JOSEPH OKONOBOH
Mathematics
Cal State Long Beach

Let $R = \mathbb{Z}_6$, the integers mod 6. Answer the following:

(1) R is a commutative ring. Tell me why.

(2) Is R an integral domain? Why or why not?

(3) Find the group of units of R.

(4) Count the number of fixed point of the matrix $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ (with entries in R) when it acts (by multiplication) on the vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ with entries in $\mathbb{R}$.

(5) Is R a field? Why or why not?

Bonus. Count the group of units of $\mathcal{M}_2(R)$, which is the ring of 2×2 matrices with entries in R.

Solution.

(1) Yes, R is a commutative ring because

- $(R, +)$ is an abelian group, and
- R is associative and closed under multiplication, has the element 1 as its multiplicative identity, and multiplication distributes over addition.

(2) R is not an integral domain because for $2, 3 \in R$ with $2 \neq 0$ and $3 \neq 0$, we have $2 \cdot 3 = 0$; i.e., R has zero divisors so that it cannot be an integral domain.

(3) The group of units of R are the elements of R with a multiplicative identity; thus the group of units of R is $\{1, 5\}$.

(4) Suppose that the matrix $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ (with entries in R) fixes some vector $\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \in R^2$. Then we must have that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 3y \\ y \end{pmatrix},$$

so that $x = x + 3y$; that is, $3y = 0$. The solutions are $y = 0$, 2, or 4. So we have 6 choices for x and 3 choices for y, and thus 18 choices for the fixed vectors.

(5) No, R is not a field because 2 has no multiplicative inverse in R. Or we can conclude from (2) that R is not a field because it is not an integral domain.