Chapter 8

Some Special Functions Selected Exercise

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List of Exercise

Lemma 1. Let $P_n(x)$ be set of polynomials of degree $3n+1$. Suppose $f(x) = e^{-1/x^2}$ for $x \neq 0$, then

$$\lim_{x\to 0} g\left(\frac{1}{x}\right) f(x) = 0$$

where $g \in P_n(x)$.

Proof. Let $x = \dfrac{1}{t}$. It is easy to show that

$$\lim_{x\to 0} g\left(\frac{1}{x}\right) f(x) = 0$$

if and only if

$$\lim_{t\to\infty} g(t) f\left(\frac{1}{t}\right) = \lim_{t\to -\infty} g(t) f\left(\frac{1}{t}\right) = 0$$

using $\varepsilon - \delta$ argument.
So, we will show that $\lim_{t\to\infty} g(t) f\left(\frac{1}{t}\right) = L$.
As $\lim_{t\to\infty} \left\{t^{3n+2} - g(t)\right\} = \infty$, there exists $C > 0$ such that $|g(t)| \leq C\, t^{3n+2}$ for all $t > 0$.

$$\begin{aligned}
|g(t)| \leq C\, t^{3n+2} &\Leftrightarrow -C\, t^{3n+2} \leq g(t) \leq C\, t^{3n+2} \\
&\Leftrightarrow -C\, t^{3n+2} e^{-t^2} \leq g(t) f\left(\frac{1}{t}\right) \leq C\, t^{3n+2} e^{-t^2} \\
&\Rightarrow \lim_{t\to\infty} -C\, t^{3n+2} e^{-t^2} \leq \lim_{t\to\infty} g(t) f\left(\frac{1}{t}\right) \leq \lim_{t\to\infty} C\, t^{3n+2} e^{-t^2}
\end{aligned}$$

By squeeze theorem, $\lim_{t\to\infty} g(t) f\left(\frac{1}{t}\right) = 0$.
The case $t \to -\infty$ is analogous, and proof is completed.

Exercise 1. Define

$$f(x) = \begin{cases} e^{-1/x^2} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Prove that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \ldots$.

Proof. Let $f(x) = e^{-1/x^2}$ for all $x \neq 0$, and $f(0) = 0$.
Then, we are enough to show that

$$f^{(n)}(x) = \begin{cases} f(x)Q_n(x) & (x > 0) \\ 0 & (x = 0) \end{cases}$$

where $P_n(x)$ a polynomial function of degree n, and $Q_n(x) = P_{3n}\left(\frac{1}{x}\right)$ fulfills the recursive definition

$$\begin{aligned} Q_0(x) &= 1 \\ Q_n(x) &= \frac{2}{x^3}Q_{n-1}(x) + Q'_{n-1}(x) \end{aligned}$$

Let us use mathematical induction.
It is easy to show when $n = 1$.
Assume $n = k$ is true.
Then, $f^{(k+1)}(x)$ is well-defined for $x \neq 0$.
More Specifically,

$$\begin{aligned} f^{(k+1)}(x) &= \left\{e^{-1/x^2}\right\}' Q_k(x) + e^{-1/x^2}\left\{Q_k(x)\right\}' \\ &= \frac{2}{x^3}\, e^{-1/x^2} Q_k(x) + e^{-1/x^2}\, Q'_k(x) \\ &= e^{-1/x^2}\left\{\frac{2}{x^3}Q_k(x) + Q'_k(x)\right\} \\ &= e^{-1/x^2} Q_{k+1}(x) \\ &= f(x)Q_{k+1}(x) \end{aligned}$$

So if we show $f^{(k+1)}(0) = 0$, then we can say that $f^{(k+1)}$ is also differentiable for $x \in \mathbb{R}$.

By definition,

$$\begin{aligned}
f^{(k+1)}(0) &= \lim_{x\to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x-0} \\
&= \lim_{x\to 0} \frac{f^{(k)}(x)}{x} && (\because\ f^{(k)}(0) = 0) \\
&= \lim_{x\to 0} f(x)\frac{Q_k(x)}{x} \\
&= \lim_{x\to 0} f(x)P_{3k+1}\left(\frac{1}{x}\right) && (\because\ Q_k(x) = P_{3k}\left(\frac{1}{x}\right)) \\
&= 0 && (\because\ \text{By Lemma 1})
\end{aligned}$$

Hence $f^{(k+1)}(0) = 0$, $f^{(k+1)}$ is also differentiable for $x \in \mathbb{R}$.
By the principle of mathematical induction, $f(x)$ is infinitely differentiable for $x \in \mathbb{R}$.

Exercise 2. Let a_{ij} be the number in the ith row and jth column of the array

$$\begin{array}{ccccc} -1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & -1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \cdots \\ \hdashline \end{array}$$

so that

$$a_{ij} = \begin{cases} 0 & (i < j), \\ -1 & (i = j), \\ 2^{j-i} & (i > j). \end{cases}$$

Prove that

$$\sum_i \sum_j a_{ij} = -2, \qquad \sum_j \sum_i a_{ij} = 0.$$

Proof. First, fix i, and define $\sum_j a_{ij} = b_i$.
By definition, $a_{ij} = 0$ if $i < j$, we are enough to calculate the value for $i \geq j$.
As $a_{ij} = 2^{j-i}$ if $i > j$,

$$b_i = \sum_{j=1}^{i-1} 2^{j-i} - 1 = \frac{2^{1-i}(2^{i-1} - 1)}{2 - 1} - 1 = -2^{1-i}.$$

Thus

$$\sum_i \sum_j a_{ij} = \sum_i b_i = \sum_i -2^{1-i} = -2.$$

In the same way, fix j, and define $\sum_i a_{ij} = c_j$.
Then,

$$c_j = \sum_{i=j+1}^{\infty} 2^{j-i} - 1 = \frac{1/2}{1 - 1/2} - 1 = 1 - 1 = 0.$$

Thus,

$$\sum_j \sum_i a_{ij} = \sum_j c_j = 0.$$

Exercise 3. Prove that

$$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$$

if $a_{ij} \geq 0$ for all i and j (the case $+\infty = +\infty$ may occur).

Proof. First, x

Exercise 11. Suppose $f \in \mathscr{R}$ on $[0,\ A]$ for all $A < \infty$, and $f(x) \to 1$ as $x \to +\infty$. Prove that

$$\lim_{t\to 0}\ t\int_0^\infty e^{-tx} f(x)\, dx = 1 \qquad (t > 0).$$

Proof. Note that $f \in \mathscr{R}$ on $[0,\ A]$ for all $A < \infty$, so we don't guarantee about existence of $\displaystyle\int_A^\infty f(x)dx$.

Also we can guarantee about $\displaystyle\int_0^A |f(x)|dx$, let assume this value k.

Since $h(x) = e^{-tx}$ be strictly increasing function for every $t > 0$,

$$\left|\int_0^A e^{-tx} f(x)dx\right| \leq \int_0^A e^{-tx}|f(x)|dx \leq e^{-tA}\int_0^A |f(x)|dx = ke^{-tA}$$

by **Theorem 6.12(b), 6.13**.

By above, we can easily know that

$$\lim_{t\to 0} t\int_0^A e^{-tx} f(x)dx = 0,$$

so if we show that $\displaystyle\lim_{t\to 0}\int_A^\infty e^{-tx} f(x)dx$ exists, which value is 1, proof is completed.

Let $\varepsilon > 0$ given.

Since $f(x) \to 1$ as $x \to \infty$, there exists t s.t. $|f(x) - 1| \leq \varepsilon$ for all $x \geq t$.

Letting $A = t$, then $1 - \varepsilon \leq f(x) \leq 1 + \varepsilon$ for all $x > A$, and $f \in \mathscr{R}$ on $[0,\ A]$.

Also $h(x) = e^{-tx}$ goes to 0 as $x \to \infty$ for $t > 0$, $\displaystyle\int_a^\infty Kte^{-tx}dx$ well defined, and value is Ke^{-ta}. Thus $1 - \varepsilon \leq f(x) \leq 1 + \varepsilon$ for every $x > A$,

$$(1-\varepsilon)e^{-tA} \leq \underline{\int_A^\infty} te^{-tx} f(x)dx \leq \overline{\int_A^\infty} te^{-tx} f(x)dx \leq (1+\varepsilon)e^{-tA} \qquad (1)$$

Letting $t \to 0$, e^{-tA} goes to 1, so left and right side on equation (1) goes to $1 - \varepsilon$ and $1 + \varepsilon$, respectively.

This shows that $\displaystyle\int_a^b te^{-tx} f(x)dx = 1$, and completes the proof.

Exercise 13. Put $f(x) = x$ if $0 \leq x < 2\pi$, and apply Parseval's theorem to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Proof.

Exercise 14. If $f(x) = (\pi - |x|)^2$ on $[-\pi,\ \pi]$, prove that

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Proof.

Exercise 22. If α is real and $-1 < x < 1$, prove Newton's binomial theorem

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n.$$

Hint : Denote the right side by $f(x)$. Prove that the series converges.
Prove that

$$(1+x)f'(x) = \alpha f(x)$$

and solve this differential equation.
Show also that

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\,\Gamma(\alpha)} x^n$$

if $-1 < x < 1$ and $\alpha > 0$.

Proof.