Lecture 10: Course Review

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May 23, 2023

Final Exam

- Nine problems
- Closed book without cheating sheets
- 9:00am-12:00pm in the morning of June 01

Course Reward

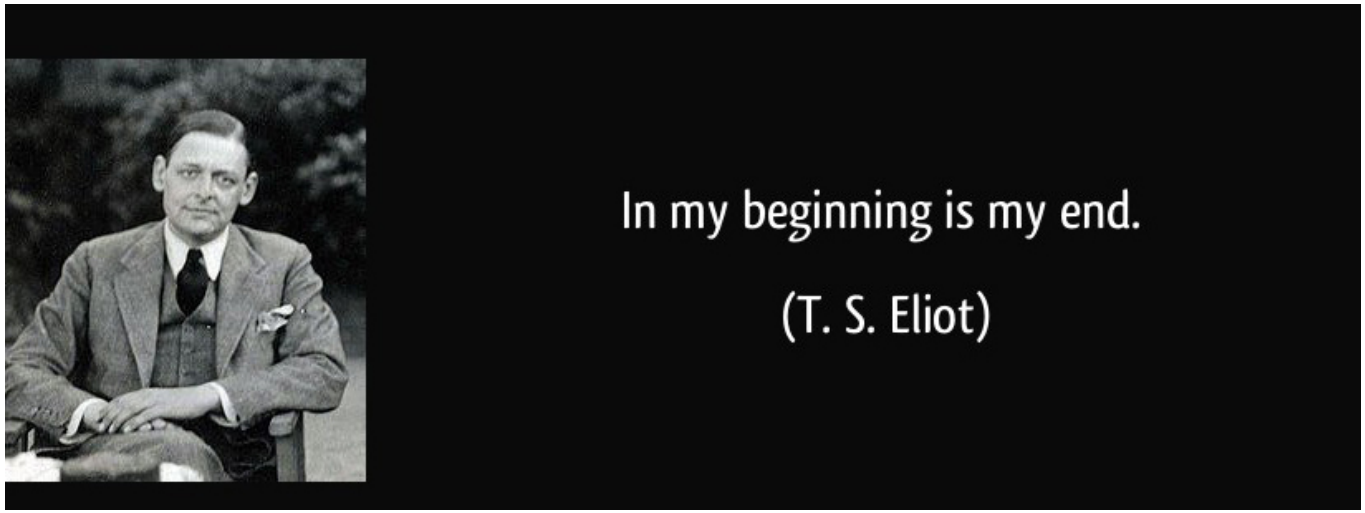
- Gauss Award
- Bernoulli Award
- Laplace Award
- Poisson Award
- Markov Award

Final Exam

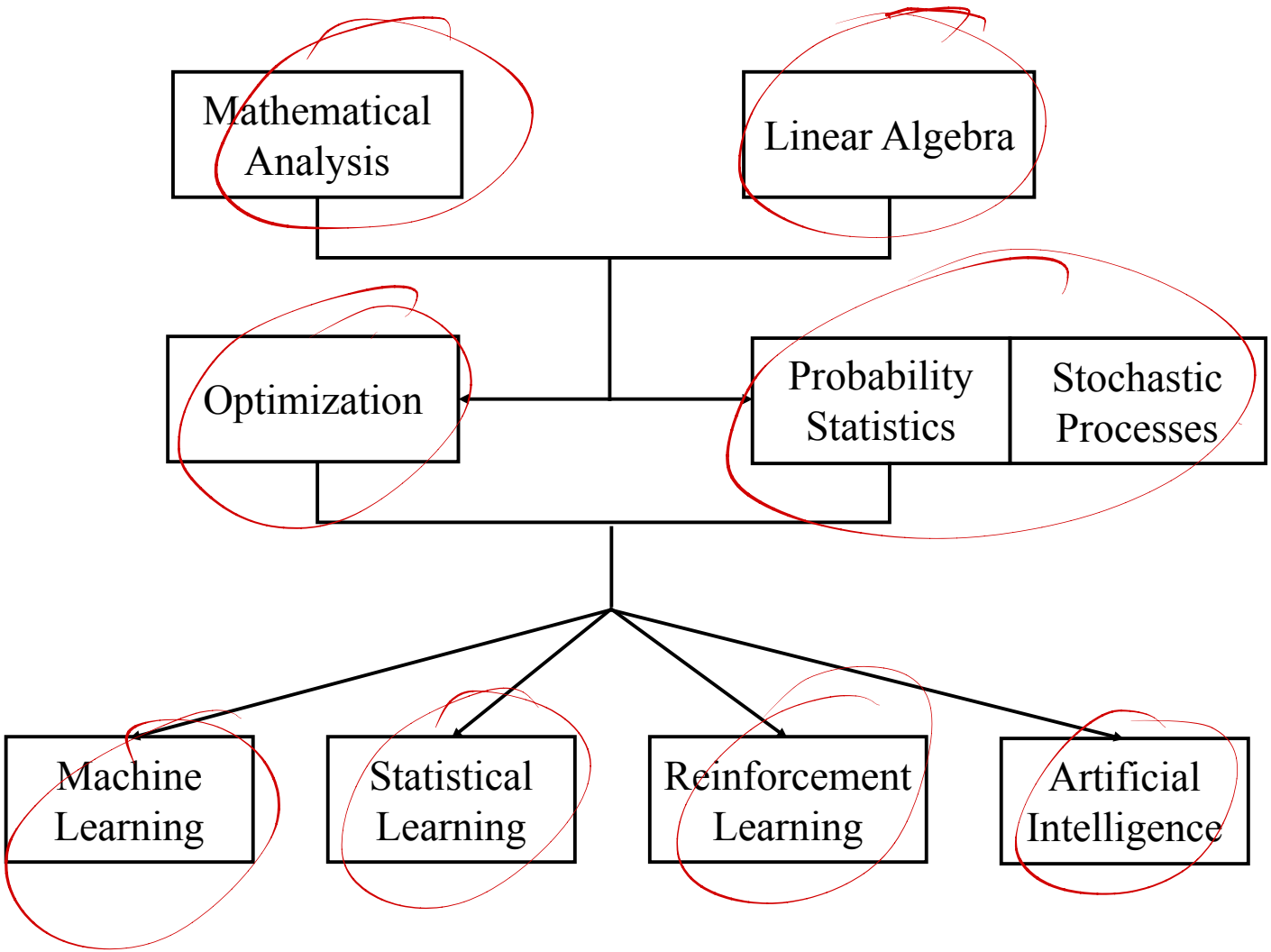
Outline

Outline

In My Beginning is My End



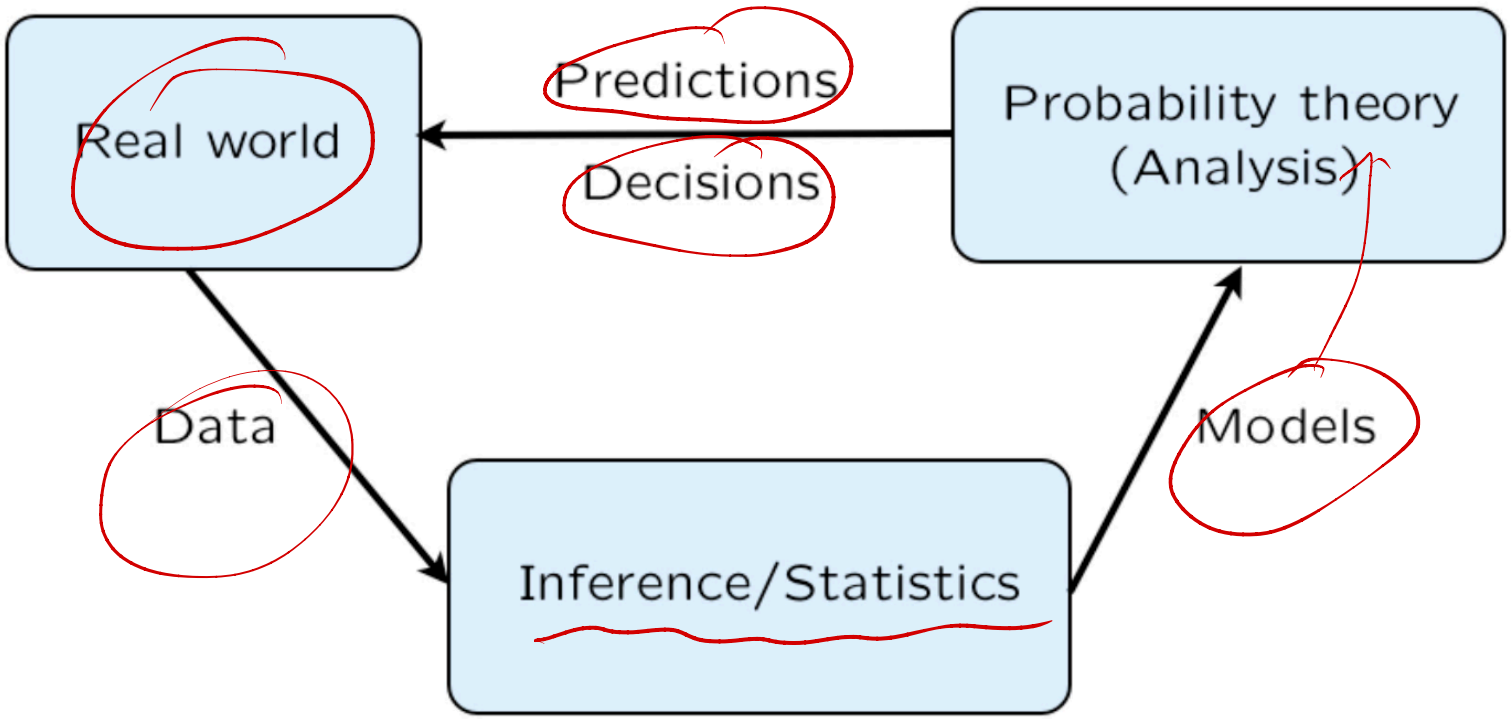
The Role of This Course



The Role of Probability & Statistics

A framework for analyzing phenomena with uncertain outcomes:

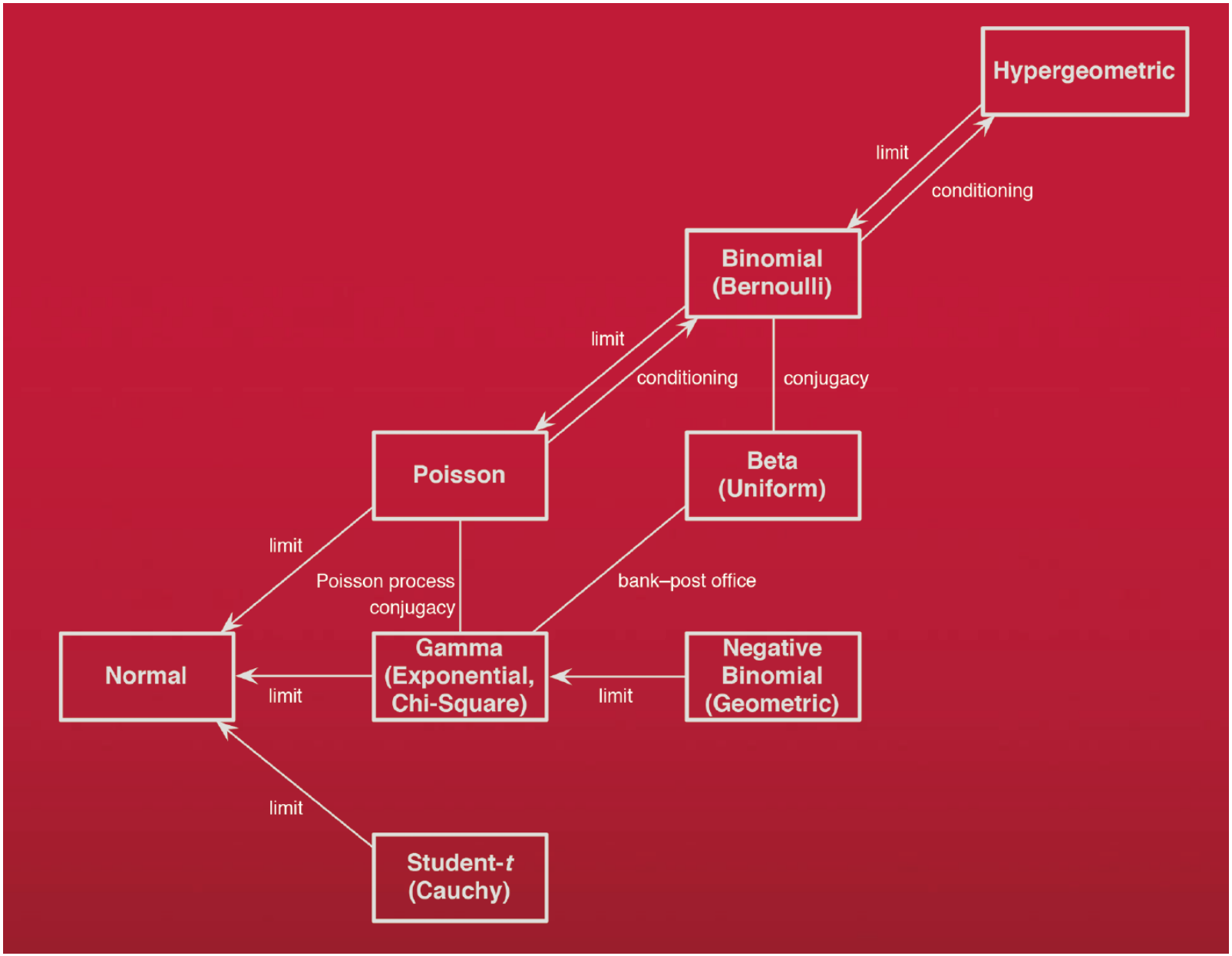
- Rules for consistent reasoning
- Used for predictions and decisions



Typical Distributions

Name	Param.	PMF or PDF	Mean	Variance
Bernoulli	p	$P(X=1)=p, P(X=0)=q$	p	pq
Binomial	n,p	$\binom{n}{k}p^k q^{n-k}$, for $k \in \{0,1,\dots,n\}$	np	npq
FS	p	pq^{k-1}, for $k \in \{1,2,\dots\}$	$1/p$	q/p^2
Geom	p	pq^k, for $k \in \{0,1,2,\dots\}$	q/p	q/p^2
NBinom	r,p	$\binom{r+n-1}{r-1}p^r q^n, n \in \{0,1,2,\dots\}$	rq/p	rq/p^2
HGeom	w,b,n	$\frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}$, for $k \in \{0,1,\dots,n\}$	$\mu = \frac{nw}{w+b}$	$(\frac{w+b-n}{w+b-1})n\frac{\mu}{n}(1-\frac{\mu}{n})$
Poisson	λ	$\frac{e^{-\lambda}\lambda^k}{k!}$, for $k \in \{0,1,2,\dots\}$	λ	λ
Uniform	$a<b$	$\frac{1}{b-a}$, for $x \in (a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal	μ,σ^2	$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$	μ	σ^2
Log-Normal	μ,σ^2	$\frac{1}{x\sigma\sqrt{2\pi}}e^{-(\log x-\mu)^2/(2\sigma^2)}, x>0$	$\theta = e^{\mu+\sigma^2/2}$	$\theta^2(e^{\sigma^2}-1)$
Expo	λ	$\lambda e^{-\lambda x}$, for $x>0$	$1/\lambda$	$1/\lambda^2$
Gamma	a,λ	$\Gamma(a)^{-1}(\lambda x)^a e^{-\lambda x}x^{-1}$, for $x>0$	a/λ	a/λ^2
Beta	a,b	$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}$, for $0<x<1$	$\mu = \frac{a}{a+b}$	$\frac{\mu(1-\mu)}{a+b+1}$
Chi-Square	n	$\frac{1}{2^{n/2}\Gamma(n/2)}x^{n/2-1}e^{-x/2}$, for $x>0$	n	$2n$
Student-t	n	$\frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\Gamma(n/2)}(1+x^2/n)^{-(n+1)/2}$	0 if $n>1$	$\frac{n}{n-2}$ if $n>2$

Relationship Among Distributions



Basic Contents: Part I

- Probability and Counting: Definition of Probability, Counting, Bose-Einstein Model, Birthday Match Problem, Hash Table.
- Conditional Probability: Bayes'rule, LOTP, Conditioning, Gambler's ruin, Simpson's Paradox, Monty Hall.
- Random Variables and Distributions: Bernoulli, Binomial, Story for distributions, entropy.
- Expectation: Indicator r.v., Geometric, Coupon Collector, Poisson, Probability Generating Function.
- Continuous Random Variables: Universality of the Uniform, Normal, Exponential, Memoryless, Moment Generating Function. Central Limit Theorem.

Basic Contents: Part II

- Joint Distributions: Joint/Marginal distribution, chicken-egg model, Poisson, meaning of conditioning on zero-probability event, Four Forms of Bayes' Rule, Four Forms of LOTP, Covariance, Correlation, Multivariate Normal Distribution.
- Transformations: Change of Variables, Jacobian Matrix, Bivariate Normal Joint PDF, Convolution, Order Statistics, Beta-Binomial Conjugacy, Dirichlet-Multinomial Conjugacy, Bayesian Ranking, Gamma, Bank–post Office model.

Basic Contents: Part III

- Monte Carlo Statistical Methods: Random Variable Generation, Acceptance-Rejection Method, Monte Carlo Integration, Importance Sampling, Law of Large Numbers, Cauchy-Schwarz Inequality, Jensen's Inequality, Kullback-Leibler Divergence (Entropy), Markov's Inequality, Chebyshev's Inequality, Chernoff's Inequality (related to MGF), Chernoff's Technique, Hoeffding Lemma, Hoeffding Bound, Parameter Estimation (confidence interval), Monte Carlo Method for Estimation π.

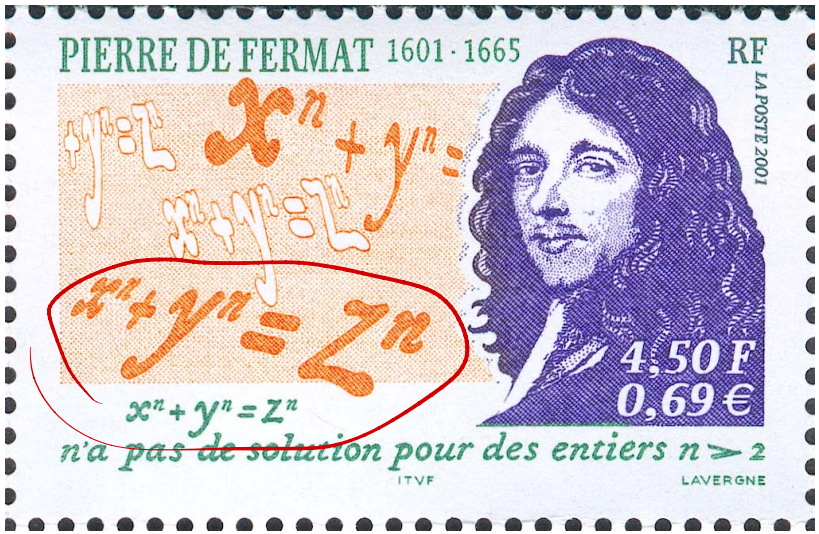
Basic Contents: Part IV

- Classical Statistical Inference: Maximum Likelihood Estimation, Confidence Interval.
- Bayesian Statistical Inference: Statistical Inference, Bayesian Inference and the Posterior Distribution, The Maximum A Posteriori Probability (MAP) Rule, Bayesian Estimation, Linear Least Square Estimate (LLSE), Minimum Mean Square Error Estimator (MMSE), Orthogonality Property of MMSE, MMSE and LLSE, MMSE for Jointly Gaussian Random Variables, Kalman Filter.
- Conditional Expectation: Conditional Expectation Given An Event , Conditional Expectation Given An R.V., LOTE, Adam' s Law, Eve' s law, Linear Regression, Projection Interpretation, Geometric Perspective, Prediction Perspective (MMSE).

Outline

Milestones in Classical Probability

- 1654: Correspondences between Blaise Pascal (1623-1662) and Pierre de Fermat (1601-1665) regarding questions on gambling by Antoine Gombaud (1607-1684).



Milestones in Classical Probability

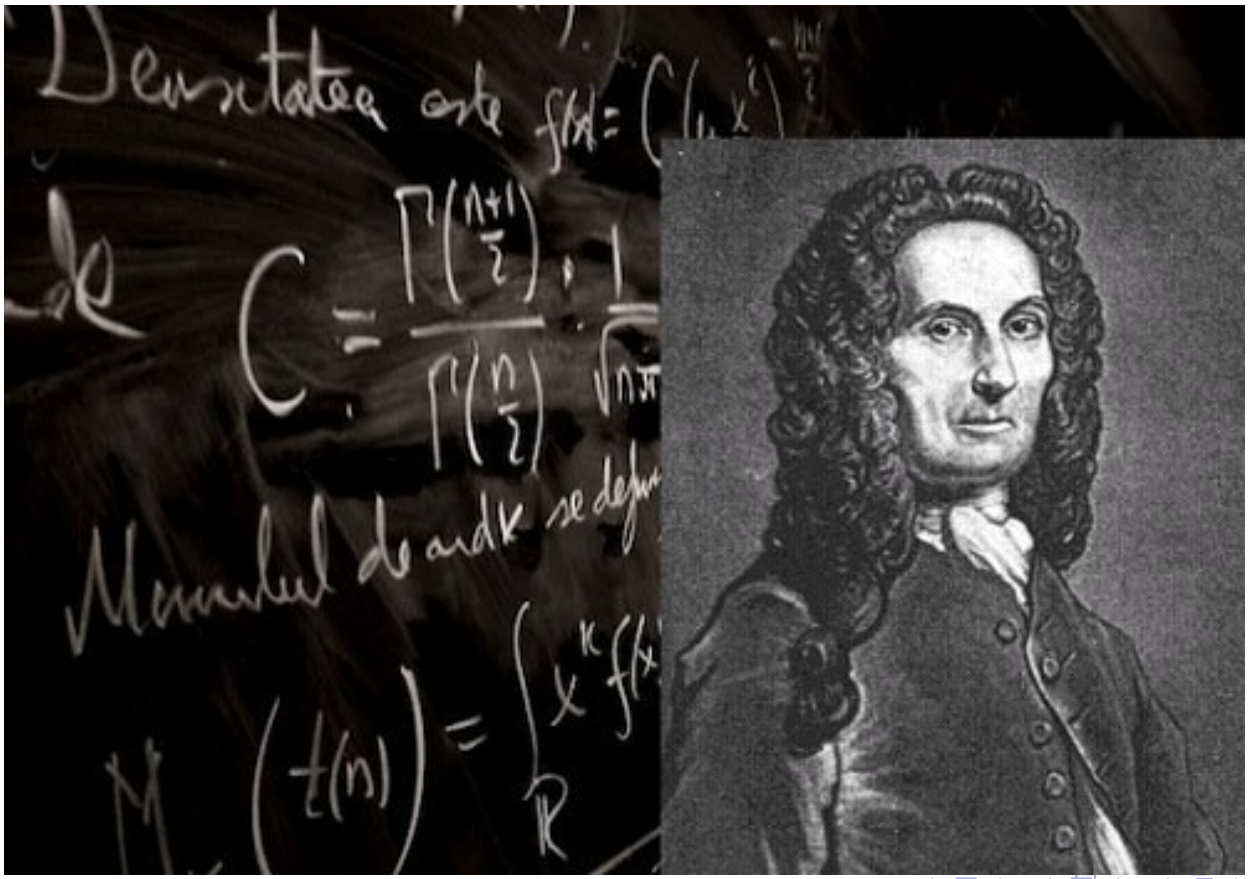
- 1657: Christiaan Huygens'(1629-1695) book, "De ratiociniis in ludo aleae (On reasoning in games of chance)."

Milestones in Classical Probability

- 1713: Jacob Bernoulli (1645-1705)'s book "Ars conjectandi (The art of conjecture)" published posthumously. "Bernoulli's Theorem" is discussed.

Milestones in Classical Probability

- 1718: Abraham de Moivre (1667-1754)'s book "Doctrine of Chances" published.
- Sharpened Bernoulli's theorem, and derived a central limit theorem by introducing the normal distribution.

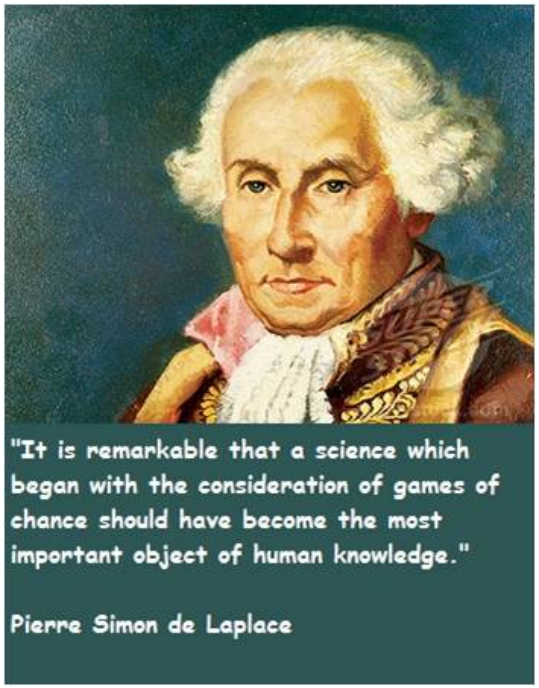


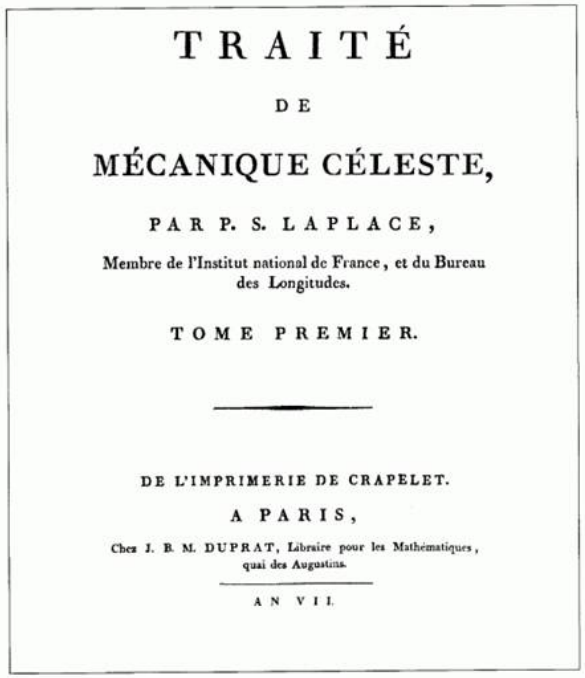
Milestones in Classical Probability

- 1764: Thomas Bayes (1701-1761), "Essay towards solving a problem in the doctrine of chances," published posthumously. "Bayes' Theorem" is discussed.

Milestones in Classical Probability

- 1783: Pierre-Simon Laplace (1749-1827) used the normal distribution to study measurement errors.
- **Pierre Simon Laplace**: the father of classical probability




TRAITÉ
DE
MÉCANIQUE CÉLESTE,
PAR P. S. LAPLACE,
Membre de l'Institut national de France, et du Bureau des Longitudes.
TOME PREMIER.
DE L'IMPRIMERIE DE CRAPELET.
A PARIS,
Chez J. B. M. DUPRAT, Libraire pour les Mathématiques, quai des Augustins.
AN VII.

Milestones in Classical Probability

- 1809: Carl Friedrich Gauss (1777-1855) used the normal distribution in the analysis of astronomical data.

Milestones in Classical Probability

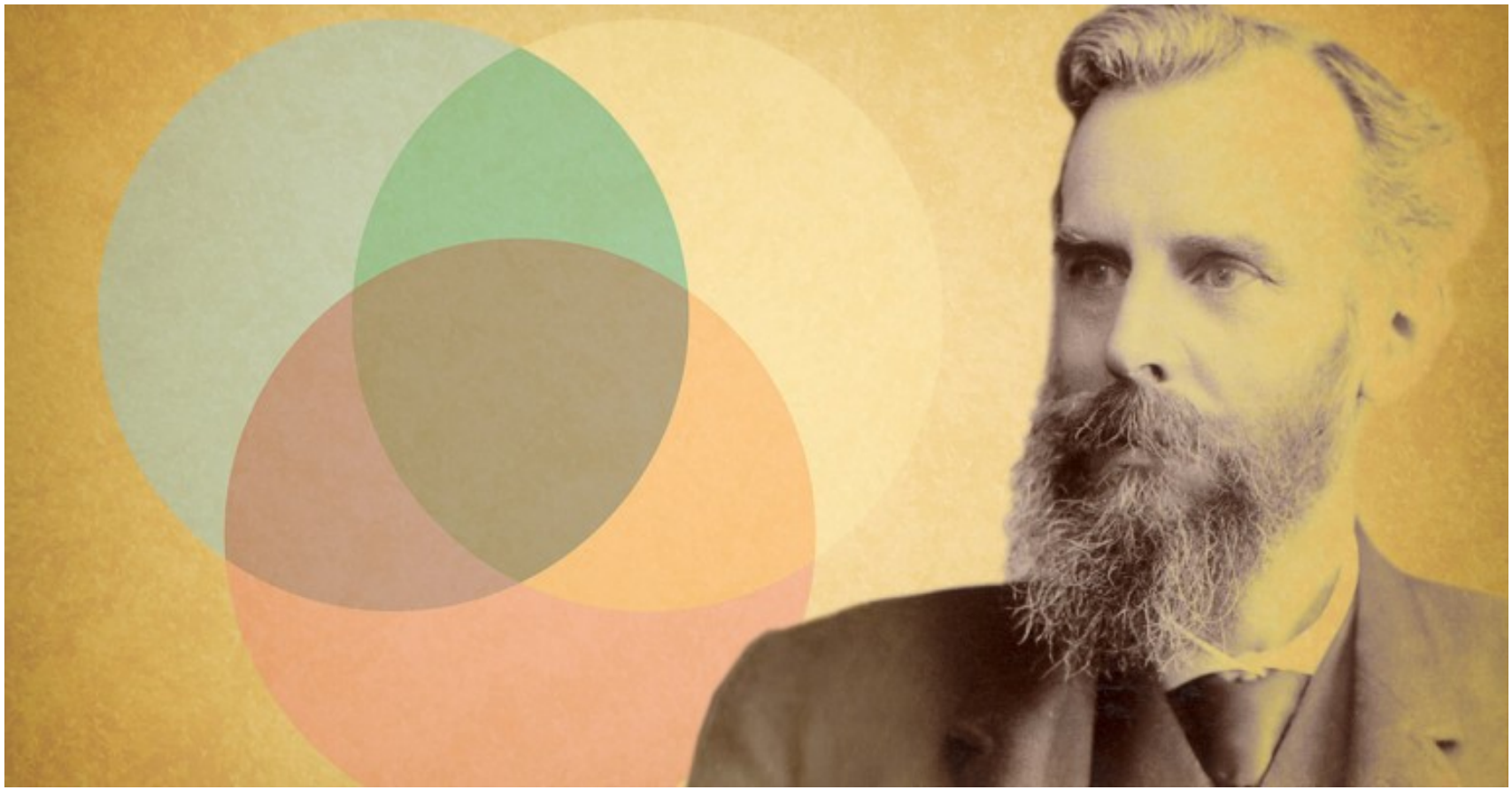
- 1812: Pierre-Simon Laplace, "Analytic theory of probability." The central limit theorem for i.i.d. (independent and identically distributed) random variables is discussed.

Milestones in Classical Probability

- 1835: Siméon-Denis Poisson (1781-1840) described Bernoulli's theorem as "La loi des grands nombres (The law of large numbers)."
- 1837: The Poisson distribution appeared in his paper "Research on the probability of judgments in criminal and civil matters."

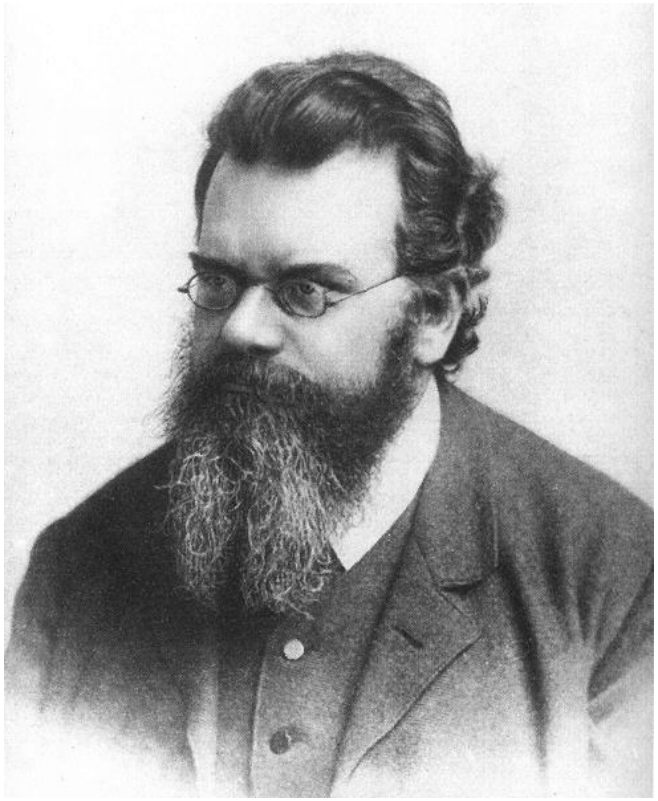
Milestones in Classical Probability

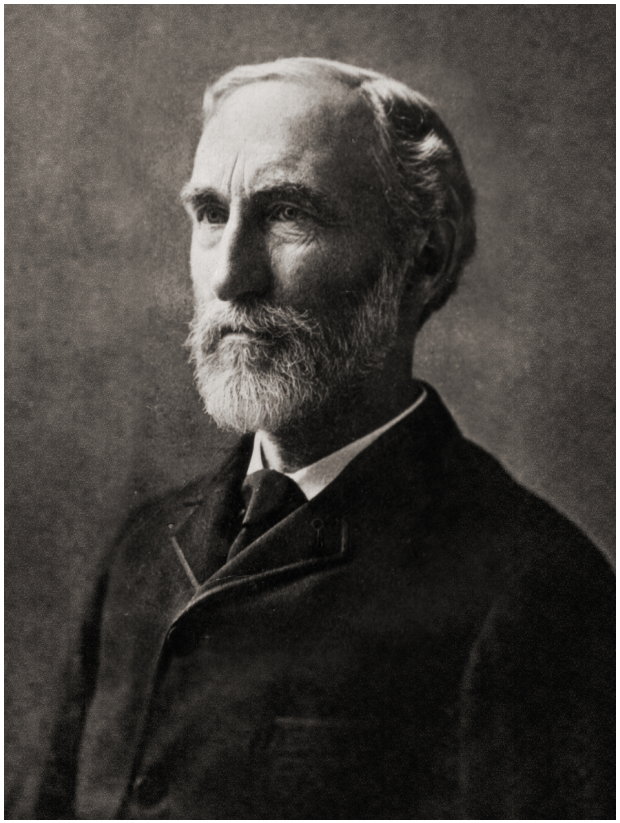
- 1866: John Venn (1834-1923) emphasized the frequency interpretation of probability in "Logic of chance." Empiricism in probability influenced the development of the theory of statistics.



Milestones in Classical Probability

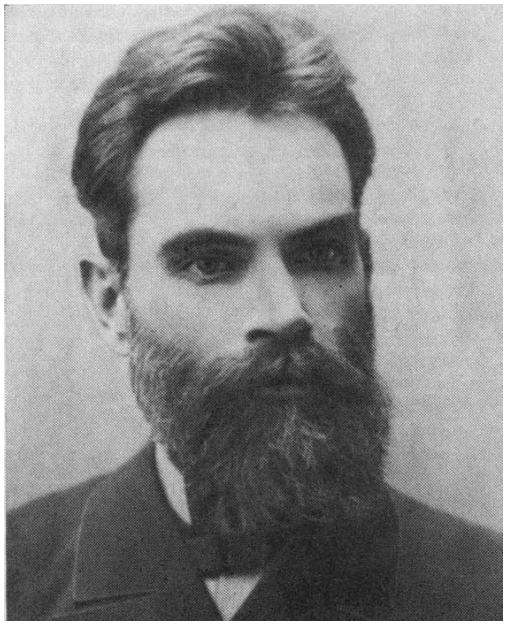
- Towards the end of the 19th century, probability began to play a fundamental role in statistical thermodynamics developed by Ludwig Boltzmann (1844-1906) and Josiah Willard Gibbs (1839-1903).





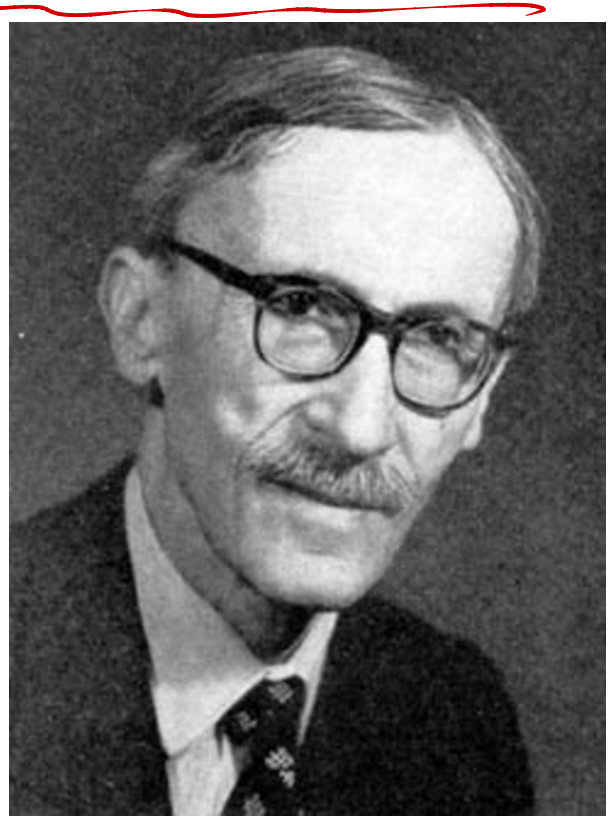
Milestones in Classical Probability

- By the late 19th century: rise of the Russian school of probability
- Pafnuty Lvovich Chebyshev (1821-1894) and his two students
- Andrei A. Markov (1856-1922)
- Aleksandr Mikhailovich Lyapunov (1857-1918)



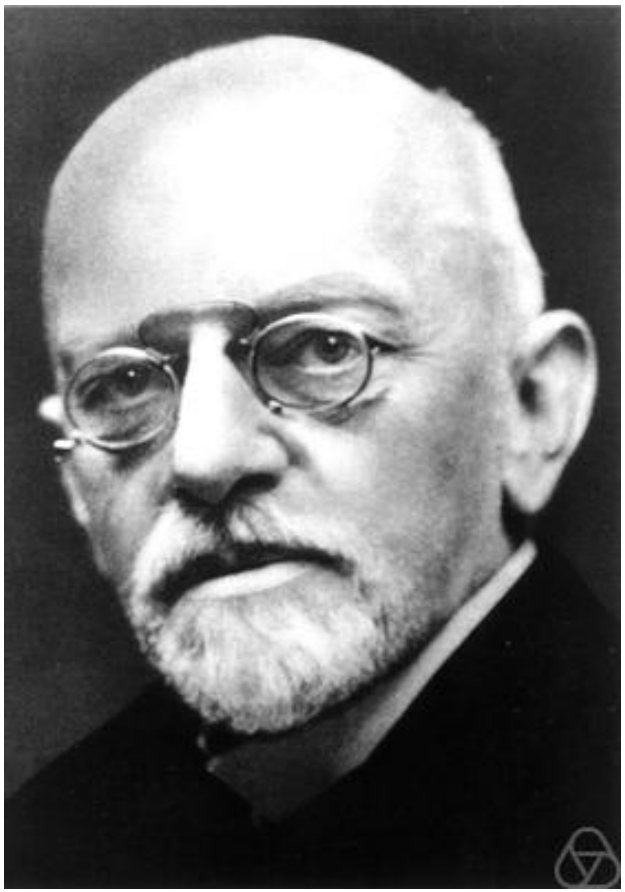
Milestones in Modern Probability

- The early twentieth century: French mathematicians regained interest in Mathematical probability
- Émile Borel (1871-1956) pioneered measure theory
- Paul Pierre Lévy (1886-1971) pioneered martingale theory



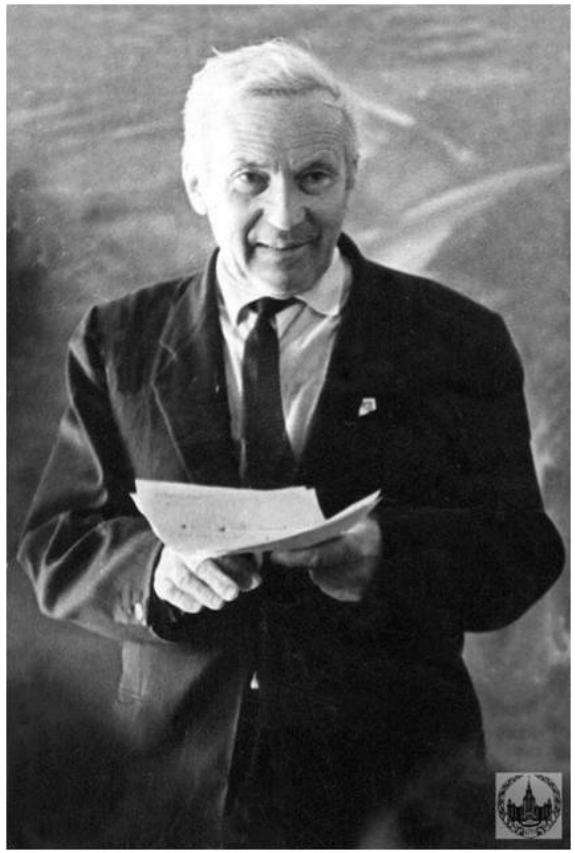
Milestones in Modern Probability

- 1900: David Hilbert (1862-1943) listed probability as a sub-discipline of his sixth problem (out of 23 open problems), i.e., axiomatic foundations for physics.



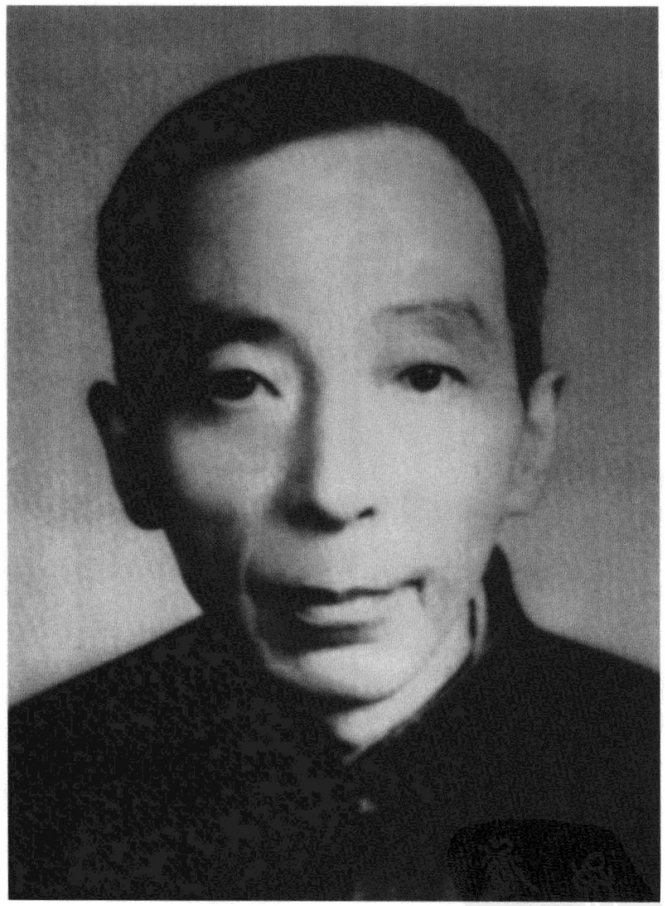
Milestones in Modern Probability

- 1933: Axiomatic-based approach built solid foundation
- **Andrey Kolmogorov**(1903-1987): the father of modern probability



Milestones in Modern Probability

- 1940s: Boris Gnedenko (1912-1995) and Pao-Lu Hsu (1910-1970) independently obtained the most general form of the Central Limit Theorem.



Milestones in Stochastic Processes

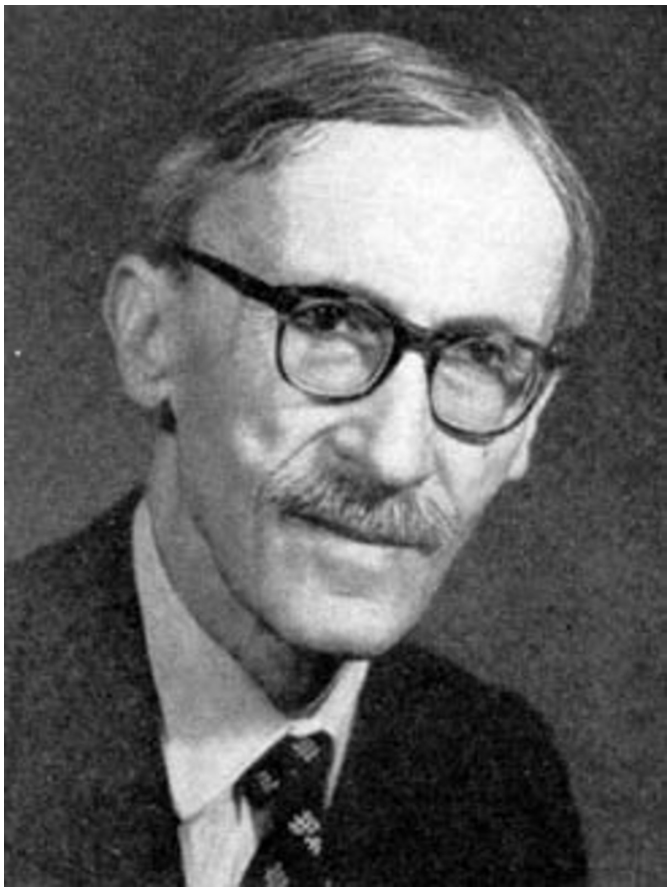
- The Poisson Process is named after Siméon-Denis Poisson (1781- 1840).

Milestones in Stochastic Processes

- 1907: Markov introduced what we now call a discrete-time Markov chain (DTMC).
- Markov chains or Markov processes have been applied to many problems: Queueing theory, Information theory, Hidden Markov model (HMM), PageRank algorithm used in Google search.

Milestones in Stochastic Processes

- The concept of martingale was introduced by Paul Pierre Lévy (1886-1971) ,and developed by Joseph Leo Doob (1910-2004).



Milestones in Stochastic Processes

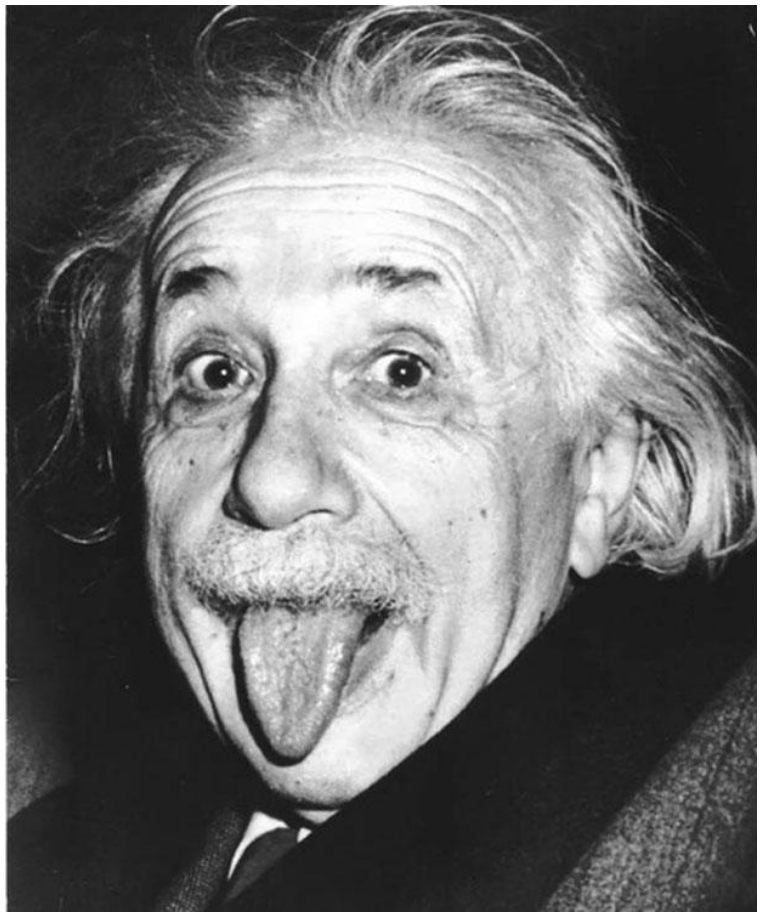
- 1827: Robert Brown (1773-1858) observed the irregular motion of pollen particles suspended in water: a phenomenon called Brownian motion.

Milestones in Stochastic Processes

- 1900: Louis Bachelier (1870-1946) gave a mathematical description of Brownian motion in his Ph.D. thesis "Théorie de la spéculation (The theory of speculation)."

Milestones in Stochastic Processes

- 1905: Albert Einstein (1879-1955) published "On the movement of small particles suspended in a stationary liquid demanded by the molecular-kinetic theory of heat"



Milestones in Stochastic Processes

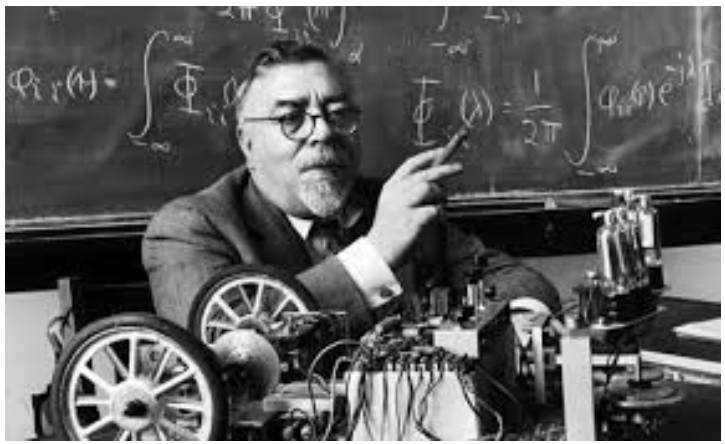
- 1906: Marion Smoluchowski (1872-1917) "Outline of the kinetic theory of Brownian motion of suspensions."

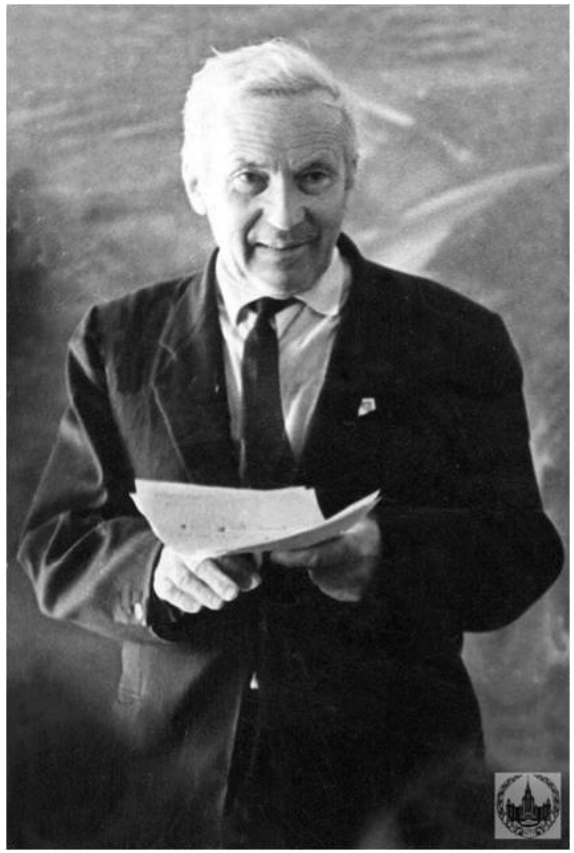
Milestones in Stochastic Processes

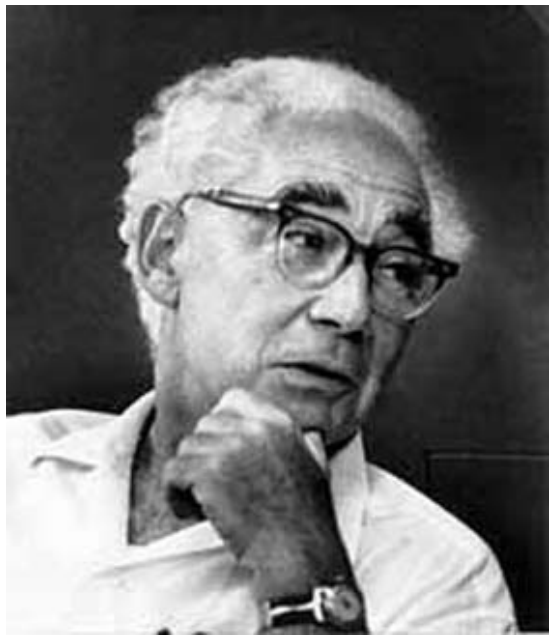
- Jean-Baptiste Perrin (1870-1942) experimentally verified the Einstein- Smoluchowski theory, and put an end to the century-long dispute about the existence of atoms and molecules. Received Nobel prize in 1926.

Milestones in Stochastic Processes

- Brownian motion was further investigated mathematically by Lévy, Nobert Wiener (1894-1964), A. N. Kolmogorov (1903-1987), and William Feller (1906- 1970).







Milestones in Stochastic Processes

- 1944: Kiyoshi Itô (1915-2008) published Stochastic Integral. The "Itô calculus" and stochastic differential equations are foundations of modern mathematical finance.

Milestones in Stochastic Processes

- The 1950-1960: Kai Lai Chung (1917- 2009) developed general theory for Markov process and Brownian motion.

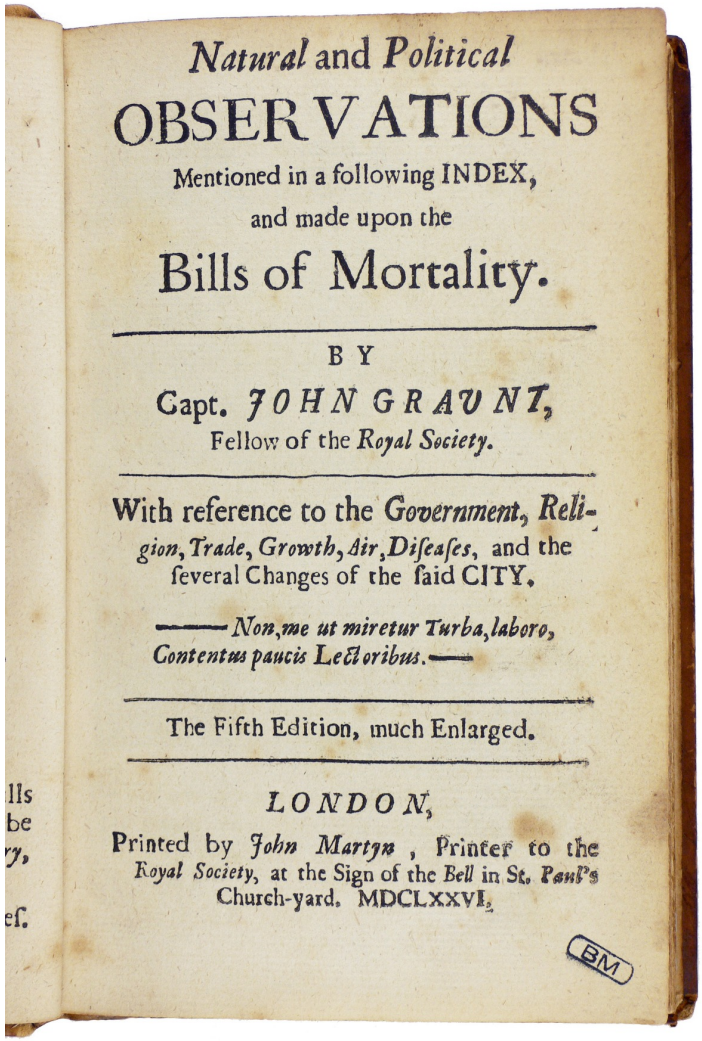
Outline

Statistics Origins

- Social statistics in Roman Empire
- Statistics (Stato in Italian)
- Statistician (Statista in Italian): the man who deals with affairs of the state

Milestones in Statistics

- John Graunt (1620-1674) is credited for the co-founder of descriptive statistics



Natural and *Political*

OBSERVATIONS

Mentioned in a following INDEX,
and made upon the

Bills of Mortality.

BY

Capt. *JOHN GRAUNT*,
Fellow of the *Royal Society*.

With reference to the *Government*, *Religion*, *Trade*, *Growth*, *Air*, *Diseases*, and the several Changes of the said CITY.

——*Non, me ut miretur Turba, laboro,*
Contentus paucis Lectoribus.——

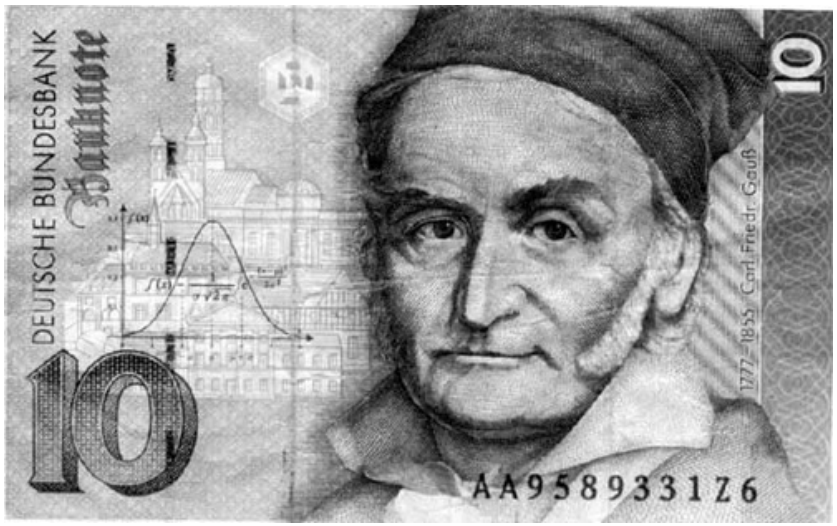
The Fifth Edition, much Enlarged.

LONDON,
Printed by *John Martyn*, Printer to the *Royal Society*, at the Sign of the *Bell* in St. *Paul's* Church-yard. MDCLXXVI.

BM

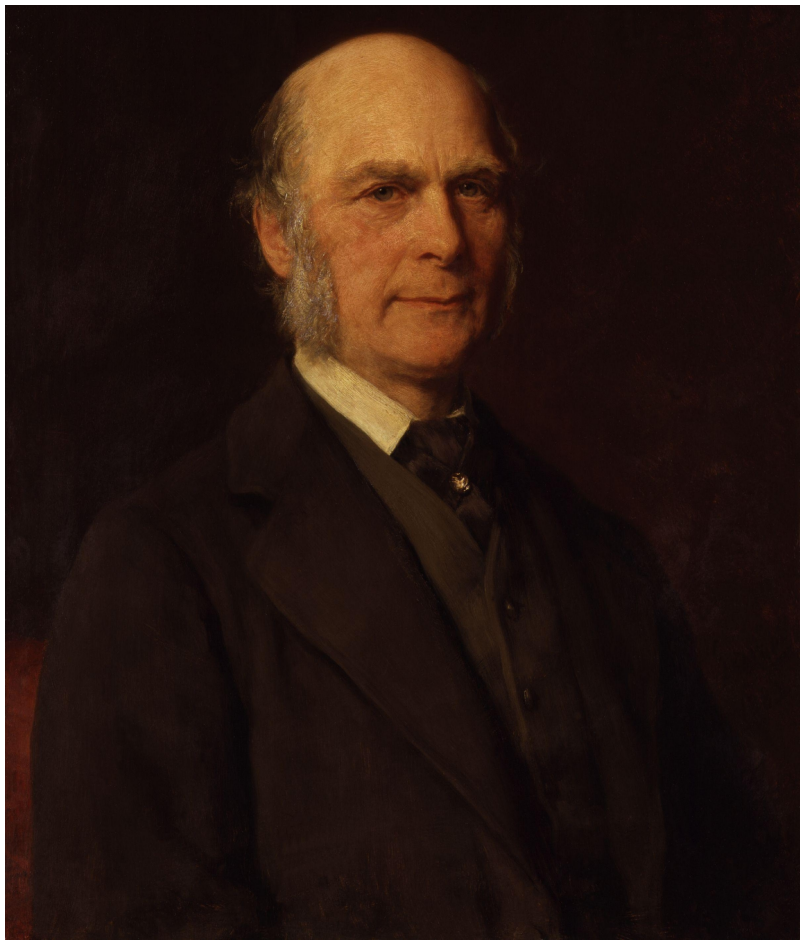
Milestones in Statistics

- 1800s: Linear Statistical Model and the method of least squares for estimation is often credited to Gauss (1777-1855) (1809), Adrien-Marie Legendre (1752-1833) (1805), Robert Adrain (1775-1843).
- Gauss also showed the optimality of the least-square approach (Gauss-Markov Theorem, 1823).



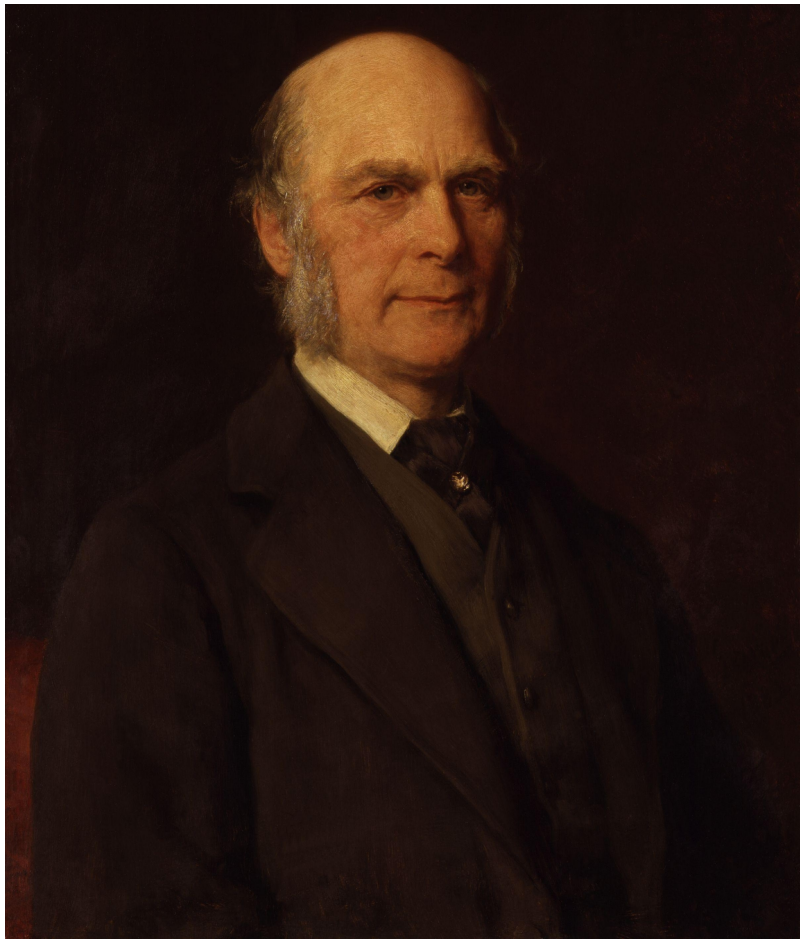
Milestones in Statistics

- 1874: Sir Francis Galton (1822-1911) developed Galton–Watson process
- 1888: Sir Francis Galton proposed the concept of correlation
- 1889: Sir Francis Galton proposed the concept of regression



Milestones in Statistics

- Francis Edgeworth (1845-1926) is credited for the contribution of theory of regression

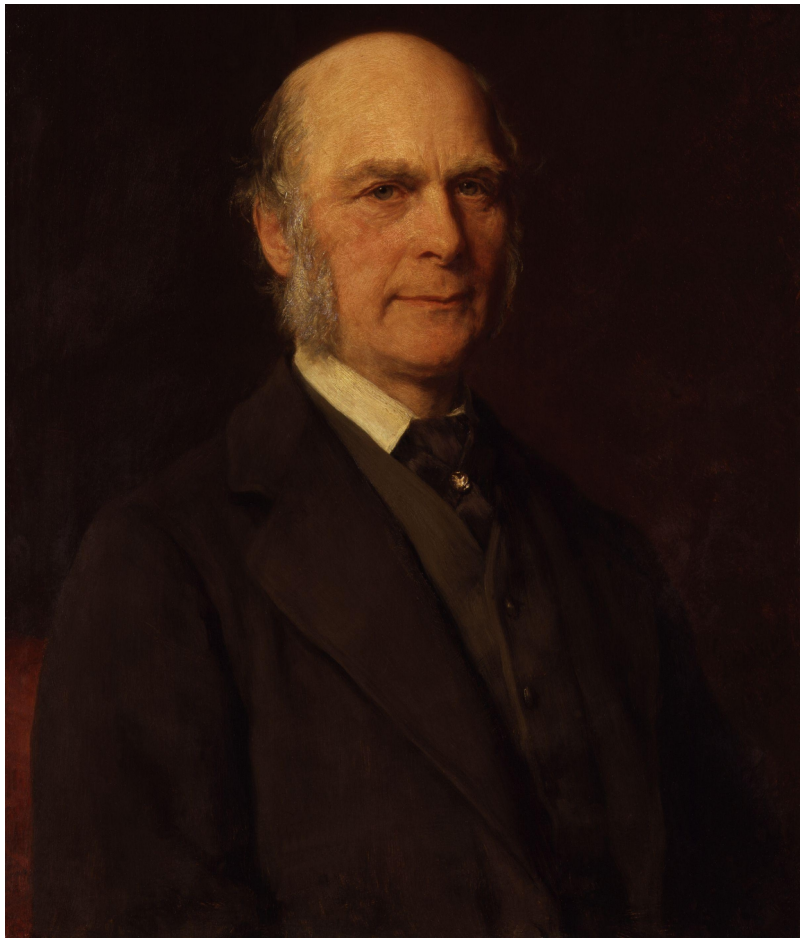


Milestones in Statistics

- Karl Pearson (1857-1936) is credited for the establishment of the discipline of statistics. He contributed to theory of linear regression, correlation, Pearson curve, chi-square test, and the method of moments for estimation.

Milestones in Statistics

- George Yule (1871-1951) is credited for the contribution of theory of regression and time-series analysis
- Fundamental work on the theory of autoregressive series



Milestones in Statistics

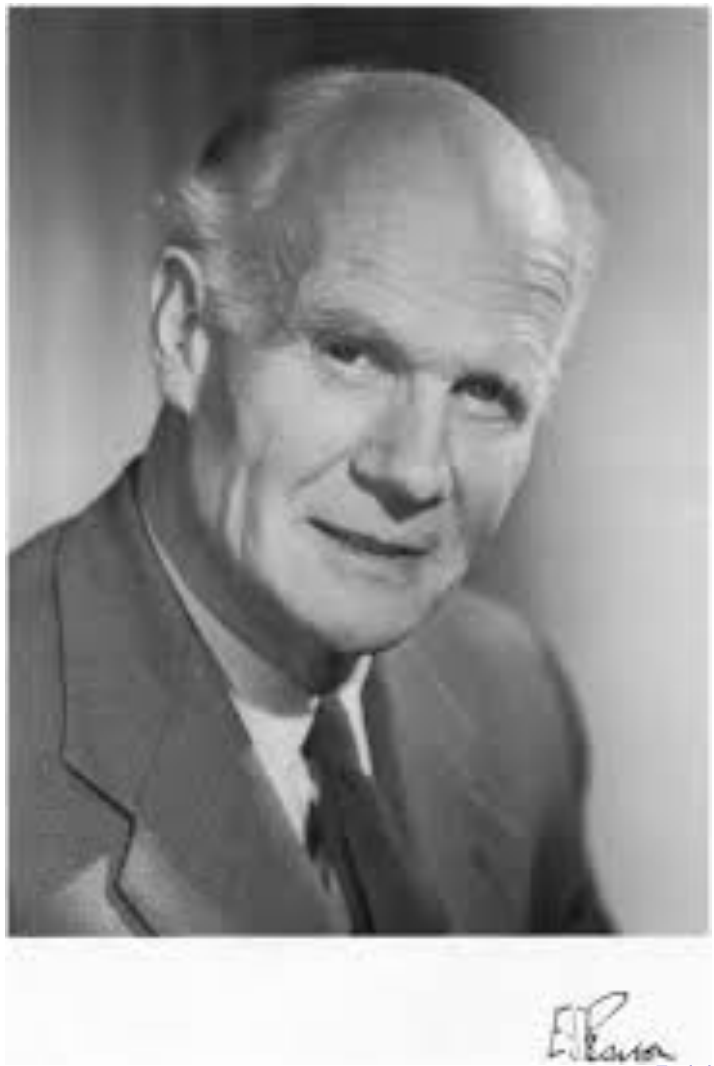
- 1908: William Gosset (Student) (1876-1937) proposed Student t-distribution and t-test statistics
- Precursor of small-sample statistics and hypothesis testing.

Milestones in Statistics

- 1912-1922: Sir Ronald Aylmer Fisher (1890-1962) developed the notion of maximum likelihood estimator.
- He also worked on the analysis of variance (ANOVA), F-distribution, Fisher information and design of experiment.
- Co-founder of Modern Statistics (Mathematical Statistics or Statistical Inference)

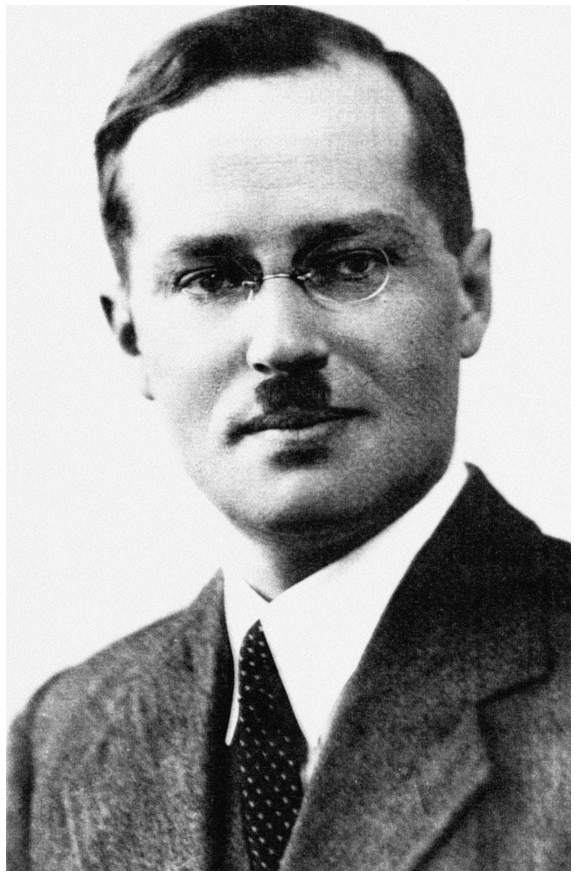
Milestones in Statistics

- Egon Sharpe Pearson (1895-1980): co-founder of Neyman-Pearson Theory for hypothesis testing.



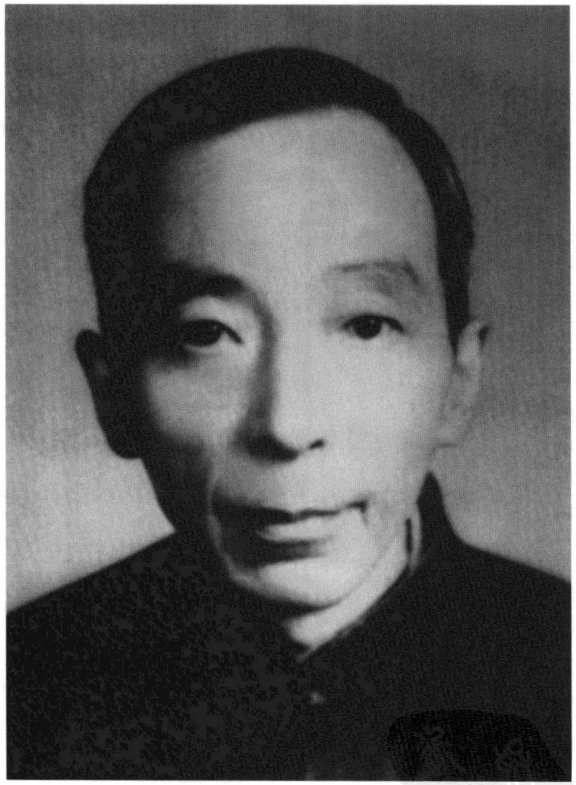
Milestones in Statistics

- Jerzy Neyman (1894-1981): Co-founder of Modern Statistics (Mathematical Statistics or Statistical Inference)
- 1928-1938: Theoretical foundations of testing hypothesis, point estimation, confidence interval and survey sampling.



Milestones in Statistics

- 1940s: Pao-Lu Hsu (1910-1970) obtains the first optimum property for the likelihood ratio test of the univariate linear hypothesis.
- He also obtained several exact or asymptotic distributions of important statistics in the theory of multivariate analysis.



Milestones in Statistics

- 1940s: Abraham Wald (1902-1950): developed the statistical decision theory and sequential testing.

Milestones in Statistics

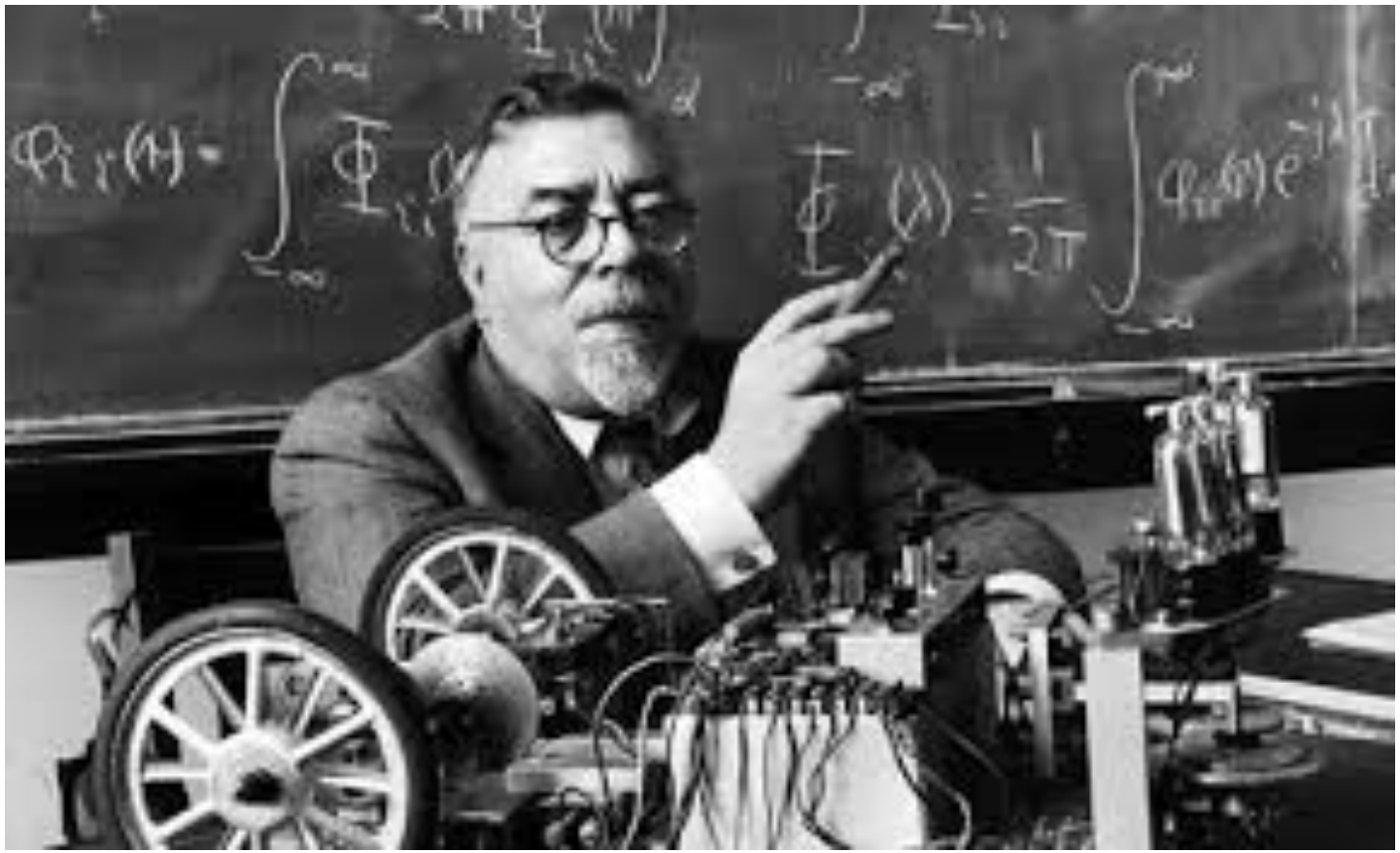
- 1945-1946: The minimum unbiased estimator bound, the Cramér-Rao bound, is due to Carl Harald Cramér (1893-1985) and Calyampudi Radhakrishna Rao (1920-).

Milestones in Statistics

- 1946-1947: Stanislaw Ulam (1909-1984) & John von Neumann(1903-1957): developed the modern Monte Carlo method.

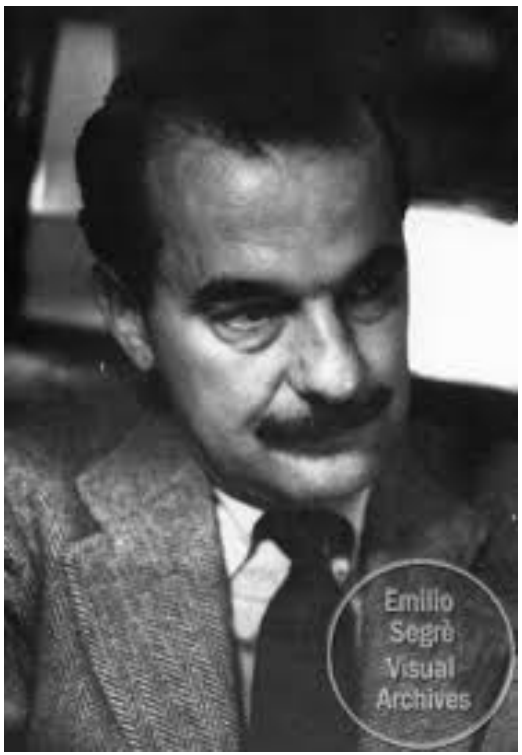
Milestones in Statistics

- 1949: Norbert Wiener (1894-1964), "Extrapolation, Interpolation, and Smoothing of Stationary Time Series."



Milestones in Statistics

- As one of the most commonly used Markov chain Monte Carlo (MCMC) method, Metropolis–Hastings algorithm was first proposed in 1953 by Nicholas Metropolis (1915-1999), and then extended in 1970 by Wilfred Keith Hastings (1930-2016).

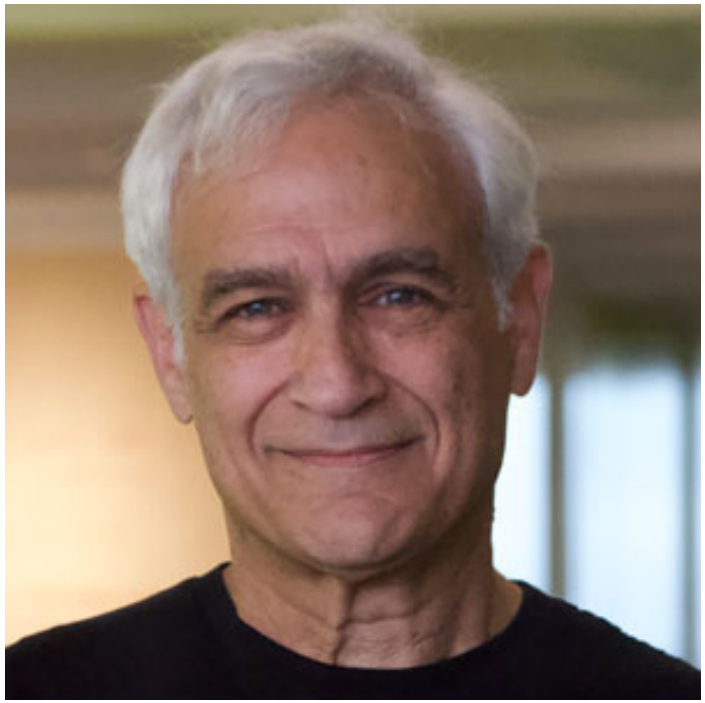


Milestones in Statistics

- 1960: Rudoph Emil Kalman (1930-2016) introduced what is known as Kalman filter.

Milestones in Statistics

- Gibbs sampling algorithm was proposed in 1984 by brothers Stuart Geman (1949-) and Donald Geman (1943-).
- Gibbs sampling is named after the physicist Josiah Willard Gibbs (1839-1903), in reference to an analogy between the sampling algorithm and statistical physics.



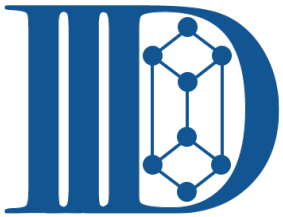
Outline

In My End is My Beginning

- Now this is not the end. It is not even the beginning of the end.
- But it is, perhaps, the end of the beginning.

The End of Adventure in Probability & Statistics

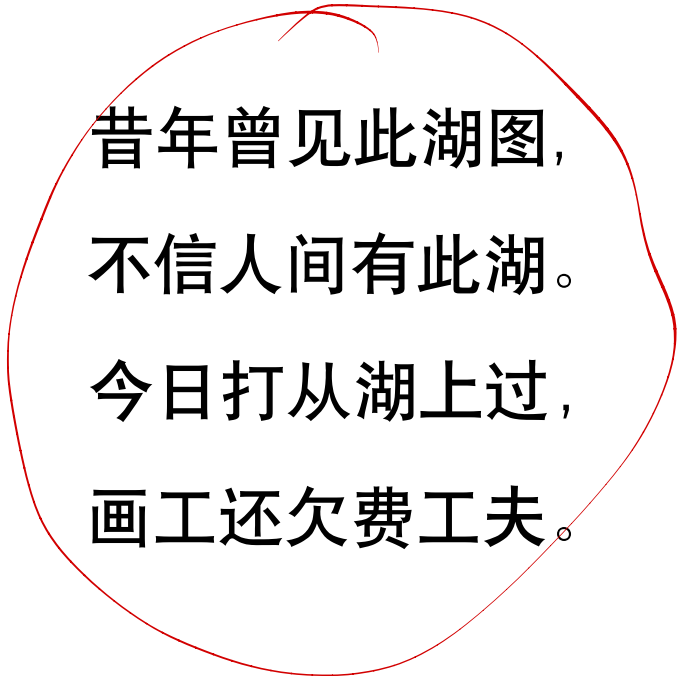
A New Adventure: Welcome to Join IID Lab!



Laboratory for Intelligence Information and Decision

智 能 信 息 与 决 策 实 验 室

In My End is My Beginning



In My End is My Beginning