CASE STUDY: Cell Towers allocation with Probability of Exceedance and VaR functions (Pr_pen, Var_risk, linear)

Background

This Case study is motivated by a "toy" problem published in the Analytics INFORMS magazine [1]. Here is the description of this problem.

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Cell towers

As the head of analytics for a cell phone company, you have been asked to optimize the location of cell towers in a new area where your company wants to provide service. The new area is made up of several neighborhoods. Each neighborhood is represented by a black house icon in the accompanying image (Figure 1).

A cell tower can be placed on any square (including squares with or without a neighborhood). Once placed, a cell tower provides service to nine squares (the eight adjacent squares surrounding it and the one it sits on). For example, if you placed a cell tower in B2, it would provide service to A1, B1, C1, A2, B2, C2, A3, B3 and C3.

The company recognizes that it may not be worthwhile to cover all neighborhoods, so it has instructed you that it needs to cover only 70 percent of the neighborhoods in the new area. Each cell tower is expensive to construct and maintain, so it is in your best interest to only use the minimum number of cell towers.

Question:

What is the minimum number of cell towers needed to provide service to at least 70 percent of the neighborhoods?

Send your answer to puzzlor@gmail.com by June 15. The winner, chosen randomly from correct answers, will receive a $25 Amazon Gift Card. Past questions and answers can be found at puzzlor.com. ORMS

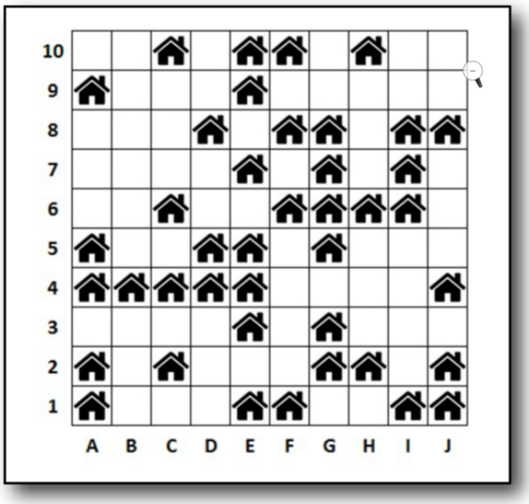

Figure 1: How many cell towers are needed?

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The considered problem is a special case of the "Partial Set Covering problem" [2, 3]. There is a set of houses. Every cell tower covers a subset of houses. We need to choose a subset of cell towers covering at least $\alpha * 100\%$ of houses, where $0 < \alpha < 1$.

Here we, also, consider some extension of the problem formulation: a reliable covering of houses. Reliability is understood as the redundancy of coverage, i.e., every house is covered by 2 or 3 Cell Towers.

This Case study suggests some compact analytic formulation of the optimization problem using a scenario based optimization framework. We use Probability of Exceedance (POE) and Value-at-Risk (VaR) functions to control percentage of uncovered houses. In our formulation $POE * 100\%$ is a percentage of uncovered houses. This value is bounded by constraint $POE \leq (1 - \alpha)$. The function VaR is equal to one minus inverse of POE function. The constraint on probability can be equivalently presented by a constraint on VaR function.

Further we consider the problem of maximizing the coverage of houses (i.e., the number of covered houses) with the constraint on the number of towers. This problem is actually a so called "Maximal Covering Location problem" discussed in [4, 5]. POE function is used in objective of the optimization problem in this formulation.

We solved the presented "toy" problem: covered 41 houses by 1 and 2 Cell Towers. Also, to demonstrate that the suggested problem statement provides a powerful approach, we generated and solved a much bigger instance (300x300) with randomly distributed 45,152 houses.

We have used PSG code with VANGRB solver which runs in the background the GUROBI MIP problem formulation. If you are an academic, you can run this problem using the PSG (free academic) license and GUROBI (free academic) license.

References

1. Toczek, J. Thinking Analytically. Cell towers. Analytics INFORMS, May/June (2016). http://www.analytics-magazine.org/may-june-2016/1618-thinking-analytically-cell-towers
2. Toregas C., Swain R., ReVelle C. and Bergman L. (1971): The location of emergency service facilities. Operations Research, 19(6), 1363-1373.
3. Moshkov M. Ju., Piliszczuk M. and Zielosko B. (2008): Partial Covers, Reducts and Decision Rules. Studies in Computational Intelligence, Springer-Verlag, V.145, 7-49.
4. Church, R., ReVelle, C. S. (1974): The Maximal Covering Location Problem. Papers in Regional Science 32 (1), 101–118.
5. Daskin, M. S. (1983): A Maximum Expected Covering Location Model: Formulation, Properties and Heuristic Solution. Transportation Science 17 (1), 48-70.

Notations

H = set of houses;

m = number of houses, $m = |H|$;

T = set of locations for placing Cell Towers;

n = number of locations for Cell Towers, $n = |T|$;

B = upper limit on a number of Cell Towers in Maximal Covering Location problem;

$T_j \subseteq T$ = subset of locations which can cover house j, where $j = 1, \dots, m$;

s_i = boolean variable indicating if Cell Tower is placed (or not placed) in location, i, where $i = 1, \dots, n$, and $s_i \in \{1,0\}$;

$\vec{s} = (s_1, \dots, s_n)$ = vector of variables s_i ;

$L_j(\vec{s}) = -\sum_{i \in T_j} s_i$ = scenario j of the Loss function $L(\vec{s})$, $j = 1, \dots, m$. It may have values 0,-1,-2,-3, … .

If $L_j(\vec{s}) \leq -1$ house j is covered at least by one Cell Tower.

If $L_j(\vec{s}) \leq -2$ house j is covered at least by two Cell Towers, and so on.

If $L_j(\vec{s}) = 0$ house j is not covered.

$p_j = 1/m$ = probability of scenarios, $= 1, \dots, m$. Probability is used to count fraction of covered (uncovered) houses.

α = minimal fraction of houses to be covered;

$VaR_{\alpha}(L(\vec{s}))$ = Value-at-Risk function with confidence level α defined on the Loss function $L(\vec{s})$. Let scenarios $\{L_{j_1}(\vec{s}), \dots, L_{j_m}(\vec{s})\}$ are ordered in ascending order. Value of $VaR_{\alpha}(L(\vec{s}))$ = value of scenario number $\lceil \alpha * n \rceil$, where $\lceil . \rceil$ is round up (ceiling) sign;

So, if $VaR_{\alpha}\big(L(\vec{s})\big) \leq -1$, then at least $\alpha * 100\%$ percent of scenarios have value less or equal to -1 and $\alpha * 100\%$ percent of houses is covered;

$Pr_w(L(\vec{s}))$ = Probability of Exceedance function with threshold, w, defined on the Loss function $L(\vec{s})$. Value of $Pr_w(L(\vec{s}))$ = fraction of scenarios having values greater than w. So, if $Pr_{-1}(L(\vec{s})) \leq (1-\alpha)$, then $\alpha * 100\%$ of scenarios have value less or equal to -1. Therefore, $\alpha * 100\%$ of houses is covered.

Optimization with Probability in constraint

Minimizing the number of placed Cell Towers

$$\min \textstyle\sum_{i=1}^{m} s_i \qquad \text{(CS.1)}$$

subject to

Constraint on covering (*threshold* $w = -1 \; or -2$)

$$Pr_w\big(L(\vec{s})\big) \leq 1-\alpha, \qquad \text{(CS.2)}$$

Binary variables

$$s_i \in \{1,0\}, \quad i = 1, \dots, n. \qquad \text{(CS.3)}$$

Optimization with VaR in constraint

Minimizing a number of placed Cell Towers

$$\min \textstyle\sum_{i=1}^{m} s_i \qquad \text{(CS.4)}$$

subject to

Constraint on covering (*upper bound* $u = -1 \; or -2$)

$$VaR_{\alpha}\big(L(\vec{s})\big) \leq u, \qquad \text{(CS.5)}$$

Binary variables

$$s_i \in \{1,0\}, \quad i = 1, \dots, n. \qquad \text{(CS.6)}$$

Optimization with Probability in objective

Minimizing the fraction of uncovered houses (*threshold* $w = -1 \; or -2$)

$$\min Pr_w\big(L(\vec{s})\big) \qquad \text{(CS.7)}$$

subject to

Constraint on the number of Cell Towers

$$\textstyle\sum_{i=1}^{m} s_i \leq B, \qquad \text{(CS.8)}$$

Binary variables

$$s_i \in \{1,0\}, \quad i = 1, \dots, n. \tag{CS.9}$$