1.3 Rational Numbers

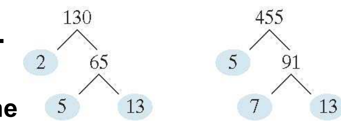
1. Define the rational numbers
2. Reduce rational numbers
3. Convert between mixed numbers and improper fractions
4. Express rational numbers as decimals.
5. Express decimals in the form a/b
6. Multiply and divide rational numbers
7. Add and subtract rational numbers
8. Use the order of operations agreement with rational numbers
9. Solve problems involving rational numbers

Defining the Rational Numbers

- The set of *rational numbers* is the set of all numbers which can be expressed in the form where *a* and *b* are integers and *b* is not equal to 0.
- The integer *a* is called the *numerator* $\frac{a}{b}$
- The integer *b* is called the *denominator*
- Examples of rational numbers: ¼, –½, ¾, 5, 0
- Equivalent Rational Numbers
  - To reduce a rational number to its lowest terms
  - divide numerator and denominator by their greatest common divisor $\frac{a\cdot c}{b\cdot c}=\frac{a}{b}$

Reducing a Rational Number

Reduce $\frac{130}{455}$ to lowest terms.



Solution: Begin by finding the Greatest Common Factor of 130 and 455.

- Thus, 130 = 2 · 5 · 13, and 455 = 5 · 7 · 13
- Divide the numerator and the denominator of the given rational number by 5 · 13 or 65

$$\frac{130}{455}=\frac{2\cdot\cancel{5}\cdot\cancel{13}}{\cancel{5}\cdot 7\cdot\cancel{13}}=\frac{2}{7} \quad \text{or} \quad \frac{130}{455}=\frac{130\div 65}{455\div 65}=\frac{2}{7}$$

- There are no common divisors of 2 and 7 other than 1.
- Thus, the rational number 2/7 is in its lowest terms.

Mixed Numbers, Improper Fractions, and Decimal

- A *mixed number* consists of the sum of an integer and a rational number, expressed without the use of an addition sign.

Example: $3\frac{4}{5}$.

The integer is 3 and the rational number is $\frac{4}{5}$. $3\frac{4}{5}$ means $3+\frac{4}{5}$.

- An *improper fraction* is a rational number whose numerator is greater than denominator $\frac{19}{5}$

19 is larger than 5

- Any rational number can be expressed as a *decimal number* by dividing the denominator into the numerator 3.8

Convert Mixed Number to Improper Fraction

1. Multiply the denominator of the rational number by the integer and add the numerator to this product.
2. Place the sum in step 1 over the denominator of the mixed number.

$$3\frac{4}{5} = \frac{5 \cdot 3 + 4}{5} = \frac{15+4}{5} = \frac{19}{5}$$

Multiply the denominator by the integer and add the numerator.

Place the sum over the mixed number's denominator.

Convert Improper Fraction to Mixed Number

1. Divide the denominator into the numerator. Record the quotient and the remainder.
2. Write the mixed number using the following form: $\text{quotient}\ \frac{\text{remainder}}{\text{original denominator}}$.

integer part

rational number part

Convert $\frac{42}{5}$ to a mixed number.

Solution:

Step 1 Divide denominator into the numerator

$$5\overline{)42} = 8,\quad 40,\quad \text{remainder } 2$$

quotient

remainder

Step 2 $\frac{42}{5} = 8\frac{2}{5}$.

remainder

original denominator

quotient

Expressing Rational Numbers as Decimals

Express each rational number as a decimal.

a. $\frac{5}{8}$

$$8\overline{)5.000} = 0.625$$

48
20
16
40
40
0

Notice the decimal stops with remainder = 0. This is a **terminating decimal**.

b. $\frac{7}{11}$

$$11\overline{)7.000} = 0.6363$$

66
40
33
70
66
40
33
70
⋮

Notice the digits 63 repeat over and over indefinitely. This is called a **repeating decimal**.

Expressing Decimals as a Fraction

Express terminating decimal as a quotient of integers:

a. 0.7 b. 0.49 c. 0.048

Solution:

a. $0.7 = \frac{7}{10}$ because the 7 is in the tenths position.

b. $0.49 = \frac{49}{100}$ because the digit on the right, 9, is in the hundredths position.

c. $0.048 = \frac{48}{1000} = \frac{48 \div 8}{1000 \div 8} = \frac{6}{125}$
because the digit on the right, 8, is thousandths position and can be reduced to lowest terms

Multiplying Rational Numbers

- **The product of two rational numbers is the product of their numerators divided by the product of their denominators.**
- **If $\frac{a}{b}$ and $\frac{c}{d}$ are multiplied, then** $\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$

$$\left(-\frac{2}{3}\right)\left(-\frac{9}{4}\right)=\frac{(-2)(-9)}{3\cdot 4}=\frac{18}{12}=\frac{3\cdot 6}{2\cdot 6}=\frac{3}{2} \text{ or } 1\frac{1}{2}$$

Multiply across. Simplify to lowest terms.

- **Pre-simplify Example**

$$\left(-\frac{\cancel{2}^{\,1}}{\cancel{3}_{\,1}}\right)\left(-\frac{\cancel{9}^{\,3}}{\cancel{4}_{\,2}}\right)=\frac{3}{2} \text{ or } 1\frac{1}{2}$$

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Dividing Rational Numbers

- **The quotient of two rational numbers is a product of the first number and the reciprocal of the second number**
- **Flip last number and multiply by first number**
- **If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, then** $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{a\cdot d}{b\cdot c}$

$$-\frac{3}{5}\div\frac{7}{11}=-\frac{3}{5}\cdot\frac{11}{7}=-\frac{3\cdot 11}{5\cdot 7}=-\frac{33}{35}$$

Change to multiplication by using the reciprocal.

Multiply across.

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Add and Subtract Rational Numbers

The sum or difference of two rational numbers with identical denominators is the sum or difference of numerators over common denominator.

If $\frac{a}{b}$ and $\frac{c}{b}$ are rational numbers, then

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b} \qquad \frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$$

Examples:

$$\frac{3}{7}+\frac{2}{7}=\frac{3+2}{7}=\frac{5}{7} \qquad \frac{11}{12}-\frac{5}{12}=\frac{11-5}{12}=\frac{6}{12}=\frac{1\cdot 6}{2\cdot 6}=\frac{1}{2}$$

$$-5\frac{1}{4}-\left(-2\frac{3}{4}\right)=-\frac{21}{4}-\left(-\frac{11}{4}\right)=-\frac{21}{4}+\frac{11}{4}=\frac{-21+11}{4}=\frac{-10}{4}=-\frac{5}{2} \text{ or } -2\frac{1}{2}$$

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Add and Subtract Rational Numbers

- **The sum or difference of two rational numbers with different denominators, we use the Least Common Multiple of their denominators to rewrite the rational numbers.**
- **The Least Common Multiple of their denominators is called the *Least Common Denominator* or *LCD*.**

$$\frac{3}{4}+\frac{1}{6}=\frac{3}{4}\cdot\frac{3}{3}+\frac{1}{6}\cdot\frac{2}{2}$$

We multiply the first rational number by 3/3 and the second one by 2/2 to obtain 12 in the denominator for each number.

$$=\frac{9}{12}+\frac{2}{12}$$

Notice, we have 12 in the denominator for each number.

$$=\frac{11}{12}$$

Add numerators and put this sum over the least common denominator.

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Exercise: Simplify using PEMDAS

$$\left(-\frac{1}{2}\right)^2-\left(\frac{7}{10}-\frac{8}{15}\right)^2(-18)$$

$$\frac{\frac{1}{2}-\frac{2}{3}}{\frac{3}{5}+\frac{1}{6}}$$ Prob 1.3.95

$$\frac{\frac{19}{5}}{3+\frac{1}{6}}-\frac{7}{8}\div\frac{3}{2}$$ Prob 1.3.99

1.4 The Irrational Numbers

1. Define the irrational numbers.
2. Simplify square roots.
3. Perform operations with square roots.
4. Rationalize the denominator.

The set of ***irrational numbers*** is the set of numbers whose decimal representations are neither terminating nor repeating.

$\pi \approx 3.1415926535897932384626433832795\ldots$

$\sqrt{2} \approx 1.414213562373095\ldots$

$\sqrt{27} \approx 5.196152422706632\ldots$

Square Roots

- The principal square root of a nonnegative number *n*, written $\sqrt{n}$, is the positive number that when multiplied by itself gives *n*.
- For example, $\sqrt{36}=6$ because 6 · 6 = 36.
- Notice that $\sqrt{36}$ is a rational number because 6 is a terminating decimal.
- ***Not all square roots are irrational.***
- For example, here are a few perfect squares:
  - $0 = 0^2$ $\sqrt{0}=0$
  - $1 = 1^2$ $\sqrt{1}=1$
  - $4 = 2^2$ $\sqrt{4}=2$
  - $9 = 3^2$ $\sqrt{9}=3$

The square root of a perfect square is a rational number

The Product Rule For Square Roots

- If *a* and *b* represent non-negative numbers

$$\sqrt{ab}=\sqrt{a}\cdot\sqrt{b} \quad \text{and} \quad \sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$$

- The square root of a product is the product of the square roots.
- Simplify, if possible:

$$\sqrt{75}=\sqrt{25\cdot 3}=\sqrt{25}\cdot\sqrt{3}=5\sqrt{3}$$

$$\sqrt{500}=\sqrt{100\cdot 5}=\sqrt{100}\cdot\sqrt{5}=10\sqrt{5}$$

Adding and Subtracting Square Roots

- The number that multiplies a square root is called the square root's ***coefficient***.
- Square roots with the same radicand can be added or subtracted by adding or subtracting their coefficients:

$$a\sqrt{c}+b\sqrt{c}=(a+b)\sqrt{c} \qquad a\sqrt{c}-b\sqrt{c}=(a-b)\sqrt{c}$$

Sum of coefficients times the common square root

Difference of coefficients times the common square root

$$7\sqrt{2}+5\sqrt{2}=(7+5)\sqrt{2}=12\sqrt{2}$$

$$2\sqrt{5}-6\sqrt{5}=(2-6)\sqrt{5}=-4\sqrt{5}$$

Product Rule Exercises

$$\sqrt{ab}=\sqrt{a}\cdot\sqrt{b} \quad \text{and} \quad \sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$$

$$\sqrt{2}\cdot\sqrt{5}=\sqrt{2\cdot 5}=\sqrt{10}$$

$$\sqrt{7}\cdot\sqrt{7}=\sqrt{49}=7$$

It is possible to multiply irrational numbers and obtain a rational number for the product.

$$\sqrt{6}\cdot\sqrt{12}=\sqrt{6\cdot 12}=\sqrt{72}=\sqrt{36\cdot 2}=\sqrt{36}\cdot\sqrt{2}=6\sqrt{2}$$

$3\sqrt{5}\cdot(2\sqrt{5}+3\sqrt{15})$ **Prob 1.4.31**

$12\sqrt{12}+3\sqrt{27}-4\sqrt{75}$ **Prob 1.4.39**

Dividing Square Roots = Quotient Rule

If *a* and *b* represent nonnegative real numbers and *b* ≠ 0, then

$$\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}} \quad \text{and} \quad \sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$$

The quotient of two square roots is the square root of the quotient.

$$\frac{\sqrt{90}}{\sqrt{2}}=\sqrt{\frac{90}{2}}=\sqrt{45}=\sqrt{9\cdot 5}=\sqrt{9}\cdot\sqrt{5}=3\sqrt{5}$$

$$\frac{\sqrt{75}}{\sqrt{3}}=\sqrt{\frac{75}{3}}=\sqrt{25}=5$$

Time is Relative

- **Planet of the Apes (1968)**
  - **Einstein's Special Relativity Equation**
    - R_a = Relative Age Astronaut
    - R_f = Relative Age Friend on Earth
    - v = Velocity
    - c = Speed of light

$$R_a=R_f\sqrt{1-\left(\frac{v}{c}\right)^2}$$