Unit-4

(10) Solve Non-Linear Programming problem by Langrangean Method

$$\text{Min } Z = x_1^2 + x_2^2 + x_3^2$$

Sub to, $4x_1 + x_2^2 + 2x_3 = 14$, $x_1, x_2, x_3 \geq 0$.

$$F(x, \lambda) = F - \lambda g$$

$$F = x_1^2 + x_2^2 + x_3^2 - \lambda(4x_1 + x_2^2 + 2x_3 - 14)$$

$$\frac{\partial F}{\partial x_1} = 2x_1 - 4\lambda = 0$$

$$x_1 = \frac{4\lambda}{2} \Rightarrow \boxed{x_1 = 2\lambda} \rightarrow (1)$$

$$\frac{\partial F}{\partial x_2} = 2x_2 - \lambda(2x_2) = 0$$

$$x_2(1-\lambda) = 0$$

$$\boxed{x_2 = 0} \text{ (or) } \boxed{\lambda = 1} \rightarrow (2)$$

$$\frac{\partial F}{\partial x_3} = 2x_3 - 2\lambda = 0$$

$$x_3 = \frac{2\lambda}{2} \Rightarrow \boxed{x_3 = \lambda} \rightarrow (3)$$

$$\frac{\partial F}{\partial \lambda} = -(4x_1 + x_2^2 + 2x_3 - 14) = 0 \rightarrow (4)$$

subs (1), (3) in (4)

$$4x_1 + x_2^2 + 2x_3 = 14$$

$$4(2\lambda) + x_2^2 + 2(\lambda) = 14$$

$$8\lambda + x_2^2 + 2\lambda = 14 \rightarrow (5)$$

$x_2^{\circ} + 10\lambda = 14$

Consider two cases,

Case (i): if $x_2 = 0$

(5) $\Rightarrow$ $x_2^2 + 10\lambda = 14$

$10\lambda = 14$

$\lambda = \frac{14}{10}$

$\lambda = 7/5$

$x_1 = 2\lambda$, $x_1 = 2(7/5) = 14/5 \Rightarrow \boxed{x_1 = 14/5}$

$x_3 = \lambda$

$\boxed{x_3 = 7/5}$

Case (ii)

If $\lambda = 1$

(5) $\Rightarrow$ $x_2^2 + 10 = 14$

$x_2^2 = 14 - 10$

$x_2^2 = 4$

$\boxed{x_2 = \pm 2}$

The value of x_2 is invalid so, the critical pts are

$x_1 = 14/5$

$x_2 = 0$

$x_3 = 7/5$

$\lambda = 7/5$

∴ The principle minor determinat

$(n-1) = 3-1 = 2$

of order 4

$$H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$= 2(0) - 0(0) + 0(0) = 0$

∴ The stationary pts are maximize

$$f_{max} = \left(\frac{14}{5}\right)^2 + 0 + \left(\frac{7}{5}\right)^2$$

$$= \frac{196}{25} + \frac{49}{25} = \frac{\cancel{245}}{\cancel{25}} = \frac{49}{5}$$

$$\boxed{f_{max} = \frac{49}{5}}$$

13. Solve NLPP by Langrangean

optimize $Z = x_1^2 + x_2^2 + x_3^2$

Sub. to

$x_1 + x_2 + 3x_3 = 2$

$5x_1 + 2x_2 + x_3 = 5$

$x_1, x_2, x_3 \geq 0$

Sol:

$F(x, \lambda) = f - \lambda_1 g - \lambda_2 g$

$F(x, \lambda) = x_1^2 + x_2^2 + x_3^2 - \lambda_1 (x_1 + x_2 + 3x_3 - 2) - \lambda_2 (5x_1 + 2x_2 + x_3 - 5)$

$$\frac{\delta F}{\delta x_1} = 2x_1 - \lambda_1 - 5\lambda_2 = 0$$

$$x_1 = \frac{\lambda_1 + 5\lambda_2}{2} \rightarrow (1)$$

$$\frac{\partial F}{\partial x_2} = 2x_2 - \lambda_1 - 2\lambda_2 \Rightarrow x_2 = \frac{\lambda_1 + 2\lambda_2}{2} \rightarrow (2)$$

$$\frac{\partial F}{\partial x_3} = 2x_3 - 3\lambda_1 - \lambda_2 = 0 \Rightarrow x_3 = \frac{3\lambda_1 + \lambda_2}{2} \rightarrow (3)$$

$$\frac{\partial F}{\partial \lambda_1} = -(x_1 + x_2 + 3x_3 - 2) = 0 \Rightarrow x_1 + x_2 + 3x_3 = 2 \rightarrow (4)$$

$$\frac{\partial F}{\partial \lambda_2} = -(5x_1 + 2x_2 + x_3 - 5) = 0 \Rightarrow$$

$$5x_1 + 2x_2 + x_3 = 5 \rightarrow (5)$$

subs. (1), (2), (3) in (4)

$$x_1 + x_2 + 3x_3 = 2$$

$$\left(\frac{\lambda_1 + 5\lambda_2}{2}\right) + \left(\frac{\lambda_1 + 2\lambda_3}{2}\right) + 3\left(\frac{3\lambda_1 + \lambda_2}{2}\right) = 2$$

$$\lambda_1 + 5\lambda_2 + \lambda_1 + 2\lambda_2 + 9\lambda_1 + 3\lambda_2 = 4$$

$$11\lambda_1 + 10\lambda_2 = 4 \rightarrow (6)$$

subs (1), (2), (6) in (5)

$$5x_1 + 2x_2 + x_3 = 5$$

$$5\left(\frac{x_1 + 5\lambda_2}{2}\right) + 2\left(\frac{\lambda_1 + 2\lambda_2}{2}\right) + \left(\frac{3\lambda_1 + \lambda_2}{2}\right) = 5$$

$$5\lambda_1 + 25\lambda_2 + 2\lambda_1 + 4\lambda_2 + 3\lambda_1 + \lambda_2 = 10$$

$$10\lambda_1 + 30\lambda_2 = 10$$

$$\lambda_1 + 3\lambda_2 = 1 \rightarrow (7)$$

solve (6) & (7)

$$\cancel{11}\lambda_1 + 10\lambda_2 = 4$$
$$11\lambda_1 + 33\lambda_2 = 33$$
$$(-)\quad -23\lambda_2 = -7$$
$$\lambda_2 = 7/23$$

$$\boxed{\lambda_2 = 0.304}$$

subs. 0.304 in (7)

$$\lambda_1 + 3(0.304) = 3$$
$$\boxed{\lambda_1 = 0.087}$$

$$x_1 = \frac{\lambda_1 + 5\lambda_2}{2} = 0.087 + 5(-0.304)/2$$
$$\boxed{x_1 = 0.804}$$

$$x_2 = \frac{\lambda_1 + 2\lambda_2}{2} = 0.087 + 2(0.304)/2$$
$$\boxed{x_2 = 0.348}$$

$$x_3 = \frac{3\lambda_1 + \lambda_2}{2} = \frac{3(0.087) + 0.304}{2}$$
$$\boxed{x_3 = 0.283}$$

$x_1 = 0.804$
$x_2 = 0.348$
$x_3 = 0.283$
$\lambda_1 = 0.087$
$\lambda_2 = 0.304$

To find maxima / minima use bordered Hessian matrix

$$H = \left[\begin{array}{c|c} 0 & P \\ \hline P^T & Q \end{array}\right]$$

$$Q = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$$

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 1 \end{bmatrix}$$

$$P^T = \begin{bmatrix} 1 & 5 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$H^B = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 5 & 2 & 1 \\ \hline 1 & 5 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 & 2 \end{array}\right]$$

Variable $n = 3$ $\quad m = 2$

equation

$n - m = 1$

$2m + 1 = 5$

Only one principle minor H^B of order 5

$$H^B = \begin{bmatrix} 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 5 & 2 & 1 \\ 1 & 5 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 & 2 \end{bmatrix}$$

$$= 1 \begin{vmatrix} 0 & 0 & 2 & 1 \\ 1 & 5 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 3 & 1 & 0 & 2 \end{vmatrix} - 1 \begin{vmatrix} 0 & 0 & 5 & 1 \\ 1 & 5 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 3 & 1 & 0 & 2 \end{vmatrix} + 3 \begin{vmatrix} 0 & 0 & 5 & 2 \\ 1 & 5 & 2 & 0 \\ 1 & 2 & 0 & 2 \\ 3 & 1 & 0 & 0 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 5 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & 5 & 0 \\ 1 & 2 & 2 \\ 3 & 1 & 0 \end{vmatrix} - 5 \begin{vmatrix} 1 & 5 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 5 & 2 \\ 1 & 2 & 0 \\ 3 & 1 & 0 \end{vmatrix}$$

$$+ 15 \begin{vmatrix} 1 & 5 & 0 \\ 1 & 2 & 2 \\ 3 & 1 & 0 \end{vmatrix} - 6 \begin{vmatrix} 1 & 5 & 2 \\ 1 & 2 & 0 \\ 3 & 1 & 0 \end{vmatrix} = 460$$

$\because$ +ve it attain minima,

the objective function

$$Z_{min} = x_1^2 + x_2^2 + x_3^2$$

$$= (0.804)^2 + (0.348)^2 + (0.283)^2 = 0.847$$

(14) NLPP

Max $z = 4x_1 - x_1^2 + 8x_2 - x_2^2$

Sub to

$$x_1 + x_2 = 2$$

$$x_1, x_2 \geq 0$$

$$F(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda g(x_1, x_2)$$

$$F(x_1, x_2, \lambda) = 4x_1 - x_1^2 + 8x_2 - x_2^2 - \lambda(x_1 + x_2 - 2)$$

$$\frac{\partial F}{\partial x_1} = 4 - 2x_1 - \lambda = 0 \qquad 4 - 2x_1 = \lambda \longrightarrow (1)$$

$$\frac{\partial F}{\partial x_2} = 8 - 2x_2 - \lambda = 0 \qquad 8 - 2x_2 = \lambda \longrightarrow (2)$$

$$\frac{\partial F}{\partial x_3} = x_1 + x_2 - 2 = 0 \qquad x_1 + x_2 = 2 \longrightarrow (3)$$

from (1) & (2)

$$4 - 2x_1 = 8 - 2x_2$$

$$-2x_1 + 2x_2 = 8 - 4$$

$$2(-x_1 + x_2) = 4$$

$$x_2 - x_1 = 2 \longrightarrow (4)$$

from (3) & (4)

$$x_1 + x_2 = 2$$
$$-x_1 + x_2 = 2$$
$$2x_2 = 4$$

$$\boxed{x_2 = 2}$$

sub $x_2 = 2$ in (4)

$$2 - x_1 = 2$$

$$\boxed{x_1 = 0}$$

sub $x_1 = 0$, $x_2 = 2$ in (1)

$4 - 0 = \lambda$

$\boxed{\lambda = 4}$

$x_1 = 0, \ x_2 = 2, \ \lambda = 4$

The objective function has max or min the solution $n - 1 = 2 - 1 = 1$

$\therefore$ The principal minor determinant order 1 and 3

(15) NLPP

optimize $Z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

Sub to

$x_1 + x_2 + x_3 = 15$

$2x_1 - x_2 + 2x_3 = 20$

$x_1, x_2, x_3 \geq 0$

Sol:

$F(x, \lambda) = f(x, \lambda) - \lambda_1 g(x, \lambda) - \lambda_2 g(x, \lambda)$

$F(x, \lambda) = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2 - \lambda_1(x_1 + x_2 + x_3 - 15) - \lambda_2(2x_1 - x_2 + 2x_3 - 20)$

$\frac{\partial F}{\partial x_1} = 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0$

$8x_1 - 4x_2 = \lambda + 2\lambda_2 \longrightarrow (1)$

$\frac{\partial F}{\partial x_2} = 4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0$

$4x_2 - 4x_1 = \lambda - \lambda_2 \longrightarrow (2)$

$\frac{\partial F}{\partial x_3} = 2x_3 - \lambda_1 - 2\lambda_2 = 0$

$2x_3 = \lambda_1 + 2\lambda_2 \longrightarrow (3)$

$$\frac{\partial F}{\partial \lambda_1} = -1(x_1 + x_2 + x_3 - 15) = 0 \rightarrow (4)$$

$$\frac{\partial F}{\partial \lambda_2} = -(2x_1 - x_2 + 2x_3 - 20) = 0 \rightarrow (5)$$

solve (1) & (2)

$$8x_1 - 4x_2 = \lambda_1 + 2\lambda_2$$

$$8x_1 + 8x_2 = 2\lambda_1 - 2\lambda_2$$

$$4x_2 = 3\lambda_1$$

$$x_2 = \frac{3\lambda_1}{4}$$

$x_2 = \frac{3\lambda_1}{4}$ in (1)

$$8x_1 - 3\lambda_1 = \lambda_1 + 2\lambda_2$$

$$8x_1 = \lambda_1 + 3\lambda_1 + 2\lambda_2$$

$$8x_1 = 4\lambda_1 + 2\lambda_2$$

$$8x_1 = 2(2\lambda_1 + \lambda_2)$$

$$x_1 = \frac{\cancel{2}(2\lambda_1 + \lambda_2)}{\cancel{8}}$$

$$\boxed{x_1 = 2\lambda_1 + \lambda_2 / 4}$$

subs. x_1, x_2, x_3 in (4)

$$x_1 + x_2 + x_3 = 15$$

$$\frac{2\lambda_1 + \lambda_2}{4} + \frac{3\lambda_1}{4} + \frac{\lambda_1 + 2\lambda_2}{2} = 15$$

$7\lambda_1 + 5\lambda_2 = 60 \longrightarrow (5)$

sub x_1, x_2, x_3 in (5)

$2x_1 - x_2 + 8x_3 = 20$

$2\left(2\lambda_1 + \lambda_2\right) - \left(\frac{3\lambda_1}{4}\right) + 2\left(\frac{\lambda_1 + 2\lambda_2}{2}\right) = 20$

$4\lambda_1 + 2\lambda_2 - 3\lambda_1 + 4\lambda_1 + 8\lambda_2 = 80$

$5\lambda_1 + 10\lambda_2 = 80$

$\lambda_1 + 2\lambda_2 = 16 \longrightarrow (6)$

solve (5) & (6)

$\cancel{7\lambda_1} + 5\lambda_2 = 60$

$\cancel{7\lambda_1} + 14\lambda_2 = 112$

$-9\lambda_2 = -52$

$\boxed{\lambda_2 = 52/9}$

sub $\lambda_2 = 52/9$ in (6)

$\lambda_1 + 2\left(\frac{52}{9}\right) = 16$

$\lambda_1 + \frac{104}{9} = 16$

$\lambda_1 = 16 - \frac{104}{9}$

$\lambda_1 = \frac{144 - 104}{9}$

$\boxed{\lambda_1 = \frac{40}{9}}$

$x_1 = \frac{2\lambda_1 + \lambda_2}{4}$

$x_1 = 2\left(\frac{40}{9}\right) + \left(\frac{52}{9}\right) / 4$

$$x_1 = \frac{80+52}{36}$$

$$x_1 = \frac{132}{36}$$

$$\boxed{x_1 = 11/3}$$

$$x_2 = \frac{3\lambda_1}{4}$$

$$= \frac{\cancel{3}\left(\frac{40}{\cancel{9}}\right)}{4}$$

$$= \frac{40}{12} = 10/3$$

$$\boxed{x_2 = 10/3}$$

$$x_3 = \frac{\lambda_1 + 2\lambda_2}{2} = \frac{40}{9} + 2\left(\frac{52}{9}\right)\Big/2$$

$$x_3 = \frac{40+104}{18} = \frac{144}{18} = 8$$

$$\boxed{x_3 = 8}$$

The Soln are,

$$\lambda_1 = 40/9$$
$$\lambda_2 = 52/9$$
$$x_1 = 11/3$$
$$x_2 = 10/3$$
$$x_3 = 8$$

To find whether the stationary is maxima or minima, apply Hessian matrix.

$$H^B = \left[\begin{array}{c|c} O & P \\ \hline P^T & Q \end{array}\right]$$

$$= \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ \hline 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{array}\right]$$

$$H^B = \begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{vmatrix}$$

$$= 1\left(\begin{vmatrix} 0 & 0 & -1 & 2 \\ 1 & 2 & -4 & 0 \\ 1 & -1 & 4 & 0 \\ 1 & 2 & 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & 0 & 2 & 2 \\ 1 & 2 & 8 & 0 \\ 1 & -1 & -4 & 0 \\ 1 & 2 & 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 0 & 0 & 2 & -1 \\ 1 & 2 & 8 & -4 \\ 1 & -1 & -4 & 4 \\ 1 & 2 & 0 & 0 \end{vmatrix}\right)$$

$$= -1\left(\begin{vmatrix} 1 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 & -4 \\ 1 & -1 & 4 \\ 1 & 2 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 & 8 \\ 1 & -1 & -4 \\ 1 & 2 & 0 \end{vmatrix}\right.$$

$$\left. + 2\begin{vmatrix} 1 & 2 & -4 \\ 1 & -1 & 4 \\ 1 & 2 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 & 8 \\ 1 & -1 & -4 \\ 1 & 2 & 0 \end{vmatrix}\right)$$

$$= 72 > 0$$

$H^B = 72$ $\therefore$ x_0 is minimum pt.

$$f_{min} = 4\left(\frac{11}{3}\right)^2 + 2\left(\frac{10}{3}\right)^2 + 8^2 - 4\left(\frac{11}{3}\right)\left(\frac{10}{3}\right)$$

$$= 820/9$$

(16)

Khun Tucker Solve NLP

Max $Z = 8x_1 + 10x_2 - x_1^2 - x_2^2$

Sub to

$3x_1 + 2x_2 \leq 6$

$x_1, x_2 \geq 0$

Khun Tucker Conditions are,

$$\frac{\partial F}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0$$

$$\frac{\partial F}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0$$

$$\lambda \, h(x) = 0$$

$$h(x) \leq 0$$

$$\lambda \geq 0$$

$f(x) = 8x_1 + 10x_2 - x_1^2 - x_2^2$

$h(x) = 3x_1 + 2x_2 - 6$

$F(x, \lambda) = f(x, \lambda) - \lambda h(x)$

$F(x, \lambda) = (8x_1 + 10x_2 - x_1^2 - x_2^2) - \lambda(3x_1 + 2x_2 - 6)$

Apply conditions.

$$\frac{\partial f}{\partial x_1} - \frac{\partial h}{\partial x_1} = 0$$

$$(8 - 2x_1) - \lambda(3) = 0 \rightarrow (1)$$

$$\frac{\partial f}{\partial x_2} - \frac{\lambda \partial h}{\partial x_2} = 0$$

$(10 - 2x_2) - \lambda(2) = 0 \longrightarrow$ (2)

$\lambda h(x) = 0$

$\lambda(3x_1 + 2x_2 - 6) = 0 \longrightarrow$ (3)

$3x_1 + 2x_2 - 6 \leq 0 \longrightarrow$ (4)

$\lambda \geq 0 \rightarrow$ (5)

Case (i)

$\lambda = 0$

$\lambda = 0$ in (1) & (2)

$8 - 2x_1 - \lambda(3) = 0 \longrightarrow$ (6)

$8 - 2x_1 = 0$

$2x_1 = 8$

$x_1 = 8/2 = 4$

$\boxed{x_1 = 4}$

$10 - 2x_2 - \lambda(2) = 0 \longrightarrow$ (7)

$10 - 2x_2 = 0$

$2x_2 = 10$

$\boxed{x_2 = 5}$

Sub $x_1 = 4, x_2 = 5$ in (4)

$3x_1 + 2x_2 = 6$

$3(4) + 2(5) = 6$

$12 + 10 = 6$

$22 \neq 6$

$\therefore x_1 = 4, x_2 = 5$ does not attain the optimal soln.

Case (ii)

$\lambda_1 \neq 0$

$3x_1 + 2x_2 - 6 = 0$

$8 - 2x_1 = 3\lambda \to$ from 6

$10 - 2x_2 = 2\lambda \to$ from 7

6 $\Rightarrow$ $2x_1 = 8 - 3\lambda$

$$\boxed{x_1 = \frac{8 - 3\lambda}{2}}$$

7 $\Rightarrow$ $2x_2 = 10 - 2\lambda \Rightarrow x_2 = \frac{10-2\lambda}{2}$

$$\boxed{x_2 = 5 - \lambda}$$

Sub x_1, x_2 in 4

$$3x_1 + 2x_2 = 6$$

$$3\left(\frac{8-3\lambda}{2}\right) + 2(5-\lambda) = 6$$

$$\frac{24 - 9\lambda}{2} + 10 - 2\lambda = 6$$

$$\frac{24 - 9\lambda}{2} + \frac{20 - 4\lambda}{2} = 6$$

$$24 - 9\lambda + 20 - 4\lambda = 6(2)$$

$$-13\lambda + 44 = 12$$

$$13\lambda = 44 - 12$$

$$\boxed{\lambda = 32/13}$$

$$x_1 = \frac{8 - 3\lambda}{2} \Rightarrow \frac{8 - 3\left(\frac{32}{13}\right)}{2} = \frac{8 - 96/13}{2}$$

$= \frac{104-96}{26} = \frac{\cancel{8}}{\cancel{26}} = 4/13$

$\boxed{x_1 = 4/13}$

$x_2 = 5 - \lambda = 5 - \frac{32}{13} = \frac{65-32}{13} = 33/13$

$\boxed{x_2 = 33/13}$

$Z_{max} = 8x_1 + 10x_2 - x_1^2 - x_2^2$

$= 8\left(\frac{4}{13}\right) + 10\left(\frac{33}{13}\right) - \left(\frac{4}{13}\right)^2 - \left(\frac{33}{13}\right)^2$

$= \frac{32}{13} + \frac{330}{13} - \left(\frac{16}{169}\right) - \left(\frac{1089}{169}\right)$

$= \frac{362}{13} - \frac{\cancel{1105}}{\cancel{169}} \frac{85}{13}$

$= \frac{362-85}{13} = \frac{277}{13}$

$\boxed{Z_{max} = \frac{277}{13}}$

(4) KT Condition solve NLPP

$\text{Max } z = 4x_1^2 + 5x_2^2 + 6x_1$

Sub to

$x_1 + 2x_2 \le 10$

$x_1 - 3x_2 \le 9$

~~$x_1 - 3x_2 \le 9$~~

$x_1, x_2 \ge 0.$

Khun Tucker Conditions are

$$\frac{\partial F}{\partial x_1} - \lambda_1 \frac{\partial h_1}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} = 0$$

$$\frac{\partial F}{\partial x_2} - \lambda_1 \frac{\partial h_1}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} = 0$$

$$\lambda_1 h_1(x) = 0$$

$$\lambda_2 h_2(x) = 0$$

$$h_1(x) \leq 0$$

$$h_2(x) \leq 0 \qquad \lambda_1, \lambda_2 \geq 0$$

$f(x) = 4x_1^2 + 6x_1 + 5x_2^2$

$h_1(x) = x_1 + 2x_2 - 10$

$h_2(x) = x_1 - 3x_2 - 9$

$$\boxed{F(x,\lambda) = f(x,\lambda) - \lambda_1 h(x) - \lambda_2 h(x)}$$

$$F(x,\lambda) = (4x_1^2 + 6x_1 + 5x_2^2) - \lambda_1(x_1 + 2x_2 - 10) - \lambda_2(x_1 - 3x_2 - 9)$$

Apply conditions:

$$\frac{\partial F}{\partial x_1} - \lambda_1 \frac{\partial h_1}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} = 0$$

$$14x_1 + 6 - \lambda_1(1) - \lambda_2(1) = 0 \rightarrow (1)$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{h_1}{h_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} = 0$$

$$10x_2 - 2\lambda_1 + 3\lambda_2 = 0 \rightarrow (2)$$

$\lambda_1 h_1(x) = 0$

$\lambda_1 (x_1 + 2x_2 - 10) = 0 \longrightarrow (3)$

$\lambda_2 h_2(x) = 0$

$\lambda_2 (x_1 - 3x_2 - 9) = 0 \longrightarrow (4)$

$h_1(x) \leq 0$

$x_1 + 2x_2 - 10 \leq 0 \longrightarrow (5)$

$h_2(x) \leq 0$

$x_1 - 3x_2 - 9 \leq 0 \longrightarrow (6)$

$\lambda_1, \lambda_2 \geq 0$

4 combinations using λ_1, λ_2

(i) $\lambda_1 = 0$; $\lambda_2 = 0$

(ii) $\lambda_1 \neq 0$; $\lambda_2 = 0$

(iii) $\lambda_1 = 0$; $\lambda_2 \neq 0$

(iv) $\lambda_1 \neq 0$; $\lambda_2 \neq 0$

Case (i)

$\lambda_1 = 0$, $\lambda_2 = 0$ in (1) & (2)

$14x_1 + 6 = 0 \Rightarrow \boxed{x_1 = -6/14}$

$10x_2 = 0$

$\boxed{x_2 = 0}$

Sub $x_2 = 0$, $x_1 = -6/14$ in (5)

$x_1 + 2x_2 - 10 \leq 0$

$(-6/14) + 0 - 10 \leq 0$

$\therefore$ It attain the infeasible solution

Case (ii) $\lambda_1 \neq 0$, $\lambda_2 = 0$

$14x_1 + 6 - \lambda_1 = 0$

$10x_2 - 2\lambda_1 = 0$

$10x_2 = 2\lambda_1$

$x_2 = \frac{2\lambda_1}{10} = \frac{\lambda_1}{5}$

$\boxed{x_2 = \frac{\lambda_1}{5}}$ $\boxed{x_1 = \frac{\lambda - 6}{14}}$

sub x_1, x_2 in (3)

$\lambda_1 \cancel{0} (x_1 + 2x_2 - 10) = 0$

$x_1 + 2x_2 - 10 = 0$

$\frac{\lambda_1 - 6}{14} + 2\left(\frac{\lambda_1}{5}\right) - 10 = 0$

$\frac{5(\lambda_1 - 6) + 28\lambda_1}{70} = \cancel{400}\ 10$

$\cancel{5(\lambda_1 - 6}\ 5\lambda_1 - 30 + 28\lambda_1 = 700$

$33\lambda_1 = 700$

$\boxed{\lambda_1 = \frac{700}{33}}$

$x_1 = \frac{730/33 - 6}{14} = \frac{38}{33}$, $x_2 = \frac{\lambda_1}{5} = \frac{700/33}{5} = \frac{146}{32}$

$\boxed{x_1 = \frac{38}{33}}$ $\boxed{x_2 = \frac{146}{32}}$

$\boxed{\lambda_2 = 0}$ $\boxed{\lambda_1 = \frac{700}{33}}$

~~$x_1 + 2x_2 = 1$~~

(5) $\Rightarrow x_1 + 2x_2 - 10 \leq 0$

$$\frac{38}{33} + 2\left(\frac{146}{33}\right) = 10$$

$$\frac{38+292}{33} = 10$$

$$330 = 330$$

It attains the feasible solution

$$Z = 7x_1^2 + 6x_1 + 5x_2^2$$

$$= 7\left(\frac{38}{33}\right)^2 + 6\left(\frac{38}{33}\right) + 5\left(\frac{146}{33}\right)^2$$

$$\boxed{Z = 114.061}$$