Probability & Statistics for EECS: Homework #09

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Problem 1

(a) X discrete, Y discrete:
With the definition of conditional probability, we could get that
$P(Y=y|X=x) = \dfrac{P(Y=y,X=x)}{P(X=x)}$,
$P(X=x|Y=y) = \dfrac{P(X=x,Y=y)}{P(Y=y)}$,
since $P(Y=y,X=x) = P(X=x,Y=y)$
so $P(Y=y|X=x)P(X=x) = P(X=x|Y=y)P(Y=y)$,
i.e. $P(Y=y|X=x) = \dfrac{P(X=x|Y=y)P(Y=y)}{P(X=x)}$.

So above all, when X discrete, Y discrete, $P(Y=y|X=x) = \dfrac{P(X=x|Y=y)P(Y=y)}{P(X=x)}$.

(b) X discrete, Y continuous:

From (a), we can get that $P(Y\in(y-\epsilon,y+\epsilon)|X=x) = \dfrac{P(X=x|Y\in(y-\epsilon,y+\epsilon))P(Y\in(y-\epsilon,y+\epsilon))}{P(X=x)}$.
And since $P(Y\in(y-\epsilon,y+\epsilon)) = \lim\limits_{\epsilon\to 0} f_Y(y)\cdot 2\epsilon$.
So $f_Y(y|X=x) = \lim\limits_{\epsilon\to 0}\dfrac{P(Y\in(y-\epsilon,y+\epsilon)|X=x)}{2\epsilon} = \lim\limits_{\epsilon\to 0}\dfrac{P(X=x|Y\in(y-\epsilon,y+\epsilon))P(Y\in(y-\epsilon,y+\epsilon))}{P(X=x)\cdot 2\epsilon}$
$= \lim\limits_{\epsilon\to 0}\dfrac{P(X=x|Y\in(y-\epsilon,y+\epsilon))\dfrac{P(Y\in(y-\epsilon,y+\epsilon))}{2\epsilon}}{P(X=x)} = \dfrac{P(X=x|Y=y)f_Y(y)}{P(X=x)}$
So above all, when X discrete, Y continuous, $f_Y(y|X=x) = \dfrac{P(X=x|Y=y)f_Y(y)}{P(X=x)}$.

(c) X continuous, Y discrete:
$P(Y=y|X=x) = \lim\limits_{\epsilon\to 0} P(Y=y|X\in(x-\epsilon,x+\epsilon))$
from (a), we can get that $P(Y=y|X\in(x-\epsilon,x+\epsilon)) = \dfrac{P(X\in(x-\epsilon,x+\epsilon)|Y=y)P(Y=y)}{P(X\in(x-\epsilon,x+\epsilon))}$

So $P(Y=y|X=x) = \lim\limits_{\epsilon\to 0}\dfrac{P(X\in(x-\epsilon,x+\epsilon)|Y=y)P(Y=y)}{P(X\in(x-\epsilon,x+\epsilon))} = \lim\limits_{\epsilon\to 0}\dfrac{\dfrac{P(X\in(x-\epsilon,x+\epsilon)|Y=y)}{2\epsilon}P(Y=y)}{\dfrac{P(X\in(x-\epsilon,x+\epsilon))}{2\epsilon}}$
$= \dfrac{f_X(x|Y=y)P(Y=y)}{f_X(x)}$.

So above all, when X continuous, Y discrete, $P(Y=y|X=x) = \dfrac{f_X(x|Y=y)P(Y=y)}{f_X(x)}$.

(d) X continuous, Y continuous:
$f_{Y|X}(y|x) = \lim\limits_{\epsilon_1\to 0,\epsilon_2\to 0}\dfrac{P(Y\in(y-\epsilon_1,y+\epsilon_1)|X\in(x-\epsilon_2,x+\epsilon_2))}{2\epsilon_1}$
from (a), we can get that
$P(Y\in(y-\epsilon_1,y+\epsilon_1)|X\in(x-\epsilon_2,x+\epsilon_2)) = \dfrac{P(X\in(x-\epsilon_2,x+\epsilon_2)|Y\in(y-\epsilon_1,y+\epsilon_1))P(Y\in(y-\epsilon_1,y+\epsilon_1))}{P(X\in(x-\epsilon_2,x+\epsilon_2))}$
so $f_{Y|X}(y|x) = \lim\limits_{\epsilon_1\to 0,\epsilon_2\to 0}\dfrac{P(X\in(x-\epsilon_2,x+\epsilon_2)|Y\in(y-\epsilon_1,y+\epsilon_1))P(Y\in(y-\epsilon_1,y+\epsilon_1))}{P(X\in(x-\epsilon_2,x+\epsilon_2))\cdot 2\epsilon_1}$
$= \lim\limits_{\epsilon_1\to 0,\epsilon_2\to 0}\dfrac{\dfrac{P(X\in(x-\epsilon_2,x+\epsilon_2)|Y\in(y-\epsilon_1,y+\epsilon_1))}{2\epsilon_2}\dfrac{P(Y\in(y-\epsilon_1,y+\epsilon_1))}{2\epsilon_1}}{\dfrac{P(X\in(x-\epsilon_2,x+\epsilon_2))}{2\epsilon_2}}$

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$= \dfrac{f_{X|Y}(x|y)f_Y(y)}{f_X(x)}$.

So above all, when X continuous, Y continuous, $f_{Y|X}(y|x) = \dfrac{f_{X|Y}(x|y)f_Y(y)}{f_X(x)}$.

Problem 2

Let $q = 1 - p$, since $X, Y \sim Geom(p)$, so $P(X = x) = q^x p, P(Y = y) = q^y p$

(a) Since $N = X + Y$, so $n = x + y$, and the joint PMF of X,Y,N is

$$P(X = x, Y = y, N = n) = P(X = x, Y = y)$$

since X,Y are i.i.d., so

$$P(X = x, Y = y) = P(X = x)P(Y = y) = (q^x p)(q^y p) = p^2 q^{x+y} = p^2(1-p)^n$$

So above all, the joint PMF of X,Y,N is that $P(X = x, Y = y, N = n) = p^2(1-p)^n$,(x,y,n are nonnegative integers).

(b) similarly with (a), we can get that

$$P(X = x, N = n) = P(X = x, Y = n - x) = P(X = x)P(Y = n - x) = (q^x p)(q^{n-x} p) = p^2 q^n = p^2(1-q)^n$$

So above all, the joint PMF if X and N is that $P(X = x, N = n) = p^2(1-p)^n$, (x,n are nonnegative integers).

(c) the conditional PMF of X given $N = n$ is that

$$P(X = x|N = n) = \frac{P(X = x, N = n)}{P(N = n)}$$

From (b) we can get that $P(X = x, N = n) = p^2(1-p)^n$.
And from LOTP, we can get that

$$P(N = n) = \sum_{k=0}^{n} P(N = n|X = k)P(X = k) = \sum_{k=0}^{n} P(N = n, X = k) = \sum_{k=0}^{n} p^2(1-p)^n = (n+1)p^2(1-p)^n$$

So

$$P(X = x|N = n) = \frac{P(X = x, N = n)}{P(N = n)} = \frac{p^2(1-p)^n}{(n+1)p^2(1-p)^n} = \frac{1}{n+1}$$

So above all, the conditional PMF of X given that $N = n$ is that $P(X = x|N = n) = \frac{1}{n+1}$.
The result says that when given $N = n$, then X could be any one of the numbers in $\{0, 1, \cdots, n\}$ with equal probabilities, i.e. $\frac{1}{n+1}$.

Problem 3

(a) Since $X \sim Expo(\lambda)$, so $F_X(x) = P(X \le x) = 1 - e^{-\lambda x}, x > 0$.
And the conditional CDF of X given $X < c$ is that $F_{X|X>c}(x) = P(X \le x|X > c) = \dfrac{P(X \le x, X > c)}{P(X > c)}$.
If $x \le c$, then $P(X \le x, X > c) = 0$,i.e. $F_{X|X>c}(x) = 0$. Then $F_{X|X>c}(x) = 0$
If $x > c$, then $P(X \le x, X > c) = P(c < X \le x) = P(X \le x) - P(X \le c) = (1 - e^{-\lambda x}) - (1 - e^{-\lambda c}) = e^{-\lambda c} - e^{-\lambda x}$.
Then $F_{X|X>c}(x) = \dfrac{e^{-\lambda c} - e^{-\lambda x}}{e^{-\lambda c}} = 1 - e^{-\lambda(x-c)}, x > c$, $F_{X|X>c}(x) = 0, x \le c$.
And the conditional PDF of X given $X < c$ is that
$f_{X|X>c}(x) = F'_{X|X>c}(x) = \lambda e^{-\lambda(x-c)}, x > c, f_{X|X>c}(x) = 0, x \le c$.

So above all, the conditional CDF of X given $X > c$ is $F_{X|X>c}(x) = 1-e^{-\lambda(x-c)}, x > c$, $F_{X|X>c}(x) = 0, x \le c$.
And the conditional PDF of X given $X > c$ is that $f_{X|X>c}(x) = \lambda e^{-\lambda(x-c)}, x > c, f_{X|X>c}(x) = 0, x \le c$.

(b) The conditional CDF of X given $X < c$ is that $F_{X|X<c}(x) = P(X \le x|X < c) = \dfrac{P(X \le x, X < c)}{P(X < c)}$.
From the support of exponential distribution, we can get that when $x \le 0, F_{X|X<c}(x) = 0$.
When $x > 0$:
If $X < c$, then $P(X \le x, X < c) = \dfrac{P(X \le x)}{P(X < c)} = \dfrac{1 - e^{-\lambda x}}{1 - e^{-\lambda c}}$.
And if $X \ge c$, then $P(X \le x, X < c) = \dfrac{P(X < c)}{P(X < c)} = 1$.
So $F_{X|X<c}(x) = 0, x \le 0, F_{X|X<c}(x) = \dfrac{1 - e^{-\lambda x}}{1 - e^{-\lambda c}}, 0 < x < c, F_{X|X<c}(x) = 1, x \ge c$.
And the conditional PDF of X given $X < c$ is that
$f_{X|X<c}(x) = \dfrac{d}{dx}F_{X|X<c}(x) = \dfrac{\lambda e^{-\lambda x}}{1 - e^{-\lambda c}}, 0 < x < c, f_{X|X<c}(x) = 0, x \le 0$, or $x \ge c$.

So above all, the conditional CDF of X given $X < c$ is that
$F_{X|X<c}(x) = 0, x \le 0, F_{X|X<c}(x) = \dfrac{1 - e^{-\lambda x}}{1 - e^{-\lambda c}}, 0 < x < c, F_{X|X<c}(x) = 1, x \ge c$.
And the conditional PDF of X given $X < c$ is that
$f_{X|X<c}(x) = \dfrac{\lambda e^{-\lambda x}}{1 - e^{-\lambda c}}, 0 < x < c, f_{X|X<c}(x) = 0, x \le 0$, or $x \ge c$.

Problem 4

(a) The marginal CDF of M is that
$F_M(m) = P(M \le m) = P(max(U_1, U_2, U_3) \le m) = P(U_1 \le m, U_2 \le m, U_3 \le m)$
since U_1, U_2, U_3 are i.i.d. Unif(0,1), so when $0 \le m \le 1$, $F_M(m) = P(U_1 \le m)P(U_2 \le m)P(U_3 \le m) = m^3$.
And when $m < 0$ or $m > 1$, $F_M(m) = 0$.
And the marginal PDF of M is that $f_M(m) = \frac{d}{dm}F_M(m) = \frac{d}{dm}m^3 = 3m^2, m \in [0, 1]$.
And $f_M(m) = 0$, otherwise.

Since U_1, U_2, U_3 are i.i.d. Unif(0,1), so
$P(L \ge l, M \le m)$
$= P(min(U_1, U_2, U_3) \ge l, max(U_1, U_2, U_3) \le m) = P(U_1 \ge l, U_2 \ge l, U_3 \ge l, U_1 \le m, U_2 \le m, U_3 \le m)$
$= P(l \le U_1 \le m)P(l \le U_2 \le m)P(l \le U_3 \le m) = (m - l)^3, m \ge l.$
And $P(L \ge l, M \le m) = 0, m < l$.

And from above, we can get that $P(M \le m) = m^3$,
and since $(L \le l) \cup (L \ge l) = R, P((L \le l) \cap (L \ge l)) = P(L = l) = 0$
so $P(M \le m) = P(L \le l, M \le m) + P(L \ge l, M \le m)$.
So the joint CDF of L,M is that
$F_{L,M}(l, m) = P(L \le l, M \le m) = P(M \le m) - P(L \ge l, M \le m) = m^3 - (m - l)^3, m \ge l$,and $l, m \in [0, 1]$,
$F_{L,M}(l, m) = 0$, otherwise.
And the joint PDf of L,M is that
$f_{L,M}(l, m) = \frac{\partial^2}{\partial l \partial m}F_{L,M}(l, m) = \frac{\partial^2}{\partial l \partial m}[m^3 - (m - l)^3] = 6(m - l),\ l, m \in [0, 1], m \ge l,$
$f_{L,M}(l, m) = 0$, otherwise

So above all, the marginal CDF of M is $F_M(m) = m^3, m \in [0, 1]$, and $F_M(m) = 0$, otherwise.
The marginal PDF of M is $f_M(m) = 3m^2, m \in [0, 1]$, and $f_M(m) = 0$, otherwise.
The joint CDF of L, M is $F_{L,M}(l, m) = m^3 - (m - l)^3, m, l \in [0, 1], m \ge l$, and $F_{L,M}(l, m) = 0$, otherwise.
The joint PDF of L, M is $f_{L,M}(l, m) = 6(m - l), m, l \in [0, 1], m \ge l$, and $f_{L,M}(l, m) = 0$, otherwise.

(b) The marginal CDF of L is that
$F_L(l) = P(L \le l) = 1 - P(L > l) = P(min(U_1, U_2, U_3) > l) = P(U_1 > l, U_2 > l, U_3 > l) = 1 - P(U_1 > l)P(U_2 > l)P(U_3 > l) = 1 - (1 - l)^3, l \in [0, 1]$,
So the mariginal PDF of L is that
$f_L(l) = \frac{d}{dl}F_L(l) = \frac{d}{dl}[1 - (1 - l)^3] = 3(1 - l)^2, l \in [0, 1],$

So the conditional PMF of M given L is that
$f_{M|L}(m|l) = \frac{f_{L,M}(l, m)}{f_L(l)} = \frac{6(m - l)}{3(1 - l)^2} = \frac{2(m - l)}{(1 - l)^2}, m, l \in [0, 1], m \ge l,$
and $f_{M|L}(m|l) = 0$, otherwise.

So above all, the conditional PMF of M given L is that
$f_{M|L}(m|l) = \frac{2(m - l)}{(1 - l)^2}, m, l \in [0, 1], m \ge l,$
and $f_{M|L}(m|l) = 0$, otherwise.

Problem 5

(a) 1. Since X and Y are i.i.d. $Geom(p)$, and let $q = 1 - p$, so the joint PMF of L and M is that
If $m < 0$ or $l < 0$, then $P_{L,M}(L = l, M = m) = 0$,
If $m \geq 0, l \geq 0$, then:
$P_{L,M}(L = l, M = m) = P(min(X, Y) = l, max(X, Y) = m)$.
If $l > m$, which means that $min(X, Y) > max(X, Y)$, which is impossible, so $P_{L,M}(L = l, M = m) = 0$.
If $l = m$, which means that $min(X, Y) = max(X, Y)$,
so $P_{L,M}(L = l, M = m) = P(X = Y = l) = P(X = l)P(Y + l) = (q^l p)^2 = p^2(1 - p)^{2l}$.
If $l < m$, which means that $min(X, Y) < max(X, Y)$,
so $P_{L,M}(L = l, M = m) = P(X = l, Y = m) + P(X = m, Y = l) =$
$P(X = l)P(Y = m) + P(X = m)P(Y = l) = 2(q^l p)(q^m p) = 2p^2(1 - p)^{l+m}$.

2. Let $l = 1, m = 0$, from the definition, we can get that
$P(L = l|M = m) = \dfrac{P(L = l, M = m)P(M = m)}{P(L = l)} = 0$,
that is because $P(L = l, M = m) = 0, P(L = l) \neq 0, P(M = m) \neq 0$.
But $P(L = l) = P(min(X, Y) = 1) = P(X = 1, Y \geq 1) + P(X \geq 2, Y = 1)$
$= (q^1 p)(1 - q^0 p) + (1 - q^0 p - q^1 p)(q^1 p) = qp - qp^2 + qp - qp^2 p - q^2 p^2$
$= p(p - 1)^2(2 - p) > 0$, since $p \in (0, 1)$.
But $P(L = l|M = m) = 0$, and $P(M = m) = q^0 p = p > 0$ so $P(L = l|M = m) \neq P(L = l)$.
So L and M are not independent.

So above all, the joint PMF of L and M is that

$$P_{L,M}(L = l, M = m) = \begin{cases} p^2(1 - p)^{2l}, & m = l \geq 0 \\ 2p^2(1 - p)^{l+m}, & m > l \geq 0 \\ 0, & otherwise \end{cases}.$$

And L and M are not independent.

(b) 1. The marginal distribution if L is that $P_L(l) = \sum\limits_m P_{L,M}(L = l, M = m)$.
If $l < 0$, then $P_L(l) = 0$,
If $l \geq 0$, then:
$P_L(l) = \sum\limits_{m=l}^{+\infty} P_{L,M}(L = l, M = m) = P_{L,M}(L = l, M = l) + \sum\limits_{m=l+1}^{+\infty} P_{L,M}(L = l, M = m)$
$= (q^l p)^2 + \sum\limits_{m=l+1}^{+\infty} 2p^2 q^{l+m} = p^2 q^{2l} + 2pq^{2l+1} = q^{2l}(p^2 + 2pq)$
$= q^{2l}(1 - q^2)$
$= p(1 - p)^{2l}(2 - p)$

2. Story:
Alice and Bob are tossing the biased coin.
For each turn, each coin has the probability of p to head, and has the probability of $q = 1 - p$ to tail.
Let X be the number of tails before the first head for Alice, and let Y be the number of tails before the first head for Bob.
Let L be the turns before at least one of them head, i.e. $L = min(X, Y)$.
When $L = l > 0$, for the first l turns, they both tails. So the probability is q^2
And for the $l + 1$-th turn, at least one of them head, so the probability is $1 - q^2$.
So $P(L = l) = (q^2)^l(1 - q^2) = q^{2l}(1 - q^2) = p(1 - p)^{2l}(2 - p)$.
And when $l < 0$, $P(L = l) = 0$.

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So above all, the mariginal distribution of L is $P(L=l)=p(1-p)^{2l}(2-p), l\geq 0$, $P(L=l)=0, l<0$ have been proved in two ways.

(c) The CDF for M is that
$F_M(m)=P(M\leq m)=P(max(X,Y)\leq m)=P(X\leq m, Y\leq m)=P(X\leq m)P(Y\leq m)$.
And $P(X\leq m)=\sum\limits_{k=0}^{m}q^kp=1-q^{m+1}$, similarly, $P(Y\leq m)=1-q^{m+1}$.
So $F_M(m)=(1-q^{m+1})^2$.
So the survival function of M is that $G_M(m)=1-F_M(m)=1-(1-q^{m+1})^2=2q^{m+1}-q^{2m+2}$.
Then $E[M]=\sum\limits_{m=0}^{+\infty}G(m)=\dfrac{2q}{1-q}-\dfrac{q^2}{1-q^2}=\dfrac{(1-p)(3-p)}{p(2-p)}$.
So above all, $E[M]=\dfrac{(1-p)(3-p)}{p(2-p)}$.

(d) 1. Let $D=M-L$ as a delta.
So the joint PMF of L and $M-L$ is that
$P_{L,M-L}(l,d)=P_{L,D}(l,d)=P(L=l,M=l+d)$.
With what we get from (b) 1.,
If $l<0, P_{L,M-L}(l,d)=0$.
If $l\geq 0:$
when $d<0$, $P_{L,M-L}(l,d)=0$,
when $d=0$, $P_{L,M-L}(l,d)=P(L=l,M=l)=p^2q^{2l}$.
when $d>0$, $P_{L,M-L}(l,d)=P(L=l,M=l+d)=2p^2q^{2l+d}$.

2. And from (b) 2. , we get that $P(L=l)=q^{2l}(1-q^2)$.
$P(D=d)=0$, when $d<0$ or $l<0$,
When $d>0, l\geq 0$: With LOTP, we can get that $P(D=d)=\sum\limits_{l=0}^{+\infty}P(L=l,D=d)=\dfrac{2p^2q^d}{1-q^2}$.
And when $d=0, l\geq 0$, with LOTP, we can get that $P(D=d)=\sum\limits_{l=0}^{+\infty}P(L=l,D=d)=\dfrac{p^2}{1-q^2}$.

So when $l\geq 0, d\geq 0$, $P(D=d)>0$, and $P(L=l)>0$.
if $d=0$,$P(L=l|D=d)=\dfrac{p^2q^{2l}}{\dfrac{p^2}{1-q^2}}=q^{2l}(1-q^2)$,
if $d>0$,$P(L=l|D=d)=\dfrac{2p^2q^{2l+d}}{\dfrac{2p^2q^d}{1-q^2}}=q^{2l}(1-q^2)$.
So $P(L=l|D=d)=P(L=l), P(D=d)>0$.
So L and D, i.e. L and $M-L$ are independent.

So above all, the joint PMF of L and $M-L$ is that
$$P_{L,M-L}(l,d)=\begin{cases}p^2q^{2l}, & d=0, l\geq 0\\ 2p^2q^{2l+d}, & d>0, l\geq 0\\ 0, & otherwise\end{cases}$$
And L and $M-L$ are independent.