7.14 Suppose $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in \mathbb{F}$, and $\epsilon > 0$. Prove that if there exists $v \in V$ such that $||v|| = 1$ and

$$||Tv - \lambda|| < \epsilon,$$

then T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$.

Proof.

7.16 Give an example of an operator T on an inner product space such that T has an invariant subspace whose orthogonal complement is not invariant under T.

Answer.

7.17 Prove that the sum of any two positive operators on V is positive.

Proof. Suppose that S and T are positive operators on V. Since S and T are both self-adjoint, it follows immediately that $S + T$ is self-adjoint because

$$(S + T)^* = S^* + T^* = S + T.$$

Now let $v \in V$. Thus

$$\begin{aligned} \langle (S+T)v, v \rangle &= \langle Sv + Tv, v \rangle \\ &= \langle Sv, v \rangle + \langle Tv, v \rangle \\ &\geq 0, && \text{[Since } Sv \geq 0, Tv \geq 0\text{]} \end{aligned}$$

so that $S + T$ is a positive operator. $\square$

7.19 Suppose that T is a positive operator on V. Prove that T is invertible if and only if

$$\langle Tv, v \rangle > 0$$

for every $v \in V \setminus \{0\}$.

Proof.

7.22 Prove that if $S \in \mathcal{L}(\mathbb{R}^3)$ is an isometry, then there exists a nonzero vector $x \in \mathbb{R}^3$ such that $S^2 x = x$.

Proof.