Convex Optimization

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Problem 1

Statement

Let $C \subset \mathbb{R}^n$ be a convex set. This means that for any two points x_1 and x_2 in C, we have $\theta x_1 + (1-\theta)x_2 \in C$ for any $0 \leq \theta \leq 1$. Extend this to k points, that is, with $x_1, ..., x_k \in C$, and let $\theta_1, ..., \theta_k \in \mathbb{R}$ satisfy $\theta_i \geq 0$, $\theta_1 + \cdots + \theta_k = 1$. Show that $\theta_1 x_1 + \cdots + \theta_k x_k \in C$

Solution

Proof. Let us user mathematical induction on k

Base case: It is trivial for $k = 2$

Induction step: Let statement be true for $k = n$

Suppose that θ_1, θ_2, ..., θ_n, $\theta_{n+1} \in \mathbb{R}$ s.t. $\theta_i \geq 0$, $\displaystyle\sum_{i=1}^{n+1} \theta_i = 1$

If $\theta_{n+1} = 1$, then $\theta_1 = \theta_2 = \cdots = \theta_n = 0$ and $\displaystyle\sum_{i=1}^{n+1} \theta_i x_i = x_{n+1} \in C$

Thus we are enough to prove when $0 < \theta_{n+1} < 1$, then $1 - \theta_{n+1} = \displaystyle\sum_{i=1}^{n} \theta_i > 0$

Let us define

$$\theta_i{}' = \frac{\theta_i}{1 - \theta_{n+1}}, \ x' = \sum_{i=1}^{n} \theta_i{}' x_i$$

Since $\displaystyle\sum_{i=1}^{n} \theta_i{}' = 1$, we conclude that $x' \in C$ using induction hypothesis

As C is convex set and x', $x_{n+1} \in C$, then $(1 - \theta_{n+1})x' + \theta_{n+1}x_{n+1} \in C$

Note that

$$x' = \sum_{i=1}^{n} \theta_i{}' x_i = \sum_{i=1}^{n} \frac{\theta_i x_i}{1 - \theta_{n+1}}$$

Finally, Substituting x' for expression,

$$\theta_1 x_1 + \cdots + \theta_{n+1} x_{n+1} \in C$$

Thus, statement hold for $k = n + 1$ with $k = n$, and the proof of the induction step is complete.

By the principle of induction, statement is true for all $k \in \mathbb{N}$ ■

Problem 2

Statement

A set C is affine if it contains the line that connects two points arbitarily chosen out of C. AN equivalent formal definition is as follows: Consider n points $x_1, ..., x_n \in C$, and some n real numbers $\theta_1, ..., \theta_n$ s.t. $\sum_{i=1}^{n} \theta_i = 1$ (no restriction on positiveness). Then C is affine if and only if $\sum_{i=1}^{n} \theta_i x_i \in C$ for all $x_1, ..., x_n \in C$.
Now the problem: consider a set of solutions to linear equations given by

$$C = \{x \mid Ax = b\}$$

Show that C is an affine set.

Solution

Proof. we have to show that set C is an affine set. i.e.

$$\sum_{i=1}^{n} \theta_i x_i \in C$$

where $x_1, ..., x_n \in C$ and $\theta_1, ..., \theta_n \in \mathbb{R}$
Since $x_1, ..., x_n \in C$,

$$A\left(\sum_{i=1}^{n} \theta_i x_i\right)$$
$$= \sum_{i=1}^{n} A(\theta_i x_i)$$
$$= \sum_{i=1}^{n} \theta_i (A x_i)$$
$$= \sum_{i=1}^{n} \theta_i b = b$$

Thus $\displaystyle\sum_{i=1}^{n} \theta_i x_i$ is also element of C, hence C is an affine set. ∎

Problem 3

Statement

asdf

Solution

Proof. sadf ■