6.6 Prove that if V is a real inner-product space, then

$$\langle u, v\rangle = \frac{||u+v||^2 - ||u-v||^2}{4}$$

for all $u.v \in V$.

Proof. Let V be a real inner-product space and let $u, v \in V$. We have that

$$\begin{aligned}
\frac{||u+v||^2 - ||u-v||^2}{4} &= \frac{\langle u+v, u+v\rangle - \langle u-v, u-v\rangle}{4} \\
&= \frac{\langle u, u+v\rangle + \langle v, u+v\rangle - \langle u, u-v\rangle - \langle -v, u-v\rangle}{4} \\
&= \frac{\langle u+v, u\rangle + \langle u+v, v\rangle - \langle u-v, u\rangle + \langle v, u-v\rangle}{4} \\
&= \frac{\langle u, u\rangle + \langle v, u\rangle + \langle u, v\rangle + \langle v, v\rangle - \langle u, u\rangle + \langle v, u\rangle + \langle u-v, v\rangle}{4} \\
&= \frac{\langle v, u\rangle + \langle u, v\rangle + \langle v, v\rangle + \langle v, u\rangle + \langle u-v, v\rangle}{4} \\
&= \frac{\langle v, u\rangle + \langle u, v\rangle + \langle v, v\rangle + \langle v, u\rangle + \langle u, v\rangle - \langle v, v\rangle}{4} \\
&= \frac{\langle v, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle u, v\rangle}{4} \\
&= \frac{\langle u, v\rangle + \langle u, v\rangle + \langle u, v\rangle + \langle u, v\rangle}{4} \\
&= \frac{4\langle u, v\rangle}{4} = \langle u, v\rangle.
\end{aligned}$$

□

6.10 On $\mathcal{P}_2(\mathbb{R})$, consider the inner product given by

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx.$$

Apply the Gram-Schmidt procedure to the basis $(1, x, x^2)$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$.

Solution.

6.13 Suppose $(e_1, \ldots, e_m)$ is an orthonormal list of vectors in V. Let $v \in V$. Prove that

$$||v||^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$$

if and only if $v \in \text{span}(e_1, \ldots, e_m)$.

Proof. Suppose $(e_1, \ldots, e_m)$ is an orthonormal list of vectors in V and let $v \in V$.

($\Leftarrow$) Assume that $v \in \text{span}(e_1, \ldots, e_m)$. Therefore $v = a_1e_1 + \cdots + a_me_m$ for some scalars $a_1, \ldots, a_m$. By the orthonomality of $(e_1, \ldots, e_m)$, it follows that $\langle v, e_j\rangle = a_j$ for all $j \in \{1, 2, \ldots, m\}$, so we have that

$$\begin{aligned} ||v||^2 &= ||a_1 e_1 + \cdots + a_m e_m||^2 \\ &= ||\langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m||^2 \\ &= |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2 && \text{[Proposition 6.15]} \end{aligned}$$

6.17 Prove that if $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and every vector in null P is orthogonal to every vector in range P, then P is an orthogonal projection.

Proof.

6.29 Suppose $T \in \mathcal{L}(\mathcal{V})$ and U is a subspace of V. Prove that U is invariant under T if and only if $U^\perp$ is invariant under $T*$.

Proof.