Probability Theory

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Problem 1

Statement

Suppose 8 identical blackboards are to be divided among 4 schools.

(a) How many divisions are possible?

(b) How many if each school must receive at least 1 blackboard?

Solution

(a) Every blackboards can be divided any school.
Hence, the total number of divisions is $\underbrace{4 \times 4 \times \cdots \times 4}_{8 \text{ blackboards}} = 4^8$

(b) Let us use inclusion-exclusion principle.
First, There are 4^8 ways on which all schools receive blackboards.
Second, There are $3^8 \times \binom{4}{1}$ ways on which only one school doesn't receive blackboard.
Third, There are $2^8 \times \binom{4}{2}$ ways on which two schools don't recevie blackboard.
Lastly, There are $\binom{4}{3}$ ways on which only one school receives blackboard.
Hence, the total number of cases is

$$4^8 - 4 \times 3^8 + 6 \times 2^8 - 4 \times 1 = 40824$$

Problem 2

Statement

A 5-card hand is dealt from a well-shuffled deck of 52 playing cards. What is the probability that the hand contains at least one card from each of the four suits?

Solution

First we know that there are 13 cards for each suits.

Thus, we can evaluate the probability by choosing a suit to be contained twice, and choosing a card for each suit.

The denominator is $\binom{52}{5}$, and the numerator is product of the number of ways of choosing one among four suits and choosing a card from each suits.

Because first is 4 and second is $13^3 \times \binom{13}{2}$, probability is

$$\begin{aligned}\frac{4 \times 13^3 \times \binom{13}{2}}{\binom{52}{5}} &= \frac{4 \times 13^3 \times 13 \times 12 \times 5!}{52 \times 51 \times 50 \times 49 \times 48 \times 2!} \\ &= \frac{2197}{8330} \approx 0.26\end{aligned}$$

Problem 3

Statement

From a group of 3 first-year students, 4 sophomores, 4 juniors, and 3 seniors, a committee of size 4 is randomly selected. Find the probability that the committee will consist of

(a) 1 from each class

(b) 2 sophomores and 2 juniors

(c) Only sophomores or juniors

Solution

First, there are $\binom{14}{4}$ ways of selecting student randomly.

(a) Let $n_1, n_2, \cdots, n_k$ be the group sizes.
If we select only one student each group, then there are $n_1 \times n_2 \times \cdots n_k$ ways by product rule.
Hence there are $3 \times 4 \times 4 \times 3 = 144$ ways of selecting member, the probability is $\dfrac{144}{1001} \approx 0.14$

(b) There are $\binom{4}{2}$ ways of selecting 2 sophomores, and $\binom{4}{2}$ ways of selecting 2 juniors.
As $\binom{4}{2} = 6$, there are $6^2 = 36$ ways on which committee will consist of 2 sophomores and 2 juniors, the probability is $\dfrac{36}{1001} \approx 0.04$.

(c) There are 8 students in sophomores and juniors, hence The number of choosing committee member is $\binom{8}{4}$
Thus, the probability is $\dfrac{70}{1001} = \dfrac{10}{143} \approx 0.07$

Problem 4

Statement

A hospital administrator codes incoming patients suffering gunshot wounds according to whether they have insurance (coding 1 if they do and 0 if they do not) and according to their condition, which is rated as good (g), fair (f), or serious (s). Consider an experiment that consists of the coding of such a patient.

(a) Give the sample space of this experiment

(b) Let A be the event that the patient is in serious condition. Specify the outcomes in A

(c) Let B be the event that the patient is uninsured. Specify the outcomes in B

(d) Give all the outcomes in the event $B^c \cup A$

Solution

Let $(a,\ b)$ be a pair where a is whether they have insuarnce, b is condition which is rated as g, f, s.

(a) $\Omega = \{(1,\ g),\ (1,\ f),\ (1,\ s)\ (0,\ g),\ (0,\ f),\ (0,\ s)\}$

(b) As patient is in serious condition, second part of pair is s
Thus (outcomes in A) $= \{(1,\ s),\ (0,\ s)\}$

(c) As patient is uninsured, first part of pair is 0
Thus (outcomes in B) $= \{(0,\ f),\ (0,\ g),\ (0,\ s)\}$

(d) Since $B = \{(0,\ f),\ (0,\ g),\ (0,\ s)\}$, $B^c = \{(1,\ f),\ (1,\ g),\ (1,\ s)\}$
Thus $B^c \cup A$ is

$$\{(1,\ f),\ (1,\ g),\ (1,\ s),\ (0,\ s)\}$$

Problem 5

Statement

Suppose that Seoul-metro consists of n cars, and k ($k \geq n$) passengers get on it by selecting one of their cars randomly. Find the probability that there will be at least one passenger in each car.

Solution

For convenience, suppose that the cars have an index $i = 1, 2, ..., n$
Let A be the event that there will be at least one passenger in each car. Then, $A = \cap_{i=1}^{n} A_i$ where A_i denotes the event at least one passenger in i-th car.
Suppose $B = A^c$, then $B = A^c = \cup_{i=1}^{n} A_i^c = \cup_{i=1}^{n} B_i$ where B_i denotes the event that there will be no passenger in car i
We want to evaluate $P(A)$, but it is difficult to get value directly. Thus we will evaluate $P(B)$ instead of $P(A)$, and will use the formula $P(A)+P(B) = 1$
To determine $P(B)$, we will use include-exclusion principle. Note that each outcome is equally likely, so the probability of each set is $1/n^k$ times the number of outcomes in the set.
Each B_i has those outcomes in which there will be no passenger in i-th car. Because there are $(n-1)$ ways for each passenger, the nubmer of cases is $(n-1)^k$, so $P(B_i) = (n-1)^k/n^k$ for each i.
Each $B_i \cap B_j$ has those outcomes in which there will be no passenger in not only i-th car but j-th car. The number of ways is $(n-2)^k$, hence $P(B_i \cap B_j) = (n-2)^k/n^k$
In same way, we see that $P(B_1 \cap B_2 \cap \cdots \cap B_\alpha) = (n-\alpha)^k/n^k$, so we can compute $P(\cup_{i=1}^{n} B_i)$ using inclusion-exclusion principle, result is

$$\begin{aligned} P(\cup_{i=1}^{n} B_i) &= \sum_{i=1}^{n} P(B_i) - \sum_{i<j} P(B_i \cap B_j) + \cdots + (-1)^{n+1} P(\cap_{i=1}^{n} B_i) \\ &= \sum_{i=1}^{n} (-1)^{n+1} \binom{n}{i} \left(1 - \frac{i}{n}\right)^k \end{aligned}$$

Finally, $P(A) = 1 - P(B) = \displaystyle\sum_{i=0}^{n} (-1)^n \binom{n}{i} \left(1 - \frac{i}{n}\right)^k$ ■

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