

Chapter 5

Differentiation

Selected Exercise homework

Exercise 5.1. Let f be defined for all real x , and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y . Prove that f is constant.

Proof.

Exercise 5.2. Suppose $f'(x) > 0$ in (a, b) . Prove that f is strictly increasing in (a, b) , and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$

Proof.

Exercise 5.3. Suppose g is a real function on \mathbb{R} , with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough.

(A set of admissible values of ε can be determined which depends only on M .)

Proof.

Exercise 5.4. If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where C_0, \dots, C_n are real constants, prove that the equation

$$C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$$

has at least one real root between 0 and 1.

Proof.

Exercise 5.5. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \rightarrow 0$ as $x \rightarrow +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \rightarrow 0$ as $x \rightarrow +\infty$.

Proof.

Exercise 5.6. Suppose

- (a) f is continuous for $x \geq 0$
- (b) $f'(x)$ exists for $x > 0$
- (c) $f(0) = 0$,
- (d) f' is monotonically increasing

Put

$$g(x) = \frac{f(x)}{x} \quad (x > 0)$$

and prove that g is monotonically increasing.

Proof.

Exercise 5.7. Suppose $f'(c)$, $g'(c)$ exist, $g'(c) \neq 0$, and $f(c) = g(c) = 0$. Prove that

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

(This holds also for complex functions.)

Proof.

Exercise 5.8. Suppose f' is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. (This could be expressed by saying that f is uniformly differentiable on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions too?

Proof.

Exercise 5.9. Let f be a continuous real function on \mathbb{R} , of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \rightarrow 3$ as $x \rightarrow 0$. Does it follow that $f'(0)$ exists?

Proof.

Exercise 5.10. Suppose f and g are complex differentiable functions on $(0, 1)$, $f(x) \rightarrow 0$, $g(x) \rightarrow 0$, $f'(x) \rightarrow A$, $g'(x) \rightarrow B$ as $x \rightarrow 0$, where A and B are complex numbers, $B \neq 0$. Prove that

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \frac{A}{B}$$

Compare with **Example 5.18**.

Hint:

$$\frac{f(x)}{g(x)} = \left\{ \frac{f(x)}{x} - A \right\} \cdot \frac{x}{g(x)} + A \cdot \frac{x}{g(x)}.$$

Proof.

Example 5.18. On the segment $(0, 1)$, define $f(x) = x$ and $g(x) = x + x^2 e^{i/x^2}$.

Since $|e^{it}| = 1$ for all real t , we see that

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = 1. \quad (36)$$

Next,

$$g'(x) = 1 + \left\{ 2x - \frac{2i}{x} e^{i/x^2} \right\} \quad (0 < x < 1).$$

so that

$$|g'(x)| \geq \left| 2x - \frac{2i}{x} \right| - 1 \geq \frac{2}{x} - 1. \quad (38)$$

Hence

$$\left| \frac{f'(x)}{g'(x)} \right| = \frac{1}{|g'(x)|} \leq \frac{x}{2-x}$$

and so

$$\lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} = 0. \quad (40)$$

By (36) and (40), L'Hospital's rule fails in this case. Note also that $g'(x) \neq 0$ on $(0, 1)$, by (38).

Exercise 5.11. Suppose f is defined in a neighborhood of x , and suppose $f''(x)$ exists. Show that

$$\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x)$$

Show by an example that the limit may exist even if $f''(x)$ does not.

Hint: Use **Theorem 5.13**.

Proof.

Exercise 5.18. Suppose f is a real function on $[a, b]$, n is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a, b]$. Let α , β , and P be as in Taylor's theorem (5.8). Define

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$$

for $t \in [a, b]$, $t \neq \beta$, differentiate

$$f(t) - f(\beta) = (t - \beta)Q(t)$$

$n - 1$ times at $t = \alpha$, and derive the following version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{n-1}(\alpha)}{(n-1)!}(\beta - \alpha)^n.$$

Proof.

Exercise 5.19. Suppose f is defined in $(-1, 1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1$, $\alpha_n \rightarrow 0$, and $\beta_n \rightarrow 0$ as $n \rightarrow \infty$. Define the difference quotients

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}.$$

Prove the following statements:

- (a) If $\alpha_n < 0 < \beta_n$, then $\lim D_n = f'(0)$
- (b) If $0 < \alpha_n < \beta_n$ and $\{\beta_n/(\beta_n - \alpha_n)\}$ is bounded, then $\lim D_n = f'(0)$.
- (c) If f' is continuous in $(-1, 1)$, then $\lim D_n = f'(0)$.

Give an example in which f is differentiable in $(-1, 1)$ (but f' is not continuous at 0) and in which α_n, β_n tend to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.

Proof.

Theorem 1. Let f be continuous mapping of $[a, b]$ to \mathbb{R}^k , $n \in \mathbb{N}$. Suppose $f^{(n-1)}$ is continuous and $f^{(n)}$ exists at $[a, b]$. Let α, β be distinct points on $[a, b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t - \alpha)^k$$

Then, there exists $c \in (\alpha, \beta)$ such that

$$\|f(\beta) - P(\beta)\| \leq \left\| \frac{f^{(n)}(c)}{n!} \right\| (\beta - \alpha)^n$$

Note that $\|\cdot\|$ is p -norm.

Exercise 5.20. Formulate and prove an inequality which follows from Taylor's theorem and which remains valid for vector-valued functions.

Proof. We will prove **Theorem 1**

Let $u = f(\beta) - P(\beta)$, $g = u \cdot f$.

Then g is continuous mapping of $[a, b]$ to \mathbb{R} .

As u is constant vector, $g^{(n)} = u \cdot f^{(n)}$. By **Theorem 5.14**, there exists $c \in (\alpha, \beta)$ such that

$$g(\beta) = P'(\beta) + \frac{g^{(n)}(c)}{n!}(\beta - \alpha)^n \quad (1)$$

$$\text{where } P'(x) = \sum_{k=0}^{n-1} \frac{u \cdot f^{(k)}(\alpha)}{k!}(x - \alpha)^k.$$

$$\begin{aligned} \therefore g(\beta) - P'(\beta) &= u \cdot f(\beta) - \sum_{k=0}^{n-1} \frac{u \cdot f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k \\ &= u \cdot \left\{ f(\beta) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(x - \alpha)^k \right\} \\ &= u \cdot \{f(\beta) - f(\alpha)\} = \|u\|^2 \\ \frac{g^{(n)}(c)}{n!}(\beta - \alpha)^n &= \frac{u \cdot f^{(n)}(c)}{n!}(\beta - \alpha)^n \\ &\leq \|u\| \cdot \left\| \frac{f^{(n)}(c)}{n!}(\beta - \alpha)^n \right\| \quad (\because \text{Cauchy-Schwarz inequality}) \end{aligned}$$

Using above formulas, (1) can be

$$\|u\|^2 \leq \|u\| \cdot \left\| \frac{f^{(n)}(c)}{n!} \right\| (\beta - \alpha)^n$$

Hence $\|u\| > 0$,

$$\|f(\beta) - P(\beta)\| \leq \left\| \frac{f^{(n)}(c)}{n!} \right\| (\beta - \alpha)^n$$

Lemma 1. Let $P_n(x)$ be set of polynomials of degree $3n+1$. Suppose $f(x) = e^{-1/x^2}$ for $x \neq 0$, then

$$\lim_{x \rightarrow 0} g\left(\frac{1}{x}\right) f(x) = 0$$

where $g \in P_n(x)$.

Proof. Let $x = \frac{1}{t}$. It is easy to show that

$$\lim_{x \rightarrow 0} g\left(\frac{1}{x}\right) f(x) = 0$$

if and only if

$$\lim_{t \rightarrow \infty} g(t) f\left(\frac{1}{t}\right) = \lim_{t \rightarrow -\infty} g(t) f\left(\frac{1}{t}\right) = 0$$

using $\varepsilon - \delta$ argument.

So, we will show that $\lim_{t \rightarrow \infty} g(t) f\left(\frac{1}{t}\right) = 0$.

As $\lim_{t \rightarrow \infty} \{t^{3n+2} - g(t)\} = \infty$, there exists $C > 0$ such that $|g(t)| \leq C t^{3n+2}$ for all $t \neq 0$.

$$\begin{aligned} |g(t)| \leq C t^{3n+2} &\Leftrightarrow -C t^{3n+2} \leq g(t) \leq C t^{3n+2} \\ &\Leftrightarrow -C t^{3n+2} e^{-t^2} \leq g(t) f\left(\frac{1}{t}\right) \leq C t^{3n+2} e^{-t^2} \\ &\Rightarrow \lim_{t \rightarrow \infty} -C t^{3n+2} e^{-t^2} \leq \lim_{t \rightarrow \infty} g(t) f\left(\frac{1}{t}\right) \leq \lim_{t \rightarrow \infty} C t^{3n+2} e^{-t^2} \end{aligned}$$

By squeeze theorem, $\lim_{t \rightarrow \infty} g(t) f\left(\frac{1}{t}\right) = 0$.

The case $t \rightarrow -\infty$ is analogous, and proof is completed.

Lemma 2. Let $f(x) = e^{-1/x^2}$ for all $x \neq 0$, and $f(0) = 0$.
Then $f(x)$ is infinitely differentiable for $x \in \mathbb{R}$.
Moreover,

$$f^{(n)}(x) = \begin{cases} f(x)Q_n(x) & (x > 0) \\ 0 & (x = 0) \end{cases}$$

where $P_n(x)$ a polynomial function of degree n , and $Q_n(x) = P_{3n}\left(\frac{1}{x}\right)$ fulfills the recursive definition

$$\begin{aligned} Q_0(x) &= 1 \\ Q_n(x) &= \frac{2}{x^3}Q_{n-1}(x) + Q'_{n-1}(x) \end{aligned}$$

Proof. Let us use mathematical induction.
It is easy to show when $n = 1$.
Assume $n = k$ is true.
Then, $f^{(k+1)}(x)$ is well-defined for $x \neq 0$.
More Specifically,

$$\begin{aligned} f^{(k+1)}(x) &= \left\{ e^{-1/x^2} \right\}' Q_k(x) + e^{-1/x^2} \{Q_k(x)\}' \\ &= \frac{2}{x^3} e^{-1/x^2} Q_k(x) + e^{-1/x^2} Q'_k(x) \\ &= e^{-1/x^2} \left\{ \frac{2}{x^3} Q_k(x) + Q'_k(x) \right\} \\ &= e^{-1/x^2} Q_{n+1}(x) \\ &= f(x)Q_{n+1}(x) \end{aligned}$$

So if we show $f^{(k+1)}(0) = 0$, then we can say that $f^{(k+1)}$ is also differentiable for $x \in \mathbb{R}$.
By definition,

$$\begin{aligned} f^{(k+1)}(0) &= \lim_{x \rightarrow 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{f^{(k)}(x)}{x} && (\because f^{(k)}(0) = 0) \\ &= \lim_{x \rightarrow 0} f(x) \frac{Q_k(x)}{x} \end{aligned}$$

$$\begin{aligned} &= \lim_{x \rightarrow 0} f(x) P_{3k+1} \left(\frac{1}{x} \right) && (\because Q_k(x) = P_{3k} \left(\frac{1}{x} \right)) \\ &= 0 && (\because \text{By Lemma 1}) \end{aligned}$$

Hence $f^{(k+1)}(0) = 0$, $f^{(k+1)}$ is also differentiable for $x \in \mathbb{R}$.

By the principle of mathematical induction, $f(x)$ is infinitely differentiable for $x \in \mathbb{R}$.

Exercise 5.21. Let E be a closed subset of \mathbb{R} . There is a real continuous function f on \mathbb{R} whose zero set is E . Is it possible, for each closed set E , to find such an f which is differentiable on \mathbb{R} , or one which is n times differentiable, or even one which has derivatives of all orders on \mathbb{R} ?

Proof. Let $f(x) = e^{-1/x^2}$ for all $x \neq 0$, and $f(0) = 0$.

By Lemma 2, $f(x)$ is infinitely differentiable, and $f^{(n)}(0) = 0$ for $n \in \mathbb{N}$.

Define function g by

$$g = (f \circ \text{ReLU})(x)$$

with $\text{ReLU}(x) = \max(0, x)$.

i.e.

$$g(x) = \begin{cases} e^{-1/x^2} & (x > 0) \\ 0 & (x \leq 0) \end{cases}$$

g is also infinitely differentiable everywhere on \mathbb{R} and whose zero set is $(-\infty, 0]$.

Let us think

$$f_{(a,b)}(x) = g(x - a)g(b - x)$$

where $-\infty \leq a < b \leq \infty$.

Note that $f_{(-\infty, b)}(x) = e^{-\frac{1}{(x-b)^2}}$ for $x > b$, and similarly when $b = \infty$.

As set E is closed, complement of E is open set consisting of a union of disjoint open intervals, so $E^c = \bigcup_i (a_i, b_i)$

Let l be collection of disjoint open intervals, and define function h by

$$h(x) = \sum_{(a,b) \in l} f_{(a,b)}(x)$$

h is well-defined since at least one of the terms in the sum isn't 0 for any $x \notin E$, infinitely differentiable on \mathbb{R} , and has zero set E .

Exercise 5.22. Suppose f is a real function on $(-\infty, \infty)$. Call x a *fixed point* of f if $f(x) = x$.

- (a) If f is differentiable and $f'(t) \neq 1$ for every real t , prove that f has at most one fixed point.
- (b) Show that the function f defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t .

- (c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t , prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, \dots$

- (d) Show that the process described in (c) can be visualized by the zig-zag path

$$(x_1, x_2) \rightarrow (x_2, x_2) \rightarrow (x_2, x_3) \rightarrow (x_3, x_3) \rightarrow (x_3, x_4) \rightarrow \dots$$

Proof.