6. Financial Mathematics

1. Calculate simple interest.
2. Use the future value formula.
3. Use the compound interest formulas.
4. Calculate present value.
5. Determine the value of an annuity.
6. Determine regular annuity payments needed to achieve a financial goal.
7. Compute the monthly payment and interest costs for a mortgage.

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6.1: Simple Interest

- ***Interest*** is the dollar amount that we get paid for lending money or pay for borrowing money
- The amount of money that we deposit or borrow is called the ***principal***
- The amount of interest depends on the principal, the interest ***rate***, which is given as a percent, and the length of time for which the money is deposited.
- ***Simple interest*** involves interest calculated only on the principal.
- The interest rate r, is expressed as a decimal when calculating simple interest.

Interest = principal × rate × time

$$I = Prt$$

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Simple Interest Exercises

You deposit $2000 in a savings account at Hometown Bank, which has a rate of 6%. Find the interest at the end of the first year.

Solution: **To find the interest at the end of the first year, we use the simple interest formula. At the end of the first year, the interest is $120.**

$$I = Prt = 2000 \cdot 0.06 \cdot 1 = 120$$

A student took out a simple interest loan for $1800 for two years at a rate of 8% to purchase a used car. Find the interest on the loan.

Solution: **To find the interest of the loan, we use the simple interest formula.The interest on the loan is $288.**

$$I = Prt = 1800 \cdot 0.08 \cdot 2 = 288$$

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Future Value = Principal + Interest

- **The *future value*, *FV* or *A*, of *P* dollars principle at simple interest rate *r* (as a decimal) for *t* years is given by:**

$$FV = A = P + I = P + Prt = P(1 + rt)$$

  - ***P* is also known as the loan's *present value*.**
- **Example: A loan of $1060 has been made at 6.5% for three months. Find the loan's future value.**

$FV = P(1 + rt)$

$FV = 1060(1 + 0.065 \cdot 0.25)$

$FV \approx \$1077.23$

Rounded to the nearest cent, the loan's future value is $1077.23. Note the t = ¼ years because 3 months

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Determining a Simple Interest Rate

You borrow \$2500 from a friend and promise to pay back \$2655 in six months. What simple interest rate will you pay? What is the interest?

Solution: Substitute
FV = \$2655
P = \$2500
t = ½ or 0.5

$$FV = P(1+rt)$$
$$2655 = 2500(1+r\cdot 0.5)$$
$$2655 = 2500+1250r$$
$$155 = 1250r$$
$$\frac{155}{1250} = \frac{1250r}{1250}$$
$$r = 0.124 = 12.4\%$$

You will pay a simple interest rate of 12.4%.

$$\text{Interest Paid} = I = (FV) - P = 2655 - 2500 = \$155$$

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Compound Interest Once per Year

- You deposit \$2000 in a savings account at Hometown Bank, which has a rate of 6%.
  - What is the amount of money in the account after three years subject to compound interest?
  - Find the accumulated interest paid after three years?
- *Solution:*
  - Principal *P* is \$2000, *r* is 6% or 0.06, and *t* is 3. Substituting this into the formula, we get $FV = P(1 + r)^t = 2000(1 + 0.06)^3 = 2000(1.06)^3 \approx \$2382.03$
  - The amount in the account after 3 years is \$2382.03. So, we take the difference of this amount and the principal to obtain the interest amount. I = FV – P = \$2382.03 – \$2000 = \$382.03

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6.2: Compound Interest

- *Compound interest* is interest computed on the original principal as well as on any accumulated interest.
- To calculate the compound interest paid once a year we use

$$FV = P(1+r)^t$$

where
- *FV* is called the account's future value
- *P* is called its present value or principle
- *r* is the interest rate per year
- *t* is the number of years

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Compound Interest < 1 year

- To calculate the compound interest paid more than once a year the formula has variable n

$$FV = P\cdot\left(1+\frac{r}{n}\right)^{nt}$$

where
- *FV* is called the account's future value
- *P* is called its present value or principle
- *r* is the interest rate per year
- *t* is the number of years
- *n* is the number of times the interest is compounded per year

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Monthly Compound Interest Example

- You deposit $7500 in a savings account that has an interest rate of 6%. Interest is compounded monthly.
  - How much money will you have after five years?
  - What is the accumulated interest after five years?
- *Solution:*
  - Principal P is $7500, r is 6% or 0.06, t is 5, and n is 12 since interest is being compounded monthly.

$$FV = P\left(1+\frac{r}{n}\right)^{nt} = 7500\left(1+\frac{0.06}{12}\right)^{12\cdot 5} = 7500(1.005)^{60} \approx \$10,116.38$$

  - The amount in the account after 5 years is $10,116.38. So, we take the difference of this amount and the principal to obtain the interest amount.

$I = FV - P = \$\ 10,116.38 - \$7500 = \$2616.38$

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Continuous Compounding Interest

- *Continuous compounding* is advertised by some banks but there is not much advantage to the depositor except the equation is easier:

$$FV = Pe^{rt}$$

where
  - FV is called the account's future value
  - P is present value =
  - r is the interest rate per year
  - t is the number of years
  - e is Euler's Constant = 2.718281828459045…
- Example: Principal P is $7500, r is 6% or 0.06, and t is 5 years

$FV = 7500e^{0.06\cdot 5} = \$10,123.94$

  - Daily Compounding = $10,123.69
  - Monthly Compounding= $10,116.38

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Purchase of Manhattan

- In 1626 Peter Minuit convinced the Wappinger Indians to sell Manhattan Island for $24. If the native Americans had put the $24 into a bank account at a 5% interest rate by the year 2016 how much would they have in the account
  - If simple interest?

$FV = P(1 + rt) = 24(1 + 0.05 \cdot 390) = \$492$

  - If compounded monthly?

$$FV = P\left(1+\frac{r}{n}\right)^{nt} = 24\left(1+\frac{0.05}{12}\right)^{12\cdot 390} = \$6,782,023,367$$

  - On Calculator: `24*(1+0.05/12)^(12*390)=`

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Planning for Future with Compound Interest

- How much money should be deposited in an account today that earns 6% compounded monthly so that it will accumulate to $20,000 in five years?
- *Solution:* We use the present value formula, where A is $20,000, r is 6% or 0.06, n is 12, and t is 5 years.

$$FV = P\cdot\left(1+\frac{r}{n}\right)^{nt},\ P = \frac{FV}{\left(1+\frac{r}{n}\right)^{nt}} = \frac{20,000}{\left(1+\frac{0.06}{12}\right)^{12\cdot 5}}$$

  - On Calculator: `20000/(1+0.06/12)^(12*5)=`
  - Approximately $14,827.45 should be invested today in order to accumulate to $20,000 in 5 years.

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6.3: Annuities

- An *annuity* is a sequence of equal payments made at equal time periods.
  - The *value of an annuity* is the sum of all deposits plus all interest paid.
  - Essentially an annuity is a contractual periodic deposit saving plan with interest
  - Typically you make monthly deposits in an annuity assuming it will grow to a large amount over several decades

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Value of an Annuity

- If *Pmt* is the **periodic deposit** made at the end of each compounding period for an annuity that pays an annual interest rate *r* (in decimal form) compounded *n* times per year, the future value, *FV*, of the annuity after *t* years is

$$FV = \frac{Pmt\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\frac{r}{n}} \qquad Pmt = \frac{FV\left(\frac{r}{n}\right)}{\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}$$

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Determining the Value of an Annuity

- You deposit \$1000 into a savings plan at the end of each year for three years. The interest rate is 8% per year compounded annually.
  - What is the value of the annuity after three years?
  - What is the interest paid?
- *Solution:*

  The value of the annuity after three years is the sum of all deposits made plus all interest paid over three years.
  - year 1 = \$1000
  - year 2 = \$1000(1 + 0.08) + \$1000= \$1080 + \$1000 = \$2080
  - year 3 = \$2080(1 + 0.08) + \$1000 = \$3246.40
  - You made three payments of \$1000 each, depositing total of \$3000. The value of the annuity is \$3246.40, the interest is *I = FV – P*3* = \$3246.40 – \$3000 = \$246.40

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Value of an Annuity Using Equation

- You deposit \$1000 into a savings plan at the end of each year for three years. The interest rate is 8% per year compounded annually.
  - What is the value of the annuity after three years?
- *Solution:*

$$FV = \frac{Pmt\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\frac{r}{n}} = \frac{1000\left[\left(1+\frac{0.08}{1}\right)^{1\cdot 3}-1\right]}{\frac{0.08}{1}}$$

- On Calculator: 1000 * ((1 + 0.08)^3 – 1) / 0.08 =
- You made three payments of \$1000 each, depositing total of \$3000.

  The value of the annuity is \$3246.40, the interest is

  I = FV – Pmt •n•t = \$3246.40 - \$3000 = \$246.40

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Individual Retirement Account = IRA Annuity

- To save for retirement, you decide to deposit \$1000 into an IRA at the end of each year for the next 30 years. How much will you have from the IRA after 30 years if:
  - the annuity interest rate = 10%?

$$FV(r=10\%)=\frac{1000\left[(1+0.10)^{30}-1\right]}{0.10}=\$\,164{,}494$$

  - the annuity interest rate = 3%?

$$FV(r=3\%)=\frac{1000\left[(1+0.03)^{30}-1\right]}{0.03}=\$\,47{,}575$$

  - the annuity interest rate = 1%?

$$FV(r=1\%)=\frac{1000\left[(1+0.01)^{30}-1\right]}{0.01}=\$\,34{,}785$$

- How much did you deposit into the account over 30 years?
  Deposit Amount = Pmt • t = \$1000 • 30 = \$30,000

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Annuity with Monthly Payments

- At age 25, to save for retirement, you decide to deposit \$200 into an account at the end of each month at an interest rate of 5% per year compounded monthly:
  - How much will the IRA when you retire at age 65?

$$FV=\frac{Pmt\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\frac{r}{n}}=\frac{200\left[\left(1+\frac{0.05}{12}\right)^{12\cdot 40}-1\right]}{\left(\frac{0.05}{12}\right)}=\$305{,}204$$

  - Calculator: 200*((1 + 0.05/12)^(12*40) – 1)/(0.05/12)=
  - What is the total interest earned during this period?

$$Deposit=Pmt\cdot n\cdot t=200\cdot 12\cdot 40=\$\,96{,}000$$

$$Interest=FV-Deposit=\$\,305{,}204-\$\,96{,}000=\$\,209{,}204$$

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Future Planning for an Annuity

- You would like to have \$20,000 for a down payment on a home in five years by making regular, end-of-the-month deposits in an annuity that pays 6% compounded monthly.
  - How much should you deposit each month?

$$Pmt=\frac{FV\left(\frac{r}{n}\right)}{\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}=\frac{20{,}000\left(\frac{0.06}{12}\right)}{\left[\left(1+\frac{0.06}{12}\right)^{12\cdot 5}-1\right]}=\$\,287$$

  - How much of the \$20,000 down payment comes from deposits?

$$Deposit=Pmt\cdot n\cdot t=\$\,287\cdot 12\cdot 5=\$17{,}220$$

  - How much comes from interest?

$$Interest=FV-Deposit=\$\,20{,}000-\$17{,}220=\$\,2780$$

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6.4: Mortgages

- A *mortgage* is a long-term loan for the purpose of buying a home.
- The *down payment* is the portion of the sale price of the home that the buyer initially pays to the seller.
- The *amount borrowed = B* is the difference between the *sale price* and the *down payment*.
- *Fixed-rate mortgages* have the same monthly payment during the entire time of the loan.
- *Variable-rate mortgages* also known as *adjustable-rate mortgages* (ARMs), have payment amounts that change from time to time depending on changes in the interest rate.

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Computations for Home Buyers

- Most lending institutions require the buyer to pay one or more *points* at the time of closing—that is, the time at which the mortgage begins.
  - A point is a one-time charge that equals 1% of the loan amount.
  - For example, two points means that the buyer must pay 2% of the loan amount at closing.
- A document, called the *Truth-in-Lending Disclosure Statement*, shows the buyer the APR for the mortgage.
- In addition, lending institutions can require that part of the monthly payment be deposited into an *escrow account*, an account used by the lender to pay real estate taxes and insurance.

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Mortgage Payment Formula

- Loan Payment Formula for Fixed Installment Loans
  - *Pmt* = payment amount made periodically
  - *B* = Borrowed amount
  - *n* = times per year periodic payment made
  - t = years of loan amount
  - r = annual interest rate

$$B = \text{Price} - \text{DownPayment}$$

$$Fee = Points \cdot 0.01 \cdot B$$

$$\text{Deposited} = Pmt \cdot n \cdot t$$

$$\text{Interest} = \text{Deposited} - B$$

$$Pmt = \frac{B\left(\frac{r}{n}\right)}{\left[1 - \left(1 + \frac{r}{n}\right)^{-nt}\right]}$$

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Monthly Payment and Interest for Mortgage

- The price of a home is $195,000. The bank requires a 10% down payment and two points at the time of closing. The cost of the home is financed with a 30-year fixed rate mortgage at 7.5%
  - Find the required down payment.
  - Find the amount to be borrowed for mortgage.
  - How much must be paid for the two points at closing?
  - Find the monthly payment (excluding escrowed taxes and insurance).
  - Find the total interest paid over 30 years.

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Computing Down Payment and Points

- The required down payment is 10% of Home Price = $195,000 or

  DownPayment = 0.10 × $195,000 = $19,500
- The amount borrowed for mortgage is the difference between the price of the home and the down payment.

  B = $195,000 – $19,500 = $175,500
- To find the cost of two points on a mortgage of $175,500, find 2% of $175,500.

  0.02 × $175,500 = $3510

  The down payment ($19,500) is paid to the seller and the cost of two points ($3510) is paid to the Bankster.

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Computing the Monthly Payment

- We need to find the monthly mortgage payment for \$175,500 at 7.5% for 30 years. We use the loan payment formula for installment loans.

$$Pmt = \frac{\$175{,}500\left(\frac{0.075}{12}\right)}{\left[1-\left(1+\frac{0.075}{12}\right)^{-12\cdot 30}\right]} = \$1227$$

The monthly mortgage payment for principal and interest is approximately \$1227.

- Total Payments

$$\text{TotalPayments} = Pmt\cdot n\cdot t = \$1227\cdot 12\cdot 30 = \$441{,}720$$

- Total Interest Paid

$$\text{I} = \text{Deposited} - B = \$441{,}720 - \$175{,}500 = \$266{,}220$$

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Comparing Car Loans

- You decide to take a \$20,000 loan for a new car. You can select one of the following loans, each requiring regular monthly payments:
  - Installment Loan A: 3-year loan at 7%.
  - Installment Loan B: 5-year loan at 9%.
- Find the monthly payments and the total interest for Loan A.
- Find the monthly payments and the total interest for Loan B.
- Compare the monthly payments and total interest for the two loans.

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Comparing Car Loans

- You decide to take a \$20,000 loan for a new car. You can select one of the following loans:
  - Installment Loan A: 3-year loan at 7%.

$$Pmt_A = \frac{20{,}000\left(\frac{0.07}{12}\right)}{\left[1-\left(1+\frac{0.07}{12}\right)^{-12\cdot 3}\right]} = \$618$$

$$Pmt_A\cdot n\cdot t = \$618\cdot 12\cdot 3 = \$22{,}248$$

$$I = \$22{,}248 - 20{,}000 = \$2{,}248$$

  - Installment Loan B: 5-year loan at 9%.

$$Pmt_B = \frac{20{,}000\left(\frac{0.09}{12}\right)}{\left[1-\left(1+\frac{0.09}{12}\right)^{-12\cdot 5}\right]} = \$415$$

$$Pmt_B\cdot n\cdot t = \$415\cdot 12\cdot 5 = \$24{,}900$$

$$I = \$24{,}900 - 20{,}000 = \$4{,}900$$

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