Probability & Statistics for EECS: Homework #08

Due on Dec 2, 2023 at 23:59

Name:

Student ID:

Problem 1

Let X and Y be two continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx^2y, & \text{if } 0 \le y \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of constant c.

(b) Find the conditional probability $P(Y \le X/4 \mid Y \le X/2)$.

Solution:

(a) According to the statement, we have

$$\begin{aligned} 1 &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1 \int_0^x cx^2y\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1 \frac{c}{2}x^4\,\mathrm{d}x \\ &= \frac{c}{10} \end{aligned} \tag{1}$$

So that, $c = 10$.

(b)

$$\begin{aligned} P\left(Y \le \frac{X}{4} \mid Y \le \frac{X}{2}\right) &= \frac{P(Y \le \frac{X}{4}, Y \le \frac{X}{2})}{P(Y \le \frac{X}{2})} \\ &= \frac{P(Y \le \frac{X}{4})}{P(Y \le \frac{X}{2})} \\ &= \frac{\int_0^1 \int_0^{\frac{x}{4}} 10x^2y\,\mathrm{d}y\,\mathrm{d}x}{\int_0^1 \int_0^{\frac{x}{2}} 10x^2y\,\mathrm{d}y\,\mathrm{d}x} \\ &= \frac{\int_0^1 \frac{x^4}{32}\,\mathrm{d}y\,\mathrm{d}x}{\int_0^1 \frac{x^4}{8}\,\mathrm{d}y\,\mathrm{d}x} \\ &= \frac{1}{4}. \end{aligned} \tag{2}$$

Problem 2

Let X and Y be two integer random variables with joint PMF

$$P_{X,Y}(x,y) = \begin{cases} \frac{1}{6\cdot 2^{\min(x,y)}}, & \text{if } x, y \geq 0, |x-y| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the marginal distributions of X and Y.

(b) Are X and Y independent?

(c) Find $P(X = Y)$.

Solution:

(a) The marginal distributions of X is

$$P_X(X) = \sum_{y=0}^{\infty} P_{X,Y}(X,Y).$$

When $X = 0$, we have

$$P(X=0) = P(X=0, Y=0) + P(X=0, Y=1) = \frac{1}{3}.$$

When $X \neq 0$, we have

$$P(X=x) = P(X=x, Y=x-1) + P(X=x, Y=x) + P(X=x, Y=x+1) = \frac{1}{6\cdot 2^{x-2}}.$$

Thus, the marginal distribution of X is

$$P_X(X) = \begin{cases} \frac{1}{3}, & x = 0 \\ \frac{1}{6\cdot 2^{x-2}}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

According to the symmetric, the marginal distribution of Y is

$$P_Y(Y) = \begin{cases} \frac{1}{3}, & y = 0 \\ \frac{1}{6\cdot 2^{y-2}}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(b) Since that

$$P_{X,Y}(0,0) = \frac{1}{6}, \tag{3}$$

and

$$P(X=0)P(Y=0) = \frac{1}{9}, \tag{4}$$

X and Y are not independent.

(c) According to symmetric, we have $P(X = Y) = P(X = Y-1) = P(X = Y+1)$ and $P(X = Y) + P(X = Y-1) + P(X = Y+1) = 1$. Thus, we have

$$P(X=Y) = \frac{1}{3}.$$

Problem 3

Let X and Y be i.i.d. $\mathcal{N}(0,1)$, and let S be a random sign (1 or -1, with equal probabilities) independent of (X, Y).

(a) Determine whether or not $(X, Y, X+Y)$ is Multivariate Normal.

(b) Determine whether or not $(X, Y, SX+SY)$ is Multivariate Normal.

(c) Determine whether or not (SX, SY) is Multivariate Normal.

Solution:

(a) Yes, $(X, Y, X+Y)$ is Multivariate Normal, because for any $a, b, c \in R$,

$$aX + bY + c(X+Y) = (a+c)X + (b+c)Y,$$

and any linear combination of independent normally distributed variables are Normal.

(b) Denote $Z = X + Y + SX + SY = (1+S)X + (1+S)Y$.
$Z = 0$ is in fact $S = -1$, hence, we have that

$$P(Z=0) = P(S=-1) = \frac{1}{2}.$$

Hence, Z is not normally distributed.

(c) Observe that random vector (X,Y) is identically distributed as (-X,-Y). So,

$$\begin{aligned}
P(SX+SY \leq k) = &\ P(SX+SY \leq k, S=1) + P(SX+SY \leq k, S=-1) \\
= &\ P(SX+SY \leq k|S=1)P(S=1) + P(SX+SY \leq k|S=-1)P(S=-1) \\
= &\ \frac{1}{2}P(X+Y \leq k) + \frac{1}{2}P(X+Y \geq -k) \\
= &\ \frac{1}{2}P(X+Y \leq k) + \frac{1}{2}P(X+Y \leq k) \\
= &\ P(X+Y \leq k).
\end{aligned}$$

So, (SX, SY) is equally distributed as (X, Y), and (X, Y) is Bivariate normal. Hence, (SX, SY) is Multivariate Normal.

Problem 4

Let Z_1, Z_2 be two i.i.d. random variables satisfying standard normal distributions, i.e., $Z_1, Z_2 \sim \mathcal{N}(0.1)$. Define

$$X = \mathbf{\Sigma}_X Z_1 + \mu_X;$$
$$Y = \mathbf{\Sigma}_Y(\rho Z_1 + \sqrt{1-\rho^2} Z_2) + \mu_Y,$$

where $\mathbf{\Sigma}_X > 0$, $\mathbf{\Sigma}_Y > 0$, $-1 < \rho < 1$.

(a) Show that X and Y are bivariate normal.

(b) Find the correlation coefficient between X and Y, i.e., $\mathrm{Corr}(X, Y)$.

(c) Find the joint PDF of X and Y.

Solution:

(a) For $a, b \in \mathbb{R}$, we have

$$aX + bY = (a\mathbf{\Sigma}_X + b\mathbf{\Sigma}_Y \rho) Z_1 + b\sqrt{1-\rho^2}\mathbf{\Sigma}_Y Z_2 + a\mu_X + b\mu_Y.$$

Since the linear combination of two Normal distribution follows Normal distribution, X and Y are bivariate normal.

(b) Since $Z_1, Z_2 \sim \mathcal{N}(0,1)$. We have $\rho Z_1 + \sqrt{1-\rho^2} Z_2 \sim \mathcal{N}(0,1)$. So $X \sim \mathcal{N}(\mu_X, \mathbf{\Sigma}_X)$, $Y \sim \mathcal{N}(\mu_Y, \mathbf{\Sigma}_Y)$. Thus, we have

$$\begin{aligned}
Cov(X, Y) &= Cov(\mathbf{\Sigma}_X Z_1 + \mu_X, \mathbf{\Sigma}_Y(\rho Z_1 + \sqrt{1-\rho^2} Z_2) + \mu_Y) \\
&= \mathbf{\Sigma}_X \mathbf{\Sigma}_Y Cov(Z_1, \rho Z_1 + \sqrt{1-\rho^2} Z_2) \\
&= \mathbf{\Sigma}_X \mathbf{\Sigma}_Y \left(\rho Var(Z_1) + \sqrt{1-\rho^2} Cov(Z_1, Z_2)\right) \\
&= \mathbf{\Sigma}_X \mathbf{\Sigma}_Y \rho.
\end{aligned}$$

Then correlation coefficient between X and y is

$$Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X) Var(Y)}} = \frac{\mathbf{\Sigma}_X \mathbf{\Sigma}_Y \rho}{\mathbf{\Sigma}_X \mathbf{\Sigma}_Y}.$$

(c) Since Z_1 and Z_2 are i.i.d., we have

$$f_{Z_1, Z_2}(z_1, z_2) = f_{Z_1}(z_1) f_{Z_2}(z_2) = \frac{1}{2\pi} e^{-\frac{z_1^2 + z_2^2}{2}}.$$

Since $X = \mathbf{\Sigma}_X Z_1 + \mu_X$, $Y = \mathbf{\Sigma}_Y(\rho Z_1 + \sqrt{1-\rho^2} Z_2) + \mu_Y$, we have

$$Z_1 = \frac{X - \mu_X}{\mathbf{\Sigma}_X}$$

and

$$Z_2 = \frac{Y - \mu_Y}{\sqrt{1-\rho^2}\mathbf{\Sigma}_Y} - \rho \frac{X - \mu_X}{\sqrt{1-\rho^2}\mathbf{\Sigma}_X}.$$

Thus,

$$
\begin{aligned}
f_{X,Y}(x,y) &= \left|\frac{\partial(Z_1,Z_2)}{\partial(X,Y)}\right| f_{Z_1,Z_2}(z_1,z_2) \\
&= \frac{1}{\begin{vmatrix} \frac{\partial z_1}{\partial x} & \frac{\partial z_1}{\partial y} \\ \frac{\partial z_2}{\partial x} & r\frac{\partial z_2}{\partial x} \end{vmatrix}} f_{Z_1,Z_2}(z_1,z_2) \\
&= \frac{1}{\begin{vmatrix} \frac{1}{\mathbf{\Sigma}_X} & 0 \\ \frac{-\rho}{\sqrt{1-\rho^2}\mathbf{\Sigma}_X} & \frac{1}{\sqrt{1-\rho^2}\mathbf{\Sigma}_Y} \end{vmatrix}} f_{Z_1,Z_2}(z_1,z_2) \\
&= \frac{1}{\mathbf{\Sigma}_X\mathbf{\Sigma}_Y\sqrt{1-\rho^2}} f_{Z_1,Z_2}(z_1,z_2) \\
&= \frac{1}{\mathbf{\Sigma}_X\mathbf{\Sigma}_Y\sqrt{1-\rho^2}} f_{Z_1,Z_2}\left(\frac{x-\mu_X}{\mathbf{\Sigma}_X}, \frac{y-\mu_Y}{\sqrt{1-\rho^2}\mathbf{\Sigma}_Y} - \rho\frac{x-\mu_X}{\sqrt{1-\rho^2}\mathbf{\Sigma}_X}\right) \\
&= \frac{1}{2\pi\mathbf{\Sigma}_X\mathbf{\Sigma}_Y\sqrt{1-\rho^2}} e^{-\frac{(\frac{x-\mu_X}{\mathbf{\Sigma}_X})^2+(\frac{y-\mu_Y}{\sqrt{1-\rho^2}\mathbf{\Sigma}_Y}-\rho\frac{x-\mu_X}{\sqrt{1-\rho^2}\mathbf{\Sigma}_X})^2}{2}} \\
&= \frac{1}{2\pi\mathbf{\Sigma}_X\mathbf{\Sigma}_Y\sqrt{1-\rho^2}} e^{-\frac{(\frac{x-\mu_X}{\mathbf{\Sigma}_X})^2-\frac{2\rho(x-\mathbf{\Sigma}_X)(Y-\mathbf{\Sigma}_Y)}{\mathbf{\Sigma}_X\mathbf{\Sigma}_Y}+(\frac{y-\mu_Y}{\mathbf{\Sigma}_Y})^2}{2(1-\rho^2)}}.
\end{aligned}
$$

Problem 5

(a) Let X and Y be i.i.d. $\mathcal{N}(0,1)$, and $Z=\frac{X}{Y}$. Find the PDF of Z.

(b) Let X and Y be i.i.d. Unif $(0,1), W=X\cdot Y$, and $Z=\frac{X}{Y}$. Find the joint PDF of (W,Z).

(c) A point (X,Y) is picked at random uniformly in the unit circle. Find the joint PDF of R and X, where $R=\sqrt{X^2+Y^2}$.

(d) A point (X,Y,Z) is picked uniformly at random inside the unit ball of $\mathbb{R}^3$. Find the joint PDF of Z and R, where $R=\sqrt{X^2+Y^2+Z^2}$.

Solution

(a) Let $X=R\cos(\Theta)$ and $Y=R\sin(\Theta)$, where R is the radial distance from the origin to the point (X,Y), and Θ is the angle formed with the positive x-axis. The variables R and Θ are given by:

$$R=\sqrt{X^2+Y^2},\Theta=\tan^{-1}\left(\frac{Y}{X}\right).$$

The joint PDF of X and Y, given that both are standard normal, is:

$$f_{X,Y}(x,y)=\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$$

To convert this joint PDF from Cartesian coordinates (x,y) to polar coordinates (r,θ), use the Jacobian of the transformation:

$$x=r\cos(\theta),\quad y=r\sin(\theta),J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\cos(\theta) & -r\sin(\theta)\\ \sin(\theta) & r\cos(\theta)\end{vmatrix}=r.$$

Substituting the polar expressions into the original joint PDF and adjusting for the Jacobian, the new joint PDF becomes:

$$f_{R,\Theta}(r,\theta)=f_{X,Y}(r\cos(\theta),r\sin(\theta))\cdot r=\frac{1}{2\pi}e^{-r^2/2}\cdot r.$$

This expression confirms that R and Θ are independent, with R following a Rayleigh distribution with scale parameter 1 and Θ being uniformly distributed from $-\pi$ to π.

Consider that the transformation $Z=\tan(\Theta)$ maps Θ to Z. To calculate the PDF of Z, we use the transformation of variables formula. The derivative of $\tan^{-1}(z)$ with respect to z is:

$$\frac{d}{dz}\tan^{-1}(z)=\frac{1}{1+z^2}$$

This derivative represents how a small change in Z corresponds to a change in Θ, factoring into the new PDF. Combining the above derivation, the PDF of Z is given by:

$$f_Z(z)=f_\Theta(\tan^{-1}(z))\left|\frac{d}{dz}\tan^{-1}(z)\right|=\frac{1}{\pi(1+z^2)}.$$

(b) Since X and Y are i.i.d. Uniform$(0,1)$, the PDF of each variable, $f_X(x)$ and $f_Y(y)$, is:

$$f_X(x)=f_Y(y)=1\text{ for }x,y\in[0,1].$$

Define the transformations:

$$W=X\cdot Y,Z=\frac{X}{Y},\Rightarrow X=\sqrt{WZ},Y=\sqrt{\frac{W}{Z}}.$$

Problem 5 continued on next page...

Computing the partial derivatives, we have:

$$\frac{\partial X}{\partial W} = \frac{1}{2}Z^{1/2}W^{-1/2}, \quad \frac{\partial X}{\partial Z} = \frac{1}{2}W^{1/2}Z^{-1/2},$$
$$\frac{\partial Y}{\partial W} = \frac{1}{2}Z^{-1/2}W^{-1/2}, \quad \frac{\partial Y}{\partial Z} = -\frac{1}{2}W^{1/2}Z^{-3/2}.$$

Thus, the Jacobian determinant is:

$$J = \begin{vmatrix} \frac{1}{2}Z^{1/2}W^{-1/2} & \frac{1}{2}W^{1/2}Z^{-1/2} \\ \frac{1}{2}Z^{-1/2}W^{-1/2} & -\frac{1}{2}W^{1/2}Z^{-3/2} \end{vmatrix} = -\frac{1}{2}Z^{-1}.$$

Therefore, the joint PDF $f_{W,Z}(w,z)$ is given by:

$$f_{W,Z}(w,z) = f_{X,Y}(x,y)|J| = 1 \cdot \left| -\frac{1}{2}z^{-1} \right| = \frac{1}{2z},$$

for $x, y \in [0,1]$ (or w, z such that $0 \le w \le 1$, $z \ge \frac{w}{1-w}$, and $z \ge \frac{1-w}{w}$). The joint PDF of (W,Z), $(x, y \ge 0$ and $x, y \le 1)$, is:

$$f_{W,Z}(w,z) = \frac{1}{2z}$$

for $w \in (0,1)$ and $z \in \left(\frac{w}{1-w}, \frac{1-w}{w}\right)$.

(c) $R = \sqrt{X^2+Y^2}$ where X and Y are uniformly distributed on a unit disk i.e. $x^2+y^2 \le 1$ and we have $0 \le r \le 1$. Use the fact that the point X, Y is picked uniformly at random and thus $f_{X,Y}(x,y) = \frac{1}{\pi}$ over unit circle. Now, we have

$$S = X, R = \sqrt{X^2+Y^2} \Rightarrow X = S, Y = \pm\sqrt{R^2-S^2}. \tag{5}$$

This implies that $|s| < r < 1$.

Thus, the Jacobian determinant is:

$$J = \begin{vmatrix} \frac{\partial X}{\partial S} & \frac{\partial X}{\partial R} \\ \frac{\partial Y}{\partial S} & \frac{\partial Y}{\partial R} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \pm\frac{-2s}{2\sqrt{r^2-s^2}} & \pm\frac{-2r}{2\sqrt{r^2-s^2}} \end{vmatrix} = \mp\frac{r}{\sqrt{r^2-s^2}}.$$

Therefore, we have the joint distribution

$$\begin{aligned} f_{R,S}(r,s) =& \ f_{X,Y}(x,y)|J| \\ =& \ f_{X,Y}(s, \sqrt{r^2-s^2})\frac{r}{\sqrt{r^2-s^2}} + f_{X,Y}(s, -\sqrt{r^2-s^2})\frac{r}{\sqrt{r^2-s^2}} \\ =& \ \frac{2r}{\pi\sqrt{r^2-s^2}}, |s| < r < 1, -1 < s < 1. \end{aligned} \tag{6}$$

(d) The point (X,Y,Z) is chosen uniformly within the unit ball, which implies that the probability density function (PDF) for (X,Y,Z) is constant inside the ball and zero outside. The volume of the unit ball in $\mathbb{R}^3$ is $\frac{4}{3}\pi$, so the uniform density inside the ball is $\frac{3}{4\pi}$.

We convert the Cartesian coordinates (X,Y,Z) into spherical coordinates (R,θ,φ), where $R = \sqrt{X^2+Y^2+Z^2}$ ranges from 0 to 1 (radius of the unit ball), φ ranges from 0 to π (polar angle), θ ranges from 0 to 2π (azimuthal angle). The relationships between Cartesian and spherical coordinates are:

$$X = R\sin\varphi\cos\theta, Y = R\sin\varphi\sin\theta, Z = R\cos\varphi.$$

Problem 5 continued on next page...

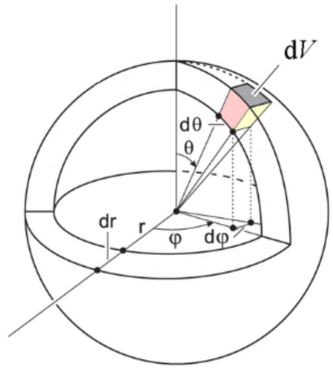

Figure 1: Spherical coordinates.

We calculate the Jacobian of the transformation from spherical coordinates to Cartesian coordinates. The determinant of the Jacobian matrix helps in finding the transformed joint PDF.

$$J = \begin{vmatrix} \frac{\partial X}{\partial R} & \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \varphi} \\ \frac{\partial Y}{\partial R} & \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \varphi} \\ \frac{\partial Z}{\partial R} & \frac{\partial Z}{\partial \theta} & \frac{\partial Z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \sin\varphi\cos\theta & -R\sin\varphi\sin\theta & R\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & R\sin\varphi\cos\theta & R\cos\varphi\sin\theta \\ \cos\varphi & 0 & -R\sin\varphi \end{vmatrix} = R^2\sin\varphi.$$

Given the uniform distribution in the unit ball, the joint PDF in spherical coordinates $f_{R,\varphi,\theta}(R,\varphi,\theta)$ is proportional to the volume element:

$$f_{R,\varphi,\theta}(r,\varphi,\theta) = \frac{3}{4\pi} r^2 \sin\varphi,$$

here we abuse the notation φ, θ as random variables and scalars at the same time.

Now, we need to find the joint PDF of R and Z. In spherical coordinates, $Z = R\cos\varphi$, so Z is directly related to R and φ, but not to θ. This allows us to integrate out θ since the distribution is symmetric around the origin and does not depend on the azimuthal angle θ. Therefore, performing the integration:

$$f_{R,\varphi}(r,\varphi) = \int_0^{2\pi} \frac{3}{4\pi} r^2 \sin\varphi \, d\theta = \frac{3}{4\pi} r^2 \sin\varphi \times 2\pi = \frac{3}{2} r^2 \sin\varphi$$

Therefore, we finally have:

$$f_{R,Z}(r,z) = f_{R,\varphi}(r,\varphi) \left| \det \begin{bmatrix} \frac{\partial r}{\partial r} & \frac{\partial r}{\partial z} \\ \frac{\partial \varphi}{\partial r} & \frac{\partial \varphi}{\partial z} \end{bmatrix} \right| = \frac{3}{2} r^2 \sin\varphi \left| \frac{\partial \varphi}{\partial z} \right| = \frac{3}{2} r^2 \sin\varphi \left| \frac{-1}{r\sin\varphi} \right| = \frac{3}{2} r, |z| \le r, r \le 1.$$

Problem 6

(Optional Challenging Problem) Let X and Y be i.i.d. Unif $(0,1)$, and $Z = \frac{X}{Y}$. Find the probability that the integer close to Z is odd.

Solution

The pdf of Z, given $Y = y$, is the pdf of X/y. Since X and Y are uniform on (0, 1), the joint pdf $f_{X,Y}(x,y) = 1$ for $x, y \in (0,1)$. Using a transformation of variables, $Z = \frac{X}{Y}$, and $X = ZY$, the Jacobian of the transformation from X, Y to Z, Y is Y. Thus, the joint pdf $f_{Z,Y}(z,y)$ is y if $0 \leq zy \leq 1$ and $0 < y \leq 1$, and zero otherwise.
Integrating out Y, we obtain the marginal pdf of Z:

$$f_Z(z) = \int_{\max(0,z)}^{1} y\, dy = \frac{1}{2}(1 - z^2) \quad \text{for } 0 \leq z \leq 1$$

For $z > 1$,

$$f_Z(z) = \int_0^{\frac{1}{z}} y\, dy = \frac{1}{2z^2}$$

The integer closest to Z is odd if Z rounds to 1, 3, 5, etc. We calculate the probability that Z falls within the intervals that round to each odd integer.
For the first few odd integers:

1. Round to 1: $0.5 \leq Z < 1.5$
2. Round to 3: $2.5 \leq Z < 3.5$, and so on.

Each probability is given by:

$$\mathbb{P}(n - 0.5 \leq Z < n + 0.5) = \int_{n-0.5}^{n+0.5} f_Z(z)\, dz$$

We calculate these probabilities for $n = 1, 3, 5, \ldots$, noting the pattern for larger n due to the rapidly decreasing pdf $f_Z(z)$. The final answer involves numerically integrating the pdf of Z over intervals corresponding to each odd integer's rounding bounds and summing these probabilities.