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Mathematics
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1. Prove that if f is continuous at x_0 and f is nonnegative then $h(x) = \sqrt{f(x)}$ is continuous at x_0.

   Proof. Let $\varepsilon > 0$. We want to find $\delta > 0$ such that if $|x - x_0| < \delta$, then $|h(x) - h(x_0)| < \varepsilon$. Now since f is continuous at x_0, it follows by definition that there exists a $\delta_1 > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta_1$. Let $x \in (x_0 - \delta_1, x_0 + \delta_1)$, with $h(x) + h(x_0) = \sqrt{f(x)} + \sqrt{f(x_0)} \neq 0$. Then it follows that

$$\begin{aligned}
|h(x) - h(x_0)| &= |\sqrt{f(x)} - \sqrt{f(x_0)}| \\
&= |\sqrt{f(x)} - \sqrt{f(x_0)}| \cdot \frac{|\sqrt{f(x)} + \sqrt{f(x_0)}|}{|\sqrt{f(x)} + \sqrt{f(x_0)}|} \\
&= \frac{|f(x) - f(x_0)|}{|\sqrt{f(x)} + \sqrt{f(x_0)}|} \\
&= \frac{|f(x) - f(x_0)|}{\sqrt{f(x)} + \sqrt{f(x_0)}} && \text{[Since } f \text{ is nonnegative]} \\
&\leq \frac{|f(x) - f(x_0)|}{\sqrt{f(x)} + \sqrt{f(x_0)} + 1}.
\end{aligned}$$

   If $\sqrt{f(x)} + \sqrt{f(x_0)} = 0$, so that $\sqrt{f(x)} = \sqrt{f(x_0)} = 0$, then the inequality above still holds. Thus if $x \in (x_0 - \delta_1, x_0 + \delta_1)$, we must have that

$$|h(x) - h(x_0)| \leq \frac{|f(x) - f(x_0)|}{\sqrt{f(x)} + \sqrt{f(x_0)} + 1}.$$

   Let $x \in (x_0 - \delta_1, x_0 + \delta_1)$. Since f is nonnegative, it follows that $\sqrt{f}$ is also nonnegative so that

$$\sqrt{f(x)} + \sqrt{f(x_0)} + 1 \geq 1,$$

   and thus

$$\frac{1}{\sqrt{f(x)} + \sqrt{f(x_0)} + 1} \leq 1. \tag{1}$$

   Multiply the inequality in (1) by the nonnegative number $|f(x) - f(x_0)|$ to get

$$\frac{|f(x) - f(x_0)|}{\sqrt{f(x)} + \sqrt{f(x_0)} + 1} \leq |f(x) - f(x_0)| < \varepsilon.$$

   We have thus shown that if $x \in (x_0 - \delta_1, x_0 + \delta_1)$, then

$$|h(x) - h(x_0)| \leq \frac{|f(x) - f(x_0)|}{\sqrt{f(x)} + \sqrt{f(x_0)} + 1} \leq |f(x) - f(x_0)| < \varepsilon.$$

   Thus it follows by definition that

$$\lim_{x \to x_0} h(x) = h(x_0),$$

   so that h is continuous at x_0. $\square$