# Applicative Functors with Strings

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April 8, 2016

## 1 Introduction

We will show how applicative functors are depicted in *string diagrams*. Don't trust my poor mathematics. Any correction is welcome at github.com/okomok/strcat.

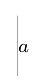
## 2 String Diagrams

We introduce *string diagrams*, which are useful for category theory. Don't be afraid. A string diagram in this document is just a kind of expression trees.

### 2.1 Vertical Composition

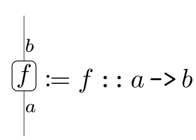
First we define how to join strings.

**Definition 2.1** A type $a$ is depicted as a string:

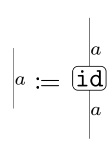



Type names are often omitted.

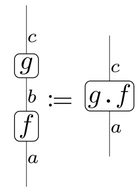
**Definition 2.2** A function is depicted as a node:




**Definition 2.3** An identity function is indistinguishable from a type:



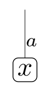
**Definition 2.4 (Vertical Composition)** The function composition joins strings:



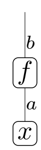

One can check that any diagram built upon these definitions has no ambiguity due to the famous laws:

$$h\,.\,(g\,.\,f) = (h\,.\,g)\,.\,f$$
$$\mathtt{id}\,.\,f = f$$
$$g\,.\,\mathtt{id} = g$$

**Definition 2.5 (Value)** Strings for the unit type `()` can be omitted so that a value $x :: a$ is represented as

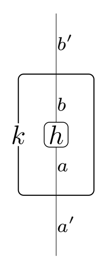



For example, a function application $f\,x$ is depicted as

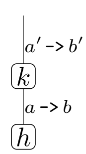



## 2.2 Functors

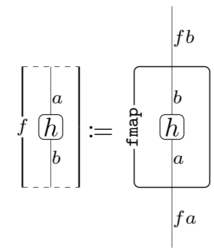
**Definition 2.6 (Functional Box)** Given a function $k :: (a \mathbin{\texttt{->}} b) \mathbin{\texttt{->}} (a' \mathbin{\texttt{->}} b')$, an application $k\,h$ can be depicted as a *box*:




rather than

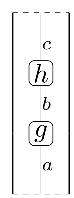


**Definition 2.7 (Functorial Tube)** Given a `Functor` $f$, an application of `fmap` can be depicted as a *tube* defined by




Tube names are often omitted.

The functor laws state that “tube then join” equals to “join then tube” so that any diagram like

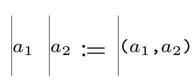



has no ambiguity.

## 2.3 Horizontal Composition

We will make string diagrams two-dimensional, equipped with the *horizontal composition*.

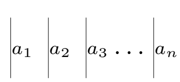
**Definition 2.8** Parallel strings are pairs.




Owing to the trivial bijections

- $\texttt{(}a_1\texttt{,(}a_2\texttt{,}a_3\texttt{))} \cong \texttt{((}a_1\texttt{,}a_2\texttt{),}a_3\texttt{)}$
- $\texttt{(}a\texttt{,())} \cong a \cong \texttt{((),}a\texttt{)}$

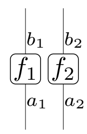
you can join any deeply nested pairs as far as their types are compatible, so that they are depicted as




without parentheses.

**Remark 2.9** Of course these bijections must be explicitly inserted to your haskell code.

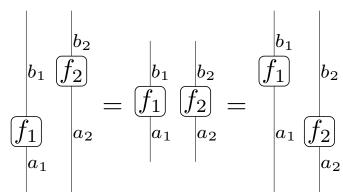
**Definition 2.10 (Horizontal Composition)** Parallel nodes are defined by




$$:= \texttt{\textbackslash(}a_1\texttt{,}a_2\texttt{) -> (}f_1\,a_1\texttt{,}f_2\,a_2\texttt{)}$$

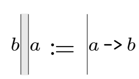
With these definitions, it is easy to check that:

**Proposition 2.11 (Sliding)**



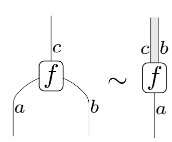

## 2.4 Currying

**Definition 2.12 (Band)** A special string for function types, a *band* is defined by




Notice that the order of types is flipped. So we often write $b \leftarrow a$ as $a$ `->` $b$.

**Definition 2.13 (Currying)** With bands, currying is represented by

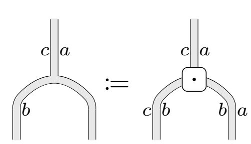



$$f \mapsto \texttt{\textbackslash}a\texttt{ -> \textbackslash}b\texttt{ -> }f\,\texttt{(}a\texttt{,}b\texttt{)}$$
$$\texttt{\textbackslash(}a\texttt{,}b\texttt{) -> }f\,a\,b \hookleftarrow f$$

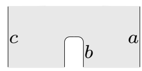
We don't distinguish these two diagrams, because "move the right-side leg up and down" works correct in any form of diagrams.

The following definitions make bands cute.

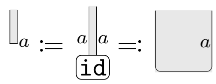
**Definition 2.14 (Function Composition)**



or you can use a *fat* form

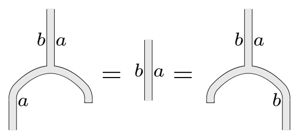



**Definition 2.15 (Identity Function)**

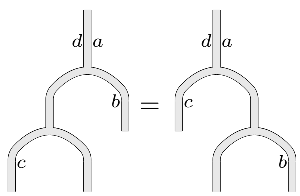



The following propositions are immediate.

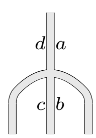
**Proposition 2.16 (Unitality)**




**Proposition 2.17 (Associativity)**



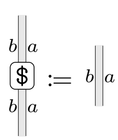

to which we assign



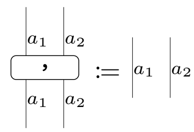

A band that has more forks is similarly defined. The equations for fat forms are left as an exercise.

For later use, we rephrase the two famous operators.

**Definition 2.18 (Apply Operator)**

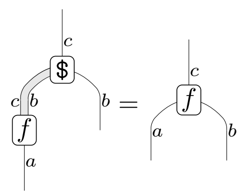


**Definition 2.19 (Comma Operator)**




One can check immediately:

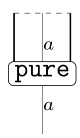
**Proposition 2.20**

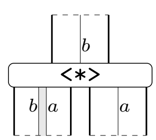



# 3 Applicative Functors

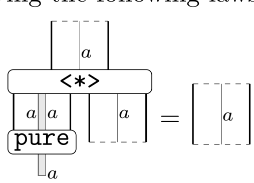
## 3.1 The Definition

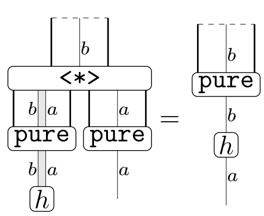
Using diagrams, an `Applicative` $f$ consists of

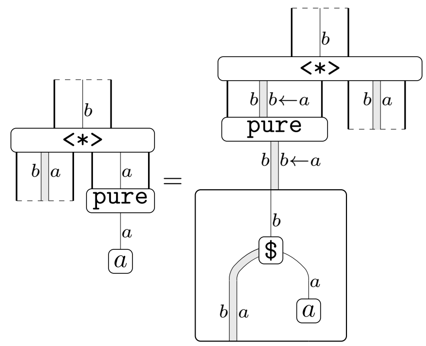
1. `Functor` $f$
2. 

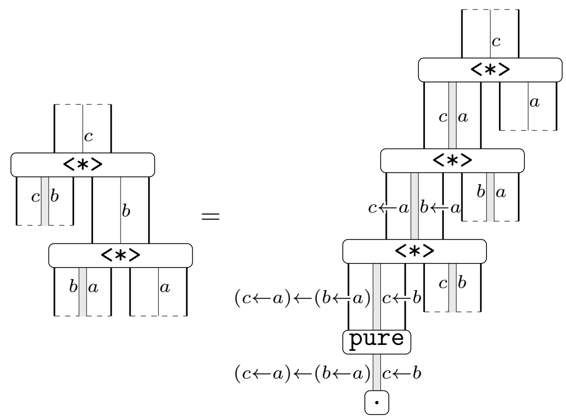
3. 

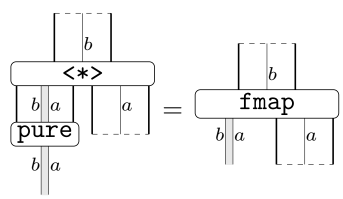

satisfying the following laws, which we can never understand,

- 


- 

- 


- 

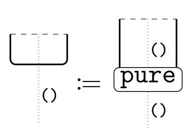

- 

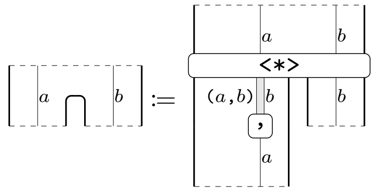

**Remark 3.1** The last law is redundant in case the *free theorem*[16] enabled.

## 3.2 Lax Functors

To depict applicative functors cuter, we represent an applicative functor as a fork-able tube, which is called a *lax functor* in category theory.

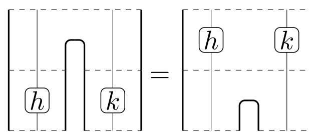
**Definition 3.2**



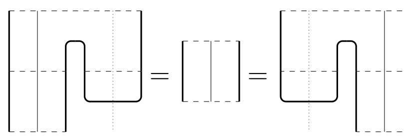



Under the applicative functor laws, one can check the following propositions that justify these pictures.

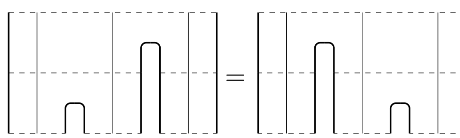
**Proposition 3.3 (Naturality)**



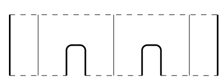

**Proposition 3.4 (Unitality)**



**Proposition 3.5 (Associativity)**



to which we assign

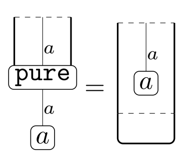


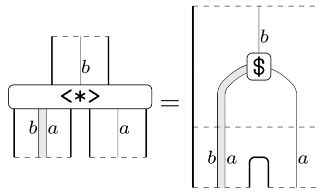
A tube that has more forks can be similarly defined without ambiguity.

**Remark 3.6** In our diagrams, cutoff lines are preferred to parentheses.

Finally one can find our goal:

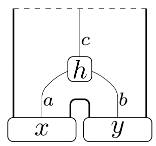
**Proposition 3.7**






Thanks to these diagrams, you can immediately prove:

**Proposition 3.8 (Lift)**




$= h \mathbin{\texttt{<\$>}} x \mathbin{\texttt{<*>}} y$

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