Revision Questions

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QUESTION 1

Derivation of Euler and Milstein Schemes for SDEs

(a) SDE:

$$dX_t = (\mu X_t + \eta)\,dt + \gamma X_t\,dW_t \quad \text{with} \quad X_0 = 1$$

Euler Scheme

Euler scheme for the SDE

$$dX_t = a(X_t)\,dt + b(X_t)\,dW_t$$

is given by:

$$Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n$$

For the given SDE:

$$a(X_t) = \mu X_t + \eta$$
$$b(X_t) = \gamma X_t$$

Euler scheme :

$$Y_{n+1} = Y_n + (\mu Y_n + \eta)\Delta t + \gamma Y_n \Delta W_n$$

Milstein Scheme

$$dX_t = a(X_t)\,dt + b(X_t)\,dW_t$$

is given by:

$$Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n + \frac{1}{2}b(Y_n)b'(Y_n)(\Delta W_n^2 - \Delta t)$$

For the given SDE:

$$a(X_t) = \mu X_t + \eta$$
$$b(X_t) = \gamma X_t$$
$$b'(X_t) = \gamma$$

Milstein scheme becomes:

$$Y_{n+1} = Y_n + (\mu Y_n + \eta)\Delta t + \gamma Y_n \Delta W_n + \frac{1}{2}\gamma^2 Y_n(\Delta W_n^2 - \Delta t)$$

(b) SDE:

$$dr_t = \gamma(\bar{r} - r_t)\,dt + \beta\,dW_t$$

Euler Scheme

For the SDE

$$dr_t = a(r_t)\,dt + b(r_t)\,dW_t$$

is given by:

$$Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n$$

For the given SDE:

$$a(r_t) = \gamma(\bar{r} - r_t)$$

$$b(r_t) = \beta$$

The Euler scheme:

$$Y_{n+1} = Y_n + \gamma(\bar{r} - Y_n)\Delta t + \beta\Delta W_n$$

Milstein Scheme

The Milstein scheme for the SDE

$$dr_t = a(r_t)\,dt + b(r_t)\,dW_t$$

is given by:

$$Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n + \frac{1}{2}b(Y_n)b'(Y_n)(\Delta W_n^2 - \Delta t)$$

For the given SDE:

$$a(r_t) = \gamma(\bar{r} - r_t)$$

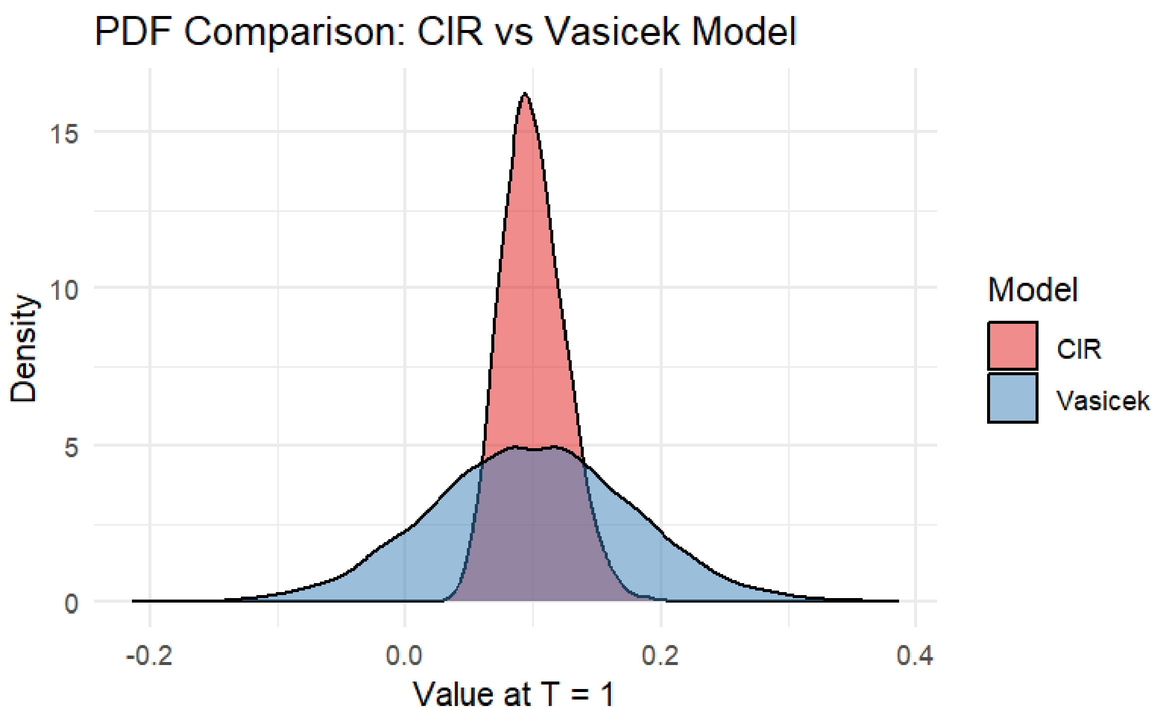
$$b(r_t) = \beta$$

$$b'(r_t) = 0$$

The Milstein scheme simplifies to the Euler scheme because $b'(r_t) = 0$:

$$Y_{n+1} = Y_n + \gamma(\bar{r} - Y_n)\Delta t + \beta\Delta W_n$$

QUESTION 2



Comparison of CIR and Vasicek Model PDFs

The plot compares the probability density functions (PDFs) of the final values of the Cox-Ingersoll-Ross (CIR) model and the Vasicek model.

Density and Value Axes

- The x-axis represents the final values of the simulated paths for both models.
- The y-axis represents the density, indicating how frequently different final values occur.

CIR Model (Blue) vs. Vasicek Model (Green)

- **CIR Model (Orange):** The orange histogram shows the distribution of final values for the CIR model. This distribution is more sharply peaked around the mean value. The CIR model constrains the values to be non-negative, leading to a higher density around the mean.

- **Vasicek Model (blue):** The blue histogram shows the distribution of final values for the Vasicek model. This distribution is more spread out and symmetric around the mean. The Vasicek model allows for a wider range of values and can even go negative, which is evident from the tails of the blue histogram.

Comparison

- **Peak Density:** The CIR model has a higher peak density, indicating that the values are more concentrated around the mean. This is due to the mean-reverting property of the CIR model and its non-negativity constraint.
- **Spread:** The Vasicek model has a broader spread, indicating that the final values are more dispersed. This model is less constrained and can generate values over a wider range, including negative values.
- **Tails:** The tails of the Vasicek model extend further into negative and positive territories compared to the CIR model. This reflects the different nature of these models in handling the variability of the underlying process.

QUESTION 3

The Heston model describes the evolution of a stock price S_t and its variance v_t through the following system of stochastic differential equations:

$$\begin{aligned} dS_t &= rS_t\,dt + \sqrt{v_t}S_t\,dW_t^1, \\ dv_t &= \kappa(\bar{v} - v_t)\,dt + \sigma\sqrt{v_t}\,dW_t^2, \end{aligned}$$

where $E[dW_t^1\,dW_t^2] = \rho\,dt$.

Decoupling the System of SDEs

To decouple the system, we express dW_t^2 in terms of dW_t^1 and an independent Brownian motion dW_t^0:

$$dW_t^2 = \rho\,dW_t^1 + \sqrt{1-\rho^2}\,dW_t^0,$$

where dW_t^0 is a standard Brownian motion independent of dW_t^1.

Rewriting the system with this transformation, we have:

$$\begin{aligned} dS_t &= rS_t\,dt + \sqrt{v_t}S_t\,dW_t^1, \\ dv_t &= \kappa(\bar{v} - v_t)\,dt + \sigma\sqrt{v_t}\left(\rho\,dW_t^1 + \sqrt{1-\rho^2}\,dW_t^0\right). \end{aligned}$$

Euler Scheme

For the Euler scheme, we approximate the SDEs using finite differences. The Euler scheme for S_t and v_t is given by:

$$
\begin{aligned}
S_{t+\Delta t} &\approx S_t + rS_t\Delta t + \sqrt{v_t}S_t\Delta W_t^1,\\
v_{t+\Delta t} &\approx v_t + \kappa(\bar{v} - v_t)\Delta t + \sigma\sqrt{v_t}\left(\rho\Delta W_t^1 + \sqrt{1-\rho^2}\Delta W_t^0\right),
\end{aligned}
$$

where ΔW_t^1 and ΔW_t^0 are independent normal random variables with mean 0 and variance Δt.

Milstein Scheme

For the Milstein scheme, include additional terms from the Itô-Taylor expansion.

Milstein Scheme for S_t

The diffusion coefficient for S_t is $g(S_t, v_t) = \sqrt{v_t}S_t$. We compute the derivatives:

$$\frac{\partial g}{\partial S} = \sqrt{v_t}, \quad \frac{\partial g}{\partial v} = \frac{S_t}{2\sqrt{v_t}}.$$

The Milstein scheme for S_t is:

$$S_{t+\Delta t} \approx S_t + rS_t\Delta t + \sqrt{v_t}S_t\Delta W_t^1 + \frac{1}{2}v_tS_t\left((\Delta W_t^1)^2 - \Delta t\right).$$

Milstein Scheme for v_t

The diffusion coefficient for v_t is $h(v_t) = \sigma\sqrt{v_t}$. We compute the derivative:

$$\frac{\partial h}{\partial v} = \frac{\sigma}{2\sqrt{v_t}}.$$

The Milstein scheme for v_t is:

$$v_{t+\Delta t} \approx v_t + \kappa(\bar{v} - v_t)\Delta t + \sigma\sqrt{v_t}\left(\rho\Delta W_t^1 + \sqrt{1-\rho^2}\Delta W_t^0\right) + \frac{1}{4}\sigma^2\left((\rho\Delta W_t^1 + \sqrt{1-\rho^2}\Delta W_t^0)^2 - \Delta t\right).$$