1 Chapter 4

1.1 Section 1

4.01 Prove that if f is continuous at x_0 then f is bounded at x_0.

Proof. Suppose that $f : A \to \mathbb{R}$ is continuous at x_0. To show that f is bounded at x_0, it suffices to show that f is bounded on $N_{x_0}(\delta) \cap A$ for some $\delta > 0$. Since f is continuous at x_0, it follows by definition that there exists $\delta_1 > 0$ such that $|f(x) - f(x_0)| < 1$ whenever $|x - x_0| < \delta_1$. Using the triangle inequality we have that

$$||f(x)| - |f(x_0)|| < |f(x) - f(x_0)| < 1,$$

so that $|f(x)| - |f(x_0)| < 1$, if $|x - x_0| < \delta_1$. Thus $|f(x)| < 1 + |f(x_0)|$, if $|x - x_0| < \delta_1$. We have thus shown that f is bounded on $N_{x_0}(\delta_1)$ by $1 + |f(x_0)|$, so that f is bounded at x_0. $\square$

4.02 Find all points of discontinuity for the following functions, classify the discontinuities as removable, jump, or second kind, and determine where the function is right- and left-continuous.

(a) $f(x) = \begin{cases} x^2 & \text{if } x < -1, \\ 2x+3 & \text{if } -1 \leq x \leq 0, \\ |x-1| & \text{if } 0 < x < 2, \\ x^3 - 7 & \text{if } 2 \leq x < 3, \\ \dfrac{x-3}{x-4} & \text{if } 3 \leq x < 4, \\ 0 & \text{if } 4 \leq x. \end{cases}$

(b) $f(x) = x + [\![-x]\!]$.

(c) $f(x) = x[\![x]\!]$.

(d) $f(x) = \operatorname{sgn}[\![|x|]\!]$.

(e) $f(x) = \begin{cases} [\![x+1]\!] \sin \dfrac{1}{x} & \text{if } x \in (-1,0) \cup (0,1) \\ 0 & \text{otherwise.} \end{cases}$

(f) $f(x) = \begin{cases} (1+x) \operatorname{sgn} x + \operatorname{sgn} |x| - 1 & \text{if } x \text{ is rational} \\ \operatorname{sgn} x & \text{if } x \text{ is irrational.} \end{cases}$

4.03 Prove that $f(x)$ is continuous on $\mathbb{R}$.

4.04 Prove that if f is continuous at x_0 and g is discontinuous at x_0 then $f + g$ must have a discontinuity at x_0.

4.05 Show that $f + g$ can be continuous at x_0 even though both f and g have discontinuities at x_0.

4.06 Show that $f \cdot g$ can be continuous at x_0 even though both f and g have discontinuities at x_0.

4.07 If f is continuous at x_0 and g is discontinuous at x_0, what can be said about continuity of the product $f \cdot g$ at x_0?

4.08 Show that the composition function $g \circ f$ can be continuous at x_0 even though f or g or both f and g are discontinuous at x_0.

4.09 Prove that the function

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

has a discontinuity of the second king at each nonzero real number.

4.10 Prove that if f is continuous at x_0 and f is nonnegative then $h(x) = \sqrt{f(x)}$ is continuous at x_0.

4.11 Find a function f which has a discontinuity of the second kind at every real number although $f \circ f$ is continuous on $\mathbb{R}$.

4.12 If f is continuous on $(0, 1)$ and $f(x) = 1 - x$ for every rational number $x \in (0, 1)$, find $f(\pi/4)$. Explain your answer.

4.13 Prove that if f and g are each continuous on (a, b) and $f(x) = g(x)$ for every rational $x \in (a, b)$ then $f(x) = g(x)$ for every $x \in (a, b)$.

4.14 Prove: f is right-continuous at x_0 if and only if $f(x_n) \to f(x_0)$ for every sequence $\{x_n\}$ in the domain of f with $x_n \to x_0$ and $x_n \geq x_0$ for $n = 1, 2, 3, ...$

4.15 Discuss one-sided continuity for the pie function.

4.16 Prove that if f is defined on $\mathbb{R}$ and continuous at $x_0 = 0$ and if $f(x_1+x_2) = f(x_1)+f(x_2)$ for each $x_1, x_2 \in \mathbb{R}$ then f is continuous on $\mathbb{R}$.

4.17 Find all functions f which are continuous on $\mathbb{R}$ and which satisfy the equation $f(x)^2 = x^2$ for each $x \in \mathbb{R}$. *Hint*: There are four possible solutions.

4.18 Prove that if g is continuous at $x_0 = 0$, $g(0) = 0$ and for some $\delta > 0$ $|f(x)| \leq |g(x)|$ for each $x \in N_\delta(0)$ then f is continuous at $x_0 = 0$.

4.19 Prove that if f is continuous on $[a, b]$ then there exists a function g continuous on $\mathbb{R}$ such that $g(x) = f(x)$ for each $x \in [a, b]$. The function g is called a *continuous extension* of f to $\mathbb{R}$.

4.20 The function $f(x) = \tan x$ defined on $(-\pi/2, \pi/2)$ clearly has no continuous extension to $\mathbb{R}$. Find a bounded continuous function on (a, b) which has no continuous extension to $\mathbb{R}$.

4.21 Assume that f is continuous on (a, b). Prove that f has a continuous extension to $\mathbb{R}$ if and only if both limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to b^-} f(x)$ exist.

4.22 Prove that if f is continuous on (a, b) and both $\lim_{x \to a^+} f(x)$ and $\lim_{x \to b^-} f(x)$ exist then f is bounded on (a, b).

4.23 Suppose f is one-to-one on (a, b) and satisfies the following property: whenever $f(x_1) \neq f(x_2)$ for $x_1 < x_2$, x_1, $x_2 \in (a, b)$ and k is any number between $f(x_1)$ and $f(x_2)$, there exists a $c \in (x_1, x_2)$ with $f(c) = k$. Prove that f is continuous on (a, b).