Problem 1

There are n balls in a jar, labeled with the numbers $1, 2, \cdots, n$. A total of k balls are drawn, one by one *with replacement*, to obtain a sequence of numbers.

(a) What is the probability that the sequence obtained is strictly increasing?

(b) What is the probability that the sequence obtained is increasing? (Note: In this book, "increasing" means "nondecreasing".)

Solution

(a) Given the problem settings, there are totally n^k feasible sequences with equal probability. There are two points needed to be mentioned. On the one hand, one strictly increasing sequence must contains all distinct numbers. On the other hand, one combination of distinct numbers can only form one strictly increasing sequence. Thus there are $\binom{n}{k}$ strictly increasing sequences among those feasible sequences. The probability is given as

$$\Pr(\text{obtaining a strictly increasing sequence}) = \frac{\binom{n}{k}}{n^k}.$$

(b) Denote how many times each number was chosen by $t_i, i = 1, 2, \ldots, n$. The problem is equivalent to the number of distinct nonnegative integer-valued vectors $(t_1, t_2, \ldots, t_n)$ that satisfy the equation

$$t_1 + t_2 + \cdots + t_n = k.$$

By Bose-Einstein, there are $\binom{n+k-1}{n-1}$ (or $\binom{n+k-1}{k}$) sequences. Thus the probability is given as

$$\Pr(\text{obtaining an increasing sequence}) = \frac{\binom{n+k-1}{n-1}}{n^k}.$$

Problem 2

Inclusion-exclusion: BH CH1 #52
Alice attends a small college in which each class meets only once a week. She is deciding between 30 non-overlapping classes. There are 6 classes to choose from for each day of the week, Monday through Friday. Trusting in the benevolence of randomness, Alice decides to register for 7 randomly selected classes out of the 30, with all choices equally likely. What is the probability that she will have classes every day, Monday through Friday? (This problem can be done either directly using the naive definition of probability, or using inclusion-exclusion.)

Solution

We will solve this both by direct counting and using inclusion-exclusion.

Direct Counting Method: There are two general ways that Alice can have classes every day: either she has 2 days with 2 classes and 3 days with 1 class, or she has 1 day with 3 classes, and has 1 class on each of the other 4 days. The number of possibilities for the former is $\binom{5}{2}\binom{6}{2}^2 6^3$ (choose the 2 days when she has 2 classes, and then select 2 classes on those days and 1 classes for the other days). The number of possibilities for the latter is $\binom{5}{1}\binom{6}{3}6^4$. So the probability is

$$\frac{\binom{5}{2}\binom{6}{2}^2 6^3 + \binom{5}{1}\binom{6}{3}6^4}{\binom{30}{7}} = \frac{114}{377} \approx 0.302$$

Inclusion-Exclusion Method: We will use inclusion-exclusion principle to find the probability of the complement event, which is the event that she has at least one day with no classes. Let B_i represent the event that there is no class on the ith day. Lat A represent the event that Alice has classes every day. Then

$$P(A^c) = Pr\{\text{at least one day with no classes}\} = P(\bigcup_{i=1}^{5} B_i) = \sum_{i=1}^{5} P(B_i) - \sum_{i<j} P(B_i \cap B_j) + \sum_{i<j<k} P(B_i \cap B_j \cap B_k)$$

where terms with the intersection of 4 or more B_i's are not needed since Alice must have at least 2 days. We have

$$P(B_1) = \frac{\binom{24}{7}}{\binom{30}{7}}, P(B_1 \cap B_2) = \frac{\binom{18}{7}}{\binom{30}{7}}, P(B_1 \cap B_2 \cap B_3) = \frac{\binom{12}{7}}{\binom{30}{7}}$$

and similarly for the other intersections. So

$$P(\bigcup_{i=1}^{5} B_i) = 5\frac{\binom{24}{7}}{\binom{30}{7}} - \binom{5}{2}\frac{\binom{18}{7}}{\binom{30}{7}} + \frac{5}{3}\frac{\binom{12}{7}}{\binom{30}{7}} = \frac{263}{377}.$$

Therefore,

$$P(A) = 1 - P(A^c) = \frac{114}{377} \approx 0.302.$$