(lecture 4)

# Flow of MEs

update B

$$\boxed{\vec{\nabla} \times \vec{E}(t) = -\frac{\partial}{\partial_t}\vec{B}(t)}$$

update H

$$\boxed{\vec{B}(t) = \bar{\bar{\mu}}(t) * \vec{H}(t)}$$

update D

$$\boxed{\vec{\nabla} \times \vec{H}(t) = \frac{\partial}{\partial t}\vec{D}(t) + \vec{J}}$$

update E

$$\boxed{\vec{D}(t) = \bar{\bar{\varepsilon}}(t) * \vec{E}(t)}$$

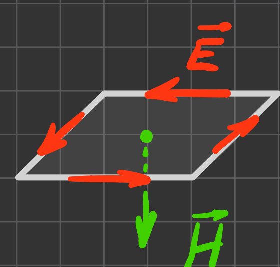
this is a multiplication in frequency domain

Assume that our material is linear isotropic non-dispersive

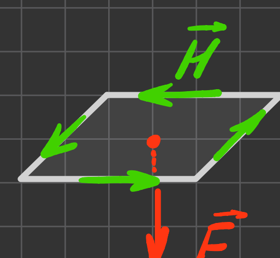
$$\bar{\bar{\mu}} * \;\rightarrow\; \mu \cdot$$

$$\bar{\bar{\varepsilon}} * \;\rightarrow\; \varepsilon \cdot$$

$$\boxed{\vec{\nabla} \times \vec{E}(t) = -\mu \cdot \frac{\partial}{\partial t}\vec{H}(t)}$$




$$\boxed{\vec{\nabla} \times \vec{H}(t) = \varepsilon \cdot \frac{\partial}{\partial t}\vec{E}(t) + \vec{J}(t)}$$



## Approximating time derivative

$$\vec{\nabla}\times\vec{E} = -\mu\frac{\partial}{\partial t}\vec{H} \longrightarrow -\frac{1}{\mu}\vec{\nabla}\times\vec{E}(t) \cong \frac{\vec{H}(t+\frac{\Delta t}{2})-\vec{H}(t-\frac{\Delta t}{2})}{\Delta t}$$

$$\vec{\nabla}\times\vec{H} = \varepsilon\frac{\partial}{\partial t}\vec{E}+\vec{J} \longrightarrow \frac{1}{\varepsilon}\vec{\nabla}\times\vec{H}(t+\frac{\Delta t}{2}) \cong \frac{\vec{E}(t+\Delta t)-\vec{E}(t)}{\Delta t}+\vec{J}(t+\frac{\Delta t}{2})$$

## Update equations

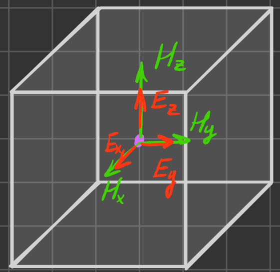
$$\boxed{\vec{H}^{t+\frac{\Delta t}{2}} = \vec{H}^{t-\frac{\Delta t}{2}} - \frac{\Delta t}{\mu}\left[\vec{\nabla}\times\vec{E}\right]^{t}}$$

$$\boxed{\vec{E}^{t+\Delta t} = \vec{E}^{t} + \frac{\Delta t}{\varepsilon}\left[\vec{\nabla}\times\vec{H}\right]^{t+\frac{\Delta t}{2}} - \frac{\Delta t}{\varepsilon}J^{t+\frac{\Delta t}{2}}}$$

$\frac{\Delta t}{\varepsilon}$, $\frac{\Delta t}{\mu}$ are update coefficents = const – will be calculated ahead of simulation.

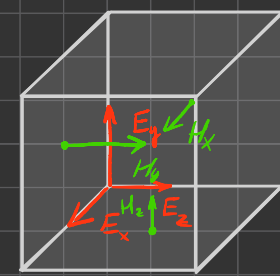
lecture 5

**Collocated grids** – when $\vec{E}$ and $\vec{H}$ are defined at the same point in a grid cell



# Staggered grids (Yee cell)

- Physical BCs are naturally satisfied
- Gauss ($\vec{D}$ and $\vec{B}$) laws are satisfied



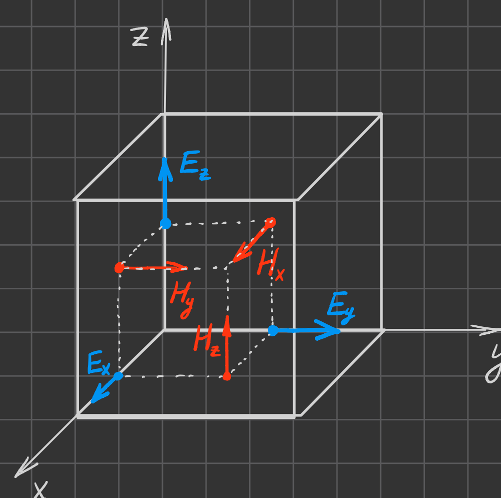

$$\vec{\nabla} \cdot \vec{D} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

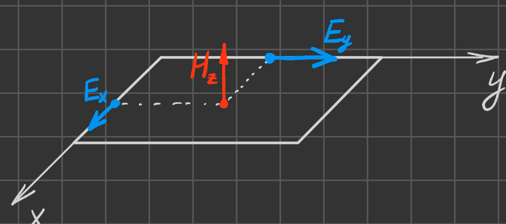
for divergence - free medium

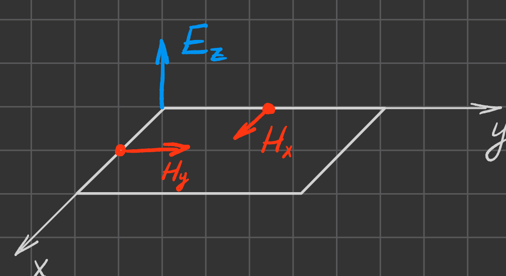
## Yee cell for 1D, 2D, and 3D

3D




2D

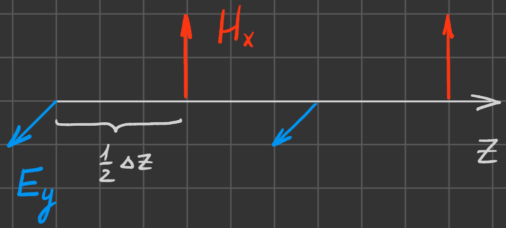


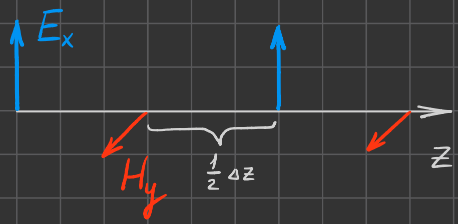




$H_z$ mode

$E_z$ mode

# 1D




$E_y$ mode

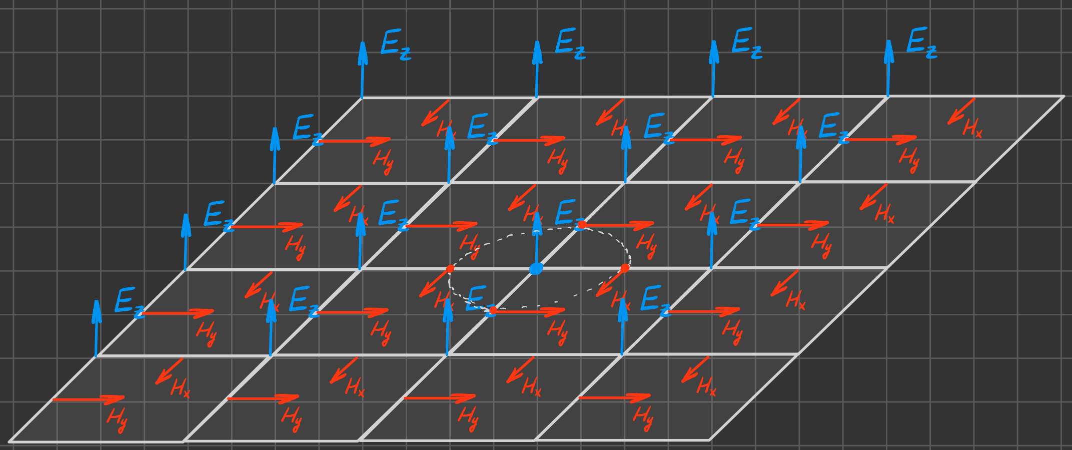
$E_x$ mode

## Consequences of Yee grid

- Field components $E_x$, $E_y$, $E_z$ may reside in different materials even being in one cell.

! • Field components will be out of phase.

Example for 2D: 4×4 grid. $E_z$ mode



# FD approximation to MEs

1. We satisfied the divergence equations (Gauss's laws) automatically by adopting Yee's grid.

2. **Normalization**

$$H = \frac{1}{\eta_0} E \quad , \quad \eta_0 \equiv \sqrt{\frac{\mu_0}{\varepsilon_0}} \simeq 377\,\Omega$$

$$\Rightarrow \quad E \simeq 377 H$$

i.e. magnitude of $H$ is always 2-3 orders lower than of $E$. Thus, we will lose 2-3 digits of accuracy in numerical simulations.

Remedy is to normalize magnetic field $H$:

$$\boxed{\vec{\tilde{H}} \equiv \sqrt{\frac{\mu_0}{\varepsilon_0}}\, \vec{H} = \eta_0 \vec{H}}$$

$$\vec{E} = \eta_0 \vec{H} \equiv \vec{\tilde{H}} \quad \Rightarrow$$ $\vec{E}$ and $\vec{\tilde{H}}$ have the same magnitude

Original magn. field $\boxed{\vec{H} \equiv \frac{1}{\eta_0} \vec{\tilde{H}}}$

Substitute it into Faraday's & Ampere's laws:

$$\begin{cases} \vec{\nabla} \times \left(\frac{1}{\eta_0}\vec{\tilde{H}}\right) = \varepsilon_0 \overline{\overline{\varepsilon}}_r \frac{\partial}{\partial t}\vec{E} + \vec{J} \\ \vec{\nabla} \times \vec{E} = -\mu_0 \overline{\overline{\mu}}_r \frac{\partial}{\partial t}\left(\frac{1}{\eta_0}\vec{\tilde{H}}\right) - \vec{M} \end{cases}$$

$$\begin{cases} \vec{\nabla} \times \vec{\tilde{H}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \varepsilon_0 \overline{\overline{\varepsilon}}_r \partial_t \vec{E} + \eta_0 \vec{J} \\ \vec{\nabla} \times \vec{E} = -\mu_0 \sqrt{\frac{\varepsilon_0}{\mu_0}} \cdot \overline{\overline{\mu}}_r \partial_t \vec{\tilde{H}} - \vec{M} \end{cases}$$

$$\frac{1}{c_0} = \sqrt{\mu_0 \varepsilon_0} \longrightarrow$$

$$\begin{cases} \vec{\nabla} \times \vec{\tilde{H}} = \frac{1}{c_0} \overline{\overline{\varepsilon}}_r \partial_t \vec{E} + \underbrace{\eta_0 \vec{J}} \\ \vec{\nabla} \times \vec{E} = -\frac{1}{c_0} \overline{\overline{\mu}}_r \partial \vec{\tilde{H}} - \vec{\mathcal{M}} \end{cases}$$

?

maybe this multiplication will be better included into Gaussian ?

We will omit ~ above $\vec{H}$ keeping in mind that it's normalized and it will be denormalized for quantitative analysis.

$$\boxed{\begin{aligned} \vec{\nabla} \times \vec{E} &= -\frac{1}{c_0} \overline{\overline{\mu}}_r \partial \vec{H} - \vec{\mathcal{M}} \\ \vec{\nabla} \times \vec{H} &= \frac{1}{c_0} \overline{\overline{\varepsilon}}_r \partial_t \vec{E} + \eta_0 \vec{J} \end{aligned}}$$

Component-wise:

$$x: \quad \partial_y E_z - \partial_z E_y = -\frac{1}{c_0}\left(\mu_{xx} \partial_t H_x + \mu_{xy} \partial_t H_y + \mu_{xz} \partial_t H_z\right) - \mathcal{M}_x$$

$$y: \quad \partial_z E_x - \partial_x E_z = -\frac{1}{c_0}\left(\mu_{yx} \partial_t H_x + \mu_{yy} \partial_t H_y + \mu_{yz} \partial_t H_z\right) - \mathcal{M}_y$$

$$z: \quad \partial_x E_y - \partial_y E_x = -\frac{1}{c_0}\left(\mu_{zx} \partial_t H_x + \mu_{zy} \partial_t H_y + \mu_{zz} \partial_t H_z\right) - \mathcal{M}_z$$

x: $\partial_y H_z - \partial_z H_y = \frac{1}{c_0}\left(\varepsilon_{xx}\partial_t E_x + \varepsilon_{xy}\partial_t E_y + \varepsilon_{xz}\partial_t E_z\right) + \gamma_0 J_x$

y: $\partial_z H_x - \partial_x H_z = \frac{1}{c_0}\left(\varepsilon_{yx}\partial_t E_x + \varepsilon_{yy}\partial_t E_y + \varepsilon_{yz}\partial_t E_z\right) + \gamma_0 J_y$

z: $\partial_x H_y - \partial_y H_x = \frac{1}{c_0}\left(\varepsilon_{zx}\partial_t E_x + \varepsilon_{zy}\partial_t E_y + \varepsilon_{zz}\partial_t E_z\right) + \gamma_0 J_z$

Assume we have only diagonally - anisotropic materials.

1 $\quad \partial_y E_z - \partial_z E_y = -\frac{1}{c_0}\mu_{xx}\partial_t H_x - M_x$

2 $\quad \partial_z E_x - \partial_x E_z = -\frac{1}{c_0}\mu_{yy}\partial_t H_y - M_y$

3 $\quad \partial_x E_y - \partial_y E_x = -\frac{1}{c_0}\mu_{zz}\partial_t H_z - M_z$

4 $\quad \partial_y H_z - \partial_z H_y = \frac{1}{c_0}\varepsilon_{xx}\partial_t E_x + \gamma_0 J_x$

5 $\quad \partial_z H_x - \partial_x H_z = \frac{1}{c_0}\varepsilon_{yy}\partial_t E_y + \gamma_0 J_y$

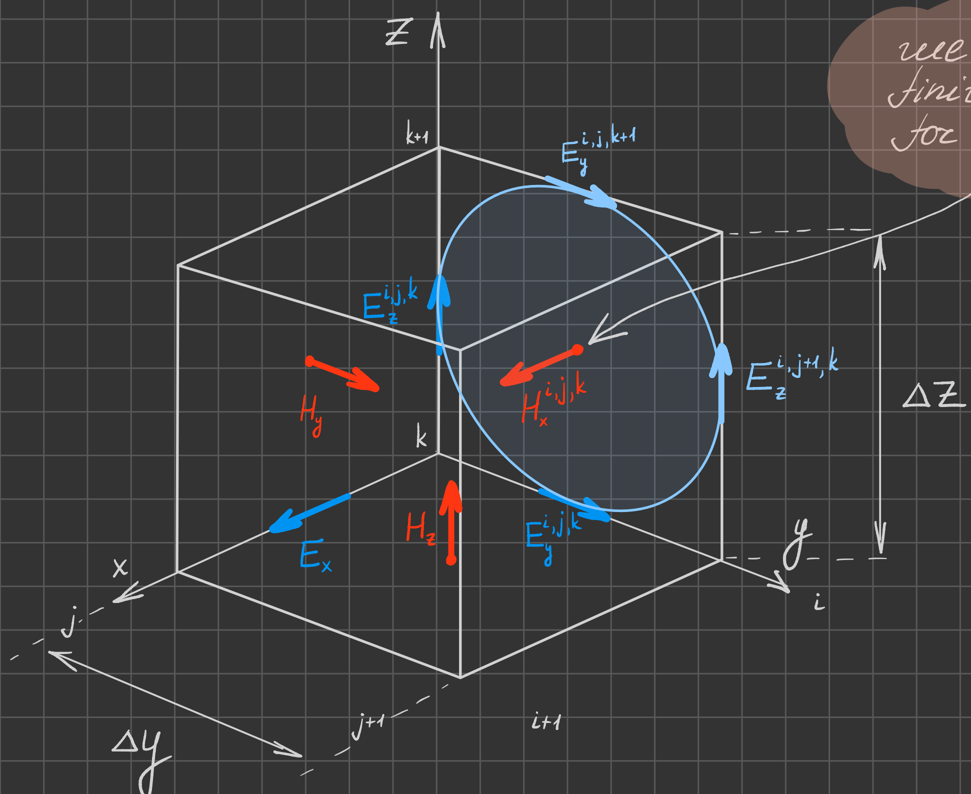
6 $\quad \partial_x H_y - \partial_y H_x = \frac{1}{c_0}\varepsilon_{zz}\partial_t E_z + \gamma_0 J_z$

We will approximate these 6 equalities on Yee grid.

(next 6 pages)

1

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{1}{c_0}\mu_{xx}\frac{\partial H_x}{\partial t} - \mathcal{M}_x$$




$$\frac{E_z^{i,j+1,k}\big|_t - E_z^{i,j,k}\big|_t}{\Delta y} - \frac{E_y^{i,j,k+1}\big|_t - E_y^{i,j,k}\big|_t}{\Delta z} = -\frac{1}{c_0}\mu_{xx}^{i,j,k}\frac{H_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$- \mathcal{M}_x^{i,j,k}\big|_t$$

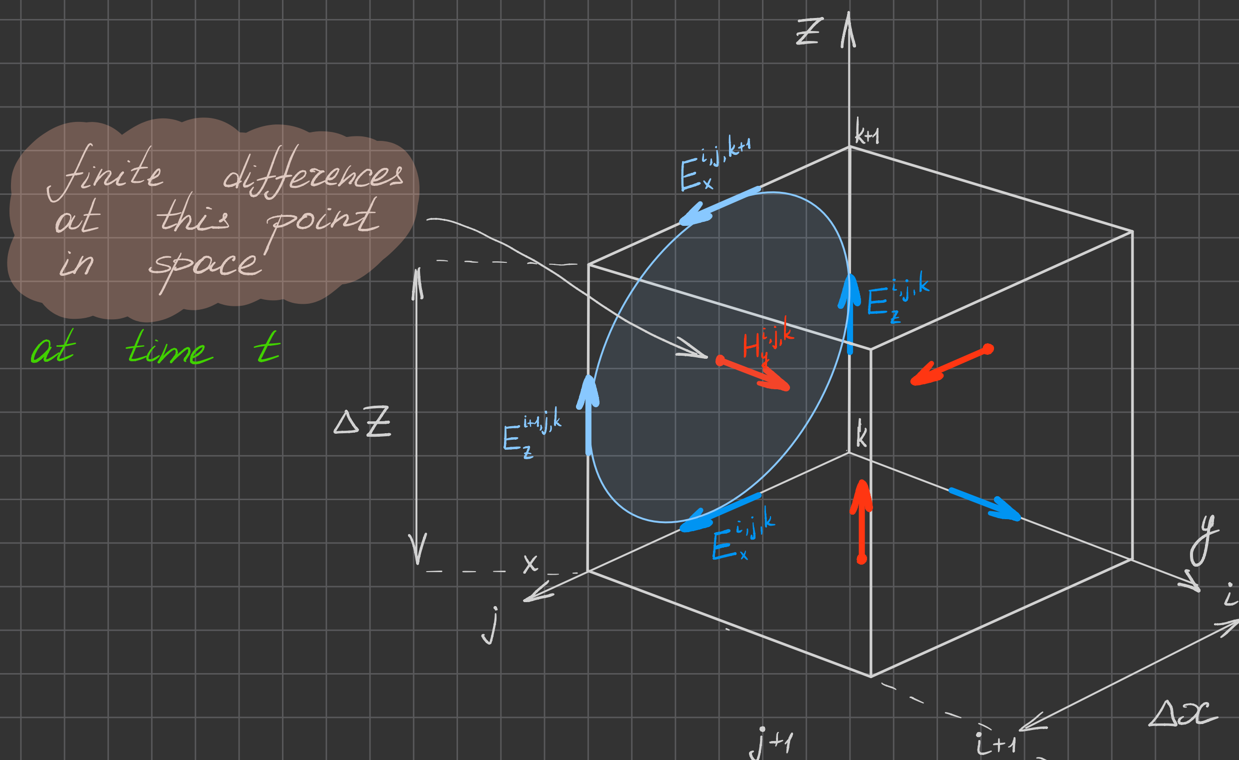
$\mu$ components may also vary in space : $\mu_{xx} \rightarrow \mu_{xx}^{i,j,k}$

$\varepsilon_{xx} \rightarrow \varepsilon_{xx}^{i,j,k}$

For the electric field derivative we need to reach to the next cells in space

2

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{1}{c_0}\mu_{yy}\frac{\partial H_y}{\partial t} - M_y$$

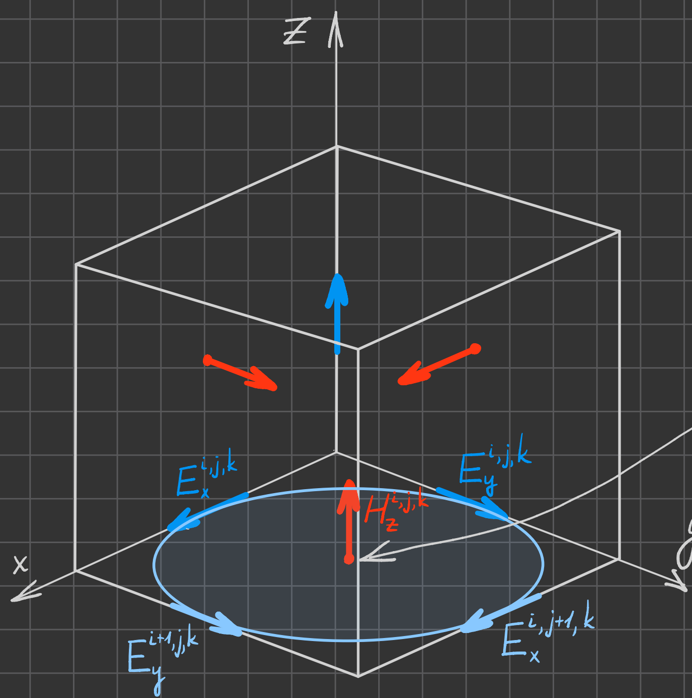



$$\frac{E_x^{i,j,k+1}\big|_t - E_x^{i,j,k}\big|_t}{\Delta z} - \frac{E_z^{i+1,j,k}\big|_t - E_z^{i,j,k}\big|_t}{\Delta x} = -\frac{1}{c_0}\mu_{yy}^{i,j,k}\frac{H_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_y^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$- M_y^{i,j,k}\big|_t$$

3

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{1}{c_0}\mu_{zz}\frac{\partial H_z}{\partial t} - M_z$$




we are approximating spatial derivatives at this point

at time $t$

$$\frac{E_y^{i+1,j,k}\big|_t - E_y^{i,j,k}\big|_t}{\Delta x} - \frac{E_x^{i,j+1,k}\big|_t - E_x^{i,j,k}\big|_t}{\Delta y} = -\frac{1}{c_0}\mu_{zz}^{i,j,k}\frac{H_z^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_z^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

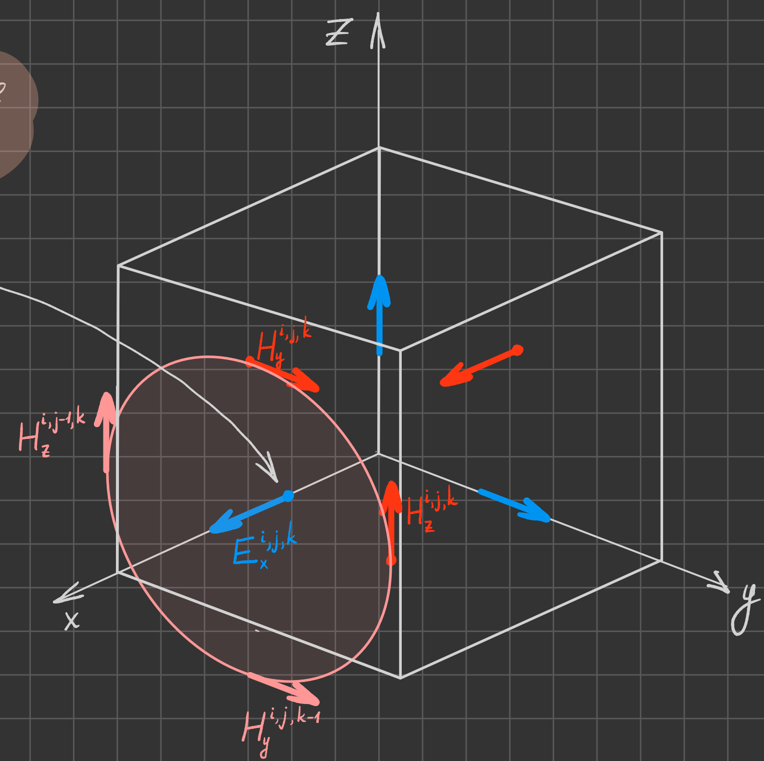
$$- M_z^{i,j,k}\big|_t$$

4

$$\partial_y H_z - \partial_z H_y = \frac{1}{c_0}\varepsilon_{xx}\partial_t E_x + \eta_0 J_x$$

finite difference equation is written for this point

at time $t+\frac{\Delta t}{2}$



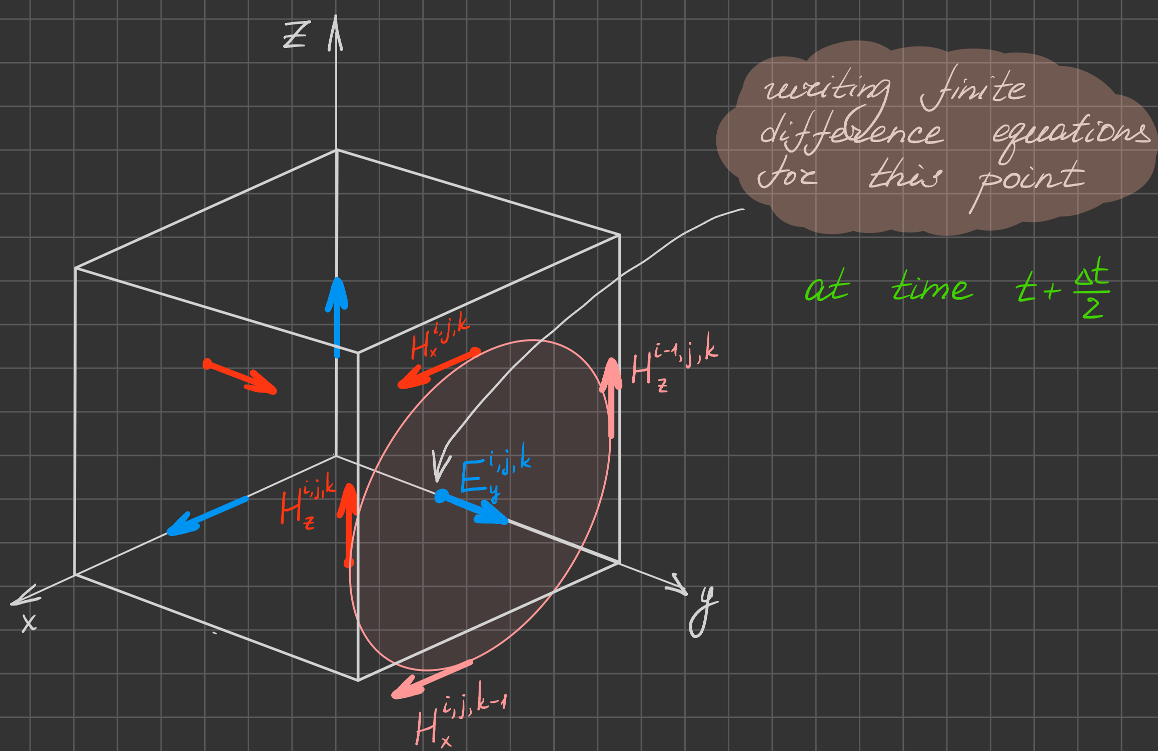

$$\frac{H_z^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_z^{i,j-1,k}\Big|_{t+\frac{\Delta t}{2}}}{\Delta y} - \frac{H_y^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_y^{i,j,k-1}\Big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{1}{c_0}\varepsilon_{xx}^{i,j,k}\,\frac{E_x^{i,j,k}\Big|_{t+\Delta t} - E_x^{i,j,k}\Big|_{t}}{\Delta t} +$$

$$+\ \eta_0 J_x^{i,j,k}\Big|_{t+\frac{\Delta t}{2}}$$

For magnetic field derivative we have to reach to the previous cell in space

5

$$\partial_z H_x - \partial_x H_z = \frac{1}{c_0}\varepsilon_{yy}\partial_t E_y + \eta_0 J_y$$




at time $t+\frac{\Delta t}{2}$

$$\frac{H_x^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k-1}\Big|_{t+\frac{\Delta t}{2}}}{\Delta z} - \frac{H_z^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_z^{i-1,j,k}\Big|_{t+\frac{\Delta t}{2}}}{\Delta x} = \frac{1}{c_0}\varepsilon_{yy}^{i,j,k}\frac{E_y^{i,j,k}\Big|_{t+\Delta t} - E_y^{i,j,k}\Big|_{t}}{\Delta t}$$

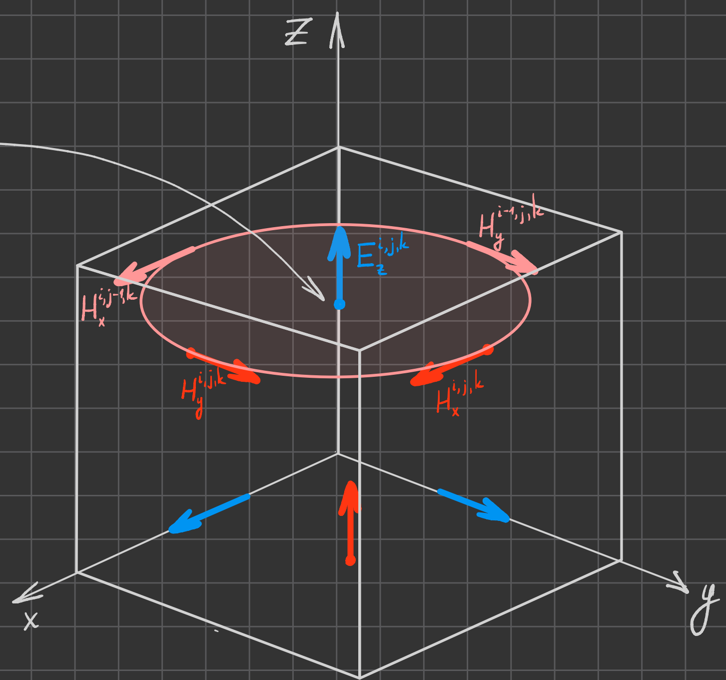
$$+ \eta_0 J_y^{i,j,k}\Big|_{t+\frac{\Delta t}{2}}$$

6

$$\partial_x H_y - \partial_y H_x = \frac{1}{c_0}\varepsilon_{zz}\partial_t E_z + \eta_0 J_z$$

we write finite difference equations at this point

at time $t+\frac{\Delta t}{2}$




$$\frac{H_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_y^{i-1,j,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta x} - \frac{H_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_x^{i,j-1,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta y} = \frac{1}{c_0}\varepsilon_{zz}^{i,j,k}\frac{E_z^{i,j,k}\big|_{t+\Delta t} - E_z^{i,j,k}\big|_{t}}{\Delta t} + \eta_0 J_z^{i,j,k}\Big|_{t+\frac{\Delta t}{2}}$$

For the same spatial cell electric $J$ and magnetic $M$ currents are also staggered in time $\frac{\Delta t}{2}$ apart. ($J$ is $\frac{\Delta t}{2}$ ahead of $M$)

$$\frac{E_z^{i,j+1,k}\Big|_t - E_z^{i,j,k}\Big|_t}{\Delta y} - \frac{E_y^{i,j,k+1}\Big|_t - E_y^{i,j,k}\Big|_t}{\Delta z} = -\frac{1}{c_0}\mu_{xx}^{i,j,k}\frac{H_x^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k}\Big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$- \mathcal{M}_x^{i,j,k}\Big|_t$$

$$\frac{E_x^{i,j,k+1}\Big|_t - E_x^{i,j,k}\Big|_t}{\Delta z} - \frac{E_z^{i+1,j,k}\Big|_t - E_z^{i,j,k}\Big|_t}{\Delta x} = -\frac{1}{c_0}\mu_{yy}^{i,j,k}\frac{H_y^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_y^{i,j,k}\Big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$- \mathcal{M}_y^{i,j,k}\Big|_t$$

$$\frac{E_y^{i+1,j,k}\Big|_t - E_y^{i,j,k}\Big|_t}{\Delta x} - \frac{E_x^{i,j+1,k}\Big|_t - E_x^{i,j,k}\Big|_t}{\Delta y} = -\frac{1}{c_0}\mu_{zz}^{i,j,k}\frac{H_z^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_z^{i,j,k}\Big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$- \mathcal{M}_z^{i,j,k}\Big|_t$$

$$\frac{H_z^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_z^{i,j-1,k}\Big|_{t+\frac{\Delta t}{2}}}{\Delta y} - \frac{H_y^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_y^{i,j,k-1}\Big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{1}{c_0}\varepsilon_{xx}^{i,j,k}\frac{E_x^{i,j,k}\Big|_{t+\Delta t} - E_x^{i,j,k}\Big|_t}{\Delta t} +$$

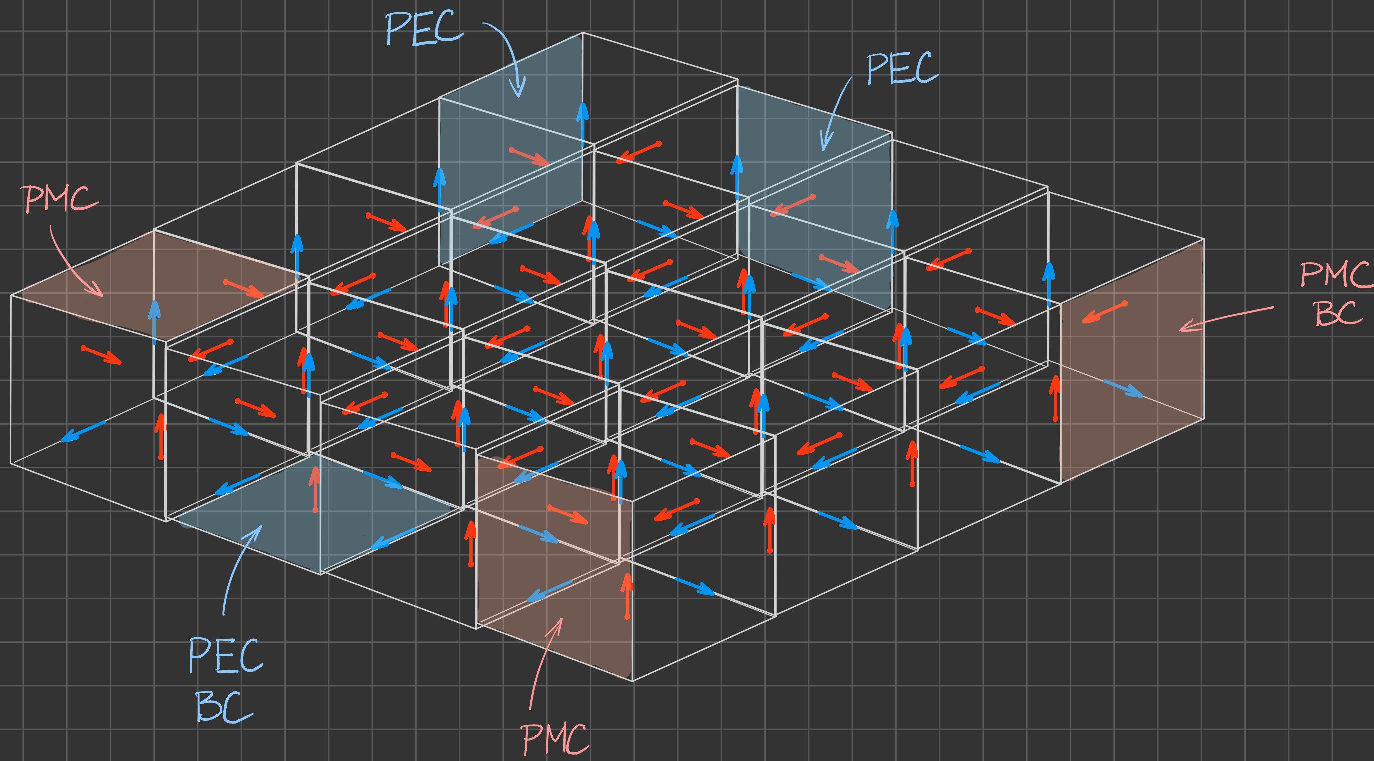
$$+ \eta_0 J_x^{i,j,k}\Big|_{t+\frac{\Delta t}{2}}$$

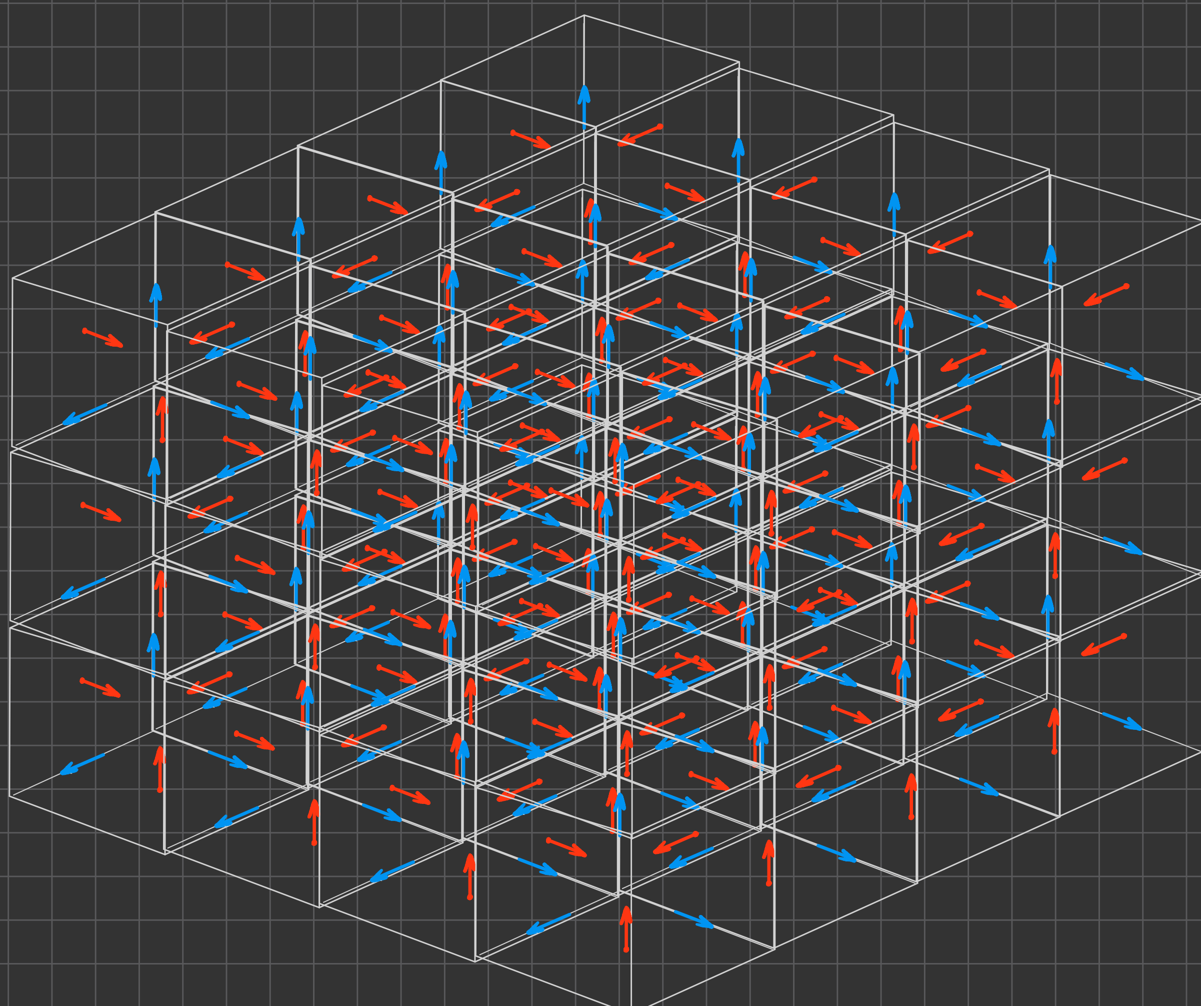
$$\frac{H_x^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k-1}\Big|_{t+\frac{\Delta t}{2}}}{\Delta z} - \frac{H_z^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_z^{i-1,j,k}\Big|_{t+\frac{\Delta t}{2}}}{\Delta x} = \frac{1}{c_0}\varepsilon_{yy}^{i,j,k}\frac{E_y^{i,j,k}\Big|_{t+\Delta t} - E_y^{i,j,k}\Big|_t}{\Delta t}$$

$$+ \eta_0 J_y^{i,j,k}\Big|_{t+\frac{\Delta t}{2}}$$

$$\frac{H_y^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_y^{i-1,j,k}\Big|_{t+\frac{\Delta t}{2}}}{\Delta x} - \frac{H_x^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H^{i,j-1,k}\Big|_{t+\frac{\Delta t}{2}}}{\Delta y} = \frac{1}{c_0}\varepsilon_{zz}^{i,j,k}\frac{E_z^{i,j,k}\Big|_{t+\Delta t} - E_z^{i,j,k}\Big|_t}{\Delta t}$$

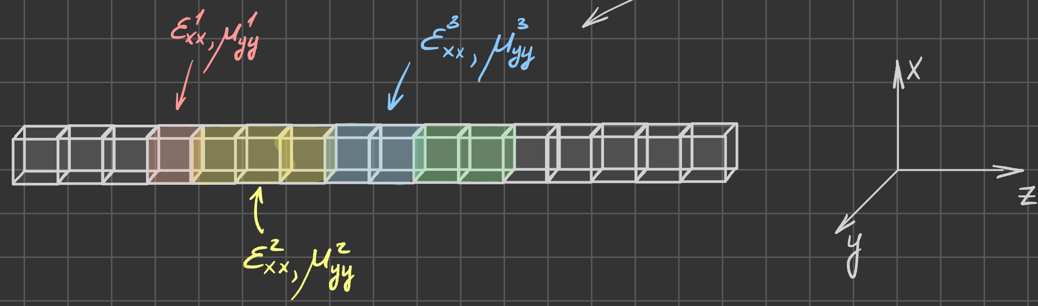
$$+ \eta_0 J_z^{i,j,k}\Big|_{t+\frac{\Delta t}{2}}$$


PEC
PEC
PMC
PMC
BC
PEC
BC
PMC



# 1D case

material parameters are functions of $z$




$$\partial_x = \partial_y = 0$$

$\Rightarrow$ We cross out corresponding terms from our finite difference equations:

$$-\frac{E_y^{i,j,k+1}\big|_t - E_y^{i,j,k}\big|_t}{\Delta z} = -\frac{1}{c_0}\mu_{xx}^{i,j,k}\frac{H_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t} - \mathcal{M}_x^{i,j,k}\big|_t$$

decoupled

$$\frac{E_x^{i,j,k+1}\big|_t - E_x^{i,j,k}\big|_t}{\Delta z} = -\frac{1}{c_0}\mu_{yy}^{i,j,k}\frac{H_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_y^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t} - \mathcal{M}_y^{i,j,k}\big|_t$$

$$H_z^{i,j,k} = 0$$

$E_x / H_y$ mode

$E_y / H_x$ mode

$$-\frac{H_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_y^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{1}{c_0}\varepsilon_{xx}^{i,j,k}\frac{E_x^{i,j,k}\big|_{t+\Delta t} - E_x^{i,j,k}\big|_t}{\Delta t} + \eta_0 J_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}}$$

decoupled

$$\frac{H_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{1}{c_0}\varepsilon_{yy}^{i,j,k}\frac{E_y^{i,j,k}\big|_{t+\Delta t} - E_y^{i,j,k}\big|_t}{\Delta t} + \eta_0 J_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}}$$

$$E_z^{i,j,k} = 0$$

If there's no anisotropy in the model, those modes have the same numerical behavior; they are physical modes and propagate independently.
We will consider $E_y / H_x$ mode.

$$-\frac{E_y^{i,j,k+1}\big|_t - E_y^{i,j,k}\big|_t}{\Delta z} = -\frac{1}{c_0}\mu_{xx}^{i,j,k}\frac{H_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t} - M_x^{i,j,k}\big|_t$$

$$\frac{H_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{1}{c_0}\varepsilon_{yy}^{i,j,k}\frac{E_y^{i,j,k}\big|_{t+\Delta t} - E_y^{i,j,k}\big|_t}{\Delta t} + \eta_0 J_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}}$$

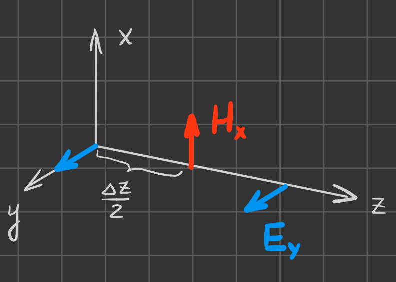
Let's derive update equations. (omit indices $i$ and $j$)

Ampere's law:

$$E_y^k\big|_{t+\Delta t} = E_y^k\big|_t + \frac{1}{\Delta z}\cdot\underbrace{\Delta t\cdot\frac{c_0}{\varepsilon_{yy}^k}}_{\equiv m_{E_y}^k}\left(H_x^k\big|_{t+\frac{\Delta t}{2}} - H_x^{k-1}\big|_{t+\frac{\Delta t}{2}}\right) - \underbrace{\Delta t\cdot\frac{c_0}{\varepsilon_{yy}^k}}_{m_{E_y}^k}\eta_0 J_y^k\big|_{t+\frac{\Delta t}{2}}$$

Faraday's law

$$H_x^k\big|_{t+\frac{\Delta t}{2}} = H_x^k\big|_{t-\frac{\Delta t}{2}} + \frac{1}{\Delta z}\cdot\underbrace{\Delta t\cdot\frac{c_0}{\mu_{xx}^k}}_{\equiv m_{H_x}^k}\cdot\left(E_y^{k+1}\big|_t - E_y^k\big|_t\right) - \underbrace{\Delta t\cdot\frac{c_0}{\mu_{xx}^k}}_{m_{H_x}^k} M_x^k\big|_t$$



# Implementation

$$\left.\begin{aligned} m_{E_y}^k &\equiv \frac{c_0 \Delta t}{\varepsilon_{yy}^k} \\ m_{H_x}^k &\equiv \frac{c_0 \Delta t}{\mu_{xx}^k} \end{aligned}\right\}$$

1D arrays since $\varepsilon_{yy}^k$ and $\mu_{xx}^k$ are the functions of $z = k \cdot \Delta z$

precomputed before the simulation

Workflow:

calculate update coefficients $m_{H_x}^k$ and $m_{E_y}^k$

↓

Initialize arrays for electric and magnetic fields: $E_y^k = 0 \quad H_x^k = 0$

↓ done?

Update $H$ from $E$

↓

Update $E$ from $H$

↓

loop over time

# Numerical boundary conditions.

## Dirichlet BC.

Faraday's law

$$H_x^k\Big|_{t+\frac{\Delta t}{2}} = H_x^k\Big|_{t-\frac{\Delta t}{2}} + \frac{1}{\Delta z}\cdot m_{H_x}^k \left(E_y^{k+1}\Big|_t - E_y^k\Big|_t\right) - m_{H_x}^k \mathcal{M}_x^k\Big|_t$$

what do we do at $k = N_z$ : $E_y^{N_z+1}$ does not exist?

Ampere's law:

$$E_y^k\Big|_{t+\Delta t} = E_y^k\Big|_t + \frac{1}{\Delta z}\cdot m_{E_y}^k \left(H_x^k\Big|_{t+\frac{\Delta t}{2}} - H_x^{k-1}\Big|_{t+\frac{\Delta t}{2}}\right) - m_{E_y}^k \eta_0 J_y^k\Big|_{t+\frac{\Delta t}{2}}$$

what do we do at $k = 0$ : $H_x^{-1}$ does not exist?

Force the values outside the grid to be zero

$k=\overline{0, N_z-1}$ $$H_x^k\Big|_{t+\frac{\Delta t}{2}} = H_x^k\Big|_{t-\frac{\Delta t}{2}} + \frac{1}{\Delta z}\cdot m_{H_x}^k \left(E_y^{k+1}\Big|_t - E_y^k\Big|_t\right) - m_{H_x}^k \mathcal{M}_x^k\Big|_t$$

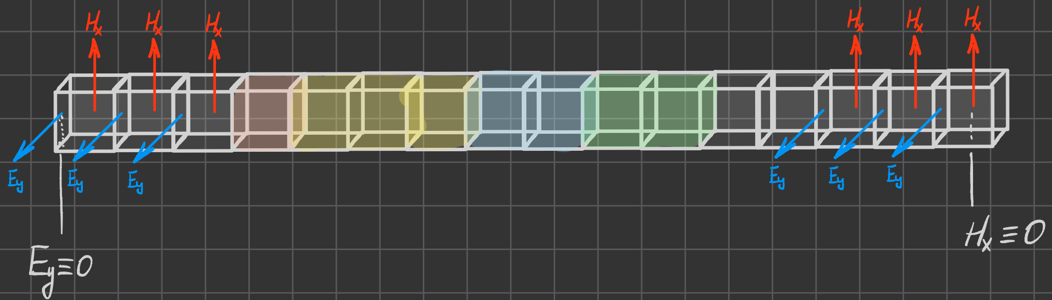
$k = N_z$ right boundary $$H_x^k\Big|_{t+\frac{\Delta t}{2}} = H_x^k\Big|_{t-\frac{\Delta t}{2}} + \frac{1}{\Delta z}\cdot m_{H_x}^k \left(0 - E_y^k\Big|_t\right) - m_{H_x}^k \mathcal{M}_x^k\Big|_t$$

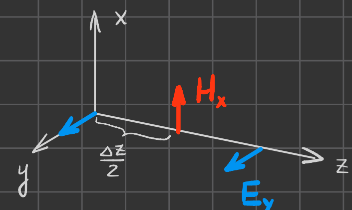
PEC

$k = \overline{1, N_z}$ $$E_y^k\Big|_{t+\Delta t} = E_y^k\Big|_t + \frac{1}{\Delta z}\cdot m_{E_y}^k \left(H_x^k\Big|_{t+\frac{\Delta t}{2}} - H_x^{k-1}\Big|_{t+\frac{\Delta t}{2}}\right) - m_{E_y}^k \eta_0 J_y^k\Big|_{t+\frac{\Delta t}{2}}$$

$k = 0$ left boundary $$E_y^k\Big|_{t+\Delta t} = E_y^k\Big|_t + \frac{1}{\Delta z}\cdot m_{E_y}^k \left(H_x^k\Big|_{t+\frac{\Delta t}{2}} - 0\right) - m_{E_y}^k \eta_0 J_y^k\Big|_{t+\frac{\Delta t}{2}}$$

PMC



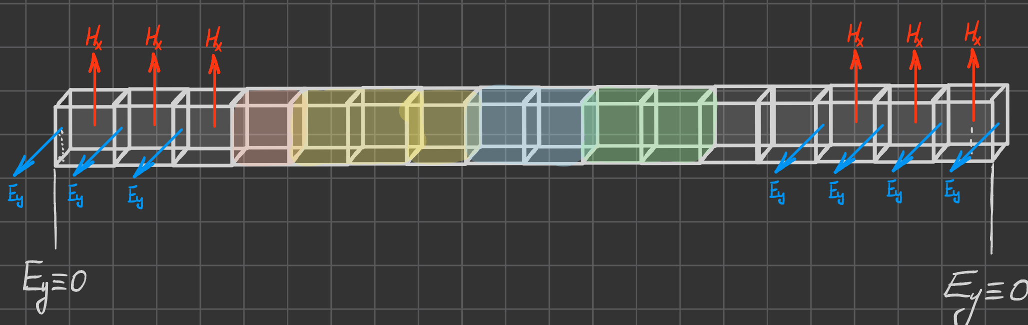

Sizes of arrays for $E_y$ and $H_x$ are the same:

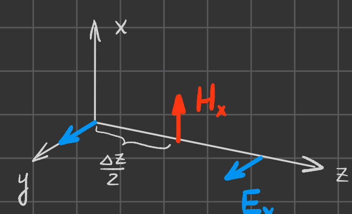
$$\boxed{\begin{aligned} E_y &= \text{array}(N_z) \\ H_x &= \text{array}(N_z) \end{aligned}}$$

## PEC boundary condition

→ values of $E_y$ are set to zero <u>at</u> the boundaries

→ we have an extra value of electric field at the right boundary




$$\boxed{\begin{aligned} E_y &= \text{array}(N_z + 1) \\ H_x &= \text{array}(N_z) \end{aligned}}$$

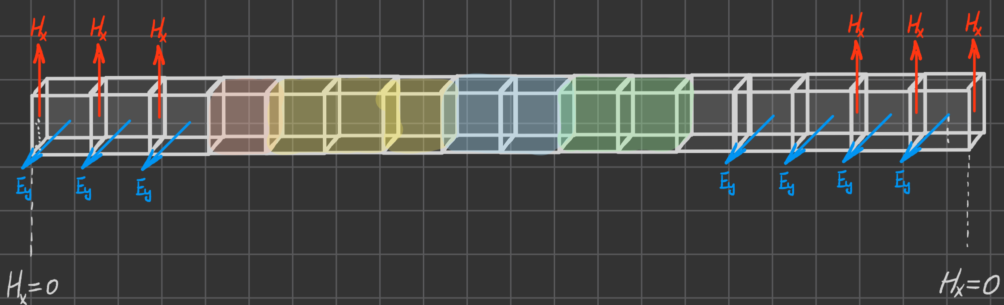
$$k = 1,\dots,N_z-1 \qquad H_x^k\Big|_{t+\frac{\Delta t}{2}} = H_x^k\Big|_{t-\frac{\Delta t}{2}} + \frac{1}{\Delta z}\cdot m_{H_x}^k \left(E_y^{k+1}\Big|_t - E_y^k\Big|_t\right) - m_{H_x}^k \mathcal{M}_x^k\Big|_t$$

$$k = 0,\dots N_z-1 \qquad E_y^k{}_{t+\Delta t} = E_y^k{}_t + \frac{1}{\Delta z}\cdot m_{E_y}^k \left(H_x^k{}_{t+\frac{\Delta t}{2}} - H_x^{k-1}{}_{t+\frac{\Delta t}{2}}\right) - m_{E_y}^k \eta_0 J_y^k{}_{t+\frac{\Delta t}{2}}$$

$$k = 0 \qquad E_y^0\Big|_{\forall t} \equiv 0$$

$$k = N_z \qquad E_y^{N_z}\Big|_{\forall t} \equiv 0$$

## PMC boundary condition





$$\boxed{\begin{aligned} H_x &= \text{array}(N_z+1) \\ E_y &= \text{array}(N_z) \end{aligned}}$$

$$k = 0,\dots N_z-1 \qquad H_x^k\Big|_{t+\frac{\Delta t}{2}} = H_x^k\Big|_{t-\frac{\Delta t}{2}} + \frac{1}{\Delta z}\cdot m_{H_x}^k \left(E_y^{k+1}\Big|_t - E_y^k\Big|_t\right) - m_{H_x}^k \mathcal{M}_x^k\Big|_t$$

$$k = 1,\dots,N_z-1 \qquad E_y^k{}_{t+\Delta t} = E_y^k{}_t + \frac{1}{\Delta z}\cdot m_{E_y}^k \left(H_x^k{}_{t+\frac{\Delta t}{2}} - H_x^{k-1}{}_{t+\frac{\Delta t}{2}}\right) - m_{E_y}^k \eta_0 J_y^k{}_{t+\frac{\Delta t}{2}}$$

$$k = 0 \qquad H_x^0\Big|_{\forall t} \equiv 0$$

$$k = N_z \qquad H_x^{N_z}\Big|_{\forall t} \equiv 0$$

# Gaussian pulse

$$f(x) = e^{-x^2}$$ – Gaussian function (base form)

$$\int_{-\infty}^{+\infty} \underbrace{e^{-x^2}}_{f(x)} dx = \sqrt{\pi}$$

$$f(x) = a \cdot \exp\left(-\frac{(x-b)^2}{2c^2}\right)$$ – with parametric extension

$$\int_{-\infty}^{+\infty} a \cdot e^{-\frac{(x-b)^2}{2c^2}} dx = a \cdot c \cdot \sqrt{2\pi} = 1$$

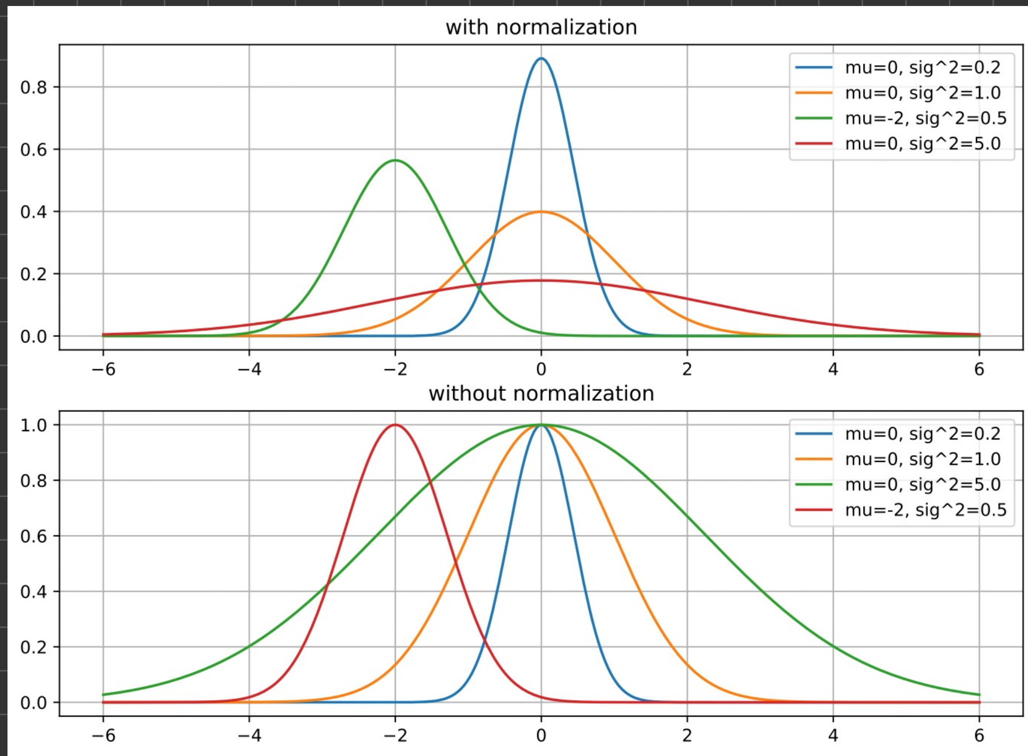
$$a \equiv \frac{1}{c\sqrt{2\pi}}$$

$$a = \frac{1}{\sigma \cdot \sqrt{2\pi}}$$

$b = \mu$ – expected value

$c = \sigma$ – variance

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$




$$g(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

In the code I used $$\boxed{g(t) = e^{-\left(\frac{t-t_0}{\tau}\right)^2}}$$

# Implementation of the Gaussian source

## Frequency content of Gaussian pulse.

Fourier transform of a Gaussian is a Gaussian:

$$g(t) = e^{-\frac{t^2}{\tau^2}} \longleftrightarrow G(f) = \frac{1}{\sqrt{\pi}B} e^{-\frac{f^2}{B^2}}$$

The frequency content of the Gaussian pulse extends from DC ( 0 Hz) up to around B

$$\boxed{B = \frac{1}{\pi \tau}}$$

✓ → the narrower the pulse ($\tau$ smaller) the more frequencies are captured => we choose $\tau$ according to the frequencies we want to have in the simulation

⊖ → The higher the frequency the shorter the time step $\Delta t$ will be.

# Designing the pulse source.

Step 1: detemine max freq. you're interested in simulating.

$$\boxed{f_{max}}$$

Step 2: compute pulse width considering upper frequency limit

$$B \equiv f_{max} = \frac{1}{\pi \tau} \longrightarrow \tau \leq \frac{1}{\pi f_{max}} ;$$

$$\boxed{\tau \cong \frac{0.5}{f_{max}}}$$

Step 3: check the time step. May need to reduce it. Gaussian pulse should be resolved by at least 10 - 20 <u>time</u> steps.

$$\Delta t = \frac{\tau}{N_t}$$

$$N_t \geqslant 10$$

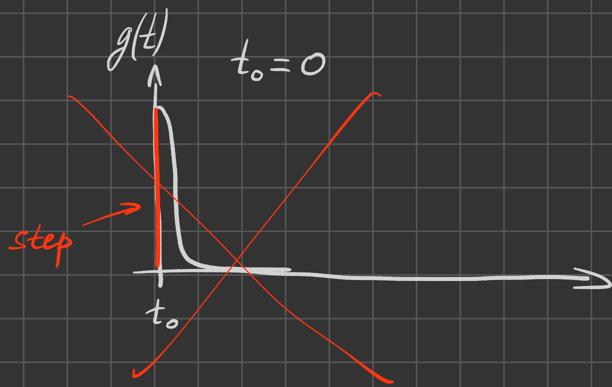
Everything is automatically satisfied if

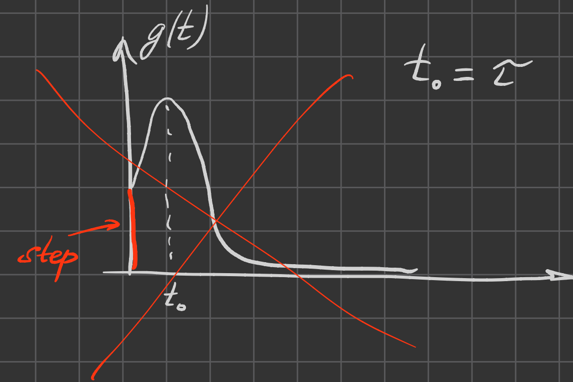
$$\boxed{\Delta t = \frac{n \Delta z}{2 c_0}} \text{ ABC}$$

Typically you determine a first $\Delta t_1$ based on CFL stability condition. Then determine $\Delta t_2$ based on resolving the maximum frequency $f_{max}$. Then choose the smallest value.

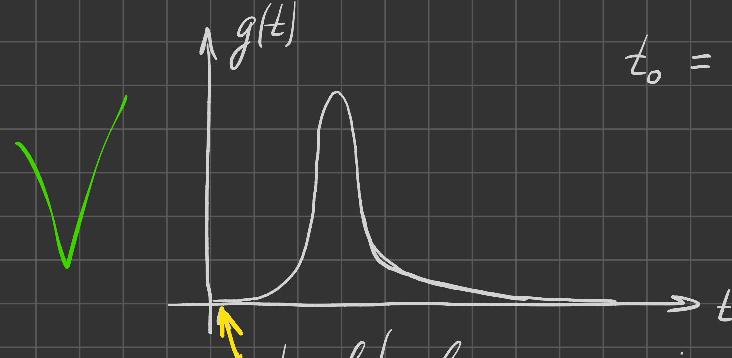
$$\Delta t \equiv \min(\Delta t_1, \Delta t_2)$$

Step 4: compute time delay $t_0$




$t_0 = 6\tau$




should be numerical zero here

The source must start at zero and gradually increase. NO STEP FUNCTIONS.

$$\boxed{t_0 \geq 6\tau}$$

why $6\tau$? what's the value $g(0)$ when $t_0 = 6\tau$?

$$g(0)\Big|_{t_0=6\tau} = e^{-\left(\frac{0-6\tau}{\tau}\right)^2} = e^{-36} \sim 2^{36} \sim 10^{-12}$$

Types of sources (see my notes on sources)

hard (voltage) soft (current)

- overwrite
- source cell is felt by the fields as either PEC or PMC

- add to
- source cell is transparent to scattered waves passing through it
- injects energy into both directions
- great for testing boundary conditions

$$\boxed{\begin{aligned} H_x^k\Big|_{t+\frac{\Delta t}{2}} &= g_H\Big|_t \\ E_y^k\Big|_{t+\Delta t} &= g_E\Big|_t \end{aligned}}$$

$$\boxed{\begin{aligned} H_x^k\Big|_{t+\frac{\Delta t}{2}} &\mathrel{+}= g_H\Big|_t \\ E_y^k\Big|_{t+\Delta t} &\mathrel{+}= g_E\Big|_t \end{aligned}}$$

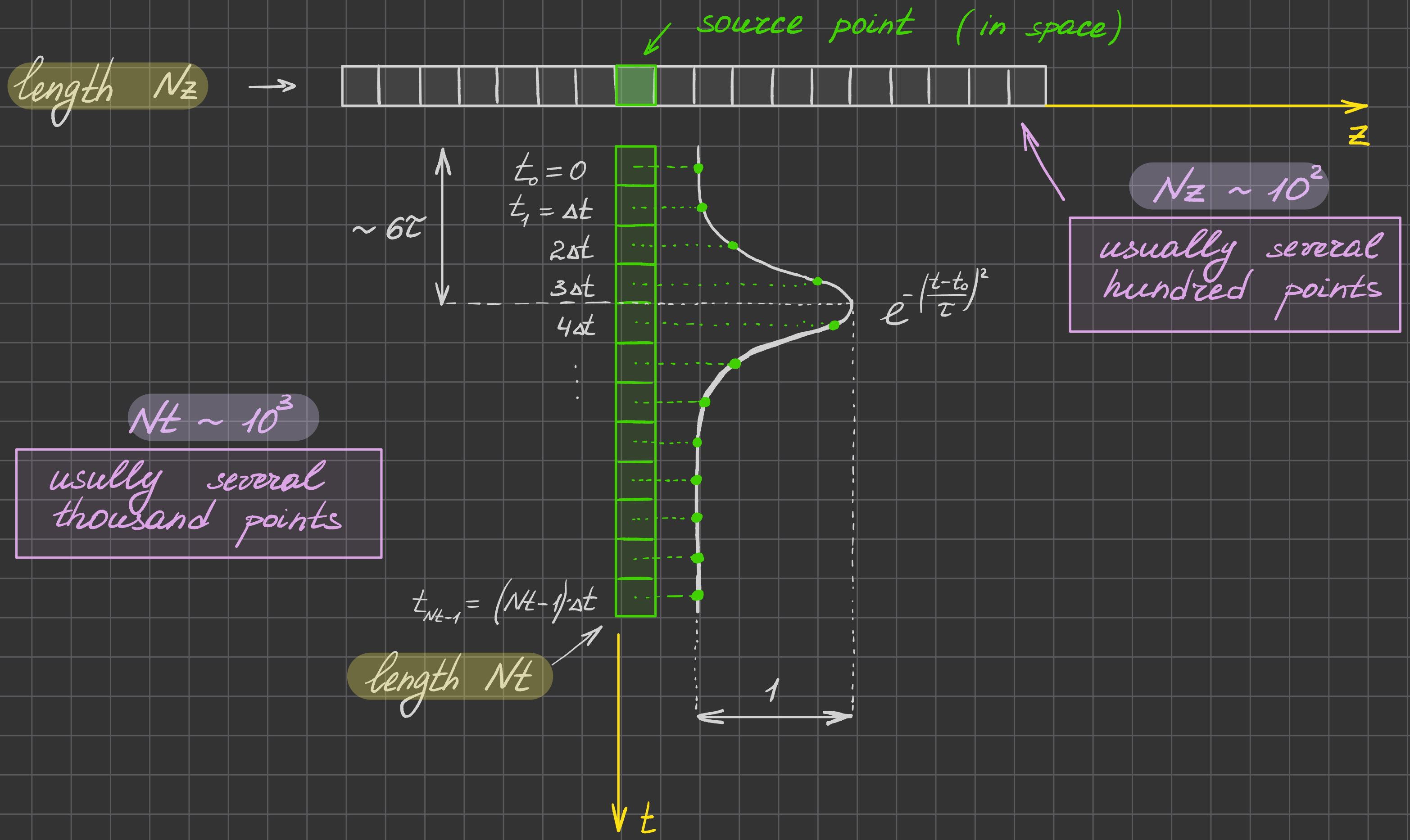
- not usually a practical source

- rarely used. Use until learn SF/TF source.

* see Kuniaki Yoshitomi's paper on this account.

## Visualizing the arrays

$E_y$, $H_x$, ($ER$, $UR$), $mEy$, $mHx$ are stored in the arrays of length $Nz$.




$g_E$ and $g_H$ are stored in the arrays of length $Nt$

$\Delta t$

# Time resolution $\Delta t$

## Courant - Friderich - Lewy stability condition

Over the duration of one time step $\Delta t$ EM disturbance will travel

- numerical distance covered in $\Delta t$ : $\Delta z$
- physical dist. covered in $\Delta t$ : $c_0 \cdot \frac{\Delta t}{n}$

In numerical algorithm it's impossible for a disturbance to travel farther than one cell $\Delta z$ in a single time step $\Delta t$.
For this reason we need to make sure that the physical wave would not propagate farther than a unit cell $\Delta z$ in one time step. $\Delta t$.

$$\underbrace{\frac{c_0}{n} \cdot \Delta t}_{\text{physical}} < \underbrace{\Delta z}_{\text{numerical}}$$

$\Rightarrow$ upper limit on time step $\Delta t$

$$\boxed{\Delta t < \frac{\Delta z}{(c_0/n)}}$$

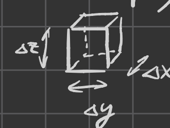
$n$ - smallest refractive idx ; $n \geq 1$ $\Rightarrow$ usually $n$ is set to 1 and dropped from the ineq.

For 2D and 3D cases:

$$\boxed{\Delta t < \frac{1}{c_0\sqrt{\frac{1}{(\Delta x)^2}+\frac{1}{(\Delta y)^2}+\frac{1}{(\Delta z)^2}}}}$$

CFL stability condition

This condition will be most restrictive along the shortest dimension of the unit cell




$$\boxed{\Delta_{min} \equiv \min\{\Delta x, \Delta y, \Delta z\}}$$

$$\Delta t < \frac{\Delta_{min}}{2c_0}$$

$\frac{1}{2}$ – safety margin

More general case:

$$\boxed{\Delta t < \frac{1}{2}\frac{n_{min}\Delta_{min}}{c_0}}$$

- grid filled with dielectric $\Rightarrow$ wave travels slower everywhere
- model includes materials with $n < 1$

For perfect boundary condition

$$\boxed{\Delta t = \frac{n_{bc}\cdot\Delta z}{2c_0}}$$

$n_{bc}$ – refr. idx at the boundaries; materials should be the same at both boundaries

Nt

# Total number of iterations Nt

Depends mainly on : → device we model

→ properties of device we calculate

**device**

- highly resonant devices typically require more iterations
- purely scattering devices require very few iterations
- more iterations needed the more waves bounce around in the grid

**information**

- calculating spectral shapes requires the most iterations: reflectance, transmittance, line shapes ...
- calculating positions of resonances requires fewer iterations

This might be very difficult in frequency domain.

## Rule of thumb

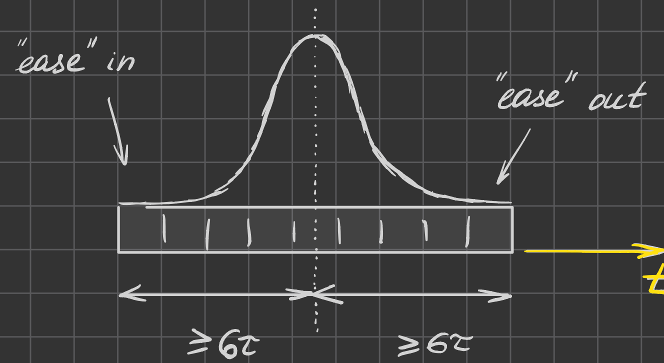
1. How long it takes a slowest wave to propagate across the grid?

$$\boxed{t_{prop} = \frac{n_{max} \overbrace{N_z \cdot \Delta z}^{\text{physical size of the grid}}}{c_0}}$$

2. Simulation time T must include the entire pulse

of duration $\tau$

$$\boxed{T \geq 6\tau + 6\tau}$$




3. Simulation time should allow for ~ 5 bounces

$$\boxed{T \gtrsim 5t_{prop}}$$

4. Rule of thumb for total simulation time

$$\boxed{T \gtrsim 12\tau + 5t_{prop}}\,^{*}$$

* for highly resonant devices this time is NOT enough

5. Given time step $\Delta t$, the total # of iterations is

$$\boxed{Nt \equiv \text{round}\!\uparrow \left[\frac{T}{\Delta t}\right] \quad (\text{ceil})}$$

this is in context of for (1, ..., Nt) loop.

Another approach is to set a while (...) loop and keep running until the power is stuck in the grid and stop iterations when the power in the domain drops below some threshold.

# Grid resolution $\Delta z$

$\Delta z$

## 1. Wavelength

- $\Delta z$ should be sufficient to resolve the shortest wavelengh

→ determine $\boxed{\lambda_{min} = \dfrac{c_0}{f_{max} \cdot n_{max}}}$

$n_{max}$ – largest refractive idx found in the grid

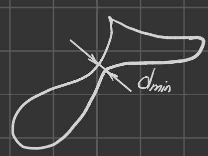
→ resolve wave with at least $N_\lambda = 10$ cells

$$\boxed{\Delta_\lambda \approx \frac{\lambda_{min}}{N_\lambda}}, \quad N_\lambda \geq 10$$

| $N_\lambda$ | material |
|---|---|
| 10 – 20 | low contrast dielectric |
| 20 – 30 | high contrast diel. |
| 40 – 60 | most metallic structures |
| 100 – 200 | plasmonic devices |

## 2. Mechanical Features

- $\Delta z$ should be sufficient to resolve the smallest mechanical feature $d_{min}$




→ want to resolve this mechanical feat. $d_{min}$ with 1 to 4 cells $\Delta_d$

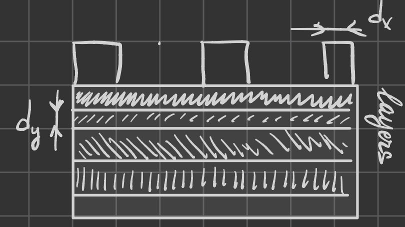
$$\boxed{\Delta_d \approx \frac{d_{min}}{N_d}}, \quad N_d \geq 1$$

3. Adjust init. resolution $\Delta Z$

$$\boxed{\Delta X = \Delta y = \Delta Z = \min\left[\Delta_\lambda , \Delta_d\right]}$$

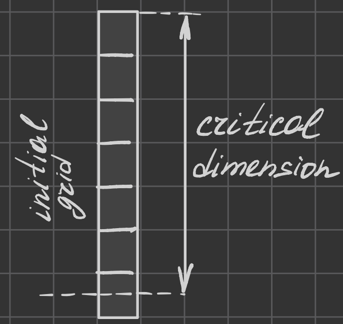

4. "Snap" grid to crytical dimensions

→ decide what dimensions along each axis are critical : $d_x$ and $d_y$



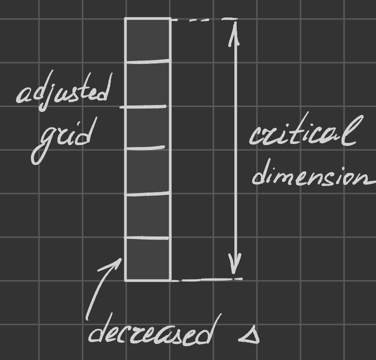

→ compute how many grid cells comprise $d_x / d_y / d_z$ and round up

$$\boxed{\begin{aligned} \mathcal{N}_x &\equiv \operatorname{ceil}\left(\frac{d_x}{\Delta_x}\right) \\ \mathcal{N}_y &\equiv \operatorname{ceil}\left(\frac{d_y}{\Delta y}\right) \end{aligned}}$$



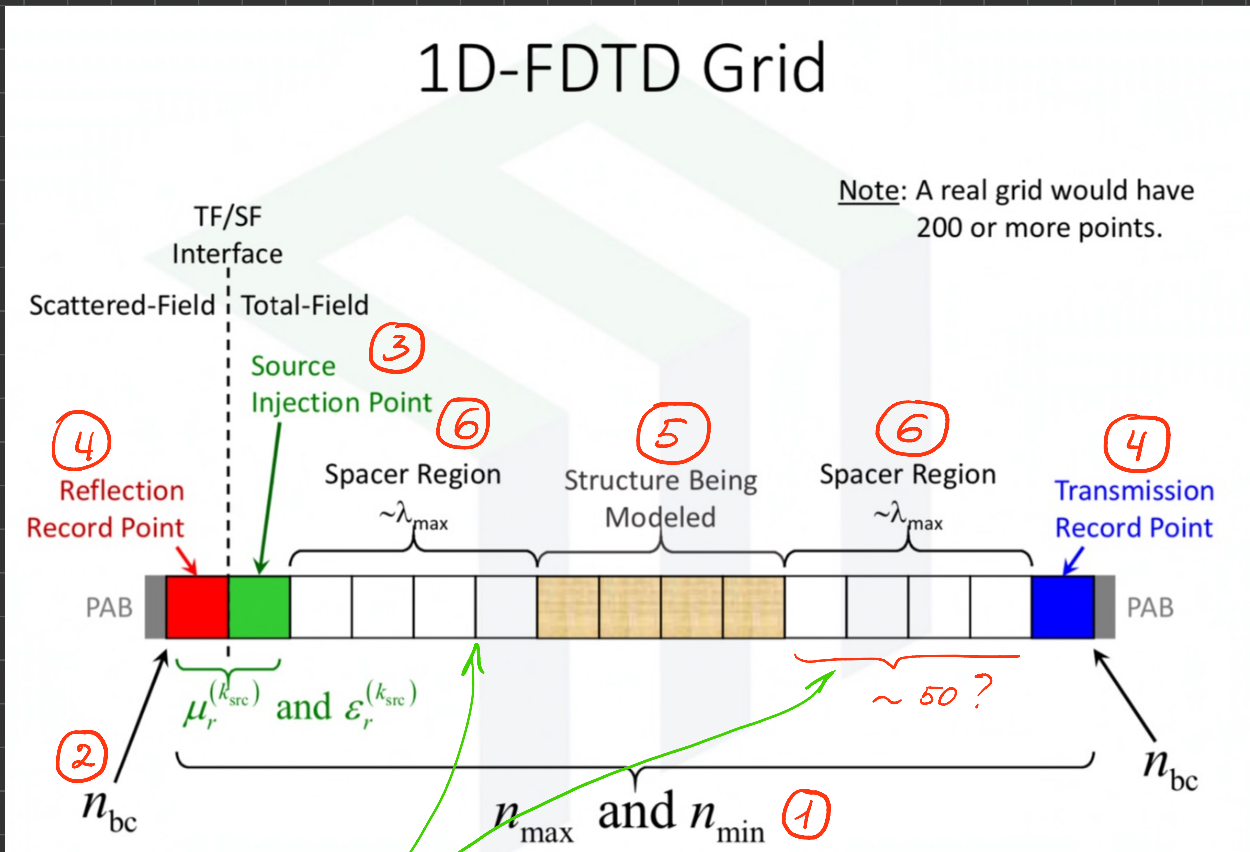

→ adjust grid resolution to fit this dimension $d_x / d_y$ in grid exactly

$$\boxed{\Delta_x \equiv \frac{d_x}{\mathcal{N}_x} \qquad \Delta_y \equiv \frac{d_y}{\mathcal{N}_y}}$$



Nz

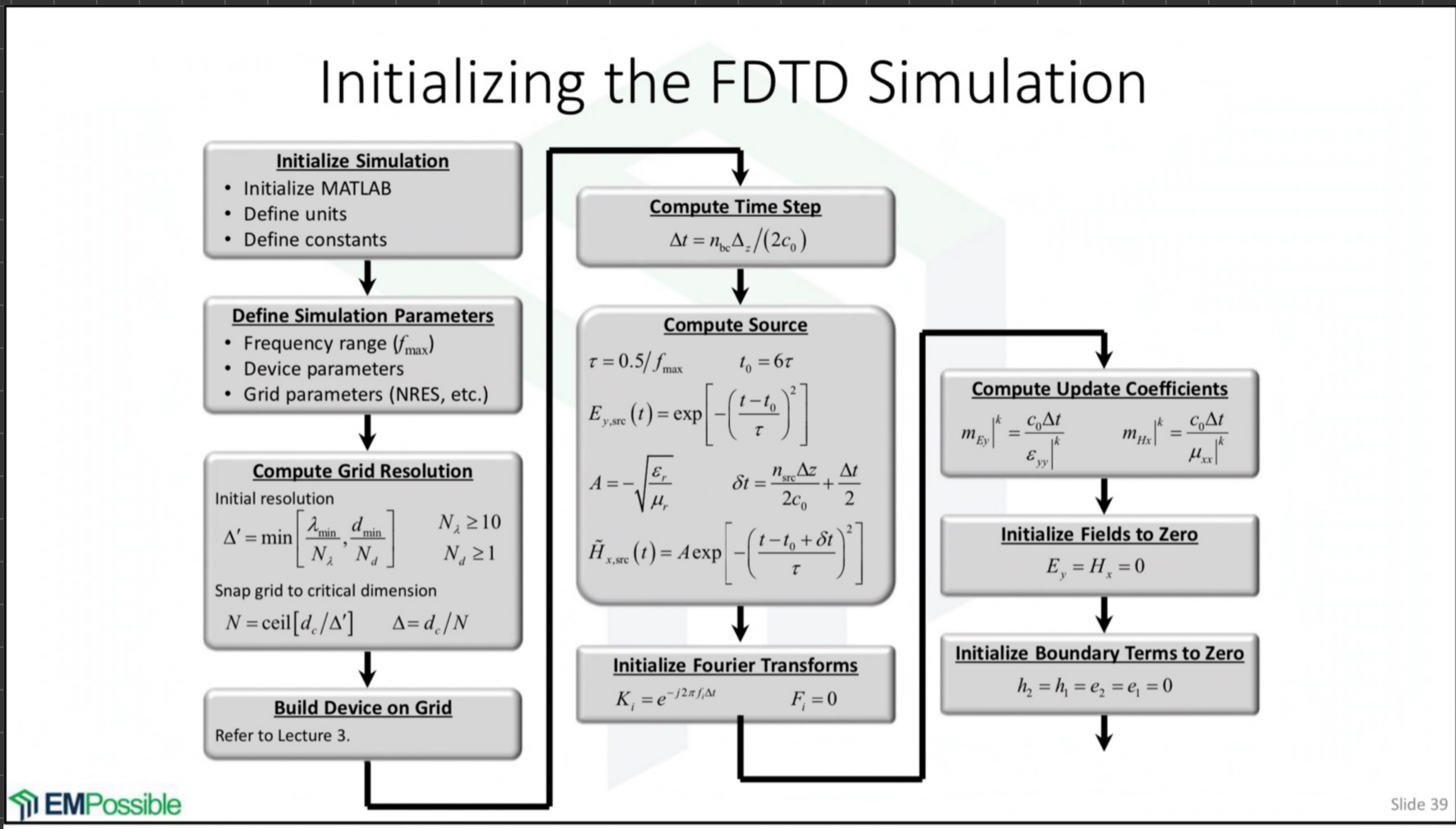
# Total number of space cells Nz



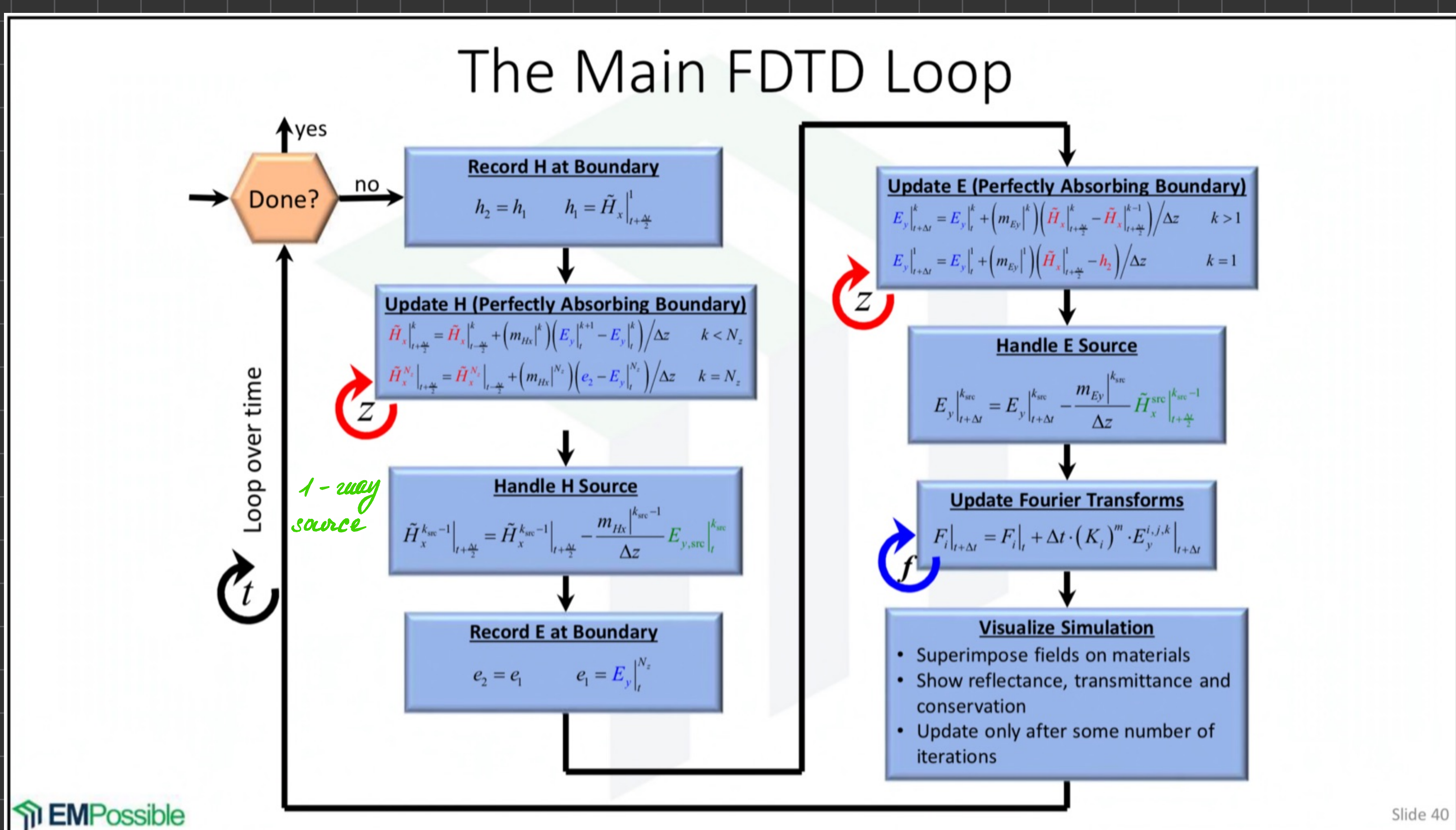

In 2D and 3D for resonant devices we will have evanescent fields around device which should not reach PML, otherwise we will have reflections.

$\Rightarrow$ need spacer regions.

## Sequence of initialization for simulation

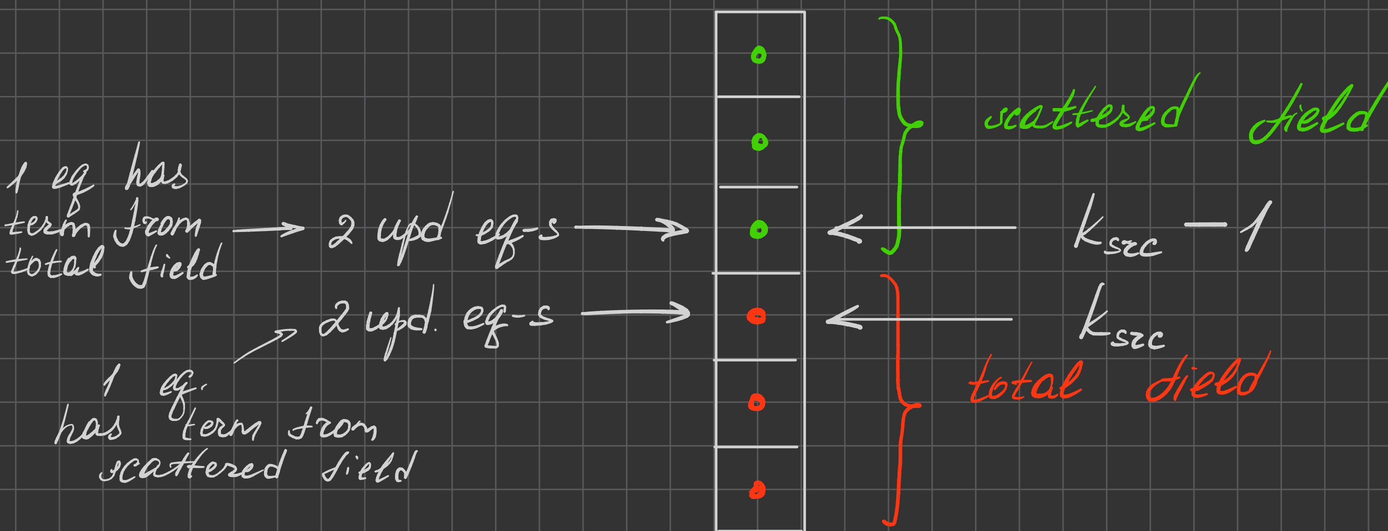



## Main loop



# Total field / Scattered field source

## 1D grid




## 1. On the scattered field side

Update equation for $H_x$ :

Faraday's law

$$H_x\Big|_{t+\frac{\Delta t}{2}}^{k_{src}-1} = H_x\Big|_{t-\frac{\Delta t}{2}}^{k_{src}-1} + \underbrace{\frac{1}{\Delta z}\cdot\Delta t\cdot\frac{c_0}{\mu_{xx}^k}}_{\equiv\, m_{H_x}^k}\cdot\left(E_y\Big|_t^{k_{src}} - E_y\Big|_t^{k_{src}-1}\right) - \underbrace{\Delta t\cdot\frac{c_0}{\mu_{xx}^k}}_{m_{H_x}^k}\mathcal{M}_x\Big|_t^{k_{src}-1}$$

on the total field side

$$E_y^{k_{src}}\Big|_t = \text{total} \doteq \text{source} + \text{scattered}$$

$\Longrightarrow$ we need to subtract source for it to be purely scattered on the scattered field side.

## Update equation with correction

$$H_x\Big|_{t+\frac{\Delta t}{2}}^{k_{src}-1} = H_x\Big|_{t-\frac{\Delta t}{2}}^{k_{src}-1} + \frac{1}{\Delta z}\cdot\underbrace{\Delta t\cdot\frac{c_0}{\mu_{xx}^k}}_{\equiv\, m_{H_x}^{k}}\cdot\left[\underbrace{\left(E_y\Big|_t^{k_{src}} - E_y^{src}\Big|_t^{k_{src}}\right)}_{E_y^{scat}\big|_t^{k_{src}}} - E_y\Big|_t^{k_{src}-1}\right] - \underbrace{\Delta t\cdot\frac{c_0}{\mu_{xx}^k}}_{m_{H_x}^k}\mathcal{M}_x\Big|_t^{k_{src}-1}$$

Let's bring this source term $E_y^{src}$ outside the brackets:

$$\underbrace{H_x\Big|_{t+\frac{\Delta t}{2}}^{k_{src}-1} = H_x\Big|_{t-\frac{\Delta t}{2}}^{k_{src}-1} + \frac{1}{\Delta z}\cdot m_{H_x}^{k-1}\cdot\left[\left(E_y\Big|_t^{k_{src}} - E_y\Big|_t^{k_{src}-1}\right] - m_{H_x}^{k-1}\cdot\mathcal{M}_x\Big|_t^{k_{src}-1}}_{\text{original equation}} \underbrace{- m_{H_x}^{k_{src}-1}\cdot\frac{E_y^{src}\big|_t^{k_{src}}}{\Delta z}}_{\text{correction}}$$

can be implemented after

## 2. On total field side : $k = k_{src}$

$$E_y\Big|_{t+\Delta t}^{k_{src}} = E_y\Big|_t^{k_{src}} + \frac{1}{\Delta z}\cdot m_{E_y}^{k_{src}}\left(H_x\Big|_{t+\frac{\Delta t}{2}}^{k_{src}} - \underbrace{H_x\Big|_{t+\frac{\Delta t}{2}}^{k_{src}-1}}_{\text{this term is from scattered field side}}\right) - m_{E_y}^{k}\,\eta_0\,J_y\Big|_{t+\frac{\Delta t}{2}}^{k}$$

$$H_x^{scat}\Big|_{t+\frac{\Delta t}{2}}^{k_{src}-1} = \text{total} - \text{source}$$

$$\Rightarrow \quad \text{total} = H_x^{scat}\Big|_{t+\frac{\Delta t}{2}}^{k_{src}-1} + H_x^{src}\Big|_{t+\frac{\Delta t}{2}}^{k_{src}-1}$$

Correct Ampere's law accordingly:

$$E_y{}^{k_{src}}_{t+\Delta t} = E_y{}^{k_{src}}_{t} + \frac{1}{\Delta z}\cdot m_{E_y}^{k_{src}}\left[H_x{}^{k_{src}}_{t+\frac{\Delta t}{2}} - \left(H_x{}^{k_{src}-1}_{t+\frac{\Delta t}{2}} + \underbrace{H_x^{src}\Big|_{t+\frac{\Delta t}{2}}^{k_{src}-1}}_{\text{take it out of the brackets:}}\right)\right] - m_{E_y}^{k}\,\eta_0\,J_y{}^{k}_{t+\frac{\Delta t}{2}}$$

$$E_y{}^{k_{src}}_{t+\Delta t} = E_y{}^{k_{src}}_{t} + \frac{1}{\Delta z}\cdot m_{E_y}^{k_{src}}\left[H_x{}^{k_{src}}_{t+\frac{\Delta t}{2}} - H_x{}^{k_{src}-1}_{t+\frac{\Delta t}{2}}\right] - m_{E_y}^{k}\,\eta_0\,J_y{}^{k}_{t+\frac{\Delta t}{2}} \underbrace{- \frac{1}{\Delta z}\, m_{E_y}^{k_{src}} H_x^{src}\Big|_{t+\frac{\Delta t}{2}}^{k_{src}-1}}_{\text{will be added after the the regular upd. equation}}$$

3. We need to calculate 2 source functions before main (time) FDTD loop.

normalized $\longrightarrow$ $\boxed{H_x^{src}\Big|_{t+\frac{\Delta t}{2}}^{k_{src}-1}}$ – for scattered field side

$\boxed{E_y^{src}\Big|_{t}^{k_{src}}}$ – for total field side

! exist
- at different time moments
- at different locations
- $H^{src}$ should be normalized before adding.

$\tilde{H} \equiv \eta_0 H$

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For $E_y / H_x$ mode:

$$E_y^{src}\Big|_t^{k_{src}} = g(t) \quad - \text{gaussian}$$

Then normalized magn. field:

$$\boxed{\tilde{H}_x^{src}\Big|_{t+\frac{\Delta t}{2}}^{k_{src}-1} = (-)\sqrt{\frac{\varepsilon_r^{k_{src}}}{\mu_r^{k_{src}}}} \cdot g\left(t + \frac{n_{src}\cdot\Delta z}{2c_0} + \frac{\Delta t}{2}\right)}$$

ahead of $E_y$ in time

because of differen phase

amplitude to satisfy Maxwell's eq-s

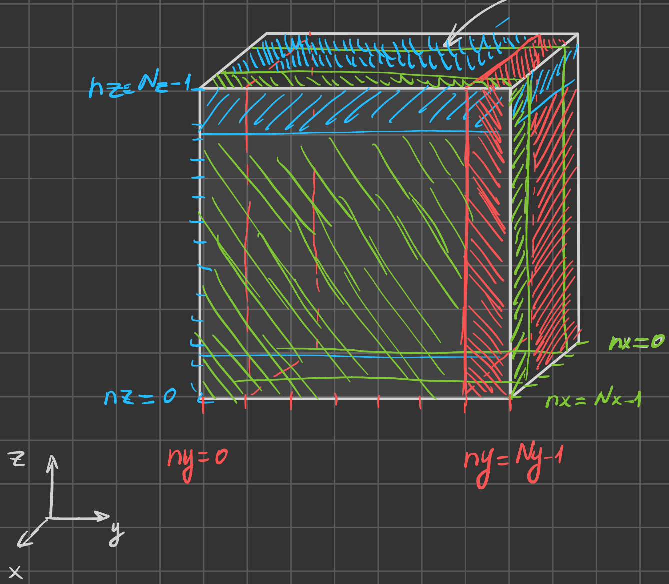
delay though half of a grid cell

half time step differen

$\mu_r^{k_{src}}$, $\varepsilon_r^{k_{src}}$, $n_{src}$ – where the source is injected.

this implementation doesn't work =(

running inside this region




$nz = Nz - 1$

$ny = Ny - 1$

$nx = Nx$ $ny = Ny$

cover

$ny = Ny$ $nz = Nz$

$nz = Nz$ $nx = Nx$

then

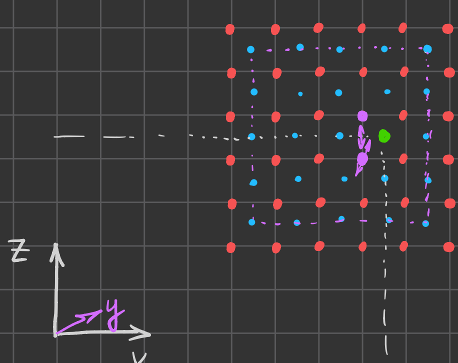
nx ny

nx nz

ny nz

E field H-field

1 upd. magnetic field




$k = 2$
$i = 3$
$j = 1$

$$\frac{c_0 \Delta t}{\mu_{xx}^{3,1,2}} \left( \frac{E_z^{3,1+1,2} - E_z^{3,1,2}}{\Delta y} - \frac{E_y^{3,1,2+1} - E_y^{3,1,2}}{\Delta z} \right) =$$

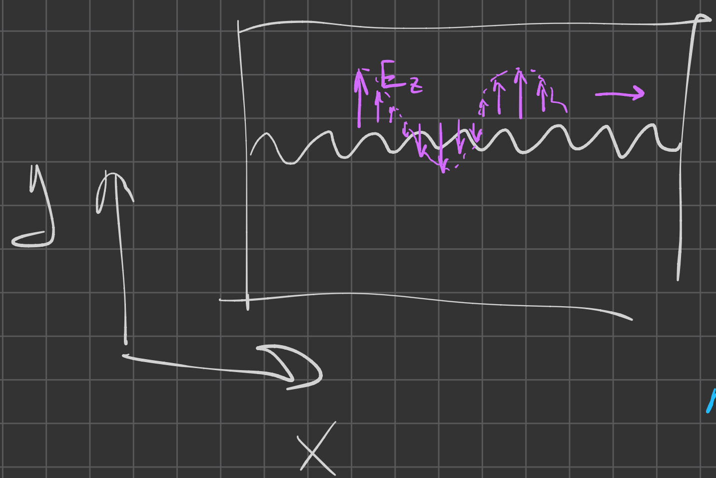
$$= H_x^{3,1,2}\Big|_{t+\frac{\Delta t}{2}} - H_x^{3,1,2}\Big|_{t-\frac{\Delta t}{2}}$$

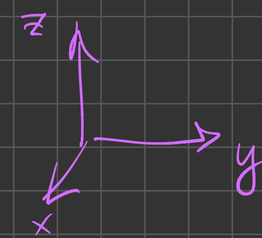
FFT

$E_x(x, y, z)$

$E_y(x, y, z)$

$E_z(x, y, z)$




need: variation of $E_z$ in x direction:

$$E_z[:, \underbrace{y\_src, z\_src}_{\text{at the source point}}]$$

need to store this for every time step:

$$\begin{array}{rl} E_z\_fft = & \left. \begin{bmatrix} \begin{bmatrix} E_z(0) \\ E_z(2) \\ E_z(3) \\ \vdots \end{bmatrix} & \begin{bmatrix} E_z(0) \\ E_z(1) \\ \vdots \end{bmatrix} & \begin{bmatrix} \\ \vdots \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \vdots \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \\ \\ \end{bmatrix} \end{bmatrix} \right\} N_x \\ & \underbrace{n=0 \quad n=1 \quad n=2 \quad n=3 \quad \cdots \quad n=N_t}_{\text{in time} \qquad \text{size } N_t} \end{array}$$

$$FFT(g) = \int_{-\infty}^{+\infty} g(t)\, e^{-j2\pi f \Delta t}\, dt$$

$g(t)$ - has $\Delta t$ vectors

$Ez\_fft$ [121, 300]

K [10]

Ez [122 ]

exponential kernel

$$Ez\_fft \mathrel{+}= K^{n \cdot dt} \cdot Ez[:, ys, zs]$$

length of frequency

$E_z$-fft - what is its size?

same spatial dimensions but

now # frequencies

Ez - spatial dim × # of time steps

$\Rightarrow$ $K$ should be # time steps $\times$

$\times$ # of frequencies

$$\underset{N_f \times N_t}{K} = e^{-j \cdot 2\pi \cdot \underbrace{[freq]}_{N_f} \cdot \underbrace{m\Delta t}_{N_t}}$$

sample it for each time step and for each freq.

$$\underset{N_f \times N_x}{E_{zf}} = \underset{N_f \times N_t}{K} \cdot \underset{N_t \times N_x}{E_z}$$

$$N_f \left\{ \underbrace{\begin{array}{lll} f_1 t_1 & f_1 t_2 & \dots \\ f_2 t_1 & & \\ f_3 t_1 & & \\ & & \end{array}}_{N_t} \right.$$

$$\frac{\partial E_z(x,y)}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_y(x,y)}{\partial x} - \frac{\partial H_x(x,y)}{\partial y}\right)$$

$$\frac{1}{\Delta t}\left(E_z^{n+1}\Big|_{i,j} - E_z^{n}\Big|_{i,j}\right) \qquad \frac{H_y^{t+\frac{1}{2}}\Big|_{i+\frac{1}{2},j} - H_y^{t+\frac{1}{2}}\Big|_{i-\frac{1}{2},j}}{\Delta x} \qquad \frac{H_x^{t+\frac{1}{2}}\Big|_{i,j+\frac{1}{2}} - H_x^{t+\frac{1}{2}}\Big|_{i,j-\frac{1}{2}}}{\Delta y}$$

From this we can write an update eq. for $E_z$ :

$$E_z^{n+1} = E_z^{n} + \ldots$$

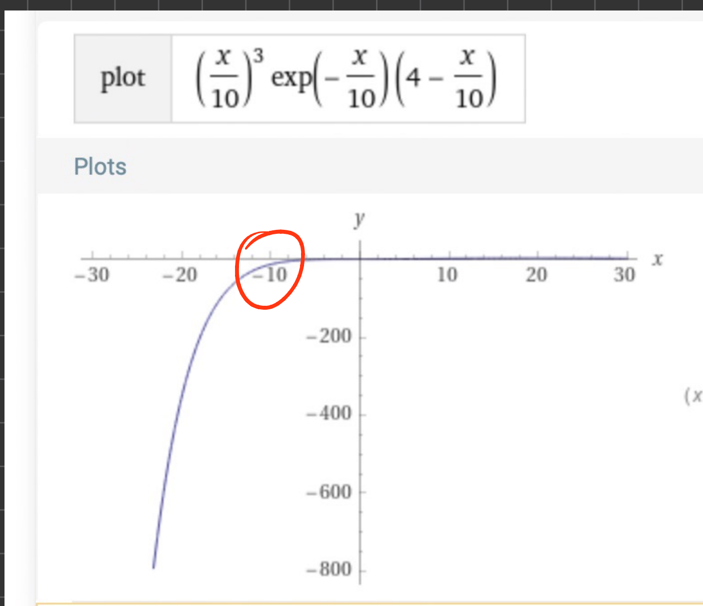
# Avoiding DC offset in current source.

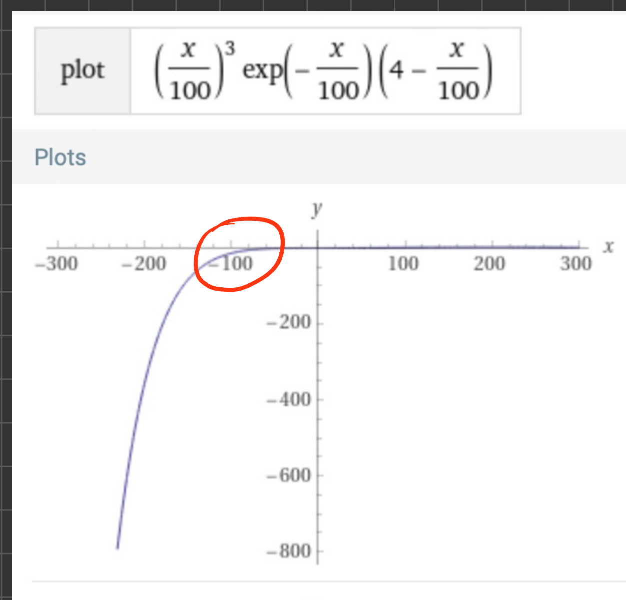
$$I(t) = \tau \frac{d}{dt}\left( \left(\frac{t}{\tau}\right)^4 \cdot e^{-t/\tau} \right) =$$

$$= \cancel{\tau}\left[ 4\left(\frac{t}{\tau}\right)^3 \cdot \cancel{\frac{1}{\tau}} \cdot e^{-t/\tau} + \left(-\cancel{\frac{1}{\tau}}\right) \cdot e^{-t/\tau} \cdot \left(\frac{t}{\tau}\right)^4 \right] =$$

$$= \boxed{\left(\frac{t}{\tau}\right)^3 e^{-t/\tau} \left(4 - \frac{t}{\tau}\right)}$$

$\tau = 10$ $\tau = 100$





Ref: Chen, Moghaddam "Model of subsurf. Interf. Radar"