Probability & Statistics for EECS: Homework #14

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Name: **Zhou Shouchen**
Student ID: 2021533042

Problem 1

(a) Let the DNA sequence be S.
And S_i be the i-th letter of the sequence, $1 \leq i \leq 115$.
Then we have $S_i \in \{A, C, G, T\}$, and S_i are independent.

Define I_j be the indecator that whether the subsequence starts at j, end at $j+5$ is "$CATCAT$", $1 \leq j \leq 110$.
So $E(I_j) = P(S_j = C, S_{j+1} = A, \cdots, S_{j+5} = T) = p_2p_1p_3p_2p_1p_3 = (p_1p_2p_3)^2$.

Let X be the number of "$CATCAT$" subsequences in S.
Then $X = \sum\limits_{j=1}^{110} I_j$.
So $E(X) = E(\sum\limits_{j=1}^{110} I_j) = \sum\limits_{j=1}^{110} E(I_j) = 110(p_1p_2p_3)^2$.

So above all, the expected number of "$CATCAT$" subsequences in S is $110(p_1p_2p_3)^2$.

(b) From what we have learned about Bayes Reference.
The prior distribution of p_2 is $p_2 \sim Unif(0,1) \sim Beta(1,1)$.

As for observation, let X_i be whether the i-th subsequence is the letter C.
So $X_i | p_2 \sim Bern(p_2)$.
And we have observate that $X_1 = 1, X_2 = 0, X_3 = 0$.
So from Beta-Binomial conjugate, we have $p_2 | X_1 = 1, X_2 = 0, X_3 = 0 \sim Beta(2,3)$.

So the posterior distribution of p_2 is $p_2 | X_1 = 1, X_2 = 0, X_3 = 0 \sim Beta(2,3)$.
So $P(S_4 = C) = E(p_2 | X_1 = 1, X_2 = 0, X_3 = 0) = \frac{2}{2+3} = \frac{2}{5}$.
This is because the expection of a Beta distribution $Beta(a,b)$ is $\frac{a}{a+b}$.

So above all, the probability that the next letter of the sequence is C is $\frac{2}{5}$.

Problem 2

(a) Suppose the CDF for all X_i is $F(x)$.
Then we can get the CDF of X_j^* with LOTP:
$F_{X_j^*}(x) = P(X_j^* \le x) = \sum\limits_{i=1}^{n} P(X_j^* \le x | X_j^* \ is \ X_i) P(X_j^* \ is \ X_i)$.
$= \sum\limits_{i=1}^{n} P(X_i \le x) \cdot \dfrac{1}{n} = F(x)$
So $F(X_j^*) = F(x)$.
i.e. $X_j^* \sim X_i$.
So $E(X_j^*) = E(X_i) = \mu$.
And $Var(X_j^*) = Var(X_i) = \sigma^2$.

So above all, $E(X_j^*) = \mu, Var(X_j^*) = \sigma^2$, for each $j \in \{1, \cdots, n\}$.

(b) <1> $E(\bar{X}^*|X_1, \cdots, X_n)$:
With the linearity of conditional expectation, we can get that
$E(\bar{X}^*|X_1, \cdots, X_n) = E(\dfrac{1}{n}\sum\limits_{i=1}^{n} X_i^*|X_1, \cdots, X_n) = \dfrac{1}{n}\sum\limits_{i=1}^{n} E(X_i^*|X_1, \cdots, X_n)$.
And for any i, from the defination of expectation, we can get that
$E(X_i^*|X_1, \cdots, X_n) = \sum\limits_{i=1}^{n} P(X_i^* = X_i|X_1, \cdots, X_n) \cdot X_i = \dfrac{1}{n}\sum\limits_{i=1}^{n} X_i$.
So $E(\bar{X}^*|X_1, \cdots, X_n) = \dfrac{1}{n}\sum\limits_{i=1}^{n} X_i$.

<2> $Var(\bar{X}^*|X_1, \cdots, X_n)$:
Let $\dfrac{1}{n}\sum\limits_{i=1}^{n} X_i = \bar{X}$.
Then $Var(\bar{X}^*|X_1, \cdots, X_n) = Var(\dfrac{1}{n}\sum\limits_{i=1}^{n} X_n^*|X_1, \cdots, X_n)$.
Since X_i are i..d. r.v.s. , so X_i^* are independent.
So $Var(\dfrac{1}{n}\sum\limits_{i=1}^{n} X_n^*|X_1, \cdots, X_n)$
$= \dfrac{1}{n^2}\sum\limits_{i=1}^{n} Var(X_i^*|X_1, \cdots, X_n)$.
$\forall j$,
$E(X_j^*|X_1, \cdots, X_n) = \sum\limits_{i=1}^{n} X_i \cdot \dfrac{1}{n} = \bar{X}$
$Var(X_i^*|X_1, \cdots, X_n) = E((X_j^* - E(X_j^*|X_1, \cdots, X_n))^2|X_1, \cdots, X_n)$
$= E((X_j^* - \bar{X})^2|X_1, \cdots, X_n) = \sum\limits_{i=1}^{n} \dfrac{1}{n} \cdot (X_i - \bar{X})^2$.
So $Var(\bar{X}^*|X_1, \cdots, X_n) = \dfrac{1}{n^2}\sum\limits_{i=1}^{n} Var(X_i^*|X_1, \cdots, X_n) = \dfrac{1}{n^2} \cdot n \sum\limits_{i=1}^{n} \dfrac{1}{n} \cdot (X_i - \bar{X})^2 = \dfrac{1}{n^2}\sum\limits_{i=1}^{n}(X_i - \bar{X})^2$.

So above all, $E(\bar{X}^*|X_1, \cdots, X_n) = \dfrac{1}{n}\sum\limits_{i=1}^{n} X_i = \bar{X}$.

And $Var(\bar{X}^*|X_1, \cdots, X_n) = \dfrac{1}{n^2}\sum\limits_{i=1}^{n}(X_i - \bar{X})^2$.

(c) <1> $E(\bar{X}^*)$:
From (b) we can get that
$E(\bar{X}^*|X_1, \cdots, X_n) = \dfrac{1}{n}\sum\limits_{i=1}^{n} X_i$ Take the expectation on both sides, we can get that
$E(E(\bar{X}^*|X_1, \cdots, X_n)) = E(\dfrac{1}{n}\sum\limits_{i=1}^{n} X_i) = \dfrac{1}{n}\sum\limits_{i=1}^{n} E(X_i) = \dfrac{1}{n} \cdot n \cdot \mu = \mu$.

And from Adam's law, we can get that
$E(E(\bar{X}^*|X_1,\cdots,X_n)) = E(\bar{X}^*)$.
So $E(\bar{X}^*) = \mu$.

<2> $Var(\bar{X}^*)$:
From the Eve's law, we can get that
$Var(\bar{X}^*) = E(Var(\bar{X}^*|X_1,\cdots,X_n)) + Var(E(\bar{X}^*|X_1,\cdots,X_n))$.
From (b), we have get that
$E(\bar{X}^*|X_1,\cdots,X_n) = \frac{1}{n}\sum\limits_{i=1}^n X_i$.
And since X_i are i.i.d. r.v.s.
So the second part $Var(E(\bar{X}^*|X_1,\cdots,X_n)) = Var(\frac{1}{n}\sum\limits_{i=1}^n X_i) = \frac{1}{n^2}\sum\limits_{i=1}^n Var(X_i) = \frac{1}{n^2}\cdot n\cdot\sigma^2 = \frac{1}{n}\sigma^2$.

And since $E(X_i)=\mu$, $Var(X_i)=\sigma^2$, so $E(X_i^2) = Var(X_i) + E(X_i)^2 = \sigma^2+\mu^2$.

As for the first part,
$E(Var(\bar{X}^*|X_1,\cdots,X_n)) = E(\frac{1}{n^2}\sum\limits_{i=1}^n (X_i-\bar{X})^2) = \frac{1}{n^2}\sum\limits_{i=1}^n E(X_i^2 - 2X_i\bar{X} + \bar{X}^2)$
$= \frac{1}{n^2}\sum\limits_{i=1}^n E(X_i^2) - 2\sum\limits_{i=1}^n E(X_i\bar{X}) + \sum\limits_{i=1}^n E(\bar{X}^2)$.
1. $\sum\limits_{i=1}^n E(X_i^2) = \sum\limits_{i=1}^n(\sigma^2+\mu^2) = n(\sigma^2+\mu^2)$.

2. Since X_i are independent, so for each i, we can get that
$E(X_i\bar{X}) = \frac{1}{n}\sum\limits_{j=1}^n E(X_iX_j) = \frac{1}{n}E(X_i^2) + \frac{1}{n}\sum\limits_{j=1,j\neq i}^n E(X_iX_j)$
$= \frac{1}{n}E(X_i^2) + \frac{1}{n}\sum\limits_{j=1,j\neq i}^n E(X_i)E(X_j) = \frac{1}{n}(\sigma^2+\mu^2) + \frac{1}{n}(n-1)\mu^2 = \frac{1}{n}(\sigma^2+n\mu^2)$.
So $2\sum\limits_{i=1}^n E(X_i\bar{X}) = 2\cdot n\cdot\frac{1}{n}(\sigma^2+n\mu^2) = 2(\sigma^2+n\mu^2)$.

3. Since $\bar{X}^2 = (\frac{1}{n}(X_1+\cdots+X_n))^2 = \frac{1}{n^2}(\sum\limits_{i=1}^n X_i^2 + 2\sum\limits_{i<j} X_iX_j)$.
So $\sum\limits_{i=1}^n E(\bar{X}^2) = n\cdot\frac{1}{n^2}\cdot(\sum\limits_{i=1}^n E(X_i^2) + 2\sum\limits_{i<j} E(X_iX_j))$
Since X_i are independent, so
$\sum\limits_{i=1}^n E(\bar{X}^2) = \frac{1}{n}\cdot(n(\sigma^2+\mu^2) + n(n-1)\mu^2) = \sigma^2+n\mu^2$.

Combine these, we can get the first part is that
$E(Var(\bar{X}^*|X_1,\cdots,X_n)) = \frac{1}{n^2}[n(\sigma^2+\mu^2) - 2(\sigma^2+n\mu^2) + \sigma^2+n\mu^2] = \frac{n-1}{n^2}\sigma^2$.

And combine the two parts, we can get that
$Var(\bar{X}^*) = \frac{n-1}{n^2}\sigma^2 + \frac{1}{n}\sigma^2$.

So above all, $E(\bar{X}^*)=\mu$, $Var(\bar{X}^*) = \frac{n-1}{n^2}\sigma^2 + \frac{1}{n}\sigma^2$.

(d) Intuitively, the variance of $\bar{X}^*$ is smaller than the variance of $\bar{X}$.
We can regard that the variance of $\bar{X}^*$ have two sources of randomness, one is the randomness on the bootstrap sample of $X_1,\cdots,X_n$ to decide which X_j^* is, and the other is the randomness on sample of each X_j^*.

But the variance of $\bar{X}$ only have one source of randomness, which is the randomness on sample of each X_i.

So we can intuitively get that $Var(\bar{X}) < Var(\bar{X}^*)$.

Problem 3

Let S be the squence of the results of the flipped coins.

(a) <1>
Let X ne the number of flips untiol the pattern HT is observed.
So with LOTE, we can get that
$E(X) = E(X|S_1 = H)P(S_1 = H) + E(X|S_1 = T)P(S_1 = T) = E(X|S_1 = H) \cdot p + E(X|S_1 = T) \cdot (1-p)$.
If $S_1 = T$, which means that it has no contributions to approaching HT, so $E(X|S_1 = T) = E(X) + 1$.
If $S_1 = H$, which means that we are closing the the pattern HT,
so with conditional LOTE, we can get that
$E(X|S_1 = H) = E(X|S_1 = H, S_2 = H)P(S_2 = H|S_1 = H) + E(X|S_1 = H, S_2 = T)P(S_2 = T|S_1 = H)$.
Since each times' fliiping are independent,
so $P(S_2 = H|S_1 = H) = P(S_2 = H) = p$, and $P(S_2 = T|S_1 = H) = P(S_2 = T) = 1 - p$.
And for $S_1 = H, S_2 = T$, which means that we get the pattern HT, so $E(X|S_1 = H, S_2 = T) = 2$.
And for $S_1 = H, S_2 = H$, which means that it is as same as $S_1 = H$,
so $E(X|S_1 = H, S_2 = H) = 1 + E(X|S_1 = H)$.

With these, we can calculate $E(X|S_1 = H)$:
$E(X|S_1 = H) = (1 + E(X|S_1 = H)) \cdot p + 2 \cdot (1-p)$
So $E(X|S_1 = H) = \dfrac{2-p}{1-p}$.
And with $E(X|S_1 = H) = \dfrac{2-p}{1-p}$, we can get that
$E(X) = \dfrac{2-p}{1-p} \cdot p + (1 + E(X)) \cdot (1-p)$
So $E(X) = \dfrac{1}{p(1-p)}$.

So above all, the expected number of flips until the pattern HT is observed is $\dfrac{1}{p(1-p)}$.

<2>
Similarly with <1>, Let X ne the number of flips untiol the pattern HH is observed.
So with LOTE, we can get that
$E(X) = E(X|S_1 = H)P(S_1 = H) + E(X|S_1 = T)P(S_1 = T) = E(X|S_1 = H) \cdot p + E(X|S_1 = T) \cdot (1-p)$.
If $S_1 = T$, which means that it has no contributions to approaching HH, so $E(X|S_1 = T) = E(X) + 1$.
If $S_1 = H$, which means that we are closing the the pattern HH,
so with conditional LOTE, we can get that
$E(X|S_1 = H) = E(X|S_1 = H, S_2 = H)P(S_2 = H|S_1 = H) + E(X|S_1 = H, S_2 = T)P(S_2 = T|S_1 = H)$.
Since each times' fliiping are independent,
so $P(S_2 = H|S_1 = H) = P(S_2 = H) = p$, and $P(S_2 = T|S_1 = H) = P(S_2 = T) = 1 - p$.
And for $S_1 = H, S_2 = H$, which means that we get the pattern HH, so $E(X|S_1 = H, S_2 = H) = 2$.
And for $S_1 = H, S_2 = T$, which means that it is has no contributions to HH again,
so $E(X|S_1 = H, S_2 = T) = 2 + E(X)$.

With these, we can calculate $E(X|S_1 = H)$:
$E(X|S_1 = H) = 2 \cdot p + (2 + E(X)) \cdot (1-p)$
So $E(X|S_1 = H) = 2 + (1-p)E(X)$.

And with $E(X|S_1 = H) = 2 + (1-p)E(X)$, we can get that

$E(X) = (2 + (1-p) \cdot E(X)) \cdot p + (1 + E(X)) \cdot (1-p)$
So $E(X) = \dfrac{1+p}{p^2}$.

So above all, the expected number of flips until the pattern HH is observed is $\dfrac{1+p}{p^2}$.

(b) Since $p \sim Beta(a,b)$, so the PDF of p is that $f_P(p) = \dfrac{1}{\beta(a,b)} p^{a-1}(1-p)^{b-1}$.
<1>
From (a)<1>, we can get that $E(X|p) = \dfrac{1}{p(p-1)}$.
So with LOTE, we can get that
$E(X) = \int_0^1 \dfrac{1}{p(1-p)} f_P(p)dp = \int_0^1 \dfrac{1}{p(1-p)} \dfrac{1}{\beta(a,b)} p^{a-1}(1-p)^{b-1}dp$.
$= \dfrac{1}{\beta(a,b)} \int_0^1 p^{a-2}(1-p)^{b-2}dp$.
Since $a,b>2$, so $\int_0^1 p^{a-2}(1-p)^{b-2}dp = \beta(a-1,b-1)$.
So $E(X) = \dfrac{1}{\beta(a,b)}\beta(a-1,b-1) = \dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \dfrac{\Gamma(a-1)\Gamma(b-1)}{\Gamma(a+b-2)}$
$= \dfrac{(a+b-1)(a+b-2)\Gamma(a+b-2)}{(a-1)\Gamma(a-1)(b-1)\Gamma(b-1)} \cdot \dfrac{\Gamma(a-1)\Gamma(b-1)}{\Gamma(a+b-2)}$
$= \dfrac{(a+b-1)(a+b-2)}{(a-1)(b-1)}$.

So above all, the expected number of flips until the pattern HT is observed when $p \sim Beta(a,b)$ is $\dfrac{(a+b-1)(a+b-2)}{(a-1)(b-1)}$.

<2>
Similarly with (b)<1>,
$E(X) = \int_0^1 \dfrac{1+p}{p^2} f_P(p)dp = \int_0^1 (\dfrac{1}{p^2} + \dfrac{1}{p}) \dfrac{1}{\beta(a,b)} p^{a-1}(1-p)^{b-1}dp$.
$= \dfrac{1}{\beta(a,b)} \int_0^1 p^{a-3}(1-p)^{b-1}dp + \dfrac{1}{\beta(a,b)} \int_0^1 p^{a-2}(1-p)^{b-1}dp$.
Since $a,b>2$, so $\int_0^1 p^{a-3}(1-p)^{b-1}dp = \beta(a-2,b)$ and $\int_0^1 p^{a-2}(1-p)^{b-1}dp = \beta(a-1,b)$.

So $E(X) = \dfrac{1}{\beta(a,b)}\beta(a-2,b) + \dfrac{1}{\beta(a,b)}\beta(a-1,b)$
$= \dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \dfrac{\Gamma(a-2)\Gamma(b)}{\Gamma(a+b-2)} + \dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \dfrac{\Gamma(a-1)\Gamma(b)}{\Gamma(a+b-1)}$
$= \dfrac{\Gamma(a-2)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b-2)} + \dfrac{\Gamma(a-1)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b-1)}$
$= \dfrac{\Gamma(a-2)(a+b-1)(a+b-2)\Gamma(a+b-2)}{(a-1)(a-2)\Gamma(a-2)\Gamma(a+b-2)} + \dfrac{\Gamma(a-1)(a+b-1)\Gamma(a+b-1)}{(a-1)\Gamma(a-1)\Gamma(a+b-1)}$
$= \dfrac{(a+b-1)(a+b-2)}{(a-1)(a-2)} + \dfrac{a+b-1}{a-1}$

So above all, the expected number of flips until the pattern HH is observed when $p \sim Beta(a,b)$ is $= \dfrac{(a+b-1)(a+b-2)}{(a-1)(a-2)} + \dfrac{a+b-1}{a-1}$

Problem 4

Let S be the squence of the rolled die numbers.

(a) Let X be the number of rolls needed to get a 1 followed right away by 2.
With the LOTE, we can get that
$E(X) = \sum\limits_{i=1}^{6} E(X|S_1 = i)P(S_1 = i) = \mathrm{E}(X|S_1 = 1)\frac{1}{6} + \sum\limits_{i=1}^{5} E(X|S_1 = i)\frac{1}{6}$.
And we have $\mathrm{E}(X|S_1 \neq 1) = 1 + \mathrm{E}(X)$.

And with conditional expectation, we have
$\mathrm{E}(X|S_1 = 1) = \sum\limits_{i=1}^{6} \mathrm{E}(X|S_1 = 1, S_2 = i)P(S_2 = i|S_1 = 1)$
Since S_i are independent, so $P(S_2 = i|S_1 = 1) = P(S_2 = i) = \frac{1}{6}$.
So $E(X|S_1 = 1) = E(X|S_1 = 1, S_2 = 1)\frac{1}{6} + E(X|S_1 = 1, S_2 = 2)\frac{1}{6} + \sum\limits_{i=3}^{6} \mathrm{E}(X|S_1 = 1, S_2 = i)\frac{1}{6}$.
Since we want to find 1 followed right away by 2,
so $E(X|S_1 = 1, S_2 = 2) = 2$, and $E(X|S_1 = 1, S_2 = 1) = 1 + E(X|S_1 = 1)$.
And for $i = 3, 4, 5, 6$, we have $E(X|S_1 = 1, S_2 = i) = 2 + E(X)$.

So $E(X|S_1 = 1) = \frac{1}{6}(1 + E(X|S_1 = 1)) + \frac{1}{6} \cdot 2 + \frac{4}{6}(2 + E(X))$
i.e. $E(X|S_1 = 1) = \frac{1}{5}(11 + 4E(X))$.

So $E(X) = \frac{1}{6} \cdot \frac{1}{5}(11 + 4E(X)) + \frac{5}{6}(1 + E(X))$
And we can calculate that $E(X) = 36$ with the equation above.

So above all, the expected number of rolls needed to get a 1 followed right away by 2 is 36.

(b) Let X be the number of rolls needed to get two consecutive 1's.
With the LOTE, we can get that
$E(X) = \sum\limits_{i=1}^{6} E(X|S_1 = i)P(S_1 = i) = \mathrm{E}(X|S_1 = 1)\frac{1}{6} + \sum\limits_{i=1}^{5} E(X|S_1 = i)\frac{1}{6}$.
And we have $\mathrm{E}(X|S_1 \neq 1) = 1 + \mathrm{E}(X)$ because of $i \neq 1$ has no contributions to the sequence 11.

As for $E(X|S_1 = 1)$, with conditional LOTE, we can get that
$E(X|S_1 = 1) = \sum\limits_{i=1}^{6} \mathrm{E}(X|S_1 = 1, S_2 = i)P(S_2 = i|S_1 = 1)$
And since S_i are independent, so $P(S_2 = i|S_1 = 1) = P(S_2 = i) = \frac{1}{6}$.
So $E(X|S_1 = 1) = \mathrm{E}(X|S_1 = 1, S_2 = 1)\frac{1}{6} + \sum\limits_{i=2}^{6} \mathrm{E}(X|S_1 = 1, S_2 = i)\frac{1}{6}$.
Since we want to find two sonsecutive 1's, so $E(X|S_1 = 1, S_2 = 1) = 2$.
And for $i = 2, 3, 4, 5, 6$, we have $E(X|S_1 = 1, S_2 = i) = 2 + E(X)$.
So we can get that $E(X|S_1 = 1) = \frac{1}{6} \cdot 2 + \frac{5}{6}(2 + E(X)) = \frac{5}{6}E(X) + 2$.

And with $E(X|S_1 = 1)$, we can get that
$E(X) = \frac{1}{6} \cdot (\frac{5}{6}E(X) + 2) + \frac{5}{6}(1 + E(X))$
Solve the equation, we can get that $E(X) = 42$.

So above all, the expected number of rolls needed to get two consecutive 1's is 42.

(c) Let X_n be the rolling times to get the consecutive same value n times.
So we can easily get that $X_1 = 1$.

And for $n > 2$, with conditional LOTE, we can get that
$E(X_{n+1}|X_n) = E(X_{n+1}|X_n, S_{X_n+1} = S_{X_n})P(S_{X_n+1} = S_{X_n}|X_n)$
$+E(X_{n+1}|X_n, S_{X_n+1} \neq S_{X_n})P(S_{X_n+1} \neq S_{X_n}|X_n)$
Since the sequence is independent with the number of rolling times,
so $P(S_{X_n+1} = S_{X_n}|X_n) = P(S_{X_n+1} = S_{X_n}), P(S_{X_n+1} \neq S_{X_n}|X_n) = P(S_{X_n+1} \neq S_{X_n})$.

If $S_{X_n+1} = S_{X_n}$, which means that the newly rolled number is the same with the prior n numbers, then $X_{n+1} = X_n + 1$.
And $P(S_{X_n+1} = S_{X_n}) = \frac{1}{6}$.

If $S_{X_n+1} \neq S_{X_n}$, which means that the newly rolled number is different with the prior n numbers, then we need to start a new consecutive sequence, so $X_{n+1} = X_n + E(X_{n+1}|X_n)$.
And $P(S_{X_n+1} \neq S_{X_n}) = \frac{5}{6}$.

So $E(X_{n+1}|X_n) = \frac{1}{6}(X_n + 1) + \frac{5}{6}(X_n + E(X_{n+1}|X_n))$.
Solve the equation, we can get that $E(X_{n+1}|X_n) = 6X_n + 1$.

And take the expectation on both sides, we can get that
$E(E(X_{n+1}|X_n)) = E(6X_n + 1) = 6E(X_n) + 1$.
From the Adam Law, we can get that
$E(E(X_{n+1}|X_n)) = E(X_{n+1})$.
So $E(X_{n+1}) = 6E(X_n) + 1$.

And since a_n is the expected number of rolls to get n consecutive same values, so $a_n = E(X_n)$.
i.e. $a_{n+1} = 6a_n + 1$.

So above all, we can get that $a_1 = 1$ and $a_{n+1} = 6a_n + 1$ for $n \geq 1$.

(d) From (c), we can get that $a_1 = 1$ and $a_{n+1} = 6a_n + 1$ for $n \geq 1$.
So $a_{n+1} + \frac{1}{5} = 6(a_n + \frac{1}{5})$.
i.e. $a_n + \frac{1}{5} = 6^{n-1}(a_1 + \frac{1}{5})$.
i.e. $a_n = \frac{6^n - 1}{5}$.

And when $n = 7$, we can calculate that $a_7 = \frac{6^7 - 1}{5} = 55987$.

So above all, $a_n = \frac{6^n - 1}{5}, \forall n \geq 1$.
And $a_7 = 55987$.

Problem 5

(a) The two Normals are linearly related.
And there correspondence $Corr(X,Y)=\rho$.
So we can have an intuitive guess that the slope of the linear relationship is ρ.

(b) Since $Y=cX+V$,
so $Corr(X,Y)=\dfrac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$.
Since $X,Y\sim N(0,1)$, so $Var(X)=Var(Y)=1$.
So $Corr(X,Y)=Cov(X,Y)=Cov(X,cX+V)=Cov(X,cX)+Cov(X,V)=cVar(X)+Cov(X,V)$.
Since $X\sim N(0,1)$, so $Var(X)=1$.
And since V is independent of X, so $Cov(X,V)=0$.
So $Corr(X,Y)=Cov(X,Y)=c$.
And since $Corr(X,Y)=\rho$, so $c=\rho$.
And $V=Y-cX=Y-\rho X$.

So above all, $c=\rho$ and $V=Y-\rho X$.

(c) Since $X=dY+W$,
so $Corr(X,Y)=\dfrac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$.
Since $X,Y\sim N(0,1)$, so $Var(X)=Var(Y)=1$.
So $Corr(X,Y)=Cov(X,Y)=Cov(dY+W,Y)=Cov(dY,Y)+Cov(W,Y)=dVar(Y)+Cov(W,Y)$.
Since $Y\sim N(0,1)$, so $Var(Y)=1$.
And since W is independent of Y, so $Cov(W,Y)=0$.
So $Corr(X,Y)=Cov(X,Y)=d$.
And since $Corr(X,Y)=\rho$, so $d=\rho$.
And $W=X-dY=X-\rho Y$.

So above all, $d=\rho$ and $W=X-\rho Y$.

(d) From (b), we can get that $Y=V+\rho X$.
So $E(Y|X)=E(V+\rho X|X)=E(V|X)+\rho E(X|X)=E(V|X)+\rho X$.
Since $X,Y\sim N(0,1)$, so $E(X)=E(Y)=0$.
Since $V=Y-\rho X$, so $E(V)=E(Y-\rho X)=E(Y)-\rho E(X)=0$.
Since V is independent of X, so $E(V|X)=E(V)=0$.
So $E(Y|X)=\rho X$.

Similarly, from (c), we can get that $X=W+\rho Y$.
So $E(X|Y)=E(W+\rho Y|Y)=E(W|Y)+\rho E(Y|Y)=E(W|Y)+\rho Y$.
Since $X,Y\sim N(0,1)$, so $E(X)=E(Y)=0$.
Since $W=X-\rho Y$, so $E(W)=E(X-\rho Y)=E(X)-\rho E(Y)=0$.
Since W is independent of Y, so $E(W|Y)=E(W)=0$.
So $E(X|Y)=\rho Y$.

So above all, $E(Y|X)=\rho X$ and $E(X|Y)=\rho Y$.

(e) Since correlation is symetric, so $Corr(X,Y)=Corr(Y,X)=\rho$.
And we could also see that in (d) that $E(Y|X)=\rho X$ and $E(X|Y)=\rho Y$.

Problem 5 continued on next page...

So using X to predict Y and using Y to predict X should have the same slope.
And since $E(X) = E(Y) = 0$,
so the best linear predictor of Y given X is the linear relation with slope ρ.