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5/23/2014
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PHYS 115

Assignment 7

Question 1 :

Part a :

```
sphericalbessel[l_, x_] := FunctionExpand[SphericalBesselJ[l, x]]
```

```
sphericalneumann[l_, x_] := FunctionExpand[SphericalBesselY[l, x]]
```

Part b :

```
sphericalbessel[0, x]
```

$$\frac{\text{Sin}[x]}{x}$$

```
sphericalbessel[1, x]
```

$$-\frac{\text{Cos}[x]}{x}+\frac{\text{Sin}[x]}{x^2}$$

```
sphericalbessel[10, x] // Simplify
```

$$-\frac{1}{x^{11}}\left(55\,x\left(11\,904\,165-1\,670\,760\,x^2+51\,597\,x^4-468\,x^6+x^8\right)\text{Cos}[x]+\left(-654\,729\,075+310\,134\,825\,x^2-18\,918\,900\,x^4+315\,315\,x^6-1485\,x^8+x^{10}\right)\text{Sin}[x]\right)$$

Part c:

```
Normal[Series[sphericalbessel[10, x], {x, 0, 14}]]
```

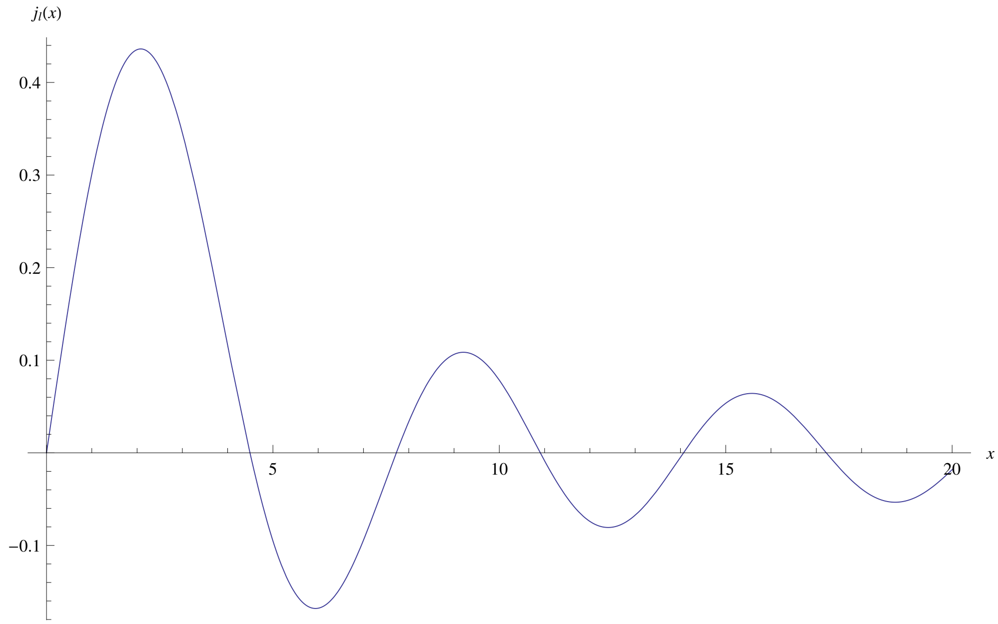
$$\frac{x^{10}}{13\,749\,310\,575}-\frac{x^{12}}{632\,468\,286\,450}+\frac{x^{14}}{63\,246\,828\,645\,000}$$

```
Normal[Series[sphericalneumann[10, x], {x, 0, -7}]]
```

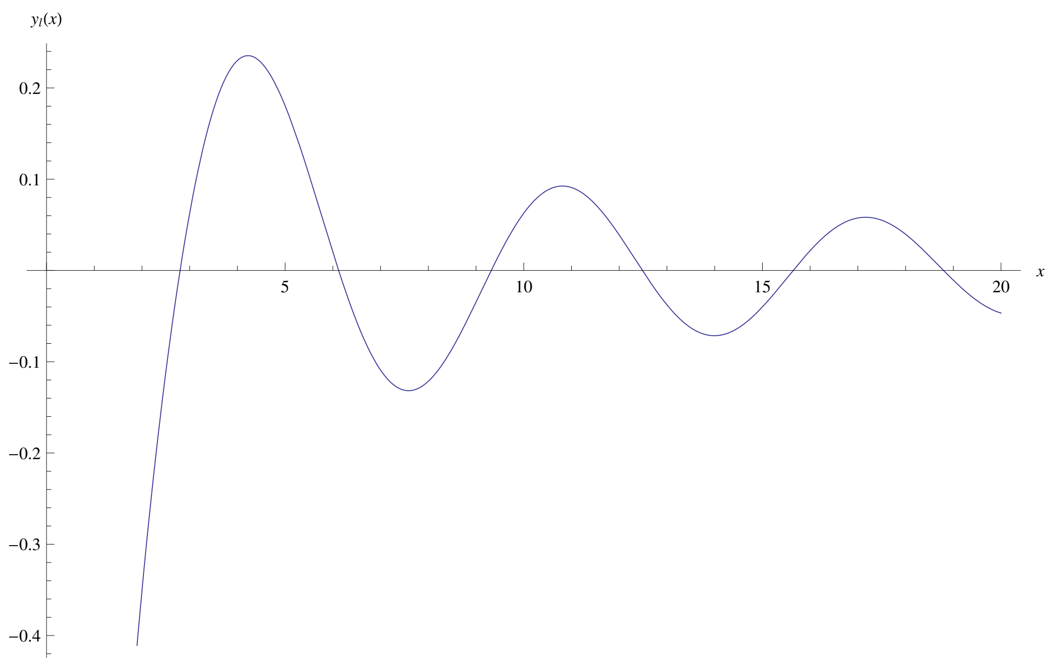
$$-\frac{654\,729\,075}{x^{11}}-\frac{34\,459\,425}{2\,x^9}-\frac{2\,027\,025}{8\,x^7}$$

Part d:

```
Plot[sphericalbessel[1, x], {x, 0, 20}, AxesLabel → {x, Subscript[j, l][x]}]
```




```
Plot[sphericalneumann[1, x], {x, 0, 20}, AxesLabel → {x, Subscript[y, l][x]}]
```



Question 2

Part a:

```
Normal[Series[Tanh[x], {x, 0, 20}]]
```

$$x-\frac{x^3}{3}+\frac{2x^5}{15}-\frac{17x^7}{315}+\frac{62x^9}{2835}-\frac{1382x^{11}}{155\,925}+\frac{21\,844x^{13}}{6\,081\,075}-\frac{929\,569x^{15}}{638\,512\,875}+\frac{6\,404\,582x^{17}}{10\,854\,718\,875}-\frac{443\,861\,162x^{19}}{1\,856\,156\,927\,625}$$

Part b :

```
Normal[InverseSeries[Series[Tanh[x], {x, 0, 20}]]]
```

$$x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\frac{x^9}{9}+\frac{x^{11}}{11}+\frac{x^{13}}{13}+\frac{x^{15}}{15}+\frac{x^{17}}{17}+\frac{x^{19}}{19}$$

Question 3

```
Clear["Global`*"]
```

Part a :

```
integral := Integrate[x^4 Exp[x] / (Exp[x] - 1) ^2, {x, 0, Infinity}]
```

```
9 (T / Subscript[Θ, D]) ^3 integral
```

$$\frac{12\pi^4 T^3}{5\Theta_D^3}$$

Part b:

Comment - In the following parts, y = T / Θ_D.

```
integral1[y_] = Integrate[x^4 Exp[x] / (Exp[x] - 1) ^2, {x, 0, y}]
```

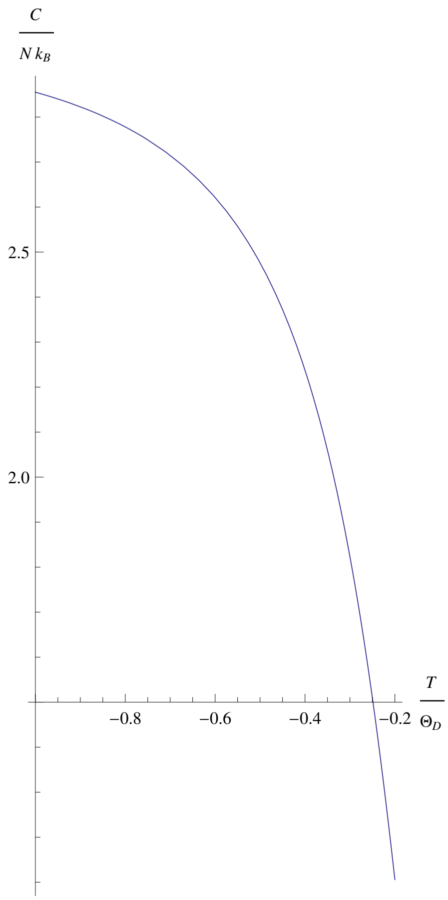
$$\text{ConditionalExpression}\Big[-\frac{4\pi^4}{15}+\frac{1}{-1+e^y}\big(-e^y y^4-4y^3\,\text{Log}[1-e^y]+4e^y y^3\,\text{Log}[1-e^y]+12(-1+e^y)y^2\,\text{PolyLog}[2,e^y]-24(-1+e^y)y\,\text{PolyLog}[3,e^y]-24\,\text{PolyLog}[4,e^y]+24e^y\,\text{PolyLog}[4,e^y]\big),\ e^y\le 1\Big]$$

```
Limit[9 / y^3 integral1[y], y → 0, Direction → 1]
```

3

Part c :

```
ParametricPlot[{1 / y, 9 / y^3 integral1[y]}, {y, -5, -1},
 AxesLabel → {T / Subscript[Θ, D], C / (N Subscript[k, B])}]
```

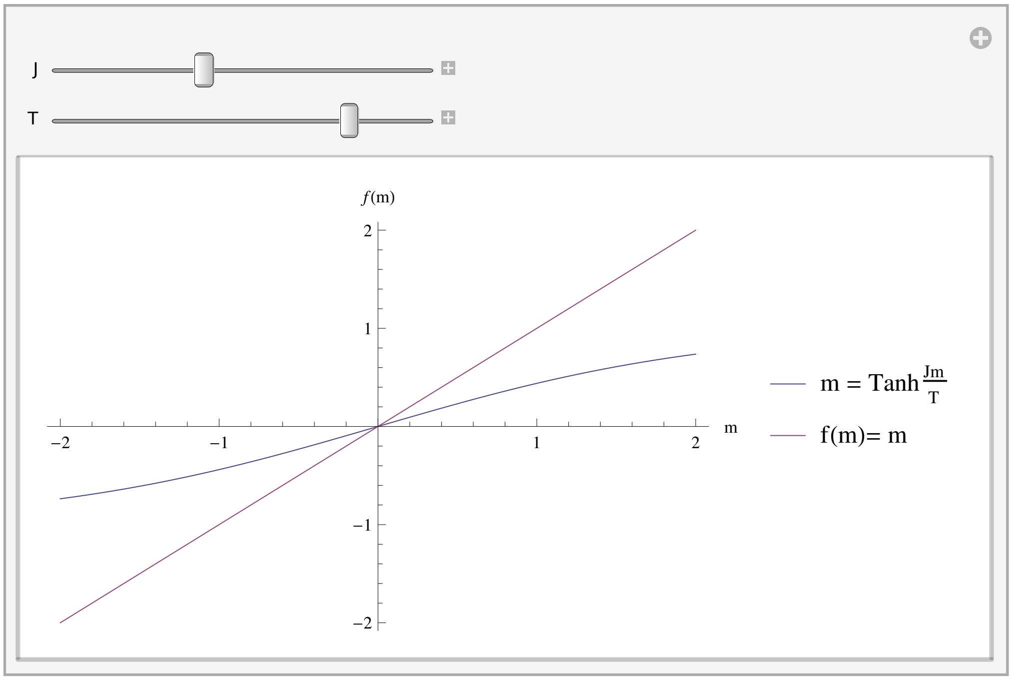


Question 4

Part a:

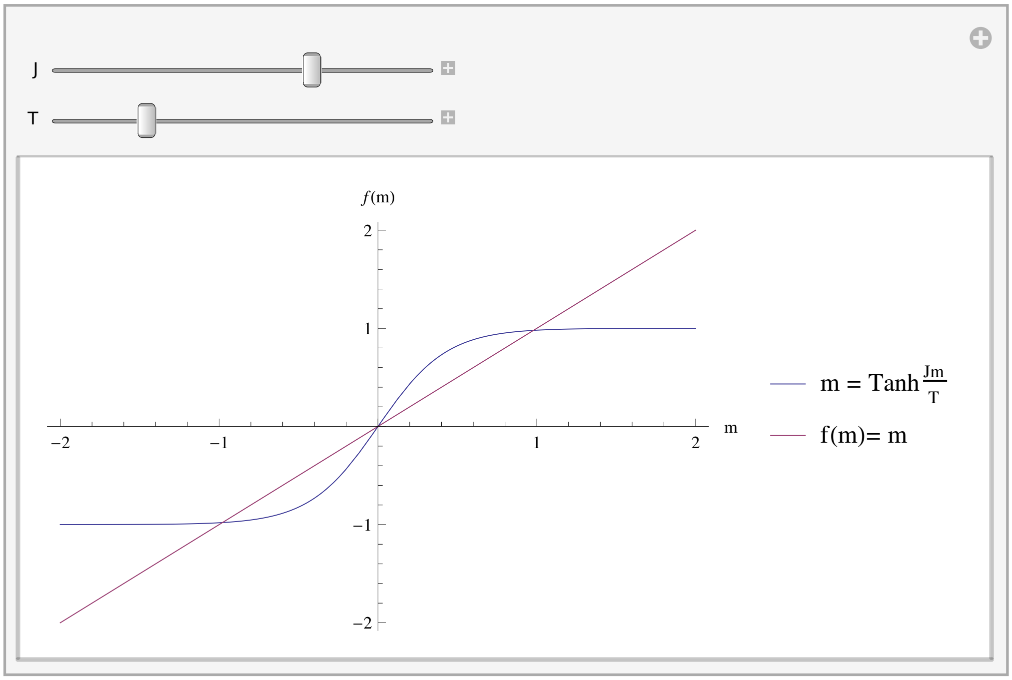
```
Clear["Global`*"]
```

```
Manipulate[Plot[{m = Tanh[J m / T], m}, {m, -2, 2}, AxesLabel → {"m", f["m"]},
  PlotLegends → {"m = Tanh Jm/T", "f(m)= m"}], {J, 0, 1}, {T, .1, 1}]
```



Part b:

```
Manipulate[Plot[{m = Tanh[J m / T], m}, {m, -2, 2}, AxesLabel → {"m", f["m"]},
  PlotLegends → {"m = Tanh Jm/T", "f(m)= m"}], {J, 0, 1}, {T, .1, 1}]
```



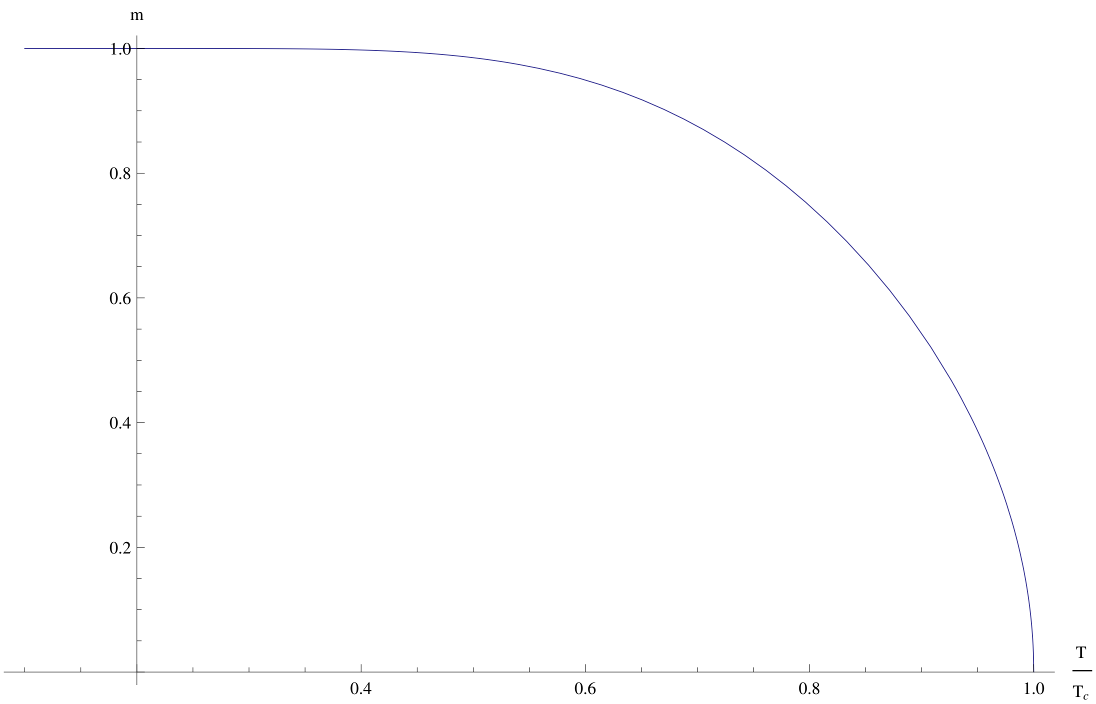
Comment: As shown above, the only solution is m=0 for T > J, while two extra symmetric solutions begin to appear when J > T.

Part c:

See back handwritten page

Part d:

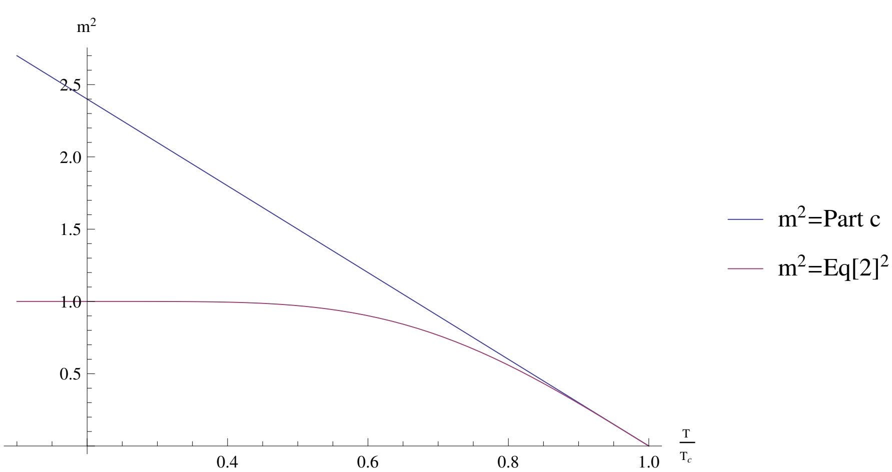
```
Plot[{Tanh[1 / t Sqrt[3 (1 - t)]]},
 {t, 0.1, 1}, AxesLabel → {"T" / Subscript["T", c], "m"}]
```



$\frac{T}{T_c}$ cannot go all the way to zero or else the magnetization will go to infinity.

Part e:

```
Plot[{3 (1 - t), Tanh[1 / t Sqrt[3 (1 - t)]]^2},
 {t, 0.1, 1}, AxesLabel → {"T" / Subscript["T", c], "m"^2},
 PlotLegends → {"m²=Part c", "m²=Eq[2]²"}]
```



As shown above, the equation for magnetization agrees with the approximation as $\frac{T}{T_c}$ approaches one.

Question 5

Part a :

```
Clear["Global`*"]

k = 0.004;
g = 9.81;
sol := NDSolve[{x''[t] == -k x'[t] Sqrt[y'[t]^2 + x'[t]^2],
   y''[t] == -k y'[t] Sqrt[y'[t]^2 + x'[t]^2] - g, x[0] == 0, y[0] == 0,
   x'[0] == v Cos[theta], y'[0] == v Sin[theta]}, {x, y}, {t, 0, 1000}]
yy[t_] := y[t] /. sol[[1]];
xx[t_] := x[t] /. sol[[1]];
tfinal[theta_] :=
 (sol; yy[t_]; xx[t_]; t /. FindRoot[yy[t], {t, 1, 1000}, MaxIterations → 50])
xfinal[theta_?NumericQ] := xx[tfinal[theta]]
xmax[vv_] := (v = vv; FindMaximum[xfinal[theta], {theta, 0, 2}][[1]])
```

The above block of code numerically solves the differential equations for a projectile and extracts the answers. It then calculates the value of theta for which the projectile travels the longest and passes this calculation along xx[t_], which in turn calculates the distance covered in that amount of time. I then created a function in which the initial velocity is variable for easy calculations as well as generality.

```
xmax[50]

150.21

xmaxnf[50]

254.842
```

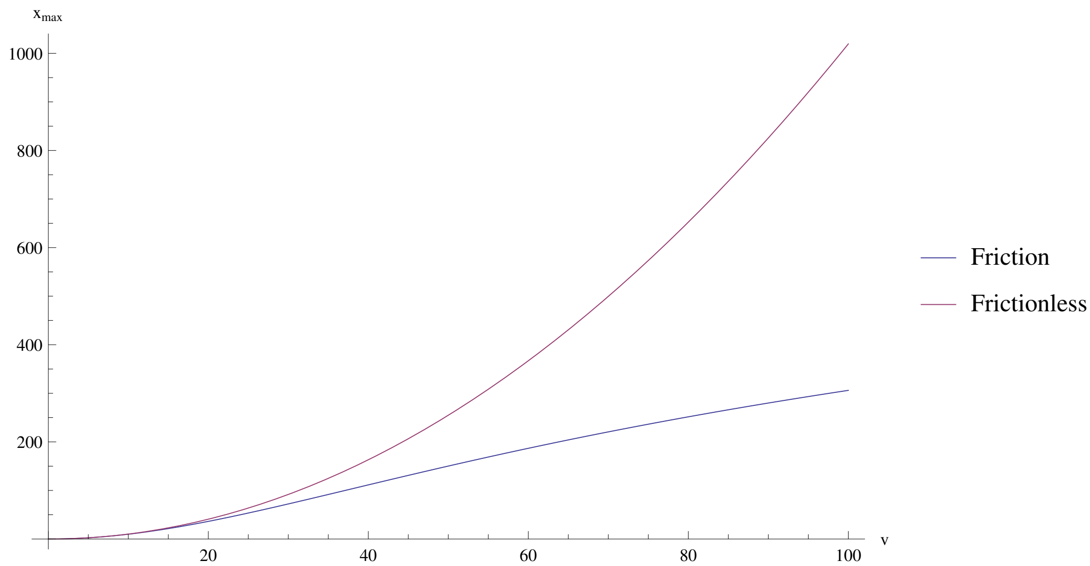
As seen above, the maximum distance the projectile travels with the absence of friction is about 100m greater than if it were to be traveling with friction present.

Part b :

```
nofriction[theta_] := v^2 Sin[2 theta] / g
xmaxnf[vv_] := (v = vv; FindMaximum[nofriction[theta], {theta, 0, 2}][[1]])
```

The above function is the distance the projectile travels in the x direction without friction for any initial velociy.

```
Plot[{xmax[v], xmaxnf[v]}, {v, 0, 100}, AxesLabel → {"v", Subscript["x", "max"]},
 PlotLegends → {"Friction", "Frictionless"}]
```



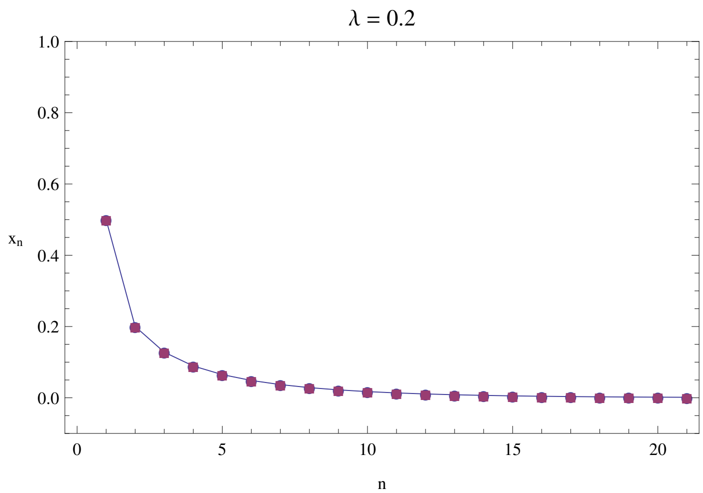
The above graph shows that the friction coefficient steadily begins to have a larger effect on the projectile after about 20m/s.

Question 6

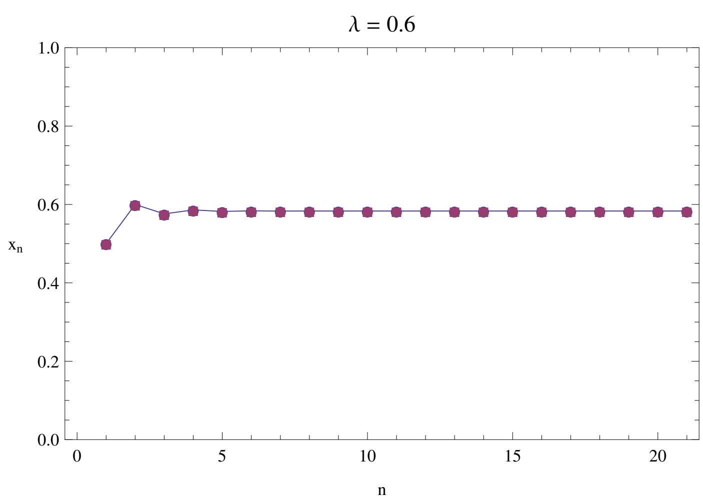
Part a :

```
f[x_] := 4 λ x (1 - x);
```

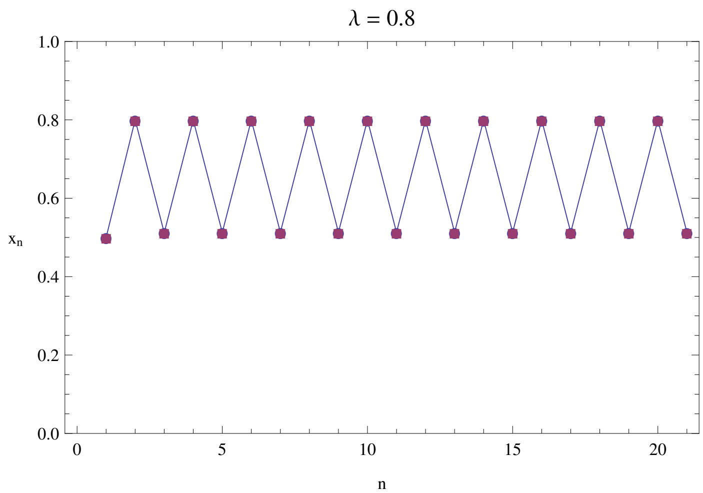
```
λ := 0.2
ListPlot[{NestList[f, 0.5, 20], NestList[f, 0.5, 20]},
 Joined → {True, False}, Axes → False, Frame → True,
 FrameLabel → {"n", Subscript["x", "n"]}, PlotRange → {-0.1, 1},
 PlotMarkers → Automatic, RotateLabel → False, PlotLabel → "λ = 0.2"]
```



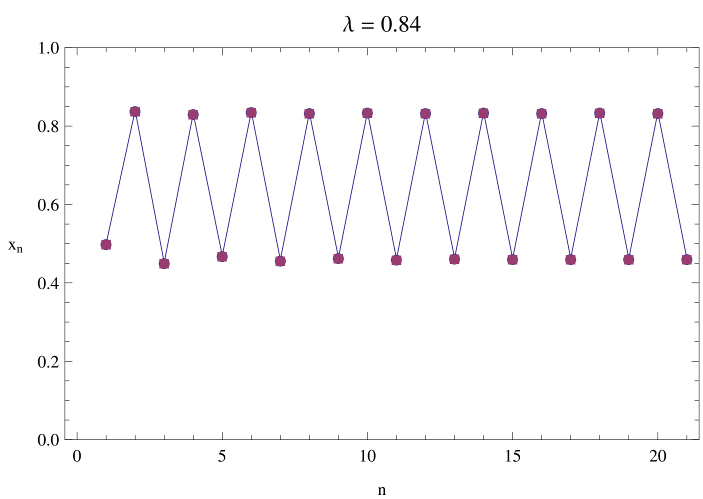

```
λ := 0.6
ListPlot[{NestList[f, 0.5, 20], NestList[f, 0.5, 20]},
 Joined → {True, False}, Axes → False, Frame → True,
 FrameLabel → {"n", Subscript["x", "n"]}, PlotRange → {0, 1},
 PlotMarkers → Automatic, RotateLabel → False, PlotLabel → "λ = 0.6"]
```



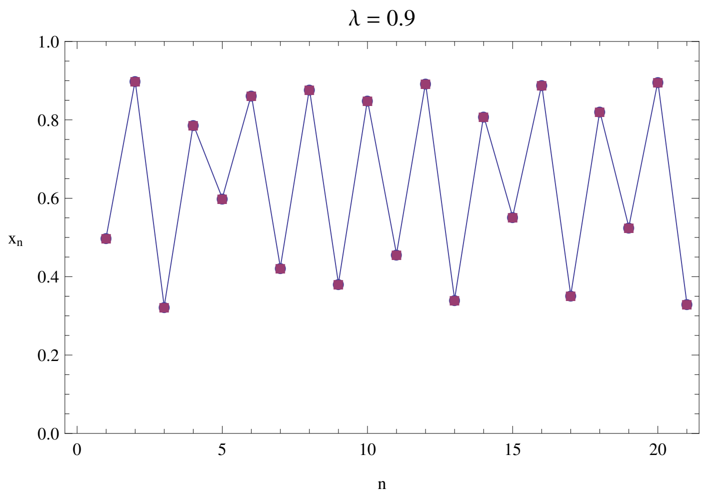
```
λ := 0.8
ListPlot[{NestList[f, 0.5, 20], NestList[f, 0.5, 20]},
 Joined → {True, False}, Axes → False, Frame → True,
 FrameLabel → {"n", Subscript["x", "n"]}, PlotRange → {0, 1},
 PlotMarkers → Automatic, RotateLabel → False, PlotLabel → "λ = 0.8"]
```



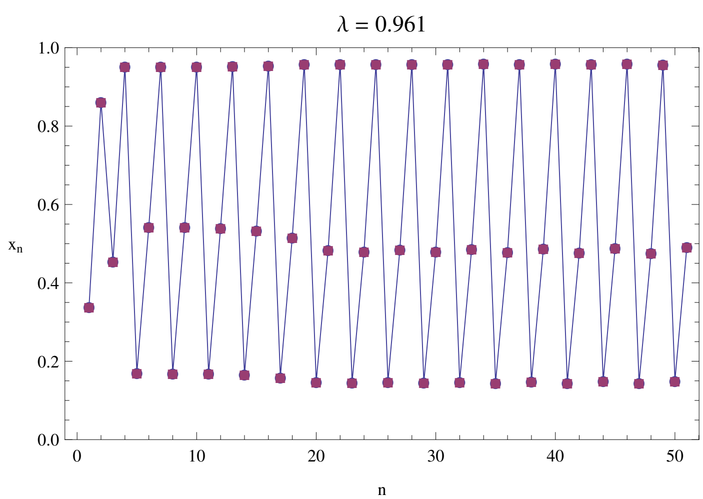

```
λ := 0.84
ListPlot[{NestList[f, 0.5, 20], NestList[f, 0.5, 20]},
 Joined → {True, False}, Axes → False, Frame → True,
 FrameLabel → {"n", Subscript["x", "n"]}, PlotRange → {0, 1},
 PlotMarkers → Automatic, RotateLabel → False, PlotLabel → "λ = 0.84"]
```



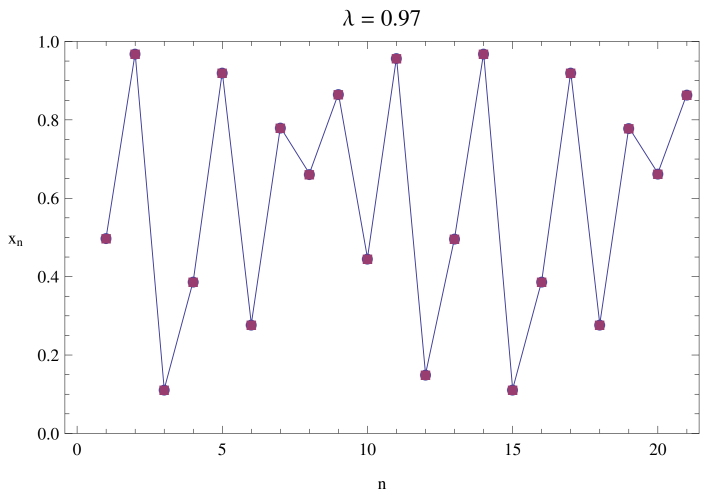
```
λ := 0.9
ListPlot[{NestList[f, 0.5, 20], NestList[f, 0.5, 20]},
 Joined → {True, False}, Axes → False, Frame → True,
 FrameLabel → {"n", Subscript["x", "n"]}, PlotRange → {0, 1},
 PlotMarkers → Automatic, RotateLabel → False, PlotLabel → "λ = 0.9"]
```



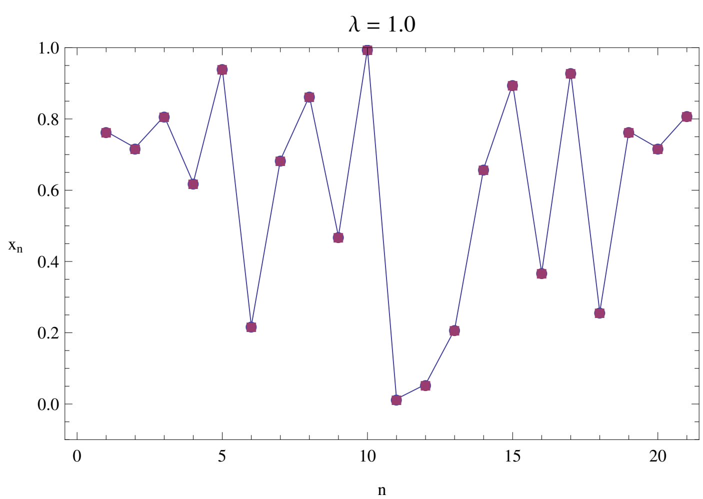

```
λ := 0.961
graph6 = ListPlot[{NestList[f, 0.34, 50], NestList[f, 0.34, 50]},
  Joined → {True, False}, Axes → False, Frame → True,
  FrameLabel → {"n", Subscript["x", "n"]}, PlotRange → {0, 1},
  PlotMarkers → Automatic, RotateLabel → False, PlotLabel → "λ = 0.961"]
```



```
λ := 0.97
ListPlot[{NestList[f, 0.5, 20], NestList[f, 0.5, 20]},
 Joined → {True, False}, Axes → False, Frame → True,
 FrameLabel → {"n", Subscript["x", "n"]}, PlotRange → {0, 1},
 PlotMarkers → Automatic, RotateLabel → False, PlotLabel → "λ = 0.97"]
```




```
λ := 1.0
ListPlot[{NestList[f, 0.765, 20], NestList[f, 0.765, 20]},
 Joined → {True, False}, Axes → False, Frame → True,
 FrameLabel → {"n", Subscript["x", "n"]}, PlotRange → {-.1, 1},
 PlotMarkers → Automatic, RotateLabel → False, PlotLabel → "λ = 1.0"]
```



λ = 0.2, 0.6: converges to fixed point
λ = 0.8, 0.84, 0.961: limit cycle
λ = 0.9, 0.97, 1.0: CHAOS!!!!!!!!!

Part b

```
lyapunov[lambda_, xinit_, n_, ninitialize_] :=
 (λ = lambda; xlist = Drop[NestList[f, xinit, n], ninitialize + 1];
  Apply[Plus, Log[Abs[f'[xlist]]]] / Length[xlist])
```

In[277]:= **λ = 0.2:**

```
lyapunov[0.2, 0.5, 50 000, 20]
```

```
-0.464708
```

λ = 0.6 :

```
lyapunov[0.6, 0.5, 50 000, 20]
```

```
-0.748128
```

λ = 0.8 :

```
lyapunov[0.8, 0.5, 50 000, 20]
```

```
-0.463043
```

λ = 0.84 :

```
lyapunov[0.84, 0.5, 50 000, 20]
```

```
-0.175468
```

λ = 0.9 :

```
lyapunov[0.9, 0.5, 50 000, 20]
```

```
lyapunov[0.9, 0.5, 50 000, 20]
```

λ = 0.961 :

```
lyapunov[0.961, 0.5, 50 000, 20]
```

```
-0.280351
```

λ = 0.97 :

```
lyapunov[0.97, 0.5, 50 000, 20]
```

```
0.462067
```

λ = 1.0 :

```
lyapunov[1.0, 0.5, 50 000, 20]
```

```
1.38629
```

If the lyapunov exponent is negative, it shows that the value of lambda iterates towards a single or multiple fixed points, whereas a positive lyapunov exponent yields represents a value of lambda which iterates towards chaos.

Part C :

```
Clear["Global`*"]
```

```
f[x_] := 4 λ x (1 - x);
f2[x_] := f[f[x]];
f4[x_] := f2[f2[x]];
f5[x_] := f[f4[x]];
```

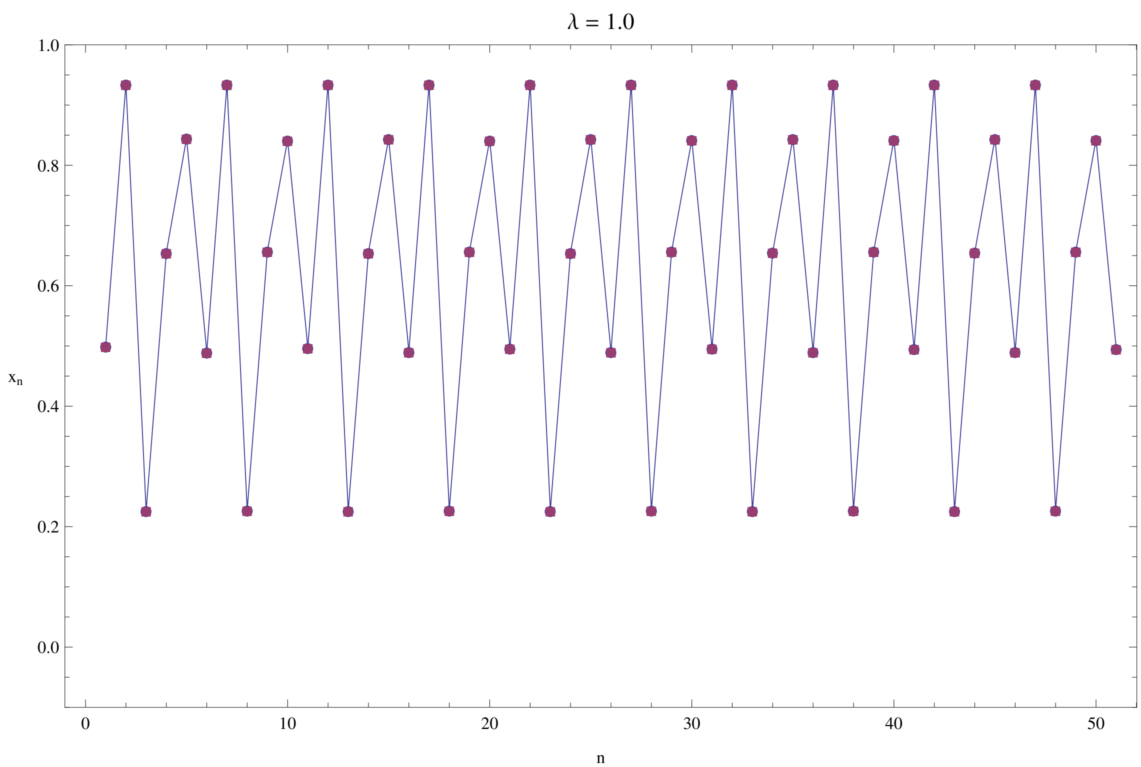
```
λ /. FindRoot[{f5[x] == x, f5'[x] == -1}, {x, 0.93}, {λ, 0.93}]
```

```
0.93528
```

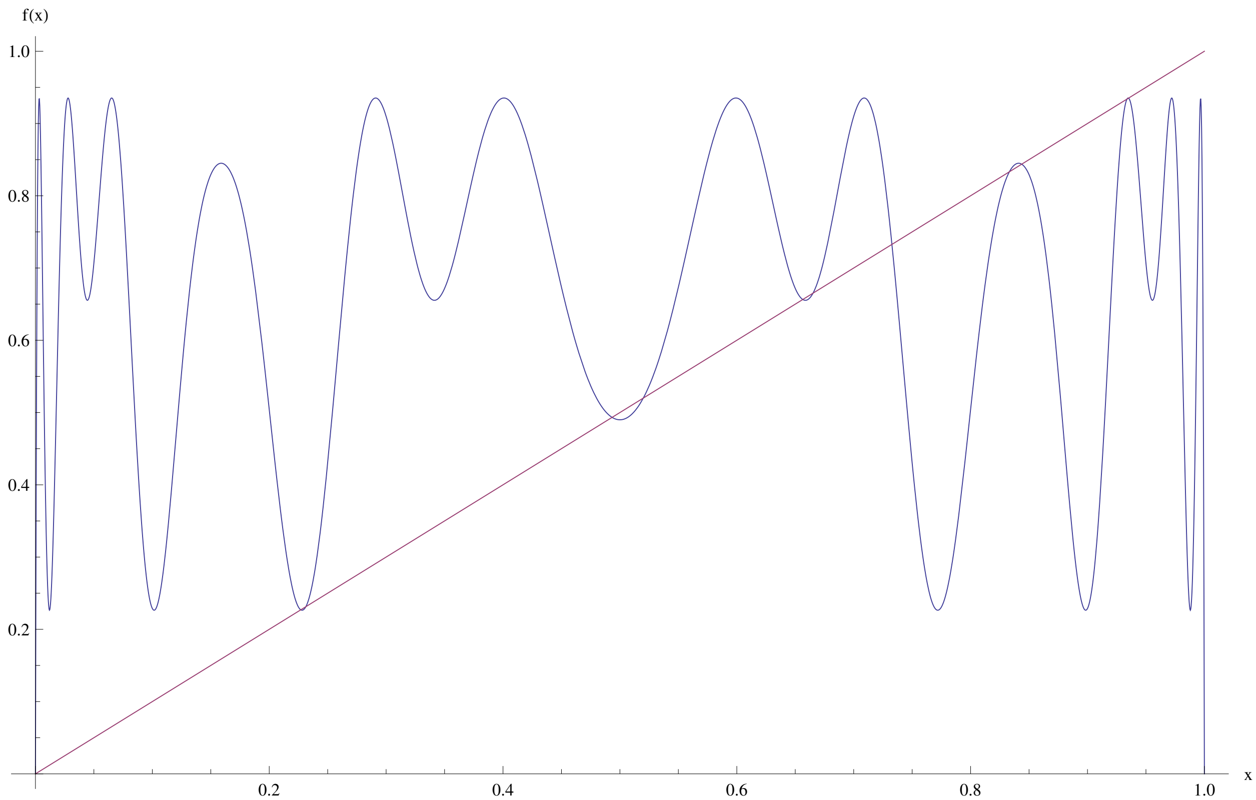
```
λ = %
```

```
0.93528
```

```
ListPlot[{NestList[f, 0.5, 50], NestList[f, 0.5, 50]},
 Joined → {True, False}, Axes → False, Frame → True,
 FrameLabel → {"n", Subscript["x", "n"]}, PlotRange → {-.1, 1},
 PlotMarkers → Automatic, RotateLabel → False, PlotLabel → "λ = 1.0"]
```



```
Plot[{f5[x], x}, {x, 0, 1}, AxesLabel → {"x", "f(x)"}, PlotLegends → {"f5(x)", "x"}]
```

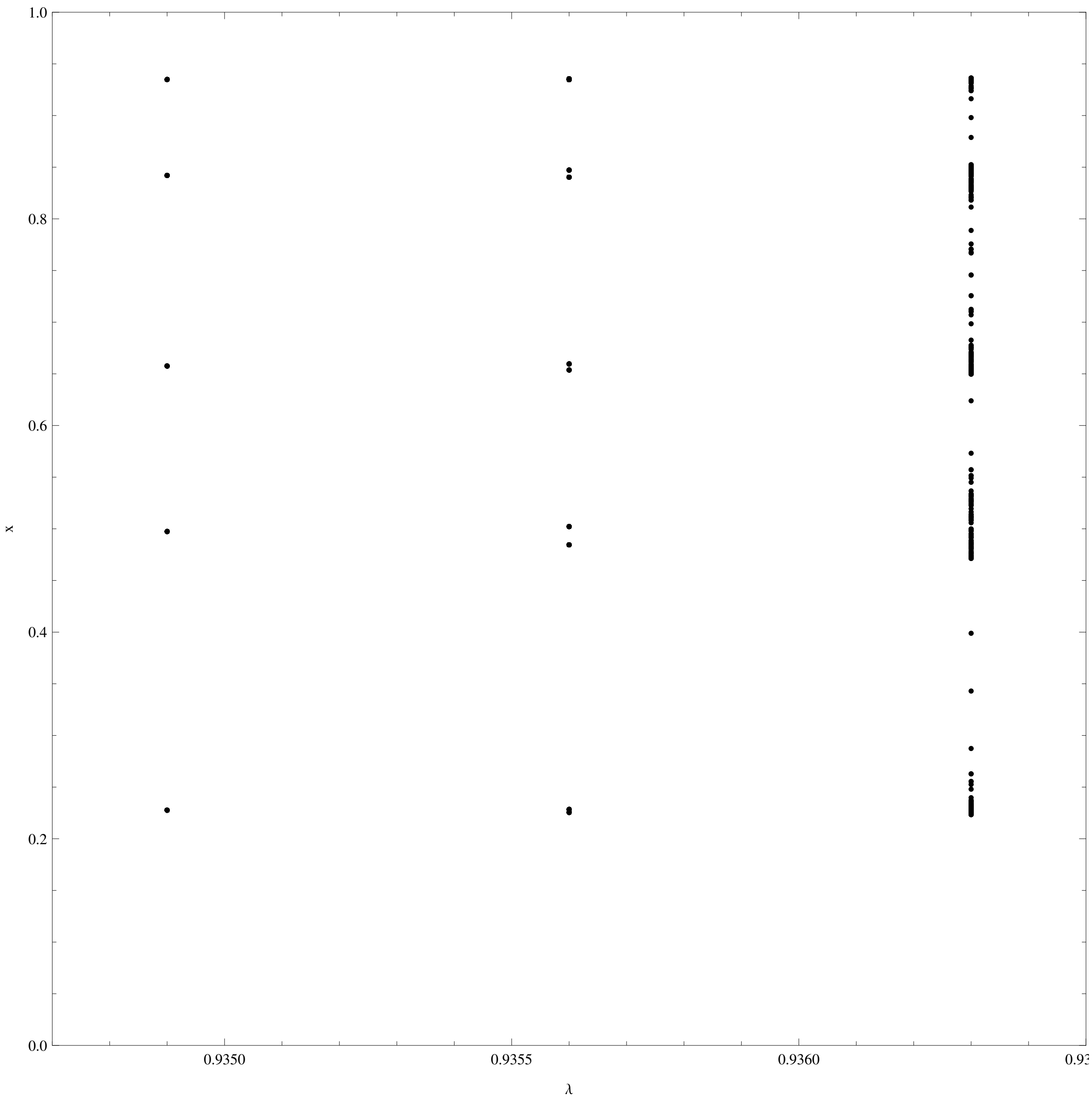


By investigating the x vs. λ plot, I was able to find a rough estimate for λ and x to use for the "FindRoot" command. By taking this value of lambda and iterating the function multiple times, I was able to visually represent the five fixed values of x to which the system oscillated. Finally, by comparing the fifth iteration of f(x) to just a straight line, I could identify the 5 stable points where the absolute value of the slope of the fifth iteration is less than one.

Part D :

```
Clear["Global`*"]

f[x_] := 4 λ x (1 - x);

iterate [ m_,  n_]  := Drop [ NestList [f, 0.5, n], m]

drawpt[y_] := Point[{λ, y}]

graph[λmin_, λmax_, nλ_, mdrop_, n_] := Graphics [ {PointSize[0.005],
   Table [ Map[drawpt, iterate [mdrop, n ] ], {λ, λmin, λmax, (λmax - λmin) / nλ} ]}]
```

```
Show[graph[0.72, 1, 400, 300, 700], Axes → False, Frame → True,
 FrameLabel → {"λ", "x"}, PlotRange → {{0.9347, 0.9365}, {0, 1}}, AspectRatio → 1]
```



When zooming in on the value of λ which corresponds to 5 fixed points, one can see that the period doubles to 10 fixed points at around λ = 0.9366. Note: The top and the bottom points are actual two distinct points which were obscured by the proximity to each other.

Part E :

```
f[x_] := 4 λ x (1 - x);
f2[x_] := f[f[x]]
f4[x_] := f2[f2[x]]
f16[x_] := f4[f4[x]]
```

λ_2 :

```
lambda[2] = λ /. FindRoot[{f2[x] == x, f2'[x] == -1}, {x, 0.5}, {λ, 0.86}]
```

```
0.862372
```

λ_3 :

```
lambda[3] = λ /. FindRoot[{f4[x] == x, f4'[x] == -1}, {x, 0.5}, {λ, 0.86}]
```

```
0.886023
```

λ_4 :

```
lambda[4] = λ /. FindRoot[{f16[x] == x, f16'[x] == -1}, {x, 0.5}, {λ, 0.86}]
```

```
0.891102
```

```
lambda[0] := 0.25
lambda[1] := 0.75
```

```
δ[k_] := (lambda[k] - lambda[k - 1]) / (lambda[k + 1] - lambda[k])
```

```
δ[1]
```

```
4.44949
```

```
δ[2]
```

```
4.75145
```

```
δ[3]
```

```
4.65625
```

Research shows that the Feigenbaum constant is about 4.669, which is the value the above deltas are oscillating towards.

Question 7

Part a :

```
ClearAll["Global`*"]
```

```
f[x_] := λ Sin[Pi x]
```

```
lyapunov[lambda_, xinit_, n_, ninitialize_] :=
 (λ = lambda; xlist = Drop[NestList[f, xinit, n], ninitialize + 1];
  Apply[Plus, Log[Abs[f'[xlist]]]] / Length[xlist])
```

```
lyapunov[0.9, 0.5, 50 000, 20]
```

```
0.349115
```

The lyapunov exponent is positive, therefore the system is chaotic.

Part b :

```
f[x_] := λ Sin[Pi x]

λ := N[9 / 10, 1000]

x0 := N[4 / 10, 1000]

N[NestList[f, x0, 5000], 1000][[5001]]

0.795585137111927326320216242974183122336542276992138037688867644646375419430395З\
 5575642364144089504099160241600115882983235818002326063902599764825338088700377З\
 95233854794860413067854011177124568816080031662867937550931275772322066102735684\
 934408484261

Precision[%]

250.985
```