1. **Quickie Queries. It is essential you put down reasons for your answers and show your work. 30 points.**

   Throughout assume $g, h \in G$, an abelian group, and that the order of g is 1000.

   1. The order of g^{2120}.
   2. The smallest n such that S_n has an element of the same order as g.
   3. The number of generators of $\langle g \rangle$.
   4. The number of subgroups of $\langle g \rangle$.
   5. The number of subgroups of $\langle g \rangle$ of order 100.
   6. The number of elements of $\langle g \rangle$ of order 100.
   7. Given that h is of order 2400, the largest possible order of an element in G (as far as you know).
   8. An element of that largest order (as in 7).

   Solution.

   1. The order of g^{2120} is
   $$\frac{1000}{\gcd(2120, 1000)} = 25.$$
   2. Since $1000 = 2^3 5^3$, it follows that $n = 2^3 + 5^3 = 133$.
   3. Let $\varphi(n)$ be the number of positive integers relatively prime to a positive integer n. Then the number of generators of $\langle g \rangle$ is $\varphi(1000) = \varphi(2^3 5^3) = \varphi(2^3)\varphi(5^3) = 400$.
   4. The number of subgroups of $\langle g \rangle$ is the number of positive divisors of 1000; since $1000 = 2^3 5^3$, it follows that we have $4 \cdot 4 = 16$ subgroups of $\langle g \rangle$.
   5. There is 1 subgroup of $\langle g \rangle$ of order 100.
   6. There are $\varphi(100) = \varphi(2^2 5^2) = \varphi(2^2)\varphi(5^2) = 40$ elements of $\langle g \rangle$ of order 100.
   7. The largest possible order of an element as far we know is
   $$\frac{1000 \cdot 2400}{\gcd(1000, 2400)} = 12000.$$
   8. The order of h^{25} is 96 and the order of g^8 is 125. Since $\gcd(96, 125) = 1$, it follows that the order of $g^8 h^{25}$ is $96 \cdot 125 = 12000$.

2. **15 points.** Recall that the centralizer of an element $a \in G$ (a group) is given by
$$C(a) = \{g \in G : ag = ga\}.$$

   Do the following:

(1) Show that $gag^{-1} = hah^{-1}$ if and only if $h^{-1}g \in C(a)$.

(2) Assume G is finite. Show that $|C(a)| \times \# = |G|$ where $\#$ is the number of conjugates of a.

Solution.

(1) **Proof.** Suppose $h^{-1}g \in C(a)$. Then

$$\begin{aligned} h^{-1}ga &= ah^{-1}g && \iff \\ ga &= hah^{-1}g && \iff \\ gag^{-1} &= hah^{-1}. \end{aligned}$$

Now suppose $gag^{-1} = hah^{-1}$. Then

$$\begin{aligned} gag^{-1} &= hah^{-1} && \iff \\ ga &= hah^{-1}g && \iff \\ h^{-1}ga &= ah^{-1}g && \iff \\ h^{-1}g &\in C(a). \end{aligned}$$

$\square$

(2) **Proof.** Let $a \in G$. We know that

$$|G_a| \cdot |Ga| = |G|,$$

where G_a is the stablilizer of a and Ga is the orbit of a (note that $\# = |Ga|$). It suffices to show that $C(a) = G_a$. Now

$$\begin{aligned} x &\in C(a) && \iff \\ xa &= ax && \iff \\ xax^{-1} &= a && \iff \\ x &\in Ga, \end{aligned}$$

so that $C(a) = Ga$, and we have that $|C(a)| \cdot |Ga| = |G_a| \cdot \# = |G|$. $\square$

3. Let A be an abelian group with identity e. **15 points.**

(1) Show that $\{a \in A : a^3 = e\}$ is a subgroup.

(2) Find the elements of this subgroup when A is the multiplicative group of nozero elements of $\mathbb{Z}_{19}$.

(3) Give necessary and sufficient conditions on the size of A in order for this subgroup to have other elements besides e, and give reasons.

Solution. Let $G = \{a \in A : a^3 = e\}$.

1. **Proof.** G is clearly associative under the operation of A since it is a subset of A, so in order to show that G is a subgroup, we need to show that it contains e and that it is closed under the operation of A and taking inverses.
   Identity. Clearly $e \in G$ since $e^3 = e$.
   Closure. Suppose $g, h \in G$. Then since G is abelian, it follows that $(gh)^3 = g^3h^3 = ee = e$, so that $gh \in G$.
   Inverse. Suppose $g \in G$. Then it follows that $ggg = g^3 = e$. Now

   $$ggg = e \Rightarrow gg = g^{-1} \Rightarrow g = (g^{-1})^2 \Rightarrow e = (g^{-1})^3 \Rightarrow g^{-1} \in G,$$

   so that G is closed under taking inverses.
   Thus we can conclude that G is a subgroup of A. □
2. We want the elements a of $\mathbb{Z}_{19}$ such that $a^3 = 1$. By computation we find that the subgroup of A that satisfies this condition is $\{1, 7, 13\}$.
3. If $a^3 = e$, then the order of a divides 3 so that the order of a is 1 or 3. So we want the order of a to be 3. Thus we must require that 3 divides $|A|$, so that by Cauchy's Theorem, an element of order 3 will be in G.

4. **On a Group.** Consider $G_3 = \left\{ \begin{pmatrix} 1 & m \\ 0 & s \end{pmatrix} : m \in \mathbb{Z}_8, s \in \{1, 3\} \right\}$. Do 1 through 6. **20 points.**

1. Show G_3 is a subgroup of $GL(2, \mathbb{Z}_8)$ and find its order.
2. Find the order of $h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and its centralizer, $C(h)$.
3. Find all the conjugates of h.
4. Show that regardless of what m is, the centralizer of $g_m = \begin{pmatrix} 1 & m \\ 0 & 3 \end{pmatrix}$, $C(g_m)$, has four elements.
5. Find the center of G_3, $Z(G)$. **Hint.** You basically already have.
6. Decide how many elements of each order there are in G_3.

Solution.

1. We have that $|G_3| = 8 \cdot 2 = 16$. To show that G_3 is a subgroup of $GL(2, \mathbb{Z}_8)$, we need only show that G_3 has an identity and that it is closed under multiplication since it is finite. Notice that G_3 is associative under multiplication since $\mathbb{Z}_8$ is associative under multiplication.
   Identity. If we let $m = 0$ and $s = 1$, we shall see that G_3 contains the identity.
   Closure. Let $\begin{pmatrix} 1 & m \\ 0 & s \end{pmatrix}, \begin{pmatrix} 1 & n \\ 0 & t \end{pmatrix} \in G_3$. Then we have that

   $$\begin{pmatrix} 1 & m \\ 0 & s \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & n + mt \\ 0 & st \end{pmatrix} \in G_3$$

because $\{1,3\}$ and $\mathbb{Z}_8$ are both closed under multiplication. Thus G_3 is closed under multiplication. Hence G_3 is a subgroup of $GL(2,\mathbb{Z}_8)$.

2 **Order of** h. Since $h^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, it follows that the order of h is 8.

Centralizer of h. Let $\begin{pmatrix} 1 & m \\ 0 & s \end{pmatrix} \in C(h)$. Then it follows that

$$\begin{pmatrix} 1 & m \\ 0 & s \end{pmatrix} h = \begin{pmatrix} 1 & 1+m \\ 0 & s \end{pmatrix} = \begin{pmatrix} 1 & s+m \\ 0 & s \end{pmatrix} = h \begin{pmatrix} 1 & m \\ 0 & s \end{pmatrix},$$

so that $1+m = s+m$. That is, $s = 1$. Thus

$$C(h) = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z}_8 \right\}.$$

3 The set of conjugates of h is $\{ghg^{-1} : g \in G_3\}$. So let $g = \begin{pmatrix} 1 & m \\ 0 & s \end{pmatrix} \in G_3$. Then we have that $ghg^{-1} = \begin{pmatrix} 1 & s^{-1} \\ 0 & 1 \end{pmatrix}$, so that the conjugates of h are $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.

4 **Proof.** Let $m \in \mathbb{Z}_8$ and $g_m = \begin{pmatrix} 1 & m \\ 0 & 3 \end{pmatrix}$.We want to show that $|C(g_m)| = 4$. Let $\begin{pmatrix} 1 & n \\ 0 & t \end{pmatrix} \in C(g_m)$. Then it follows that

$$\begin{pmatrix} 1 & n \\ 0 & t \end{pmatrix} g_m = \begin{pmatrix} 1 & m+3n \\ 0 & s \end{pmatrix} = \begin{pmatrix} 1 & mt+n \\ 0 & s \end{pmatrix} = g_m \begin{pmatrix} 1 & n \\ 0 & t \end{pmatrix},$$

so that $m + 3n = mt + n$; that is, $m + 2n = mt$.
Case 1. $t = 1$. Then we have that $m + 2n = m$ so that $2n = 0$; that is $n \in \{0, 4\}$.
Case 2. $t = 3$. Then we have that $m + 2n = 3m$ so that $2m = 2n$. Notice that since $m \in \mathbb{Z}_8$, then $2m \in \{0, 2, 4, 6\}$. For each value of $2m$, the equation $2m = 2n$ has exactly two solutions for n and they are

$2m$	n
0	{0, 4}
2	{1, 5}
4	{2, 6}
6	{3, 7}

.

Thus for each value of t, n has exactly two values, so that there are 4 matrices in $C(g_m)$. □

5 The center of G_3 is the set of elements of G_3 that commute with all the elements of G_3. Notice that if an element is in the center of G_3, then it must be in the

centralizer of all elements of G_3. By (4), we know that

$$h^8 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } h^4 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$$

are the only elements in $\bigcap_{m \in \mathbb{Z}_8} C(g_m)$ (the intersection of the centralizers of matrices of the form $\begin{pmatrix} 1 & m \\ 0 & 3 \end{pmatrix}$). Since the remaining matrices (matrices of the form $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$) are all powers of h, it follows that h^8 and h^4 also commute with them. Thus the center of G_3 consists of h^8 and h^4.

(6)

Matrix	Order
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1
$\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 6 \\ 0 & 3 \end{pmatrix}$	2
$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 7 \\ 0 & 3 \end{pmatrix}$	4
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}$	8

.

5. **On the Same Group.** Let G_3 be the group from the previous exercise. Let it act on the set X of vectors of size 2: $\begin{pmatrix} x \\ y \end{pmatrix}$ with entries in $\mathbb{Z}_8$. Do the following: **25 points.**

(1) Count the number of fixed points of $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.

(2) Count the number of fixed points of $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$.

(3) Count the number of fixed points of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(4) Finish filling the table below with the number of fixed points below each of the respective matrices:

$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}$
16	32	8	16	8	8	16	8
$\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 3 \\ 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 5 \\ 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 6 \\ 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 7 \\ 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
8	8	16	8	16	8	16	64

.

(5) Use Burnside's Lemma to count the number of orbits.

(6) Find the stabilizer of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and use it to find the size of its orbit.

7 Find the stabilizer of $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$, and use it to find the size of its orbit.

8 Find the stabilizer of $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$, and use it to find the size of its orbit.

Solution.

1 We want to find all x, $y \in \mathbb{Z}_8$ such that

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

From the above, we shall get

$$\begin{pmatrix} x + 2y \\ 3y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

so that $x + 2y = x$ and $2y = 0$. These equations both simplify to $2y = 0$; thus $y \in \{0, 4\}$. So we have 8 choices for x and 2 choices for y for a total of 16 choices. The number of fixed points of $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ is 32.

2 We want to find all x, $y \in \mathbb{Z}_8$ such that

$$\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

From the above, we shall get

$$\begin{pmatrix} x + 4y \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

so that $x + 4y = x$. That is $4y = 0$, so that $y \in \{0, 2, 4, 6\}$. So we have 8 choices for x and 4 choices for y for a total of 32 choices. The number of fixed points of $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ is thus 32.

3 Proceed as in 1 and 2 above to get $x + y = x$, so that $y = 0$; so we have 8 choices for x and 1 choices for y for a total of 8 choices. The number of fixed points of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is thus 8.

4 The table has been filled above.

5 According to Burnside's Lemma, the number of orbits is

$$\frac{16 + 32 + 8 + 16 + 8 + 8 + 16 + 8 + 8 + 8 + 16 + 8 + 16 + 8 + 16 + 64}{16} = 16.$$

6 Let $X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We want to find all $\begin{pmatrix} 1 & m \\ 0 & s \end{pmatrix} \in G_3$ such that

$$\begin{pmatrix} 1 & m \\ 0 & s \end{pmatrix} X_1 = X_1.$$

From the above equation, we shall get $X_1 = X_1$, so that all of G_3 stabilizes X_1. That is, $|\text{Stabilizer}(X_1)| = |G_3|$. Recall that

$$|\text{Stabilizer}(X_1)| \cdot |\text{Orbit}(X_1)| = |G_3|.$$

Thus $|\text{Orbit}(X_1)| = 1$.

7 Similarly let $X_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. We want to find all $\begin{pmatrix} 1 & m \\ 0 & s \end{pmatrix} \in G_3$ such that

$$\begin{pmatrix} 1 & m \\ 0 & s \end{pmatrix} X_2 = X_2.$$

From the above equation, we shall get

$$\begin{pmatrix} 1+4m \\ 4s \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix},$$

so that all $4m = 0$ and $4s = 4$. Thus $m \in \{0, 2, 4, 6\}$ and $s \in \{1, 3\}$. Thus $|\text{Stabilizer}(X_2)| = 8$. Since

$$|\text{Stabilizer}(X_2)| \cdot |\text{Orbit}(X_2)| = |G_3|,$$

it follows that $|\text{Orbit}(X_2)| = 2$.

8 Similarly let $X_3 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. We want to find all $\begin{pmatrix} 1 & m \\ 0 & s \end{pmatrix} \in G_3$ such that

$$\begin{pmatrix} 1 & m \\ 0 & s \end{pmatrix} X_3 = X_3.$$

From the above equation, we shall get

$$\begin{pmatrix} 2+2m \\ 2s \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix},$$

so that all $2m = 0$ and $2s = 2$. Thus $m \in \{0, 4\}$ and $s = 1$. Thus $|\text{Stabilizer}(X_3)| = 2$. Since

$$|\text{Stabilizer}(X_3)| \cdot |\text{Orbit}(X_3)| = |G_3|,$$

it follows that $|\text{Orbit}(X_3)| = 8$.

6. **True or False.** Consider the veracity or falsehood of each of the following statements, and argue as well as you can for those that you believe are true while providing a counterexample for those that you believe are false. There is partial credit for just the correct answer. **20 points**

1. Every abelian group of order 525 has an element of order 21.
2. The permutation (123)(2465)(5674)(12578)(456312).
3. $S = \left\{ \begin{pmatrix} a & b \\ -3b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ is an integral domain.
4. A subgroup of a cyclic group is cyclic.
5. It is possible for an infinite group to have subgroups of finite index.
6. Any group of order 12 has an element of order 6.
7. Every group of order 1001 has an element of order 7.
8. Every field is an integral domain.
9. Let G and H be groups. Let $f : G \to H$ be a homomorphism onto H. If G is abelian, then so is H.
10. Let G and H be groups. Let $f : G \to H$ be a homomorphism onto H. If H is abelian, then so is G.

Solution.

1. True.
   Proof. Let G be an abelian group of order 525. Because the primes 3 and 7 both divide $|G|$, it follows by Cauchy's Theorem that G has an element h of order 3 and an element g of order 7. Since 3 and 7 are relatively prime and since G is abelian, it must have an element of order $3 \cdot 7 = 21$. ☐
2. False because the sign of the permutation is

   $$\text{sign}(123)\cdot\text{sign}(2465)\cdot\text{sign}(5674)\cdot\text{sign}(12578)\cdot\text{sign}(456312) = 1\cdot-1\cdot-1\cdot 1\cdot -1 = -1.$$

3. True.
   Proof. First observe that the determinant of a matrix in S is 0 if and only it is the zero matrix. Also we know that S is a commutative ring. Now let $A_1, A_2 \in S$. Suppose $A_1 A_2 = \mathbf{0}$. Then we must have that

   $$0 = \det(\mathbf{0}) = \det(A_1 A_2) = \det(A_1)\det(A_2),$$

   so that $\det(A_1) = 0$ or $\det(A_2) = 0$. That is $A_1 = \mathbf{0}$ or $A_2 = \mathbf{0}$, so that S is an integral domain. ☐
4. True. Let $G = \langle g \rangle$ be a cyclic group and let H be a subgroup of G. Then H is also cyclic since it is generated by g^r where r is the smallest positive integer such that $g^r \in H$.
5. True. Although $\mathbb{Z}$ is infinite, its subgroup, $60\mathbb{Z}$, has index 60.
6. False because A_4 has order 12 but it has no element of order 6 (since A_4 doesn't even have a subgroup of order 6).

(7) True.

Proof. Let G be a group of order 1001. Since 7 is prime and since 7 divides 1001, it follows at once by Cauchy's Theorem that G has an element of order 7. $\square$

(8) True.

Proof. Let F be a field. Suppose for some $a, b \in F$, with $a \neq 0$, we have $ab = 0$. Since F is a field and since a is nonzero, it follows that a^{-1} exists. Thus we have that $0 = ab = a^{-1}(ab) = (a^{-1}a)b = b$, so that F is an integral domain. $\square$

(9) True.

Proof. Let $x, y \in H$. Then since f is onto, there exist $a, b \in G$ such that $f(a) = x$ and $f(b) = y$. Thus

$$xy = f(a)f(b) = f(ab) = f(ba) = f(b)f(a) = yx,$$

so that H is also abelian. $\square$

(10) False.

Counterexample. Consider $f : S_6 \to \{e\}$. This map is onto and $\{e\}$ is trivially abelian. Also f is a homomorphism, but S_6 is not abelian.