Analysis of Errors in Finite Difference Methods for Pricing European Options

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1 Introduction

This analysis focuses on three FDM schemes: Explicit, Implicit, and Crank-Nicolson, and examines the errors between these methods and the exact Black-Scholes price under various conditions.

The analysis investigates the sensitivity of these methods to changes in time steps (M), space steps (N), and volatility (σ). By systematically varying these parameters, we aim to understand which method offers the best trade-off between accuracy and computational efficiency.

2 Error Analysis with Different Time Steps (M)

To evaluate the effect of time discretization on the accuracy of each method, we compare the errors between the Black-Scholes price and the FDM prices using $N = 100$ space steps while varying the number of time steps between $M = 30$ and $M = 500$. Figure 1 shows the results.

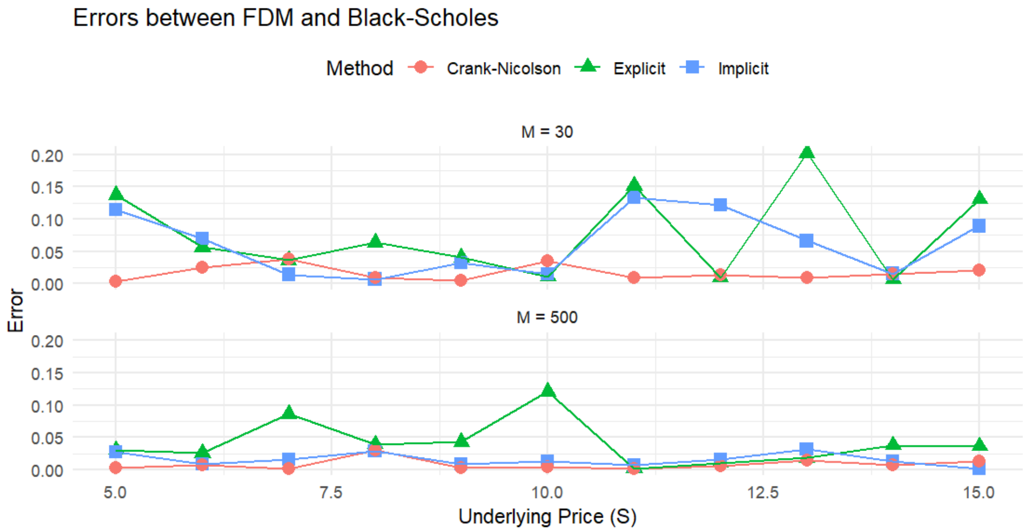

Figure 1: Errors between FDM and Black-Scholes prices with varying time steps (M). The plot shows the error for Explicit (green triangles), Implicit (blue squares), and Crank-Nicolson (red circles) methods.

2.1 Observations

- **For $M = 30$ time steps:** The Explicit method (green triangles) shows significant errors, particularly at lower underlying prices where the errors reach a maximum of approximately 0.15. The Implicit method (blue squares) displays moderate errors, with the highest deviations occurring near S = 5 and S = 12. The Crank-Nicolson method (red circles) demonstrates minimal errors across all underlying prices, suggesting superior accuracy and stability.

- **For $M = 500$ time steps:** As the number of time steps increases, all methods show reduced errors, indicating better accuracy with finer time discretization. The Explicit method still shows relatively higher errors compared to the other two methods, but these errors are significantly reduced compared to the $M = 30$ case. Both the Implicit and Crank-Nicolson methods show minimal errors, with Crank-Nicolson remaining the most stable.

The Crank-Nicolson method provides the most stable and accurate results with minimal errors, irrespective of the number of time steps. The Explicit method is highly sensitive to the number of time steps, showing significant errors at lower values of M. Increasing the number of time steps enhances the accuracy of all methods, but the Crank-Nicolson method remains the most reliable choice for varying time step sizes.

3 Error Analysis with Different Volatility Levels

The impact of volatility on the errors of the FDM methods. The errors are calculated for volatilities of $\sigma = 0.2$ and $\sigma = 0.4$ using $N = 200$ space steps and $M = 30$ time steps. Figure 2 presents the errors for each method under different volatility levels.

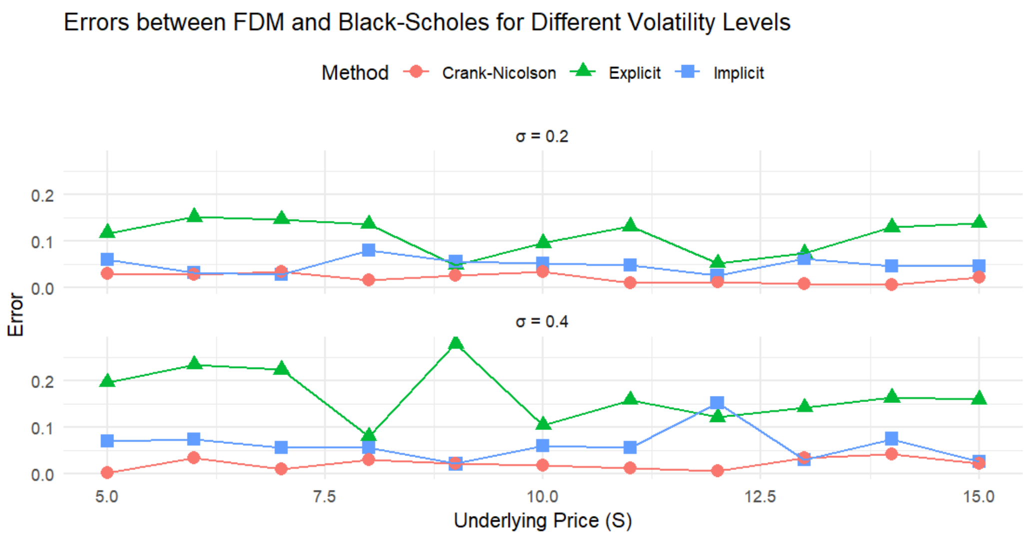

Figure 2: Errors between FDM and Black-Scholes prices for different volatility levels ($\sigma = 0.2$ and $\sigma = 0.4$).

3.1 Observations

- **At $\sigma = 0.2$:** The Explicit method shows higher errors across all underlying prices compared to the Implicit and Crank-Nicolson methods. The Crank-Nicolson method again shows the least error, suggesting robustness to changes in volatility.
- **At $\sigma = 0.4$:** The increase in volatility leads to higher errors for all methods. The Explicit method exhibits the most significant increase in error, with errors peaking at around S = 10. The Implicit method

shows a moderate increase in error, while the Crank-Nicolson method maintains relatively low errors, indicating its superior stability.

The Explicit method is highly sensitive to volatility changes, showing the largest errors at higher volatility levels. The Crank-Nicolson method remains the most stable across different volatilities, making it a preferable choice in scenarios with high market volatility.

4 Effect of Varying Space Steps (N) on Errors

Figure 3 illustrates the effect of varying the number of space steps (N) from 300 to 700 on the errors for each method, with a fixed number of time steps ($M = 30$).

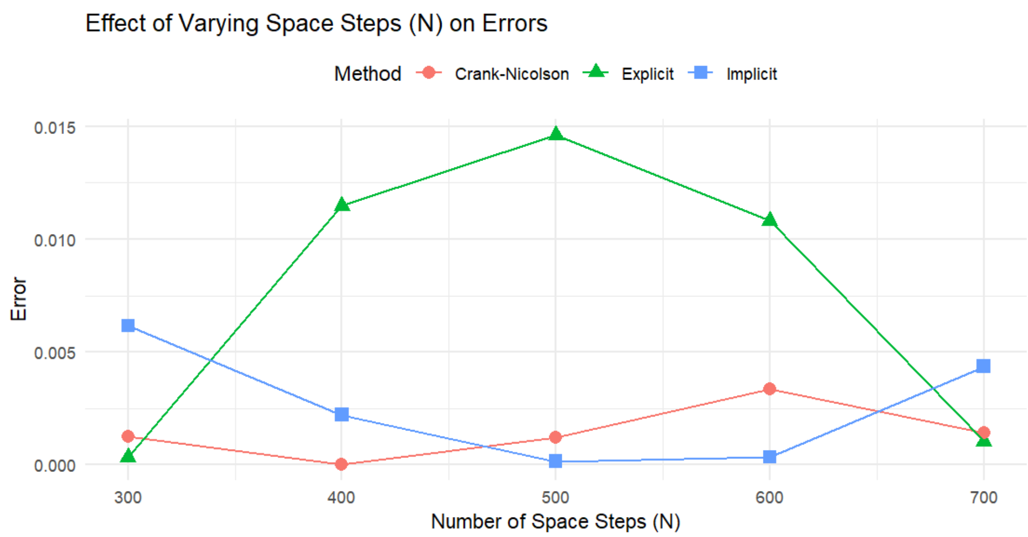

Figure 3: Effect of varying space steps (N) on errors for different methods.

4.1 Observations

- The Crank-Nicolson method maintains the lowest error across all space steps, indicating a high level of numerical stability.
- The Explicit method shows a peak error at $N = 500$, suggesting that this specific discretization level causes numerical instability. As the number of space steps increases beyond 500, the errors decrease, indicating a recovery in stability.

- The Implicit method displays a steady increase in error as the number of space steps increases, but the errors remain lower than those of the Explicit method at most points.

5 Effect of Increasing volatility

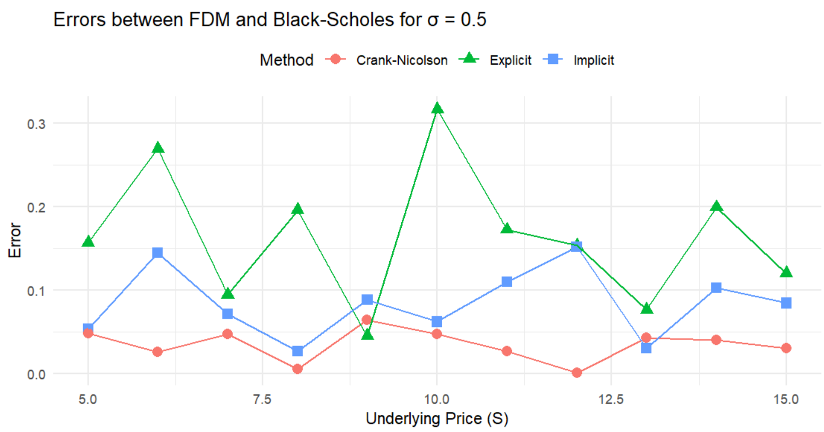

Figure 4: Errors between FDM and Black-Scholes for $\sigma = 0.5$

Figure 4

1. **Explicit method (green triangles):**
    - Exhibits the highest overall errors
    - Shows significant fluctuations
    - Peaks at approximately 0.32 when S = 10.0
2. **Implicit method (blue squares):**
    - Generally demonstrates lower errors than the Explicit method
    - Higher errors compared to the Crank-Nicolson method
    - Shows less volatility than the Explicit method
3. **Crank-Nicolson method (red circles):**
    - Consistently shows the lowest errors across all underlying prices
    - Errors remain below 0.1 throughout the price range
    - Appears to be the most stable and accurate method

5.1 Conclusion

The Crank-Nicolson method demonstrates consistent performance with minimal errors across varying space steps, confirming its stability. The Explicit method's errors show instability at certain discretization levels, suggesting that careful selection of space steps is crucial for maintaining accuracy. The Implicit method shows a more gradual error trend but remains less stable than Crank-Nicolson.