1. A box contains 4 red balls, 3 white balls, 1 blue ball, and 2 green balls. Seven balls are selected with replacement. Find

   (a) the probability of selecting 3 red balls, 2 white balls, 1 blue ball, and 1 green ball.

   (b) the expectation and variance of the number of red balls selected.

   (c) the covariance of the number of red balls selected and the number of white balls selected

   Solution.

   (a) $p(3r, 2w, 1b, 1g) = \frac{7!}{3!2!1!1!}(0.4)^3(0.3)^2(0.1)^1(0.2)^1 = 0.048384.$

   (b) $E[\text{red}] = 7 \cdot 0.4 = 2.8$, $V(\text{red}) = 7 \cdot 0.4 \cdot 0.6 = 1.68.$

   (c) $\text{Cov}(\text{red, white}) = -7 \cdot 0.4 \cdot 0.3 = -0.84.$

2. Suppose in a certain (large) community, 40% of the population is under 30 years of age, 30% of the population is between 30 and 50 years of age, 20% of the population is between 50 and 70 years of age, and 10% of the population is over 70. If eight people are randomly selected from the community, find the probability that

   (a) exactly four are under 30, exactly two are between 30 and 50, exactly one is between 50 and 70, and exactly one is over 70.

   (b) exactly three are under 30 and exactly two are between 50 and 70.

   (c) exactly five are under 50.

   Solution.

   (a) $P = \frac{8!}{4!2!1!1!}(0.4)^4(0.3)^2(0.2)^1(0.1)^1 = 0.0387072.$

   (b) $P = \frac{8!}{3!2!3!}(0.4)^3(0.2)^2(0.4)^3 = 0.0917504.$

   (c) $P = \frac{8!}{5!3!}(0.7)^5(0.3)^3 = 0.25412.$

3. Suppose Y has a normal distribution with parameters $\mu = M$ and $\sigma^2 = 1$ where M has a gamma distribution with parameters $\alpha = 3$ and $\beta = 2$. Find (a) $E[Y]$ and (b) $V(Y)$.

   Solution.

   (a) $E[Y] = E[E(Y|M)] = E[M] = 3 \cdot 2 = 6.$

   (b) $V(Y) = E[V(Y|M)] + V[E(Y|M)] = E[\sigma^2] + V[M] = \sigma^2 E[1] + \alpha\beta = 13.$

4. Suppose Y is an exponential random variable with parameter Λ and Λ itself is a geometric random variable with parameter $p = 0.4$. Find $E[Y]$ and $V(Y)$.

   Solution.

   (a) $E[Y] = E[E(Y|\Lambda)] = E[\Lambda] = 1/0.4 = 2.5.$

(b)

$$\begin{aligned} V(Y) &= E[V(Y|\Lambda)] + V(E[Y|\Lambda]) \\ &= E[\Lambda^2] + V(\Lambda) \\ &= 2V(\Lambda) + E[\Lambda]^2 \\ &= 2\frac{1-0.4}{0.4^2} + 2.5^2 = 13.75. \end{aligned}$$

5. Let Y be exponentially distributed with mean 3.

   (a) Use the method of distribution functions to find the density function of Y^2.

   (b) Find a transformation G such that $G(Y)$ is uniformly distributed on (1, 2).

Solution.

(a) Let $U = Y^2$. We have that $F_U(u) = 0$ if $u \le 0$. Now if $u > 0$, we have that

$$\begin{aligned} F_U(u) &= P(U \le u) = P(Y^2 \le u) \\ &= P(-\sqrt{u} \le Y \le \sqrt{u}) \\ &= \frac{1}{3}\int_{-\sqrt{u}}^{\sqrt{u}} e^{-y/3} dy \\ &= e^{\sqrt{u}/3} - e^{-\sqrt{u}/3}, \end{aligned}$$

so that

$$f_U(u) = \frac{dF_U(u)}{du} = \begin{cases} \frac{1}{6\sqrt{u}}\left(e^{\sqrt{u}/3} + e^{-\sqrt{u}/3}\right) & \text{if } u > 0 \\ 0 & \text{if otherwise} \end{cases}$$

(b) The distribution of Y is given by

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ 1 - e^{-y/3} & \text{if } y \ge 0. \end{cases}$$

Let U be a uniformly distributed random variable on (1, 2). Then the distribution of U is given by

$$F_U(u) = \begin{cases} 0 & \text{if } u < 1, \\ u - 1 & \text{if } 1 \le u \le 2, \\ 1 & \text{if } u > 2. \end{cases}$$

Let $1 \le u < 2$. We want to find $H(u)$ such that $P(Y \le H(u)) = u - 1$. Thus we have that

$$u - 1 = P(Y \le H(u)) = 1 - e^{-H(u)/3}$$

so that $H(u) = -3\ln(2-u)$. Now

$$\begin{aligned} u - 1 &= P(Y \leq H(u)) \\ &= P(Y \leq -3\ln(2-u)) \\ &= P\left(-\frac{1}{3}Y \geq \ln(2-u)\right) \\ &= P\left(2 - e^{-1/3Y} \leq u\right) \end{aligned}$$

if $1 \leq u < 2$. So let $G(Y) = 2 - e^{-1/3Y}$.

6. Suppose Y has a probability density function

$$f(y) = \begin{cases} 3y^2 & \text{if } 0 \leq y \leq 1 \\ y^3 & \text{if otherwise} \end{cases}$$

Use the method of transformation to find the probability density functions of
a) $U_1 = 3Y - 1$ and b) $U_2 = Y^3$.

Solution.

(a) We are interested in the function $h(y) = 3y - 1$, an increasing function. Let $u_1 = 3y - 1$ so that

$$y = h^{-1}(u_1) = \frac{u_1 + 1}{3} \text{ and } \frac{dh^{-1}}{du_1} = \frac{1}{3}.$$

Thus

$$\begin{aligned} f_{U_1}(u_1) &= f_Y[h^{-1}(u_1)]\frac{dh^{-1}}{du_1} \\ &= \begin{cases} \dfrac{(u_1+1)^2}{9} & \text{if } -1 \leq u_1 \leq 2, \\ \dfrac{(u_1+1)^3}{81} & \text{otherwise .} \end{cases} \end{aligned}$$

(b) Let $u_2 = h(y) = y^3$. Thus

$$y = h^{-1}(u_2) = u_2^{1/3} \text{ and } \frac{dh^{-1}}{du_2} = \frac{1}{3}u_2^{-2/3}.$$

Thus

$$\begin{aligned} f_{U_2}(u_2) &= f_Y[h^{-1}(u_2)]\frac{dh^{-1}}{du_2} \\ &= \begin{cases} 1 & \text{if } 0 \leq u_2 \leq 1, \\ \dfrac{1}{3}u_2^{1/3} & \text{otherwise .} \end{cases} \end{aligned}$$

7. Let Y_1 and Y_2 have joint probability density function

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2) & \text{if } 0 < y_1 \le y_2 \le 1 \\ 0 & \text{if otherwise} \end{cases}$$

Use the method of transformation to find the probability density functions of $U = Y_1/Y_2$.

Solution. Let Y_1 be fixed for some $y_1 > 0$. Then $U = y_1/Y_2$, so that $h(y_2) = y_1/y_2$, a decreasing function. Let $g(y_1, u)$ denote the joint density of Y_1 and U, with

$$y_2 = y_1/u = h^{-1}(u).$$

Thus

$$\begin{aligned} g(y_1, u) &= \begin{cases} f[y_1, h^{-1}(u)] \left| \dfrac{dh^{-1}}{du} \right| & \text{if } 0 < y_1 \le y_1/u \le 1, \\ 0 & \text{otherwise .} \end{cases} \\ &= \begin{cases} 6\left(1 - \dfrac{y_1}{u}\right) \dfrac{y_1}{u^2} & \text{if } 0 < y_1 \le u \le 1, \\ 0 & \text{otherwise .} \end{cases} \end{aligned}$$

It follows that

$$\begin{aligned} f_U(u) &= \int_{-\infty}^{\infty} g(y_1, u) dy_1 \\ &= \int_0^u 6\left(1 - \frac{y_1}{u}\right) \frac{y_1}{u^2} dy_1 \\ &= \begin{cases} 1 & \text{if } 0 < u \le 1, \\ 0 & \text{otherwise .} \end{cases} \end{aligned}$$

8. Suppose Y_1 and Y_2 are two independent exponentially distributed random variables each with mean β. Use the method of moment generating functions to obtain the probability density function of $Y_1 + Y_2$.

Solution. Let $U = Y_1 + Y_2$. Let $m_U(t)$, $m_{Y_1}(t)$, and $m_{Y_2}(t)$ denote the moment generating functions of U, Y_1 and Y_2. Since Y_1 and Y_2 are independent, it follows from Theorem 6.2 that $m_U(t) = m_{Y_1}(t) \times m_{Y_2}(t)$. We know from discussions in class that $m_{Y_1}(t) = m_{Y_2}(t) = (1 - \beta t)^{-1}$. Thus $m_U(t) = (1 - \beta t)^{-2}$, so that U has gamma distribution and we have that

$$f_U(u) = \begin{cases} \dfrac{1}{\beta^2} u e^{-u/\beta} & \text{if } 0 < u, \\ 0 & \text{otherwise .} \end{cases}$$