Probability & Statistics for EECS: Homework #11 Solution

Problem 1

Let X and Y be i.i.d. $\mathcal{N}(0,1)$, and let S be a random sign (1 or -1, with equal probabilities) independent of (X,Y).

(a) Determine whether or not $(X,Y,X+Y)$ is MVN.

(b) Determine whether or not $(X,Y,SX+SY)$ is MVN.

(c) Determine whether or not (SX,SY) is MVN.

Solution

(a) Yes, since $aX+bY+c(X+Y)=(a+c)X+(b+c)Y$ is Normal for any a,b,c.

(b) No, since $X+Y+(SX+SY)=(1+S)X+(1+S)Y$ is 0 with probability 1/2 (since it is 0 if $S=-1$ and a non-degenerate Normal if $S=1$).

(c) Yes. To prove this, let's show that any linear combination

$$a(SX)+b(SY)=S(aX+bY)$$

is Normal. We already know that

$$aX+bY\sim\mathcal{N}\left(0,a^2+b^2\right)$$

By the symmetry of the Normal, as discussed in Example 7.5.2, $SZ\sim\mathcal{N}(0,1)$ if $Z\sim\mathcal{N}(0,1)$ and S is a random sign independent of Z. Letting

$$Z=\frac{aX+bY}{\sqrt{a^2+b^2}},$$

we have

$$S(aX+bY)=\sqrt{a^2+b^2}\cdot SZ\sim\mathcal{N}\left(0,a^2+b^2\right)$$

Problem 2

Let X and Y be i.i.d. $\mathcal{N}(0,1)$ r.v.s, $T = X + Y$, and $W = X - Y$. Show that T and W are independent using two methods: 1) properties of MVN and 2) change of variables.

Solution

1. Properties of MVN: Since $(X+Y, X-Y)$ is Bivariate Normal and

   $$\operatorname{Cov}(X+Y, X-Y) = \operatorname{Var}(X) - \operatorname{Cov}(X,Y) + \operatorname{Cov}(Y,X) - \operatorname{Var}(Y) = 0$$

   $X+Y$ is independent of $X-Y$. Furthermore, they are i.i.d. $\mathcal{N}(0,2)$. By the same method, we have that if $X \sim \mathcal{N}\left(\mu_1, \sigma^2\right)$ and $Y \sim \mathcal{N}\left(\mu_2, \sigma^2\right)$ are independent (with the same variance), then $X+Y$ is independent of $X-Y$.

   It can be shown that the independence of the sum and difference is a unique characteristic of the Normal! That is, if X and Y are i.i.d. and $X+Y$ is independent of $X-Y$, then X and Y must have Normal distributions.

2. Change of variables: Let $t = x+y, w = x-y$, so $x = (t+w)/2, y = (t-w)/2$. The Jacobian matrix is

   $$\frac{\partial(x,y)}{\partial(t,w)} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

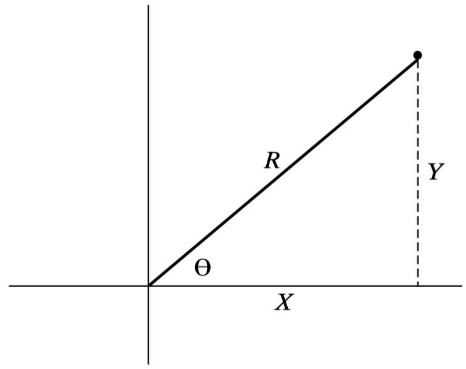
   which has absolute determinant $1/2$. By the change of variables formula, the joint PDF of T and W is

   $$\begin{aligned} f_{T,W}(t,w) &= f_{X,Y}\left(\frac{t+w}{2}, \frac{t-w}{2}\right) \cdot \frac{1}{2} \\ &= \frac{1}{4\pi} \exp\left(-(t+w)^2/8 - (t-w)^2/8\right) \\ &= \frac{1}{4\pi} \exp\left(-t^2/4\right) \exp\left(-w^2/4\right) \end{aligned}$$

   which shows that T and W are i.i.d. $\mathcal{N}(0,2)$.

Problem 3

Let (X, Y) denote a random point in the plane, and assume that the rectangular coordinates X and Y are i.i.d. $\mathcal{N}(0, 1)$ r.v.s. Find the joint distribution of R and Θ (shown in the following figure). Are R and Θ independent?



Solution

Since $X = R\cos\theta$, $Y = R\sin\theta$,

$$\begin{aligned} f_{R,\Theta}(r,\theta) &= \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| f_{X,Y}(x,y) \\ &= \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} f_{X,Y}(x,y) \\ &= r f_{X,Y}(x,y) \end{aligned}$$

Since X, Y are i.i.d $\mathcal{N}(0, 1)$, we can solve the joint PDF of X, Y:

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} = \frac{1}{2\pi} e^{-\frac{r^2}{2}}$$

Therefore, the joint PDF of R, Θ is:

$$f_{R,\Theta}(r,\theta) = \frac{1}{2\pi} r e^{-\frac{r^2}{2}}$$

Since $\int_0^{2\pi} \frac{1}{2\pi} d\theta = 1$, $\int_0^{+\infty} r e^{-\frac{r^2}{2}} dr = 1$, by pattern matching, we can conclude that $f_R(r) = r e^{-\frac{r^2}{2}}, r \in [0, +\infty]$, and $f_\Theta(\theta) = \frac{1}{2\pi}, \theta \in [0, 2\pi)$.

Problem 4

(a) Let X and Y be i.i.d. Expo(λ), and transform them to $T = X+Y$, $W = X/Y$. Find the marginal PDFs of T and W, and the joint PDF of T and W.

(b) Let X, Y, Z be i.i.d. Unif$(0,1)$, and $W = X+Y+Z$. Find the PDF of W using convolution.

(c) Let X and Y be i.i.d. Expo(λ) r.v.s and $M = \max(X,Y)$. Show that M has the same distribution as $X + \frac{1}{2}Y$ using two methods: 1) properties of the Exponential and 2) convolution.

Solution

(a) We can use the change of variables formula, but it is faster to relate this problem to the bank-post office story. Let $U = X/(X+Y)$. By the bank-post office story, T and U are independent, with $T \sim \text{Gamma}(2,\lambda)$ and $U \sim \text{Unif}(0,1)$. But

$$W = \frac{X/(X+Y)}{Y/(X+Y)} = \frac{U}{1-U}$$

is a function of U. So T and W are independent. The CDF of W is

$$P(W \le w) = P(U \le w/(w+1)) = w/(w+1)$$

for $w > 0$ (and 0 for $w \le 0$). So the PDF of W is

$$f_W(w) = \frac{(w+1)-w}{(w+1)^2} = \frac{1}{(w+1)^2}$$

for $w \ge 0$. And since $T \sim \text{Gamma}(2,\lambda)$, the marginal PDF of T is

$$f_T(t) = (\lambda t)^2 e^{-\lambda t}\frac{1}{t}$$

for $t \ge 0$.

Since T and W are independent, their joint PDF is

$$f_{T,W}(t,w) = (\lambda t)^2 e^{-\lambda t}\frac{1}{t} \cdot \frac{1}{(w+1)^2},$$

for $t \ge 0$ and $w \ge 0$.

(b) Hint: We already know the PDF of $X+Y$. Be careful about limits of integration in the convolution integral; there are 3 cases that should be considered separately.

Solution: Let $T = X+Y$. As shown in Example 8.2.5, T has a triangle-shaped density:

$$f_T(t) = \begin{cases} t, & \text{if } 0 < t \le 1 \\ 2-t, & \text{if } 1 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

We will find the PDF of $W = T+Z$ using a convolution:

$$f_W(w) = \int_{-\infty}^{\infty} f_T(t) f_Z(w-t)dt = \int_{w-1}^{w} f_T(t)dt,$$

Problem 4 continued on next page...

since $f_Z(w-t)$ is 0 except when $0 < w - t < 1$, which is equivalent to $t > w - 1, t < w$. Consider the 3 cases $0 < w < 1, 1 < w < 2$, and $2 < w < 3$ separately.

Case 1: $0 < w < 1$ In this case,

$$\int_{w-1}^{w} f_T(t)dt = \int_0^w t dt = \frac{w^2}{2}$$

Case 2: $1 < w < 2$. In this case,

$$\int_{w-1}^{w} f_T(t)dt = \int_{w-1}^{1} t dt + \int_1^w (2-t)dt = -w^2 + 3w - \frac{3}{2}.$$

Case 3: $2 < w < 3$. In this case,

$$\int_{w-1}^{w} f_T(t)dt = \int_{w-1}^{2} (2-t)dt = \frac{w^2 - 6w + 9}{2}.$$

Thus, the PDF of W is the piecewise quadratic function

$$f_W(w) = \begin{cases} \frac{w^2}{2}, & \text{if } 0 < w \le 1 \\ -w^2 + 3w - \frac{3}{2}, & \text{if } 1 < w \le 2 \\ \frac{(w-3)^2}{2}, & \text{if } 2 < w < 3 \\ 0, & \text{otherwise.} \end{cases}$$

(c) 1) Properties of the Exponential: As in Example 7.3.6, imagine that two students are independently trying to solve a problem. Suppose that X and Y are the times required. Let $L = \min(X, Y)$, and write $M = L + (M - L)$. $L \sim \text{Expo}(2\lambda)$ is the time it takes for the first student to solve the problem and then by the memoryless property, the additional time until the second student solves the problem is $M - L \sim \text{Expo}(\lambda)$, independent of L. Since $\frac{1}{2}Y \sim \text{Expo}(2\lambda)$ is independent of $X \sim \text{Expo}(\lambda), M = L + (M - L)$ has the same distribution as $\frac{1}{2}Y + X$.

2) Convolution: Since $X \sim \text{Expo}(\lambda), Y/2 \sim \text{Expo}(2\lambda)$, so we have the PDF of $Z = X + Y/2$:

$$\begin{aligned} f_Z(z) &= \int_0^z f_X(x) f_{Y/2}(z-x)dx \\ &= \int_0^z \lambda e^{-\lambda x} \times 2\lambda e^{-2\lambda(z-x)}dx \\ &= 2\lambda e^{-2\lambda z}(e^{\lambda z} - 1) \end{aligned}$$

where $z \in [0, +\infty]$. And the CDF of M is:

$$F_M(m) = F_X(m)F_Y(m) = (1 - e^{-\lambda m})^2$$

So the PDF of M is:

$$f_M(m) = \frac{dF_M(m)}{dm} = 2\lambda e^{-2\lambda m}(e^{\lambda m} - 1)$$

where $m \in [0, +\infty]$. Therefore, M has the same distribution as $\frac{1}{2}Y + X$.

Problem 5

Programming Assignment:

(a) Use the Box-Muller Method to obtain the samples from the standard normal distribution $\mathcal{N}(0,1)$. You need to plot the pictures of both histogram and the theoretical PDF.

(b) Based on (a), generate samples from the standard bivariate Normal distribution, where the correlation is $\rho \in (-1,1)$, and the marginal PDFs are both $\mathcal{N}(0,1)$.

(c) According to the following picture format, plot the joint PDFs and the corresponding contours of standard bivariate Normal distribution with correlation $\rho = 0, 0.3, 0.5, 0.7, 0.9$.