TA Lecture - HW6

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Outline

HW 6

Challenge Question of HW6

Problem 1

The Cauchy distribution has PDF

$$f(x) = \frac{1}{\pi\left(1 + x^2\right)}$$

for all x. Find the CDF of a random variable with the Cauchy PDF. Hint: Recall that the derivative of the inverse tangent function $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$.

Problem 1 Solution

Problem 2

The Pareto distribution with parameter $a > 0$ has PDF

$$f(x) = \frac{a}{x^{a+1}}$$

for $x \geq 1$ (and 0 otherwise). This distribution is often used in statistical modeling. Find the CDF of a Pareto r.v. with parameter a; check that it is a valid CDF.

Problem 2 Solution

Problem 3

The *Beta distribution* with parameters $a = 3$, $b = 2$ has PDF

$$f(x) = 12x^2(1 - x), \text{ for } 0 < x < 1.$$

Let X have this distribution.

(a) Find the CDF of X.

(b) Find $P(0 < X < 1/2)$.

(c) Find the mean and variance of X (without quoting results about the Beta distribution).

Problem 3 Solution

Problem 4

The Exponential is the analog of the Geometric in continuous time. This problem explores the connection between Exponential and Geometric in more detail, asking what happens to a Geometric in a limit where the Bernoulli trials are performed faster and faster but with smaller and smaller success probabilities.

Suppose that Bernoulli trials are being performed in continuous time; rather than only thinking about first trial, second trial, etc., imagine that the trials take place at points on a timeline. Assume that the trials are at regularly spaced times $0, \Delta t, 2\Delta t, \ldots$, where Δt is a small positive number. Let the probability of success of each trial be $\lambda \Delta t$, where λ is a positive constant. Let G be the number of failures before the first success (in discrete time), and T be the time of the first success (in continuous time).

Problem 4 Continued

(a) Find a simple equation relating G to T. Hint: Draw a timeline and try out a simple example.

(b) Find the CDF of T. Hint: First find $P(T > t)$.

(c) Show that as $\Delta t \to 0$, the CDF of T converges to the Expo(λ) CDF, evaluating all the CDFs at a fixed $t \geq 0$.

Problem 4 Solution

Problem 5

Let $Z \sim \mathcal{N}(0, 1)$, and c be a nonnegative constant. Find $E(\max(Z - c, 0))$, in terms of the standard Normal CDF Φ and PDF φ.

Problem 5 Solution

Outline

HW 6

Challenge Question of HW6

Gaussian Bounds

(Optional Challenging Problem) Let $X \sim \mathcal{N}(0, 1)$, its corresponding CDF is denoted as Φ and the corresponding PDF is denoted as φ.

(a) If $x > 0$, show the following inequality holds:

$$\frac{x}{x^2+1}\varphi(x) \leq 1 - \Phi(x) \leq \frac{1}{x}\varphi(x).$$

Solution

Gaussian Bounds

(b) Define the function $g(x)$ as follows:

$$g(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt, \forall x \geq 0.$$

Show the following inequality holds:

$$g(x) \leq e^{-x^2}, \forall x \geq 0.$$

Solution