Probability & Statistics for EECS: Homework #3 Solution

Problem 1

A system composed of 5 homogeneous devices is shown in the following figure. It is said to be functional when there exists at least one end-to-end path that devices on such path are all functional. For such a system, if each device, which is independent of all other devices, functions with probability p, then what is the probability that the system functions? Such a probability is also called the system reliability.

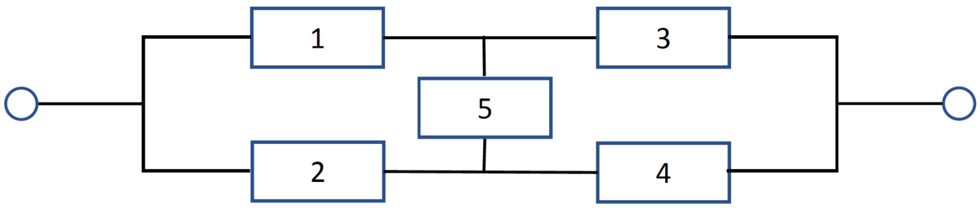

Figure 1: An illustration of the system composed of 5 homogeneous devices.

Solution

Method 1: In order to calculate the probability of system activation, we can classify according to the number of activated devices:

- If two devices are activated (1/3, 2/4): $P_2 = 2 \times p^2(1-p)^3$
- If three devices are activated (all cases except 1/2/5, 3/4/5): $P_3 = (C_5^3 - 2) \times p^3(1-p)^2 = 8p^3(1-p)^2$
- If four devices are activated (Four of any five devices are activated): $P_4 = 5 \times p^4(1-p)$
- If five devices are activated: $P_5 = p^5$

So the probability that the system is activated is:

$$\begin{aligned} P &= P_2 + P_3 + P_4 + P_5 \\ &= 2p^5 - 5p^4 + 2p^3 + 2p^2 \end{aligned}$$

Method 2: There are two situations whether the device 5 is functional.

1. ***case 1:*** device 5 is functional, the probability is p, the system is functional if and only if device 1 or device 2 is functional and device 3 or device 4 is functional, the probability is $(p+p-p^2)*(p+p-p^2)$. Therefore, the probability that the system functions is $p*(p+p-p^2)*(p+p-p^2)$.
2. ***case 2:*** device 5 is not functional, the probability is $1-p$, the system is functional if and only if both device 1 and device 3 are functional or both device 2 and device 4 are functional, the probability is $p^2+p^2-p^4$. Therefore, the probability that the system functions is $(1-p)*(p^2+p^2-p^4)$.

In conclusion, taking both case 1 and case 2 into consideration, the probability that the system functions is $p*(p+p-p^2)*(p+p-p^2)+(1-p)*(p^2+p^2-p^4) = 2p^5 - 5p^4 + 2p^3 + 2p^2$.

Problem 2

(a) Suppose that in the population of gamers, being good at Genshin Impact is independent of being good at Apex (with respect to some measure of "good"). A certain school club has a simple admission procedure: admit an applicant if and only if the applicant is good at Genshin Impact or is good at Apex.

Intuitive explain why it makes sense that among gamers that the club admits, being good at Apex is negatively associated with being good at Genshin Impact, *i.e.*, conditioning on being good at Apex decreases the chance of being good at Genshin Impact.

(b) (Berkson's paradox) Show that if A and B are independent and $C = A \cup B$, then A and B are conditionally dependent given C (as long as $P(A \cap B) > 0$ and $P(A \cup B) < 1$), with

$$P(A \mid B \cap C) < P(A \mid C).$$

Solution

(a) Intuitively, suppose we divide the people who are admitted into three types:

- Good at Apex, but not Genshin
- Good at Genshin, but not Apex
- Good at both Apex and Genshin

So if we are given the condition that the person is good at Apex, then the person can only in the first and third type. However, if we don't have the condition, the person can in all three types. So, the condition of the person good at Apex decrease the chance of being good at Genshin Impact.

(b) The inequality which should be proved is equal to the inequality as below:

$$\begin{aligned}
& P(A|B \cap C) < P(A|C) && (1)\\
\iff & \frac{P(A,B|C)}{P(B|C)} < P(A|C) && (2)\\
\iff & P(A,B|C) < P(A|C)P(B|C) && (3)\\
\iff & \frac{P(A,B,C)}{P(C)} < \frac{P(A,C)}{P(C)}\frac{P(B,C)}{P(C)} && (4)\\
\iff & \frac{P(A,B)}{P(C)} < \frac{P(A)}{P(C)}\frac{P(B)}{P(C)} && (5)\\
\iff & \frac{P(A)P(B)}{P(C)} < \frac{P(A)}{P(C)}\frac{P(B)}{P(C)} && (6)\\
\iff & P(C) < 1 && (7)
\end{aligned}$$

where (5) is because $C = A \cup B$, and (6) is because A, B are unconditionally independent. Because $P(C) = P(A \cup B)$, so the ineuqation of $P(A|B \cap C) < P(A|C)$ holds.

Problem 3

A fair die is rolled repeatedly, and a running total is kept (which is, at each time, the total of all the rolls up until that time). Let p_n be the probability that the running total is ever exactly n (assume the die will always be rolled enough times so that the running total will eventually exceed n, but it may or may not ever equal n).

(a) Write down a recursive equation for p_n. Your equation should be true for all positive integers n, so give a definition of p_0 and p_k for $k < 0$ so that the recursive equation is true for small values of n.

(b) Find p_7.

(c) Give an intuitive explanation for the fact that $p_n \to 1/3.5 = 2/7$ as $n \to \infty$.

Solution

(a) We will find something to condition on to reduce the case of interest to earlier, simpler cases. This is achieved by the useful strategy of first step anaysis. Let p_n be the probability that the running total is ever exactly n. Note that if, for example, the first throw is a 3, then the probability of reaching n exactly is p_{n-3} since starting from that point, we need to get a total of $n-3$ exactly. So

$$p_n = \frac{1}{6}\left(p_{n-1} + p_{n-2} + p_{n-3} + p_{n-4} + p_{n-5} + p_{n-6}\right)$$

where we define $p_0 = 1$ (which makes sense anyway since the running total is 0 before the first toss) and $p_k = 0$ for $k < 0$. (b) Using the recursive equation in (a), we have

$$\begin{aligned} &p_1 = \tfrac{1}{6}, \quad p_2 = \tfrac{1}{6}\left(1+\tfrac{1}{6}\right), \quad p_3 = \tfrac{1}{6}\left(1+\tfrac{1}{6}\right)^2 \\ &p_4 = \tfrac{1}{6}\left(1+\tfrac{1}{6}\right)^3, \quad p_5 = \tfrac{1}{6}\left(1+\tfrac{1}{6}\right)^4, \quad p_6 = \tfrac{1}{6}\left(1+\tfrac{1}{6}\right)^5 \end{aligned}$$

Hence,

$$p_7 = \frac{1}{6}\left(p_1 + p_2 + p_3 + p_4 + p_5 + p_6\right) = \frac{1}{6}\left(\left(1+\frac{1}{6}\right)^6 - 1\right) \approx 0.2536$$

(c) An intuitive explanation is as follows. The average number thrown by the die is $(total\ of\ dots)/6$, which is $21/6 = 7/2$, so that every throw adds on an average of $7/2$. We can therefore expect to land on 2 out of every 7 numbers, and the probability of landing on any particular number is $2/7$. A mathematical derivation (which was not requested in the problem) can be given as follows:

$$\begin{aligned} &p_{n+1} + 2p_{n+2} + 3p_{n+3} + 4p_{n+4} + 5p_{n+5} + 6p_{n+6} \\ =&p_{n+1} + 2p_{n+2} + 3p_{n+3} + 4p_{n+4} + 5p_{n+5} \\ &+ p_n + p_{n+1} + p_{n+2} + p_{n+3} + p_{n+4} + p_{n+5} \\ =&p_n + 2p_{n+1} + 3p_{n+2} + 4p_{n+3} + 5p_{n+4} + 6p_{n+5} \\ =&\cdots \\ =&p_{-5} + 2p_{-4} + 3p_{-3} + 4p_{-2} + 5p_{-1} + 6p_0 = 6. \end{aligned}$$

Taking the limit of the lefthand side as n goes to ∞, we have

$$(1+2+3+4+5+6)\lim_{n\to\infty} p_n = 6,$$

so $\lim_{n\to\infty} p_n = 2/7$

Problem 4

There are n types of toys, which you are collecting one by one. Each time you buy a toy, it is randomly determined which type it has, with equal probabilities. Let $p_{i,j}$ be the probability that just after you have bought your ith toy, you have exactly j toy types in your collection, for $i \geq 1$ and $0 \leq j \leq n$.

(a) Find a recursive equation expressing $p_{i,j}$ in terms of $p_{i-1,j}$ and $p_{i-1,j-1}$, for $i \geq 2$ and $1 \leq j \leq n$.

(b) Describe how the recursion from (a) can be used to calculate $p_{i,j}$.

Solution

(a) There are two ways to have exactly j toy types just after buying your ith toy: either you have exactly $j-1$ toy types just after buying your $i-1st$ toy and then the ith toy you buy is of a type you don't already have, or you already have exactly j toy types just after buying your $i-1st$ toy and then the ith toy you buy is of a type you do already have. Conditioning on how many toy types you have just after buying your $i-1st$ toy,

$$p_{ij} = p_{i-1,j-1}\frac{n-j+1}{n} + p_{i-1,j}\frac{j}{n}$$

for $i \geq 2$ and $1 \leq j \leq n$, with $j \leq i$ and

$$p_{i,j} = 0, \forall j > i.$$

(b) First note that $p_{11} = 1$, and $p_{ij} = 0$ for $j = 0$ or $j > i$. Now suppose that we have computed $p_{i-1,1}, p_{i-1,2}, ..., p_{i-1,i-1}$ for some $i2$. Then we can compute $p_{i,1}, p_{i,2}, ..., p_{i,i}$ using the recursion from (a). We can then compute $p_{i+1,1}, p_{i+1,2}, ..., p_{i+1,i+1}$ using the recursion from (a), and so on. By induction, it follows that we can obtain any desired p_{ij} recursively by this method.

Problem 5

Link is an immortal drunk man who wanders around randomly on the integers. He starts at the origin, and at each step he moves 1 unit to the right or 1 unit to the left, with probabilities p and $q = 1 - p$ respectively, independently of all his previous steps. Let S_n be his position after n steps.

(a) Let p_k be the probability that the drunk ever reaches the value k, for all $k \geq 0$. Write down a recursive equation for p_k (you do not need to solve it for this part).

(b) Find p_k, fully simplified; be sure to consider all 3 cases: $p < 1/2, p = 1/2$, and $p > 1/2$.

Solution

(a) Conditioning on the first step,

$$p_k = pp_{k-1} + qp_{k+1}$$

for all $k \geq 1$, with $p_0 = 1$.

(b) For fixed k and any positive integer j, let A_j be the event that the drunk reaches k before ever reaching $-j$. Then $A_j \subseteq A_{j+1}$ for all j since the drunk would have to walk past $-j$ to reach $-j-1$. Also, $\cup_{j=1}^{\infty} A_j$ is the event that the drunk ever reaches k, since if he reaches $-j$ before k for all j, then he will never have time to get to k. Now we just need to find $\lim_{j\to\infty} P(A_j)$, where we already know $P(A_j)$ from the result of the gambler's ruin problem. For $p = 1/2$:

$$P(A_j) = \frac{j}{j+k} \to 1 \text{ as } j \to \infty$$

For $p > 1/2$:

$$P(A_j) = \frac{1 - \left(\frac{q}{p}\right)^j}{1 - \left(\frac{q}{p}\right)^{j+k}} \to 1 \text{ as } j \to \infty,$$

since $(q/p)^j \to 0$. For $p < 1/2$:

$$P(A_j) = \frac{1 - \left(\frac{q}{p}\right)^j}{1 - \left(\frac{q}{p}\right)^{j+k}} \to \left(\frac{p}{q}\right)^k \text{ as } j \to \infty,$$

since $(q/p)^j \to \infty$ so the 1's in the numerator and denominator become negligible as $j \to \infty$.