Probability & Statistics for EECS: TA Lecture #04 - Expectation

Problem 1 (BH CH4 #63)

A group of n people play "Secret Santa" as follows: each puts his or her name on a slip of paper in a hat, picks a name randomly from the hat (without replacement), and then buys a gift for that person. Unfortunately, they overlook the possibility of drawing one's own name, so some may have to buy gifts for themselves (on the bright side, some may like self-selected gifts better). Assume $n \geq 2$.

(a) Find the expected number of people who pick their own names.

(b) Find the expected number of pairs of people, A and B, such that A picks B 's name and B picks A 's name (where $A \neq B$ and order doesn't matter).

(c) Denote X as the number of people who pick their own names. What is the approximate distribution of X if n is large (specify the parameter value or values)? What does $P(X = 0)$ converge to as $n \to \infty$?

Solution

(a) Let I_j be the indicator r.v. for the jth person picking his or her own name. Then

$$E\left[I_j\right] = P\left(I_j = 1\right) = \frac{1}{n}.$$

By linearity, the expected number is

$$E\left[\sum_j I_j\right] = \sum_j E\left[I_j\right] = n \cdot E\left[I_j\right] = 1.$$

(b) Let I_{ij} the the indicator r.v. for the ith and jth persons having such a swap (for $i < j$). Then

$$E\left[I_{ij}\right] = P(i \text{ picks } j)P(j \text{ picks } i \mid i \text{ picks } j) = \frac{1}{n(n-1)}.$$

Alternatively, we can get this by counting: there are $n!$ permutations for who picks whom, of which $(n-2)!$ have i pick j and j pick i, giving $\frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$. So by linearity, the expected number is

$$\binom{n}{2} \cdot \frac{1}{n(n-1)} = \frac{1}{2}.$$

(c) By the Poisson paradigm, X is approximately Pois(1) for large n. As $n \to \infty$, $P(X = 0) \to \frac{1}{e}$, which is the probability of a Pois(1) r.v. being 0.