1. Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false.

(1) Every non-constant complex polynomial has a complex root.

(2) Conjugation of complex numbers is a field automorphism of the complex numbers.

(3) Let $x, y \in R$, a finite ring. If $x * y = 1$, then $y * x = 1$ also.

(4) There are exactly four quadratics in $\mathbb{Z}_2[x]$.

(5) If $p(x)$ is a real polynomial, then it either has a real root or there is a quadratic polynomial with real coefficients that divides it.

Solution.

(1) True.
This follows from the Fundamental Theorem of Algebra.

(2) True.
Proof. We want to show that

$$f : \mathbb{C} \to \mathbb{C},\ a + bi \mapsto a - bi$$

is an isomorphism. So we have that

$$\begin{aligned} f((a+bi)(c+di)) &= f(ac - bd + (ad+bc)i) \\ &= ac - bd - (ad+bc)i \\ &= ac - adi - bci - bd \\ &= a(c - di) - bi(c - di) \\ &= (a - bi)(c - di) \\ &= f(a+bi)f(c+di), \text{ and} \end{aligned}$$

$$\begin{aligned} f((a+bi)+(c+di)) &= f((a+c)+(b+d)i) \\ &= (a+c) - (b+d)i \\ &= a - bi + c - di \\ &= f(a+bi) + f(c+di). \end{aligned}$$

Thus conjugation of complex numbers is a field automorphism.

(3) True.
Proof. Let R be a finite ring, and consider $x, y \in R$ such that $x * y = 1$. The map $f : R \to R$, $r \mapsto r * x$ is bijective because for $r_1, r_2 \in R$ with $f(r_1) = f(r_2)$, we have that $r_1 * x = r_2 * x$. We then cancel x on both sides by multiplying each side on the right by y to get $r_1 = r_2$; thus f is injective, and since R is finite, we can conclude that f is also bijective. Thus there exists $r_3 \in R$ such that $r_3 * x = 1$. Mutltiply the preceding equality on the right by y to get $r_3 = y$. □

4 False.

There are exactly 8 quadratics in $\mathbb{Z}_2[x]$, and they are

$$0, 1, x, x + 1, x^2, x^2 + 1, x^2 + x, x^2 + x + 1.$$