# Stretched coordinate PML – FDTD implementation

"Non-physical" split-fields used to get comp. eff-ent approach.

$$\vec{\nabla}_s \times \vec{E} = S_x^{-1} \partial_x (\hat{x} \times E) + S_y^{-1} \partial_y (\hat{y} \times \vec{E}) + S_z^{-1} \partial_z (\hat{z} \times \vec{E})$$

$$\vec{\nabla}_s \times \vec{E} = -j\omega\mu \vec{H}$$

## Split field for $\vec{H}$ :

$$\vec{H} = \vec{H}_{sx} + \vec{H}_{sy} + \vec{H}_{sz}$$

s means "split"

Break Faraday's law into 3 vector eq-s as

$$S_x^{-1} \partial_x (\hat{x} \times \vec{E}) = -j\omega\mu \vec{H}_{sx}$$
$$S_y^{-1} \partial_y (\hat{y} \times \vec{E}) = -j\omega\mu \vec{H}_{sy}$$
$$S_z^{-1} \partial_z (\hat{z} \times \vec{E}) = -j\omega\mu \vec{H}_{sz}$$

For simple time-domain implement-on, choose:

$$S_x = 1 - j \frac{\sigma_x}{\omega \varepsilon}$$

$$S_y = \dots - \frac{\sigma_y}{\dots}$$

$$S_z = \dots \frac{\sigma_z}{\dots}$$

Go to time domain

$$\boxed{\begin{aligned}
\partial_x(\hat{x}\times\vec{E}) &= -\mu\partial_t\vec{H}_{sx} - \sigma_x\mu\varepsilon^{-1}\vec{H}_{sx}\\
\partial_y(\hat{y}\times\vec{E}) &= -\mu\partial_t\vec{H}_{sy} - \sigma_y\mu\varepsilon^{-1}\vec{H}_{sy}\\
\partial_z(\hat{z}\times\vec{E}) &= -\mu\partial_t\vec{H}_{sz} - \sigma_z\mu\varepsilon^{-1}\vec{H}_{sz}
\end{aligned}}$$

Same treatment for Ampere's law:

s means "split" here

Split field: $\vec{E} \equiv \vec{E}_{sx} + \vec{E}_{sy} + \vec{E}_{sz}$

$$\partial_x(\hat{x}\times\vec{H}) = -\varepsilon\partial_t\vec{E}_{sx} - \sigma_x\vec{E}_{sx}$$

$$\partial_y(\hat{y}\times\vec{H}) = -\varepsilon\partial_t\vec{E}_{sy} - \sigma_y\vec{E}_{sy}$$

$$\partial_z(\hat{z}\times\vec{H}) = -\varepsilon\partial_t\vec{E}_{sz} - \sigma_z\vec{E}_{sz}$$

Use Yee's scheme to discretize

2D $TM_z$ analysis: $\vec{E} = \hat{z}E_z$

$\begin{cases} H_z = 0 \\ E_x, E_y = 0 \end{cases}$ $\vec{H} = \hat{x}H_x + \hat{y}H_y$

$\Rightarrow$ $\vec{H}_{sz} = 0$ from Faraday's law

$\vec{E}_{sz} = 0$ from Ampere's law.

From Faraday's law we get:

y - component $\partial_x E_z = \mu\partial_t H_y + \sigma_x\mu\varepsilon^{-1}H_y$ (1)

x - component $\partial_y E_z = -\mu\partial_t H_x - \sigma_y\mu\varepsilon^{-1}H_x$ (2)

From Ampere's law:

? what does $s_x$ mean in $\vec{E}$ ?

$\vec{E}_{sx} = \hat{z} E_{sx,z}$

$\vec{E}_{sy} = \hat{z} E_{sy,z}$

$\vec{E}/_{s_y}$ ?

$$\Rightarrow \quad \partial_x H_y = \varepsilon \partial_t E_{sx,z} + \sigma_x E_{sx,z} \tag{3}$$

$$\partial_y H_x = -\varepsilon \partial_y E_{sy,z} - \sigma_y E_{sy,z} \tag{4}$$

Time stepping formulas:

$$a_{x(y)} = \left[ \frac{\varepsilon}{\Delta t} + \frac{\sigma_{x(y)}}{2} \right]^{-1}$$

$$b_{x(y)} = \left[ \frac{\varepsilon}{\Delta t} - \frac{\sigma_{x(y)}}{2} \right]$$

$$H_{sx}^{n+\frac{1}{2}}(i, j+\tfrac{1}{2}) = a_y(i, j+\tfrac{1}{2}) \Big\{ b(i, j+\tfrac{1}{2}) H_x^{n-\frac{1}{2}}(i, j+\tfrac{1}{2}) - \frac{\varepsilon}{\mu \Delta y} \left[ E_z^n(i, j+1) - E_z^n(i,j) \right] \Big\}$$

Similar for $H_{sy}^{n+\frac{1}{2}}(i+\tfrac{1}{2}, j)$

$$E_{sx,z}^{n+1}(i,j) = a_x(i,j) \Big\{ b_x(i,j) E_{sx,z}^n(i,j) + \frac{1}{\Delta x} \left[ H_y^{n+\frac{1}{2}}(i+\tfrac{1}{2}, j) - H_y^{n+\frac{1}{2}}(i-\tfrac{1}{2}, j) \right] \Big\}$$

Similar for $E_{sy,z}^{n+1}$

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# Anisotropic Absorber version.

Coordinate stretching PML is equivalent to a uniaxial medium w/ specially designed parameters.

**Goal:** convert stretched form MEs into smth that looks like regular MEs.

new fields: stretched

$$\bar{E}^a = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{pmatrix} \cdot E^s$$

Similar for $\vec{H}^a$ and $\vec{H}^s$.

Want to convert

$$\vec{\nabla}_s \times \vec{E}^s = -j\omega\mu \vec{H}^s$$

into eq-n involving $\vec{E}^a$ and $\vec{H}^a$

$$\vec{E}^a_z = s_z \vec{E}^s_z$$

Start with LHS:

$$\vec{\nabla}_s \times \vec{E}^s = \begin{pmatrix} s_y^{-1}\partial_y E_z^s - s_z^{-1}\partial_z E_y^s \\ s_z^{-1}\partial_z E_x^s - s_x^{-1}\partial_x E_z^s \\ s_x^{-1}\partial_x E_y^s - s_y^{-1}\partial_y E_x^s \end{pmatrix} = \begin{pmatrix} (s_y s_z)^{-1}(\partial_y E_z^a - \partial_z E_y^a) \\ (s_z s_x)^{-1}(\partial_z E_x^a - \partial_x E_z^a) \\ (s_x s_y)^{-1}(\partial_x E_y^a - \partial_y E_x^a) \end{pmatrix}$$

$$= \begin{pmatrix} (s_y s_z)^{-1} & 0 & 0 \\ 0 & (s_z s_x)^{-1} & 0 \\ 0 & 0 & (s_x s_y)^{-1} \end{pmatrix} \cdot \vec{\nabla} \times \vec{E}^a$$

Using this, we can write Faraday's law:

$$\begin{pmatrix} (s_y s_z)^{-1} & 0 & 0 \\ 0 & (s_x s_z)^{-1} & 0 \\ 0 & 0 & (s_x s_y)^{-1} \end{pmatrix} \cdot \vec{\nabla} \times \vec{E}^a = -j\omega\mu \, \mathrm{diag}\{s_x^{-1}, s_y^{-1}, s_z^{-1}\} \cdot \vec{H}^a$$

$$\vec{\nabla} \times \vec{E}^a = -j\omega\mu \, \bar{\bar{\Lambda}} \cdot \vec{H}^a$$

$$\bar{\bar{\Lambda}} = \begin{pmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{pmatrix}$$

Repeat process for the rest of MEs to get:

$$\boxed{\begin{aligned} &\vec{\nabla} \times \vec{H}^a = j\omega\varepsilon \, \bar{\bar{\Lambda}} \, \vec{E}^a \\ &\vec{\nabla} \cdot \left(\varepsilon \, \bar{\bar{\Lambda}} \cdot \vec{E}^a\right) = 0 \\ &\vec{\nabla} \cdot \left(\mu \, \bar{\bar{\Lambda}} \, \vec{H}^a\right) = 0 \end{aligned}}$$

anisotropic medium MEs

Main difficulty $\quad s_x = 1 - j\frac{\sigma_x}{\omega\varepsilon}$ leads to complex frequency dependence of $\bar{\Lambda}$ in edges to corners that is difficult to work in time-domain. - motivation why to use stretched-coord. PML vs uniaxial PML

Circumvent the issue by defining a set of auxiliary vectors.

not exactly flux densities!!!

$$\bar{D} = \varepsilon \underbrace{\begin{pmatrix} s_z/s_x & 0 & 0 \\ 0 & s_x/s_y & 0 \\ 0 & 0 & s_y/s_z \end{pmatrix}}_{[s]} \cdot \vec{E}$$

$$\bar{B} = \mu \cdot \mathrm{diag}\left\{ \frac{s_z}{s_x}, \frac{s_x}{s_y}, \frac{s_y}{s_z} \right\} \cdot \vec{H}$$

with these auxiliary vectors, we get:

$$\vec{\nabla} \times \vec{E} = -j\omega \, \mathrm{diag}\{s_y, s_z, s_x\} \cdot \vec{B} \quad (5)$$

$$\vec{\nabla} \times \vec{H} = +j\omega \, \mathrm{diag}\{s_y, s_z, s_x\} \cdot \vec{D} \quad (6)$$

Single stretching parameter makes conversion into TD easy.

Rearrange Auxiliary vector eq-ns as:

$$\mathrm{diag}\{s_x, s_y, s_z\} \cdot \vec{D} = \varepsilon \, \mathrm{diag}\{s_z, s_x, s_y\} \cdot \vec{E} \quad (7)$$

$$\mathrm{diag}\{s_x, s_y, s_z\} \cdot \vec{B} = \mu\, \mathrm{diag}\{s_z, s_x, s_y\} \cdot \vec{H} \qquad (8)$$

$x$ - components of (5) - (8):

$$((5)\cdot\hat{x}) : \quad \partial_y E_z - \partial_z E_y = -\partial_t B_x - \frac{\sigma_y}{\varepsilon} B_x$$

$$((6)\cdot\hat{x}) : \quad \partial_y H_z - \partial_z H_y = -\partial_t D_x - \frac{\sigma_y}{\varepsilon} D_x$$

Convert into time - stepping formula on <u>Yee's grid.</u>

$\vec{E}$ and $\vec{D}$ on same points of <u>primal</u> grid

$\vec{H}$ and $\vec{B}$ on ———— <u>dual</u> grid

When converting to time - stepping formula, use averages in time for components like

$$\underbrace{B_x^n}_{\text{unknown on stag. grid}} = \frac{1}{2}\underbrace{\left(B_x^{n+\frac{1}{2}} + B_x^{n-1/2}\right)}_{\text{known values on staggered grid}}$$

$$((7)\cdot\hat{x}) \qquad \partial_t D_x + \frac{\sigma_x}{\varepsilon} D_x = \varepsilon\, \partial_t E_x + \sigma_z E_x$$

$$((8)\cdot\hat{x}) \qquad \partial_t B_x + \frac{\sigma_x}{\varepsilon} B_x = \mu\, \partial_t H_x + \mu \frac{\sigma_z}{\varepsilon} H_x$$

$\hat{x}$ component led to 4 eq-s;

for $\hat{y}$ and $\hat{z}$ we will get 8 more equation

## Leapfrog update strategy:

1). (5) at $\vec{E}^{n}$ to compute $B_x^{n+\frac{1}{2}}$, $B_y^{n+\frac{1}{2}}$, $B_z^{n+\frac{1}{2}}$

2). use $\vec{B}^{n+\frac{1}{2}}$ in (8) to compute $\vec{H}^{n+\frac{1}{2}}$

3). Use $\vec{H}^{n+\frac{1}{2}}$ in (6) to compute $D_x^{n+1}$

4). Use $\vec{D}^{n+1}$ in (7) to compute $\vec{E}^{n+1}$

## Concluding remarks

PMLs can be systematically improved

10 cell thick PML

$$\sigma(z) = \frac{\sigma_{max} |z - z_m|^m}{L^m}$$

thickness of PML

$m = 4$

PML's not perfect, can absorb plane waves but not truly arbitrary fields.

Substituting

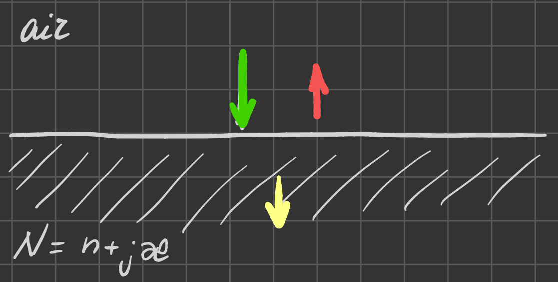
# PWL notes Prof. Raymond Rumph

## Complex refractive index

$$\boxed{N = n + j\varkappa}$$

ordinary refract index

extinction coefficient




## Reflection coefficient

$$\boxed{R = \frac{(1-n)^2 + \varkappa^2}{(1+n)^2 + \varkappa^2}}$$

loss contributes to the reflection !

## Reflection and transmission. Anisotropic case.

### TE polariz.

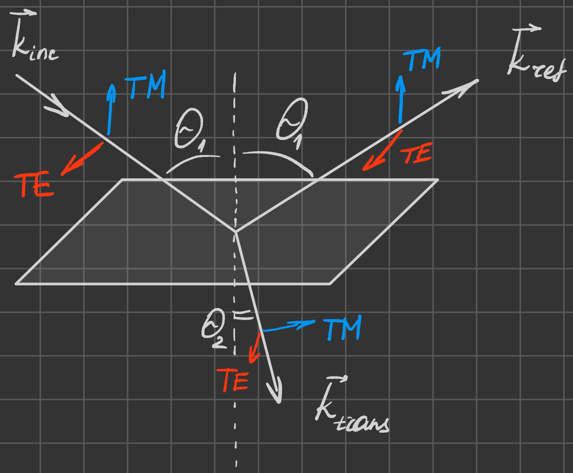
$$r_{TE} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

$$t_{TE} = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

### TM polariz.

$$r_{TM} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}$$

$$t_{TM} = \frac{2\eta_2 \cos\theta_1}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2}$$




## Maxwell's equations in anisotropic media

$$\vec{\nabla} \times \vec{H}(\vec{r},\omega) = j\omega\varepsilon_0 \cdot \bar{\bar{\varepsilon}}_r \cdot \vec{E}(\vec{r},\omega)$$

$$\vec{\nabla} \times \vec{E}(\vec{r},\omega) = -j\omega\mu_0 \cdot \bar{\bar{\mu}}_r \cdot \vec{H}(\vec{r},\omega)$$

In tensor form

$$\begin{pmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix} \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = j\omega\varepsilon_0 \cdot \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = -j\omega\mu_0 \cdot \begin{pmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{pmatrix} \cdot \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}$$

## Three basic types of anisotropic media

isotropic uniaxial biaxial

$$\begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} \qquad \begin{pmatrix} \varepsilon_o & 0 & 0 \\ 0 & \varepsilon_o & 0 \\ 0 & 0 & \varepsilon_e \end{pmatrix} \qquad \begin{pmatrix} \varepsilon_a & 0 & 0 \\ 0 & \varepsilon_b & 0 \\ 0 & 0 & \varepsilon_c \end{pmatrix}$$

Note: off-diagonal terms arize when tensor is rotated with respect to the coordinate system.

## Two different conventions for incorporating loss

$$(\varepsilon_r', \varepsilon_r'') \quad \text{vs} \quad (\varepsilon_r, \sigma)$$

- very low frequencies (quasi-static)
- time domain analysis

- high frequencies
- frequency domain

$$\vec{\nabla} \times \vec{H}(\vec{r},\omega) = j\omega \vec{D}(\vec{r},\omega) = j\omega \tilde{\varepsilon}_r \vec{E}$$

$$\vec{\nabla} \times \vec{H}(\vec{r},\omega) = j\omega \vec{D}(\vec{r},\omega) + \vec{J}(\vec{r},\omega) = j\omega \varepsilon_r \vec{E} + \sigma \vec{E}$$

conduction current is negligible

$$\boxed{\tilde{\varepsilon}_r = \varepsilon_r + \frac{\sigma}{j\omega}}$$

note: it doesn't make sense to have a complex permittivity $\tilde{\varepsilon}_r$ and a conductivity $\sigma$.

## Maxwell's equations in bianisotropic media

$$\begin{pmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix} \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = j\omega\varepsilon_0 \cdot \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = -j\omega\mu_0 \cdot \begin{pmatrix} \mu_{xx} & 0 & 0 \\ 0 & \mu_{yy} & 0 \\ 0 & 0 & \mu_{zz} \end{pmatrix} \cdot \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}$$

Incorporating loss

$$\begin{pmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix} \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = j\omega\varepsilon_0 \cdot \begin{pmatrix} \varepsilon_{xx} + \frac{\sigma_x^E}{j\omega} & 0 & 0 \\ 0 & \varepsilon_{yy} + \frac{\sigma_y^E}{j\omega} & 0 \\ 0 & 0 & \varepsilon_{zz} + \frac{\sigma_z^E}{j\omega} \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = -j\omega\mu_0 \cdot \begin{pmatrix} \mu_{xx} + \frac{\sigma_x^H}{j\omega} & 0 & 0 \\ 0 & \mu_{yy} + \frac{\sigma_y^H}{j\omega} & 0 \\ 0 & 0 & \mu_{zz} + \frac{\sigma_z^H}{j\omega} \end{pmatrix} \cdot \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}$$

## Scattering at a doubly-diagonal anisotropic interface.

$$\bar{\bar{\mu}}_r = \bar{\bar{\varepsilon}}_r = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

refraction

$$\sin\theta_1 = \sqrt{bc}\,\sin\theta_2$$

reflection

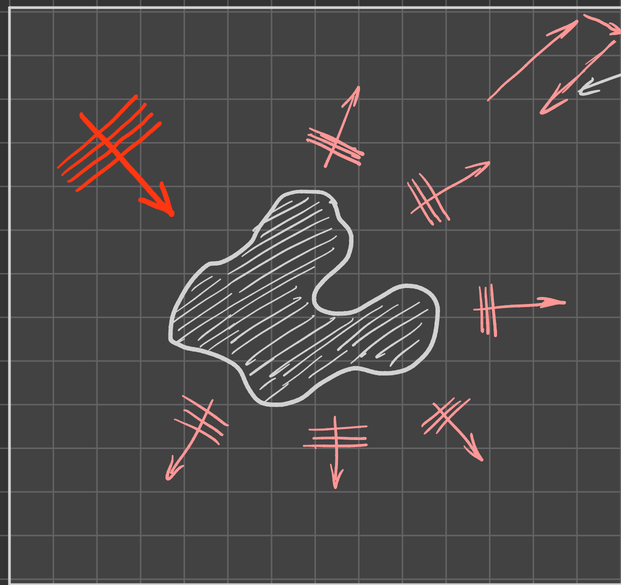
$$r_{TE} = \frac{\sqrt{a}\cos\theta_1 - \sqrt{b}\cos\theta_2}{\sqrt{a}\cos\theta_1 + \sqrt{b}\cos\theta_2}$$

$$r_{TM} = \frac{-\sqrt{a}\cos\theta_1 + \sqrt{b}\cos\theta_2}{\sqrt{a}\cos\theta_1 + \sqrt{b}\cos\theta_2}$$

* from air to anisotropic medium

- change of impedance causes reflection
- Snell's law quantifies angle of transmission relative to incidence; angles $\theta_1$ of reflection and $\theta_2$ of refraction do not depend on polarization.
- Fresnel equations quantify the amount of reflection and transmission; this amount depends on the polarization.

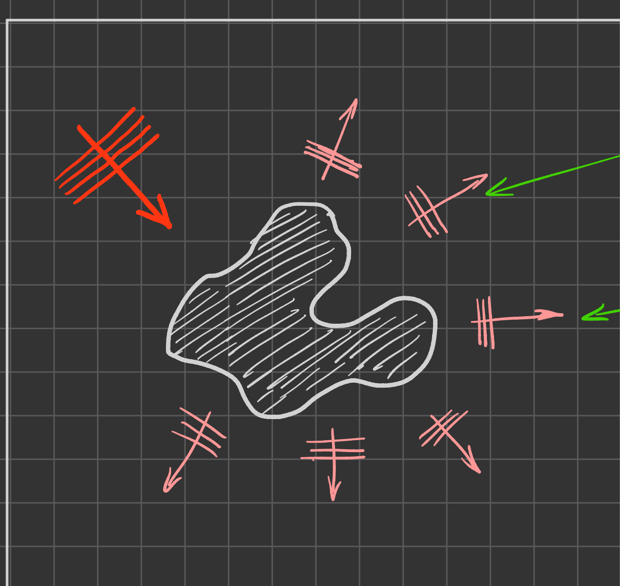
## Boundary condition for scattering problem



? how to prevent reflection back into the domain

For 1D we can use perfectly absorbing boundary (PAB); requirement to use it is: wave have to travel one cell $\Delta x$ in two time steps $\Delta t$ : $2 \cdot \Delta x = c \cdot \Delta t \rightarrow \boxed{\Delta t = 2\frac{\Delta x}{c}}$.

In 2D, for 2 waves moving in different directions we can't ensure this time step relation.

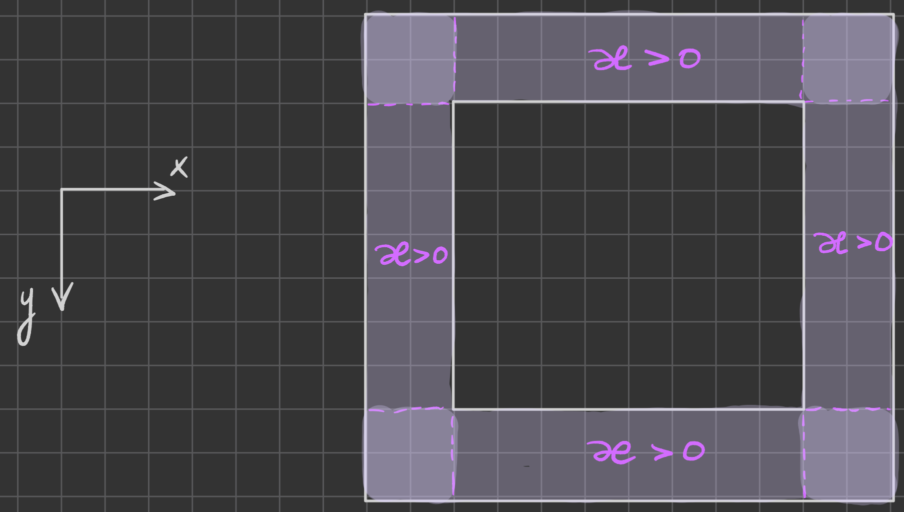


moves slower $\Delta t > 2\frac{\Delta x}{c}$

moves faster $\Delta t = 2\frac{\Delta x}{c}$

$\Rightarrow$ PAB can be satisfied only in one direction.

Let's introduce loss at the boundary of the domain.




Loss corresponding to æ absorbs the waves. But!
it's also in the reflectance equation

$$R = \frac{(1-n)^2 + \varkappa^2}{(1+n)^2 + \varkappa^2}$$

Match the impedance!

1. loss to absorb outgoing waves
2. match the impedance of the problem space to prevent reflections.

$$\boxed{\tilde{\varepsilon}_r = \varepsilon_r' + j\,\varepsilon_r''} \quad \in \mathbb{C}$$

control impedance

add loss

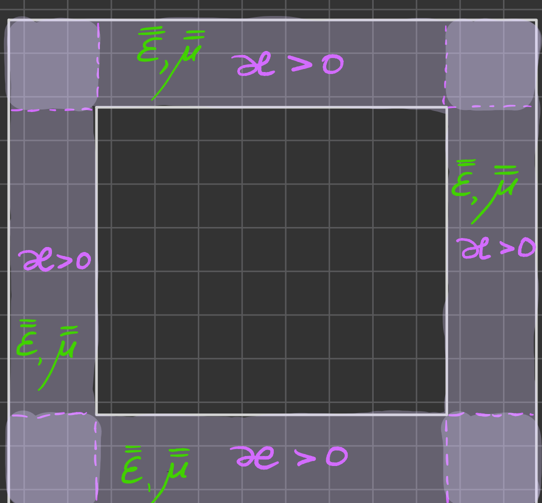
Examine Fresnel equations

$$r_{TE} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2} = 0 \quad \longrightarrow \quad \eta_2 = \eta_1 \frac{\cos\theta_2}{\cos\theta_1}$$

$$r_{TM} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2} = 0 \quad \longrightarrow \quad \eta_2 = \eta_1 \frac{\cos\theta_1}{\cos\theta_2}$$

we cannot prevent reflection for all angles of incidence

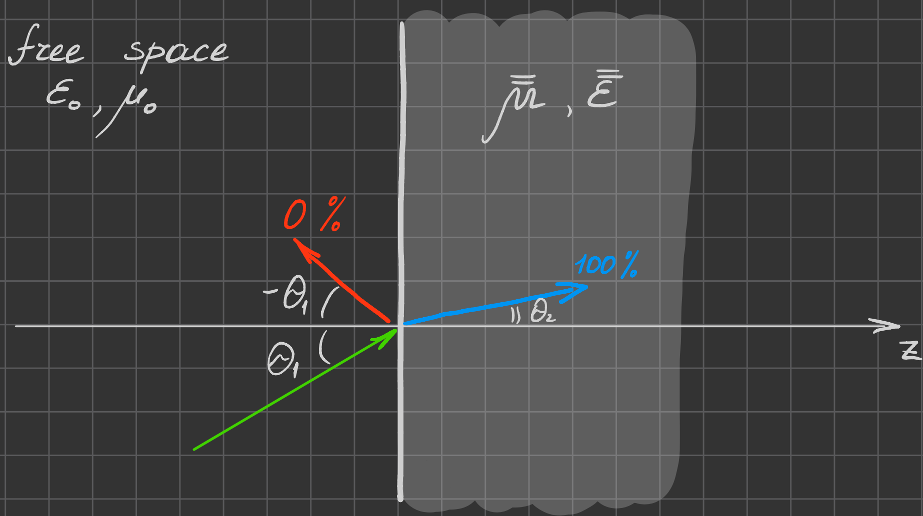
Anisotropy is to help!




$$\begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} \qquad \begin{pmatrix} \mu_{xx} & 0 & 0 \\ 0 & \mu_{yy} & 0 \\ 0 & 0 & \mu_{zz} \end{pmatrix}$$

Can prevent reflections at all angles and for all polarizations by allowing absorbing material to be doubly-diagonally anisotropic (biaxial)

## Problem statement for PML (uniaxial)




- for any angle of incidence $\theta_1$
- for any material parameters $\bar{\bar{\varepsilon}}, \bar{\bar{\mu}}$
- for any polarization

We need to perfectly match the impedance of the grid (simulation domain) to the impedance of the absorbing region.

For air / medium interface

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Ensure $\eta$ is matced to air

$$\bar{\bar{S}} \equiv \bar{\bar{\mu}}_r = \bar{\bar{\varepsilon}}_r = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

Choose $\sqrt{bc} = 1$ (to avoid refraction)

refraction (Snell's)

$$\sin\theta_1 = \sqrt{bc}\,\sin\theta_2 = \sin\theta_2$$

$\boxed{\theta_1 = \theta_2}$ no refraction

reflection (Fresnel)

$$r_{TE} = \frac{\sqrt{a}\cos\theta_1 - \sqrt{b}\cos\theta_2}{\sqrt{a}\cos\theta_1 + \sqrt{b}\cos\theta_2} = \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}}$$

$$r_{TM} = \frac{-\sqrt{a}\cos\theta_1 + \sqrt{b}\cos\theta_2}{\sqrt{a}\cos\theta_1 + \sqrt{b}\cos\theta_2} = \frac{-\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}}$$

do not depend on angle

Choose $a = b$ (thus the name "uniaxial")

$$r_{TE} = \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}} = 0$$

$$r_{TM} = \frac{-\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}} = 0$$

no reflection

Conditions:
$$\boxed{\begin{gathered}\sqrt{bc} = 1 \\ a = b\end{gathered}}$$

PML parameters

$$\overline{\overline{S}} \equiv \overline{\overline{\mu}}_z = \overline{\overline{\varepsilon}}_z = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

$$a = b = \frac{1}{c}$$

Since $a = b$, we can write $\bar{\bar{\varepsilon}}_r$ and $\bar{\bar{\mu}}_r$ in terms of a single (uni) parameter

$$\bar{\bar{S}}_z \equiv \begin{pmatrix} S_z & 0 & 0 \\ 0 & S_z & 0 \\ 0 & 0 & \frac{1}{S_z} \end{pmatrix}$$

This form of the tensor is the reason why we call it uniaxial

$$S_z \equiv \alpha - j\beta \in \mathbb{C}$$

for a wave travelling in $+\hat{z}$ direction incident on z-axis boundary

What are those parameters $s_x$, $s_y$, $s_z$?

$$\tilde{\varepsilon} = \varepsilon + \frac{\sigma}{j\omega} \quad \Rightarrow \quad \tilde{\varepsilon}_r = \varepsilon_r + \frac{\sigma}{j\omega\varepsilon_0}$$

The PML parameters $s$ act like complex relative permittivity $\tilde{\varepsilon}$ => we know how to control the loss:

$$s_z = \varepsilon_r' + \frac{\sigma'}{j\omega\varepsilon_0}$$

$\varepsilon_r'$, $\sigma'$ – PML fictitious rel. permittivity and conductivity

In general, PML implementations use $\varepsilon_r' \neq 1$ in order to better handle evanescent fields, but we set $\varepsilon_r' = 1$ for simplicity.

$$s_z = 1 + \frac{\sigma'}{j\omega\varepsilon_0}$$

⚠ But waves can travel in all 3 directions, hence we can introduce $\bar{\bar{\varepsilon}}_r$ and $\bar{\bar{\mu}}_r$ for the other 2 directions as well

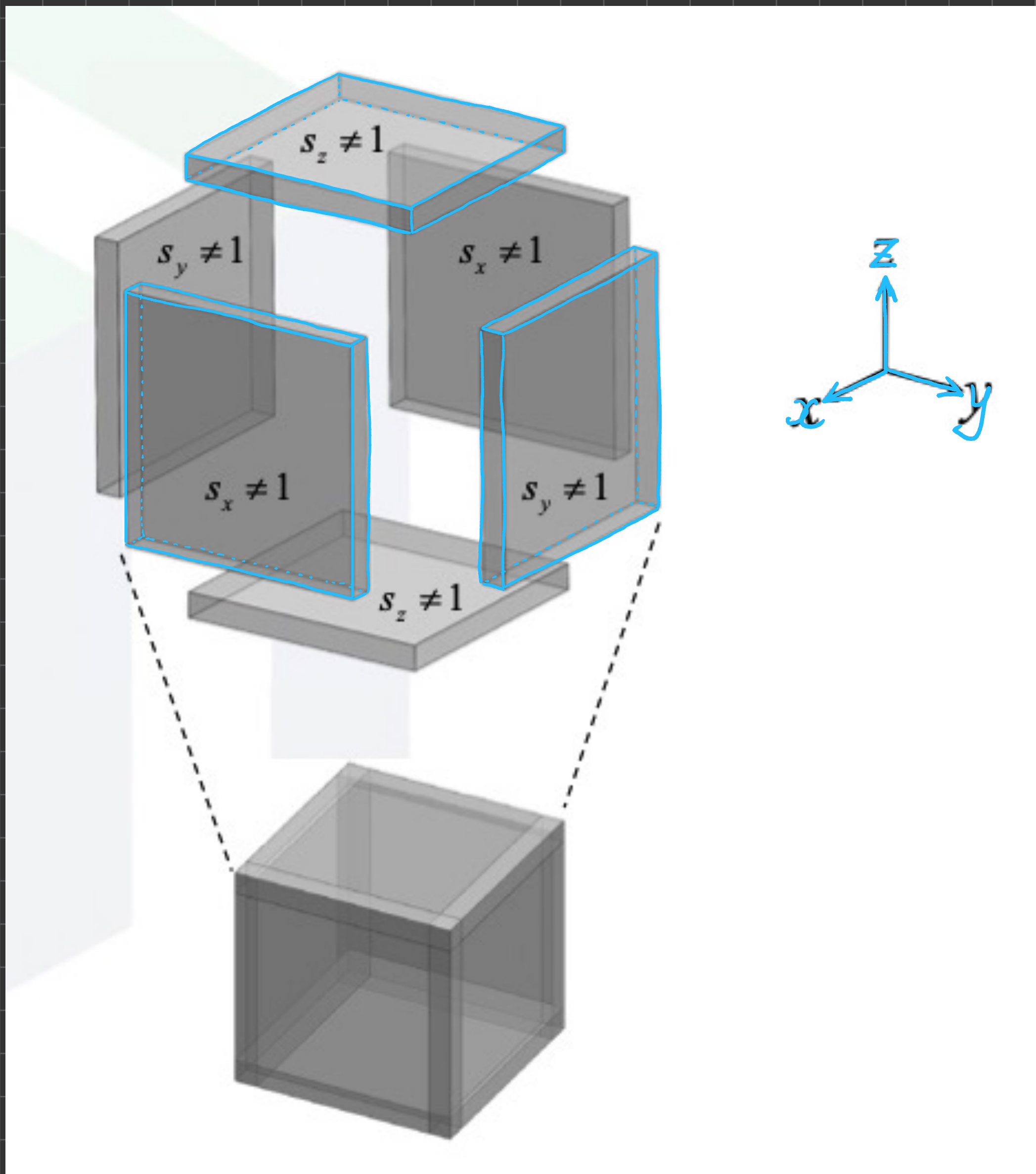
$$\bar{\bar{S}}_x \equiv \begin{pmatrix} S_x & 0 & 0 \\ 0 & S_x & 0 \\ 0 & 0 & \frac{1}{S_x} \end{pmatrix} \qquad \bar{\bar{S}}_y \equiv \begin{pmatrix} S_y & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & \frac{1}{S_y} \end{pmatrix} \qquad \bar{\bar{S}}_z \equiv \begin{pmatrix} S_z & 0 & 0 \\ 0 & S_z & 0 \\ 0 & 0 & \frac{1}{S_z} \end{pmatrix}$$

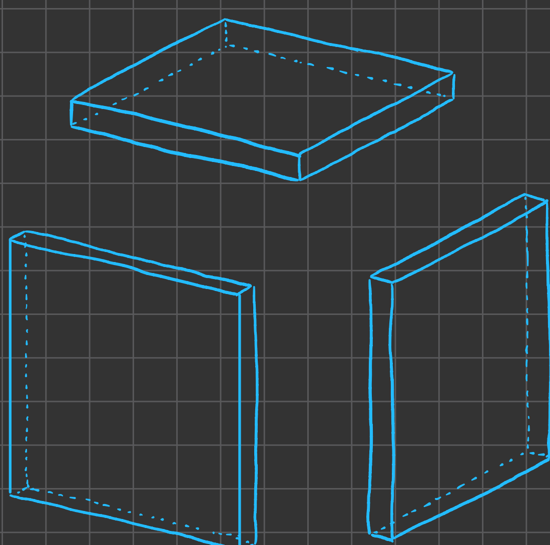
Each of these 3 tensors represent uniaxial anisotropy => uniaxial PML (UPML).

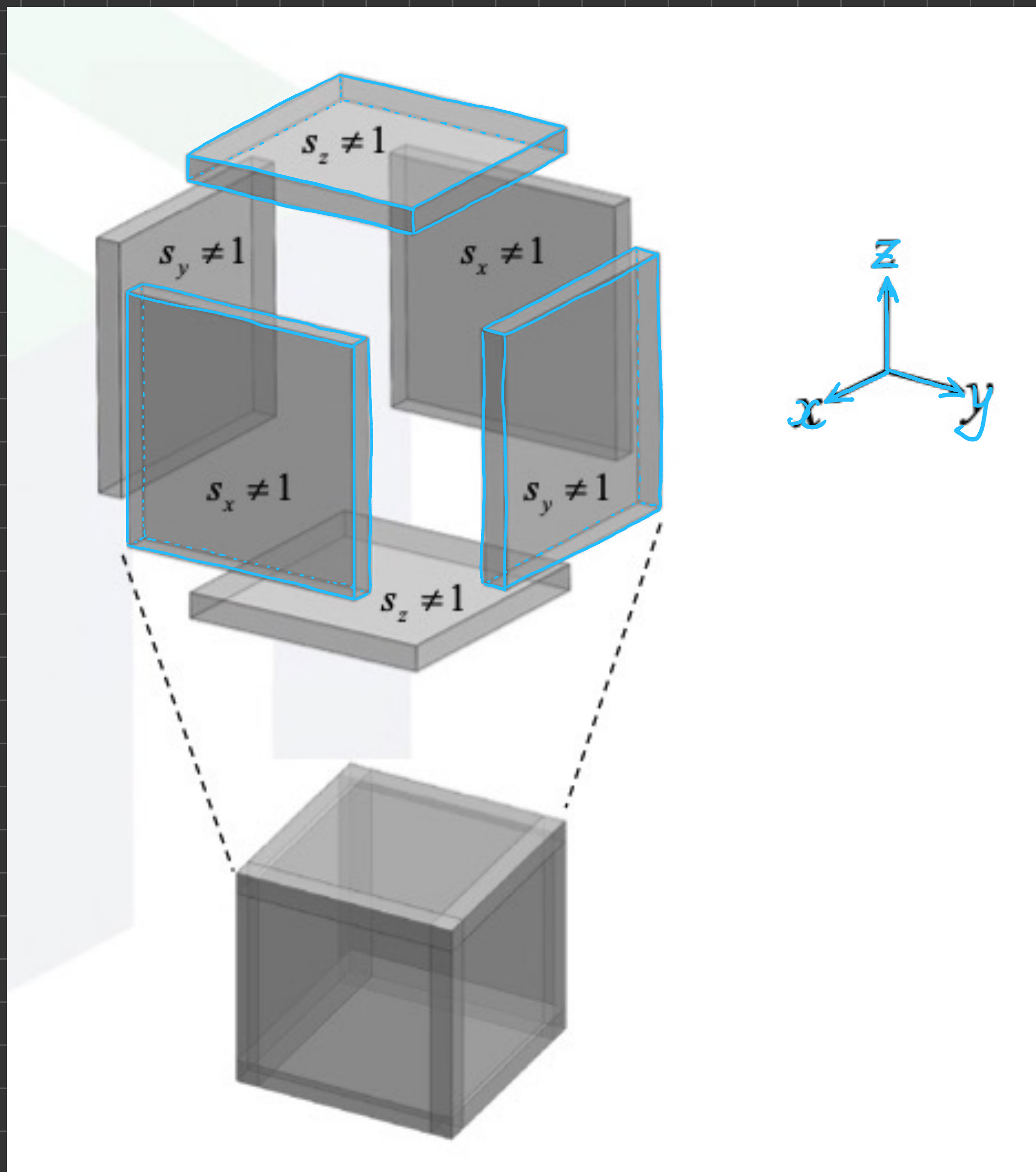
We can combine tensors $\bar{\bar{S}}_x$, $\bar{\bar{S}}_y$, and $\bar{\bar{S}}_z$ into a single tensor

$$\bar{\bar{S}} \equiv \bar{\bar{S}}_x \cdot \bar{\bar{S}}_y \cdot \bar{\bar{S}}_z = \begin{pmatrix} \frac{S_y S_z}{S_x} & 0 & 0 \\ 0 & \frac{S_x S_z}{S_y} & 0 \\ 0 & 0 & \frac{S_x S_y}{S_z} \end{pmatrix}$$

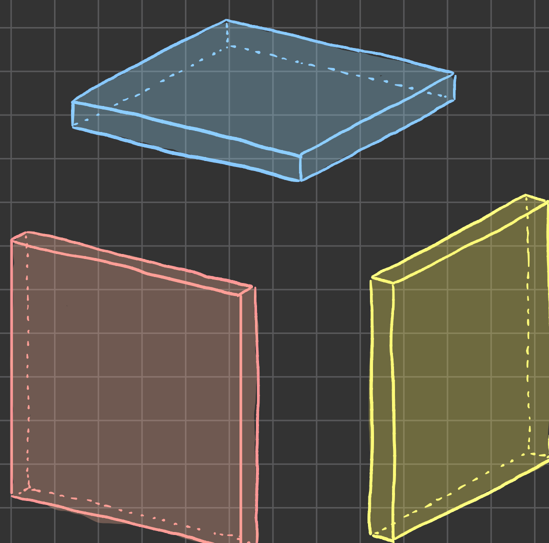
3D PML can be visualized as

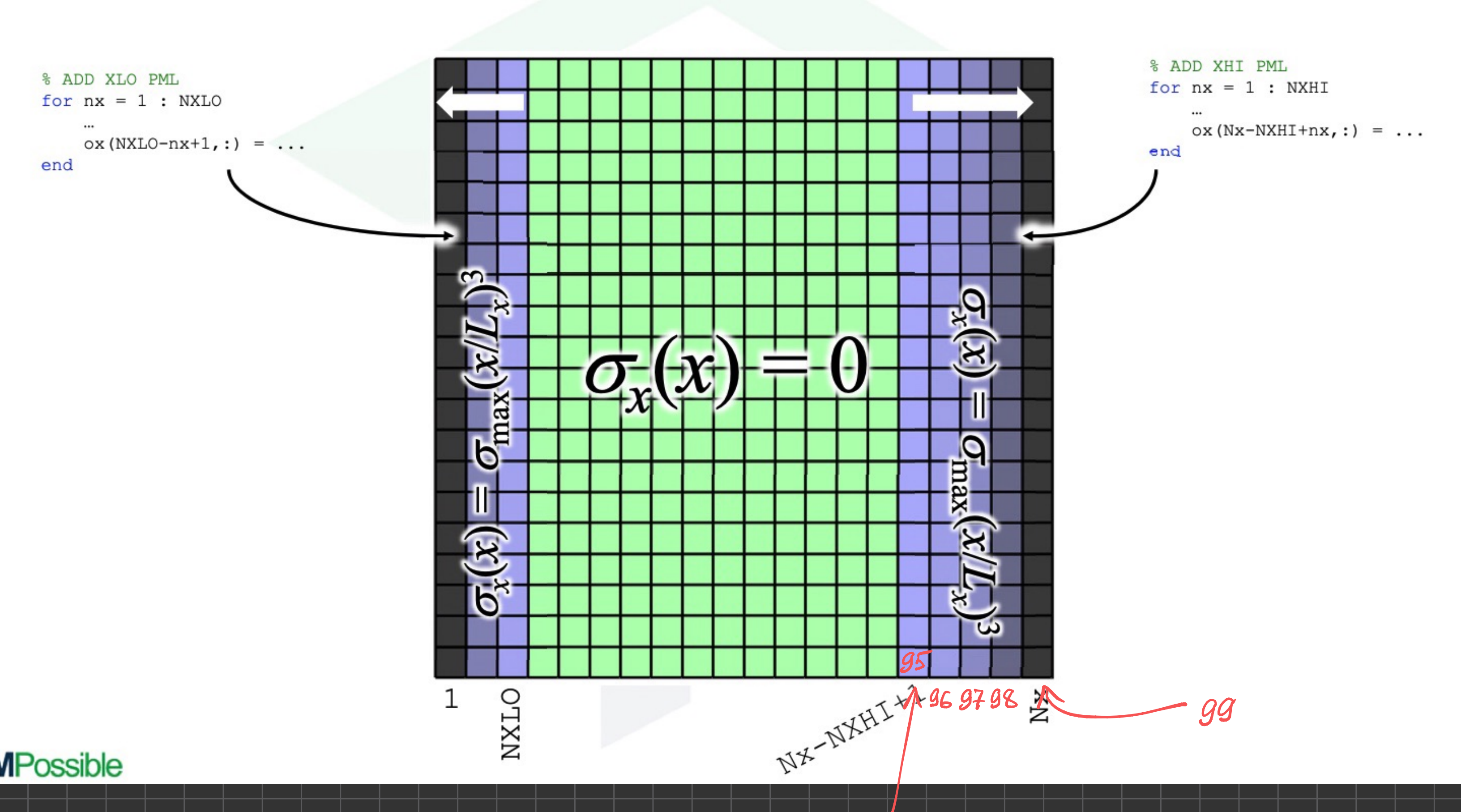





$S_z$
$S_y$
$S_x$






$Nx = 100$

$Nx\_lo = 3$

$Nx\_hi = 5$

for high side :

$Nx - (Nx\_hi - nx)$

$Nx - Nx\_hi$

$100 - (5-0)$; :

but it runs to 99

| | |
|---|---|
| $100 - (5-0)$ | 95 |
| $100 - (5-1)$ | 96 |
| $100 - (5-2)$ | 97 |
| $100 - (5-3)$ | 98 |
| $100 - (5-4)$ | 99 |

nx in range (5)

## Derivation of stretced coordinate PML (SCPML) from uniaxial PML (UPML)

Maxwell's equations in UPML

$$\vec{\nabla} \times \vec{E}(\vec{r},\omega) = k_0 \cdot \bar{\bar{\mu}}_r \cdot \left(\bar{\bar{S}}\, \tilde{\vec{H}}(\vec{r},\omega)\right)$$

$$\vec{\nabla} \times \tilde{\vec{H}}(\vec{r},\omega) = k_0 \cdot \bar{\bar{\varepsilon}}_r \cdot \left(\bar{\bar{S}}\, \vec{E}(\vec{r},\omega)\right)$$

Assuming that $\bar{\bar{S}}$ is invertible

($\bar{\bar{\mu}}_r$ and $\bar{\bar{\varepsilon}}_r$ are diagonal, thus commute with $\bar{\bar{S}}$)

$$\bar{\bar{S}}^{-1} \cdot \vec{\nabla} \times \vec{E}(\vec{r},\omega) = k_0 \cdot \bar{\bar{\mu}}_r \cdot \tilde{\vec{H}}(\vec{r},\omega)$$

$$\bar{\bar{S}}^{-1} \cdot \vec{\nabla} \times \tilde{\vec{H}}(\vec{r},\omega) = k_0 \cdot \bar{\bar{\varepsilon}}_r \cdot \vec{E}(\vec{r},\omega)$$

$$\bar{\bar{S}}^{-1} = \begin{pmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{s_x}{s_y s_z} & 0 & 0 \\ 0 & \frac{s_y}{s_x s_z} & 0 \\ 0 & 0 & \frac{s_z}{s_x s_y} \end{pmatrix}$$

$$\bar{\bar{S}}^{-1} \cdot \vec{\nabla} \times = \begin{pmatrix} \frac{s_x}{s_y s_z} & 0 & 0 \\ 0 & \frac{s_y}{s_x s_z} & 0 \\ 0 & 0 & \frac{s_z}{s_x s_y} \end{pmatrix} \cdot \begin{pmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix} \overset{\text{uniaxial PML}}{=}$$

$$= \begin{pmatrix} 0 & -\frac{s_x}{s_y}\left(\frac{1}{s_z}\frac{\partial}{\partial z}\right) & \frac{s_x}{s_z}\left(\frac{1}{s_y}\frac{\partial}{\partial y}\right) \\ \frac{s_y}{s_x}\left(\frac{1}{s_z}\frac{\partial}{\partial z}\right) & 0 & -\frac{s_y}{s_z}\left(\frac{1}{s_x}\frac{\partial}{\partial x}\right) \\ -\frac{s_z}{s_x}\left(\frac{1}{s_y}\frac{\partial}{\partial y}\right) & \frac{s_z}{s_y}\left(\frac{1}{s_x}\frac{\partial}{\partial x}\right) & 0 \end{pmatrix}$$

$s_x, s_y, s_z \in \mathbb{C}$ are effectively "stretching" the coordinates, but they are "stretching" them into a complex space.

PML

Berenger, 1994

**Split-field PML**

splitting MEs into two sets of equations

introduce fictitious E&M

artificial anisotropic absorbing material

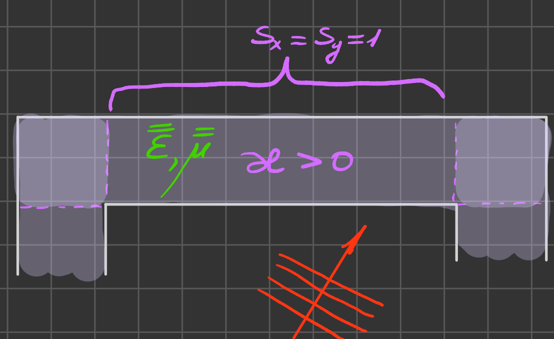
**Uniaxial PML**

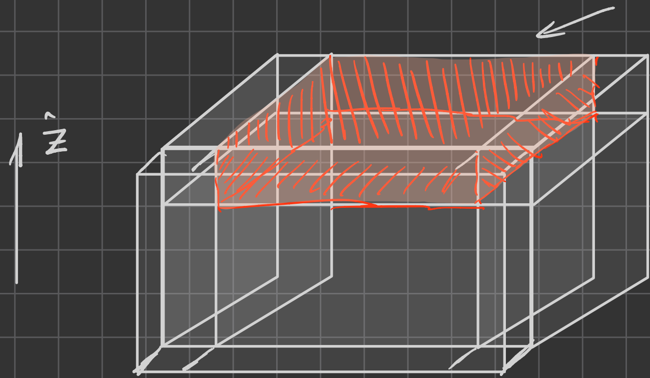
equivalent to a much more elegant and general approach

stretched-coordinate PML

Perfectly Matched Layers were shown to correspond to a coordinate transformation in which one or more coordinates are mapped to complex numbers. This is an analytic continuation of the wave equation into complex coordinates, replacing propagating (unattenuated) oscillations by exponentially decaying waves.
This viewpoint allows derivation of PML for inhomogeneous media (like waveguides) and for other CSs and wave equations.

We can drop non-stretching terms. Suppose the wave hits the PML in z direction.






where $s_x = s_y = 1$

in red is what called above as PML in z direction

Inside this z - PML $\quad s_x = s_y = 1$ (no PML overlap).

x - PML $\quad s_y = s_z = 1$

y - PML $\quad s_x = s_z = 1$

$$\bar{\bar{s}}^{-1} \cdot \vec{\nabla} \times = \begin{pmatrix} 0 & -\left(\frac{1}{s_z}\frac{\partial}{\partial z}\right) & \left(\frac{1}{s_y}\frac{\partial}{\partial y}\right) \\ \left(\frac{1}{s_z}\frac{\partial}{\partial z}\right) & 0 & -\left(\frac{1}{s_x}\frac{\partial}{\partial x}\right) \\ -\left(\frac{1}{s_y}\frac{\partial}{\partial y}\right) & \left(\frac{1}{s_x}\frac{\partial}{\partial x}\right) & 0 \end{pmatrix}$$

Now this is a stretched coordinate PML.

Avoid overlaping PML with evanescent fields (from resonant cavity or waveguide modes) — make computational cell sufficiently large

Rule of thumb

PML thickness $\sim \frac{\lambda_{max}}{2}$

- explain pros / cons UPML vs SC PML
- justify dropping $\frac{s_x}{s_y}$ ... factors from the $\overline{\overline{S}}$ tensor

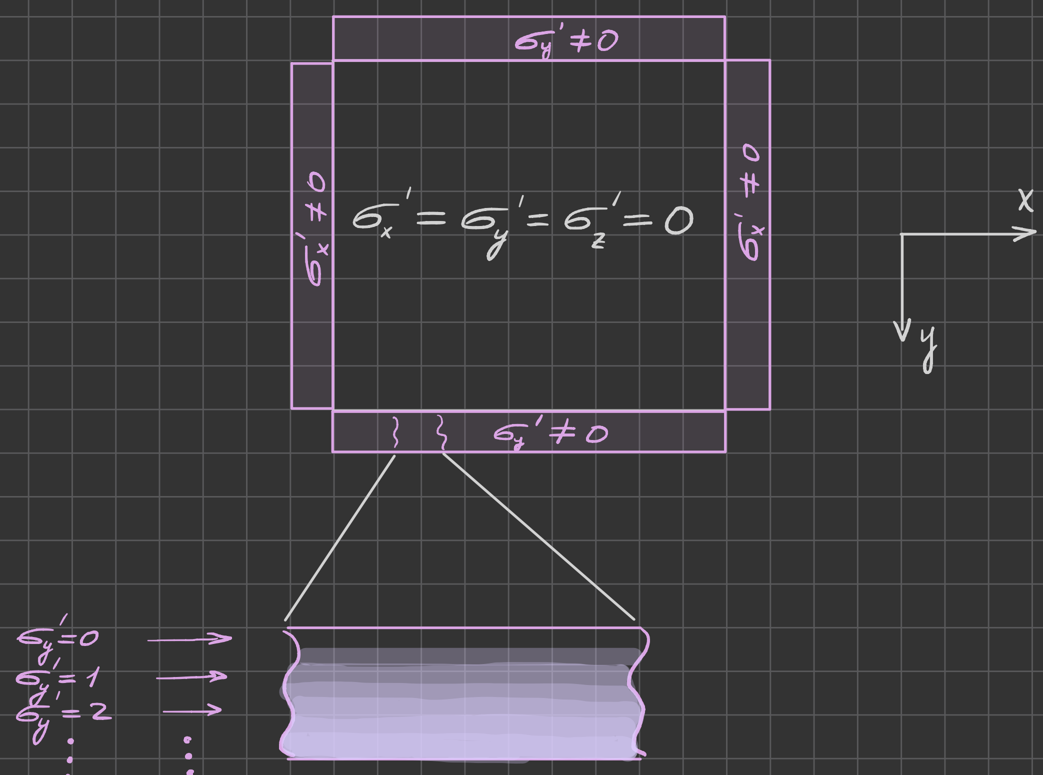
1. finish UPML derivation

2. then go to SC PML and derive it from UPML with justification (+ from lect 10)

3. Back to UPML : incorporate it into Maxwell's equation

How to calculate PML parameters?

Introduce loss through conductivity terms: fictitious conductivity (primed).




$\sigma_y'$ is graded in the PML region. In mathematical formulation it's not necessary, but we do it to prevent numerical reflections.

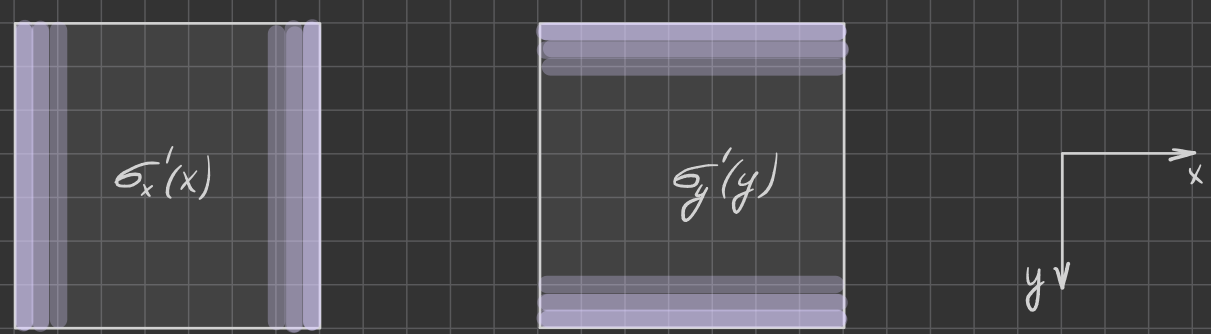
$$s_x(x) = 1 + \frac{\sigma_x'(x)}{j\omega\varepsilon_0} \qquad \sigma_x'(x) = \frac{\varepsilon_0}{2\Delta t}\left(\frac{x}{L_x}\right)^3$$

$$s_y(y) = 1 + \frac{\sigma_y'(y)}{j\omega\varepsilon_0} \qquad \sigma_y'(y) = \frac{\varepsilon_0}{2\Delta t}\left(\frac{y}{L_y}\right)^3$$

$$s_z(z) = 1 + \frac{\sigma_z'(z)}{j\omega\varepsilon_0} \qquad \sigma_z'(z) = \frac{\varepsilon_0}{2\Delta t}\left(\frac{z}{L_z}\right)^3$$

other than 1 if the wave is evanescent in computational domain.

these are arrays with the size of the whole computational domain.




Coding up the tapering of fictitious conductivity.

$x, y, z$ – positions inside PML

$L_x, L_y, L_z$ – thicknesses of PMLs

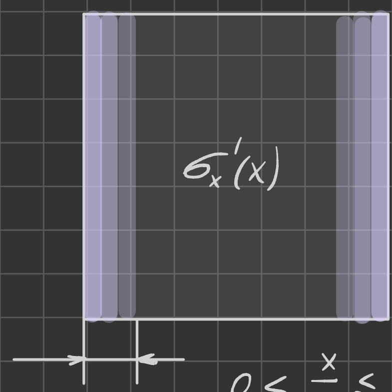
$\frac{x}{L_x}, \frac{y}{L_y}, \frac{z}{L_z}$ grows inside PML in the directions $\pm x, \pm y, \pm z$

These ratios can be expressed in terms of grid's indices.

$$\frac{x}{L_x} \approx \frac{n_x}{N_x^{low}} \quad \text{or} \quad \frac{n_x}{N_x^{high}}$$

← index of PML layer

← total # of layers




$$0 < \frac{x}{L_x} \leq 1$$

$N_x = 3 \quad \Rightarrow \quad \dfrac{n_x}{N_x} = \cancel{\dfrac{0}{3}},\ \dfrac{1}{3},\ \dfrac{2}{3},\ \dfrac{3}{3}$

same for $y$ and $z$ axes

Continue here with initialization of PLL parameters and then derivation of update equations.

# How to validate Hertzian dipole

$$\vec{J}(\vec{r}', \overset{t_n}{n\Delta t}) = \hat{z}\, I \underset{\Delta z}{\ell} \cdot \delta(\vec{r}') = \hat{z} \cdot I \cdot \sin(\omega \cdot n \cdot \Delta t) \cdot \Delta z \cdot \delta(\vec{r}')$$

$\Delta z$ discrete case

$\vec{r}'$ - source coordinates

$$I(t) = \sin(\omega t) = \sin(2\pi f \cdot n \cdot \Delta t)$$

$f$ - known frequency (300 MHz)

$\lambda = 1m$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathcal{J}(\vec{r}', t)}{|\vec{r} - \vec{r}'|} d\vec{r}' \overset{\text{for Hertzian dipole}}{=}$$

$V'$ ← over the source points

$$= \frac{\mu_0}{4\pi} \int \frac{\hat{z} \sin(\omega t') \cdot \Delta z \cdot I\, \delta(\vec{r}')}{|\vec{r} - \vec{r}'|} d\vec{r}'$$

$$\boxed{= \frac{\mu_0}{4\pi} \frac{\hat{z} \cdot I \cdot \sin(\omega t') \cdot \Delta z}{|\vec{r} - \vec{r}_{src}|} = \vec{A}(\vec{r}, t)}$$

known point

substitute specific form of potential

$$\vec{E}(\vec{r}, t) = -\vec{\nabla}\Phi(\vec{r}, t) - \frac{\partial}{\partial t} \vec{A}(\vec{r}, t)$$

$$\vec{\nabla}\Phi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \rho(\vec{r}',t')\cdot\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|}\,d\vec{r}' \quad =$$

$$-\frac{\partial}{\partial t'}\rho(\vec{r}',t') = \vec{\nabla}'\cdot\vec{J}(\vec{r}',t')$$

$$\rho(\vec{r}',t') = -\int_0^{t'}\vec{\nabla}'\cdot\vec{J}(\vec{r}',t'')\,dt''$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{V'}\int_0^{t'}\left(\vec{\nabla}'\cdot\vec{J}(\vec{r}',t'')\,dt''\right)\cdot\left(\underbrace{\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)}_{-\frac{1}{|\vec{r}-\vec{r}'|^2}}\right)d\vec{r}' =$$

$$=$$

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(Top)

Magnetic vector potential

Gauge choices

Maxwell's equations in terms of vector potential

**Calculation of potentials from source distributions**

Depiction of the A-field

Electromagnetic four-potential

See also

Notes

References

External links

In other gauges, the equations are different. A different notation to write these same equations (using

## Calculation of potentials from source distributions [ edit ]

*Main article: Retarded potential*

The solutions of Maxwell's equations in the Lorenz gauge (see Feynman[3] and Jackson[5]) with the b
potentials go to zero sufficiently fast as they approach infinity are called the retarded potentials, which
potential **A**(**r**, *t*) and the electric scalar potential $\varphi$(**r**, *t*) due to a current distribution of current density
and volume Ω, within which $\rho$ and **J** are non-zero at least sometimes and some places):

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_\Omega \frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}'$$

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int_\Omega \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}'$$

where the fields at position vector **r** and time *t* are calculated from sources at distant position **r**′ at an
a source point in the charge or current distribution (also the integration variable, within volume Ω). Th
*retarded time*, and calculated as

$$t' = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}.$$

There are a few notable things about **A** and $\varphi$ calculated in this way:

- The Lorenz gauge condition: $\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$ is satisfied.
- The position of **r**, the point at which values for $\varphi$ and **A** are found, only enters the equation as pa
to **r**. The direction from **r**′ to **r** does not enter into the equation. The only thing that matters about a
it is.
- The integrand uses *retarded time*, *t*′. This simply reflects the fact that changes in the sources pro
Hence the charge and current densities affecting the electric and magnetic potential at **r** and *t*, fro
be at some prior time *t*′.
- The ~~equation for **A** is a vector equation. In~~ Cartesian coordinates, the equation separates into thre

$$t' = t - \frac{|\vec{r}-\vec{r}'|}{c} \quad \text{retarded time}$$

for free space

$$\vec{E} = -\frac{\partial}{\partial t}\vec{A}(\vec{r},t) = -\frac{\partial}{\partial t}\left(\frac{\mu_0}{4\pi}\,\frac{\hat{z}\cdot I\cdot\sin(\omega t')\cdot\Delta z}{|\vec{r}-\vec{r}_{src}|}\right) =$$

$$= -\frac{\partial t'}{\partial t}\,\frac{\partial}{\partial t'}\left(\frac{\mu_0}{4\pi}\,\frac{\hat{z}\,I\cdot\Delta z}{|\vec{r}-\vec{r}_{src}|}\cdot\sin(\omega t')\right)$$

$$= - \underbrace{\frac{\partial\left(t - \frac{|\vec{r}-\vec{r}_{src}|}{c}\right)}{\partial t}}_{1} \cdot \frac{\mu_0}{4\pi} \frac{\hat{z} I \Delta z}{|\vec{r}-\vec{r}_{src}|} \cdot \omega \cdot \cos(\omega t')$$

$$\boxed{\vec{E}(\vec{r},t) = - \frac{\mu_0}{4\pi} \frac{I \Delta z \cdot \hat{z}}{|\vec{r}-\vec{r}_{src}|} \cdot \omega \cdot \cos\left(\omega \left[t - \frac{|\vec{r}-\vec{r}_{src}|}{c}\right]\right)}$$

$$\vec{r}_{src} = \begin{pmatrix} x\_src \\ y\_src \\ z\_src \end{pmatrix}$$

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$ – any point in simulation domain

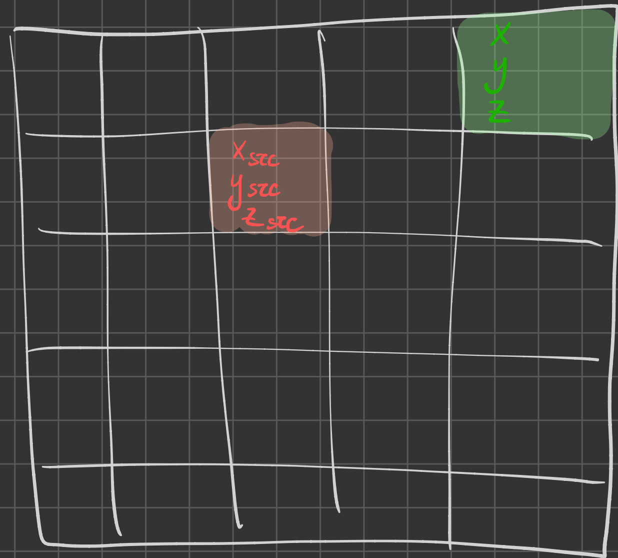
$$I = 5$$

$$\Delta z = dz$$

$$\omega = 2\pi f = 2\pi \cdot 300\ MHz$$

$$t \rightarrow n \cdot \Delta t \quad \text{for} \quad n = \overline{0, Nt}$$

$\vec{r} - \vec{r}_{src}$ for all $N_x \cdot N_y \cdot N_z$ cells




$$r^*[i,j,k] = \text{norm}\begin{pmatrix} X[i] - x\_src \\ Y[j] - y\_src \\ Z[k] - z\_src \end{pmatrix}$$