Category Theory with Strings

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1 Introduction

This is a supplemental document for introductory books of category theory ([1], [2], [3], [8]) using *string diagrams*. Don't trust my poor mathematics. Any correction is welcome at github.com/okomok/strcat.

2 Preliminaries

2.1 Lambda Expressions

Definition 2.1 (Lambda Expression) Following famous symbols like Σ, define $\Lambda_x y$ as an anonymous function $x \mapsto y$. We casually call any form of anonymous functions a *lambda expression*.

Definition 2.2 (Lambda-Tasted Form) Given a function Γ whose domain is a set of functions, you can choose a short form of $\Gamma(\Lambda_x y)$ from the following *lambda-tasted* forms

1. $\Gamma_x y$
2. $\Gamma x.y$
3. $(\Gamma x)(y)$
4. Γxy

Definition 2.3 (Placeholder Expression) For simple lambda expressions, you may use *placeholders* for example,

$$\texttt{?}{+}1 \coloneqq \Lambda_n n{+}1$$

Placeholder symbols can vary: `?`, `-`, `1`, etc.

2.2 Universality

Definition 2.4 (Predicate) We call a Boolean-valued function a *predicate*.

Definition 2.5 (Universal Quantifier) Given a predicate P, we define a Boolean value $\forall P$ as "anything satisfies P".

Definition 2.6 (Existential Quantifier) Given a predicate P, we define a Boolean value $\exists P$ as "something satisfies P".

Definition 2.7 (Uniqueness) Given a predicate P, a predicate $!P$ is defined by

$$!P(a) \coloneqq P(a) \land (\forall a')(P(a') \implies a = a')$$

using the third lambda-tasted form, meaning that "a is the unique thing that satisfies P".

Definition 2.8 (Unique Existential Quantifier) The *unique existential quantifier* $\exists!$ is defined as $\exists \circ !$, where $\circ$ is the function composition. Spelling out the detail,

$$(\exists! a)(P(a)) = (\exists a)(!P(a))$$

meaning that "there exists a unique thing that satisfies P".

Remark 2.9 On the other hand, $(\exists a)(!P(a) \land Q(a))$ states "there exists a unique a that satisfies P. Furthermore, the a satisfies Q".

Definition 2.10 Given a predicate P and a set X,

$$\begin{aligned}(\forall x \in X)(P(x)) &\coloneqq (\forall x)(x \in X \implies P(x))\\ (\exists x \in X)(P(x)) &\coloneqq (\exists x)(x \in X \land P(x))\end{aligned}$$

Definition 2.11 (Universality) Given a binary predicate P, we boldly call a statement of the form

$$(\forall x \in X)(\exists! y \in Y)(P(x, y))$$

the *universality* of P.

Proposition 2.12 (Functional Universality) Given a binary predicate P,

$$\begin{aligned}&(\forall x \in X)(\exists! y \in Y)(P(x, y))\\ \iff &(\exists f : X \to Y)(\forall x \in X)(\forall y \in Y)(P(x, y) \iff y = f(x))\end{aligned}$$

Proof. ($\implies$) by the axiom of choice. ($\impliedby$) immediate. ∎

Definition 2.13 (Functional Bijectivity) Given a function $g : Y \to X$, the statement

$$(\exists f : X \to Y)(\forall x \in X)(\forall y \in Y)(x = g(y) \iff y = f(x))$$

is known as the *bijectivity* of f and g. This is a special case of universality where $P(x, y)$ is $x = g(y)$.

2.3 Families

Syntax of function applications is world-standard:

$$f(x)$$

but sometimes you might want cuter syntax like that

$$\boxed{x}$$

Definition 2.14 (Family) A *family declaration* is a way to provide a function with arbitrary application syntax. Its usage is clear from an example

$$(\boxed{x} \in Y)_{x \in X}$$

We call it a *family* of Y. Furthermore, a function body can be placed like that

$$(\boxed{x} := x^2 \in Y)_{x \in X}$$

Example 2.15 The most-used family declaration is the subscript style $(a_i)_i$. You can view a tuple $(a_1, a_2, \ldots, a_n)$ to be an abbreviation of $(a_i)_{i \in \{1,2,\ldots,n\}}$. Subscripts are often omitted.

Families can do more.

Definition 2.16 (Dependent Function) Let F a set-valued function. A family

$$(f(x) \in F(x))_{x \in X}$$

defines a function

$$f : X \to \bigcup_{x \in X} F(x)$$

such that

$$(\forall x \in X)(f(x) \in F(x))$$

We call such f a *dependent function*, for the $F(x)$ depends on x. In case F is a constant function, f is a normal function $X \to Y$.

2.4 Coherence

It is sure you write

$$3 + 1 + 2$$

rather than

$$(3 + (0 + 1)) + 2$$

because you know the arithmetic laws

$$x + (y + z) = (x + y) + z$$
$$0 + x = x = x + 0$$

disambiguate unparenthesized expressions. Informally laws to introduce natural syntax are called *coherence conditions* or shortly *coherence*.

3 Categories

3.1 The Definition

Definition 3.1 (Category) A *category* $\mathcal{C}$ consists of

1. *objects*: a class $\mathrm{Ob}(\mathcal{C})$
2. *morphisms* or *hom-sets*: a family of sets $(\mathcal{C}(A,B))_{A,B\in\mathrm{Ob}(\mathcal{C})}$
3. *compositions*: a family of functions

$$(\circ : \mathcal{C}(B,C) \times \mathcal{C}(A,B) \to \mathcal{C}(A,C))_{A,B,C\in\mathrm{Ob}(\mathcal{C})}$$

4. *identities* or *units*: a family of morphisms

$$(\mathrm{id}_A \in \mathcal{C}(A,A))_{A\in\mathrm{Ob}(\mathcal{C})}$$

satisfying the following coherence conditions

1. *associativity*: for any $f \in \mathcal{C}(A,B)$, $g \in \mathcal{C}(B,C)$, and $h \in \mathcal{C}(C,D)$,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

2. *unitality*: for any $f \in \mathcal{C}(A,B)$,

$$\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$$

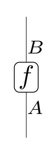
A morphism $f \in \mathcal{C}(A,B)$ is often denoted as $f : A \to B$.

3.2 String Diagrams

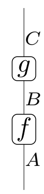
From now on, we will introduce *string diagrams* to complement(or hopefully replace) commutative diagrams.

Given a category $\mathcal{C}$, an object A is depicted as an optionally-tagged string

A morphism $f : A \to B$ is depicted as a node

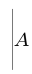


The composition joins two strings.



$$:= g \circ f$$

Identity morphisms are indistinguishable from objects.



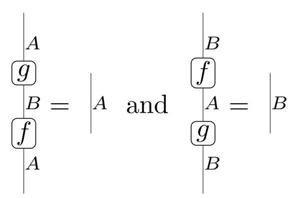
$$:= \mathrm{id}_A$$

Check these diagrams create no ambiguity thanks to the coherence.

Definition 3.2 (Isomorphism) We call a pair of morphisms

$$f : A \to B$$
$$g : B \to A$$

an *isomorphism* or shortly *iso* provided that the *invertibility*



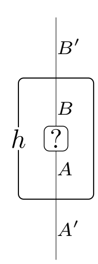
is satisfied. A morphism that is a part of an isomorphism is called *invertible*.

Remark 3.3 Most people call each morphism of this pair an isomorphism.

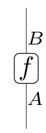
Definition 3.4 (Functional Box) Given categories $\mathcal{C}$ and $\mathcal{C}'$, a function

$$h : \mathcal{C}(A, B) \to \mathcal{C}'(A', B')$$

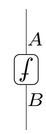
is depicted as a *functional box*



Definition 3.5 (Opposite Category) Given a category $\mathcal{C}$ and a morphism



you can construct a category with strings upside down:



which is denoted as $\mathcal{C}^{\mathrm{op}}$, the *opposite category* of $\mathcal{C}$.

Definition 3.6 (Discrete Category) A category $\mathcal{C}$ such that

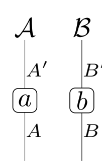
$$\begin{aligned} A = B &\implies \mathcal{C}(A, B) = \{\mathrm{id}_A\} \\ A \neq B &\implies \mathcal{C}(A, B) = \emptyset \end{aligned}$$

is called a *discrete category*. Any set can be represented as a discrete category.

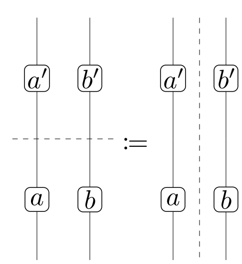
Definition 3.7 (Product Category) Given two categories $\mathcal{A}$ and $\mathcal{B}$, the *product category*

$$\mathcal{A} \times \mathcal{B}$$

is depicted as parallel strings

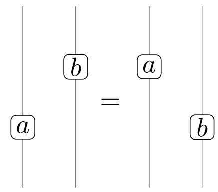


The composition, which joins parallel strings, is defined by



Identity morphisms are trivially

By these definitions,



4 Functors

4.1 The Definition

Definition 4.1 (Functor) A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of

1. *domain*: a category $\mathcal{C}$
2. *codomain*: a category $\mathcal{D}$
3. a family of objects $(FA \in \mathrm{Ob}(\mathcal{D}))_{A \in \mathrm{Ob}(\mathcal{C})}$
4. families of morphisms

$$\big((F(f) \in \mathcal{D}(FA, FB))_{f \in \mathcal{C}(A,B)}\big)_{A,B \in \mathrm{Ob}(\mathcal{C})}$$

satisfying the *functoriality*:

1. *composition-compatibility*: for any $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$,

$$F(g \circ f) = F(g) \circ F(f)$$

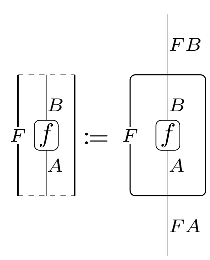
2. *unit-compatibility*: for any $A \in \mathrm{Ob}(\mathcal{C})$,

$$F(\mathrm{id}_A) = \mathrm{id}_{FA}$$

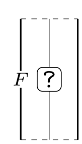
Definition 4.2 (Infrafunctor) An *infrafunctor* is a functor without the requirement of functoriality.

4.2 Functorial Tubes

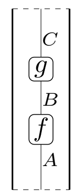
In string diagrams, a functor is represented as a tube



Placeholders make it simple:

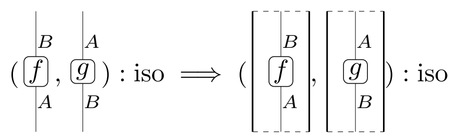


One can check the functoriality ensures any tube like



be unambiguous. "Join then tube" is the same as "tube then join".

Proposition 4.3 Any functor preserves isomorphisms meaning that



Proof. Immediate by functoriality, which inheres in tubes. ∎

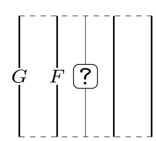
Definition 4.4 (Composite Functor) For any two functors

$$F : \mathcal{A} \to \mathcal{B}$$
$$G : \mathcal{B} \to \mathcal{C}$$

the *composite functor* of F and G

$$G \circ F : \mathcal{A} \to \mathcal{C}$$

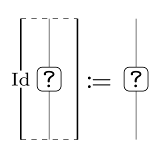
is defined as



Definition 4.5 (Identity Functor) Given a category $\mathcal{C}$, the *identity functor* on $\mathcal{C}$

$$\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$$

is defined by

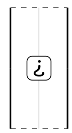


Definition 4.6 (Contravariant Functor) A functor whose domain is an opposite category

$$F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$$

is called *contravariant*, while a normal functor is called *covariant*.

A contravariant functor is depicted as



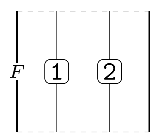
Definition 4.7 (Variant) Given a statement regarding functors, you can obtain a corresponding one regarding contravariant functors and vice versa. We call such a statement the *variant* of the original one.

Definition 4.8 (Binary Functor) A functor whose domain is a product category

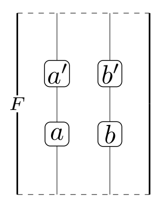
$$F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$$

is called a *binary functor* or *bifunctor*.

With numbered placeholders, it is depicted as



Spelling out the definition of functoriality, one can check a diagram like

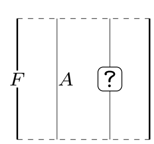


is unambiguous.

Definition 4.9 (Partial Application) Given a binary functor $F : \mathcal{A}\times\mathcal{B} \to \mathcal{C}$, a *partially applied* functor

$$\Lambda_B F(A, B) : \mathcal{B} \to \mathcal{C} \text{ or shortly}$$
$$F(A, \texttt{?}) : \mathcal{B} \to \mathcal{C}$$

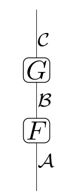
is defined as



The definition of $F(\texttt{?}, B)$ is an exercise.

Definition 4.10 (Small Category) A category $\mathcal{C}$ is called *small* when $\mathrm{Ob}(\mathcal{C})$ is a set.

Definition 4.11 (Category of Small Categories) The *category of small categories* **Cat** is the category whose objects are all small categories and whose morphisms are functors:



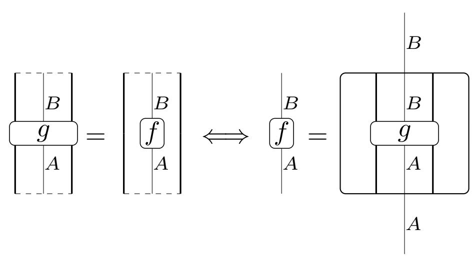
where composite functors join strings.

Definition 4.12 (Full and Faithful Functor) A functor $F : \mathcal{C} \to \mathcal{D}$ is called *full and faithful* if for each object A and B in $\mathcal{C}$, the family

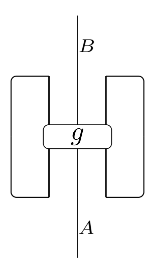
$$(F(f) : FA \to FB)_{f:A\to B}$$

is bijective.

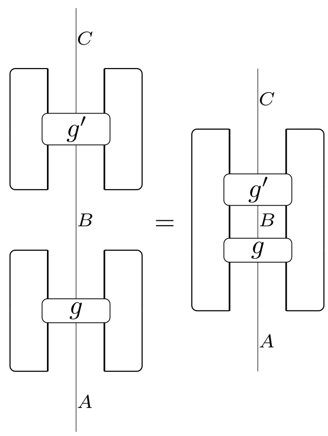
In other words, there is a functional box such that



One can make this box better-looking

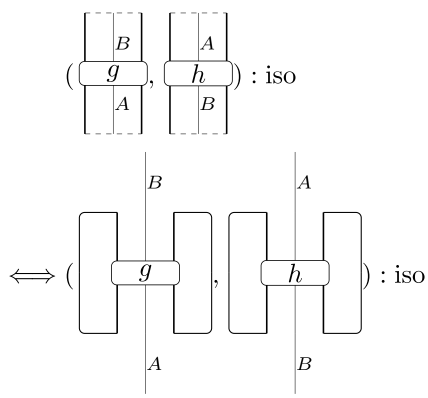


Proposition 4.13 This box has a functoriality-like property:



Combined with proposition 4.3,

Proposition 4.14



5 Natural Transformations

5.1 The Definition

Definition 5.1 (Naturality) Given two infrafunctors

$$F, G : \mathcal{C} \to \mathcal{D}$$

a family of morphisms

$$(\tau_A \in \mathcal{D}(FA, GA))_{A \in \mathrm{Ob}(\mathcal{C})}$$

is called *natural* when for any $f \in \mathcal{C}(A, B)$,

$$\tau_B \circ F(f) = G(f) \circ \tau_A$$

In case parentheses are cumbersome, you can say "τ_A is *natural in* A".

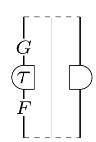
Definition 5.2 (Natural Transformation) Furthermore, in particular case F and G are functorial(then they are functors), τ is denoted as a *natural transformation* $\tau : F \to G$.

Remark 5.3 In this document, naturality is explicitly defined to be orthogonal to functoriality.

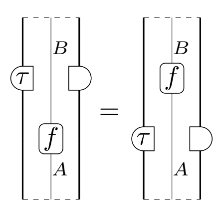
Proposition 5.4 Let $F : \mathcal{C} \to \mathcal{D}$ be an infrafunctor. Recall this is by definition a family of functions $(F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB))_{A,B}$. Then $F_{A,B}$ is natural in A *or* B if and only if F is composition-compatible.

5.2 Natural Connectors

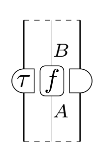
In string diagrams, a natural transformation is a connector of two tubes



because the naturality states a node can travel between tubes



This inspires us to assign



Definition 5.5 (Vertical Composition) Given three functors

$$F, G, H : \mathcal{C} \to \mathcal{D}$$

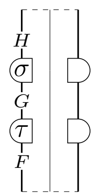
and two natural transformations

$$\tau : F \to G$$
$$\sigma : G \to H$$

the *vertical composition* of τ and σ

$$\sigma \circ \tau : F \to H$$

is defined as



Definition 5.6 (Horizontal Composition) Given four functors

$$F, G : \mathcal{A} \to \mathcal{B}$$
$$H, K : \mathcal{B} \to \mathcal{C}$$

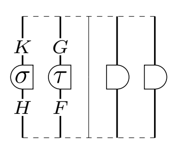
and two natural transformations

$$\tau : F \to G$$
$$\sigma : H \to K$$

the *horizontal composition* of τ and σ

$$\sigma\tau : H \circ F \to K \circ G$$

is defined as



You can easily check the naturality. Travel by car ferry.

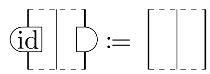
Definition 5.7 (Identity Natural Transformation) Given a functor

$$F : \mathcal{C} \to \mathcal{D}$$

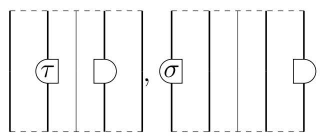
the *identity natural transformation*

$$\mathrm{id}_F : F \to F$$

is defined by



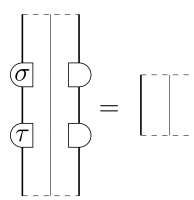
Definition 5.8 (Whiskering) A *whiskering* is a horizontal composition with identity natural transformations:

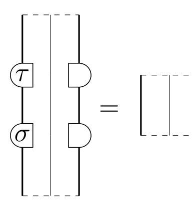


Definition 5.9 (Natural Isomorphism) We call a pair of natural transformations

$$\tau : F \to G$$
$$\sigma : G \to F$$

a *natural isomorphism* or shortly *natural iso* provided that the *invertibility*





is satisfied.

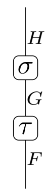
Remark 5.10 Most people call each morphism of this pair a natural isomorphism.

Proposition 5.11 For any natural transformation τ,

$$(\forall A)(\tau_A : \text{invertible})$$

is enough to build the other natural σ.

Definition 5.12 (Functor Category) Given a small category $\mathcal{C}$ and a category $\mathcal{D}$, the functor category $[\mathcal{C}, \mathcal{D}]$ is a category whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are natural transformations:



where the vertical composition joins strings.

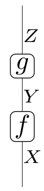
Definition 5.13 For the later use, define a lambda-tasted form for a set of natural transformations:

$$\mathrm{Nat}_A(FA, GA) \coloneqq \mathrm{Nat}(F, G) \coloneqq [\mathcal{C}, \mathcal{D}](F, G)$$

6 Category of Sets

6.1 The Definition

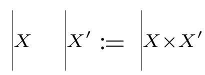
Definition 6.1 (Category of Sets) The *category of sets* **Set** is a category whose objects are sets and whose morphisms are functions:



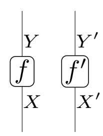
where strings are joined by the function composition.

A category is essentially one-dimensional so far: the vertical composition only. Here we introduce the horizontal composition for **Set**.

Definition 6.2 (Monoidal Category of Sets) Parallel strings are defined by

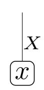


The *horizontal composition* of functions is defined by



$$:= \Lambda_{x,x'}(f(x), f'(x'))$$

Strings for the singleton set $\{*\}$ is omitted so that an element of a set is represented as



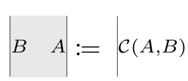
One can check any string diagram built upon these definitions is unambiguous due to the trivial bijections

$$X \times (X' \times X'') \cong (X \times X') \times X''$$
$$X \times \{*\} \cong X$$

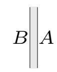
Informally such two-dimensional categories are called *monoidal*.

6.2 Hom-Set Bands

Given a category $\mathcal{C}$, a special string, a *band*, is introduced for hom-sets:

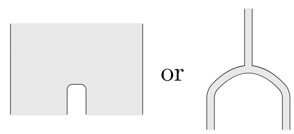


A space-saving form is depicted as

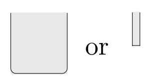


Remark 6.3 Note that the order of objects is flipped. This is resulting from an unfortunate convention that one write "$b = h(a)$" but not "$h : B \leftarrow A$". By the way, "B^A" is fine.

The composition of morphisms can be depicted as



Identity morphisms can be depicted as

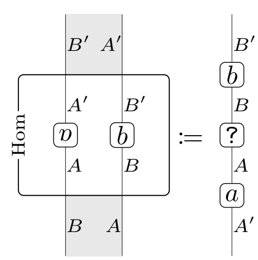


As an exercise, write down the associativity and unitality using these diagrams.

Definition 6.4 (Hom-Functor) Hom-sets can be extended to the *Hom-functor*

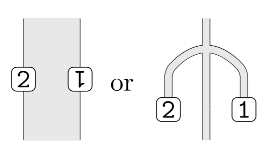
$$
\begin{aligned}
\Lambda_{A,B}\mathcal{C}(A, B) &: \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set} \text{ or shortly} \\
\mathcal{C}(-, +) &: \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set} \text{ or shortly} \\
\mathrm{Hom}_{\mathcal{C}} &: \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}
\end{aligned}
$$

defined by



where the world in the box is product category $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$.

This definition inspires us to depict hom-functors as

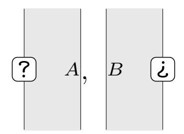


that looks topologically equivalent.

Definition 6.5 (Unary Hom-Functor) Due to definition 4.9,

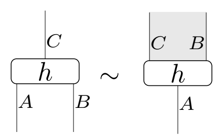
$$\mathcal{C}(A, +) : \mathcal{C} \to \mathbf{Set}$$
$$\mathcal{C}(-, B) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

are respectively depicted as



Definition 6.6 (Currying) In particular case $\mathcal{C} = \mathbf{Set}$, the *curry bijection* is defined by

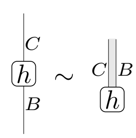
$$\begin{aligned} \mathbf{Set}(A \times B, C) &\cong \mathbf{Set}(A, \mathbf{Set}(B, C)) \\ h &\mapsto (a \mapsto b \mapsto h(a, b)) \\ ((a, b) \mapsto h(a)(b)) &\leftarrow\!\shortmid h \end{aligned}$$



We don't distinguish these two diagrams, for the naturality of this bijection ensures "move the right-side leg up and down" works correct.

Proposition 6.7 The curry bijection is natural in all three variables.

Definition 6.8 (Naming) In case A is the singleton set, which is omitted in diagrams, a currying



is trivial. We call it a *naming*.

Proposition 6.9 Currying *preserves naturality*, meaning that given a family of functions $(f_X : FX \times B \to GX)_X$ with infrafunctors F and G, f_X is natural in X if and only if $\mathrm{curry}(f_X)$ is. So does uncurrying.

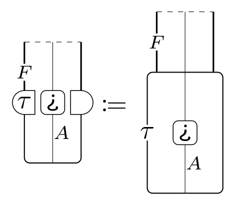
Remark 6.10 In general, natural bijections have similar properties so that you don't bother with proof of naturality ([6]).

7 The Yoneda Lemma

Definition 7.1 Given a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and an object A in $\mathcal{C}$, a natural transformation of the form

$$(\tau_X : \mathcal{C}(X, A) \to FX)_X$$

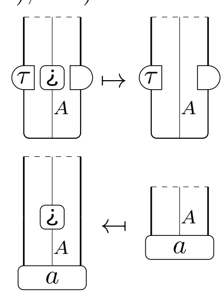
can be depicted as



owing to the naturality.

Definition 7.2 (Yoneda Bijection) The *Yoneda bijection* is defined by

$$\mathrm{Nat}_X(\mathcal{C}(X, A), FX) \cong FA$$

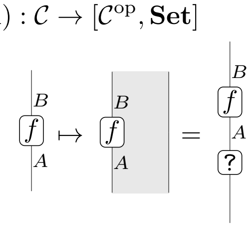


Lemma 7.3 (Yoneda Lemma) The Yoneda bijection is actually bijective and natural in F and A.

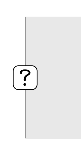
Proof. Now the proof is on my soul trivial! ∎

Definition 7.4 (Yoneda Embedding) The *Yoneda embedding* is defined by

$$\Lambda_A \Lambda_X \mathcal{C}(X, A) : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$$



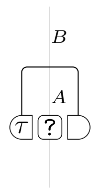
using the diagram of hom-functors. In short,



Definition 7.5 A natural transformation of the form

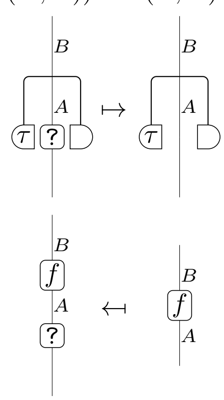
$$(\tau_X : \mathcal{C}(X, A) \to \mathcal{C}(X, B))_X$$

can be depicted as



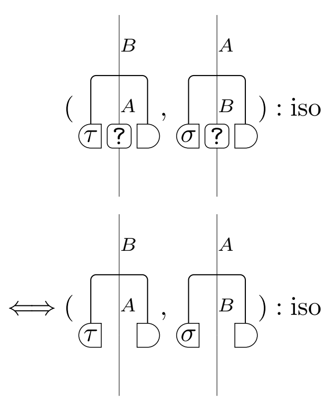
Definition 7.6 (Yoneda Embedding Bijection) In special case $F \coloneqq \mathcal{C}(-, B)$, the Yoneda bijection is expanded to

$$\mathrm{Nat}_X(\mathcal{C}(X, A), \mathcal{C}(X, B)) \cong \mathcal{C}(A, B)$$



The second mapping is the Yoneda embedding so that it is full and faithful. Combined with proposition 4.14,

Proposition 7.7 (Yoneda Principle)

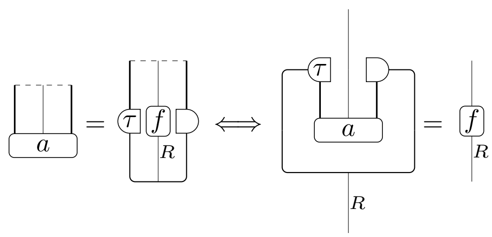


8 Representations

Definition 8.1 (Representation) Given a functor $H : \mathcal{C} \to \mathbf{Set}$, a *representation* of H is a pair of

1. an object R in $\mathcal{C}$
2. a natural bijection $(\tau_X : HX \cong \mathcal{C}(R, X))_X$

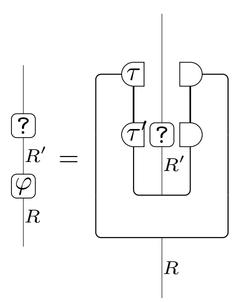
This bijectivity can be expressed using the weird boxes



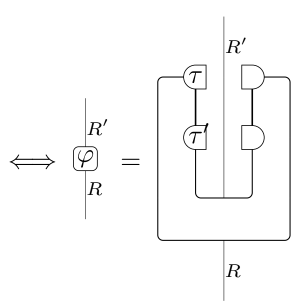
thanks to the naturality. The following proposition allows us to call it *the* representation of H, denoted as rep$H$.

Proposition 8.2 (Uniqueness of Representations) Representations are unique up to unique isomorphism.

Proof. Let (R', τ') be another representation. By the variant of proposition 7.7,



: invertible



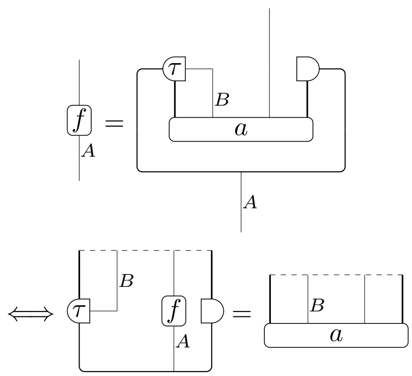
: invertible

∎

Definition 8.3 Given a functor $H : \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Set}$, a natural bijection of the form

$$(\tau_X : H(B, X) \cong \mathcal{A}(A, X))_X$$

can be expressed by



Proposition 8.4 (Parameterized Representations) Let $H : \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Set}$ be a functor. Given a family of objects $(SB)_B$ and a family of representations

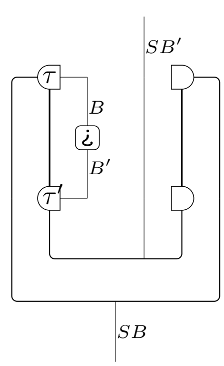
$$((\tau^B_X : H(B, X) \cong \mathcal{A}(SB, X))_X)_B$$

there exists a unique family

$$(S(f) \in \mathcal{A}(SB, SB'))_{f \in \mathcal{B}(B,B')}$$

such that τ is natural in B. Furthermore, S is functorial.

Proof. Define S as



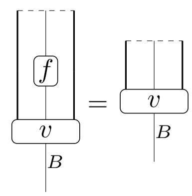
□

9 Limits

Definition 9.1 (Cone) Given a functor $F : \mathcal{A} \to \mathcal{B}$, a cone of F consists of

1. an object B in $\mathcal{B}$
2. a natural transformation $(v_X : B \to FX)_X$

Definition 9.2 (Conicality) We may explicitly call naturality of cones *conicality*, which can be expressed as



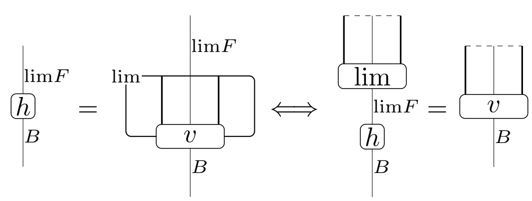
like a magical box any morphism can appear from.

Remark 9.3 Vertical and horizontal composition preserve conicality, a special case of naturality.

Definition 9.4 (Limit) Given a functor $F : \mathcal{A} \to \mathcal{B}$, a limit of F is a pair of

1. an object in $\mathcal{B}$ denoted as $\lim F$
2. a natural bijection $(\mathcal{B}(B, \lim F) \cong \mathrm{Nat}_X(B, FX))_B$

Definition 9.5 (Limiting Cone) The limit bijectivity, thanks to its naturality, can be expressed as



where $\boxed{\text{lim}}$ is a cone called a *limiting cone* of F.

The following proposition justifies the notation $\lim F$, *the* limit of F.

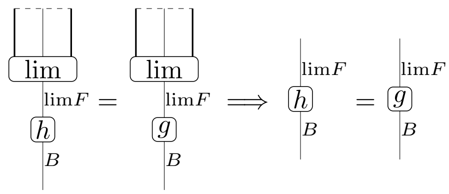
Proposition 9.6 Limits are unique up to isomorphism.

PROOF. Immediate by proposition 8.2, because a limit is nothing but a contravariant representation

$$\mathrm{rep}_B \, \mathrm{Nat}_X(B, FX)$$

∎

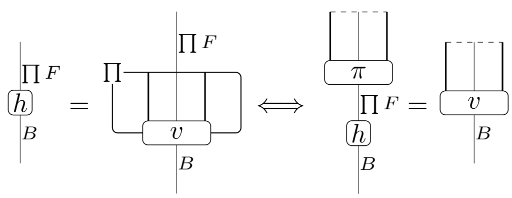
Proposition 9.7 A limiting cone is *monic* meaning that



Proof. Immediate by the limit bijectivity. ∎

Definition 9.8 (Product) In particular case the domain of a functor $F : \mathcal{A} \to \mathcal{B}$ is discrete, the limit of F is called the *product* of F, denoted as $\prod F$.

Definition 9.9 (Projection) Spelling out the product bijectivity,



where $\boxed{\ \pi\ }$ is called the *projection* of F.

Remark 9.10 Conicality has no concern here, because any family of the form

$$(v_X : B \to FX)_{X \in \mathrm{Ob}(\mathcal{A})}$$

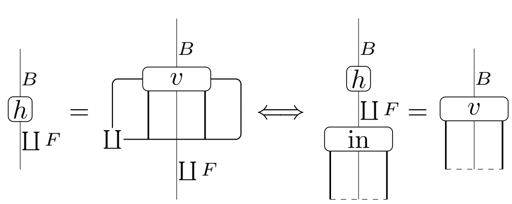
is always natural in case $\mathcal{A}$ is discrete.

Example 9.11 In case F is a functor $X \to \mathcal{S}et$ with a set X(as a discrete category), the product of F is a set of dependent functions

$$\textstyle\prod_x F(x) \cong \{f \mid (f(x) \in F(x))_x\}$$

Definition 9.12 (Dual) Given a statement containing string diagrams, by flipping the diagrams upside down, a corresponding statement is obtained. It is called the *dual* of the original one.

Definition 9.13 (Coproduct) A *coproduct* is a structure obtained from the product bijectivity flipped.



Remark 9.14 Informally the dual makes a codomain opposite, while the variant does for a domain.

Definition 9.15 (Preservation of Limits) Given a functor $F : \mathcal{A} \to \mathcal{B}$ and a limiting cone of F

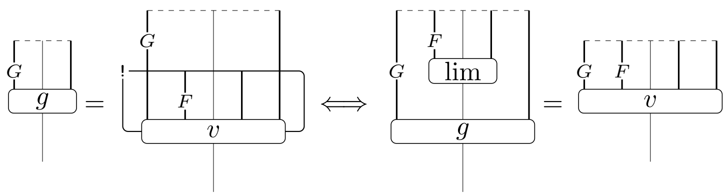
$$(\lim_X : \lim F \to FX)_X$$

a functor $G : \mathcal{B} \to \mathcal{C}$ *preserves limits* of F provided that

$$(G(\lim_X) : G\lim F \to GFX)_X$$

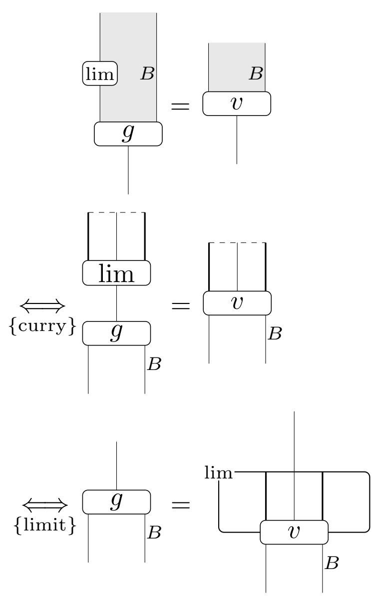
is a limiting cone of $G \circ F$.

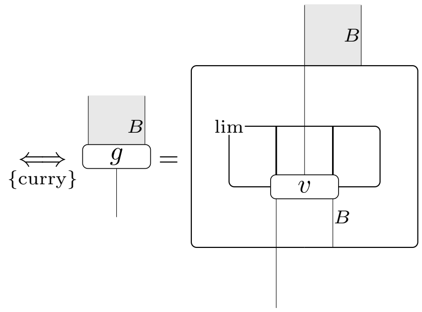
In diagrams, G is such that there exists some box ! satisfying



Proposition 9.16 (HFPL) Hom-functors preserve limits, meaning that given a functor $F : \mathcal{A} \to \mathcal{B}$ and an object B in $\mathcal{B}$, the covariant hom-functor $\mathcal{B}(B, +) : \mathcal{B} \to \mathbf{Set}$ preserves limits of F.

Proof.





$\square$

10 Adjunctions

Definition 10.1 (Adjunction) Given two categories $\mathcal{C}$ and $\mathcal{D}$, an *adjunction*

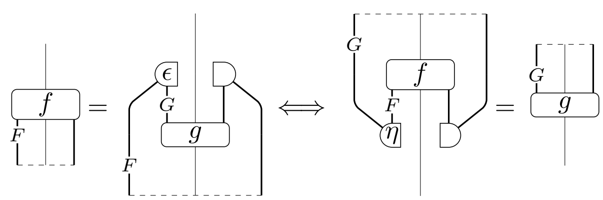
$$F \dashv G$$

consists of

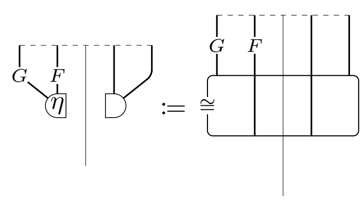
1. *left adjoint*: a functor $F : \mathcal{C} \to \mathcal{D}$
2. *right adjoint*: a functor $G : \mathcal{D} \to \mathcal{C}$
3. *adjunct*: a natural bijection

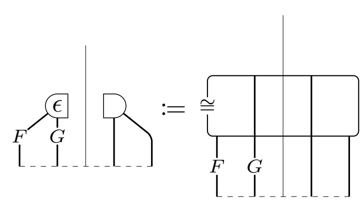
$$(\mathcal{D}(FC, D) \cong \mathcal{C}(C, GD))_{C,D}$$

A nice consequence is that this bijectivity needs no boxes, expressed by natural transformations only.



where





called the *unit* and *counit* respectively.

Proposition 10.2 Given a functor $G : \mathcal{D} \to \mathcal{C}$, a family of natural bijections

$$((\mathcal{C}(C, GD) \cong \mathcal{D}(F_c, D))_D)_C$$

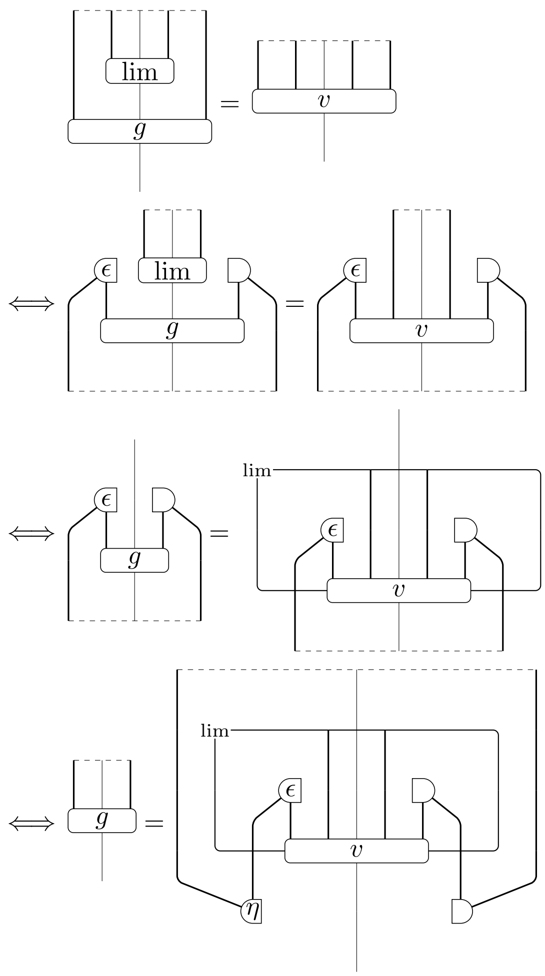
is enough to construct the adjunction $F \dashv G$.

Proof. Immediate by proposition 8.4 with $H(C, D) \coloneqq \mathcal{C}(C, GD)$. $\square$

Proposition 10.3 (RAPL) Right adjoints preserve limits, meaning that given an adjunction $F \dashv (G : \mathcal{D} \to \mathcal{C})$ and a functor $T : \mathcal{B} \to \mathcal{D}$,

$$
\begin{aligned}
&(\lim_X : \lim T \to TX)_X : \text{limiting cone} \\
\Longrightarrow\ &(G(\lim_X) : G\lim T \to GTX)_X : \text{limiting cone}
\end{aligned}
$$

Proof.



□

11 Monads

11.1 The Definition

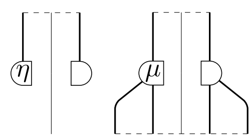
Definition 11.1 (Monad) Given a category $\mathcal{C}$, a *monad* on $\mathcal{C}$ consists of

1. a functor $T : \mathcal{C} \to \mathcal{C}$
2. *unit*: a natural transformation $\eta : \mathrm{Id}_T \to T$
3. *multiplication*: a natural transformation $\mu : T \circ T \to T$

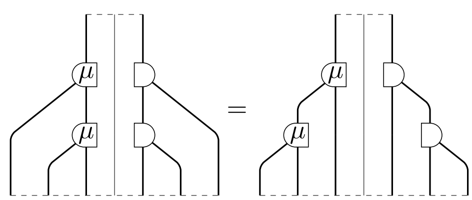
satisfying the coherence conditions

1. *associativity*: $\mu \circ T\mu = \mu \circ \mu T$
2. *unitality*: $\mu \circ T\eta = \mathrm{Id}_T = \mu \circ \eta T$

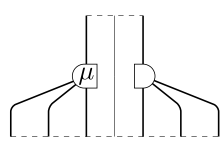
A unit and multiplication are depicted respectively as



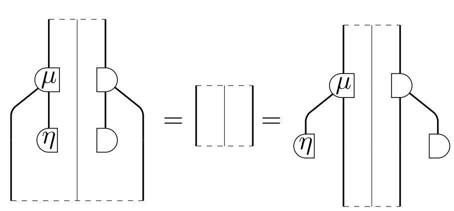
The associativity is depicted as



This inspires us to assign



The unitality is



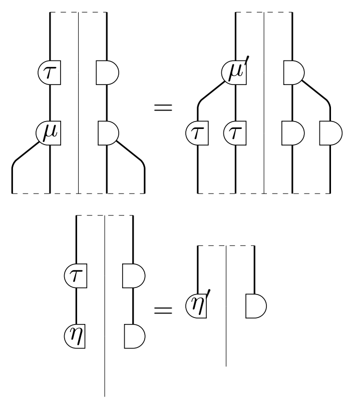
Definition 11.2 (Monad Morphism) Given a category $\mathcal{C}$, a *monad morphism* consists of

1. *domain*: a monad (T, η, μ) on $\mathcal{C}$
2. *codomain*: a monad (T', η', μ') on $\mathcal{C}$
3. a natural transformation $\tau : T \to T'$

satisfying the coherence conditions

1. *multiplication-compatibility*: $\tau \circ \mu = \mu' \circ \tau\tau$
2. *unit-compatibility*: $\tau \circ \eta = \eta'$

The coherence is depicted as

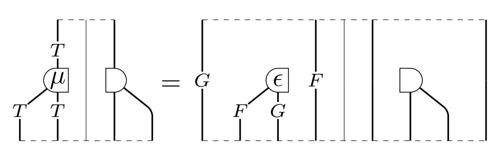


Definition 11.3 (Category of Monads) Given a category $\mathcal{C}$, the *category of monads* $\mathbf{Mnd}(\mathcal{C})$ is a category whose objects are monads and whose morphisms are monad morphisms.

Definition 11.4 (Monad-Associated Adjunction) Given a monad (T, η, μ), we say an adjunction $F \dashv G$ is T-*associated* provided that

1. $T = G \circ F$
2. $\mu = G\epsilon F$

This condition can be depicted as

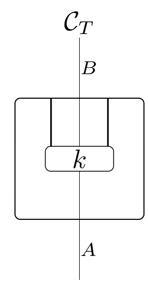


11.2 Kleisli Categories

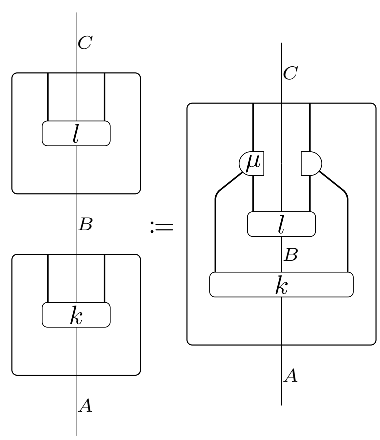
Definition 11.5 (Kleisli Category) Given a monad (T, η, μ) on $\mathcal{C}$, the *Kleisli category* of T, denoted as $\mathcal{C}_T$, is a category consisting of

1. $\mathrm{Ob}(\mathcal{C}_T) \coloneqq \mathrm{Ob}(\mathcal{C})$
2. $\mathcal{C}_T(A, B) \coloneqq \mathcal{C}(A, TB)$
3. $l \circ k \coloneqq \mu \circ T(l) \circ k$
4. $\mathrm{id}_A \coloneqq \eta_A$

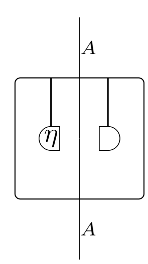
In diagrams, a morphism in $\mathcal{C}_T$ is depicted as a *Kleisli box*



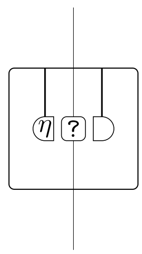
The composition is defined by



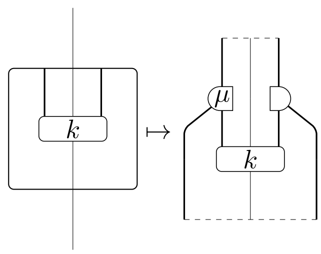
Identity morphisms are defined by



Definition 11.6 (Kleisli Adjunction) Define a functor $L : \mathcal{C} \to \mathcal{C}_T$ as



$K : \mathcal{C}_T \to \mathcal{C}$ as



then they constitute the *Kleisli adjunction* $L \dashv K$ whose adjunct is the Kleisli boxing. This adjunction is T-associated.

11.3 EM Categories

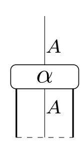
Definition 11.7 (Monad Algebra) Given a monad (T, η, μ) on $\mathcal{C}$, a *monad algebra*, denoted as T-algebra, consists of

1. an object $A \in \mathcal{C}$
2. a morphism $\alpha : TA \to A$

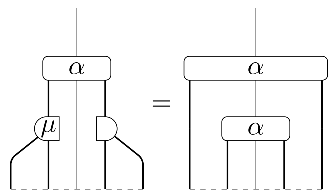
satisfying the coherence

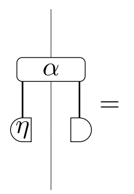
1. *associativity*: $\alpha \circ \mu = \alpha \circ T(\alpha)$
2. *unitality*: $\alpha \circ \eta = \mathrm{id}$

A T-algebra is depicted as



The coherence can be depicted as



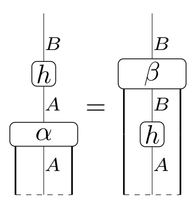


Definition 11.8 (EM Category) Given a monad (T, η, μ), the *Eilenberg-Moore(EM) category* of T, denoted as $\mathcal{C}^T$, is a category whose objects are T-algebras and whose morphisms are those of the form $h : A \to B$ such that

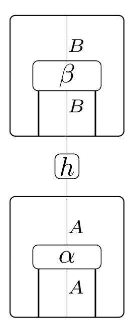
$$h \circ \alpha = \beta \circ T(h)$$

where (A, α) and (B, β) are T-algebras.

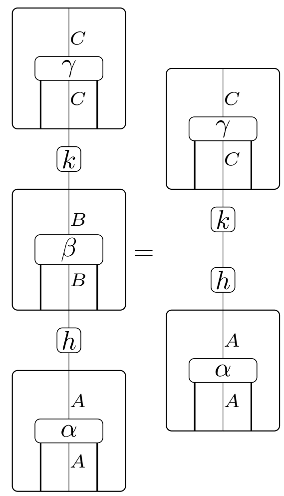
This condition is depicted as



A morphism in $\mathcal{C}^T$ is by compromise depicted as

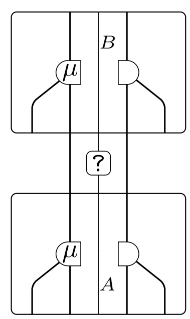


Boxes are objects. The composition can be depicted as

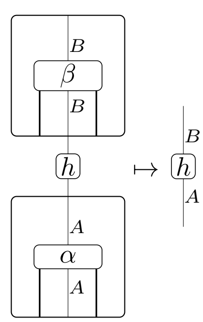


A diagram for an identity morphism is left as an exercise.

Definition 11.9 (EM Adjunction) Given a monad (T, η, μ) on $\mathcal{C}$, define a functor $M : \mathcal{C} \to \mathcal{C}^T$ as

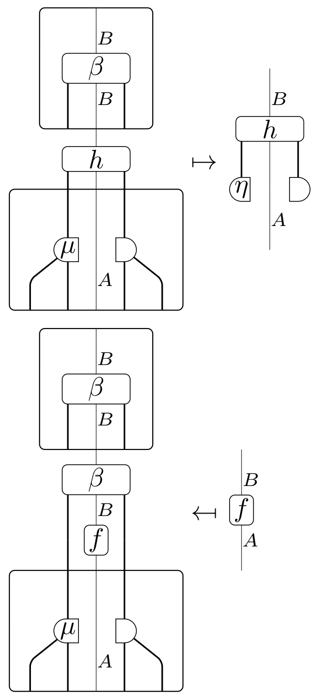


a functor $U : \mathcal{C}^T \to \mathcal{C}$ as



They constitute the *EM adjunction* $M \dashv U$ whose adjunct is defined by

$$\mathcal{C}^T(MA, (B, \beta)) \cong \mathcal{C}(A, U(B, \beta))$$



This adjunction is T-associated.

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