Probability & Statistics for EECS: Homework #07

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Problem 1

(a) $F(x) = \dfrac{2}{\pi} sin^{-1}(\sqrt{x}), x \in (0, 1)$.
1.Since at the range of $[0, 1]$, $\sqrt{x}$ and $sin^{-1}(x) = arcsin(x)$ are continuous, so $sin^{-1}(\sqrt{x})$ is continuous.
So $F(x)$ is continuous.
So $\lim\limits_{x \to 0^+} F(x) = \dfrac{2}{\pi} sin^{-1}(0) = \dfrac{2}{\pi} \cdot 0 = 0$.
And $\lim\limits_{x \to 1^-} F(1) = \dfrac{2}{\pi} sin^{-1}(1) = \dfrac{2}{\pi} \cdot \dfrac{\pi}{2} = 1$.
And we are given that $F(x) = 0, x \le 0$, and $F(x) = 1, x \ge 1$, so $F(x)$ is a continuous in the domain. And $\lim\limits_{x \to -\infty} F(x) = 0$, $\lim\limits_{x \to +\infty} F(x) = 1$

2.Also, $f(x) = F'(x) = \dfrac{1}{\pi\sqrt{x(1-x)}}, x \in (0, 1)$.
$f(x) = 0, x \in (-\infty, 0], [1, +\infty)$.
But $\lim\limits_{x \to +0^+} f(x) \to +\infty$ and $\lim\limits_{x \to +1^-} f(x) \to +\infty$.
So only for $x = 0$ and $x = 1$, $f(x)$ is not continuous.
i.e. $F(x)$ only have two endpoints($x = 0, x = 1$) that is continuous but not differentiable. And for other period, $F(x)$ is differentiable.

3.from 2. we know that $f(x) = \dfrac{1}{\pi\sqrt{x(1-x)}}, x \in (0, 1)$, and for other points, $f(x) = 0$.
Since $x \in (0, 1)$, so $x(1 - x) > 0$, so $F'(x) = f(x) > 0$.
So for all points in the domain, we have $f(x) \ge 0$. i.e. the PDF is valid.
And in the period of $(0, 1)$, the CDF F is increasing.

So combine the above three parts, we have $F(x)$ is a continuous function in the domain, have finite endpoints not differentiable, and have a valid PDF.

So above all, F is a valid CDF,
and the correspinding PDF is $f(x) = \dfrac{1}{\pi\sqrt{x(1-x)}}, x \in (0, 1)$; $f(x) = 0, otherwise$.

(b) Although $\lim\limits_{x \to +0^+} f(x) \to +\infty$ and $\lim\limits_{x \to +1^-} f(x) \to +\infty$.
But the probability at these points are 0.
i.e. the small integral at that part is 0.
Proof: We already know that $F(x)$ is a continuous function in the domain.
So $\forall x_0 \in R, \lim\limits_{\delta \to 0} |F(x + \delta) - F(x)| = 0$.
And the small integral is that $\lim\limits_{\delta \to 0^+} \int_0^\delta f(x) = \lim\limits_{\delta \to 0^+} F(\delta) - F(0) = 0$, $\lim\limits_{\delta \to 1^-} \int_\delta^1 f(x) = \lim\limits_{\delta \to 1^-} F(1) - F(\delta) = 0$, so the probability at these points are 0.

So above all, the probability that $x \to 0$ and $x \to 1$ is 0.
So the PDF is valid.

Problem 2

Since μ is the mean of the distribution with CDF F.
So $\mu = \int_{-\infty}^{+\infty} xf(x)dx$, where $f(x)$ is the PDF of the distribution.
Since F is the CDF of the distribution, and f is the PDF of the distribution. So $f(x) = F'(x)$
Since F is continuous and strickly increasing, so its quantile function is injective.
So let $u = F(x)$, then we can get that $x = F^{-1}(u)$, and $du = dF(x) = f(x)dx$.
With this mapping, we can get that when $x \in (-\infty, +\infty)$, $u \in (0, 1)$.
In other word, when $u \in (0, 1)$, $x \in (-\infty, +\infty)$
From the beginning, we know that $\int_{-\infty}^{+\infty} xf(x)dx = \mu$,
so the area under the curve of the quantile function from 0 to 1 is that
$\int_0^1 F^{-1}(u)du = \int_{-\infty}^{+\infty} xf(x)dx = \mu$.
So above all, the area under the curve of the quantile function from 0 to 1 is μ.

Problem 3

Suppose that the CDF of X is $F(x)$.
Since $X = max(U_1, \cdots, U_n)$, and since $U_i \sim Unif(0,1)$, so $\forall x \leq 0, F(x) = 0$, and $\forall x \geq 1, F(x) = 1$.
Then for $x \in (0,1)$:
$F(x) = P(X \leq x) = P(max(U_1, \cdots, U_n) \leq x) = P(U_1 \leq x, \cdots, U_n \leq x)$.
Since $U_1, \cdots, U_n$ are i.i.d. So $P(U_1 \leq x, \cdots, U_n \leq x) = P(U_1 \leq x)P(U_2 \leq x) \cdots P(U_n \leq x)$.
And because $U_i \sim Unif(0,1)$, so $P(U_i \leq x) = x$.
So $F(x) = x^n$.
So the CDF of X is $F(x) = x^n, x \in (0,1)$. And $\forall x \leq 0, F(x) = 0$, $\forall x \geq 1, F(x) = 1$
So the PDF of X is $f(x) = F'(x) = nx^{n-1}, x \in (0,1)$. $f(x) = 0$, otherwise.

Let the survival function of X be $G(x) = 1 - F(x)$.
Then $S(x) = 1 - x^n, x \in (0,1)$. $G(x) = 0, x \in [1, +\infty)$.
Since X is nonnegative r.v.
so $E(X) = \int_0^{+\infty} G(x)dx = \int_0^1 (1 - x^n)dx = 1 - \dfrac{1}{n+1} = \dfrac{n}{n+1}$.

So above all, the PDF of X is $f(x) = nx^{n-1}, x \in (0,1)$. $f(x) = 0$, otherwise.
And $E(X) = \dfrac{n}{n+1}$.

Problem 4

(a) Since $R = \frac{X}{Y}$, where X is the shorter piece, and Y is the longer case, so $0 < R < 1$.
Let $F(r)$ be the CDF of R. Then $\forall r \le 0, F(r) = 0$, and $\forall x \ge 1, F(r) = 1$.
As for $r \in (0,1)$, $r = \frac{X}{Y}$, so $X = r \cdot Y$. And since $X + Y = 1$, so $X = \frac{r}{r+1}, Y = \frac{1}{r+1}$.
Let $U \sim Unif(0,1)$. So when $u \in (0,1), P(U \le u) = u$, and with the symmetry, $P(1 - U \le u) = u$.
Suppose the the $U = u$ is the break point of the stick.
So $F(r) = P(R \le r) = P(X \le \frac{r}{r+1}) = P(u \le \frac{r}{r+1} \ or\ 1 - u \le \frac{r}{r+1}) = \frac{2r}{r+1}$.

And let $f(r)$ be the PDF of R.
Then when $r \in (0,1), f(r) = F'(r) = \frac{2}{(r+1)^2}$. And $f(r) = 0$, otherwise.

So above all, the PDF of R is $f(r) = \frac{2}{(r+1)^2}, r \in (0,1)$, and $f(r) = 0$, otherwise.

And the CDF of R is $F(r) = \frac{2r}{r+1}$. And $\forall r \le 0, F(r) = 0$, $\forall x \ge 1, F(r) = 1$.

(b) $E(R) = \int_{-\infty}^{+\infty} rf(r)dr = \int_0^1 \frac{2r}{(r+1)^2} = \int_0^1 \frac{2(r+1)-2}{(r+1)^2} d(r+1) = \int_1^2 \frac{2x-2}{x^2} dx = \int_1^2 (\frac{2}{x} - \frac{2}{x^2})dx$
$= (2ln|x| + \frac{2}{x})\Big|_{x=1}^{2} = 2ln2 - 1$.

So above all, the expected value of R is $E(R) = 2ln2 - 1$.

(c) With LOTUS, we can get that
$E(\frac{1}{R}) = \int_{-\infty}^{+\infty} \frac{1}{r} f(r)dr = \int_0^1 \frac{2}{r(r+1)^2} dr = \int_0^1 (\frac{2}{r} - \frac{2}{r+1} - \frac{2}{(r+1)^2})dr = (2lnr - 2ln(r+1) + \frac{2}{r+1})\Big|_{r=0}^{1}$
$= \lim_{\epsilon \to 0^+} (-2ln2 - 1 - 2ln\epsilon) \to \infty$.

So above all, the expected value of $\frac{1}{R}$ is not exitst.

Problem 5

(a) Since T is the first time that success, so at the time T, we totally failed G times and successed 1 time.
So we faced totally $G + 1 - 1 = G$ trails, and each trail have the time of Δt.
So $T = G \cdot \Delta t$.
So above all, $T = G\Delta t$.

(b) From the description, we could know that $G \sim Geom(\lambda\Delta t)$.
Let $p = \lambda\Delta t$, and let $q = 1 - p$. From what we have learned about Geometry distribution, we can get that the PDF of G is $P(G = g) = q^g \cdot p, g \geq 0$.
So its CDF is $P(G \leq g) = \sum\limits_{k=0}^{g} q^k p = p \cdot \dfrac{1(1 - q^g)}{1 - q} = 1 - (1 - \lambda\Delta t)^g$.
And from (a) we know that $T = G\Delta t$, so the PDF of T is that $P(T = t) = P(G = \lfloor\dfrac{t}{\Delta t}\rfloor), t \geq 0$.
And there exist a round down $\lfloor\dfrac{t}{\Delta t}\rfloor$, because of G is a discrete r.v. , so it must be integer.
So the CDF of T is $P(T \leq t) = P(G \leq \lfloor\dfrac{t}{\Delta t}\rfloor) = 1 - (1 - \lambda\Delta t)^{\lfloor\frac{t}{\Delta t}\rfloor}$.

So above all, the CDF of T is $P(T \leq t) = 1 - (1 - \lambda\Delta t)^{\lfloor\frac{t}{\Delta t}\rfloor}, t \geq 0$.

(c) From what we have learned in mathematical analysis, we know that $\lim\limits_{x\to+\infty}(1 + \dfrac{1}{x})^x = e$.
So $\lim\limits_{x\to+\infty}(1 - \dfrac{1}{x})^x = \lim\limits_{x\to+\infty}(\dfrac{x-1}{x})^x = \dfrac{1}{\lim\limits_{x\to+\infty}(\dfrac{x}{x-1})^x}$.
And $\lim\limits_{x\to+\infty}(\dfrac{x}{x-1})^x = \lim\limits_{x\to+\infty}(\dfrac{x-1+1}{x-1})^x = \lim\limits_{x\to+\infty}(1+\dfrac{1}{x-1})^x = \lim\limits_{x\to+\infty}(1+\dfrac{1}{x-1})^{x-1}(1+\dfrac{1}{x-1}) = \dfrac{1}{e}\cdot 1 = \dfrac{1}{e}$
With the property of round down, we could know that $\dfrac{t}{\Delta t} - 1 < \lfloor\frac{t}{\Delta t}\rfloor \leq \dfrac{t}{\Delta t}$.
From (b), we can get that the CDF of T is $F(t) = 1 - (1 - \lambda\Delta t)^{\lfloor\frac{t}{\Delta t}\rfloor}, t \geq 0$.
And from monotonicity of exponential function, we can get that
$1 - (1 - \lambda\Delta t)^{\frac{t}{\Delta t} - 1} < 1 - (1 - \lambda\Delta t)^{\lfloor\frac{t}{\Delta t}\rfloor} \leq 1 - (1 - \lambda\Delta t)^{\frac{t}{\Delta t}}$.

Let $x = \dfrac{1}{\lambda\Delta t}$, and since $\Delta t > 0$, so when $\Delta t \to 0, x \to +\infty$.
Since $\lim\limits_{\Delta t\to 0} 1 - (1 - \lambda\Delta t)^{\frac{t}{\Delta t}} = \lim\limits_{x\to+\infty} 1 - (1 - \dfrac{1}{x})^{x\cdot\lambda t} = 1 - (\dfrac{1}{e})^{\lambda t} = 1 - e^{-\lambda t}$,
and similarly, $\lim\limits_{\Delta t\to 0} 1 - (1 - \lambda\Delta t)^{\frac{t}{\Delta t} - 1} = \lim\limits_{x\to+\infty} 1 - \dfrac{(1 - \frac{1}{x})^{x\cdot\lambda t}}{1 - \frac{1}{x}} = 1 - \dfrac{(\frac{1}{e})^{\lambda t}}{1} = 1 - e^{-\lambda t}$.

According to the Squeeze Theorem, when $\Delta t \to 0$,
since $1 - (1 - \lambda\Delta t)^{\frac{t}{\Delta t} - 1} < 1 - (1 - \lambda\Delta t)^{\lfloor\frac{t}{\Delta t}\rfloor} \leq 1 - (1 - \lambda\Delta t)^{\frac{t}{\Delta t}}$,
and $\lim\limits_{\Delta t\to 0} 1 - (1 - \lambda\Delta t)^{\frac{t}{\Delta t} - 1} = 1 - e^{-\lambda t}$,
and $\lim\limits_{\Delta t\to 0} 1 - (1 - \lambda\Delta t)^{\frac{t}{\Delta t}} = 1 - e^{-\lambda t}$,
so $\lim\limits_{\Delta t\to 0} 1 - (1 - \lambda\Delta t)^{\lfloor\frac{t}{\Delta t}\rfloor} = 1 - e^{-\lambda t}$.

So we get $\lim\limits_{\Delta t\to 0} F(t) = \lim\limits_{\Delta t\to 0} 1 - (1 - \lambda\Delta t)^{\lfloor\frac{t}{\Delta t}\rfloor} = 1 - e^{-\lambda t}, t \geq 0$.
From what we have learned, the CDF of the Exponential distribution $Expo(\lambda)$ is that $F(x) = 1 - e^{-\lambda x}, x \geq 0$.

So above all, as $\Delta t \to 0$, the CDF of T converges to the $Expo(\lambda)$.
And the CDF at fixed $t \geq 0$ is that $F(t) = 1 - e^{-\lambda t}$.

Problem 6

With LOTUS, we can get that
$E[max(Z-c),0] = \int_{-\infty}^{+\infty} max(z-c,0)\varphi(z)dz = \int_{c}^{+\infty}(z-c)\varphi(z)dz = \int_{c}^{+\infty} z\varphi(z)dz - c\int_{c}^{+\infty}\varphi(z)dz$
From we have learned, we can get that the PDF of the standard distribution is that $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$.
So $\int_{c}^{+\infty} z\varphi(z)dz = \int_{c}^{+\infty} z\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz = \int_{c}^{+\infty}\frac{1}{2\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz^2$
$= \int_{c^2}^{+\infty}\frac{1}{2\sqrt{2\pi}}e^{-\frac{x}{2}}dx = \frac{1}{2\sqrt{2\pi}}\cdot(-2)\ e^{-\frac{x}{2}}\Big|_{x=c^2}^{+\infty} = \frac{1}{\sqrt{2\pi}}e^{-\frac{c^2}{2}}$.

And since $\varphi(x)$ is the PDF of standard normal distribution, and $\Phi(x)$ is its CDF.
So $c\int_{c}^{+\infty}\varphi(z)dz = \lim_{z\to+\infty} c[\Phi(z)-\Phi(c)] = c[1-\Phi(c)]$.

So above all, combine the two parts, we can get that $E[max(Z-c,0)] = \frac{1}{\sqrt{2\pi}}e^{-\frac{c^2}{2}} - c[1-\Phi(c)]$.