1. Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false. Let $G = \langle g \rangle$ have order 300.

   (1) There are exactly 80 generators of G.

   (2) G has only one element of order 3.

   (3) G can be embedded in S_{30}.

   (4) G has a subgroup of order 20.

   (5) G has a totality of 18 subgroups.

   Solution.

   (1) True. Since $G = \langle g \rangle$ is cyclic and since $|g| = 300$, it follows that the number of generators of G is the number of positive integers relatively prime to 300, which is 80.

   (2) False.
   Counterexample. We have that $g^{100} \neq g^{200}$ (since $|g| = 300$) and

   $$|g^{100}| = \frac{300}{\gcd(300, 100)} = 3 = \frac{300}{\gcd(300, 200)} = |g^{200}|.$$

   (3) False.
   Proof. It suffices to show that S_{30} has no element of order 300. Suppose to the contrary that $\sigma \in S_{30}$ has order 300. Then we can write σ as a product of disjoint cycles (each of length greater than 1)

   $$\sigma = \alpha_1 \alpha_2 \cdots \alpha_n$$

   so that $|\sigma| = \operatorname{lcm}(|\alpha_1|, |\alpha_2|, \cdots, |\alpha_n|) = 300$. Now since $5^2 \mid 300$, it follows that 5^2 must divide the order of at least one of the cycles. We can assume without loss that $5^2 \mid |\alpha_1|$. Thus α_1 must be a 25-cycle. By a similar argument, it follows that 2^2 must divide the order of at least one of the cycles. Assume without loss that $2^2 \mid |\alpha_2|$. Since there are 25 elements in α_1, there can be at most 5 elements in α_2, so that α_2 is a 4-cycle. Thus

   $$\sigma = \alpha_1 \alpha_2,$$

   a contradiction since $\operatorname{lcm}(|\alpha_1|, |\alpha_2|) = 100 \neq 300$. $\square$

   (4) True. This subgroup of G, $\langle g^{15} \rangle$, has 20 elements.

   (5) True. Since the number of positive divisors of 300 is 18, it follows that G has exactly 18 subgroups.

2. Let G be an abelian group and let $a, b \in G$ be of order 120 and 72 respectively. Do the following:

(1) Find an element of order 15.

(2) What is the order of b^{10}?

(3) Find an element of as large an order as you can.

Solution.

(1) The element a^8 has order 15.

(2) $|b^{10}| = \dfrac{72}{\gcd(72, 10)} = 36.$

(3) The element a^{24} has order 5; since $\gcd(5, 72) = 1$, it follows that $a^{24}b$ has order 360.

3. Consider the non-abelian group of order 55 from **Homework #4**. View this group as acting on all column vectors of size 2 (with entries in $\mathbb{Z}_{11}$).

(1) Find the number of fixed points of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(2) Find the number of fixed points of $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$.

(3) Decide on the number of fixed elements each of the elements of the group has.

(4) Use Burnside's Lemma to count the orbits.

Solution.

(1) Suppose $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ for some $a, b \in \mathbb{Z}_{11}$. Then it follows that

$$\begin{pmatrix} a+b \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix},$$

so that $b = 0$. Thus the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ fixes 11 elements and they are

$$\left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} : a \in \mathbb{Z}_{11} \right\}.$$

(2) Suppose $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ for some $a, b \in \mathbb{Z}_{11}$. Then it follows that

$$\begin{pmatrix} 3a \\ 4b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix},$$

so that $2a = 0$ and $3b = 0$. Multiply the former equality by 6 and the latter by 4 to get $a = b = 0$.Thus the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ fixes only 1 element, the zero vector.

(3) There are 44 matrices of the form $\begin{pmatrix} b & x \\ 0 & b^{-1} \end{pmatrix}$, with $b \neq 1$, and each only fixes the zero vector. The identity matrix fixes all the vectors (121 of them), while each of the remaining 10 matrices fixes exactly 11 vectors.

(4) Let n be the number of orbits. Using our results from (3) and Burnside's Lemma, it follows that that

$$n \cdot 55 = 44 \cdot 1 + 1 \cdot 121 + 10 \cdot 11 = 275,$$

so that $n = 5$.

4. Let the vertices of the cube be given as follows:

$$1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, 2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, 3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, 4 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, 5 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, 6 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, 7 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$

and $8 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$.

(1) Label the faces A, A', B, B', C, and C' (where the prime means opposite), and give each as a set of four vertices. Let A be the intersection with the plane $x = 1$, B with the plane $y = 1$ and C with $z = 1$.

(2) Find 24 3×3 matrices of determinant 1 that are isometries of the cube, and write each as a permutation in S_8 (of the eight vertices) and also as a permutation of the faces. **Hint:** Start with the six permutation matrices of size 3.

Assume these 24 matrices form a group G. **Bonus.** Prove this.
Assume $G \simeq S_4$. **Bonus.** Prove this.

(3) Find the conjugacy classes of G.

(4) Find the number of ways to color a cube with two colors.

(5) Find the number of ways to color a cube with three colors.

Bonus. Find the number of ways to color the cube with n colors.

Solution.

(1) We have (Right Hand Coordinate System)

$$\begin{aligned} A &= \{1, 2, 3, 5\} \\ A' &= \{4, 6, 7, 8\} \\ B &= \{1, 2, 4, 6\} \\ B' &= \{3, 5, 7, 8\} \\ C &= \{1, 3, 4, 7\} \\ C' &= \{2, 3, 5, 8\} \end{aligned}$$

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$$\begin{aligned}
\text{P(faces)} &= \text{Permutation of faces}\\
\text{P(vertices)} &= \text{Permutation of vertices}\\
\text{ccw} &= \text{counterclockwise}\\
\overrightarrow{ab} &= \text{line that passes through vertices } a \text{ and } b.\\
B(\overrightarrow{abcd}) &= \text{line that bisects egdes } ab \text{ and } cd.
\end{aligned}$$

	Matrix	P(vertices)	P(faces)	Rotation
1	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	(1)	(A)	Identity
2	$\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}$	(1 4 7 3)(2 6 8 5)	$(ABA'B')$	90° ccw across z-axis.
3	$\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	(1 7)(2 8)(3 4)(5 6)	$(AA')(BB')$	180° ccw across z-axis.
4	$\begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix}$	(1 3 7 4)(2 5 8 6)	$(AB'A'B)$	270° ccw across z-axis.
5	$\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}$	(1 3 5 2)(4 7 8 6)	$(BCB'C')$	90° ccw across x-axis.
6	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$	(1 5)(2 3)(4 8)(6 7)	$(BB')(CC')$	180° ccw across x-axis.
7	$\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}$	(1 2 5 3)(4 6 8 7)	$(BC'B'C)$	270° ccw across x-axis.
8	$\begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix}$	(1 2 6 4)(3 5 8 7)	$(AC'A'C)$	90° ccw across y-axis.

	Matrix	P(vertices)	P(faces)	Rotation
9	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	(1 6)(2 4)(3 8)(5 7)	$(AA')(CC')$	180° ccw across y-axis.
10	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	(1 4 6 2)(3 7 8 5)	$(ACA'C')$	270° ccw across y-axis.
11	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	(2 3 4)(5 7 6)	$(ACB)(A'C'B')$	120° ccw across $\overrightarrow{18}$.
12	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	(2 4 3)(5 6 7)	$(ABC)(A'B'C')$	240° ccw across $\overrightarrow{18}$.
13	$\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	(1 7 6)(2 3 8)	$(AB'C')(BCA')$	120° ccw across $\overrightarrow{45}$.
14	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$	(1 6 7)(2 8 3)	$(AC'B')(A'CB)$	240° ccw across $\overrightarrow{45}$.
15	$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$	(1 6 5)(3 4 8)	$(ABC')(A'B'C)$	120° ccw across $\overrightarrow{27}$.
16	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$	(1 5 6)(3 8 4)	$(AC'B)(A'CB')$	240° ccw across $\overrightarrow{27}$.

	Matrix	P(vertices)	P(faces)	Rotation
17	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$	(1 7 5)(2 4 8)	$(ACB')(BA'C')$	120° ccw across $\overrightarrow{36}$.
18	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	(1 5 7)(2 8 4)	$(AB'C)(A'BC')$	240° ccw across $\overrightarrow{36}$.
19	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	(1 8)(2 7)(3 5)(4 6)	$(AB')(BA')(CC')$	180° ccw across $B(\overrightarrow{4635})$.
20	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	(1 2)(3 6)(4 5)(7 8)	$(AB)(CC')(A'B')$	180° ccw across $B(\overrightarrow{1278})$.
21	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$	(1 8)(2 6)(3 7)(4 5)	$(AA')(BC')(B'C)$	180° ccw across $B(\overrightarrow{2637})$.
22	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	(1 4)(2 7)(3 6)(5 8)	$(AA')(BC)(B'C')$	180° ccw across $B(\overrightarrow{1458})$.
23	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	(1 8)(2 5)(3 6)(4 7)	$(AC')(A'C)(BB')$	180° ccw across $B(\overrightarrow{6813})$.
24	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	(1 3)(2 7)(4 5)(6 8)	$(AC)(BB')(A'C')$	180° ccw across $B(\overrightarrow{7452})$.

Bonus. Show that G is a group.

Proof. The set G is associative under multiplication because matrix multiplication is associative under multiplication. The set G contains the 3×3 identity so that G has an identity. Since G is finite, we need only show that it is closed under multiplication to complete the proof. Let A and B be two matrices in G. Then A and B correspond to some permutations σ_A and σ_B of the vertices of the cube. Thus AB corresponds to $\sigma_A \circ \sigma_B$ which is also a permutation of the vertices of the cube. Since σ_A and σ_B are rotations, $\sigma_A \circ \sigma_B$ must also be a rotation so that $AB \in G$. Thus G is a group under multiplication. ☐

3. Let M_i denote the matrix on line i from our results in 1. Since $G \simeq S_4$, it follows that the conjugacy classes of G are:

$$\begin{aligned} 1+1+1+1 &= \{M_1\} \\ 2+1+1 &= \{M_{19}, M_{20}, M_{21}, M_{22}, M_{23}, M_{24}\} \\ 3+1 &= \{M_{11}, M_{12}, M_{13}, M_{14}, M_{15}, M_{16}, M_{17}, M_{18}\} \\ 2+2 &= \{M_3, M_6, M_9\} \\ 4 &= \{M_2, M_4, M_5, M_7, M_8, M_{10}\} \end{aligned}$$

4. Setting $n = 3$ in the equation in the Bonus below we have that there are 57 ways to color a cube with three colors.

(5) Setting $n = 2$ in the equation in the Bonus below we have that there are 10 ways to color a cube with three colors.

Bonus.

Conjugacy Class	Representative g	#g	# of elements
$1+1+1+1$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	n^6	1
$2+1+1$	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	n^3	6
$3+1$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	n^2	8
$2+2$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	n^4	3
4	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	n^3	6

Thus the number of ways to color a cube with n colors is

$$\frac{n^6 + 6n^3 + 8n^2 + 3n^4 + 6n^3}{24} = \frac{n^6 + 3n^4 + 12n^3 + 8n^2}{24}.$$