Probability & Statistics for EECS: Homework #5 Solution

Problem 1

A treasure is randomly placed in one of the nine realms (numbered from 1 to 9) attached to the Yggdrasill. Kratos searches for the treasure by asking Mimir yes-no questions. Find the expected number of questions until Kratos is sure about the location of the treasure, under each of the following strategies.

1. An enumeration strategy: Kratos asks questions of the form "is it in realm k?".
2. A bisection strategy: Kratos eliminates as close to half of the remaining realms as possible by asking questions of the form "is it in a realm numbered less than or equal to k?".

Solution

(a) Suppose that a random variable X represents the location of the treasure.
So, by enumeration strategy, we have the distribution:

$$P(X = n) = \frac{1}{9},\ n = 1, \ldots, 9.$$

Assume that $f(X)$ represents the number of problems needed to be sure about the location of the treasure in X.

$$f(X) = \begin{cases} X & X = 1, \ldots 8 \\ 8 & X = 9 \end{cases}$$

Therefore, the expected number of questions until Kratos is sure about the location is:

$$E(f(X)) = \sum_{n=1}^{9} P(X = n) f(n) = \sum_{n=1}^{8} \frac{n}{9} + \frac{8}{9} = \frac{44}{9}$$

(b) Assume that $f(X)$ represents the number of problems needed to be sure about the location of the treasure in X. By bisection strategy, we have:

$$f(X) = \begin{cases} 4 & X = 1, 2 \\ 3 & X = 3, \ldots, 9 \end{cases}$$

Therefore, the expected number of questions until Kratos is sure about the location is:

$$E(f(X)) = \sum_{n=1}^{9} P(X = n) f(n) = \sum_{n=1}^{2} \frac{4}{9} + \sum_{n=3}^{9} \frac{3}{9} = \frac{29}{9}$$

Problem 2

A particular Youtuber is known for his arbitrary steak-eating habit during live streaming. In each live streaming day, he orders a steak with doneness as one of well done, medium well, medium, medium rare, and rare (with equal probability, independent of other live streaming days). How many days of live streaming do you expect to watch before you see him eating steaks with each possible doneness at least one once (suppose you are a big fan who watches every live streaming)?

Solution

Let's denote the number of days of live streaming we need to watch all the doneness levels as X. Then, X can be expressed as the sum of X_1, X_2, X_3, X_4, and X_5, where X_i is the number of days of live streaming needed from seeing a steak with the $(i-1)$th doneness level for the first time to the steak with ith doneness levels for the first time.
So, it is easily to see that $X_i \sim FS(\frac{5-(i-1)}{5}) = FS(\frac{6-i}{5})$ $(i = 1, 2, 3, 4, 5)$.
And the expectation of X is:

$$\begin{aligned}
E(X) &= E(X_1 + X_2 + X_3 + X_4 + X_5) \\
&= E(X_1) + E(X_2) + E(X_3) + E(X_4) + E(X_5) \\
&= \sum_{i=1}^{5} \frac{1}{(6-i)/5} \\
&= \frac{137}{12} \\
&\approx 11.42
\end{aligned}$$

Problem 3

Mario and Zelda are independently performing independent Bernoulli trials. For concreteness, assume that Mario is flipping a nickel with probability p_1 of Heads and Zelda is flipping a penny with probability p_2 of Heads. Let $X_1, X_2, \cdots$ be Mario's results and $Y_1, Y_2, \cdots$ be Zelda's results, with $X_i \sim \text{Bern}(p_1)$ and $Y_j \sim \text{Bern}(p_2)$.

(a) Find the distribution and expected value of the first time at which they are simultaneously successful, *i.e.*, the smallest n such that $X_n = Y_n = 1$.

(b) Find the expected time until at least one has a success (including the success).

(c) For $p_1 = p_2$, find the probability that their first successes are simultaneous, and use this to find the probability that Mario's first success precedes Zelda's.

Solution

(a) Let N be the time at which they are simultaneously successful. Then $N - 1 \sim \text{Geom}(p_1 p_2)$ by the story of the Geometric (in other words, N has a First Success distribution with parameter $p_1 p_2$). So the PMF is $P(N = n) = p_1 p_2 (1 - p_1 p_2)^{n-1}$ for $n = 1, 2, \ldots$, and the mean is $E(N) = 1/(p_1 p_2)$.

(b) Let T be the time this happens, and let $q_1 = 1 - p_1, q_2 = 1 - p_2$. Define a new sequence of Bernoulli trials by saying that the jth trial is a success if at least one of the two people succeeds in the jth trial. These trials have probability $1 - q_1 q_2$ of success, which implies that $T - 1 \sim \text{Geom}(1 - q_1 q_2)$. Therefore, $E(T) = 1/(1 - q_1 q_2) = \frac{1}{p_1 + p_2 - p_1 p_2}$.

(c) Let T_1 and T_2 be the first times at which Mario and Zelda are successful, respectively. Let $p = p_1 = p_2$ and $q = 1 - p$. Then

$$P(T_1 = T_2) = \sum_{n=1}^{\infty} P(T_1 = n \mid T_2 = n) P(T_2 = n) = \sum_{n=1}^{\infty} p^2 q^{2(n-1)} = \frac{p^2}{1 - q^2} = \frac{p}{2 - p}.$$

By symmetry,

$$1 = P(T_1 < T_2) + P(T_2 < T_1) + P(T_1 = T_2) = 2P(T_1 < T_2) + P(T_1 = T_2).$$

So

$$P(T_1 < T_2) = \frac{1}{2}\left(1 - \frac{p}{2 - p}\right) = \frac{1 - p}{2 - p}$$

Problem 4

Let X and Y be independent geometric random variables, where X has parameter p and Y has parameter q.

(a) What is the probability that $X = Y$?

(b) What is $E[\max(X, Y)]$?

(c) What is $P(\min(X, Y) = k)$?

(d) What is $E[X|X \leq Y]$?

Solution

(a)

$$\begin{aligned}
\mathbb{P}[X = Y] &= \sum_x (1-p)^x p(1-q)^x q \\
&= \sum_x [(1-p)(1-q)]^x pq \\
&= \frac{1-(1-p)(1-q)}{pq} \\
&= \frac{pq}{p+q-pq}
\end{aligned}$$

(b) Because $\max(X, Y) = X + Y - \min(X, Y)$ so $\mathbb{E}[\max(X, Y)] = \mathbb{E}[X] + \mathbb{E}[Y] - \mathbb{E}[\min(X, Y)]$. From below, in part (c), we know that $\min(X, Y)$ is a geometric random variable mean $p + q - pq$. Therefore, $\mathbb{E}[\min(X, Y)] = \frac{1+pq-p-q}{p+q-pq}$, and we get

$$\mathbb{E}[\max(X, Y)] = \frac{1-p}{p} + \frac{1-q}{q} - \frac{1+pq-p-q}{p+q-pq}$$

(c) We split this event into two disjoint events.

$$\begin{aligned}
\mathbb{P}[\min(X, Y) = k] &= \mathbb{P}[X = k, Y \geq k] + \mathbb{P}[X > k, Y = k] \\
&= \mathbb{P}[X = k]\mathbb{P}[Y \geq k] + \mathbb{P}[X > k]\mathbb{P}[Y = k]
\end{aligned}$$

because $\mathbb{P}[X > k] = 1 - \mathbb{P}[x \leq k] = 1 - \sum_{x=0}^{k} (1-p)^x p = (1-p)^k(1-p)$. Finally, we get

$$\begin{aligned}
\mathbb{P}[\min(X, Y) = k] &= (1-p)^k p(1-q)^k + (1-p)^k(1-p)(1-q)^k q \\
&= [(1-p)(1-q)]^k(p + (1-p)q) \\
&= [(1-p)(1-q)]^k(p + q - pq)
\end{aligned}$$

(d)

$$\begin{aligned}
\mathbb{E}[X \mid X \leq Y] &= \sum_{x \geq 0} x\mathbb{P}[X = x \mid x \leq Y] \\
&= \sum_{x \geq 0} x\frac{\mathbb{P}[X = x, x \leq Y]}{\mathbb{P}[X \leq Y]}
\end{aligned}$$

First, let's consider the denominator.

$$\begin{aligned}
\mathbb{P}[X \leq Y] &= \sum_{z\geq 0} \mathbb{P}[X = z, z \leq Y] \\
&= \sum_{z} \mathbb{P}[X = z]\mathbb{P}[Y \geq z] \\
&= \sum_{z} (1-p)^z p(1-q)^z \\
&= \sum_{z} [(1-p)(1-q)]^z p \\
&= p\sum_{z} [(1-p-q+pq)]^z \\
&= \frac{p}{p+q-pq}
\end{aligned}$$

Now we can compute the whole equation.

$$\begin{aligned}
\mathbb{E}[X \mid X \leq Y] &= \frac{p+q-pq}{p} \sum_{x\geq 0} x\mathbb{P}[X = x]\mathbb{P}[x \leq Y] \\
&= \frac{p+q-pq}{p} \sum_{x} x(1-p)^x p(1-q)^x \\
&= (p+q-pq)\sum_{x} x(1+pq-p-q)^x
\end{aligned}$$

This is equal to the expectation of a geometric random variable with mean $p+q-pq$. Therefore

$$\mathbb{E}[X \mid X \leq Y] = \frac{1+pq-p-q}{p+q-pq}$$

Problem 5

You plan to eat m meals at a certain restaurant, where you have never eaten before. Each time, you will order one dish (without replacement).
The restaurant has n dishes on the menu, with $n \geq m$. Assume that if you had tried all the dishes, you would have a definite ranking of them from 1 (your least favorite) to n (your favorite). If you knew which your favorite was, you would be happy to order it always (you never get tired of it).
Before you've eaten at the restaurant, this ranking is completely unknown to you. After you've tried some dishes, you can rank those dishes amongst themselves, but don't know how they compare with the dishes you haven't yet tried. There is thus an *exploration-exploitation trade-off*: should you try new dishes, or should you order your favorite among the dishes you have tried before?
A natural strategy is to have two phases in your series of visits to the restaurant: an exploration phase, where you try different dishes each time, and an exploitation phase, where you always order the best dish you obtained in the exploration phase. Let k be the length of the exploration phase (so $m-k$ is the length of the exploitation phase). Your goal is to maximize the expected sum of the ranks of the dishes you eat there (the rank of a dish is the true rank from 1 to n that you would give that dish if you could try all the dishes). Show that the optimal choice is

$$k=\sqrt{2(m+1)}-1$$

or this rounded up or down to an integer if needed. Do this in the following steps:

(a) Let X be the rank of the best dish that you find in the exploration phase. Find the expected sum of the ranks of the dishes you eat, in terms of $E[X]$.

(b) Find the PMF of X, as a simple expression in terms of binomial coefficients.

(c) Show that

$$E[X]=\frac{k(n+1)}{k+1}.$$

(d) Use calculus to find the optimal value of k.

Solution

(a) Let R_j be the rank of the jth dish that you try, and R be the sum of the ranks. Then

$$R=R_1+\cdots+R_k+(m-k)X$$

and

$$E\left(R_1\right)=\frac{1}{n} \sum_{i=1}^n i=(n+1)/2.$$

The result for $E\left(R_1\right)$ can be obtained by summing the arithmetic series (see the math appendix) or using indicator r.v.s. For the latter, note that R_1 is 1 plus the number of dishes ranked below the first dish. Create an indicator for each of the $n-1$ other dishes of whether it is ranked below the first dish. Each of those indicators has expected value 1/2 by symmetry, so by linearity we again have $E\left(R_j\right)=1+(n-1)/2=(n+1)/2$. Therefore,

$$E(R)=k(n+1)/2+(m-k)E(X)$$

(b) The support is $\{k, k+1, \ldots, n\}$. The PMF is given by

$$P(X = j) = \frac{\left(\begin{array}{c} j-1 \\ k-1 \end{array} \right)}{\left(\begin{array}{c} n \\ k \end{array} \right)}$$

for j in the support, since $X = j$ is equivalent to getting the dish with rank j and $k-1$ worse dishes. (To see that this is a valid PMF, we can use the hockey stick identity referred to in the hint for the next part.)
(c) Using the identities in the hint and then writing the binomial coefficients in terms of factorials, we have

$$E(X) = \frac{1}{\left(\begin{array}{c} n \\ k \end{array} \right)} \sum_{j=k}^{n} j \left(\begin{array}{c} j-1 \\ k-1 \end{array} \right) = \frac{k}{\left(\begin{array}{c} n \\ k \end{array} \right)} \sum_{j=k}^{n} \left(\begin{array}{c} j \\ k \end{array} \right) = \frac{k \left(\begin{array}{c} n+1 \\ k+1 \end{array} \right)}{\left(\begin{array}{c} n \\ k \end{array} \right)} = \frac{k(n+1)}{k+1}$$

(d) Replacing the integer k by a real variable x and using the above results, we obtain

$$g(x) = \frac{x}{2} + \frac{(m-x)x}{x+1}$$

as the function to maximize (after dividing by the constant n+1). Then

$$g'(x) = \frac{1}{2} + \frac{(x+1)(m-2x) - (m-x)x}{(x+1)^2} = \frac{1}{2} + \frac{m - (x+1)^2 + 1}{(x+1)^2} = \frac{m+1}{(x+1)^2} - \frac{1}{2}.$$

Setting $g'(x) = 0$, we have

$$x_0 = \sqrt{2(m+1)} - 1$$

as the positive solution. Since g is increasing from 0 to x_0 and decreasing from x_0 to m, the local and absolute maximum of g on $[0, m]$ is at x_0, and the optimal k is x_0, rounded up or down to an integer if needed.