

# Category Theory with Strings

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## 1 Introduction

This is a supplemental document for introductory books of category theory ([1], [2], [3], [8]) using *string diagrams*. Don't trust my poor mathematics. Any correction is welcome at [github.com/okomok/strcat](https://github.com/okomok/strcat).

## 2 Preliminaries

### 2.1 Lambda Expressions

**Definition 2.1 (Lambda Expression)** Following famous symbols like  $\Sigma$ , define  $\Lambda_x y$  as an anonymous function  $x \mapsto y$ . We casually call any form of anonymous functions a *lambda expression*.

**Definition 2.2 (Lambda-Tasted Form)** Given a function  $\Gamma$  whose domain is a set of functions, you can choose a short form of  $\Gamma(\Lambda_x y)$  from the following *lambda-tasted* forms

1.  $\Gamma_x y$
2.  $\Gamma x.y$
3.  $(\Gamma x)(y)$
4.  $\Gamma xy$

**Definition 2.3 (Placeholder Expression)** For simple lambda expressions, you may use *placeholders* for example,

$$?+1 := \Lambda_n n+1$$

Placeholder symbols can vary:  $?$ ,  $-$ ,  $1$ , etc.

## 2.2 Universality

**Definition 2.4 (Predicate)** We call a Boolean-valued function a *predicate*.

**Definition 2.5 (Universal Quantifier)** Given a predicate  $P$ , we define a Boolean value  $\forall P$  as “anything satisfies  $P$ ”.

**Definition 2.6 (Existential Quantifier)** Given a predicate  $P$ , we define a Boolean value  $\exists P$  as “something satisfies  $P$ ”.

**Definition 2.7 (Uniqueness)** Given a predicate  $P$ , a predicate  $!P$  is defined by

$$!P(a) := P(a) \wedge (\forall a')(P(a') \implies a = a')$$

using the third lambda-tasted form, meaning that “ $a$  is the unique thing that satisfies  $P$ ”.

**Definition 2.8 (Unique Existential Quantifier)** The *unique existential quantifier*  $\exists!$  is defined as  $\exists \circ !$ , where  $\circ$  is the function composition. Spelling out the detail,

$$(\exists!a)(P(a)) = (\exists a)(!P(a))$$

meaning that “there exists a unique thing that satisfies  $P$ ”.

**Remark 2.9** On the other hand,  $(\exists a)(!P(a) \wedge Q(a))$  states “there exists a unique  $a$  that satisfies  $P$ . Furthermore, the  $a$  satisfies  $Q$ ”.

**Definition 2.10** Given a predicate  $P$  and a set  $X$ ,

$$\begin{aligned} (\forall x \in X)(P(x)) &:= (\forall x)(x \in X \implies P(x)) \\ (\exists x \in X)(P(x)) &:= (\exists x)(x \in X \wedge P(x)) \end{aligned}$$

**Definition 2.11 (Universality)** Given a binary predicate  $P$ , we boldly call a statement of the form

$$(\forall x \in X)(\exists!y \in Y)(P(x, y))$$

the *universality* of  $P$ .

**Proposition 2.12 (Functional Universality)** Given a binary predicate  $P$ ,

$$\begin{aligned} &(\forall x \in X)(\exists!y \in Y)(P(x, y)) \\ \iff &(\exists f : X \rightarrow Y)(\forall x \in X)(\forall y \in Y)(P(x, y) \iff y = f(x)) \end{aligned}$$

PROOF. ( $\implies$ ) by the axiom of choice. ( $\impliedby$ ) immediate.  $\square$

**Definition 2.13 (Functional Bijectivity)** Given a function  $g : Y \rightarrow X$ , the statement

$$(\exists f : X \rightarrow Y)(\forall x \in X)(\forall y \in Y)(x = g(y) \iff y = f(x))$$

is known as the *bijectivity* of  $f$  and  $g$ . This is a special case of universality where  $P(x, y)$  is  $x = g(y)$ .

## 2.3 Families

Syntax of function applications is world-standard:

$$f(x)$$

but sometimes you might want cuter syntax like that

$$\langle x \rangle$$

**Definition 2.14 (Family)** A *family declaration* is a way to provide a function with arbitrary application syntax. Its usage is clear from an example

$$(\langle x \rangle \in Y)_{x \in X}$$

We call it a *family* of  $Y$ . Furthermore, a function body can be placed like that

$$(\langle x \rangle := x^2 \in Y)_{x \in X}$$

**Example 2.15** The most-used family declaration is the subscript style  $(a_i)_i$ . You can view a tuple  $(a_1, a_2, \dots, a_n)$  to be an abbreviation of  $(a_i)_{i \in \{1, 2, \dots, n\}}$ . Subscripts are often omitted.

Families can do more.

**Definition 2.16 (Dependent Function)** Let  $F$  a set-valued function. A family

$$(f(x) \in F(x))_{x \in X}$$

defines a function

$$f : X \rightarrow \bigcup_{x \in X} F(x)$$

such that

$$(\forall x \in X)(f(x) \in F(x))$$

We call such  $f$  a *dependent function*, for the  $F(x)$  depends on  $x$ . In case  $F$  is a constant function,  $f$  is a normal function  $X \rightarrow Y$ .

## 2.4 Coherence

It is sure you write

$$3 + 1 + 2$$

rather than

$$(3 + (0 + 1)) + 2$$

because you know the arithmetic laws

$$\begin{aligned} x + (y + z) &= (x + y) + z \\ 0 + x &= x = x + 0 \end{aligned}$$

disambiguate unparenthesized expressions. Informally laws to introduce natural syntax are called *coherence conditions* or shortly *coherence*.

### 3 Categories

#### 3.1 The Definition

**Definition 3.1 (Category)** A *category*  $\mathcal{C}$  consists of

1. *objects*: a class  $\text{Ob}(\mathcal{C})$
2. *morphisms* or *hom-sets*: a family of sets  $(\mathcal{C}(A, B))_{A, B \in \text{Ob}(\mathcal{C})}$
3. *compositions*: a family of functions

$$(\circ : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, C))_{A, B, C \in \text{Ob}(\mathcal{C})}$$

4. *identities* or *units*: a family of morphisms

$$(\text{id}_A \in \mathcal{C}(A, A))_{A \in \text{Ob}(\mathcal{C})}$$

satisfying the following coherence conditions

1. *associativity*: for any  $f \in \mathcal{C}(A, B)$ ,  $g \in \mathcal{C}(B, C)$ , and  $h \in \mathcal{C}(C, D)$ ,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

2. *unitality*: for any  $f \in \mathcal{C}(A, B)$ ,

$$\text{id}_B \circ f = f = f \circ \text{id}_A$$

A morphism  $f \in \mathcal{C}(A, B)$  is often denoted as  $f : A \rightarrow B$ .

#### 3.2 String Diagrams

From now on, we will introduce *string diagrams* to complement (or hopefully replace) commutative diagrams.

Given a category  $\mathcal{C}$ , an object  $A$  is depicted as an optionally-tagged string

$$\begin{array}{c} \mathcal{C} \\ | \\ A \end{array}$$

A morphism  $f : A \rightarrow B$  is depicted as a node

$$\begin{array}{c} B \\ | \\ \boxed{f} \\ | \\ A \end{array}$$

The composition joins two strings:

$$\begin{array}{c} C \\ | \\ \boxed{g} \\ | \\ B \\ | \\ \boxed{f} \\ | \\ A \end{array} := g \circ f$$

Identity morphisms are indistinguishable from objects:

$$\begin{array}{c} | \\ A \end{array} := \text{id}_A$$

Check these diagrams create no ambiguity thanks to the coherence.

**Definition 3.2 (Isomorphism)** We call a pair of morphisms

$$\begin{array}{l} f : A \rightarrow B \\ g : B \rightarrow A \end{array}$$

an *isomorphism* or shortly *iso* provided that the *invertibility*

$$\begin{array}{c} A \\ | \\ \boxed{g} \\ | \\ B \\ | \\ \boxed{f} \\ | \\ A \end{array} = \begin{array}{c} | \\ A \end{array} \quad \text{and} \quad \begin{array}{c} B \\ | \\ \boxed{f} \\ | \\ A \\ | \\ \boxed{g} \\ | \\ B \end{array} = \begin{array}{c} | \\ B \end{array}$$

is satisfied. A morphism that is a part of an isomorphism is called *invertible*.

**Remark 3.3** Most people call each morphism of this pair an isomorphism.

**Definition 3.4 (Functional Box)** Given categories  $\mathcal{C}$  and  $\mathcal{C}'$ , a function

$$h : \mathcal{C}(A, B) \rightarrow \mathcal{C}'(A', B')$$

is depicted as a *functional box*

$$\begin{array}{c} B' \\ | \\ \boxed{\begin{array}{c} B \\ | \\ \boxed{?} \\ | \\ A \end{array}} \\ | \\ A' \end{array} \quad h$$

**Definition 3.5 (Opposite Category)** Given a category  $\mathcal{C}$  and a morphism



you can build a category with strings upside down:



which is denoted as  $\mathcal{C}^{\text{op}}$ , the *opposite category* of  $\mathcal{C}$ .

**Definition 3.6 (Discrete Category)** A category  $\mathcal{C}$  such that

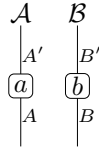
$$\begin{aligned} A = B &\implies \mathcal{C}(A, B) = \{\text{id}_A\} \\ A \neq B &\implies \mathcal{C}(A, B) = \emptyset \end{aligned}$$

is called a *discrete category*. Any set can be represented as a discrete category.

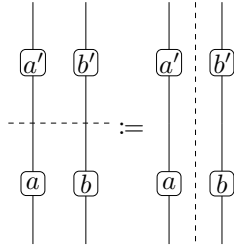
**Definition 3.7 (Product Category)** Given two categories  $\mathcal{A}$  and  $\mathcal{B}$ , the *product category*

$$\mathcal{A} \times \mathcal{B}$$

is depicted as parallel strings



A composition, which joins parallel strings, is defined by



An identity is trivially



By these definitions,

$$\begin{array}{c} | \\ \boxed{a} \\ | \end{array} = \begin{array}{c} | \\ \boxed{b} \\ | \end{array}$$

## 4 Functors

### 4.1 The Definition

**Definition 4.1 (Functor)** A *functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  consists of

1. *domain*: a category  $\mathcal{C}$
2. *codomain*: a category  $\mathcal{D}$
3. a family of objects  $(FA \in \text{Ob}(\mathcal{D}))_{A \in \text{Ob}(\mathcal{C})}$
4. families of morphisms

$$\left( (F(f) \in \mathcal{D}(FA, FB))_{f \in \mathcal{C}(A, B)} \right)_{A, B \in \text{Ob}(\mathcal{C})}$$

satisfying the *functoriality*:

1. *composition-compatibility*: for any  $f \in \mathcal{C}(A, B)$  and  $g \in \mathcal{C}(B, C)$ ,

$$F(g \circ f) = F(g) \circ F(f)$$

2. *unit-compatibility*: for any  $A \in \text{Ob}(\mathcal{C})$ ,

$$F(\text{id}_A) = \text{id}_{FA}$$

**Definition 4.2 (Infrafunctor)** An *infrafunctor* is a functor without the requirement of functoriality.

### 4.2 Functorial Tubes

In string diagrams, a functor is represented as a tube

$$\left[ \begin{array}{c} B \\ \boxed{f} \\ A \end{array} \right]_F := \left[ \begin{array}{c} B \\ \boxed{f} \\ A \end{array} \right]_F$$

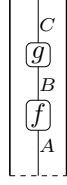
$\begin{array}{c} FB \\ | \\ \boxed{f} \\ | \\ FA \end{array}$

Placeholders make it simple:

$$\left[ \begin{array}{c} \\ \boxed{?} \\ \end{array} \right]_F$$



One can check the functoriality ensures any tube like



be unambiguous. “Join then tube” is the same as “tube then join”.

**Proposition 4.3** Any functor preserves isomorphisms meaning that

$$\left( \begin{array}{c|c} B & A \\ \hline \boxed{f} & \boxed{g} \\ \hline A & B \end{array} \right) : \text{iso} \implies \left( \begin{array}{c|c} B & A \\ \hline \boxed{f} & \boxed{g} \\ \hline A & B \end{array} \right) : \text{iso}$$

PROOF. Immediate by functoriality, which inheres in tubes.  $\square$

**Definition 4.4 (Composite Functor)** For any two functors

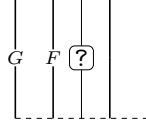
$$F : \mathcal{A} \rightarrow \mathcal{B}$$

$$G : \mathcal{B} \rightarrow \mathcal{C}$$

the *composite functor* of  $F$  and  $G$

$$G \circ F : \mathcal{A} \rightarrow \mathcal{C}$$

is defined as



**Definition 4.5 (Identity Functor)** Given a category  $\mathcal{C}$ , the *identity functor* on  $\mathcal{C}$

$$\text{Id}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$$

is defined by

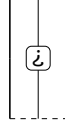
$$\left( \begin{array}{c|c} & \\ \hline \text{Id} & \boxed{?} \\ \hline & \end{array} \right) := \boxed{?}$$

**Definition 4.6 (Contravariant Functor)** A functor whose domain is an opposite category

$$F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$$

is called *contravariant*, while a normal functor is called *covariant*.

A contravariant functor is depicted as



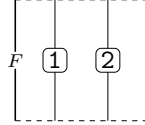
**Definition 4.7 (Variant)** Given a statement regarding functors, you can obtain a corresponding one regarding contravariant functors and vice versa. We call such a statement the *variant* of the original one.

**Definition 4.8 (Binary Functor)** A functor whose domain is a product category

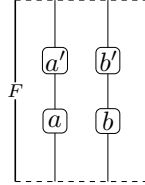
$$F : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$$

is called a *binary functor* or *bifunctor*.

With numbered placeholders, it is depicted as



Spelling out the definition of functoriality, one can check a diagram like

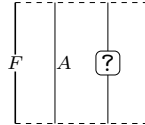


is unambiguous.

**Definition 4.9 (Partial Application)** Given a binary functor  $F : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ , a *partially applied* functor

$$\Lambda_B F(A, B) : \mathcal{B} \rightarrow \mathcal{C} \text{ or shortly } F(A, ?) : \mathcal{B} \rightarrow \mathcal{C}$$

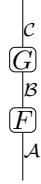
is defined as



The definition of  $F(?, B)$  is an exercise.

**Definition 4.10 (Small Category)** A category  $\mathcal{C}$  is called *small* when  $\text{Ob}(\mathcal{C})$  is a set.

**Definition 4.11 (Category of Small Categories)** The *category of small categories*  $\mathbf{Cat}$  is the category whose objects are all small categories and whose morphisms are functors:



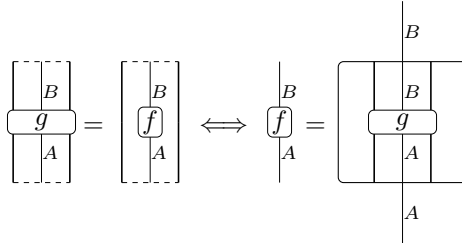
where composite functors join strings.

**Definition 4.12 (Full and Faithful Functor)** A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called *full and faithful* if for each object  $A$  and  $B$  in  $\mathcal{C}$ , the family

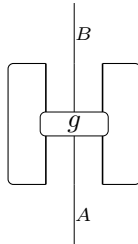
$$(F(f) : FA \rightarrow FB)_{f:A \rightarrow B}$$

is bijective.

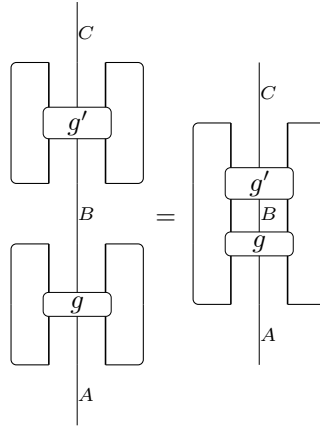
In other words, there is a functional box such that



One can make the box better-looking

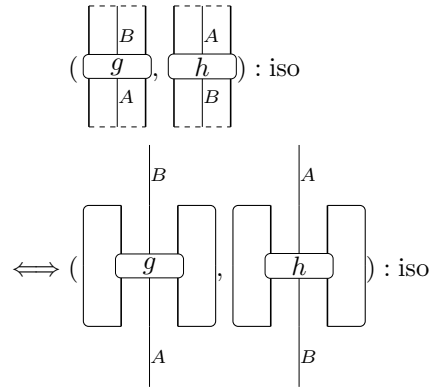


**Proposition 4.13** This box has a functoriality-like property:



Combined with proposition 4.3,

**Proposition 4.14**



## 5 Natural Transformations

### 5.1 The Definition

**Definition 5.1 (Naturality)** Given two infrafunctors

$$F, G : \mathcal{C} \rightarrow \mathcal{D}$$

a family of morphisms

$$(\tau_A \in \mathcal{D}(FA, GA))_{A \in \text{Ob}(\mathcal{C})}$$

is called *natural* when for any  $f \in \mathcal{C}(A, B)$ ,

$$\tau_B \circ F(f) = G(f) \circ \tau_A$$

In case parentheses are cumbersome, you can say “ $\tau_A$  is *natural in A*”.

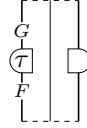
**Definition 5.2 (Natural Transformation)** Furthermore, in particular case  $F$  and  $G$  are functorial (then they are functors),  $\tau$  is denoted as a *natural transformation*  $\tau : F \rightarrow G$ .

**Remark 5.3** In this document, naturality is explicitly defined to be orthogonal to functoriality.

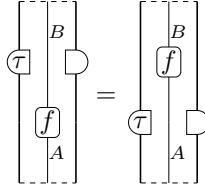
**Proposition 5.4** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be an infrafunctor. Recall it is by definition a family of functions  $(F_{A,B} : \mathcal{C}(A, B) \rightarrow \mathcal{D}(FA, FB))_{A,B}$ . Then  $F_{A,B}$  is natural in  $A$  or  $B$  if and only if  $F$  is composition-compatible.

### 5.2 Natural Connectors

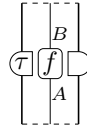
In string diagrams, a natural transformation is a connector of two tubes



because the naturality states a node can travel between tubes



This inspires us to assign



**Definition 5.5 (Vertical Composition)** Given three functors

$$F, G, H : \mathcal{C} \rightarrow \mathcal{D}$$

and two natural transformations

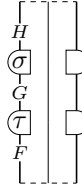
$$\tau : F \rightarrow G$$

$$\sigma : G \rightarrow H$$

the *vertical composition* of  $\tau$  and  $\sigma$

$$\sigma \circ \tau : F \rightarrow H$$

is defined as



**Definition 5.6 (Horizontal Composition)** Given four functors

$$F, G : \mathcal{A} \rightarrow \mathcal{B}$$

$$H, K : \mathcal{B} \rightarrow \mathcal{C}$$

and two natural transformations

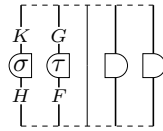
$$\tau : F \rightarrow G$$

$$\sigma : H \rightarrow K$$

the *horizontal composition* of  $\tau$  and  $\sigma$

$$\sigma \tau : H \circ F \rightarrow K \circ G$$

is defined by



You can easily check the naturality. Travel by car ferry.

**Definition 5.7 (Identity Natural Transformation)** Given a functor

$$F : \mathcal{C} \rightarrow \mathcal{D}$$

the *identity natural transformation*

$$\text{id}_F : F \rightarrow F$$

is defined as

$$\boxed{\text{id}} \boxed{\phantom{\text{id}}} := \boxed{\phantom{\text{id}}} \boxed{\phantom{\text{id}}}$$

**Definition 5.8 (Whiskering)** A *whiskering* is a horizontal composition with identity natural transformations:

$$\boxed{\tau} \boxed{\phantom{\tau}}, \boxed{\sigma} \boxed{\phantom{\sigma}}$$

**Definition 5.9 (Natural Isomorphism)** We call a pair of natural transformations

$$\begin{aligned} \tau : F &\rightarrow G \\ \sigma : G &\rightarrow F \end{aligned}$$

a *natural isomorphism* or shortly *natural iso* provided that the *invertibility*

$$\begin{aligned} \boxed{\sigma} \boxed{\tau} &= \boxed{\phantom{\sigma}} \boxed{\phantom{\tau}} \\ \boxed{\tau} \boxed{\sigma} &= \boxed{\phantom{\tau}} \boxed{\phantom{\sigma}} \end{aligned}$$

is satisfied. The same symbol is often used for them.

**Remark 5.10** Most people call each morphism of this pair a natural isomorphism.

**Proposition 5.11** For any natural transformation  $\tau$ ,

$$(\forall A)(\tau_A : \text{invertible})$$

is enough to build the other natural  $\sigma$ .

**Definition 5.12 (Functor Category)** Given a small category  $\mathcal{C}$  and a category  $\mathcal{D}$ , the functor category  $[\mathcal{C}, \mathcal{D}]$  is a category whose objects are functors from  $\mathcal{C}$  to  $\mathcal{D}$  and whose morphisms are natural transformations:



where the vertical composition joins the strings.

**Definition 5.13** For the later use, define a lambda-tasted form for a set of natural transformations:

$$\text{Nat}_A(FA, GA) := \text{Nat}(F, G) := [\mathcal{C}, \mathcal{D}](F, G)$$



## 6 Category of Sets

### 6.1 The Definition

**Definition 6.1 (Category of Sets)** The *category of sets* **Set** is a category whose objects are sets and whose morphisms are functions:

$$\begin{array}{c} | \\ Z \\ \boxed{g} \\ | \\ Y \\ \boxed{f} \\ | \\ X \end{array}$$

where strings are joined by the function composition.

A category is essentially one-dimensional so far: the vertical composition only. Here we introduce the horizontal composition for functions.

**Definition 6.2 (Monoidal Category of Sets)** Parallel strings are defined by

$$\begin{array}{c} | \\ X \end{array} \begin{array}{c} | \\ X' \end{array} := \begin{array}{c} | \\ X \times X' \end{array}$$

The *horizontal composition* of functions is defined by

$$\begin{array}{c} | \\ Y \\ \boxed{f} \\ | \\ X \end{array} \begin{array}{c} | \\ Y' \\ \boxed{f'} \\ | \\ X' \end{array} := \Lambda_{x,x'}(f(x), f'(x'))$$

Strings for the singleton set  $\{*\}$  is omitted so that an element of a set is represented as

$$\begin{array}{c} | \\ X \\ \boxed{x} \end{array}$$

One can check any string diagram built upon these definitions is unambiguous due to the trivial bijections

$$\begin{aligned} X \times (X' \times X'') &\cong (X \times X') \times X'' \\ X \times \{*\} &\cong X \end{aligned}$$

Informally such two-dimensional categories are called *monoidal*.

### 6.2 Hom-Set Bands

Given a category  $\mathcal{C}$ , a special string, a *band*, is introduced for hom-sets:

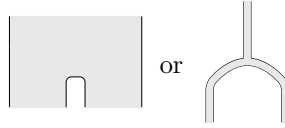
$$\begin{array}{c} \boxed{B \quad A} \\ | \end{array} := \begin{array}{c} | \\ \mathcal{C}(A, B) \end{array}$$

A space-saving form is depicted as



**Remark 6.3** Note that the order of objects is flipped. This is resulting from an unfortunate convention that one write “ $b = h(a)$ ” but not “ $h : B \leftarrow A$ ”. By the way, “ $B^A$ ” is fine.

The composition of morphisms can be depicted as



Identity morphisms can be depicted as



As an exercise, write down the associativity and unitality using these diagrams.

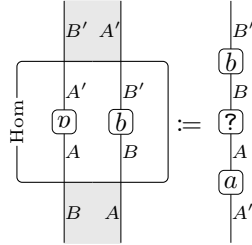
**Definition 6.4 (Hom-Functor)** Hom-sets can be extended to the *Hom-functor*

$$\Lambda_{A,B}\mathcal{C}(A, B) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set} \text{ or shortly}$$

$$\mathcal{C}(-, +) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set} \text{ or shortly}$$

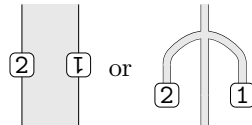
$$\text{Hom}_{\mathcal{C}} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}$$

defined by



where the world in the box is product category  $\mathcal{C}^{\text{op}} \times \mathcal{C}$ .

This definition inspires us to depict hom-functors as

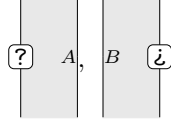


that looks topologically equivalent.

**Definition 6.5 (Unary Hom-Functor)** Due to definition 4.9,

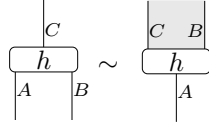
$$\begin{aligned}\mathcal{C}(A, +) &: \mathcal{C} \rightarrow \mathbf{Set} \\ \mathcal{C}(-, B) &: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}\end{aligned}$$

are respectively depicted as



**Definition 6.6 (Currying)** In particular case  $\mathcal{C} = \mathbf{Set}$ , there exists the *curry bijection*

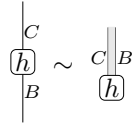
$$\begin{aligned}\mathbf{Set}(A \times B, C) &\cong \mathbf{Set}(A, \mathbf{Set}(B, C)) \\ h &\mapsto (a \mapsto b \mapsto h(a, b)) \\ ((a, b) \mapsto h(a)(b)) &\leftarrow h\end{aligned}$$



We don't distinguish these two diagrams, for the naturality of this bijection ensures “move the right-side leg up and down” works correct.

**Proposition 6.7** The curry bijection is natural in all three variables.

**Definition 6.8 (Naming)** In case  $A$  is the singleton set, which is omitted in diagrams, a currying



is trivial. We call it a *naming*.

**Proposition 6.9** Currying *preserves naturality*, meaning that given a family of functions  $(f_X : FX \times B \rightarrow GX)_X$  with infrafunctors  $F$  and  $G$ ,  $f_X$  is natural in  $X$  if and only if  $\text{curry}(f_X)$  is. So does uncurrying.

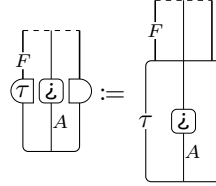
**Remark 6.10** In general, natural bijections have similar properties so that you don't bother with proof of naturality ([6]).

## 7 The Yoneda Lemma

**Definition 7.1** Given a functor  $F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  and an object  $A$  in  $\mathcal{C}$ , a natural transformation of the form

$$(\tau_X : \mathcal{C}(X, A) \rightarrow FX)_X$$

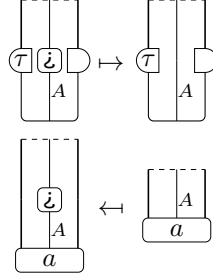
can be depicted as



owing to the naturality.

**Definition 7.2 (Yoneda Bijection)** The *Yoneda bijection* is defined by

$$\text{Nat}_X(\mathcal{C}(X, A), FX) \cong FA$$

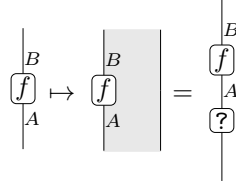


**Lemma 7.3 (Yoneda Lemma)** The Yoneda bijection is actually bijective and natural in  $F$  and  $A$ .

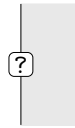
PROOF. Now the proof is on my soul trivial!  $\square$

**Definition 7.4 (Yoneda Embedding)** The *Yoneda embedding* is defined by

$$\Lambda_A \Lambda_X \mathcal{C}(X, A) : \mathcal{C} \rightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$$



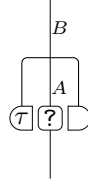
using the diagram of hom functors. In short,



**Definition 7.5** A natural transformation of the form

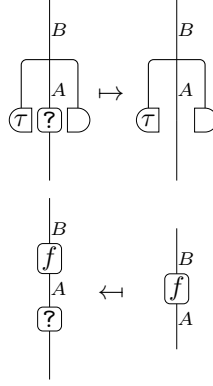
$$(\tau_X : \mathcal{C}(X, A) \rightarrow \mathcal{C}(X, B))_X$$

can be depicted as



**Definition 7.6 (Yoneda Embedding Bijection)** In special case  $F := \mathcal{C}(-, B)$ , the Yoneda bijection is expanded to

$$\text{Nat}_X(\mathcal{C}(X, A), \mathcal{C}(X, B)) \cong \mathcal{C}(A, B)$$



The second mapping is the Yoneda embedding so that it is full and faithful. Combined with proposition 4.14,

**Proposition 7.7 (Yoneda Principle)**

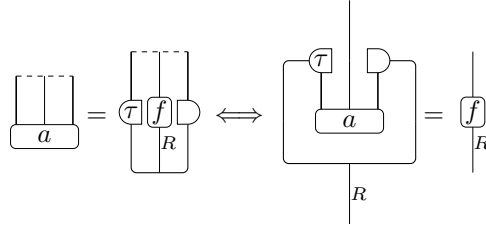
$$\begin{aligned} & \left( \begin{array}{c} B \\ \boxed{A} \\ \tau \quad ? \end{array} , \begin{array}{c} A \\ \boxed{B} \\ \sigma \quad ? \end{array} \right) : \text{isomorphism} \\ \iff & \left( \begin{array}{c} B \\ \boxed{A} \\ \tau \end{array} , \begin{array}{c} A \\ \boxed{B} \\ \sigma \end{array} \right) : \text{isomorphism} \end{aligned}$$

## 8 Representations

**Definition 8.1 (Representation)** Given a functor  $H : \mathcal{C} \rightarrow \mathbf{Set}$ , a *representation* of  $H$  is a pair of

1. an object  $R$  in  $\mathcal{C}$
2. a natural bijection  $(\tau_X : HX \cong \mathcal{C}(R, X))_X$

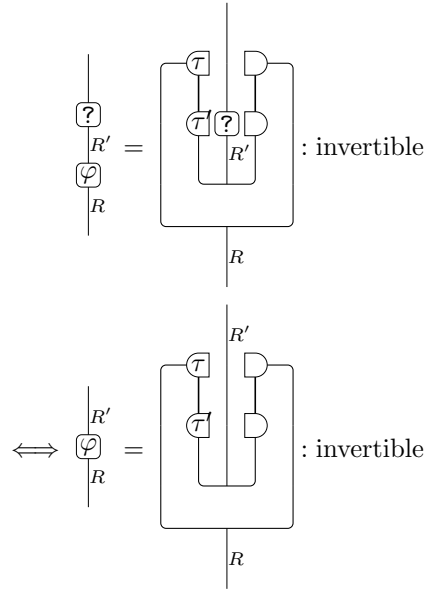
This bijectivity can be expressed using the weird boxes



thanks to the naturality. The following proposition allows us to call it *the* representation of  $H$ , denoted as  $\text{rep}H$ .

**Proposition 8.2 (Uniqueness of Representations)** Representations are unique up to unique isomorphism.

PROOF. Let  $(R', \tau')$  be another representation. By the variant of proposition 7.7,

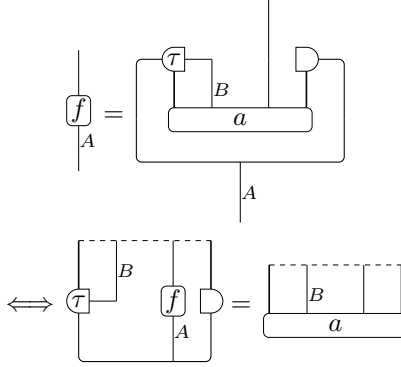


□

**Definition 8.3** Given a functor  $H : \mathcal{B}^{\text{op}} \times \mathcal{A} \rightarrow \mathbf{Set}$ , a natural bijection of the form

$$(\tau_X : H(B, X) \cong \mathcal{A}(A, X))_X$$

can be expressed by



**Proposition 8.4 (Parameterized Representations)** Let  $H : \mathcal{B}^{\text{op}} \times \mathcal{A} \rightarrow \mathbf{Set}$  be a functor. Given a family of objects  $(SB)_B$  and a family of representations

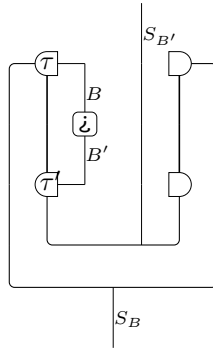
$$((\tau_X^B : H(B, X) \cong \mathcal{A}(SB, X))_X)_B$$

there exists a unique family

$$(S(f) \in \mathcal{A}(SB, SB'))_{f \in \mathcal{B}(B, B')}$$

such that  $\tau$  is natural in  $B$ . Furthermore,  $S$  is functorial.

PROOF. Define  $S$  as



□

## 9 Limits

**Definition 9.1 (Cone)** Given a functor  $F : \mathcal{A} \rightarrow \mathcal{B}$ , a cone of  $F$  consists of

1. an object  $B$  in  $\mathcal{B}$
2. a natural transformation  $(v_X : B \rightarrow FX)_X$

**Definition 9.2 (Conicality)** We may explicitly call naturality of cones *conicality*, which can be expressed as

like a magical box any morphism can appear from.

**Remark 9.3** Vertical and horizontal composition preserve conicality, a special case of naturality.

**Definition 9.4 (Limit)** Given a functor  $F : \mathcal{A} \rightarrow \mathcal{B}$ , a limit of  $F$  is a pair of

1. an object in  $\mathcal{B}$  denoted as  $\lim F$
2. a natural bijection  $(\mathcal{B}(B, \lim F) \cong \text{Nat}_X(B, FX))_B$

**Definition 9.5 (Limiting Cone)** The limit bijectivity, thanks to its naturality, can be expressed as

where  $\boxed{\lim}$  is a cone called a *limiting cone* of  $F$ .

The following proposition justifies the notation  $\lim F$ , *the limit of  $F$* .

**Proposition 9.6** Limits are unique up to isomorphism.

**PROOF.** Immediate by proposition 8.2, because a limit is nothing but a contravariant representation

$$\text{rep}_B \text{Nat}_X(B, FX)$$

□



**Proposition 9.7** A limiting cone is *monic* meaning that

PROOF. Immediate by the limit bijectivity.  $\square$

**Definition 9.8 (Product)** In particular case the domain of a functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is discrete, the limit of  $F$  is called the *product* of  $F$  denoted as  $\prod F$ .

**Definition 9.9 (Projection)** Spelling out the product bijectivity,

where  $\boxed{\pi}$  is called the *projection* of  $F$ .

**Remark 9.10** Conicality has no concern here, because any family of the form

$$(v_X : B \rightarrow FX)_{X \in \text{Ob}(\mathcal{A})}$$

is always natural in case  $\mathcal{A}$  is discrete.

**Example 9.11** In case  $F$  is a functor  $X \rightarrow \mathcal{S}et$  with a set  $X$  (as a discrete category), the product of  $F$  is a set of dependent functions

$$\prod_x F(x) \cong \{f \mid (f(x) \in F(x))_x\}$$

**Definition 9.12 (Dual)** Given a statement containing string diagrams, by flipping the diagrams upside down, a corresponding statement is obtained. It is called the *dual* of the original one.

**Definition 9.13 (Coproduct)** A *coproduct* is a structure obtained from that product bijectivity flipped.

**Remark 9.14** Informally the dual makes a codomain opposite, while the variant does for a domain.

**Definition 9.15 (Preservation of Limits)** Given a functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  and a limiting cone of  $F$

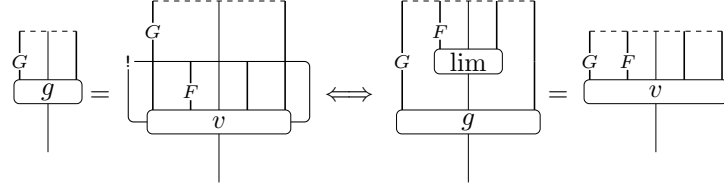
$$(\lim_X : \lim F \rightarrow FX)_X$$

a functor  $G : \mathcal{B} \rightarrow \mathcal{C}$  *preserves limits* of  $F$  when

$$(G(\lim_X) : G\lim F \rightarrow GFX)_X$$

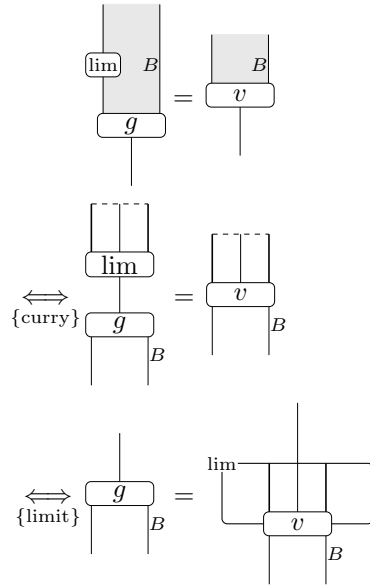
is a limiting cone of  $G \circ F$ .

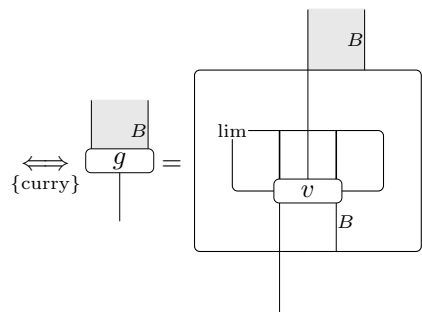
In diagrams,  $G$  is such that there exists some box  $!$  satisfying



**Proposition 9.16 (HFPL)** Hom-functors preserve limits, meaning that given a functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  and an object  $B$  in  $\mathcal{B}$ , the covariant hom-functor  $\mathcal{B}(B, +) : \mathcal{B} \rightarrow \mathbf{Set}$  preserves limits of  $F$ .

PROOF.





□

## 10 Adjunctions

**Definition 10.1 (Adjunction)** Given two categories  $\mathcal{C}$  and  $\mathcal{D}$ , an *adjunction*

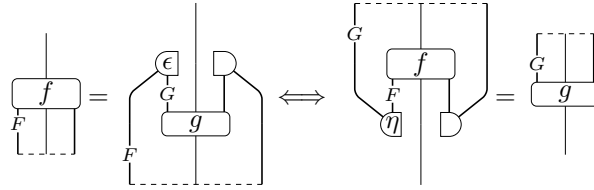
$$F \dashv G$$

consists of

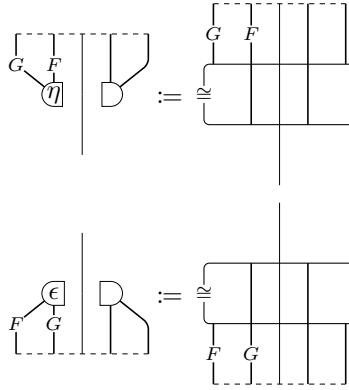
1. *left adjoint*: a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$
2. *right adjoint*: a functor  $G : \mathcal{D} \rightarrow \mathcal{C}$
3. *adjunct*: a natural bijection

$$(\mathcal{D}(FC, D) \cong \mathcal{C}(C, GD))_{C,D}$$

A nice consequence is that this bijectivity needs no boxes, expressed by natural transformations only.



where



called respectively the *unit* and *counit*.

**Proposition 10.2** Given a functor  $G : \mathcal{D} \rightarrow \mathcal{C}$ , a family of natural bijections

$$((\mathcal{C}(C, GD) \cong \mathcal{D}(F_c, D))_D)_C$$

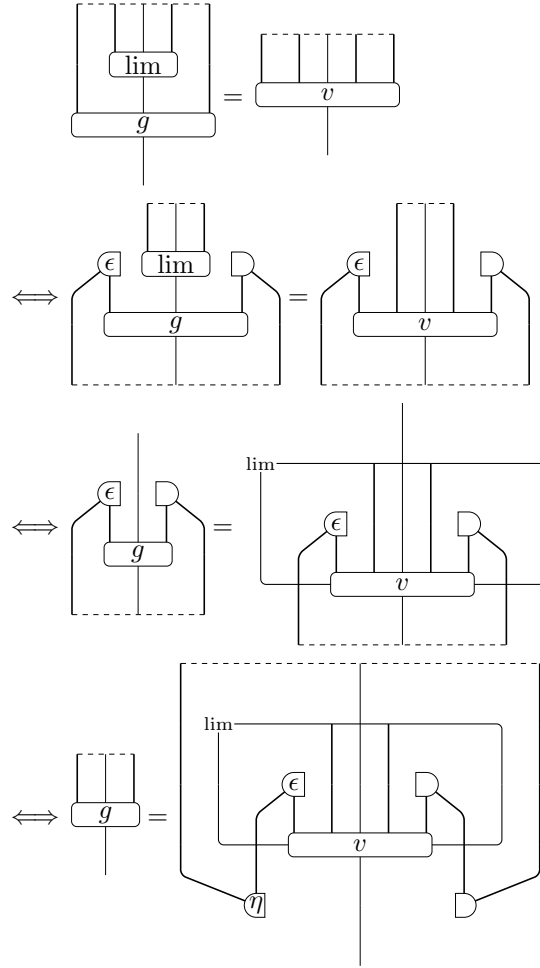
is enough to construct the adjunction  $F \dashv G$ .

PROOF. Immediate by proposition 8.4 with  $H(C, D) := \mathcal{C}(C, GD)$ .  $\square$

**Proposition 10.3 (RAPL)** Right adjoints preserve limits, meaning that given an adjunction  $F \dashv (G : \mathcal{D} \rightarrow \mathcal{C})$  and a functor  $T : \mathcal{B} \rightarrow \mathcal{D}$ ,

$$\begin{aligned} & (\lim_X : \lim T \rightarrow TX)_X : \text{limiting cone} \\ \implies & (G(\lim_X) : G\lim T \rightarrow GTX)_X : \text{limiting cone} \end{aligned}$$

PROOF.



□

## 11 Monads

### 11.1 The Definition

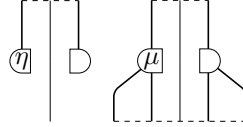
**Definition 11.1 (Monad)** Given a category  $\mathcal{C}$ , a *monad* on  $\mathcal{C}$  consists of

1. a functor  $T : \mathcal{C} \rightarrow \mathcal{C}$
2. *unit*: a natural transformation  $\eta : \text{Id}_T \rightarrow T$
3. *multiplication*: a natural transformation  $\mu : T \circ T \rightarrow T$

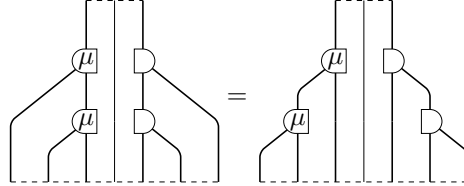
satisfying the coherence conditions

1. *associativity*:  $\mu \circ T\mu = \mu \circ \mu T$
2. *unitality*:  $\mu \circ T\eta = \text{Id}_T = \mu \circ \eta T$

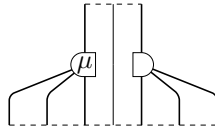
A unit and multiplication are depicted respectively as



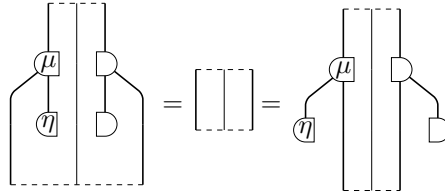
The associativity is depicted as



This inspires us to assign



The unitality is



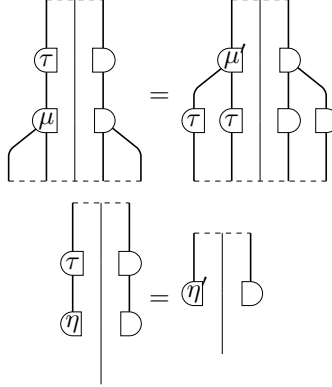
**Definition 11.2 (Monad Morphism)** Given a category  $\mathcal{C}$ , a *monad morphism* consists of

1. *domain*: a monad  $(T, \eta, \mu)$  on  $\mathcal{C}$
2. *codomain*: a monad  $(T', \eta', \mu')$  on  $\mathcal{C}$
3. a natural transformation  $\tau : T \rightarrow T'$

satisfying the coherence conditions

1. *multiplication-compatibility*:  $\tau \circ \mu = \mu' \circ \tau \tau$
2. *unit-compatibility*:  $\tau \circ \eta = \eta'$

The coherence is depicted as

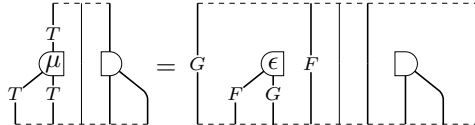


**Definition 11.3 (Category of Monads)** Given a category  $\mathcal{C}$ , the *category of monads*  $\mathbf{Mnd}(\mathcal{C})$  is a category whose objects are monads and whose morphisms are monad morphisms.

**Definition 11.4 (Monad-Associated Adjunction)** Given a monad  $(T, \eta, \mu)$ , we call an adjunction  $F \dashv G$  *T-associated* when

1.  $T = G \circ F$
2.  $\mu = G\epsilon F$

This condition can be depicted as

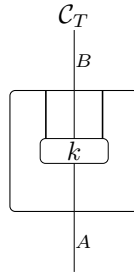


## 11.2 Kleisli Categories

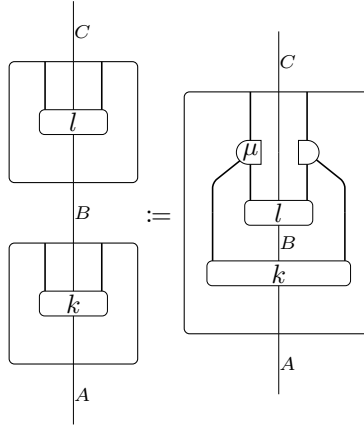
**Definition 11.5 (Kleisli Category)** Given a monad  $(T, \eta, \mu)$  on  $\mathcal{C}$ , the *Kleisli category* of  $T$ , denoted as  $\mathcal{C}_T$ , is a category consisting of

1.  $\text{Ob}(\mathcal{C}_T) := \text{Ob}(\mathcal{C})$
2.  $\mathcal{C}_T(A, B) := \mathcal{C}(A, TB)$
3.  $l \circ k := \mu \circ T(l) \circ k$
4.  $\text{id}_A := \eta_A$

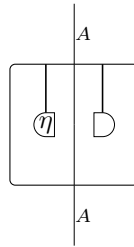
In diagrams, a morphism in  $\mathcal{C}_T$  is depicted as a *Kleisli box*



The composition is defined by

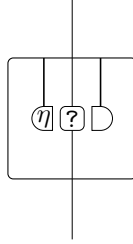


Identity morphisms are defined by

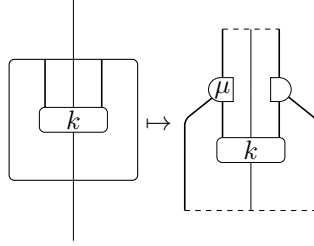




**Definition 11.6 (Kleisli Adjunction)** Define a functor  $L : \mathcal{C} \rightarrow \mathcal{C}_T$  as



$K : \mathcal{C}_T \rightarrow \mathcal{C}$  as



then they constitute the *Kleisli adjunction*  $L \dashv K$  whose adjunct is the Kleisli boxing. This adjunction is  $T$ -associated.

### 11.3 EM Categories

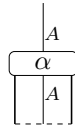
**Definition 11.7 (Monad Algebra)** Given a monad  $(T, \eta, \mu)$  on  $\mathcal{C}$ , a *monad algebra*, denoted as  $T$ -algebra, consists of

1. an object  $A \in \mathcal{C}$
2. a morphism  $\alpha : TA \rightarrow A$

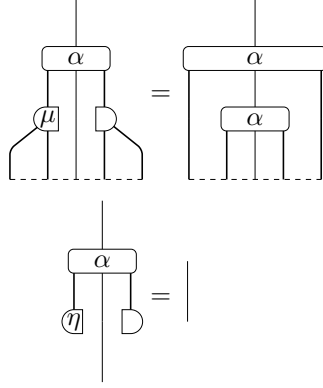
satisfying the coherence

1. *associativity*:  $\alpha \circ \mu = \alpha \circ T(\alpha)$
2. *unitality*:  $\alpha \circ \eta = \text{id}$

A  $T$ -algebra is depicted as



The coherence can be depicted as

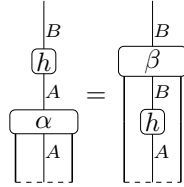


**Definition 11.8 (EM Category)** Given a monad  $(T, \eta, \mu)$ , the *Eilenberg-Moore (EM) category* of  $T$ , denoted as  $\mathcal{C}^T$ , is a category whose objects are  $T$ -algebras and whose morphisms are those of the form  $h : A \rightarrow B$  such that

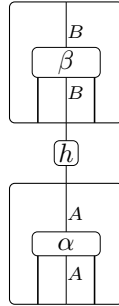
$$h \circ \alpha = \beta \circ T(h)$$

where  $(A, \alpha)$  and  $(B, \beta)$  are  $T$ -algebras.

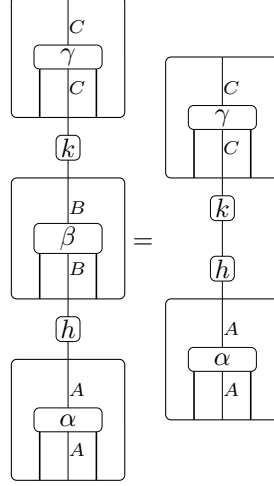
This condition is depicted as



A morphism in  $\mathcal{C}^T$  is by compromise depicted as

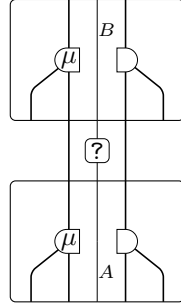


Boxes are objects. The composition can be depicted as

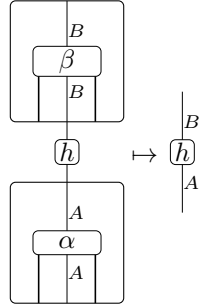


A diagram for identity morphisms is left as an exercise.

**Definition 11.9 (EM Adjunction)** Given a monad  $(T, \eta, \mu)$  on  $\mathcal{C}$ , define a functor  $M : \mathcal{C} \rightarrow \mathcal{C}^T$  as

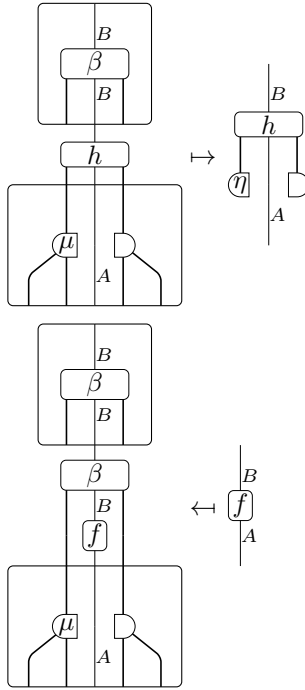


a functor  $U : \mathcal{C}^T \rightarrow \mathcal{C}$  as



They constitute the *EM adjunction*  $M \dashv U$  whose adjunct is defined by

$$\mathcal{C}^T(MA, (B, \beta)) \cong \mathcal{C}(A, U(B, \beta))$$



This adjunction is  $T$ -associated.

## References

- [1] Jiri Adamek, Horst Herrlich, and George E Strecker. *Abstract and Concrete Categories: The Joy of Cats (Dover Books on Mathematics)*. Dover Publications, 8 2009.
- [2] Steve Awodey. *Category Theory (Oxford Logic Guides)*. Oxford University Press, 2 edition, 8 2010.
- [3] Francis Borceux. *Handbook of Categorical Algebra: Volume 1, Basic Category Theory (Encyclopedia of Mathematics and its Applications)*. Cambridge University Press, 1 edition, 4 2008.
- [4] John Bourke and Micah Blake McCurdy. Frobenius morphisms of bicategories, 2009.
- [5] Ralf Hinze. Kan extensions for program optimisation or: Art and dan explain an old trick. In *Mathematics of Program Construction*, pages 324–362. Springer, 2012.
- [6] Max Kelly. *Basic Concepts of Enriched Category Theory (London Mathematical Society Lecture Note Series)*. Cambridge University Press, 4 1982.
- [7] Aleks Kissinger. Pictures of processes: automated graph rewriting for monoidal categories and applications to quantum computing. *arXiv preprint arXiv:1203.0202*, 2012.
- [8] Saunders Mac Lane. *Categories for the Working Mathematician (Graduate Texts in Mathematics)*. Springer, 2nd ed. 1978. softcover reprint of the original 2nd ed. 1978 edition, 11 2010.
- [9] Dan Marsden. Category theory using string diagrams. *CoRR*, abs/1401.7220, 2014.
- [10] Micah Blake McCurdy. Strings and stripes, graphical calculus for monoidal functors and monads, 2010.
- [11] Paul-André Mellies. Functorial boxes in string diagrams. In *Computer science logic*, pages 1–30. Springer, 2006.
- [12] Peter Selinger. A survey of graphical languages for monoidal categories. In *New structures for physics*, pages 289–355. Springer, 2010.
- [13] Daniele Turi. Category theory lecture notes. *Laboratory for Foundations of Computer Science, University of Edinburgh*, 2001.