Chapter 5

Differentation
Selected Exercise homework

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Exercise 5.1. Let f be defined for all real x, and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y. Prove that f is constant.

Proof. Let x, y be real number with $x \neq y$.
As $|x - y| > 0$,

$$|f(x) - f(y)| \leq (x - y)^2 \Rightarrow \left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|$$

By squeeze theorem, $\lim_{x \to y} \frac{f(x) - f(y)}{x - y}$ exists, and that is 0.
By definition of differentinate, $f'(y) = 0$ for all $y \in \mathbb{R}$.
Hence **Theorem 5.9(c)**, $f(y)$ is constant function for $y \in \mathbb{R}$.

Exercise 5.2. Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \qquad (a < x < b).$$

Proof. As $f'(x) > 0$ in (a, b), then f is monotonically increasing function, *i.e.* f is one-to-one corresponding.
So we can assume that g is also not only one-to-one but continuous in $(f(a), f(b))$
Let c be real number in $(f(a), f(b))$.
By definition of continuous, $\lim_{x\to c} g(x) = g(c)$.
More Specifically, there exists unique x^*, a^* in (a, b) such that $f(x^*) = x$, $g(c^*) = c$, and

$$\lim_{x\to c} x^* = c^* \tag{1}$$

Also, $\dfrac{1}{f'(c^*)}$ is well-defined since $f'(c^*) > 0$, so

$$\lim_{x^*\to c^*} \frac{1}{\dfrac{f(x^*) - f(c^*)}{x^* - c^*}} = \frac{1}{f'(c^*)} \tag{2}$$

Let $\varepsilon > 0$.

There exists $\delta_1 > 0$ such that $\left| \dfrac{1}{\dfrac{f(x^*) - f(c^*)}{x^* - c^*}} - \dfrac{1}{f(c^*)} \right| < \varepsilon$ for $|x^* - c^*| < \delta_1$ at (2).
Also, there exists $\delta_2 > 0$ such that $|x^* - c^*| < \delta_1$ for $|x - c| < \delta_2$ at (1).
Since $f(x^*) = x$ and $f(c^*) = c$, $g(x) = x^*$ and $g(c) = c^*$
By above formulas, there exists $\delta_2 > 0$ such that $\left| \dfrac{g(x) - g(c)}{x - c} - \dfrac{1}{f'(c^*)} \right| < \varepsilon$ for $|x - c| < \delta_2$.
By definition, $\lim_{x\to c} \dfrac{g(x) - g(c)}{x - c} = g'(c)$, so $g'(c) = g'(f(c^*)) = \dfrac{1}{f'(c^*)}$ for all $c^* \in (a, b)$, and proof is completed.

Exercise 5.3. Suppose g is a real function on $\mathbb{R}$, with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough.
(A set of admissible values of ε can be determined which depends only on M.)

Proof. Since x, $g(x)$ are differentiable on $\mathbb{R}$, $f(x) = x + \varepsilon g(x)$ is also differentiable on $\mathbb{R}$ by **Theorem 5.3**.
Let $0 < \varepsilon < \dfrac{1}{2M}$.

$$\begin{aligned} f'(x) = 1 + \varepsilon g'(x) &\geq 1 - \varepsilon |g'(x)| \\ &\geq 1 - \varepsilon M \\ &> 1 - \frac{1}{2M} \cdot M = 1 - \frac{1}{2} = \frac{1}{2} \end{aligned}$$

Thus $f'(x) \geq \dfrac{1}{2} > 0$, $f(x)$ is monotonically increasing function, one-to-one.

Exercise 5.4. If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where $C_0, ..., C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

Proof. Define function g by

$$g(x) = \sum_{i=1}^{n+1} \frac{C_{i-1}}{i} x^i$$

Note that $g'(x) = \sum_{i=0}^{n} C_i x^i$. By assumption, $g(1) = \sum_{i=1}^{n+1} \frac{C_{i-1}}{i} = 0$.
Also $g(0) = 0$, there exsits $c \in (0, 1)$ such that $g'(c) = 0$ by MVT.
Thus $g'(x)$ has at least one real root between 0 and 1.

Exercise 5.5. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

Proof. Let $\varepsilon > 0$.
As $\lim\limits_{x\to\infty} f'(x) = 0$, there exists $N \in \mathbb{R}$ such that

$$|f'(x)| < \varepsilon \tag{1}$$

for $x > N$.
Let $x > N$.
By MVT,

$$\exists c \in (x,\ x+1)\ :\ |g(x)| = |f'(c)|$$

By (1), $|f'(c)| < \varepsilon$ since $c > x > N$.
Thus there exists $N \in \mathbb{R}$ for all $\varepsilon > 0$ such that $|g(x)| < \varepsilon$ for $x > N$, $\lim\limits_{x\to\infty} g(x) = 0$

Exercise 5.6. Suppose

(a) f is continuous for $x \geq 0$

(b) $f'(x)$ exists for $x > 0$

(c) $f(0) = 0$,

(d) f' is monotonically increasing

Put

$$g(x) = \frac{f(x)}{x} \qquad (x > 0)$$

and prove that g is monotonically increasing.

Proof. By MVT,

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = f'(c) \tag{1}$$

where $c \in (0,\, x)$ for arbitary $x > 0$.
By (d), f' is monotonically increasing, $f'(c) < f'(x)$ at (1).

$$\therefore\ \forall x > 0\ :\ xf'(x) - f(x) > 0$$

Note that x and $f(x)$ is differentiable for $x > 0$ which differentation of x not be 0.
Thus $g'(x) = \dfrac{xf'(x) - f(x)}{x^2} > 0$, g is monotonically increasing function.

Exercise 5.7. Suppose $f'(c)$, $g'(c)$ exist, $g'(c) \neq 0$, and $f(c) = g(c) = 0$. Prove that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

(This holds also for complex functions.)

Proof.

$$\begin{aligned}
\lim_{x \to c} \frac{f(x)}{g(x)} &= \lim_{x \to c} \frac{f(x) - f(c)}{g(x) - g(c)} && (\because\ f(c) = g(c) = 0) \\
&= \lim_{x \to c} \frac{\dfrac{f(x) - f(c)}{x - c}}{\dfrac{g(x) - g(c)}{x - c}} \\
&= \frac{\displaystyle\lim_{x \to c} \frac{f(x) - f(c)}{x - c}}{\displaystyle\lim_{x \to c} \frac{g(x) - g(c)}{x - c}} && (\because\ f'(c) \text{ exists},\ g'(c) \neq 0) \\
&= \frac{f'(c)}{g'(c)}
\end{aligned}$$

Exercise 5.8. Suppose f' is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. (This could be expressed by saying that f is uniformly differentiable on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions too?

Proof. Let $\varepsilon > 0$. Since f' is continuous, there exists $\delta' > 0$ such that

$$|f'(c) - f'(x)| < \varepsilon \tag{1}$$

for $|c - x| < \delta', \ a \leq c \leq b, \ a \leq x \leq b$.
Assume $\delta = \delta'$
By M.V.T, there exists $k \in B(x, \ d(t, \ x))$ such that

$$\frac{f(t) - f(x)}{t - x} = f'(k)$$

Note that $|k - x| < |t - x| < \delta$.
By (1), $|f'(k) - f'(x)| < \varepsilon$, proof is completed.
Also, this holds for vectror-valued function as calculate independent.

Exercise 5.9. Let f be a continuous real function on $\mathbb{R}$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?

Proof. First, we know that $\lim_{x\to 0} f'(x) = 3$.
By definition,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x}$$

Since f is continuous real function,

$$f'(0) = \lim_{x\to 0} f'(x)$$

by L'Hospital rule.
By assumption, there exists $f'(0)$, which value is 3.

Exercise 5.10. Suppose f and g are complex differentiable functions on $(0,\,1)$, $f(x)\to 0$, $g(x)\to 0$, $f'(x)\to A$, $g'(x)\to B$ as $x\to 0$, where A and B are complex numbers, $B\neq 0$. Prove that

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\frac{A}{B}$$

Compare with **Example 5.18**.
Hint:

$$\frac{f(x)}{g(x)}=\left\{\frac{f(x)}{x}-A\right\}\cdot\frac{x}{g(x)}+A\cdot\frac{x}{g(x)}.$$

Proof. Let $Re\,:\,\mathbb{C}\to\mathbb{R}$, $Im\,:\,\mathbb{C}\to\mathbb{R}$ which each returns real part and imagine part of complex number z. Since f and g are complex differentiable function, f and g can be expressed

$$\begin{aligned}f(x)&=f_r(x)+i\cdot f_i(x)\\ g(x)&=g_r(x)+i\cdot g_i(x)\end{aligned}$$

where $f_r,\ f_i,\ g_r,\ g_i$ are real differentiable functions. Since $f'(x)\to A$ and $g'(x)\to B$ as $x\to 0$,

$$\begin{aligned}f_r(x)\to Re(A),\ f_i(x)\to Im(A)\\ g_r(x)\to Re(B),\ g_i(x)\to Im(B)\end{aligned}$$

as $x\to 0$.
By **Exercise 5.9**,

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)}{x}&=\lim_{x\to 0}\frac{f_r(x)+i\cdot f_i(x)}{x}\\ &=Re(A)+i\cdot Im(A)=A\\ \lim_{x\to 0}\frac{g(x)}{x}&=\lim_{x\to 0}\frac{g_r(x)+i\cdot g_i(x)}{x}\\ &=Re(B)+i\cdot Im(B)=B\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)}{g(x)}&=\lim_{x\to 0}\left[\left\{\frac{f(x)}{x}-A\right\}\cdot\frac{x}{g(x)}+A\cdot\frac{x}{g(x)}\right]\\ &=\lim_{x\to 0}\left\{\frac{f(x)}{x}-A\right\}\cdot\frac{x}{g(x)}+A\cdot\lim_{x\to 0}\frac{x}{g(x)}\\ &=0\cdot\frac{1}{B}+\frac{A}{B}=\frac{A}{B}\end{aligned}$$

Example 5.18. On the segment (0, 1), define $f(x) = x$ and $g(x) = x + x^2 e^{i/x^2}$.
Since $|e^{it}| = 1$ for all real t, we see that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 1. \tag{36}$$

Next,

$$g'(x) = 1 + \left\{ 2x - \frac{2i}{x} e^{i/x^2} \right\} \qquad (0 < x < 1).$$

so that

$$|g'(x)| \geq \left| 2x - \frac{2i}{x} \right| - 1 \geq \frac{2}{x} - 1. \tag{38}$$

Hence

$$\left| \frac{f'(x)}{g'(x)} \right| = \frac{1}{|g'(x)|} \leq \frac{x}{2 - x}$$

and so

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = 0. \tag{40}$$

By (36) and (40), L'Hospital's rule fails in this case. Note also that $g'(x) \neq 0$ on (0, 1), by (38).

Exercise 5.11. Suppose f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Show that

$$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}=f''(x)$$

Show by an example that the limit may exist even if $f''(x)$ does not.
Hint: Use **Theorem 5.13**.

Proof.

$$\begin{aligned}\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} &= \frac{1}{2}\cdot\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{h}\\ &= \frac{1}{2}\cdot\lim_{h\to 0}\left\{\frac{f'(x+h)-f'(0)}{h}+\frac{f'(x-h)-f'(0)}{-h}\right\}\\ &= \frac{1}{2}\cdot\left\{\lim_{h\to 0}\frac{f'(x+h)-f'(0)}{h}+\lim_{h\to 0}\frac{f'(x-h)-f'(0)}{-h}\right\}\\ &= \frac{1}{2}\left\{f''(x)+f''(x)\right\}=f''(x)\end{aligned}$$

By L'Hospital rule,

$$\begin{aligned}&\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}\\ &=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} && \text{(By L'Hospital rule)}\\ &=f''(x)\end{aligned}$$

Suppose $f(x)=sgn(x)$ which returns sign of x to $\{-1,\ 0,\ 1\}$.
Since f is discontinuous at $x=0$, $f''(0)$ doesn't defined, but limit exists which value 0.

Exercise 5.18. Suppose f is a real function on $[a, b]$, n is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a, b]$. Let α, β, and P be as in Taylor's theorm (5.8). Define

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$$

for $t \in [a, b]$, $t \neq \beta$, differentiate

$$f(t) - f(\beta) = (t - \beta)Q(t)$$

$n - 1$ times at $t = \alpha$, and derive the following version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{n-1}(\alpha)}{(n-1)!}(\beta - \alpha)^n.$$

Proof. First, we will show that $f^{(N)}(t) = (t - \beta)Q^{(N)}(t) + nQ^{(N-1)}(t) \ldots (1)$
Let us use mathematical induction.
It is trivial when $k = 1$.
Assume that $N = k$ $(k \leq n - 2)$ is true. *i.e.*,

$$f^{(k)}(t) = (t - \beta)Q^{(k)}(t) + kQ^{(k-1)}(t).$$

So, $f^{(k)}(t)$ is differentiable for every $t \in [a, b]$,

$$\begin{aligned} f^{(k+1)}(t) &= Q^{(k)}(t) + (t - \beta)Q^{(k+1)}(t) + kQ^{(k)}(t) \\ &= (t - \beta)Q^{(k+1)}(t) + (k+1)Q^{(k)}(t) \end{aligned}$$

Thus $N = k + 1$ is also true whenever $N = k$ is true, and (1) is true for $N < n$
At (1), multiply $\dfrac{(\beta - \alpha)^N}{N!}$ and subsitute α for t both sides. Then,

$$\begin{aligned} \frac{N!}{f^N(\alpha)}(\beta - \alpha)^N &= \frac{Q^{N-1}(\alpha)}{(N-1)!}(\beta - \alpha)^N - \frac{Q^N(\alpha)}{N!}(\beta - \alpha)^{N+1} \\ &= A_N - A_{N+1} \end{aligned} \tag{2}$$

where $A_N = \dfrac{Q^{(N-1)(\alpha)}}{(N-1)!}(\beta - \alpha)^N$.
Summation both sides from $N = 1$ to $n - 1$,

$$\sum_{N=1}^{n-1} \frac{f^N(\alpha)}{N!}(\beta - \alpha)^N = A_1 - A_2 + A_2 - A_3 + \cdots + A_{n-1} - A_n$$

$$= A_1 - A_n$$

$$\begin{aligned}
\therefore\ P(\beta) - f(\alpha) &= Q(\alpha)(\beta-\alpha) - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n \\
&= \frac{f(\alpha)-f(\beta)}{\alpha-\beta}(\beta-\alpha) - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n \\
&= f(\beta) - f(\alpha) - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n
\end{aligned}$$

Thus $f(\beta) = P(\beta) + \dfrac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n$, proof is completed.

Exercise 5.19. Suppose f is defined in $(-1, 1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1$, $\alpha_n \to 0$, and $\beta_n \to 0$ as $n \to \infty$. Define the difference quotients

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}.$$

Prove the following statements:

(a) If $\alpha_n < 0 < \beta_n$, then $\lim D_n = f'(0)$

(b) If $0 < \alpha_n < \beta_n$ and $\{\beta_n/(\beta_n - \alpha_n)\}$ is bounded, then $\lim D_n = f'(0)$.

(c) If f' is continuous in $(-1, 1)$, then $\lim D_n = f'(0)$.

Give an example in which f is differentiable in $(-1, 1)$ (but f' is not continuous at 0) and in which α_n, β_n tend to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.

Proof. (a)
$\forall \varepsilon > 0 \exists N_1$ such that $\left|\frac{f(\beta_n) - f(0)}{\beta_n} - f'(0)\right| < \varepsilon$ for $n > N_1$.
$\forall \varepsilon > 0 \exists N_2$ such that $\left|\frac{f(\alpha_n) - f(0)}{\alpha_n} - f'(0)\right| < \varepsilon$ for $n > N_2$.
Let $N = \max(N_1,\ N_2)$. Then,

$$\begin{aligned}
\left|\frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} - f'(0)\right| &= \left|\frac{f(\beta_n) - f(0)}{\beta_n - \alpha_n} + \frac{f(\alpha_n) - f(0)}{\beta_n - \alpha_n} - f'(0)\right| \\
&= \frac{1}{\beta_n - \alpha_n} |\{f(\beta_n) - f(0) - \beta_n f'(0)\} \\
&\qquad - \{f(\alpha_n) - f(0) - \alpha_n f'(0)\}| \\
&\le \frac{1}{\beta_n - \alpha_n} \{|f(\beta_n) - f(0) - \beta_n f'(0)| \\
&\qquad + |f(\alpha_n) - f(0) - \alpha_n f'(0)|\} \\
&= \frac{1}{\beta_n - \alpha_n} \left\{|\beta_n| \cdot \left|\frac{f(\beta_n) - f(0)}{\beta_n} - f'(0)\right|\right. \\
&\qquad \left. + |\alpha_n| \cdot \left|\frac{f(\alpha_n) - f(0)}{\alpha_n} - f'(0)\right|\right\} \\
&< \frac{1}{\beta_n - \alpha_n} \{|\beta_n| \cdot \varepsilon + |\alpha_n| \cdot \varepsilon\} = \varepsilon
\end{aligned}$$

(b)
Since $\left\{\dfrac{\beta_n}{\beta_n-\alpha_n}\right\}$ is bounded, there exists $M>0$ such that $\left|\dfrac{\beta_n}{\beta_n-\alpha_n}\right|<M$.
Let $\varepsilon>0$.
By $\varepsilon-\delta$ argument,

$$\exists N_1\ :\ \left|\frac{f(\beta_n)-f(0)}{\beta_n}-f'(0)\right|<\frac{\varepsilon}{2M}\text{ for }n>N_1$$
$$\exists N_2\ :\ \left|\frac{f(\alpha_n)-f(0)}{\alpha_n}-f'(0)\right|<\frac{\varepsilon}{2M}\text{ for }n>N_2$$

Let $N=\max(N_1,\ N_2)$.

$$\begin{aligned}\left|\frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}-f'(0)\right| &= \left|\frac{f(\beta_n)-f(0)}{\beta_n-\alpha_n}+\frac{f(\alpha_n)-f(0)}{\beta_n-\alpha_n}-f'(0)\right|\\ &= \frac{\beta_n}{\beta_n-\alpha_n}\left|\left\{\frac{f(\beta_n)-f(0)}{\beta_n}-f'(0)\right\}\right.\\ &\qquad\left.-\left\{\frac{f(\alpha_n)-f(0)}{\beta_n}-\frac{\alpha_n f'(0)}{\beta_n}\right\}\right|\\ &\le \frac{\beta_n}{\beta_n-\alpha_n}\left\{\left|\frac{f(\beta_n)-f(0)}{\beta_n}-f'(0)\right|\right.\\ &\qquad\left.+\left|\frac{f(\alpha_n)-f(0)}{\beta_n}-\frac{\alpha_n f'(0)}{\beta_n}\right|\right\}\end{aligned}\tag{1}$$

Hence $0<\alpha_n<\beta_n$,

$$\left|\frac{f(\alpha_n)-f(0)}{\beta_n}-\frac{\alpha_n f'(0)}{\beta_n}\right|<\left|\frac{f(\alpha_n)-f(0)}{\alpha_n}-f'(0)\right|$$

At (1),

$$\begin{aligned}&\frac{\beta_n}{\beta_n-\alpha_n}\left\{\left|\frac{f(\beta_n)-f(0)}{\beta_n}-f'(0)\right|+\left|\frac{f(\alpha_n)-f(0)}{\beta_n}-\frac{\alpha_n f'(0)}{\beta_n}\right|\right\}\\ &\le \frac{\beta_n}{\beta_n-\alpha_n}\left\{\left|\frac{f(\beta_n)-f(0)}{\beta_n}-f'(0)\right|+\left|\frac{f(\alpha_n)-f(0)}{\alpha_n}-f'(0)\right|\right\}\\ &< M\times\left(\frac{\varepsilon}{2M}+\frac{\varepsilon}{2M}\right)=\varepsilon\end{aligned}$$

(c) As f' is continuous, we can use M.V.T on function f.
By M.V.T, there exists $c_n \in (\alpha_n,\ \beta_n)$ such that

$$\frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = f'(c_n)$$

Thus $\{c_n\}$ converges to 0 and $f'(0)$ exists,

$$\begin{aligned}\lim_{n\to\infty} \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} &= \lim_{n\to\infty} f'(c_n) \\ &= f'\left(\lim_{n\to\infty} c_n\right) = f'(0)\end{aligned}$$

....

Suppose $f(x) = x^2 \sin\frac{1}{x}$, $\alpha_n = \frac{2}{(4n+1)\pi}$, $\beta_n = \frac{1}{2n\pi}$.
As $\sin\left(2n + \frac{1}{2}\right)\pi = 1$,$\sin(2n\pi) = 0$ for $n \in \mathbb{N}$, $\sin\alpha_n = \left\{\frac{2}{(4n+1)\pi}\right\}^2$ and $\sin\beta_n = 0$ for $n \in \mathbb{N}$.
Also $\beta_n - \alpha_n = \left(\frac{1}{2n} - \frac{2}{4n+1}\right)\frac{1}{\pi} = \frac{2}{4n(4n+1)}\frac{1}{\pi}$
Thus

$$\begin{aligned}\lim_{n\to\infty} \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} &= \lim_{n\to\infty} \left\{\frac{-\left(\frac{2}{(4n+1)\pi}\right)^2}{2\cdot\frac{1}{4n(4n+1)}\frac{1}{\pi}}\right\} \\ &= \lim_{n\to\infty} -\frac{2\cdot 4n}{(4n+1)\pi} = -\frac{2}{\pi}\end{aligned}$$

But, $f'(0) = 0$.

Theorem 1. Let f be continuous mapping of $[a, b]$ to $\mathbb{R}^k$, $n \in \mathbb{N}$.
Suppose $f^{(n-1)}$ is continuous and $f^{(n)}$ exists at $[a, b]$.
Let α, β be distinct points on $[a, b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$$

Then, there exists $c \in (\alpha, \beta)$ such that

$$\|f(\beta) - P(\beta)\| \leq \left\| \frac{f^{(n)}(c)}{n!} \right\| (\beta - \alpha)^n$$

Note that $\| \cdot \|$ is p-norm.

Exercise 5.20. Formulate and prove an inequality which follows from Taylor's theorem and which remains valid for vector-valued functions.

Proof. We will prove **Theorem 1**
Let $u = f(\beta) - P(\beta)$, $g = u \cdot f$.
Then g is continuous mapping of $[a, b]$ to $\mathbb{R}$.
As u is constant vector, $g^{(n)} = u \cdot f^{(n)}$. By **Theorem 5.14**, there exists $c \in (\alpha, \beta)$ such that

$$g(\beta) = P'(\beta) + \frac{g^{(n)}(c)}{n!}(\beta - \alpha)^n \tag{1}$$

where $P'(x) = \displaystyle\sum_{k=0}^{n-1} \frac{u \cdot f^{(k)}(\alpha)}{k!}(x - \alpha)^k$.

$$\begin{aligned}
\therefore\ g(\beta) - P'(\beta) &= u \cdot f(\beta) - \sum_{k=0}^{n-1} \frac{u \cdot f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k \\
&= u \cdot \left\{ f(\beta) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(x - \alpha)^k \right\} \\
&= u \cdot \{f(\beta) - f(\alpha)\} = \|u\|^2 \\
\frac{g^{(n)}(c)}{n!}(\beta - \alpha)^n &= \frac{u \cdot f^{(n)}(c)}{n!}(\beta - \alpha)^n \\
&\leq \|u\| \cdot \|\frac{f^{(n)}(c)}{n!}(\beta - \alpha)^n\| \qquad (\because \text{ Cauchy-Schwarz inequality})
\end{aligned}$$

Using above formulas, (1) can be

$$\|u\|^2 \leq \|u\| \cdot \left\| \frac{f^{(n)}(c)}{n!} \right\| (\beta - \alpha)^n$$

Hence $\|u\| > 0$,

$$\|f(\beta) - P(\beta)\| \leq \left\| \frac{f^{(n)}(c)}{n!} \right\| (\beta - \alpha)^n$$

Lemma 1. Let $P_n(x)$ be set of polynomials of degree $3n+1$. Suppose $f(x) = e^{-1/x^2}$ for $x \neq 0$, then

$$\lim_{x\to 0} g\left(\frac{1}{x}\right) f(x) = 0$$

where $g \in P_n(x)$.

Proof. Let $x = \dfrac{1}{t}$. It is easy to show that

$$\lim_{x\to 0} g\left(\frac{1}{x}\right) f(x) = 0$$

if and only if

$$\lim_{t\to\infty} g(t) f\left(\frac{1}{t}\right) = \lim_{t\to -\infty} g(t) f\left(\frac{1}{t}\right) = 0$$

using $\varepsilon - \delta$ argument.

So, we will show that $\lim_{t\to\infty} g(t) f\left(\frac{1}{t}\right) = L$.

As $\lim_{t\to\infty} \left\{t^{3n+2} - g(t)\right\} = \infty$, there exists $C > 0$ such that $|g(t)| \leq C\,t^{3n+2}$ for all $t > 0$.

$$\begin{aligned} |g(t)| \leq C\,t^{3n+2} &\Leftrightarrow -C\,t^{3n+2} \leq g(t) \leq C\,t^{3n+2} \\ &\Leftrightarrow -C\,t^{3n+2} e^{-t^2} \leq g(t) f\left(\frac{1}{t}\right) \leq C\,t^{3n+2} e^{-t^2} \\ &\Rightarrow \lim_{t\to\infty} -C\,t^{3n+2} e^{-t^2} \leq \lim_{t\to\infty} g(t) f\left(\frac{1}{t}\right) \leq \lim_{t\to\infty} C\,t^{3n+2} e^{-t^2} \end{aligned}$$

By squeeze theorem, $\lim_{t\to\infty} g(t) f\left(\frac{1}{t}\right) = 0$.

The case $t \to -\infty$ is analogous, and proof is completed.

Lemma 2. Let $f(x) = e^{-1/x^2}$ for all $x \neq 0$, and $f(0) = 0$.
Then $f(x)$ is infinitely differentiable for $x \in \mathbb{R}$.
Moreover,

$$f^{(n)}(x) = \begin{cases} f(x)Q_n(x) & (x > 0) \\ 0 & (x = 0) \end{cases}$$

where $P_n(x)$ a polynomial function of degree n, and $Q_n(x) = P_{3n}\left(\frac{1}{x}\right)$ fulfills the recursive definition

$$\begin{aligned} Q_0(x) &= 1 \\ Q_n(x) &= \frac{2}{x^3}Q_{n-1}(x) + Q'_{n-1}(x) \end{aligned}$$

Proof. Let us use mathematical induction.
It is easy to show when $n = 1$.
Assume $n = k$ is true.
Then, $f^{(k+1)}(x)$ is well-defined for $x \neq 0$.
More Specifically,

$$\begin{aligned} f^{(k+1)}(x) &= \left\{e^{-1/x^2}\right\}' Q_k(x) + e^{-1/x^2}\left\{Q_k(x)\right\}' \\ &= \frac{2}{x^3}\, e^{-1/x^2} Q_k(x) + e^{-1/x^2}\, Q'_k(x) \\ &= e^{-1/x^2}\left\{\frac{2}{x^3}Q_k(x) + Q'_k(x)\right\} \\ &= e^{-1/x^2}Q_{k+1}(x) \\ &= f(x)Q_{k+1}(x) \end{aligned}$$

So if we show $f^{(k+1)}(0) = 0$, then we can say that $f^{(k+1)}$ is also differentiable for $x \in \mathbb{R}$.
By definition,

$$\begin{aligned} f^{(k+1)}(0) &= \lim_{x \to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{f^{(k)}(x)}{x} && (\because\ f^{(k)}(0) = 0) \\ &= \lim_{x \to 0} f(x)\frac{Q_k(x)}{x} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{x\to 0} f(x) P_{3k+1}\left(\frac{1}{x}\right) && (\because\ Q_k(x) = P_{3k}\left(\frac{1}{x}\right)) \\
&= 0 && (\because\ \text{By Lemma 1})
\end{aligned}
$$

Hence $f^{(k+1)}(0) = 0$, $f^{(k+1)}$ is also differentiable for $x \in \mathbb{R}$.
By the principle of mathematical induction, $f(x)$ is infinitely differentiable for $x \in \mathbb{R}$.

Exercise 5.21. Let E be a closed subset of $\mathbb{R}$. There is a real continuous function f on $\mathbb{R}$ whose zero set is E. Is it possible, for each closed set E, to find such an f which is differentiable on $\mathbb{R}$, or one which is n times differentiable, or even one which has derivatives of all orders on $\mathbb{R}$?

Proof. Let $f(x) = e^{-1/x^2}$ for all $x \neq 0$, and $f(0) = 0$.
By Lemma 2, $f(x)$ is infinitely differentiable, and $f^{(n)}(0)$ for $n \in \mathbb{N}$.
Define function g by

$$g = (f \circ ReLU)(x)$$

with $ReLU(x) = \max(0,\, x)$.
i.e.

$$g(x) = \begin{cases} e^{-1/x^2} & (x > 0) \\ 0 & (x \leq 0) \end{cases}$$

g is also infinitely differentiable everywhere on $\mathbb{R}$ and whose zero set is $(-\infty,\ 0]$.
Let us think

$$f_{(a,\,b)}(x) = g(x-a)g(b-x)$$

where $-\infty \leq a < b \leq \infty$.
Note that $f_{(-\infty,\,b)}(x) = e^{-\frac{1}{(x-b)^2}}$ for $x > b$, and similarly when $b = \infty$.
As set E is closed, complement of E is open set consisting of a union of disjoint open intervals, so $E^c = \bigcup_i\, (a_i,\ b_i)$
Let l be collection of disjoint open intervals, and define function h by

$$h(x) = \sum_{(a,\,b)\,\in\, l} f_{(a,\,b)}(x)$$

h is well-defined since at least one of the terms in the sum isn't 0 for any $x \notin E$, infinitely differentiable on $\mathbb{R}$, and has zero set E.

Exercise 5.22. Suppose f is a real function on $(-\infty,\ \infty)$. Call x a *fixed point* of f if $f(x) = x$.

(a) If f is differentiable and $f\,'(t) \neq 1$ for every real t, prove that f has at most one fixed point.

(b) Show that the function f defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f\,'(t) < 1$ for all real t.

(c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitary real number and

$$x_{n+1} = f(x_n)$$

for $n = 1,\ 2,\ 3, ...$

(d) Show that the process described in (c) can be visualized by the zig-zag path

$$(x_1,\ x_2) \to (x_2,\ x_2) \to (x_2,\ x_3) \to (x_3,\ x_3) \to (x_3,\ x_4) \to \cdots$$

Proof. (a) Assume that f has two fixed point $x_1,\ x_2$.
i.e., $f(x_1) = x_1,\ f(x_2) = x_2$.
By MVT, there exists $c \in (x_1,\ x_2)$ such that

$$x_2 - x_1 = f(x_2) - f(x_1) = (x_2 - x_1)f'(c)$$

Hence $f'(c) = 1$ for $c \in \mathbb{R}$, that is contradiction by assumption.
(b) If f has fixed point x_k, then $f(x_k) = x_k = x_k + (1+e^{x_k})^{-1}$, *i.e.* $(1+e^{x_k}) = 0$.
But $(1 + e^x)^{-1} > 0$ for all $x \in \mathbb{R}$, that is contradiction.
(c) Let $\{x_n\}_{n=1}^{\infty}$ be sequence, where x_1 is an arbitary real number.
First, we will show that $|x_{n+1} - x_n| \leq A^n|x_2 - x_1|$ for $n \in \mathbb{N}$. . . . .(1)
Let us use indcution, and base case$(n = 1)$ is trivial.
Assume that $n = k$ is true.

$$\begin{aligned}|x_{k+2}-x_{k+1}| &= |f(x_{k+1})-f(x_k)| \\ &\le |(x_{k+1}-x_k)f'(c)| && (\exists c\in(x_k,\ x_{k+1}) \text{ by MVT}) \\ &\le A|(x_{k+1}-x_k) \le A^{k+1}|x_2-x_1|\end{aligned}$$

By principle of mathematical induction, (1) is true for $n\in\mathbb{N}$.
Anyway, It holds

$$|x_n-x_m| \le \sum_{i=m}^{n-1}|x_{i+1}-x_i|$$

by triangular inequlaity for $n>m$.
Therefore,

$$\begin{aligned}|x_n-x_m| &\le \sum_{i=m}^{n-1}|x_{i+1}-x_i| \\ &\le \sum_{i=m}^{n-1}A^i|x_2-x_1| \\ &\le \sum_{i=m}^{\infty}A^i|x_2-x_1| \\ &= \frac{A^m}{1-A}|x_2-x_1| \qquad (2)\end{aligned}$$

is holds, and (2)'s RHS goes to 0 as $m\to 0$, so LHS goes to 0 by squeeze theorem.
In other words, $\{x_n\}_{n=1}^{\infty}$ is cauchy sequence on real number, so $\{x_n\}_{n=1}^{\infty}$ converges to real number. Let us called this value x.
Thus, $\sum_{n\to\infty}|x_{n+1}-x_n|=0$, and $x_{n+1}=f(x_n)$, so $\lim_{n\to\infty}\{f(x_n)-x_n\}=0$
As $x=\lim_{n\to\infty}x_n=x$, $\lim_{n\to\infty}f(x_n)=f\left(\lim_{n\to\infty}x_n\right)=f(x)$
Finally,

$$\begin{aligned}\lim_{n\to\infty}\{f(x_n)-x_n\} &= \lim_{n\to\infty}f(x_n)-\lim_{n\to\infty}x_n \\ &= f(x)-x=0\end{aligned}$$

, which implies x is *fixed point* of f.

(d) Suppose $f(x) = \dfrac{1}{1+e^{-x}}$(Sigmoid function).