Probability & Statistics for EECS: Homework #6 Solutions

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Problem 1

The Cauchy distribution has PDF

$$f(x) = \frac{1}{\pi\left(1 + x^2\right)}$$

for all x. Find the CDF of a random variable with the Cauchy PDF. Hint: Recall that the derivative of the inverse tangent function $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$.

Solution

Given that the PDF of the Cauchy distribution is

$$f(x) = \frac{1}{\pi(1 + x^2)},$$

and the hint that the derivative of the inverse tangent function $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$, we can calculate the CDF of the Cauchy distribution by definition, *i.e.*, integrating the PDF over range $(-\infty, x]$.
Therefore, the CDF of the Cauchy distribution $F(x)$ is as follows:

$$F(x) = \int_{-\infty}^{x} f(t)dt = \frac{1}{\pi}\tan^{-1}(t)\Big|_{-\infty}^{x} = \frac{1}{\pi}\tan^{-1}(x) - \frac{1}{\pi}(-\frac{\pi}{2}) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x),$$

where $x \in (-\infty, \infty)$.

Problem 2

The Pareto distribution with parameter $a > 0$ has PDF

$$f(x) = \frac{a}{x^{a+1}}$$

for $x \geq 1$ (and 0 otherwise). This distribution is often used in statistical modeling. Find the CDF of a Pareto r.v. with parameter a; check that it is a valid CDF.

Solution

Given that the PDF of the Pareto distribution with parameter $\alpha > 0$ is

$$f(x) = \begin{cases} \frac{a}{x^{a+1}}, & x \geq 1 \\ 0, & \text{Otherwise} \end{cases},$$

we can calculate the CDF of the Pareto distribution by definition, *i.e.*, integrating the PDF over range $(-\infty, x]$.

Therefore, the CDF of the Pareto distribution $F(x)$ is as follows:

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{1}^{x} \frac{a}{t^{a+1}} dt = -\frac{1}{t^a}\Big|_1^x = 1 - \frac{1}{x^a},$$

where $x \in [1, \infty)$. When $x \in (-\infty, 1)$, by definition, $F(x) = 0$.

We then check if $F(x)$ is a valid CDF as follows:

- Increasing: Due to the fact that $\frac{1}{x^a}, a > 0$ is decreasing over $[1, \infty)$, CDF $F(x) = 1 - \frac{1}{x^a}$ is increasing over the corresponding support $[1, \infty)$.
- Right-continuous: Due to the fact that $1 - \frac{1}{x^a}, a > 0$ is continuous over $[1, \infty)$, CDF $F(x)$ is right-continuous over the corresponding support $[1, \infty)$.
- Convergence to 0 and 1 in the limits: Due to the fact that $F(x) = 0, x < 1$ and $\lim_{x\to\infty} \frac{1}{x^a} = 0$ when $a > 0$, CDF $F(x)$ have its limits as follows:

$$\lim_{x\to-\infty} F(x) = 0, \lim_{x\to\infty} F(x) = 1 - 0 = 1.$$

In summary, the CDF $F(x)$ is valid.

Problem 3

The *Beta distribution* with parameters $a = 3$, $b = 2$ has PDF

$$f(x) = 12x^2(1 - x),\ \text{for} 0 < x < 1.$$

Let X have this distribution.

(a) Find the CDF of X.

(b) Find $P(0 < X < 1/2)$.

(c) Find the mean and variance of X (without quoting results about the Beta distribution).

Solution

(a) The CDF of X is

$$\begin{aligned} F(X) &= \int_0^x f(t)dt = \int_0^x 12t^2(1-t)dt \\ &= \int_0^x 12t^2 dt - \int_0^x 12t^3 dt \\ &= 4t^3|_0^x - 3t^4|_0^x \\ &= x^3(4 - 3x), \quad \text{for } 0 < x < 1 \end{aligned}$$

(b) According to CDF $F(x)$, $P(0 < x < 1/2) = F(1/2) = \frac{5}{16}$.

(c) According to PDF, the mean of X is

$$\begin{aligned} E(X) &= \int_0^1 xf(x)dx = \int_0^1 12x^2(1-x)dx \\ &= \int_0^1 12x^3 dx - \int_0^1 12x^4 dx \\ &= \frac{3}{5} \end{aligned}$$

We have

$$\begin{aligned} E(X^2) &= \int_0^1 x^2 f(x)dx = \int_0^1 12x^4(1-x)dx \\ &= \int_0^1 12x^4 dx - \int_0^1 12x^5 dx \\ &= \frac{2}{5} \end{aligned}$$

Thus, we have

$$Var(X) = E(X^2) - EX^2 = \frac{1}{25}$$

Problem 4

The Exponential is the analog of the Geometric in continuous time. This problem explores the connection between Exponential and Geometric in more detail, asking what happens to a Geometric in a limit where the Bernoulli trials are performed faster and faster but with smaller and smaller success probabilities.
Suppose that Bernoulli trials are being performed in continuous time; rather than only thinking about first trial, second trial, etc., imagine that the trials take place at points on a timeline. Assume that the trials are at regularly spaced times 0, Δt, $2\Delta t$,... , where Δt is a small positive number. Let the probability of success of each trial be $\lambda\Delta t$, where λ is a positive constant. Let G be the number of failures before the first success (in discrete time), and T be the time of the first success (in continuous time).

(a) Find a simple equation relating G to T. Hint: Draw a timeline and try out a simple example.

(b) Find the CDF of T. Hint: First find $P(T > t)$.

(c) Show that as $\Delta t \to 0$, the CDF of T converges to the Expo(λ) CDF, evaluating all the CDFs at a fixed $t \geq 0$.

Solution

(a) $T = G\Delta t$.

(b) For $t \geq 0$, $P(T > t) = P\left(G > \frac{t}{\Delta t}\right) = P\left(\text{no success in the first } \left\lfloor \frac{t}{\Delta t} \right\rfloor \text{ trials}\right) = (1 - \lambda\Delta t)^{\left\lfloor \frac{t}{\Delta t} \right\rfloor + 1}$. Thus The CDF of T is

$$P(T \leq t) = 1 - P(T > t) = 1 - (1 - \lambda\Delta t)^{\left\lfloor \frac{t}{\Delta t} \right\rfloor + 1}.$$

(c) As $\Delta t \to 0$,

$$\begin{aligned}\lim_{\Delta t \to 0} P(T \leq t) &= \lim_{\Delta t \to 0} \left[1 - (1 - \lambda\Delta t)^{\left\lfloor \frac{t}{\Delta t} \right\rfloor + 1}\right] = 1 - \lim_{\Delta t \to 0} (1 - \lambda\Delta t)^{\frac{t}{\Delta t}} \\ &= 1 - \lim_{\Delta t \to 0} \left[(1 - \lambda\Delta t)^{\frac{1}{\lambda\Delta t}}\right]^{\lambda t} = 1 - e^{-\lambda t}.\end{aligned}$$

Thus for $t \geq 0$, the CDF of T converges to the Expo(λ) CDF as $\Delta t \to 0$.

Problem 5

Let $Z \sim \mathcal{N}(0,1)$, and c be a nonnegative constant. Find $E(\max(Z-c,0))$, in terms of the standard Normal CDF Φ and PDF φ.

Let φ be the PDF of $\mathcal{N}(0,1)$, then we have

$$
\begin{aligned}
E(\max(Z-c,0)) &= \int_{-\infty}^{\infty} \max(z-c,0)\varphi(z)\,\mathrm{d}z \\
&= \int_{c}^{\infty} (z-c)\varphi(z)\,\mathrm{d}z \\
&= \int_{c}^{\infty} z\varphi(z)\,\mathrm{d}z - c\int_{c}^{\infty} \varphi(z)\,\mathrm{d}z \\
&= \frac{-1}{\sqrt{2\pi}} e^{-z^2/2}\bigg|_{c}^{\infty} - c(1-\Phi(c)) \\
&= \frac{1}{\sqrt{2\pi}} e^{-c^2/2} - c(1-\Phi(c)) \\
&= \varphi(c) + c\Phi(c) - c
\end{aligned} \tag{1}
$$

Problem 6

(Optional Challenging Problem) Let $X \sim \mathcal{N}(0,1)$, its corresponding CDF is denoted as Φ and the corresponding PDF is denoted as φ.

(a) If $x > 0$, show the following inequality holds:

$$\frac{x}{x^2+1}\varphi(x) \leq 1 - \Phi(x) \leq \frac{1}{x}\varphi(x).$$

(b) Define the function $g(x)$ as follows:

$$g(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt, \forall x \geq 0.$$

Show the following inequality holds:

$$g(x) \leq e^{-x^2}, \forall x \geq 0.$$

Solution

(a) Let X be a standard normal random variable. These notes present upper and lower bounds for the complementary cumulative distribution function

$$\Phi^c(x) = P(X > x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt.$$

An upper bound is easy to obtain. Since $t/x > 1$ for t in (x, ∞), we have

$$\begin{aligned}
\Phi^c(x) &= \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt \\
&< \frac{1}{\sqrt{2\pi}} \int_t^{\infty} \frac{t}{x} e^{-t^2/2} dt \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}.
\end{aligned}$$

We can also show there is a lower bound

$$\Phi^c(x) > \frac{1}{\sqrt{2\pi}} \frac{x}{x^2+1} e^{-x^2/2}.$$

To prove this lower bound, define

$$g(x) = \Phi^c(x) - \frac{1}{\sqrt{2\pi}} \frac{x}{x^2+1} e^{-x^2/2}.$$

We will show that $g(x)$ is always positive. Clearly $g(x) > 0$. From the derivative

$$g'(x) = -\frac{2}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{(x^2+1)^2}$$

we see that g is strictly decreasing. Since the limit of $g(x)$ as x goes infinity vanishes, g must always be positive.

Combining the inequalities above we have

$$\frac{x}{x^2+1} < \sqrt{2\pi} e^{x^2/2} \Phi^c(x) < \frac{1}{x}.$$

(b) The function $g(x)$ as defined is related to the complementary error function, which is used in statistics to describe the tail distribution of the normal curve. To prove that $g(x) \leq e^{-x^2}$ for all $x \geq 0$, we can start by expressing $g(x)$ in terms of the error function and then use known inequalities to establish the desired result.

Let's first write down $g(x)$:

$$g(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

We know that the complementary error function, denoted as $\operatorname{erfc}(x)$, is defined as:

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

So we can say that $g(x) = \operatorname{erfc}(x)$.

The inequality to prove is:

$$\operatorname{erfc}(x) \leq e^{-x^2}, \text{ for } x \geq 0$$

To prove this, we'll use a standard approach that involves comparing the rate of decrease of both sides as x increases. Specifically, we'll look at the derivatives of both sides with respect to x and show that the derivative of $\operatorname{erfc}(x)$ is always less than or equal to the derivative of e^{-x^2}, and that $\operatorname{erfc}(x)$ and e^{-x^2} are equal at $x = \infty$, from which the result will follow.

The derivative of e^{-x^2} with respect to x is:

$$\frac{d}{dx} e^{-x^2} = -2xe^{-x^2}$$

And the derivative of $\operatorname{erfc}(x)$ is:

$$\frac{d}{dx} \operatorname{erfc}(x) = -\frac{2}{\sqrt{\pi}} e^{-x^2}$$

We can now compare the absolute values of the derivatives for $x \geq 0$:

$$2xe^{-x^2} \text{ versus } \frac{2}{\sqrt{\pi}} e^{-x^2}$$

Since $x \geq 0$ and $\sqrt{\pi} > 1$, it is clear that:

$$-2xe^{-x^2} \leq -\frac{2}{\sqrt{\pi}} e^{-x^2}$$

This shows that the rate at which e^{-x^2} decreases is always greater than or equal to the rate at which $\operatorname{erfc}(x)$ decreases for $x \geq 0$. Since both functions tend to 0 as x approaches infinity, and e^{-x^2} decreases faster, it follows that for all $x \geq 0$:

$$\operatorname{erfc}(x) \leq e^{-x^2}.$$