Probability & Statistics for EECS: Homework #4 Solution

Problem 1

Consider the original Monty Hall problem, except that Monty enjoys opening door 2 more than he enjoys opening door 3, and if he has a choice between opening these two doors, he opens door 2 with probability p, where $\frac{1}{2} \leq p \leq 1$.
To recap: there are three doors, behind one of which there is a car (which you want), and behind the others there are goats (which you don't want). Initially, all possibilities are equally likely for where the car is. You choose a door, which for concreteness we assume is door 1. Monty (knows which door has the car) then opens a door to reveal a goat, and offers you the option of switching.

(a) Find the unconditional probability that the strategy of always switching succeeds (unconditional in the sense that we do not condition on which of doors 2 or 3 Monty opens).

(b) Find the probability that the strategy of always switching succeeds, given that Monty opens door 2 (assume we always choose door 1 first).

(c) Find the probability that the strategy of always switching succeeds, given that Monty opens door 3 (assume we always choose door 1 first).

Solution

(a) Let C_j be the event that the car is hidden behind door j and let W be the event that we win using the switching strategy. Using the law of total probability, we can find the unconditional probability of winning in the same way as in class:

$$\begin{aligned} P(W) &= P(W \mid C_1) P(C_1) + P(W \mid C_2) P(C_2) + P(W \mid C_3) P(C_3) \\ &= 0 \cdot 1/3 + 1 \cdot 1/3 + 1 \cdot 1/3 = 2/3. \end{aligned}$$

(b) A tree method works well here (delete the paths which are no longer relevant after the conditioning, and reweight the remaining values by dividing by their sum), or we can use Bayes' rule and the law of total probability (as below).
Let D_i be the event that Monty opens Door i. Note that we are looking for $P(W \mid D_2)$, which is the same as $P(C_3 \mid D_2)$ as we first choose Door 1 and then switch to Door 3. By Bayes' rule and the law of total probability,

$$\begin{aligned} P(C_3 \mid D_2) &= \frac{P(D_2 \mid C_3) P(C_3)}{P(D_2)} \\ &= \frac{P(D_2 \mid C_3) P(C_3)}{P(D_2 \mid C_1) P(C_1) + P(D_2 \mid C_2) P(C_2) + P(D_2 \mid C_3) P(C_3)} \\ &= \frac{1 \cdot 1/3}{p \cdot 1/3 + 0 \cdot 1/3 + 1 \cdot 1/3} \\ &= \frac{1}{1+p}. \end{aligned}$$

(c) The structure of the problem is the same as part (b) (except for the condition that $p \geq 1/2$, which was no needed above). Imagine repainting doors 2 and 3, reversing which is called which. By part (b) with $1-p$ in place of p, $P(C_2 \mid D_3) = \frac{1}{1+(1-p)} = \frac{1}{2-p}$.

Problem 2

(a) Is there a discrete distribution with support $\{1, 2, 3, \dots\}$, such that the value of the PMF at n is proportional to $1/n$?

(b) Is there a discrete distribution with support $\{1, 2, 3, \dots\}$, such that the value of the PMF at n is proportional to $1/n^2$?

Solution

(a) No, since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so it is not possible to find a constant c such that $\sum_{n=1}^{\infty} \frac{c}{n}$ converges to 1.

(b) Yes, since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Let a be the sum of the series (it turns out that $a = \pi^2/6$) and $b = 1/a$. Then $p(n) = b/n^2$ for $n = 1, 2, \dots$ is a valid PMF.

Problem 3

Let X be a random day of the week, coded so that Monday is 1, Tuesday is 2, *etc.* (so X takes values $1, 2, \ldots, 7$ with equal probabilities). Let Y be the next day after X. Do X and Y have the same distribution? What is $P(X < Y)$?

Solution

Yes, X and Y have the same distribution, since Y is also equally likely to represent any day of the week. However, X is likely to be less than Y. Specifically,

$$P(X < Y) = P(X \neq 7) = \frac{6}{7}$$

In general, if Z and W are independent r.v.s with the same distribution, then $P(Z < W) = P(W < Z)$ by symmetry. Here though, X and Y are dependent, and we have $P(X < Y) = 6/7, P(X = Y) = 0, P(Y < X) = 1/7$.

Problem 4

There are two coins, one with probability p_1 of Heads and the other with probability p_2 of Heads. One of the coins is randomly chosen (with equal probabilities for the two coins). It is then flipped $n \geq 2$ times. Let X be the number of times it lands Heads.

(a) Find the PMF of X.

(b) What is the distribution of X if $p_1 = p_2$?

(c) Give an intuitive explanation of why X is not Binomial for $p_1 \neq p_2$.

Solution

(a) By LOTP, conditioning on which coin is chosen, we have

$$P(X = k) = \frac{1}{2} \binom{n}{k} p_1^k (1 - p_1)^{n-k} + \frac{1}{2} \binom{n}{k} p_2^k (1 - p_2)^{n-k},$$

for $k = 0, 1, \ldots, n$.

(b) For $p_1 = p_2$, the above expression reduces to the $\text{Bin}(n, p_1)$ PMF.

(c) A mixture of two Binomials is not Binomial (except in the degenerate case $p_1 = p_2$).
Marginally, each toss has probability $(p_1 + p_2)/2$ of landing Heads, but the tosses are not independent since earlier tosses give information about which coin was chosen, which in turn gives information about later tosses.
Let n be large, and imagine repeating the entire experiment many times (each repetition consists of choosing a random coin and flipping it n times). We would expect to see either approximately np_1 Heads about half the time, and approximately np_2 Heads about half the time. In contrast, with a $\text{Bin}(n, p)$ distribution we would expect to see approximately np Heads; no fixed choice of p can create the behavior described above.

Problem 5

For x and y binary digits (0 or 1), let $x \bigoplus y$ be 0 if $x = y$ and 1 if $x \neq y$ (this operation is called exclusive or (often abbreviated to XOR), or addition mod 2).

(a) Let $X \sim \text{Bern}(p)$ and $Y \sim \text{Bern}(1/2)$, independently. What is the distribution of $X \bigoplus Y$?

(b) With notation as in sub-problem (a), is $X \bigoplus Y$ independent of X? Is $X \bigoplus Y$ independent of Y? Be sure to consider both the case $p = 1/2$ and the case $p \neq 1/2$.

(c) Let $X_1, \ldots, X_n$ be i.i.d. (*i.e.*, independent and identically distributed) Bern(1/2) R.V.s. For each nonempty subset J of $\{1, 2, \ldots, n\}$, let

$$Y_J = \bigoplus_{j \in J} X_j.$$

Show that $Y_J \sim \text{Bern}(1/2)$ and that these $2^n - 1$ R.V.s are pairwise independent, but not independent.

Solution

(a) The distribution of $X \oplus Y$ is Bern(1/2), no matter what p is:

$$\begin{aligned} P(X \oplus Y = 1) &= P(X \oplus Y = 1 \mid X = 1)P(X = 1) + P(X \oplus Y = 1 \mid X = 0)P(X = 0) \\ &= P(Y = 0)P(X = 1) + P(Y = 1)P(X = 0) \\ &= p/2 + (1 - p)/2 \\ &= 1/2 \end{aligned}$$

(b) The conditional distribution of $X \oplus Y \mid (X = x)$ is Bern(1/2) , as shown within the above calculation. This conditional distribution does not depend on x, so $X \oplus Y$ is independent of X. This result and the result from (a) make sense intuitively: adding Y destroys all information about X, resulting in a fair coin flip independent of X. Note that given $X = x$, $X \oplus Y$ is x with probability 1/2 and $1 - x$ with probability 1/2, which is another way to see that $X \oplus Y \mid (X = x) \sim \text{Bern}(1/2)$.

If $p = 1/2$, then the above reasoning shows that $X \oplus Y$ is independent of Y. So $X \oplus Y$ is independent of X and independent of Y, even though it is clearly not independent of the pair (X, Y).

But if $p \neq 1/2$, then $X \oplus Y$ is not independent of Y. The conditional distribution of $X \oplus Y \mid (Y = y)$ is $\text{Bern}(p(1 - y) + (1 - p)y)$, since

$$P(X \oplus Y = 1 \mid Y = y) = P(X \oplus y = 1) = P(X \neq y) = p(1 - y) + (1 - p)y.$$

(c) These r.v.s are not independent since, for example, if we know $Y_{\{1\}}$ and $Y_{\{2\}}$, then we know $Y_{\{1,2\}}$ via $Y_{\{1,2\}} = Y_1 + Y_2$. But they are pairwise independent. To show this, let's use the notation and approach from the hint. We can write Y_J and Y_K in their stated forms by partitioning $J \cup K$ (the set of indices that appear in Y_J or Y_K) into the sets $J \cap K$, $J \cap K^c$, and $J^c \cap K$.
Assume that $J \cap K$ is nonempty (the case where it is empty was handled in the hint). By (a), $A \sim \text{Bern}(1/2)$. Then for $y \in \{0, 1\}, z \in \{0, 1\}$,

$$
\begin{aligned}
P\left(Y_J=y, Y_K=z\right) &= \frac{1}{2}P(A \oplus B = y, A \oplus C = z \mid A=1) + \frac{1}{2}P(A \oplus B = y, A \oplus C = z \mid A=0) \\
&= \frac{1}{2}P(1 \oplus B = y)P(1 \oplus C = z) + \frac{1}{2}P(0 \oplus B = y)P(0 \oplus C = z) \\
&= \frac{1}{8} + \frac{1}{8} \\
&= \frac{1}{4} \\
&= P\left(Y_J = y\right)P\left(Y_K = z\right),
\end{aligned}
$$

using the fact that A, B, C are independent and the fact that A, B, C, Y_J, Y_K are Bern(1/2). Thus, Y_J and Y_K are independent.