ commute under composition. (This is an extension of **Problem 25**.)
Hints: The result is trivial if each operator has only one eigenvalue. Otherwise, establish the general result by mathematical induction on $\dim(\mathsf{V})$, using the fact that V is the direct sum of the eigenspaces of some operator in $\mathcal{C}$ that has more than one eigenvalue.

Proof.

Problem 40. Let B_1, B_2, $\ldots$, B_k be square matrices with entries in the smae field, and let $A = B_1 \oplus B_2 \oplus \cdots \oplus B_k$. Prove that the characteristic polynomial of A is the product of the characteristic polynomials of the B_i's.

Proof.

Problem 41. Let

$$A = \begin{pmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & & \vdots \\ n^2-n+1 & n^2-n+2 & \cdots & n^2 \end{pmatrix}.$$

Find the characteristic polynomial of A.
Hint: First prove that A has rank 2 and $\operatorname{span}(\{(1,\ 1,\ \ldots,\ 1),\ (1,\ 2,\ \ldots,\ n)\})$ is L_A-invariant.

Proof.

Problem 42. Let $A \in \mathsf{M}_{n\times n}(R)$ be the matrix defined by $A_{ij} = 1$ for all i and j. Find the characteristic polynomial of A.

Proof.