08 Properties of Relation

Relations on Sets

The Less-than Relation for Real Numbers

Define a relation L from $\mathbf{R}$ to $\mathbf{R}$ as follows: For all real numbers x and y,

$$x\ L\ y \Leftrightarrow x < y.$$

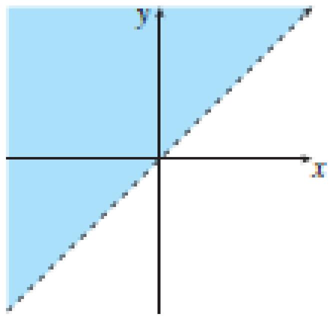
a. Is $57\ L\ 53$? b. Is $(-17)\ L\ (-14)$? c. Is $143\ L\ 143$? d. Is $(-35)\ L\ 1$?

e. Draw the graph of L as a subset of the Cartesian plane $\mathbf{R} \times \mathbf{R}$

Solution

a. No, $57 > 53$ b. Yes, $-17 < -14$ c. No, $143 = 143$ d. Yes, $-35 < 1$

e. For each value of x, all the points (x, y) with $y > x$ are on the graph. So the graph consists of all the points above the line $x = y$.



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Example 2

The Congruence Modulo 2 Relation

Define a relation E from $\mathbf{Z}$ to $\mathbf{Z}$ as follows: For all $(m, n) \in \mathbf{Z} \times \mathbf{Z}$,

$$m\ E\ n \quad \Leftrightarrow \quad m - n \text{ is even.}$$

a. Is $4\ E\ 0$? Is $2\ E\ 6$? Is $3\ E\ (-3)$? Is $5\ E\ 2$?

b. List five integers that are related by E to 1.

c. Prove that if n is any odd integer, then $n\ E\ 1$.

Solution

a. Yes, $4\ E\ 0$ because $4 - 0 = 4$ and 4 is even.
Yes, $2\ E\ 6$ because $2 - 6 = -4$ and -4 is even.
Yes, $3\ E\ (-3)$ because $3 - (-3) = 6$ and 6 is even.
No, $5\ \not{E}\ 2$ because $5 - 2 = 3$ and 3 is not even.

b. There are many such lists. One is

$$\begin{array}{rl} 1 & \text{because } 1 - 1 = 0 \text{ is even,} \\ 3 & \text{because } 3 - 1 = 2 \text{ is even,} \\ 5 & \text{because } 5 - 1 = 4 \text{ is even,} \\ -1 & \text{because } -1 - 1 = -2 \text{ is even,} \\ -3 & \text{because } -3 - 1 = -4 \text{ is even.} \end{array}$$

c. **Proof:** Suppose n is any odd integer. Then $n = 2k + 1$ for some integer k. Now by definition of E, $n\ E\ 1$ if, and only if, $n - 1$ is even. But by substitution,

$$n - 1 = (2k + 1) - 1 = 2k,$$

and since k is an integer, $2k$ is even. Hence $n\ E\ 1$ *[as was to be shown]*.

Example 3

A Relation on a Power Set

Let $X = \{a, b, c\}$. Then $\mathscr{P}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Define a relation **S** from $\mathscr{P}(X)$ to **Z** as follows: For all sets A and B in $\mathscr{P}(X)$ (i.e., for all subsets A and B of X),

$$A \; \mathbf{S} \; B \Leftrightarrow A \text{ has at least as many elements as } B.$$

a. Is $\{a, b\} \; \mathbf{S} \; \{b, c\}$? b. Is $\{a\} \; \mathbf{S} \; \emptyset$? c. Is $\{b, c\} \; \mathbf{S} \; \{a, b, c\}$? d. Is $\{c\} \; \mathbf{S} \; \{a\}$?

Solution

a. Yes, both sets have two elements.

b. Yes, $\{a\}$ has one element and $\emptyset$ has zero elements, and $1 \geq 0$.

c. No, $\{b, c\}$ has two elements and $\{a, b, c\}$ has three elements and $2 < 3$.

d. Yes, both sets have one element. ■

Inverse Relation

If R is a relation from A to B, then a relation R^{-1} from B to A can be defined by interchanging the elements of all the ordered pairs of R.

• Definition

Let R be a relation from A to B. Define the inverse relation R^{-1} from B to A as follows:

$$R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}.$$

This definition can be written operationally as follows:

$$\text{For all } x \in A \text{ and } y \in B, \quad (y, x) \in R^{-1} \quad \Leftrightarrow \quad (x, y) \in R.$$

Example 1

The Inverse of a Finite Relation

Let $A = \{2, 3, 4\}$ and $B = \{2, 6, 8\}$ and let R be the "divides" relation from A to B: For all $(x, y) \in A \times B$,

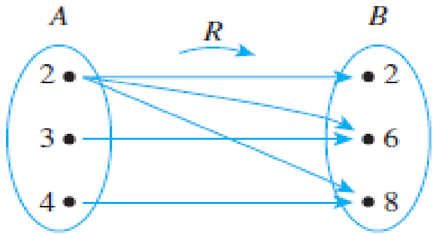
$$x\ R\ y \quad \Leftrightarrow \quad x \mid y \qquad \text{x divides y.}$$

a. State explicitly which ordered pairs are in R and R^{-1}, and draw arrow diagrams for R and R^{-1}.

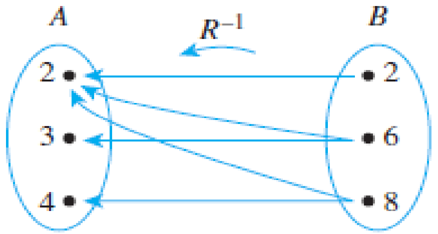
b. Describe R^{-1} in words.

Solution

a. $R = \{(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)\}$
$R^{-1} = \{(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)\}$

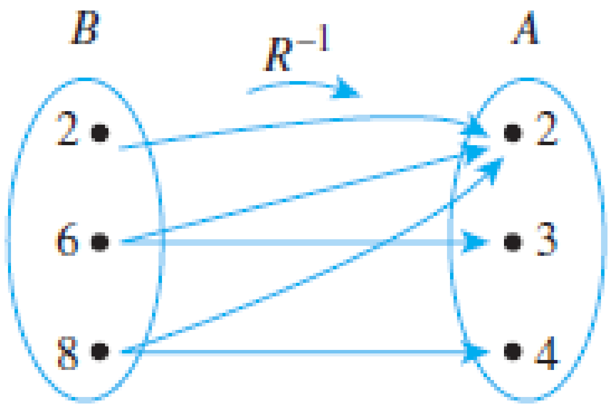


To draw the arrow diagram for R^{-1}, you can copy the arrow diagram for R but reverse the directions of the arrows.



Example 1(Continue)

Or you can redraw the diagram so that B is on the left.



b. R^{-1} can be described in words as follows: For all $(y, x) \in B \times A$,

$$y\ R^{-1}\ x \quad \Leftrightarrow \quad y \text{ is a multiple of } x.$$

Properties of Relations

1. Reflexive
2. Symmetric
3. Transitivity

• Definition

Let R be a relation on a set A.

1. R is **reflexive** if, and only if, for all $x \in A$, $x \; R \; x$.
2. R is **symmetric** if, and only if, for all $x, y \in A$, *if* $x \; R \; y$ then $y \; R \; x$.
3. R is **transitive** if, and only if, for all $x, y, z \in A$, *if* $x \; R \; y$ and $y \; R \; z$ then $x \; R \; z$.

Because of the equivalence of the expressions $x\ R\ y$ and $(x, y) \in R$ for all x and y in A, the reflexive, symmetric, and transitive properties can also be written as follows:

1. R is reflexive $\Leftrightarrow$ for all x in A, $(x, x) \in R$.
2. R is symmetric $\Leftrightarrow$ for all x and y in A, *if* $(x, y) \in R$ then $(y, x) \in R$.
3. R is transitive $\Leftrightarrow$ for all x, y and z in A, *if* $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

In informal terms, properties (1)–(3) say the following:

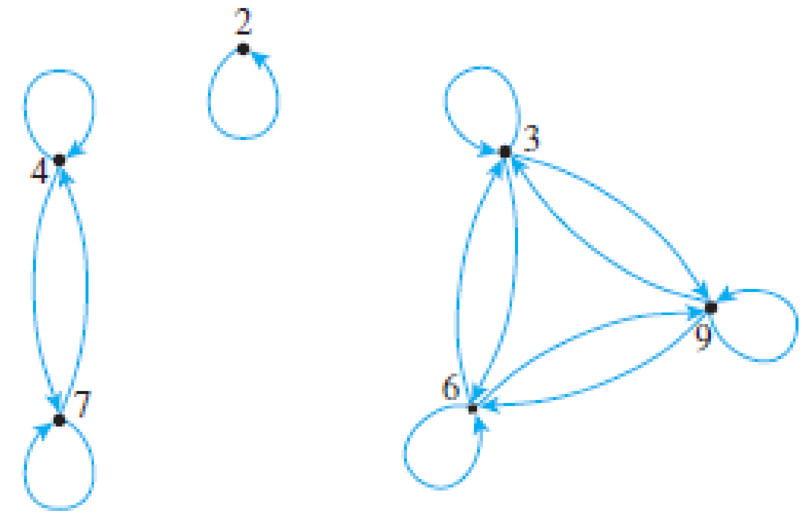
1. **Reflexive:** Each element is related to itself.
2. **Symmetric:** If any one element is related to any other element, then the second element is related to the first.
3. **Transitive:** If any one element is related to a second and that second element is related to a third, then the first element is related to the third.

Example 1

Let $A = \{2, 3, 4, 6, 7, 9\}$ and define a relation R on A as follows: For all $x, y \in A$,

$$x\ R\ y \quad \Leftrightarrow \quad 3 \mid (x - y).$$

Then $2\ R\ 2$ because $2 - 2 = 0$, and $3 \mid 0$. Similarly, $3\ R\ 3$, $4\ R\ 4$, $6\ R\ 6$, $7\ R\ 7$, and $9\ R\ 9$. Also $6\ R\ 3$ because $6 - 3 = 3$, and $3 \mid 3$. And $3\ R\ 6$ because $3 - 6 = -(6 - 3) = -3$, and $3 \mid (-3)$. Similarly, $3\ R\ 9$, $9\ R\ 3$, $6\ R\ 9$, $9\ R\ 6$, $4\ R\ 7$, and $7\ R\ 4$. Thus the directed graph for R has the appearance shown below.



Example 1 (Continued)

This graph has three important properties:

1. Each point of the graph has an arrow looping around from it back to itself.
2. In each case where there is an arrow going from one point to a second, there is an arrow going from the second point back to the first.
3. In each case where there is an arrow going from one point to a second and from the second point to a third, there is an arrow going from the first point to the third. That is, there are no "incomplete directed triangles" in the graph.

Properties (1), (2), and (3) correspond to properties of general relations called *reflexivity, symmetry,* and *transitivity*.

Example 2

Properties of Relations on Finite Sets

Let $A = \{0, 1, 2, 3\}$ and define relations R, S, and T on A as follows:

$$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\},$$
$$S = \{(0, 0), (0, 2), (0, 3), (2, 3)\},$$
$$T = \{(0, 1), (2, 3)\}.$$

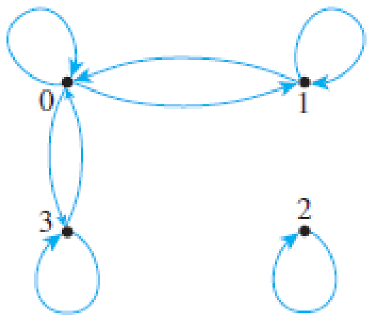
a. Is R reflexive? symmetric? transitive?

b. Is S reflexive? symmetric? transitive?

c. Is T reflexive? symmetric? transitive?

Solution

a. The directed graph of R has the appearance shown below.

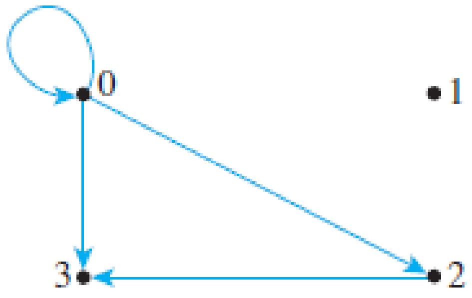


***R is reflexive*:** There is a loop at each point of the directed graph. This means that each element of A is related to itself, so R is reflexive.

***R* is symmetric**: In each case where there is an arrow going from one point of the graph to a second, there is an arrow going from the second point back to the first. This means that whenever one element of A is related by R to a second, then the second is related to the first. Hence R is symmetric.

***R* is not transitive**: There is an arrow going from 1 to 0 and an arrow going from 0 to 3, but there is no arrow going from 1 to 3. This means that there are elements of A—0, 1, and 3—such that $1 \, R \, 0$ and $0 \, R \, 3$ but $1 \not R \, 3$. Hence R is not transitive.

b. The directed graph of S has the appearance shown below.



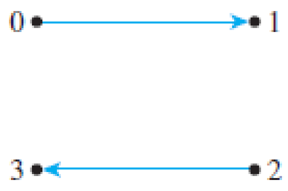
S is not reflexive: There is no loop at 1, for example. Thus $(1, 1) \notin S$, and so S is not reflexive.

S is not symmetric: There is an arrow from 0 to 2 but not from 2 to 0. Hence $(0, 2) \in S$ but $(2, 0) \notin S$, and so S is not symmetric.

S is transitive: There are three cases for which there is an arrow going from one point of the graph to a second and from the second point to a third: Namely, there are arrows going from 0 to 2 and from 2 to 3; there are arrows going from 0 to 0 and from 0 to 2; and there are arrows going from 0 to 0 and from 0 to 3. In each case there is an arrow going from the first point to the third. (Note again that the "first," "second," and "third" points need not be distinct.) This means that whenever $(x, y) \in S$ and $(y, z) \in S$, then $(x, z) \in S$, for all $x, y, z \in \{0, 1, 2, 3\}$, and so S is transitive.

$(x, z) \in S$, for all $x, y, z \in \{0, 1, 2, 3\}$, and so S is transitive.

c. The directed graph of T has the appearance shown below.



***T is not reflexive*:** There is no loop at 0, for example. Thus $(0, 0) \notin T$, so T is not reflexive.

***T is not symmetric*:** There is an arrow from 0 to 1 but not from 1 to 0. Thus $(0, 1) \in T$ but $(1, 0) \notin T$, and so T is not symmetric.

***T is transitive*:** The transitivity condition is vacuously true for T. To see this, observe that the transitivity condition says that

$$\text{For all } x, y, z \in A, \quad \text{if } (x, y) \in T \text{ and } (y, z) \in T \text{ then } (x, z) \in T.$$

The only way for this to be false would be for there to exist elements of A that make the hypothesis true and the conclusion false. That is, there would have to be elements x, y, and z in A such that

$$(x, y) \in T \quad \text{and} \quad (y, z) \in T \quad \text{and} \quad (x, z) \notin T.$$

In other words, there would have to be two ordered pairs in T that have the potential to "link up" by having the *second* element of one pair be the *first* element of the other pair. But the only elements in T are (0, 1) and (2, 3), and these do not have the potential to link up. Hence the hypothesis is never true. It follows that it is impossible for T *not* to be transitive, and thus T is transitive. ■

Example 3

Properties of Equality

Define a relation R on $\mathbf{R}$ (the set of all real numbers) as follows: For all real numbers x and y.

$$x \; R \; y \quad \Leftrightarrow \quad x = y.$$

a. Is R reflexive? b. Is R symmetric? c, Is R transitive?

Solution

a. ***R is reflexive*:** R is reflexive if, and only if, the following statement is true:

$$\text{For all } x \in \mathbf{R}, \quad x \; R \; x.$$

Since $x \; R \; x$ just means that $x = x$, this is the same as saying

$$\text{For all } x \in \mathbf{R}, \quad x = x.$$

But this statement is certainly true; every real number is equal to itself.

b. ***R is symmetric*:** R is symmetric if, and only if, the following statement is true:

$$\text{For all } x, y \in \mathbf{R}, \quad \text{if } x \; R \; y \text{ then } y \; R \; x.$$

By definition of R, $x\ R\ y$ means that $x = y$ and $y\ R\ x$ means that $y = x$. Hence R is symmetric if, and only if,

$$\text{For all } x, y \in \mathbf{R}, \quad \text{if } x = y \text{ then } y = x.$$

But this statement is certainly true; if one number is equal to a second, then the second is equal to the first.

c. ***R is transitive*:** R is transitive if, and only if, the following statement is true:

$$\text{For all } x, y, z \in \mathbf{R}, \quad \text{if } x\ R\ y \text{ and } y\ R\ z \text{ then } x\ R\ z.$$

By definition of R, $x\ R\ y$ means that $x = y$, $y\ R\ z$ means that $y = z$, and $x\ R\ z$ means that $x = z$. Hence R is transitive if, and only if, the following statement is true:

$$\text{For all } x, y, z \in \mathbf{R}, \quad \text{if } x = y \text{ and } y = z \text{ then } x = z.$$

But this statement is certainly true: If one real number equals a second and the second equals a third, then the first equals the third. ■

Equivalence Relation

Definition of an Equivalence Relation

A relation on a set that satisfies the three properties of reflexivity, symmetry, and transitivity is called an *equivalence relation*.

• Definition

Let A be a set and R a relation on A. R is an **equivalence relation** if, and only if, R is reflexive, symmetric, and transitive.

Example 1

An Equivalence Relation on a Set of Subsets

Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then

$$X = \{\{1\}, \{2\}, \{3\}, \{1,\ 2\}, \{1,\ 3\}, \{2,\ 3\}, \{1,\ 2,\ 3\}\}$$

Define a relation **R** on X as follows: For all A and B in X,

$$A \mathbf{R} B \Leftrightarrow \text{the least element of } A \text{ equals the least element of } B.$$

Prove that **R** is an equivalence relation on X.

Solution

R *is reflexive*: Suppose A is a nonempty subset of $\{1, 2, 3\}$. *[We must show that A **R** A.]* It is true to say that the least element of A equals the least element of A. Thus, by definition of R, A **R** A.

R *is symmetric*: Suppose A and B are nonempty subsets of $\{1, 2, 3\}$ and A **R** B. *[We must show that B **R** A.]* Since A **R** B, the least element of A equals the least element of B. But this implies that the least element of B equals the least element of A, and so, by definition of **R**, B **R** A.

R *is transitive*: Suppose A, B, and C are nonempty subsets of $\{1, 2, 3\}$, A **R** B, and B R C. *[We must show that A **R** C.]* Since A **R** B, the least element of A equals the least element of B and since B **R** C, the least element of B equals the least element of C. Thus the least element of A equals the least element of C, and so, by definition of **R**, A **R** C. ■

Example 2

Properties of "Less Than"

Define a relation R on $\mathbf{R}$ (the set of all real numbers) as follows: For all $x, y \in R$,

$$x \; R \; y \quad \Leftrightarrow \quad x < y.$$

a. Is R reflexive? b. Is R symmetric? c. Is R transitive?

Solution

a. ***R is not reflexive:*** R is reflexive if, and only if, $\forall x \in \mathbf{R}, x \; R \; x$. By definition of R, this means that $\forall x \in \mathbf{R}, x < x$. But this is false: $\exists x \in \mathbf{R}$ such that $x \not< x$. As a counterexample, let $x = 0$ and note that $0 \not< 0$. Hence R is not reflexive.

b. ***R is not symmetric:*** R is symmetric if, and only if, $\forall x, y \in \mathbf{R}$, if $x \; R \; y$ then $y \; R \; x$. By definition of R, this means that $\forall x, y \in \mathbf{R}$, if $x < y$ then $y < x$. But this is false: $\exists x, y \in \mathbf{R}$ such that $x < y$ and $y \not< x$. As a counterexample, let $x = 0$ and $y = 1$ and note that $0 < 1$ but $1 \not< 0$. Hence R is not symmetric.

c. ***R is transitive:*** R is transitive if, and only if, for all $x, y, z \in \mathbf{R}$, if $x \; R \; y$ and $y \; R \; z$ then $x \; R \; z$. By definition of R, this means that for all $x, y, z \in \mathbf{R}$, if $x < y$ and $y < z$, then $x < z$. But this statement is true by the transitive law of order for real numbers (Appendix A, T18). Hence R is transitive. ■