Question 2:

$$X_t = 0.01 + 0.2\, X_{t-2} + a_t$$

where a_t is a Gaussian white noise series with mean zero and variance $0{,}02$.

ⓐ What are the mean and variance of the return series X_t?

$$E(X_t) = E(0.01 + 0.2\, X_{t-2} + a_t)$$
$$= 0.01 + E(0.2\, X_{t-2}) + E(a_t)$$
$$E(a_t) = 0$$
$$\text{Stationary} \longrightarrow E(X_t) = E(X_{t-2}) = \mu$$
$$\mu = 0.01 + 0.2\,\mu$$
$$0{,}8\,\mu = 0{,}01$$

Mean $\Rightarrow$ $\boxed{\mu = 0{,}0125}$

$$X_t = a_0 + a_1 X_{t-1} + a_2 X_{t-2} + \varepsilon_t$$
$$\mu = a_0 + \mu a_1 + \mu a_2$$
$$\mu = \frac{a_0}{1 - a_1 - a_2}$$

$$X_t = \mu(1 - a_1 - a_2) + a_1 X_{t-1} + a_2 X_{t-2} + a_t$$

$$\ast \to X_t - \mu = a_1(X_{t-1} - \mu) - a_2(X_{t-2} - \mu) + \varepsilon_t$$

$$Var(X_t) = a_1^2\, Var(X_{t-1}) - a_2^2\, Var(X_{t-2}) + \sigma_\varepsilon^2$$

Stationary Cond:

$$Var(X_t) = Var(X_{t-1}) = Var(X_{t-2})$$
$$Var(X_t) - a_1^2\, Var(X_{t-1}) - a_2^2\, Var(X_{t-2}) = \sigma_\varepsilon^2$$
$$Var(X_t)(1 - a_1^2 - a_2^2) = \sigma_\varepsilon^2 \qquad \boxed{Var(X_t) = \frac{\sigma_\varepsilon^2}{1 - a_1^2 - a_2^2}}$$

$$Var(X_t) = \frac{0.02}{1 - 0 - (0.2)^2} = \underline{\underline{0.02083}}$$

(b) Compute the lag-1 and lag-2 autocorrelation of X_t

Multiplying Eq. (★) by $(X_{t-l} - \mu)$

$$(X_{t-l} - \mu) \cdot (X_t - \mu) = a_1 (X_{t-1} - \mu) \cdot (X_{t-l} - \mu) + a_2 \cdot (X_{t-2} - \mu) \cdot (X_{t-l} - \mu)$$
$$+ \varepsilon_t \cdot (X_{t-l} - \mu)$$

Taking Expectation:

$$E((X_{t-l} - \mu) \cdot (X_t - \mu)) = a_1 \cdot E((X_{t-1} - \mu) \cdot (X_{t-l} - \mu))$$
$$+ a_2 \cdot E((X_{t-2} - \mu)(X_{t-l} - \mu))$$
$$+ E(\varepsilon_t \cdot (X_{t-l} - \mu))$$

$$Cov(X_t, X_{t-l}) = a_1 \cdot Cov(X_{t-1}, X_{t-l}) + a_2 \cdot Cov(X_{t-2}, X_{t-l})$$
$$+ E(\varepsilon_t (X_{t-l} - \mu))$$

$$\gamma_l = a_1 \gamma_{l-1} + a_2 \cdot \gamma_{l-2} + 0 \rightarrow r_l = \frac{\gamma_l}{\gamma_0} = a_1 r_{l-1}$$

$$r_l = a_1 r_{l-1} + a_2 r_{l-2} \quad l \geq 0$$

for lag-1:

$$r_1 = a_1 \cdot r_0 + a_2 r_{-1} \qquad r_0 = 1 \qquad a_1 = 0$$
$$r_1 = a_1 + a_2 r_{-1} \longrightarrow r_1 = r_{-1}$$
$$r_1 - a_2 r_1 = a_1$$
$$r_1 = \frac{a_1}{1 - a_2} = \frac{0}{1 - 0.2} = 0 \qquad \boxed{r_1 = 0}$$

for lag-2:

$$r_2 = a_1 \cdot r_1 + a_2 \cdot r_0$$
$$r_2 = 0 + 0.2 \cdot 1 \qquad \boxed{r_2 = 0.2}$$

(c) Assume that $X_{100} = -0.01$ and $X_{99} = 0.02$.
Compute the 1- and 2-step-ahead forecast of the return series at the forecast origin $t=100$

$$X_t = a_0 + a_1 X_{t-1} + a_2 X_{t-2} + \varepsilon_t$$

$t = 101$

$$X_{101} = a_0 + a_1 \cdot X_{100} + a_2 X_{99} + \varepsilon_t$$

$$X_{101} = 0.01 + 0 \cdot (-0.01) + 0.2 \cdot (0.02) + \varepsilon_t$$

$$X_{101} = \underline{\underline{0.014 + \varepsilon_t}}$$

$$X_{102} = a_0 + a_1 X_{101} + a_2 X_{100} + \varepsilon_t$$

$$X_{102} = 0.01 + 0 \cdot (0.014 + \varepsilon_t) + 0.2 \cdot (-0.01) + \varepsilon_t$$

$$X_{102} = 8 \cdot 10^{-3} + \varepsilon_t$$