Probability & Statistics for EECS: Homework #04

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Problem 1

Let $C_i:$ the car is behind the door i, $i = 1, 2, 3$
$M_i:$ Monty opended the door i, $i = 1, 2, 3$
$A:$ we get the car after switching the door.

(a) Since there is no condition on which of doors 2 or 3 Monty opened, so with LOTP, we can get that

$$P(A) = P(A|C_1)P(C_1) + P(A|C_2)P(C_2) + P(A|C_3)P(C_3)$$

Since the car behind each door with equal probability, so $P(C_1) = P(C_2) = P(C_3) = \frac{1}{3}$. Also, if C_1 happened, then we cannot get the car after switching, so $P(A|C_1) = 0$.
And if C_2 happened, then Monty must open the door 3, so we must get the car after switching, so $P(A|C_2) = 1$.
Similarly, $P(A|C_3) = 1$.
So

$$\begin{aligned} P(A) &= 0 * \frac{1}{3} + 1 * \frac{1}{3} + 1 * \frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

So above all, the unconditional probability is $P(A) = \frac{2}{3}$.

(b) Since Monty opens the door 2, so with LOTP with extra conditioning, we can get that

$$P(A|M_2) = P(A|C_1, M_2)P(C_1|M_2) + P(A|C_2, M_2)P(C_2|M_2) + P(A|C_3, M_2)P(C_3|M_2)$$

Since that if C_1 happens, then we must not get the car after switching, so $P(A|C_1, M_2) = 0$.
And if C_2 happens, it is impossible for Monty to open the door 2, so $P(M_2|C_2) = 0$, and from the Bayes' Rule, we know that $P(C_2|M_2) = \frac{P(M_2|C_2)P(C_2)}{P(M_2)}$, since $P(M_2|C_2) = 0$, so $P(C_2|M_2) = 0$.
And if C_3 happens, then we must get the car after switching, so $P(A|C_3, M_2) = 1$.
So

$$\begin{aligned} P(A|M_2) &= 0 \cdot P(C_1|M_2) + P(A|C_2, M_2) \cdot 0 + 1 \cdot P(C_3|M_2) \\ &= P(C_3|M_2) \end{aligned}$$

With Bayes' Rule, we get that

$$= \frac{P(M_2|C_3)P(C_3)}{P(M_2)}$$

Using LOTP, we can get that

$$\begin{aligned} P(M_2) &= P(M_2|C_1)P(C_1) + P(M_2|C_2)P(C_2) + P(M_2|C_3)P(C_3) \\ &= p \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} \\ &= \frac{1}{3}(1 + p) \end{aligned}$$

After getting $P(M_2) = \frac{1}{3}(1 + p)$, we can get that

$$P(A|M_2) = \frac{1 \cdot \frac{1}{3}}{\frac{1}{3}(1 + p)}$$

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$$= \frac{1}{1+p}$$

So above all, the probability given that Monty opens door 2 is $P(A|M_2) = \dfrac{1}{1+p}$.

(c) Since Monty opens the door 3, so with LOTP with extra conditioning, we can get that

$$P(A|M_3) = P(A|C_1, M_3)P(C_1|M_3) + P(A|C_2, M_3)P(C_2|M_3) + P(A|C_3, M_3)P(C_3|M_3)$$

Since that if C_1 happens, then we must not get the car after switching, so $P(A|C_1, M_3) = 0$.
And if C_2 happens, then we must get the car after switching, so $P(A|C_2, M_3) = 1$.
And if C_3 happens, it is impossible for Monty to open the door 2, so $P(M_3|C_3) = 0$, and from the Bayes' Rule, we know that $P(C_3|M_3) = \dfrac{P(M_3|C_3)P(C_3)}{P(M_3)}$, since $P(M_3|C_3) = 0$, so $P(C_3|M_3) = 0$.
So

$$P(A|M_3) = 0 \cdot P(C_1|M_3) + 1 \cdot P(C_2|M_3) + P(A|C_3, M_3) \cdot 0$$
$$= P(C_2|M_3)$$

With Bayes' Rule, we get that

$$= \frac{P(M_3|C_2)P(C_2)}{P(M_3)}$$

Using LOTP, we can get that

$$P(M_3) = P(M_3|C_1)P(C_1) + P(M_3|C_2)P(C_2) + P(M_3|C_3)P(C_3)$$
$$= (1-p) \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}$$
$$= \frac{1}{3}(2-p)$$

After getting $P(M_3) = \dfrac{1}{3}(2-p)$, we can get that

$$P(A|M_3) = \frac{1 \cdot \frac{1}{3}}{\frac{1}{3}(2-p)}$$
$$= \frac{1}{2-p}$$

So above all, the probability given that Monty opens door 3 is $P(A|M_3) = \dfrac{1}{2-p}$.

Problem 2

(a) No.

Since the value of the PMF at n is proportional to $\frac{1}{n}$, so let $P(x=n) = k \cdot \frac{1}{n}$, where k is a constant.

According to the PMF's property, since the support of the distribution is $\{1, 2, 3, \cdots\}$, so $\sum\limits_{n=1}^{\infty} P(x=n) = 1$, so $\sum\limits_{n=1}^{\infty} k \cdot \frac{1}{n} = 1$, i.e. $k \cdot \sum\limits_{n=1}^{\infty} \frac{1}{n} = 1$.

However, from the knowledge that we have learn about infinite series in mathematical analysis, $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is divergent, is it is impossible to find a number k, s.t. $k \cdot \sum\limits_{n=1}^{\infty} \frac{1}{n} = 1$.

So above all, there do not have such distribution.

(b) Yes.

Since the value of the PMF at n is proportional to $\frac{1}{n^2}$, so let $P(x=n) = k \cdot \frac{1}{n^2}$, where k is a constant.

According to the PMF's property, since the support of the distribution is $\{1, 2, 3, \cdots\}$, so $\sum\limits_{n=1}^{\infty} P(x=n) = 1$, so $\sum\limits_{n=1}^{\infty} k \cdot \frac{1}{n^2} = 1$, i.e. $k \cdot \sum\limits_{n=1}^{\infty} \frac{1}{n^2} = 1$.

From the knowledge that we have learn about infinite series in mathematical analysis, $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent to $\frac{\pi^2}{6}$,

so $k \cdot \frac{\pi^2}{6} = 1$, i.e. $k = \frac{6}{\pi^2}$.

So there exist a $k = \frac{6}{\pi^2}$, s.t. $k \cdot \sum\limits_{n=1}^{\infty} \frac{1}{n^2} = 1$.

So above all, there have such distribution, and its PMF is that $P(x=n) = \frac{6}{\pi^2 n^2}$

Problem 3

Since Y is the next day of X, so

$$Y = \begin{cases} X+1, & X \le 6 \\ 1, & X = 7 \end{cases}$$

And since X takes values with equal probabilities, so Y also takes values with equal probabilities.

So $P(X = i) = \frac{1}{7}, P(Y = i) = \frac{1}{7}$, and X, Y have the same support $C = \{1, 2, 3, 4, 5, 6, 7\}$.

So $X \sim DUnif(C), Y \sim DUnif(C)$.

So X, Y have the same distribution.

As for $P(X < Y)$, from the relation between X, Y that we have mentioned above, we could know that $P(X < Y) = \sum\limits_{x=1}^{6} P(X = x, Y = x+1) = 6 \cdot \frac{1}{7} = \frac{6}{7}$.

So above all, X, Y have the same distribution. And $P(X < Y) = \frac{6}{7}$.

Problem 4

(a) Since the coins are randomly chosen, so each coin has the probability of $\frac{1}{2}$ to be chosen.
Suppose that Y is the number that the first coin heads, and Z is the number that the second coin heads.
So $Y \sim Bin(n, p_1)$, and $Z \sim Bin(n, p_2)$
Suppose that there are i times head. Then $P(Y = i) = \binom{n}{i} p_1^i (1 - p_1)^{n-i}, P(Z = i) = \binom{n}{i} p_2^i (1 - p_2)^{n-i}$.
Since the coins are randomly chosen with equal probability, so we have $\frac{1}{2}$'s probability to choose the first coin as well as the second coin.
So $P(X = i) = \frac{1}{2}P(Y = i) + \frac{1}{2}P(Z = i) = \frac{1}{2}\binom{n}{i} p_1^i (1 - p_1)^{n-i} + \frac{1}{2}\binom{n}{i} p_2^i (1 - p_2)^{n-i}$
So above all, the PMF of X is

$$P(X = i) = \frac{1}{2}\binom{n}{i} p_1^i (1 - p_1)^{n-i} + \frac{1}{2}\binom{n}{i} p_2^i (1 - p_2)^{n-i}$$

And its support is $\{0, 1, 2, \cdots, n\}$.

(b) Since $p_1 = p_2$, so with what we have in (a), let $p = p_1 = p_2$, then we can get that

$$P(X = i) = \binom{n}{i} p^i (1 - p)^{n-i}$$

And its support is $\{0, 1, 2, \cdots, n\}$.
From what we have learned, we can find that it is the same as $X \sim Bin(n, p)$.
So above all, if $p_1 = p_2 = p$, the distribution of X is $Bin(n, p)$.

(c) Intuitively, when $p_1 \neq p_2$, if we flip n times, and i times it heads.
For the first coin, $P(X = i) = \binom{n}{i} p_1^i (1 - p_1)^{n-i}$.
And for the second coin $P(Y = i) = \binom{n}{i} p_2^i (1 - p_2)^{n-i}$.
As n get bigger, for a fixed i, $\frac{P(X = i)}{P(Y = i)} = (\frac{p_1}{p_2} \cdot \frac{1 - p_2}{1 - p_1})^i \cdot (\frac{1 - p_1}{1 - p_2})^n$.
the rate of $P(X = i)$ and $P(Y = i)$ is getting extrame to 0 or ∞. The difference between them are getting bigger.
And it is much harder(impossible) to find a p s.t. $\binom{n}{i} p^i (1 - p)^{n-i} = \frac{1}{2}\binom{n}{i} p_1^i (1 - p_1)^{n-i} + \frac{1}{2}\binom{n}{i} p_2^i (1 - p_2)^{n-i}$.
Because as n get bigger, the difference bewteen two parts that are needed to be averaged are also getting bigger, so no p are suitable to average the two parts.
So intuitively, X is not Binomial for $p_1 \neq p_2$.

Problem 5

(a) Since $X \sim Bern(p), Y \sim Bern(\frac{1}{2})$, so $P(X=1)=p, P(X=0)=1-p, P(Y=1)=\frac{1}{2}, P(Y=0)=\frac{1}{2}$
And since X and Y are independent, so
$P(X \oplus Y=1) = P(X \neq Y) = P(X=1,Y=0)+P(X=0,Y=1) = P(X=1)P(Y=0)+P(X=0)P(Y=1) = p \cdot \frac{1}{2}+(1-p)\cdot\frac{1}{2} = \frac{1}{2}$.
$P(X \oplus Y=0) = P(X = Y) = P(X=1,Y=1)+P(X=0,Y=0) = P(X=1)P(Y=1)+P(X=0)P(Y=0) = p \cdot \frac{1}{2}+(1-p)\cdot\frac{1}{2} = \frac{1}{2}$.
So $P(X \oplus Y=1)=\frac{1}{2}, P(X \oplus Y=0)=\frac{1}{2}$
The PMF is same as what we have learned that $X \oplus Y \sim Bern(\frac{1}{2})$.
So above all, the distribution of $X \oplus Y$ is that $Bern(\frac{1}{2})$.

(b) From (a) we know that $P(X \oplus Y=1)=P(X \oplus Y=0)=\frac{1}{2}$.
$P(X \oplus Y=1, X=1) = P(X=1,Y=0) = P(X=1)P(Y=0) = \frac{1}{2}p$, and $P(X \oplus Y=1)P(X=1)=\frac{1}{2}p$.
$P(X \oplus Y=1, X=0) = P(X=0,Y=1) = P(X=0)P(Y=1) = \frac{1}{2}(1-p)$, and $P(X \oplus Y=0)P(X=0) = \frac{1}{2}(1-p)$.
$P(X \oplus Y=0, X=1) = P(X=1,Y=1) = P(X=1)P(Y=1) = \frac{1}{2}p$, and $P(X \oplus Y=1)P(X=1)=\frac{1}{2}p$.
$P(X \oplus Y=0, X=0) = P(X=0,Y=0) = P(X=0)P(Y=0) = \frac{1}{2}(1-p)$, and $P(X \oplus Y=0)P(X=0) = \frac{1}{2}(1-p)$.

So for all $X \oplus Y, X$, we have $P(X \oplus Y=a, X=b) = P(X \oplus Y=a)P(X=b)$, where $a,b=0,1$.
So $X \oplus Y, X$ are independent.

As for $X \oplus Y$ and X
$P(X \oplus Y=1, Y=1) = P(X=0,Y=1) = P(X=0)P(Y=1) = \frac{1}{2}p$, and $P(X \oplus Y=1)P(Y=1)=\frac{1}{4}$.
$P(X \oplus Y=1, Y=0) = P(X=1,Y=0) = P(X=1)P(Y=0) = \frac{1}{2}(1-p)$, and $P(X \oplus Y=1)P(Y=0)=\frac{1}{4}$.
$P(X \oplus Y=0, Y=1) = P(X=1,Y=1) = P(X=1)P(Y=1) = \frac{1}{2}p$, and $P(X \oplus Y=0)P(Y=1)=\frac{1}{4}$.
$P(X \oplus Y=0, Y=0) = P(X=0,Y=0) = P(X=0)P(Y=0) = \frac{1}{2}(1-p)$, and $P(X \oplus Y=0)P(Y=0)=\frac{1}{4}$.
If $p=\frac{1}{2}$, then for all $X \oplus Y, Y$, we have $P(X \oplus Y=a, Y=b) = P(X \oplus Y=a)P(Y=b)$, for all $a,b=0,1$.
If $p\neq\frac{1}{2}$, then for all $X \oplus Y, Y$, we have $P(X \oplus Y=a, Y=b) \neq P(X \oplus Y=a)P(Y=b)$, for all $a,b=0,1$.
So when $p=\frac{1}{2}$, $X \oplus Y, Y$ are independent, and when $p\neq\frac{1}{2}$, $X \oplus Y, Y$ are not independent.

So above all, $X \oplus Y, X$ are independent.
When $p=\frac{1}{2}$, $X \oplus Y, Y$ are independent.
When $p\neq\frac{1}{2}$, $X \oplus Y, Y$ are not independent.

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(c) <1>. We can prove that $Y_J \sim Bern(\frac{1}{2})$ with Mathematical induction.
Since all $X_i \sim Bern(\frac{1}{2}), i = 1, 2, \cdots, n$. So $P(X_i = 1) = P(X_i = 0) = \frac{1}{2}$.
When $|J| = 1$, $P(Y_J = 1) = P(X_j = 1, j \in J) = \frac{1}{2}$. Similarly, $P(Y_J = 0) = \frac{1}{2}$.

When $|J| = 2$, since $X_1, \cdots, X_n$ are i.i.d.
So $P(Y_J = 1) = P(X_{j_1} \oplus X_{j_2} = 1) = P(X_{j_1} = 1)P(X_{j_2} = 0) + P(X_{j_1} = 0)P(X_{j_2} = 1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.
Similarly, $P(Y_J = 0) = \frac{1}{2}$.

And when $|J'| = k$, $k = 1, 2, \cdots, n-1$, for a specific J'.
Let $J = J' \cup \{i\}$, where $i \notin J'$. So $|J| = k + 1$.
And we know that $P(Y_{J'} = 1) = P(Y_{J'} = 0) = \frac{1}{2}$.
So for all $i \notin J'$,$P(Y_J = 1) = P(X_i \oplus Y_{J'} = 1) = P(X_i = 1)P(Y_{J'} = 0) + P(X_i = 0)P(Y_{J'} = 1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.
Similarly, $P(Y_J = 0) = \frac{1}{2}$.
And for all J', all the above arguments are valid.
So for all J, $P(Y_J = 1) = P(Y_J = 0) = \frac{1}{2}$.
So $Y_J \sim Bern(\frac{1}{2})$.

<2>. Let J, K be two of the $2^n - 1$ R.V.s, and $J \neq K$.
From the Venn diagram, we can devide the $J \cup K$ part into three parts.
Let $A = J \cap K, B = J \cap K^c, C = J^c \cap K$.
So $J = A \cup B, K = A \cup C$.
1. If $A = \emptyset$, then $B = J, C = K$, i.e. $J \cap K = \emptyset$, so it is obvious that J, K are independent since they do not have any intersection.

As for $A \neq \emptyset$.
2. If $J \subset K$ or $K \subset J$, without loss of generality, take $J \subset K$.
And let $K = J \cup C, J \cap C = \emptyset$, so $Y_K = Y_J \oplus Y_C$.
Since $Y_J, Y_C \sim Bern(\frac{1}{2})$,
so $P(Y_J = a, Y_K = b) = P(Y_J = a, Y_C = (a \oplus b))$
since $J \cap C = \emptyset$, so Y_J and Y_C are independent, so $P(Y_J = a, Y_K = b) = P(Y_J = a)P(Y_C = (a \oplus b)) = \frac{1}{4}$.
And also $P(Y_J = a)P(Y_K = b) = \frac{1}{4}$.
So $P(Y_J = a, Y_K = b) = P(Y_J = a)P(Y_K = b)$
So the R.V.s are pairwise independent.

3. If $J \not\subset K$ and $K \not\subset J$, then $Y_J = Y_A \oplus Y_B, Y_K = Y_A \oplus Y_C$. And $Y_A, Y_B, Y_C \sim Bern(\frac{1}{2})$.With LOTP, we can get that

$$\begin{gathered} P(Y_J = a, Y_K = b) \\ = P(Y_A \oplus Y_B = a, Y_A \oplus Y_C = b | Y_A = 1)P(Y_A = 1) + P(Y_A \oplus Y_B = a, Y_A \oplus Y_C = b | Y_A = 0)P(Y_A = 0) \end{gathered}$$

Since A, B, C are devided to three parts, so A, B, C are independent. So the original formula

$$= \frac{1}{2}P(1 \oplus Y_B = a)P(1 \oplus Y_C = b) + \frac{1}{2}P(0 \oplus Y_B = a)P(0 \oplus Y_C = b)$$

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And since the XOR equaltion has the unique solution, and $Y_B, Y_C \sim Bern(\frac{1}{2})$, so the original formula

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$$
$$= \frac{1}{4}$$

And since $P(Y_J = a)P(Y_K = b) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$,
so for all $a, b = 0, 1$, we have $P(Y_J = a, Y_K = b) = P(Y_J = a)P(Y_K = b)$.
So above all, the R.V.s are pairwise independent.

<3>. For a situation that J, K, L are three of the $2^n - 1$ R.V.s, and $J \neq K, J \neq L, K \neq L$.
Let $J \cap K = \emptyset$, and $L = J \cup K$, then $Y_L = Y_J \oplus Y_K$.
And $Y_J, Y_K, Y_L \sim Bern(\frac{1}{2})$.
But $P(Y_J = 1, Y_K = 1, Y_L = 1) = 0$, since when $Y_J = Y_K = 1, Y_L = Y_J \oplus Y_K = 0 \neq 1$.
However, $P(Y_J = 1)P(Y_K = 1)P(Y_L = 1) = (\frac{1}{2})^3$,
so $P(Y_J = 1, Y_K = 1, Y_L = 1) \neq P(Y_J = 1)P(Y_K = 1)P(Y_L = 1)$ in this situation.
So the R.V.s are not independent.

So above all, $Y_J \sim Bern(\frac{1}{2})$, the R.V.s are pairwise independent, and not independent had all been proved,