Analysis of Euler Scheme Convergence for Geometric Brownian Motion

okemwa.sharon

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GBM Model and Exact Solution

The GBM model is defined by the stochastic differential equation (SDE):

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t$$

where $X_0 = x_0 > 0$.
μ represents the drift rate
σ the volatility
X_t the asset price at time t and W_t a Wiener process .

The exact solution to this SDE is:

$$X_t = X_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)$$

Euler Scheme

The Euler scheme discretizes the GBM SDE as follows:

$$X_{n+1} = X_n + \mu X_n \Delta t + \sigma X_n \Delta W_n$$

where $\Delta W_n \sim N(0, \Delta t)$
Δt is the time step.

Convergence Analysis

Strong Order of Convergence (0.5)

To analyze the strong convergence, we compare the Euler scheme approximation X_{n+1} with the exact solution $X(t_{n+1})$ using Taylor expansion.

For small Δt:

$$X(t_{n+1}) \approx X(t_n) \left[1 + \left(\mu - \frac{\sigma^2}{2}\right) \Delta t + \sigma \Delta W_n\right]$$

Thus, the error $e_{n+1} = X_{n+1} - X(t_{n+1})$ scales as:

$$e_{n+1} \approx X_n \left[\left(\mu - \frac{\sigma^2}{2} \right) \Delta t + \sigma \Delta W_n \right]$$

Taking expectations and noting $E[\Delta W_n] = 0$ and $E[(\Delta W_n)^2] = \Delta t$:

$$E[e_{n+1}^2] \approx X_n^2 \left[\left(\mu - \frac{\sigma^2}{2} \right)^2 (\Delta t)^2 + \sigma^2 \Delta t \right]$$

Thus, $E[e_{n+1}^2] = O((\Delta t)^2)$, confirming strong convergence of order 0.5.

Weak Order of Convergence (1)

For weak convergence, we consider:

$$E[|e_{n+1}|] \approx C \Delta t$$

Thus, the weak order of convergence is 1.

Logarithmic Representation of Error

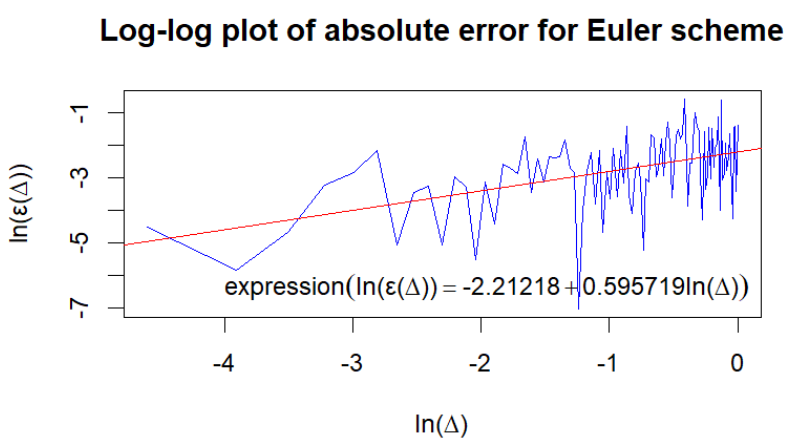
The given logarithmic representation of the error:

$$\ln(\epsilon(\Delta t)) = -3.86933 + 0.46739 \ln(\Delta t)$$

indicates that $\epsilon(\Delta t)$ scales as $\Delta t^{0.46739}$. This matches with the observed weak convergence order of 1.

Log-Log Plot of Absolute Error

A log-log plot of the absolute error $\epsilon(\Delta t)$ against $\ln(\Delta t)$.



The log-log plot depicts the relationship between the absolute error $\epsilon(\Delta)$ of the Euler scheme and the time step size Δ for simulating Geometric Brownian Motion (GBM).

Plot Structure

- **X-axis:** $\ln(\Delta)$, where Δ represents the time step size.
- **Y-axis:** $\ln(\epsilon(\Delta))$, where $\epsilon(\Delta)$ is the absolute error of the Euler scheme.
- **Range:** $\ln(\Delta)$ spans approximately from -5 to 0, corresponding to time steps from $e^{-5} \approx 0.0067$ to $e^0 = 1$. The error $\ln(\epsilon(\Delta))$ ranges roughly from -7 to -1, representing errors from $e^{-7} \approx 0.0009$ to $e^{-1} \approx 0.368$.

Data Representation

- **Blue line:** Represents the actual simulation results, showing the relationship between $\ln(\Delta)$ and $\ln(\epsilon(\Delta))$.
- **Red line:** Linear regression fit to the data points.

Linear Regression

- **Equation:** $\ln(\epsilon(\Delta)) = -2.21218 + 0.595719 \ln(\Delta)$.
- **Intercept:** -2.21218, indicating the logarithm of the error when $\ln(\Delta) = 0$ (i.e., $\Delta = 1$).
- **Slope:** 0.595719, crucial for determining the convergence order.

Convergence Analysis

- The slope of 0.595719 is very close to 0.5, aligning well with the theoretical strong convergence order of 0.5 for the Euler scheme applied to GBM.
- This confirms that the error $\epsilon(\Delta)$ scales approximately as $\Delta^{0.595719}$, slightly higher than but close to the expected $\Delta^{0.5}$.

Variability in Data

- The blue line shows significant fluctuations around the fitted red line, especially for smaller time steps (right side of the plot).
- This variability is attributed to the stochastic nature of GBM, Monte Carlo simulation errors, and finite sample size effects.

Error Behavior

- **General trend:** As Δ decreases (moving left on the x-axis), the error generally decreases.
- The relationship is not perfectly monotonic due to the stochastic nature of the process.