Lemma 7.1 *If $T \in \mathcal{L}(V)$ is normal, then null T = null T^*.*

Proof. Assume that $T \in \mathcal{L}(V)$ is normal. Thus

$$\begin{aligned} x \in \text{null } T &\iff Tx = 0 \\ &\iff \langle Tx, Tx \rangle = 0 \\ &\iff \langle T^*x, T^*x \rangle = 0 && \text{[Proposition 7.6]} \\ &\iff T^*x = 0 \\ &\iff x \in \text{null } T^*, \end{aligned}$$

so that null T = null T^*. □

7.1 Make $\mathcal{P}_2(\mathbb{R})$ into an inner-product space by defining

$$\langle p, q \rangle = \int_0^1 p(x)q(x)\, dx.$$

Define $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ by $T(a_0 + a_1x + a_2x^2) = a_1x$.

(a) Show that T is not self-adjoint.

(b) The matrix of T with respect to the basis $(1, x, x^2)$ is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix equals its conjugate transpose, even though T is not self-adjoint. Explain why this is not a contradiction.

Solution.

(a) Suppose to the contrary that T is self-adjoint. Consider $x, 1 \in \mathcal{P}_2(\mathbb{R})$. Then we must have that $\langle T1, x \rangle = \langle T^*1, x \rangle = \langle 1, Tx \rangle$. But

$$\begin{aligned} \langle T1, x \rangle &= \langle 0, x \rangle \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \langle 1, Tx \rangle &= \langle 1, x \rangle \\ &= \int_0^1 x\, dx \\ &= \frac{1}{2}, \end{aligned}$$

so that $\langle T1, x \rangle \neq \langle 1, Tx \rangle$; i.e., T is not self-adjoint.

(b) Let N be the conjugate transpose of $\mathcal{M}(T)$. It is clear that $N = \mathcal{M}(T)$. We know that T is self-adjoint if and only if $\mathcal{M}(T) = \mathcal{M}(T^*)$ if and only if $N = \mathcal{M}(T^*)$. Since T is not self-adjoint it follows that $N \neq \mathcal{M}(T^*)$. This makes sense because the equality $N = \mathcal{M}(T^*)$ is guaranteed to be true (by Proposition 6.47) if $(1, x, x^2)$ is an orthonormal basis for $\mathcal{P}_2(\mathbb{R})$, but this is not the case since $\langle 1, x\rangle \neq 0$. So although we have that the matrix of T equals its conjugate transpose, it does not equal the matrix of T^*, so we have no contradiction.

7.4 Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that P is an orthogonal projection if and only if P is self-adjoint.

Proof. Let $P \in \mathcal{L}(V)$ such that $P^2 = P$.

($\Longrightarrow$) Suppose that P is an orthogonal projection. Then we have that $P = P_{U,U^\perp}$ for some subspace U of V. To prove that P is self-adjoint, we must show that $Px = P^*x$ for all $x \in V$. So let $v_1, v_2 \in V$. There exist unique $u_1, u_2 \in U$ and $u_1', u_2' \in U^\perp$ such that $v_1 = u_1 + u_1'$ and $v_2 = u_2 + u_2'$. Thus

$$\begin{aligned}
\langle Pv_1 - P^*v_1, v_2\rangle &= \langle Pv_1, v_2\rangle - \langle P^*v_1, v_2\rangle \\
&= \langle u_1, v_2\rangle - \langle v_1, Pv_2\rangle \\
&= \langle u_1, v_2\rangle - \langle v_1, u_2\rangle \\
&= \langle u_1, u_2 + u_2'\rangle - \langle u_1 + u_1', u_2\rangle \\
&= \langle u_1, u_2\rangle + \langle u_1, u_2'\rangle - \langle u_1, u_2\rangle - \langle u_1', u_2\rangle \\
&= \langle u_1, u_2'\rangle - \langle u_1', u_2\rangle \\
&= 0 - 0 = 0. \qquad [\text{Since } u_1, u_2 \in U, u_1', u_2' \in U^\perp]
\end{aligned}$$

Since v_2 was arbitrary, we have that $\langle Pv_1 - P^*v_1, v_2\rangle = 0$ for all $v_2 \in V$. Setting $v_2 = Pv_1 - P^*v_1$ gives us that $\langle Pv_1 - P^*v_1, Pv_1 - P^*v_1\rangle = 0$, so that $Pv_1 - P^*v_1 = 0$; i.e., $Pv_1 = P^*v_1$. Thus P is self-adjoint.

($\Longleftarrow$) Suppose that P is self-adjoint. We have that

$$\begin{aligned}
\text{null } P &= (\text{range } P*)^\perp && [\text{Proposition 6.46}] \\
&= (\text{range } P)^\perp && [P \text{ is self-adjoint}],
\end{aligned}$$

so that every vector in null P is orthogonal to every vector in range P. It follows by Homework 5, Problem 6.17 that P is an orthogonal projection of V onto range P, with null space null P. □

7.6 Prove that if $T \in \mathcal{L}(V)$ is normal, then

$$\text{range } T = \text{range } T^*.$$

Proof. Assume that $T \in \mathcal{L}(V)$ is normal. We have that

$$\begin{aligned}
\text{range } T &= (\text{null } T^*)^\perp && [\text{Proposition 6.46}] \\
&= (\text{null } T)^\perp && [\text{Lemma 7.1}] \\
&= \text{range } T^*, && [\text{Proposition 6.46}]
\end{aligned}$$

which is what we wanted to prove. □

7.7 Prove that if $T \in \mathcal{L}(V)$ is normal, then

$$\text{null } T^k = \text{null } T \text{ and range } T^k = \text{range } T$$

for every positive integer k.

Proof. We shall first proceed by induction on k to show that null $T^k =$ null T.

Base Case. $k = 1$. It follows trivially that null $T^1 =$ null T.

Inductive Hypothesis. Suppose that null $T^n =$ null T for some positive integer n.

Now we shall show that null $T^{n+1} =$ null T. We have that

$$\begin{aligned} x \in \text{null } T &\Longrightarrow x \in \text{null } T^n && \text{[Inductive Hypothesis]} \\ &\Longrightarrow T^n x = 0 \\ &\Longrightarrow T(T^n x) = T(0) = 0 \\ &\Longrightarrow T^{n+1} x = 0 \\ &\Longrightarrow x \in \text{null } T^{n+1}, \end{aligned}$$

so that null $T \subseteq$ null T^{n+1}. Now let $x \in$ null T^{n+1}. By Problem 7.6, we know that range $T =$ range T^*; thus $Tx = T^*y'$ for some $y' \in V$. Now

$$\begin{aligned} x \in \text{null } T^{n+1} &\Longrightarrow T^{n+1} x = 0 \\ &\Longrightarrow TT^n x = 0 \\ &\Longrightarrow \langle TT^n x, T^{n-1} y' \rangle = 0 \\ &\Longrightarrow \langle T^n x, T^* T^{n-1} y' \rangle = 0 \\ &\Longrightarrow \langle T^n x, T^{n-1} T^* y' \rangle = 0 && [T \text{ is normal}] \\ &\Longrightarrow \langle T^n x, T^{n-1} T x \rangle \\ &\Longrightarrow \langle T^n x, T^n x \rangle = 0 \\ &\Longrightarrow T^n x = 0 \\ &\Longrightarrow x \in \text{null } T^n \\ &\Longrightarrow x \in \text{null } T, && \text{[Inductive Hypothesis]} \end{aligned}$$

so that null $T^{n+1} \subseteq$ null T. We have thus shown that null $T =$ null T^{n+1}. It follows by Mathematical Induction that null $T^k =$ null T for each positive integer k. Let m be a positive integer. Observe that T^m is also normal; thus

$$\begin{aligned} \text{range } T^m &= (\text{null } (T^m)^*)^\perp && \text{[Proposition 6.46]} \\ &= (\text{null } T^m)^\perp && \text{[Lemma 7.1]} \\ &= (\text{null } T)^\perp && [\text{null } T^m = \text{null } T] \\ &= \text{range } T^* && \text{[Proposition 6.46]} \\ &= \text{range } T, && \text{[Problem 7.6]} \end{aligned}$$

which is what we wanted to show. □

7.9 Prove that a normal operator on a complex inner-product space is self-adjoint if and only if all its eigenvalues are real.

Proof. Let $T \in \mathcal{L}(V)$ be normal, where V is a complex inner-product space.

($\Longrightarrow$) Suppose that T is self-adjoint. It follows by Proposition 7.1 that all the eigenvalues of T are real.

($\Longleftarrow$) Suppose that all the eigenvalues of T are real. Note that if u is an eigenvector of T with eigenvalue λ, then by Corollary 7.7, we have that $T^*u = \overline{\lambda}u$. To show that T is self-adjoint, we must show that $Tx = T^*x$ for all $x \in V$. So let $y \in V$. By the Spectral Theorem, V has an orthonormal basis $(v_1, \dots v_n)$ consisting of eigenvectors of T. Let λ_i be the eigenvalue corresponding to v_i. We know that $y = a_1v_1 + \cdots + a_nv_n$ for some unique scalars. Thus

$$\begin{aligned}
Ty &= T(a_1v_1 + \cdots + a_nv_n) \\
&= a_1Tv_1 + \cdots + a_nTv_n \\
&= a_1\lambda_1v_1 + \cdots + a_n\lambda_nv_n \\
&= a_1\overline{\lambda_1}v_1 + \cdots + a_n\overline{\lambda_n}v_n && \text{[Eigenvalues of } T \text{ are real]} \\
&= a_1T^*v_1 + \cdots + a_nT^*v_n \\
&= T^*(a_1v_1 + \cdots + a_nv_n) \\
&= T^*y,
\end{aligned}$$

so that $T = T^*$. $\square$