Applicative Functors with Strings

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1 Introduction

We will show how applicative functors are depicted in *string diagrams*. Don't trust my poor mathematics. Any correction is welcome at github.com/okomok/strcat.

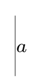
2 String Diagrams

We introduce *string diagrams*, which are useful for category theory. Don't be afraid. A string diagram in this document is just a kind of expression trees.

2.1 Vertical Composition

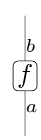
First we define how to join strings.

Definition 2.1 A type a is depicted as a string:



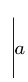
Type names are often omitted.

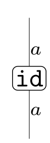
Definition 2.2 A function is depicted as a node:



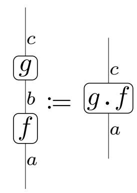
$:=$ `f :: a -> b`

Definition 2.3 An identity function is indistinguishable from a type:



$:=$


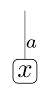
Definition 2.4 (Vertical Composition) The function composition joins strings:



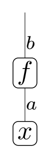
One can check that any diagram built upon these definitions has no ambiguity due to the famous laws:

$$h\,.\,(g\,.\,f) = (h\,.\,g)\,.\,f$$
$$\mathtt{id}\,.\,f = f$$
$$g\,.\,\mathtt{id} = g$$

Definition 2.5 (Value) Strings for the unit type `()` can be omitted so that a value $x :: a$ is represented as

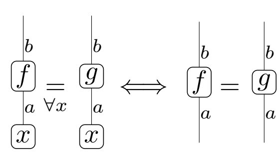


For example, a function application $f\,x$ is depicted as



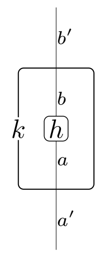
Due to the following definition, equations containing diagrams can often be simplified, known as *pointfree* style.

Definition 2.6 (Function Equality)

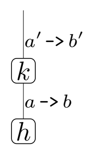


2.2 Functors

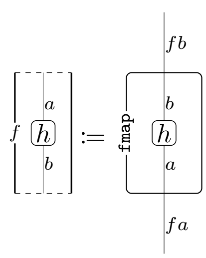
Definition 2.7 (Functional Box) Given a function $k :: (a \mathbin{\texttt{->}} b) \mathbin{\texttt{->}} (a' \mathbin{\texttt{->}} b')$, an application $k\,h$ can be depicted as a *box*:



rather than

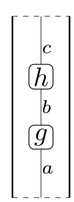


Definition 2.8 (Functorial Tube) Given a `Functor` f, an application of `fmap` can be depicted as a *tube* defined by



Tube names are often omitted.

The functor laws state that "tube then join" equals to "join then tube" so that any diagram like

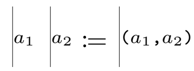


has no ambiguity.

2.3 Horizontal Composition

We will make string diagrams two-dimensional, equipped with the *horizontal composition*.

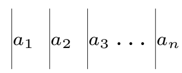
Definition 2.9 Parallel strings are pairs.



Owing to the trivial bijections

- (a_1,(a_2,a_3)) $\cong$ ((a_1,a_2),a_3)
- (a,()) $\cong a \cong$ ((),a)

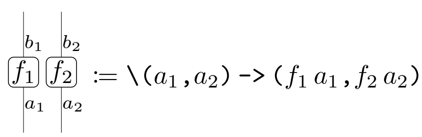
you can join any deeply nested pairs as far as their types are compatible, so that they are depicted as



without parentheses.

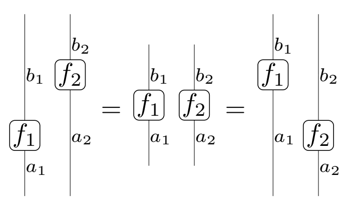
Remark 2.10 Of course these bijections must be explicitly inserted to your haskell code.

Definition 2.11 (Horizontal Composition) Parallel nodes are defined by



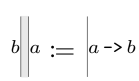
With these definitions, it is easy to check that:

Proposition 2.12 (Sliding)



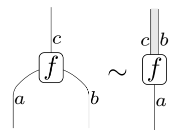
2.4 Currying

Definition 2.13 (Band) A special string for function types, a *band* is defined by



Notice that the order of types is flipped. So we often write $b \leftarrow a$ as a -> b.

Definition 2.14 (Currying) With bands, currying is represented by

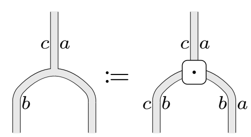


$$f \mapsto \backslash a \mathrel{\texttt{->}} \backslash b \mathrel{\texttt{->}} f\,(a,b)$$
$$\backslash(a,b) \mathrel{\texttt{->}} f\,a\,b \mapsfrom f$$

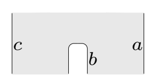
We don't distinguish these two diagrams, because "move the right-side leg up and down" works correct in any form of diagrams.

The following definitions make bands cute.

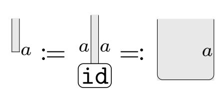
Definition 2.15 (Function Composition)



or you can use a *fat* form

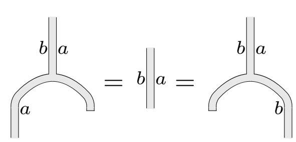


Definition 2.16 (Identity Function)

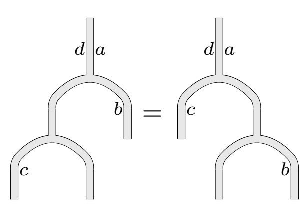


The following propositions are immediate.

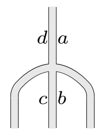
Proposition 2.17 (Unitality)



Proposition 2.18 (Associativity)



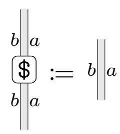
to which we assign



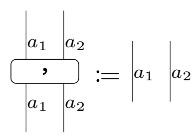
A band that has more forks is similarly defined. The equations for fat forms are left as an exercise.

For later use, we note the two famous operators.

Definition 2.19 (Apply Operator)

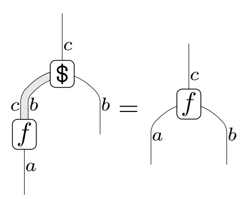


Definition 2.20 (Comma Operator)



One can check immediately:

Proposition 2.21

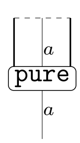


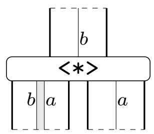
3 Applicative Functors

3.1 The Definition

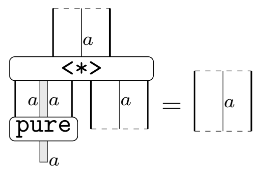
Using diagrams, an `Applicative` f consists of

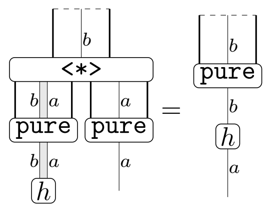
1. `Functor` f

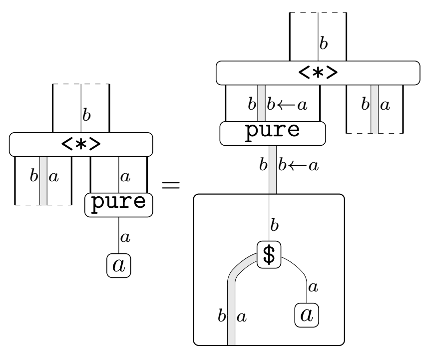
2. 

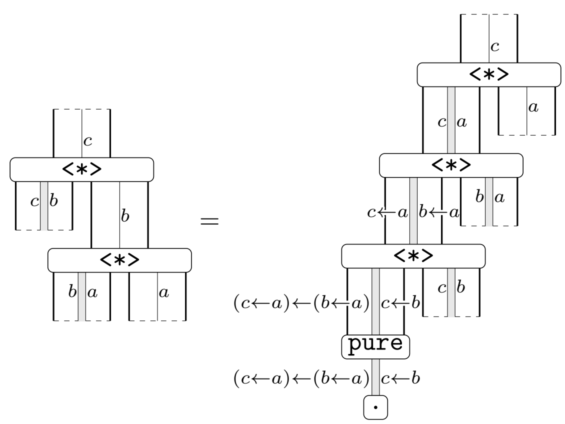
3. 

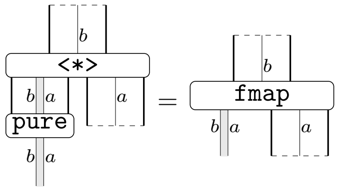
satisfying the following laws, which we can never understand,

- *identity*: 

- *homomorphism*: 

- *interchange*: 

- *composition*: 

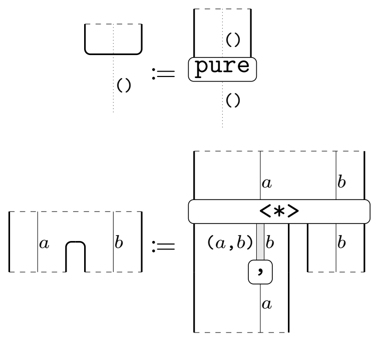
• 

Remark 3.1 The last law is redundant in case the *free theorem*[16] enabled.

3.2 Lax Functors

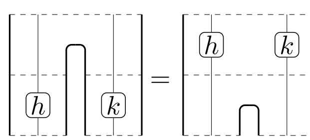
To depict applicative functors cuter, we represent an applicative functor as a fork-able tube, which is called a *lax functor* in category theory.

Definition 3.2

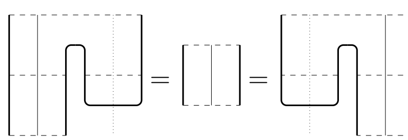


Under the applicative functor laws, one can check the following propositions that justify these pictures.

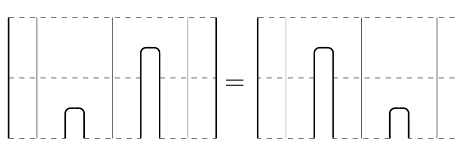
Proposition 3.3 (Naturality)



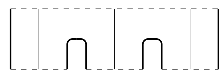
Proposition 3.4 (Unitality)



Proposition 3.5 (Associativity)



to which we assign

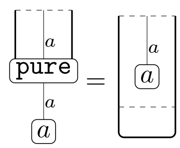


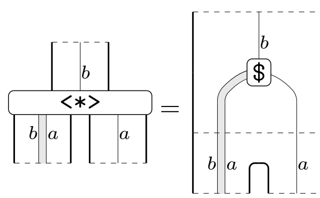
A tube that has more forks can be similarly defined without ambiguity.

Remark 3.6 In our diagrams, cutoff lines are preferred to parentheses.

Finally one can find our goal:

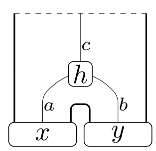
Proposition 3.7





Thanks to these diagrams, you can immediately prove:

Proposition 3.8 (Lift)



$=h$ `<$>` x `<*>` y

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