

Lemma 7.1 *If $T \in \mathcal{L}(V)$ is normal, then $\text{null } T = \text{null } T^*$.*

Proof. Assume that $T \in \mathcal{L}(V)$ is normal. Thus

$$\begin{aligned} x \in \text{null } T &\iff Tx = 0 \\ &\iff \langle Tx, Tx \rangle = 0 \\ &\iff \langle T^*x, T^*x \rangle = 0 && [\text{Proposition 7.6}] \\ &\iff T^*x = 0 \\ &\iff x \in \text{null } T^*, \end{aligned}$$

so that $\text{null } T = \text{null } T^*$. □

7.1 Make $\mathcal{P}_2(\mathbb{R})$ into an inner-product space by defining

$$\langle p, q \rangle = \int_0^1 p(x)q(x) \, dx.$$

Define $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ by $T(a_0 + a_1x + a_2x^2) = a_1x$.

- (a) Show that T is not self-adjoint.
- (b) The matrix of T with respect to the basis $(1, x, x^2)$ is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix equals its conjugate transpose, even though T is not self-adjoint. Explain why this is not a contradiction.

Solution.

- (a) Suppose to the contrary that T is self-adjoint. Consider $x, 1 \in \mathcal{P}_2(\mathbb{R})$. Then we must have that $\langle T1, x \rangle = \langle T^*1, x \rangle = \langle 1, Tx \rangle$. But

$$\begin{aligned} \langle T^*1, x \rangle &= \langle T1, x \rangle \\ &= \langle 0, x \rangle \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \langle 1, Tx \rangle &= \langle 1, x \rangle \\ &= \int_0^1 x \, dx \\ &= \frac{1}{2}, \end{aligned}$$

so that $\langle T1, x \rangle \neq \langle 1, Tx \rangle$; i.e., T is not self-adjoint.

7.4 Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that P is an orthogonal projection if and only if P is self-adjoint.

Proof. Let $P \in \mathcal{L}(V)$ such that $P^2 = P$.

(\Rightarrow) Suppose that P is an orthogonal projection. Then we have that $P = P_{U, U^\perp}$ for some subspace U of V . To prove that P is self-adjoint, we must show that $Px = P^*x$ for all $x \in V$. So let $v_1, v_2 \in V$. There exist unique $u_1, u_2 \in U$ and $u'_1, u'_2 \in U^\perp$ such that $v_1 = u_1 + u'_1$ and $v_2 = u_2 + u'_2$. Thus

$$\begin{aligned} \langle Pv_1 - P^*v_1, v_2 \rangle &= \langle Pv_1, v_2 \rangle - \langle P^*v_1, v_2 \rangle \\ &= \langle u_1, v_2 \rangle - \langle v_1, Pv_2 \rangle \\ &= \langle u_1, v_2 \rangle - \langle v_1, u_2 \rangle \\ &= \langle u_1, u_2 + u'_2 \rangle - \langle u_1 + u'_1, u_2 \rangle \\ &= \langle u_1, u_2 \rangle + \langle u_1, u'_2 \rangle - \langle u_1, u_2 \rangle - \langle u'_1, u_2 \rangle \\ &= \langle u_1, u'_2 \rangle - \langle u'_1, u_2 \rangle \\ &= 0 - 0 = 0. \end{aligned}$$

Since v_2 was arbitrary, we have that $\langle Pv_1 - P^*v_1, v_2 \rangle = 0$ for all $v_2 \in V$. Setting $v_2 = Pv_1 - P^*v_1$ gives us that $\langle Pv_1 - P^*v_1, Pv_1 - P^*v_1 \rangle = 0$, so that $Pv_1 - P^*v_1 = 0$; i.e., $Pv_1 = P^*v_1$. Thus P is self-adjoint.

(\Leftarrow) Suppose that P is self-adjoint. We have that

$$\begin{aligned} \text{null } P &= (\text{range } P^*)^\perp && [\text{Proposition 6.46}] \\ &= (\text{range } P)^\perp && [P \text{ is self-adjoint}], \end{aligned}$$

so that every vector in $\text{null } P$ is orthogonal to every vector in $\text{range } P$. It follows by Homework 5, Problem 6.17 that P is an orthogonal projection of V onto $\text{range } P$, with null space $\text{null } P$. \square

7.6 Prove that if $T \in \mathcal{L}(V)$ is normal, then

$$\text{range } T = \text{range } T^*.$$

Proof. Assume that $T \in \mathcal{L}(V)$ is normal. We have that

$$\begin{aligned} \text{range } T &= (\text{null } T^*)^\perp && [\text{Proposition 6.46}] \\ &= (\text{null } T)^\perp && [\text{Lemma 7.1}] \\ &= \text{range } T^*, && [\text{Proposition 6.46}] \end{aligned}$$

which is what we wanted to prove. \square

7.7 Prove that if $T \in \mathcal{L}(V)$ is normal, then

$$\text{null } T^k = \text{null } T \text{ and } \text{range } T^k = \text{range } T$$

for every positive integer k .

Proof. We shall first proceed by induction on k to show that $\text{null } T^k = \text{null } T$.

Base Case. $k = 1$. It follows trivially that $\text{null } T^1 = \text{null } T$.

Inductive Hypothesis. Suppose that $\text{null } T^n = \text{null } T$ for some positive integer n .

Now we shall show that $\text{null } T^{n+1} = \text{null } T$. We have that

$$\begin{aligned} x \in \text{null } T &\implies x \in \text{null } T^n && [\text{Inductive Hypothesis}] \\ &\implies T^n x = 0 \\ &\implies T(T^n x) = T(0) = 0 \\ &\implies T^{n+1} x = 0 \\ &\implies x \in \text{null } T^{n+1}, \end{aligned}$$

so that $\text{null } T \subseteq \text{null } T^{n+1}$. Now let $x \in \text{null } T^{n+1}$. By Problem 7.6, we know that $\text{range } T = \text{range } T^*$; thus $Tx = T^*y'$ for some $y' \in V$. Now

$$\begin{aligned} x \in \text{null } T^{n+1} &\implies T^{n+1}x = 0 \\ &\implies TT^n x = 0 \\ &\implies \langle TT^n x, T^{n-1}y' \rangle = 0 \\ &\implies \langle T^n x, T^*T^{n-1}y' \rangle = 0 \\ &\implies \langle T^n x, T^{n-1}T^*y' \rangle = 0 && [T \text{ is normal}] \\ &\implies \langle T^n x, T^{n-1}Tx \rangle \\ &\implies \langle T^n x, T^n x \rangle = 0 \\ &\implies T^n x = 0 \\ &\implies x \in \text{null } T^n \\ &\implies x \in \text{null } T, && [\text{Inductive Hypothesis}] \end{aligned}$$

so that $\text{null } T^{n+1} \subseteq \text{null } T$. We have thus shown that $\text{null } T = \text{null } T^{n+1}$. It follows by Mathematical Induction that $\text{null } T^k = \text{null } T$ for each positive integer k . Let m be a positive integer. Observe that T^m is also normal; thus

$$\begin{aligned} \text{range } T^m &= (\text{null } (T^m)^*)^\perp && [\text{Proposition 6.46}] \\ &= (\text{null } T^m)^\perp && [\text{Lemma 7.1}] \\ &= (\text{null } T)^\perp && [\text{null } T^m = \text{null } T] \\ &= \text{range } T^* && [\text{Proposition 6.46}] \\ &= \text{range } T, && [\text{Problem 7.6}] \end{aligned}$$

which is what we wanted to show. \square

7.9 Prove that a normal operator on a complex inner-product space is self-adjoint if and only if all its eigenvalues are real.

Proof. Let $T \in \mathcal{L}(V)$ be normal, where V is a complex inner-product space.

(\implies) Suppose that T is self-adjoint. It follows by Proposition 7.1 that all the eigenvalues of T are real.

(\impliedby) Suppose that all the eigenvalues of T are real. Note that if u is an eigenvector of T with eigenvalue λ , then by Corollary 7.7, we have that $T^*u = \bar{\lambda}u$. To show that T is self-adjoint, we must show that $Tx = T^*x$ for all $x \in V$. So let $y \in V$. By the Spectral Theorem, V has an orthonormal basis (v_1, \dots, v_n) consisting of eigenvectors of T . Let λ_i be the eigenvalue corresponding to v_i . We know that $y = a_1v_1 + \dots + a_nv_n$ for some unique scalars. Thus

$$\begin{aligned}
 Ty &= T(a_1v_1 + \dots + a_nv_n) \\
 &= a_1Tv_1 + \dots + a_nTv_n \\
 &= a_1\lambda_1v_1 + \dots + a_n\lambda_nv_n \\
 &= a_1\bar{\lambda}_1v_1 + \dots + a_n\bar{\lambda}_nv_n && \text{[Eigenvalues of } T \text{ are real]} \\
 &= a_1T^*v_1 + \dots + a_nT^*v_n \\
 &= T^*(a_1v_1 + \dots + a_nv_n) \\
 &= T^*y,
 \end{aligned}$$

so that $T = T^*$. □