(lecture 4)

Flow of MEs

update B

$$\vec{\nabla} \times \vec{E}(t) = -\frac{\partial}{\partial t}\vec{B}(t)$$

update H

$$\vec{B}(t) = \bar{\bar{\mu}}(t) * \vec{H}(t)$$

$$\vec{\nabla} \times \vec{H}(t) = \frac{\partial}{\partial t}\vec{D}(t) + \vec{J}$$

update D

$$\vec{D}(t) = \bar{\bar{\varepsilon}}(t) * \vec{E}(t)$$

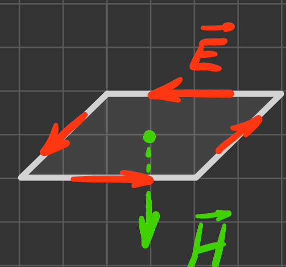
update E

this is a multiplication in frequency domain

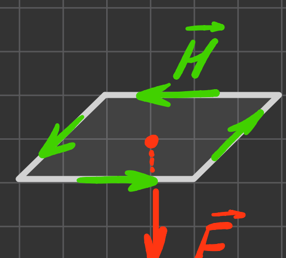
Assume that our material is linear, isotropic, non-dispersive

$$\bar{\bar{\mu}} * \;\rightarrow\; \mu \cdot$$
$$\bar{\bar{\varepsilon}} * \;\rightarrow\; \varepsilon \cdot$$

$$\vec{\nabla} \times \vec{E}(t) = -\mu \cdot \frac{\partial}{\partial t}\vec{H}(t)$$



$$\vec{\nabla} \times \vec{H}(t) = \varepsilon \cdot \frac{\partial}{\partial t}\vec{E}(t) + \vec{J}(t)$$



Approximating time derivative

$$\vec{\nabla}\times\vec{E} = -\mu\frac{\partial}{\partial t}\vec{H} \longrightarrow -\frac{1}{\mu}\vec{\nabla}\times\vec{E}(t) \cong \frac{\vec{H}(t+\frac{\Delta t}{2})-\vec{H}(t-\frac{\Delta t}{2})}{\Delta t}$$

$$\vec{\nabla}\times\vec{H} = \varepsilon\frac{\partial}{\partial t}\vec{E}+\vec{J} \longrightarrow \frac{1}{\varepsilon}\vec{\nabla}\times\vec{H}(t+\frac{\Delta t}{2}) \cong \frac{\vec{E}(t+\Delta t)-\vec{E}(t)}{\Delta t}+\vec{J}(t+\frac{\Delta t}{2})$$

Update equations

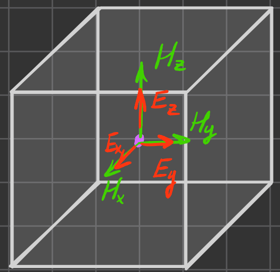
$$\boxed{\vec{H}^{t+\frac{\Delta t}{2}} = \vec{H}^{t-\frac{\Delta t}{2}} - \frac{\Delta t}{\mu}\left[\vec{\nabla}\times\vec{E}\right]^{t}}$$

$$\boxed{\vec{E}^{t+\Delta t} = \vec{E}^{t} + \frac{\Delta t}{\varepsilon}\left[\vec{\nabla}\times\vec{H}\right]^{t+\frac{\Delta t}{2}} - \frac{\Delta t}{\varepsilon}J^{t+\frac{\Delta t}{2}}}$$

$\frac{\Delta t}{\varepsilon}, \frac{\Delta t}{\mu}$ are update coefficents = const – will be calculated ahead of simulation.

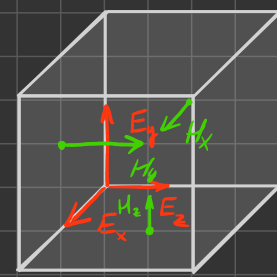
lecture 5

Collocated grids – when $\vec{E}$ and $\vec{H}$ are defined at the same point in a grid cell



Staggered grids (Yee cell)

- Physical BCs are naturally satisfied
- Gauss ($\vec{D}$ and $\vec{B}$) laws are satisfied



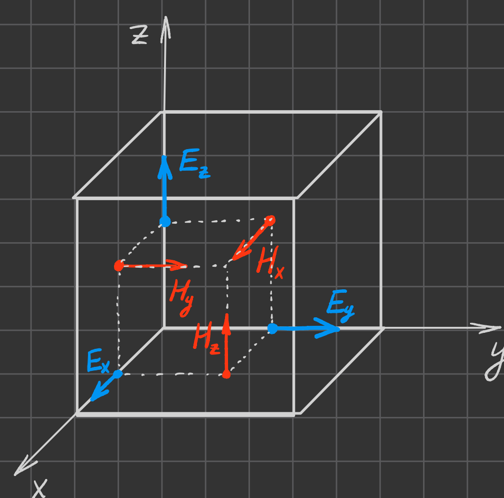
$$\vec{\nabla} \cdot \vec{D} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

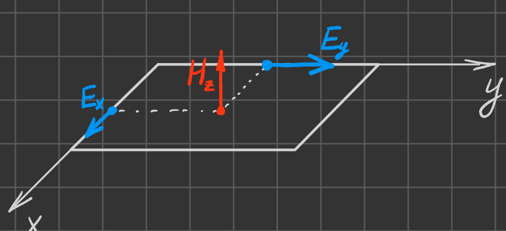
for divergence-free medium

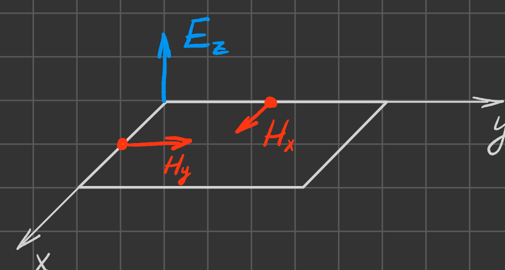
Yee cell for 1D, 2D, and 3D

3D



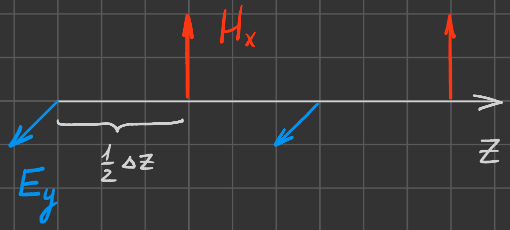
2D

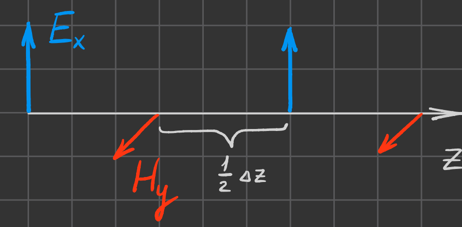




H_z mode $\qquad$ E_z mode

1D



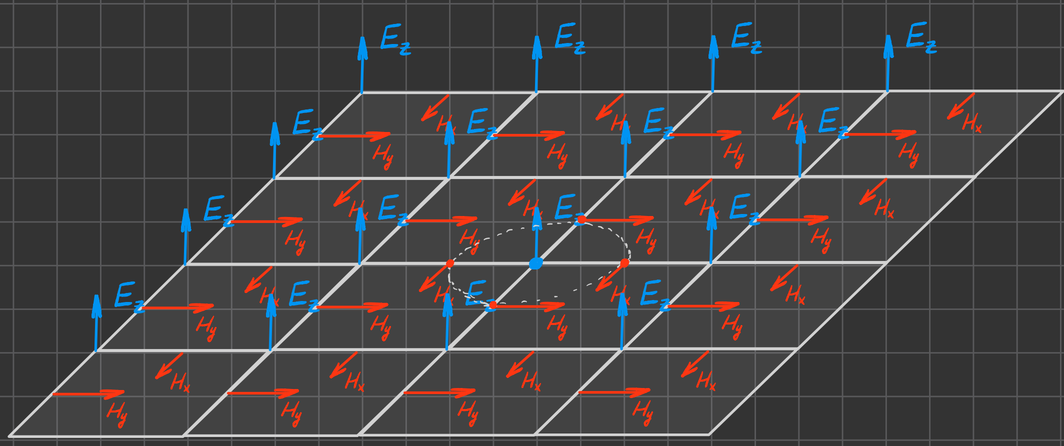


E_y mode $\qquad$ E_x mode

Consequences of Yee grid

- Field components E_x, E_y, E_z may reside in different materials even being in one cell.
- ! Field components will be out of phase.

Example for 2D: 4×4 grid. E_z mode



FD approximation to MEs.

1. We satisfied the divergence equations (Gauss's laws) automatically by adopting Yee's grid.

2. Normalization

$$H = \frac{1}{\eta_0} E \quad , \quad \eta_0 \equiv \sqrt{\frac{\mu_0}{\varepsilon_0}} \simeq 377\,\Omega$$

$$\Longrightarrow \quad E \simeq 377 H$$

i.e. magnitude of H is always 2-3 orders lower than of E. Thus, we will lose 2-3 digits of accuracy in numerical simulations.

Remedy is to normalize magnetic field H:

$$\boxed{\vec{\tilde{H}} \equiv \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\vec{H} = \eta_0 \vec{H}}$$

$\vec{E} = \eta_0 \vec{H} \equiv \vec{\tilde{H}} \Longrightarrow$ $\vec{E}$ and $\vec{\tilde{H}}$ have the same magnitude

Original magn. field $\boxed{\vec{H} \equiv \frac{1}{\eta_0} \vec{\tilde{H}}}$

Substitute it into Faraday's & Ampere's laws:

$$\begin{cases} \vec{\nabla} \times \left(\frac{1}{\eta_0}\tilde{\vec{H}}\right) = \varepsilon_0 \bar{\bar{\varepsilon}}_r \frac{\partial}{\partial t}\vec{E} + \vec{J} \\ \vec{\nabla} \times \vec{E} = -\mu_0 \bar{\bar{\mu}}_r \frac{\partial}{\partial t}\left(\frac{1}{\eta_0}\tilde{\vec{H}}\right) - \vec{M} \end{cases} \qquad \begin{cases} \vec{\nabla} \times \tilde{\vec{H}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \varepsilon_0 \bar{\bar{\varepsilon}}_r \partial_t \vec{E} + \eta_0 \vec{J} \\ \vec{\nabla} \times \vec{E} = -\mu_0 \sqrt{\frac{\varepsilon_0}{\mu_0}} \cdot \bar{\bar{\mu}}_r \partial_t \tilde{\vec{H}} - \vec{M} \end{cases}$$

$$\frac{1}{c_0} = \sqrt{\mu_0 \varepsilon_0} \longrightarrow \qquad \begin{cases} \vec{\nabla} \times \tilde{\vec{H}} = \frac{1}{c_0} \bar{\bar{\varepsilon}}_r \partial_t \vec{E} + \eta_0 \vec{J} \\ \vec{\nabla} \times \vec{E} = -\frac{1}{c_0} \bar{\bar{\mu}}_r \partial \tilde{\vec{H}} - \vec{\mathcal{M}} \end{cases}$$

?

maybe this multiplication will be better included into Gaussian ?

We will omit ~ above $\vec{H}$ keeping in mind that it's normalized and it will be denormalized for quantitative analysis.

$$\boxed{\begin{aligned} \vec{\nabla} \times \vec{E} &= -\frac{1}{c_0} \bar{\bar{\mu}}_r \partial \vec{H} - \vec{\mathcal{M}} \\ \vec{\nabla} \times \vec{H} &= \frac{1}{c_0} \bar{\bar{\varepsilon}}_r \partial_t \vec{E} + \eta_0 \vec{J} \end{aligned}}$$

Component-wise:

x: $$\partial_y E_z - \partial_z E_y = -\frac{1}{c_0}\left(\mu_{xx} \partial_t H_x + \mu_{xy} \partial_t H_y + \mu_{xz} \partial_t H_z\right) - \mathcal{M}_x$$

y: $$\partial_z E_x - \partial_x E_z = -\frac{1}{c_0}\left(\mu_{yx} \partial_t H_x + \mu_{yy} \partial_t H_y + \mu_{yz} \partial_t H_z\right) - \mathcal{M}_y$$

z: $$\partial_x E_y - \partial_y E_x = -\frac{1}{c_0}\left(\mu_{zx} \partial_t H_x + \mu_{zy} \partial_t H_y + \mu_{zz} \partial_t H_z\right) - \mathcal{M}_z$$

x: $\partial_y H_z - \partial_z H_y = \frac{1}{c_0}\left(\varepsilon_{xx}\partial_t E_x + \varepsilon_{xy}\partial_t E_y + \varepsilon_{xz}\partial_t E_z\right) + \eta_0 J_x$

y: $\partial_z H_x - \partial_x H_z = \frac{1}{c_0}\left(\varepsilon_{yx}\partial_t E_x + \varepsilon_{yy}\partial_t E_y + \varepsilon_{yz}\partial_t E_z\right) + \eta_0 J_y$

z: $\partial_x H_y - \partial_y H_x = \frac{1}{c_0}\left(\varepsilon_{zx}\partial_t E_x + \varepsilon_{zy}\partial_t E_y + \varepsilon_{zz}\partial_t E_z\right) + \eta_0 J_z$

Assume we have only diagonally-anisotropic materials.

1 $\quad \partial_y E_z - \partial_z E_y = -\frac{1}{c_0}\mu_{xx}\partial_t H_x - M_x$

2 $\quad \partial_z E_x - \partial_x E_z = -\frac{1}{c_0}\mu_{yy}\partial_t H_y - M_y$

3 $\quad \partial_x E_y - \partial_y E_x = -\frac{1}{c_0}\mu_{zz}\partial_t H_z - M_z$

4 $\quad \partial_y H_z - \partial_z H_y = \frac{1}{c_0}\varepsilon_{xx}\partial_t E_x + \eta_0 J_x$

5 $\quad \partial_z H_x - \partial_x H_z = \frac{1}{c_0}\varepsilon_{yy}\partial_t E_y + \eta_0 J_y$

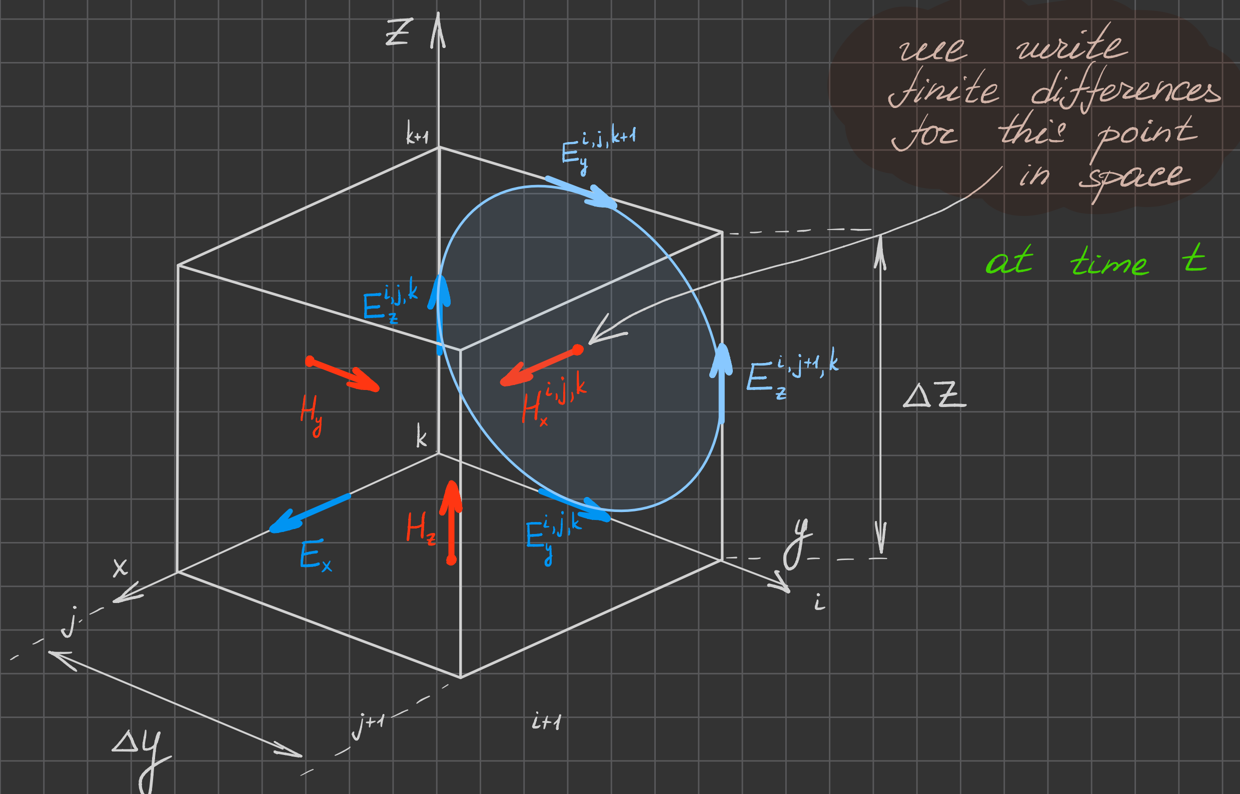
6 $\quad \partial_x H_y - \partial_y H_x = \frac{1}{c_0}\varepsilon_{zz}\partial_t E_z + \eta_0 J_z$

We will approximate these 6 equalities on Yee grid.

(next 6 pages)

1

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{1}{c_0}\mu_{xx}\frac{\partial H_x}{\partial t} - \mathcal{M}_x$$



$$\frac{E_z^{i,j+1,k}\Big|_t - E_z^{i,j,k}\Big|_t}{\Delta y} - \frac{E_y^{i,j,k+1}\Big|_t - E_y^{i,j,k}\Big|_t}{\Delta z} = -\frac{1}{c_0}\mu_{xx}^{i,j,k}\frac{H_x^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k}\Big|_{t-\frac{\Delta t}{2}}}{\Delta t} - \mathcal{M}_x^{i,j,k}\Big|_t$$

μ components may also vary in space : $\mu_{xx} \rightarrow \mu_{xx}^{i,j,k}$

$\varepsilon_{xx} \rightarrow \varepsilon_{xx}^{i,j,k}$

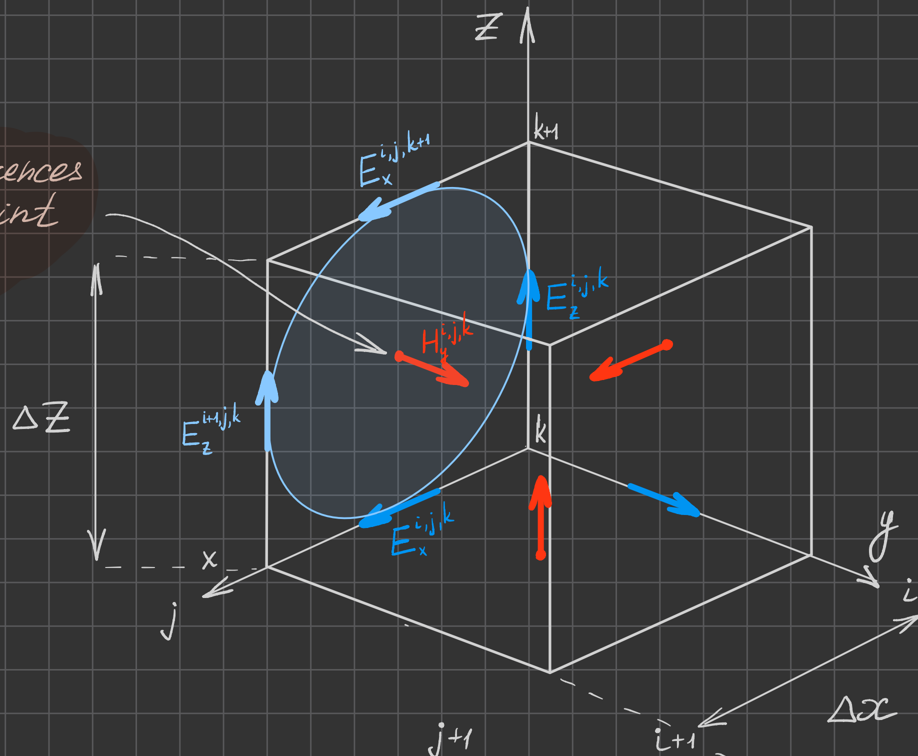
For the electric field derivative we need to reach to the next cells in space

2

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{1}{c_0}\mu_{yy}\frac{\partial H_y}{\partial t} - M_y$$

finite differences at this point in space

at time t

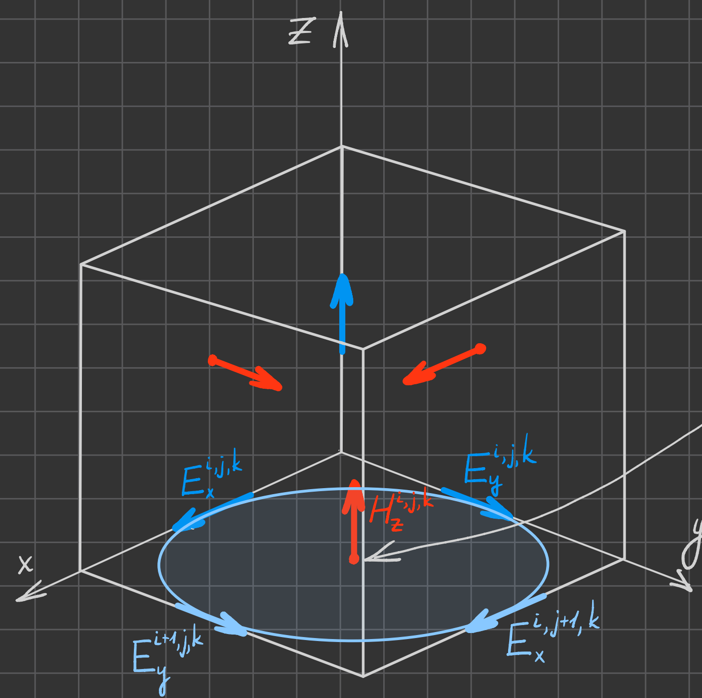


$$\frac{E_x^{i,j,k+1}\big|_t - E_x^{i,j,k}\big|_t}{\Delta z} - \frac{E_z^{i+1,j,k}\big|_t - E_z^{i,j,k}\big|_t}{\Delta x} = -\frac{1}{c_0}\mu_{yy}^{i,j,k}\frac{H_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_y^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$- M_y^{i,j,k}\big|_t$$

3

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{1}{c_0}\mu_{zz}\frac{\partial H_z}{\partial t} - \mathcal{M}_z$$



we are approximating spatial derivatives at this point

at time t

$$\frac{E_y^{i+1,j,k}\Big|_t - E_y^{i,j,k}\Big|_t}{\Delta x} - \frac{E_x^{i,j+1,k}\Big|_t - E_x^{i,j,k}\Big|_t}{\Delta y} = -\frac{1}{c_0}\mu_{zz}^{i,j,k}\frac{H_z^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_z^{i,j,k}\Big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

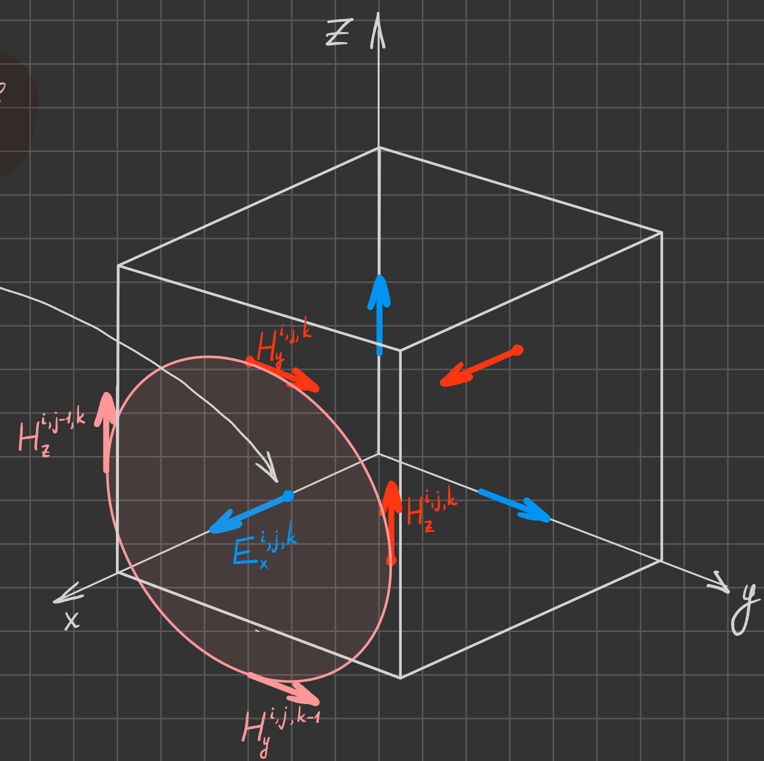
$$- \mathcal{M}_z^{i,j,k}\Big|_t$$

4

$$\partial_y H_z - \partial_z H_y = \frac{1}{c_0}\varepsilon_{xx}\partial_t E_x + \eta_0 J_x$$

finite difference equation is written for this point

at time $t+\frac{\Delta t}{2}$



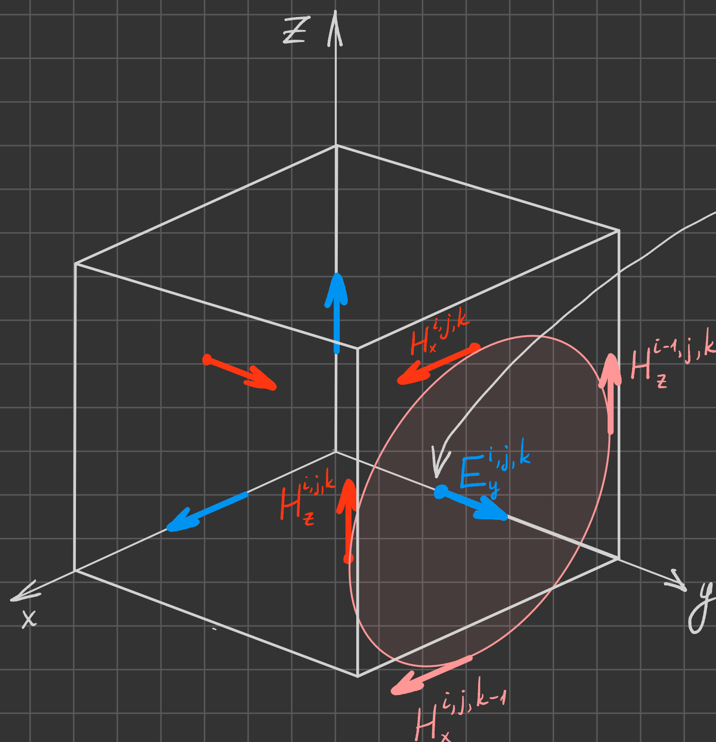
$$\frac{H_z^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_z^{i,j-1,k}\Big|_{t+\frac{\Delta t}{2}}}{\Delta y} - \frac{H_y^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_y^{i,j,k-1}\Big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{1}{c_0}\varepsilon_{xx}^{i,j,k}\frac{E_x^{i,j,k}\Big|_{t+\Delta t} - E_x^{i,j,k}\Big|_{t}}{\Delta t} +$$

$$+\ \eta_0 J_x^{i,j,k}\Big|_{t+\frac{\Delta t}{2}}$$

For magnetic field derivative we have to reach to the previous cell in space

5

$$\partial_z H_x - \partial_x H_z = \frac{1}{c_0}\varepsilon_{yy}\partial_t E_y + \eta_0 J_y$$



writing finite difference equations for this point

at time $t+\frac{\Delta t}{2}$

$$\frac{H_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} - \frac{H_z^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_z^{i-1,j,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta x} = \frac{1}{c_0}\varepsilon_{yy}^{i,j,k}\frac{E_y^{i,j,k}\big|_{t+\Delta t} - E_y^{i,j,k}\big|_{t}}{\Delta t}$$

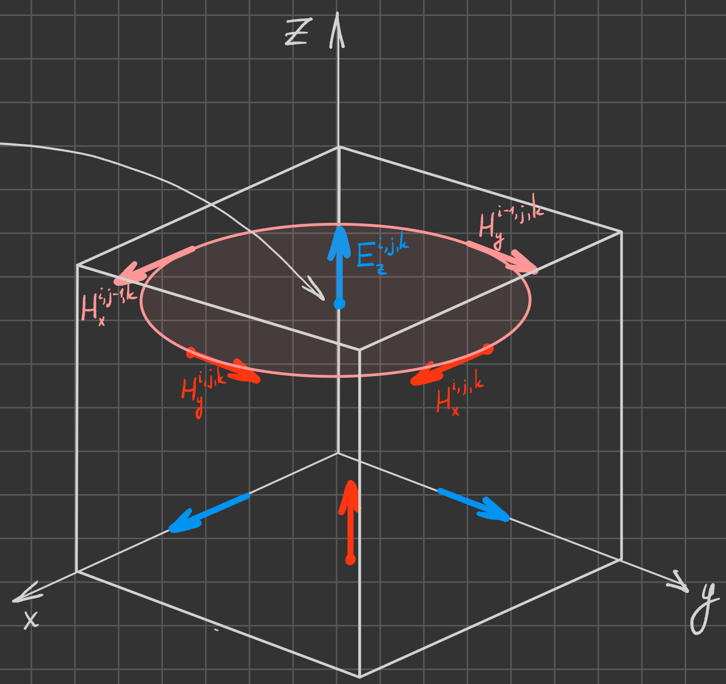
$$+ \eta_0 J_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}}$$

6

$$\partial_x H_y - \partial_y H_x = \frac{1}{c_0}\varepsilon_{zz}\partial_t E_z + \eta_0 J_z$$

we write finite difference equations at this point

at time $t+\frac{\Delta t}{2}$



$$\frac{H_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_y^{i-1,j,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta x} - \frac{H_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H^{i,j-1,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta y} = \frac{1}{c_0}\varepsilon_{zz}^{i,j,k}\frac{E_z^{i,j,k}\big|_{t+\Delta t} - E_z^{i,j,k}\big|_{t}}{\Delta t} + \eta_0 J_z^{i,j,k}\Big|_{t+\frac{\Delta t}{2}}$$

For the same spatial cell electric J and magnetic M currents are also staggered in time $\frac{\Delta t}{2}$ apart. (J is $\frac{\Delta t}{2}$ ahead of M)

$$\frac{E_z^{i,j+1,k}\big|_t - E_z^{i,j,k}\big|_t}{\Delta y} - \frac{E_y^{i,j,k+1}\big|_t - E_y^{i,j,k}\big|_t}{\Delta z} = -\frac{1}{c_0}\mu_{xx}^{i,j,k}\frac{H_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t} - \mathcal{M}_x^{i,j,k}\big|_t$$

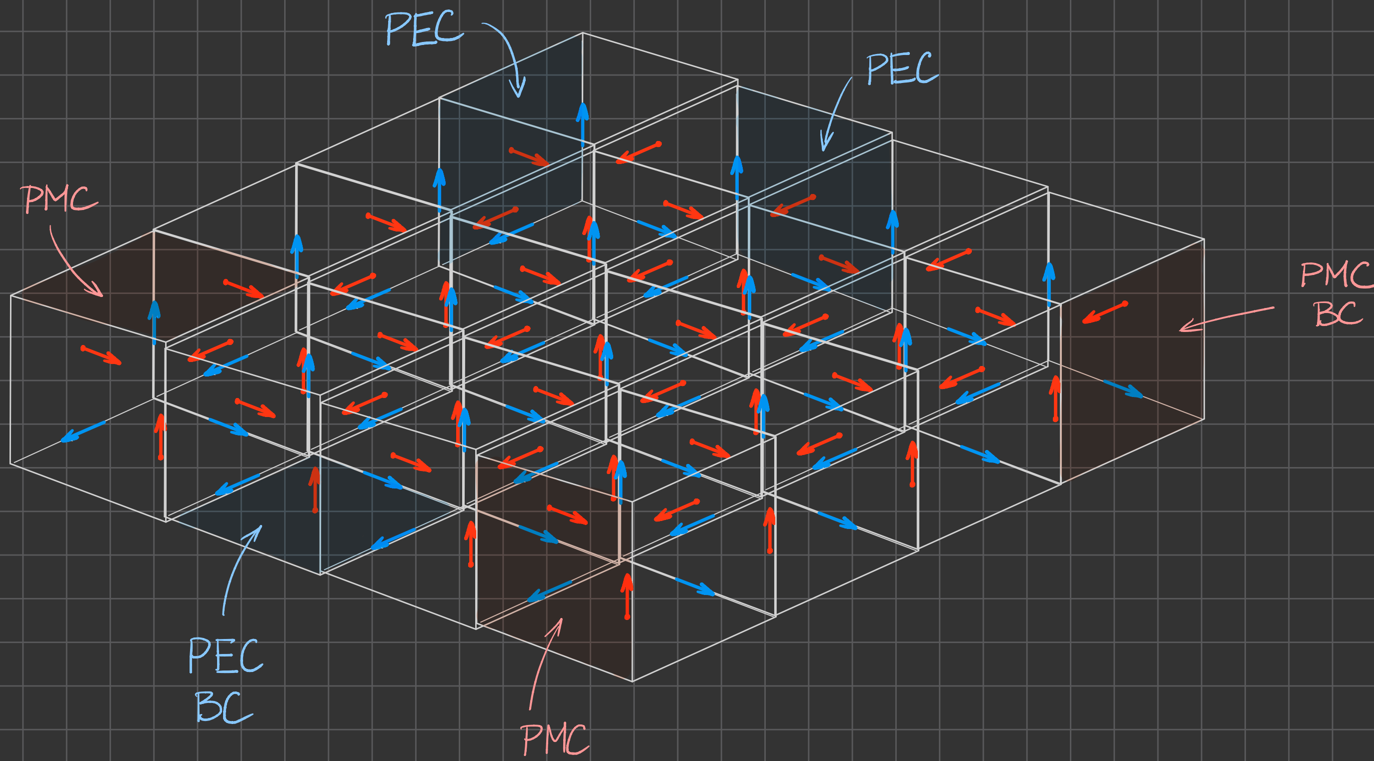
$$\frac{E_x^{i,j,k+1}\big|_t - E_x^{i,j,k}\big|_t}{\Delta z} - \frac{E_z^{i+1,j,k}\big|_t - E_z^{i,j,k}\big|_t}{\Delta x} = -\frac{1}{c_0}\mu_{yy}^{i,j,k}\frac{H_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_y^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t} - \mathcal{M}_y^{i,j,k}\big|_t$$

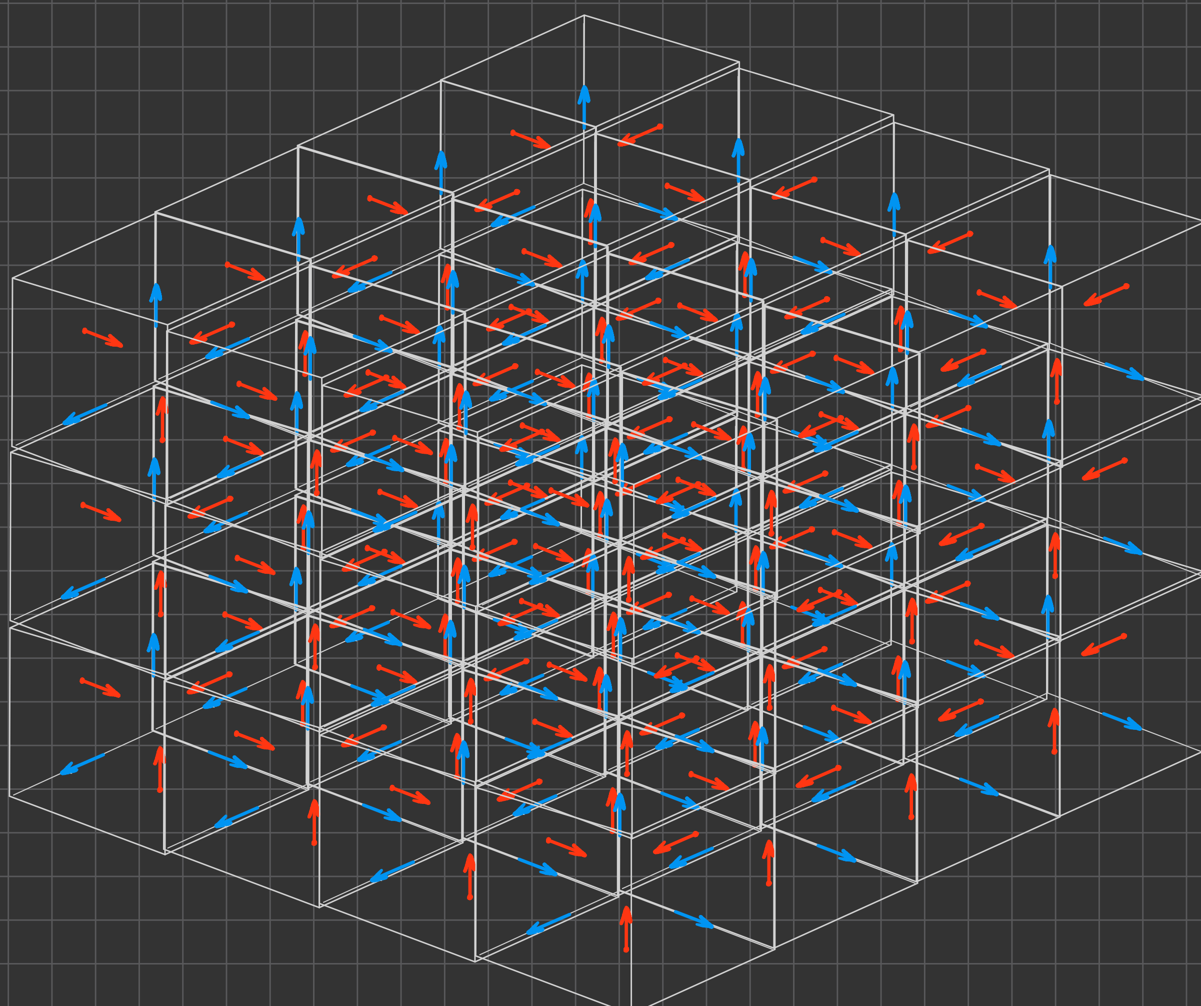
$$\frac{E_y^{i+1,j,k}\big|_t - E_y^{i,j,k}\big|_t}{\Delta x} - \frac{E_x^{i,j+1,k}\big|_t - E_x^{i,j,k}\big|_t}{\Delta y} = -\frac{1}{c_0}\mu_{zz}^{i,j,k}\frac{H_z^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_z^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t} - \mathcal{M}_z^{i,j,k}\big|_t$$

$$\frac{H_z^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_z^{i,j-1,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta y} - \frac{H_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_y^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{1}{c_0}\varepsilon_{xx}^{i,j,k}\frac{E_x^{i,j,k}\big|_{t+\Delta t} - E_x^{i,j,k}\big|_t}{\Delta t} + \eta_0 J_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}}$$

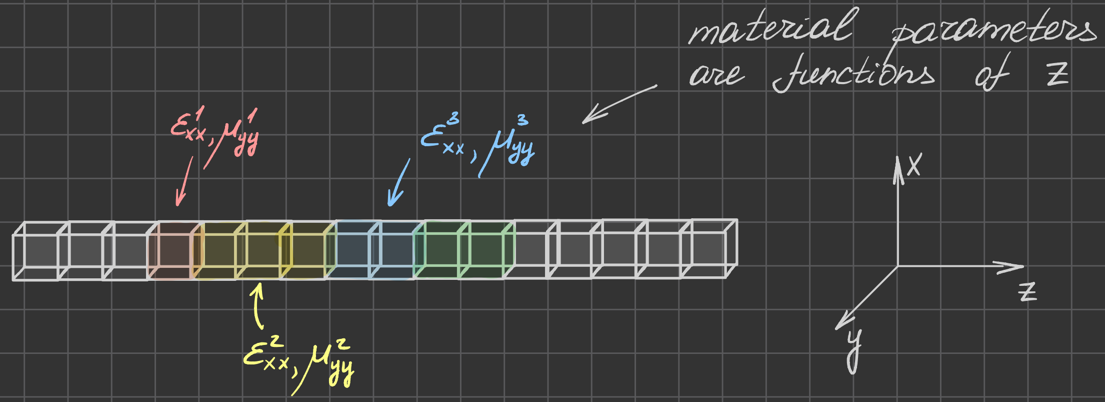
$$\frac{H_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} - \frac{H_z^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_z^{i-1,j,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta x} = \frac{1}{c_0}\varepsilon_{yy}^{i,j,k}\frac{E_y^{i,j,k}\big|_{t+\Delta t} - E_y^{i,j,k}\big|_t}{\Delta t} + \eta_0 J_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}}$$

$$\frac{H_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_y^{i-1,j,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta x} - \frac{H_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H^{i,j-1,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta y} = \frac{1}{c_0}\varepsilon_{zz}^{i,j,k}\frac{E_z^{i,j,k}\big|_{t+\Delta t} - E_z^{i,j,k}\big|_t}{\Delta t} + \eta_0 J_z^{i,j,k}\big|_{t+\frac{\Delta t}{2}}$$


PEC
PEC
PMC
PMC
BC
PEC
BC
PMC



1D case



$$\partial_x = \partial_y = 0$$

$\Rightarrow$ We cross out corresponding terms from our finite difference equations:

$$-\frac{E_y^{i,j,k+1}\big|_t - E_y^{i,j,k}\big|_t}{\Delta z} = -\frac{1}{c_0}\mu_{xx}^{i,j,k}\frac{H_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t} - \mathcal{M}_x^{i,j,k}\big|_t$$

decoupled

$$\frac{E_x^{i,j,k+1}\big|_t - E_x^{i,j,k}\big|_t}{\Delta z} = -\frac{1}{c_0}\mu_{yy}^{i,j,k}\frac{H_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_y^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t} - \mathcal{M}_y^{i,j,k}\big|_t$$

E_x/H_y mode

$$H_z^{i,j,k} = 0$$

E_y/H_x mode

$$-\frac{H_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_y^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{1}{c_0}\varepsilon_{xx}^{i,j,k}\frac{E_x^{i,j,k}\big|_{t+\Delta t} - E_x^{i,j,k}\big|_t}{\Delta t} + \eta_0 J_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}}$$

decoupled

$$\frac{H_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{1}{c_0}\varepsilon_{yy}^{i,j,k}\frac{E_y^{i,j,k}\big|_{t+\Delta t} - E_y^{i,j,k}\big|_t}{\Delta t} + \eta_0 J_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}}$$

$$E_z^{i,j,k} = 0$$

If there's no anisotropy in the model, those modes have the same numerical behavior; they are physical modes and propagate independently. We will consider E_y / H_x mode.

$$-\frac{E_y^{i,j,k+1}\big|_t - E_y^{i,j,k}\big|_t}{\Delta z} = -\frac{1}{c_0}\mu_{xx}^{i,j,k}\frac{H_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t} - \mathcal{M}_x^{i,j,k}\Big|_t$$

$$\frac{H_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{1}{c_0}\varepsilon_{yy}^{i,j,k}\frac{E_y^{i,j,k}\big|_{t+\Delta t} - E_y^{i,j,k}\big|_t}{\Delta t} + \eta_0 J_y^{i,j,k}\Big|_{t+\frac{\Delta t}{2}}$$

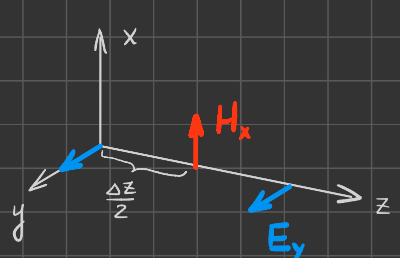
Let's derive update equations. (omit indices i and j)

Ampere's law:

$$E_y^k\big|_{t+\Delta t} = E_y^k\big|_t + \frac{1}{\Delta z}\cdot\underbrace{\Delta t\cdot\frac{c_0}{\varepsilon_{yy}^k}}_{\equiv\, m_{E_y}^k}\left(H_x^k\big|_{t+\frac{\Delta t}{2}} - H_x^{k-1}\big|_{t+\frac{\Delta t}{2}}\right) - \underbrace{\Delta t\cdot\frac{c_0}{\varepsilon_{yy}^k}}_{m_{E_y}^k}\eta_0 J_y^k\big|_{t+\frac{\Delta t}{2}}$$

Faraday's law

$$H_x^k\big|_{t+\frac{\Delta t}{2}} = H_x^k\big|_{t-\frac{\Delta t}{2}} + \frac{1}{\Delta z}\cdot\underbrace{\Delta t\cdot\frac{c_0}{\mu_{xx}^k}}_{\equiv\, m_{H_x}^k}\cdot\left(E_y^{k+1}\big|_t - E_y^k\big|_t\right) - \underbrace{\Delta t\cdot\frac{c_0}{\mu_{xx}^k}}_{m_{H_x}^k}\mathcal{M}_x^k\big|_t$$



Implementation

$$m_{E_y}^k \equiv \frac{c_0 \Delta t}{\varepsilon_{yy}^k}, \qquad m_{H_x}^k \equiv \frac{c_0 \Delta t}{\mu_{xx}^k}$$

1D arrays since ε_{yy}^k and μ_{xx}^k are the functions of $z = k \cdot \Delta z$

precomputed before the simulation

Workflow:

calculate update coefficients $m_{H_x}^k$ and $m_{E_y}^k$

↓

Initialize arrays for electric and magnetic fields: $E_y^k = 0 \quad H_x^k = 0$

↓ done?

Update H from E

↓

Update E from H

loop over time

Numerical boundary conditions.

Dirichlet BC.

Faraday's law

$$H_x^k\Big|_{t+\frac{\Delta t}{2}} = H_x^k\Big|_{t-\frac{\Delta t}{2}} + \frac{1}{\Delta z}\cdot m_{H_x}^k \left(E_y^{k+1}\Big|_t - E_y^k\Big|_t\right) - m_{H_x}^k \mathcal{M}_x^k\Big|_t$$

what do we do at $k = N_z$: $E_y^{N_z+1}$ does not exist.?

Ampere's law :

$$E_y^k\Big|_{t+\Delta t} = E_y^k\Big|_t + \frac{1}{\Delta z}\cdot m_{E_y}^k \left(H_x^k\Big|_{t+\frac{\Delta t}{2}} - H_x^{k-1}\Big|_{t+\frac{\Delta t}{2}}\right) - m_{E_y}^k \eta_0 J_y^k\Big|_{t+\frac{\Delta t}{2}}$$

what do we do at $k = 0$: H_x^{-1} does not exist.?

Force the values outside the grid to be zero

PEC

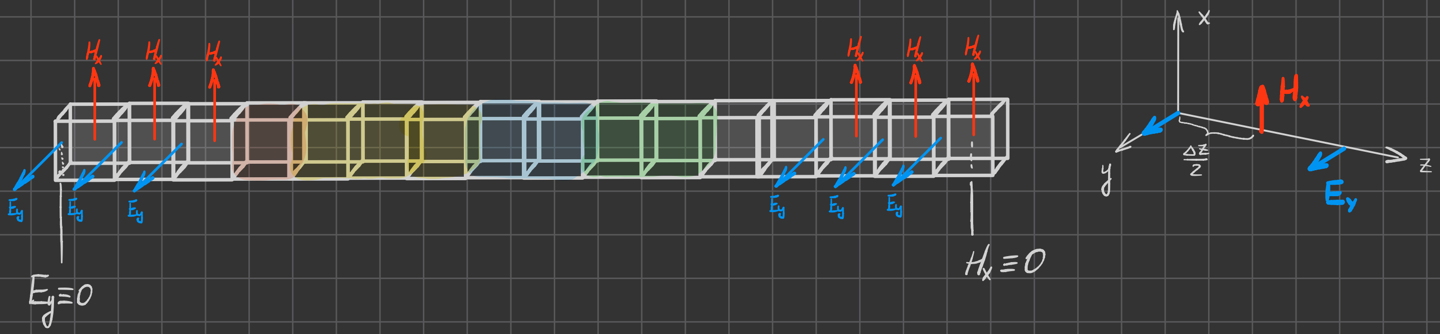
$k=\overline{0, N_z-1}$ $$H_x^k\Big|_{t+\frac{\Delta t}{2}} = H_x^k\Big|_{t-\frac{\Delta t}{2}} + \frac{1}{\Delta z}\cdot m_{H_x}^k \left(E_y^{k+1}\Big|_t - E_y^k\Big|_t\right) - m_{H_x}^k \mathcal{M}_x^k\Big|_t$$

$k=N_z$ right boundary $$H_x^k\Big|_{t+\frac{\Delta t}{2}} = H_x^k\Big|_{t-\frac{\Delta t}{2}} + \frac{1}{\Delta z}\cdot m_{H_x}^k \left(0 - E_y^k\Big|_t\right) - m_{H_x}^k \mathcal{M}_x^k\Big|_t$$

PMC

$k=\overline{1, N_z}$ $$E_y^k\Big|_{t+\Delta t} = E_y^k\Big|_t + \frac{1}{\Delta z}\cdot m_{E_y}^k \left(H_x^k\Big|_{t+\frac{\Delta t}{2}} - H_x^{k-1}\Big|_{t+\frac{\Delta t}{2}}\right) - m_{E_y}^k \eta_0 J_y^k\Big|_{t+\frac{\Delta t}{2}}$$

$k=0$ left boundary $$E_y^k\Big|_{t+\Delta t} = E_y^k\Big|_t + \frac{1}{\Delta z}\cdot m_{E_y}^k \left(H_x^k\Big|_{t+\frac{\Delta t}{2}} - 0\right) - m_{E_y}^k \eta_0 J_y^k\Big|_{t+\frac{\Delta t}{2}}$$



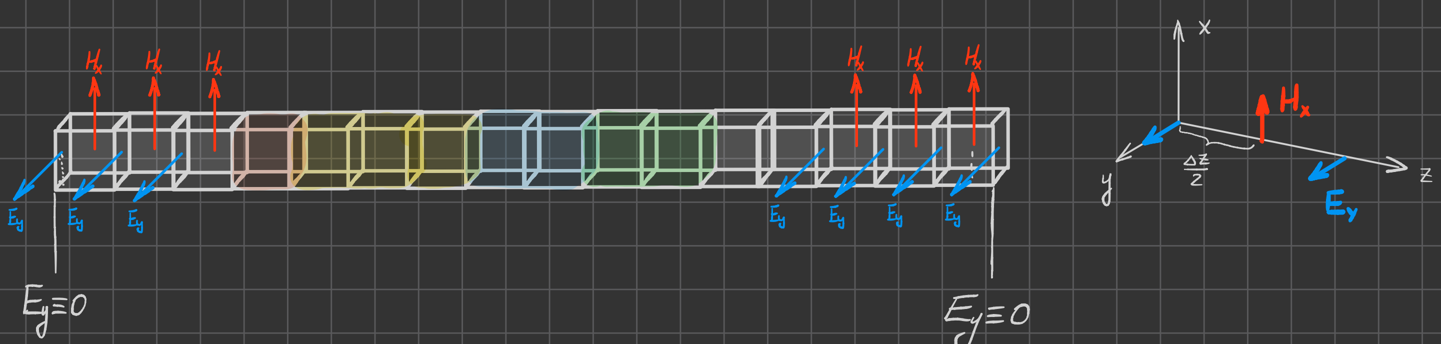
Sizes of arrays for E_y and H_x are the same :

$$\boxed{\begin{aligned} E_y &= \text{array}(Nz) \\ H_x &= \text{array}(Nz) \end{aligned}}$$

PEC boundary condition

→ values of E_y are set to zero <u>at</u> the boundaries

→ we have an extra value of electric field at the right boundary



$$\boxed{\begin{aligned} E_y &= \text{array}(Nz + 1) \\ H_x &= \text{array}(Nz) \end{aligned}}$$

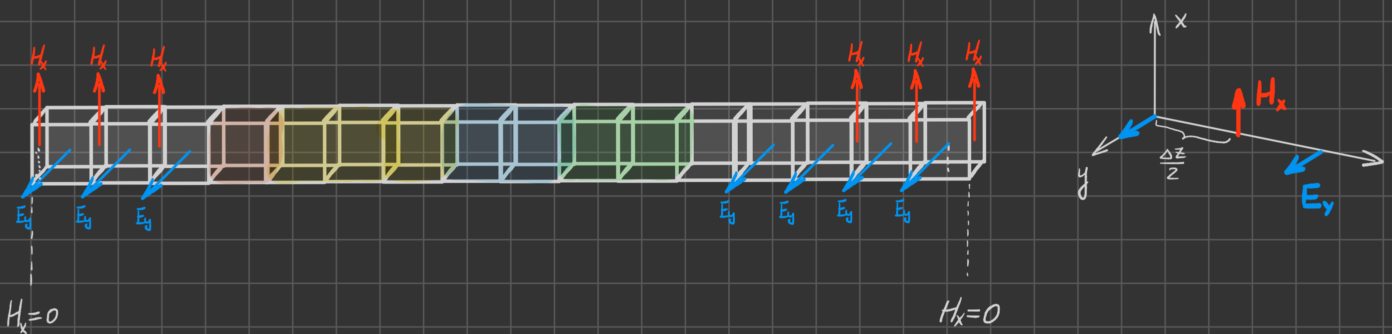
$$k = 1, \dots, N_z - 1 \qquad H_x^k\Big|_{t+\frac{\Delta t}{2}} = H_x^k\Big|_{t-\frac{\Delta t}{2}} + \frac{1}{\Delta z} \cdot m_{H_x}^k \left(E_y^{k+1}\Big|_t - E_y^k\Big|_t \right) - m_{H_x}^k \mathcal{M}_x^k\Big|_t$$

$$k = 0, \dots N_z - 1 \qquad E_y^k\Big|_{t+\Delta t} = E_y^k\Big|_t + \frac{1}{\Delta z} \cdot m_{E_y}^k \left(H_x^k\Big|_{t+\frac{\Delta t}{2}} - H_x^{k-1}\Big|_{t+\frac{\Delta t}{2}} \right) - m_{E_y}^k \eta_0 \mathcal{J}_y^k\Big|_{t+\frac{\Delta t}{2}}$$

$$k = 0 \qquad E_y^0\Big|_{\forall t} \equiv 0$$

$$k = N_z \qquad E_y^{N_z}\Big|_{\forall t} \equiv 0$$

PMC boundary condition



$$\boxed{\begin{array}{l} H_x = \text{array}(N_z + 1) \\ E_y = \text{array}(N_z) \end{array}}$$

$$k = 0, \dots N_z - 1 \qquad H_x^k\Big|_{t+\frac{\Delta t}{2}} = H_x^k\Big|_{t-\frac{\Delta t}{2}} + \frac{1}{\Delta z} \cdot m_{H_x}^k \left(E_y^{k+1}\Big|_t - E_y^k\Big|_t \right) - m_{H_x}^k \mathcal{M}_x^k\Big|_t$$

$$k = 1, \dots, N_z - 1 \qquad E_y^k\Big|_{t+\Delta t} = E_y^k\Big|_t + \frac{1}{\Delta z} \cdot m_{E_y}^k \left(H_x^k\Big|_{t+\frac{\Delta t}{2}} - H_x^{k-1}\Big|_{t+\frac{\Delta t}{2}} \right) - m_{E_y}^k \eta_0 \mathcal{J}_y^k\Big|_{t+\frac{\Delta t}{2}}$$

$$k = 0 \qquad H_x^0\Big|_{\forall t} \equiv 0$$

$$k = N_z \qquad H_x^{N_z}\Big|_{\forall t} \equiv 0$$

Gaussian pulse

$$f(x) = e^{-x^2}$$ – Gaussian function (base form)

$$\int_{-\infty}^{+\infty} \underbrace{e^{-x^2}}_{f(x)} dx = \sqrt{\pi}$$

$$f(x) = a \cdot \exp\left(-\frac{(x-b)^2}{2c^2}\right)$$ – with parametric extension

$$\int_{-\infty}^{+\infty} a \cdot e^{-\frac{(x-b)^2}{2c^2}} dx = a \cdot c \cdot \sqrt{2\pi} = 1$$

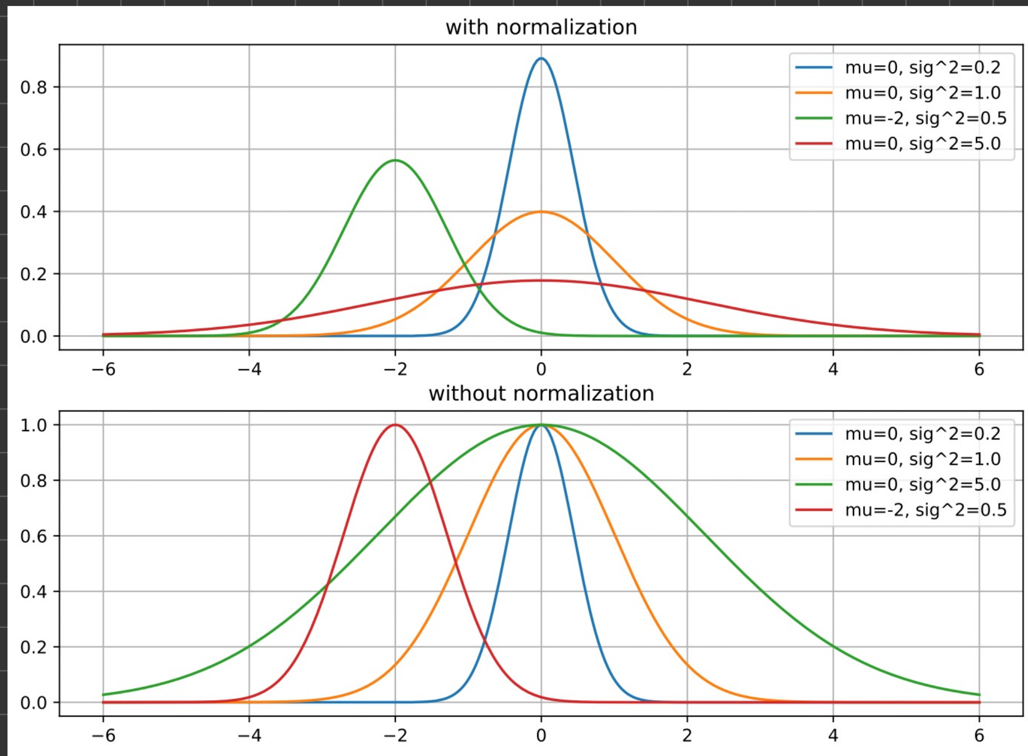
$$a \equiv \frac{1}{c\sqrt{2\pi}}$$

$$a = \frac{1}{\sigma \cdot \sqrt{2\pi}}$$

$b = \mu$ – expected value

$c = \sigma$ – variance

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



$$g(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

In the code I used $\boxed{g(t) = e^{-\left(\frac{t-t_0}{\tau}\right)^2}}$

Implementation of the Gaussian source

Frequency content of Gaussian pulse.

Fourier transform of a Gaussian is a Gaussian:

$$g(t) = e^{-\frac{t^2}{\tau^2}} \quad \longleftrightarrow \quad G(f) = \frac{1}{\sqrt{\pi} B} e^{-\frac{f^2}{B^2}}$$

The frequency content of the Gaussian pulse extends from DC (0 Hz) up to around B

$$\boxed{B = \frac{1}{\pi \tau}}$$

✓ → the narrower the pulse (τ smaller) the more frequencies are captured ⇒ we choose τ according to the frequencies we want to have in the simulation

⊖ → The higher the frequency the shorter the time step Δt will be.

Designing the pulse source.

Step 1: detemine max freq. you're interested in simulating.

$$\boxed{f_{max}}$$

Step 2: compute pulse width considering upper frequency limit

$$B \equiv f_{max} = \frac{1}{\pi \tau} \rightarrow \tau \leq \frac{1}{\pi f_{max}};$$

$$\boxed{\tau \cong \frac{0.5}{f_{max}}}$$

Step 3: check the time step. May need to reduce it. Gaussian pulse should be resolved by at least 10 - 20 time steps.

$$\Delta t = \frac{\tau}{N_t}$$

$$N_t \geq 10$$

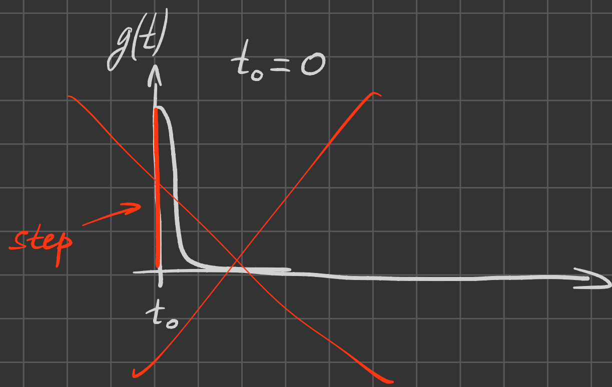
Typically you determine a first Δt_1 based on CFL stability condition. Then determine Δt_2 based on resolving the maximum frequency f_{max}. Then choose the smallest value.

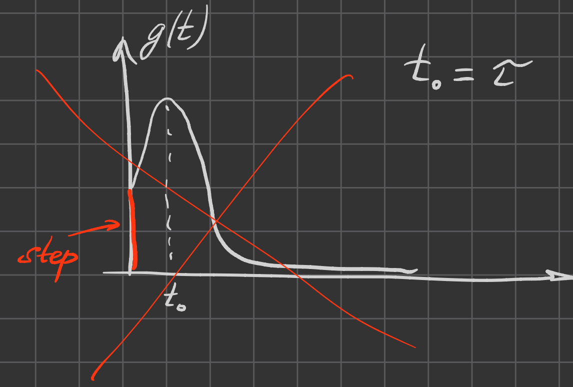
$$\Delta t \equiv \min(\Delta t_1, \Delta t_2)$$

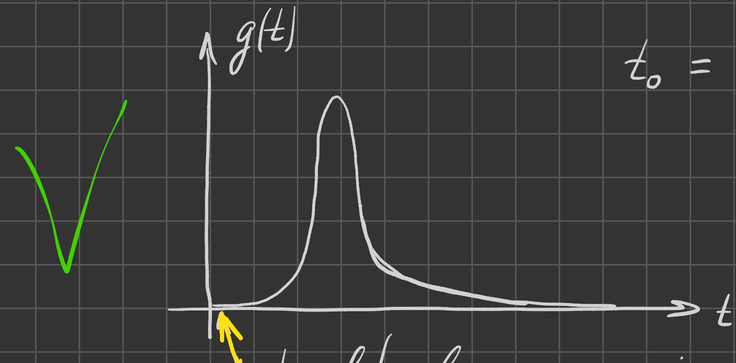
Everything is automatically satisfied if

$$\boxed{\Delta t = \frac{n \Delta z}{2 c_0}} \text{ ABC}$$

Step 4: compute time delay t_0







The source must start at zero and gradually increase. NO STEP FUNCTIONS.

$$\boxed{t_0 \geq 6\tau}$$

why 6τ? what's the value $g(0)$ when $t_0 = 6\tau$?

$$g(0)\Big|_{t_0=6\tau} = e^{-\left(\frac{0-6\tau}{\tau}\right)^2} = e^{-36} \sim 2^{36} \sim 10^{-12}$$

Types of sources (see my notes on sources)

hard (voltage) soft (current)

- overwrite
- source cell is felt by the fields as either PEC or PMC

$$H_x^k \Big|_{t+\frac{\Delta t}{2}} = g_H \Big|_t$$

$$E_y^k \Big|_{t+\Delta t} = g_E \Big|_t$$

- not usually a practical source

- add to
- source cell is transparent to scattered waves passing through it
- injects energy into both directions
- great for testing boundary conditions

$$H_x^k \Big|_{t+\frac{\Delta t}{2}} += g_H \Big|_t$$

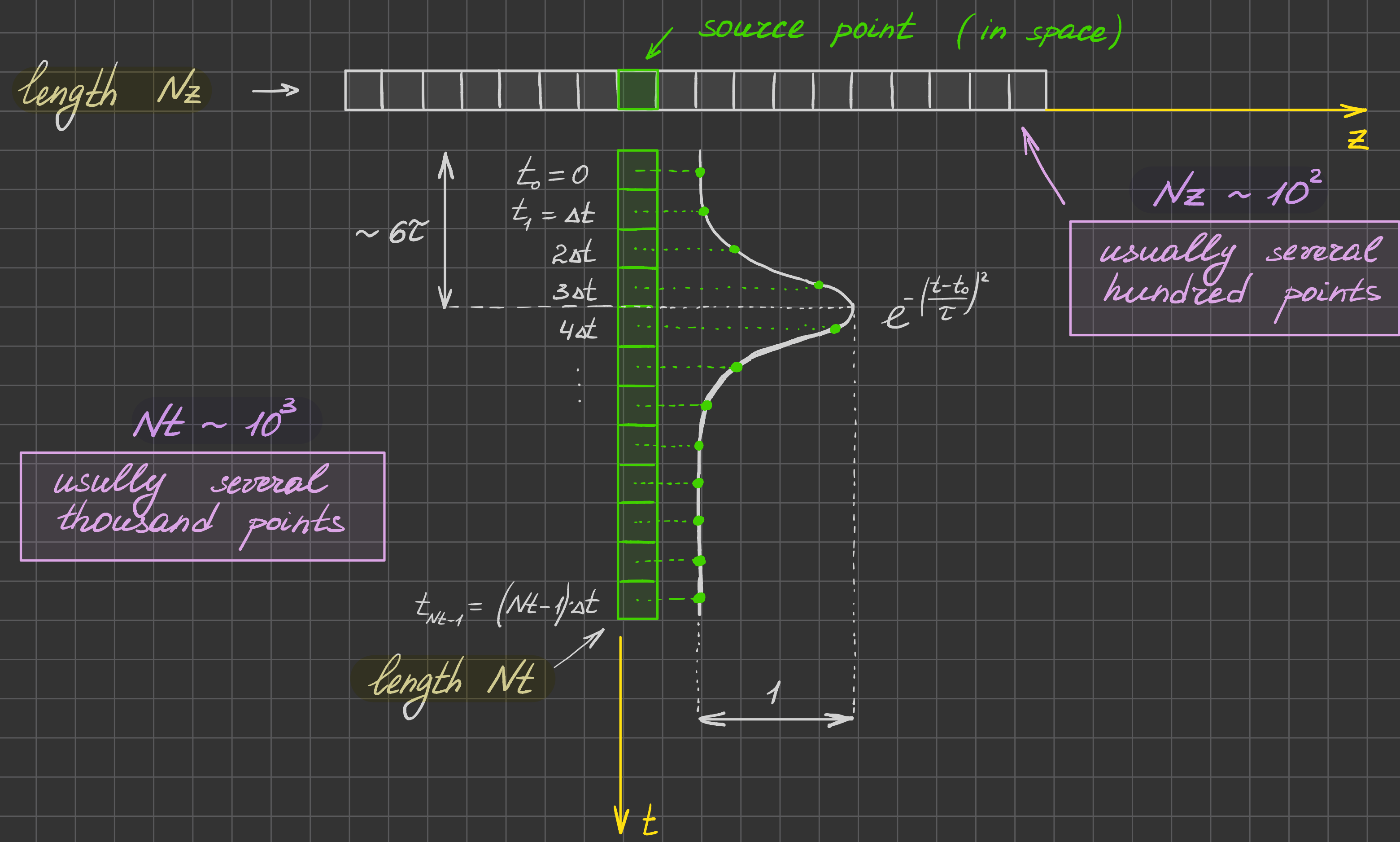
$$E_y^k \Big|_{t+\Delta t} += g_E \Big|_t$$

- rarely used. Use until learn SF/TF source.

* see Kuniaki Yoshitomi's paper on this account.

Visualizing the arrays

E_y, H_x, (ER, UR), mEy, mHx are stored in the arrays of length Nz.



g_E and g_H are stored in the arrays of length Nt

Δt

Time resolution Δt

Courant - Friderich - Lewy stability condition

over the duration of one time step Δt EM disturbance will travel

- numerical distance covered in Δt : Δz
- physical dist. covered in Δt : $c_0 \cdot \frac{\Delta t}{n}$

In numerical algorithm it's impossible for a disturbance to travel farther than one cell Δz in a single time step Δt.

For this reason we need to make sure that the physical wave would not propagate farther than a unit cell Δz in one time step. Δt.

$$\underbrace{\frac{c_0}{n} \cdot \Delta t}_{\text{physical}} < \underbrace{\Delta z}_{\text{numerical}}$$

$\Rightarrow$ upper limit on time step Δt

$$\boxed{\Delta t < \frac{\Delta z}{(c_0/n)}}$$

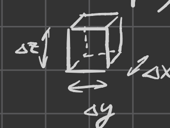
n - smallest refractive idx ; $n \geq 1 \Rightarrow$ usually n is set to 1 and dropped from the ineq.

For 2D and 3D cases:

$$\boxed{\Delta t < \frac{1}{c_0\sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2}}}}$$

CFL stability condition

This condition will be most restrictive along the shortest dimension of the unit cell



$$\boxed{\Delta_{min} \equiv \min\{\Delta x, \Delta y, \Delta z\}}$$

$$\Delta t < \frac{\Delta_{min}}{2c_0}$$

$\frac{1}{2}$ – safety margin

More general case:

$$\boxed{\Delta t < \frac{1}{2}\frac{n_{min}\Delta_{min}}{c_0}}$$

- grid filled with dielectric $\Rightarrow$ wave travels slower everywhere
- model includes materials with $n < 1$

For perfect boundary condition

$$\boxed{\Delta t = \frac{n_{bc} \cdot \Delta z}{2c_0}}$$

n_{bc} – refr. idx at the boundaries; materials should be the same at both boundaries

Nt

Total number of iterations Nt

Depends mainly on : → device we model
→ properties of device we calculate

device

- highly resonant devices typically require more iterations
- purely scattering devices require very few iterations
- more iterations needed the more waves bounce around in the grid

information

- calculating spectral shapes requires the most iterations : reflectance, transmittance, line shapes ...
- calculating positions of resonances requires fewer iterations

This might be very difficult in frequency domain.

Rule of thumb

1. How long it takes a slowest wave to propagate across the grid?

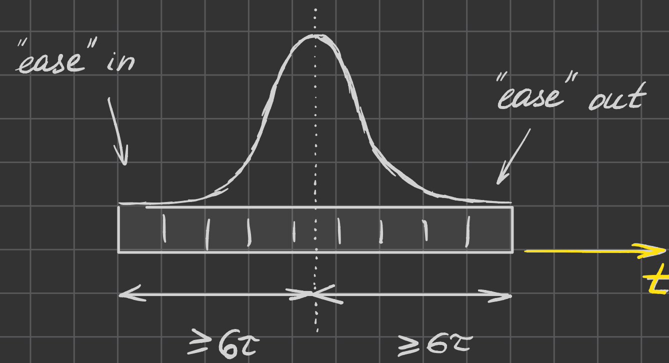
$$\boxed{t_{prop} = \frac{n_{max} \overbrace{N_z \cdot \Delta z}}{c_0}}$$

physical size of the grid

2. Simulation time T must include the entire pulse

of duration τ

$$\boxed{T \geq 6\tau + 6\tau}$$



3. Simulation time should allow for ~ 5 bounces

$$\boxed{T \gtrsim 5t_{prop}}$$

4. Rule of thumb for total simulation time

$$\boxed{T \gtrsim 12\tau + 5t_{prop}}^{*}$$

* for highly resonant devices this time is NOT enough

5. Given time step Δt, the total # of iterations is

$$\boxed{Nt \equiv \text{round}\uparrow\left[\frac{T}{\Delta t}\right] \quad (\text{ceil})}$$

this is in context of for (1, ..., Nt) loop.

Another approach is to set a while (...) loop and keep running until the power is stuck in the grid and stop iterations when the power in the domain drops below some threshold.

ΔZ

Grid resolution ΔZ

1. Wavelength

- ΔZ should be sufficient to resolve the shortest wavelegh

→ determine
$$\boxed{\lambda_{min} = \frac{c_0}{f_{max} \cdot n_{max}}}$$

n_{max} – largest refractive idx found in the grid

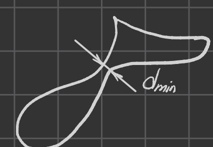
→ resolve wave with at least $N_\lambda = 10$ cells

$$\boxed{\Delta_\lambda \approx \frac{\lambda_{min}}{N_\lambda}} \quad , \quad N_\lambda \geq 10$$

N_λ	material
10 – 20	low contrast dielectric
20 – 30	high contrast diel.
40 – 60	most metallic structures
100 – 200	plasmonic devices

2. Mechanical Features

- ΔZ should be sufficient to resolve the smallest mechanical feature d_{min}

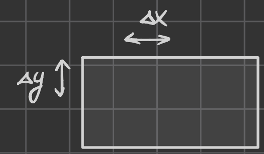


→ want to resolve this mechanical feat. d_{min} with 1 to 4 cells Δ_d

$$\boxed{\Delta_d \approx \frac{d_{min}}{N_d}} \quad , \quad N_d \geq 1$$

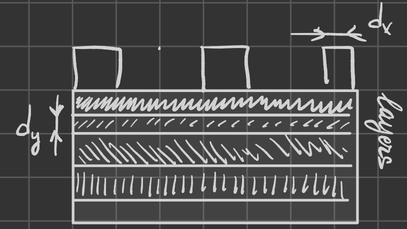
3. Adjust init. resolution ΔZ

$$\boxed{\Delta X = \Delta y = \Delta Z = \min\left[\Delta_\lambda , \Delta_d\right]}$$



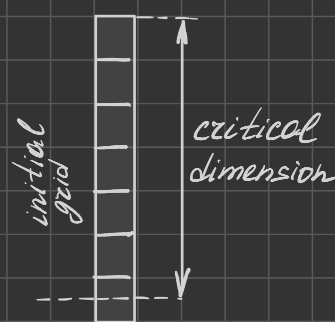
4. "Snap" grid to crytical dimensions

→ decide what dimensions along each axis are critical : d_x and d_y



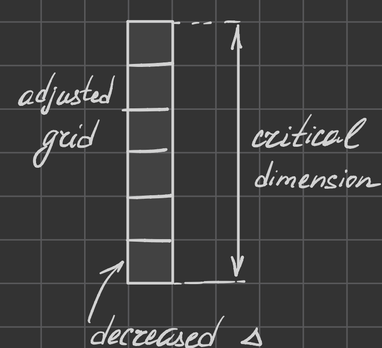
→ compute how many grid cells comprise $d_x/d_y/d_z$ and round up

$$\boxed{\begin{aligned} M_x &\equiv \operatorname{ceil}\left(\frac{d_x}{\Delta_x}\right) \\ M_y &\equiv \operatorname{ceil}\left(\frac{d_y}{\Delta y}\right) \end{aligned}}$$



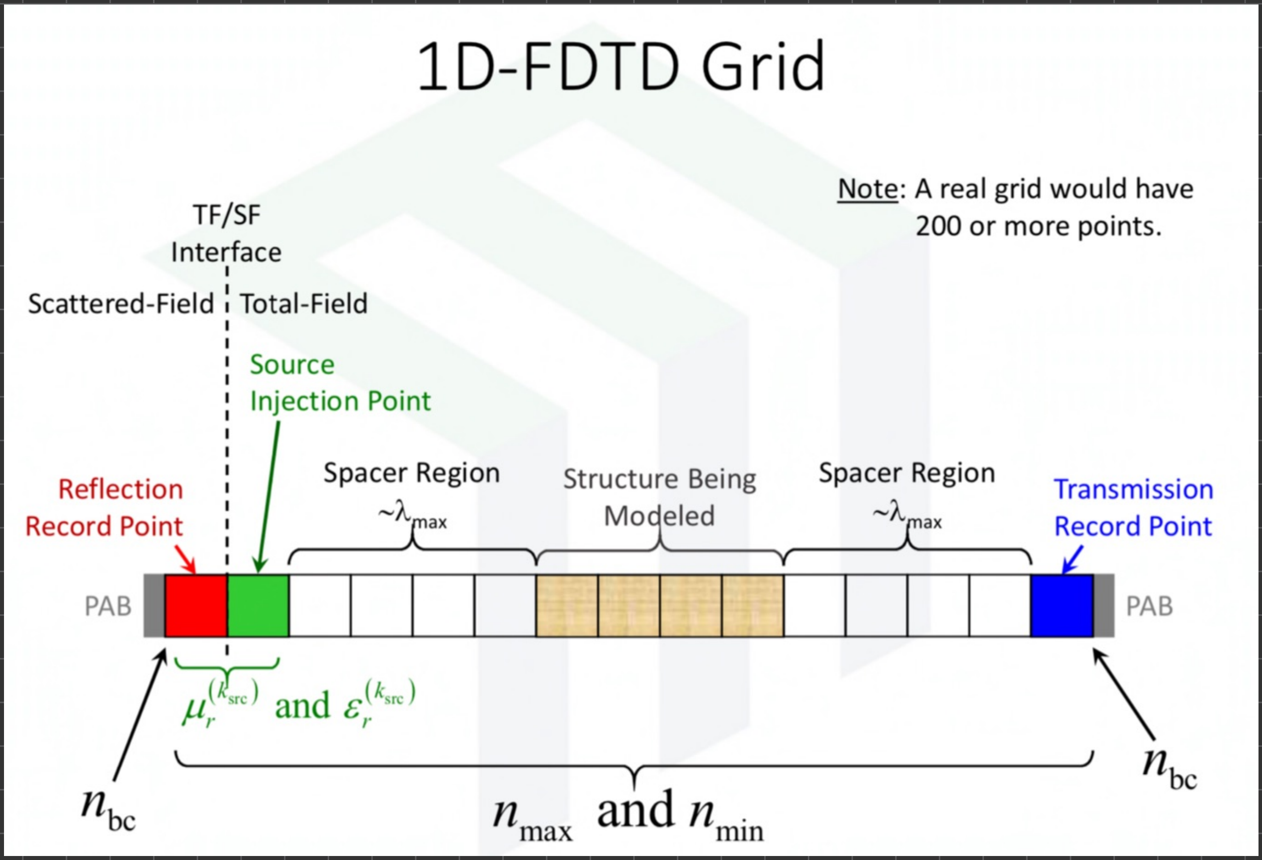
→ adjust grid resolution to fit this dimension d_x/d_y in grid exactly

$$\boxed{\Delta_x \equiv \frac{d_x}{M_x} \qquad \Delta_y \equiv \frac{d_y}{M_y}}$$



Total number of space cells Nz

Nz



Revised FDTD