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1. If Z is a standard normal variable, find

   (a) $P(Z^2 < 1)$ $\qquad$ (b) $P(Z^2 > 3.84146)$.

   Solution.

   (a) We have that

   $$P(Z^2 < 1) = P(-1 < Z < 1) = 1 - 2 \cdot P(Z > 1) \approx 0.6826,$$

   and

   (b)

   $$P(Z^2 > 3.84146) = 2 \cdot P(Z > \sqrt{3.84146}) \approx 2 \cdot P(Z > 1.96) \approx 0.05.$$

2. If Y is a normal random variable with $\mu = 20$ and variance $\sigma^2 = 4$, i.e., $Y \sim N(20, 4)$, find

   (a) $P(16 \leq Y \leq 22)$ $\qquad$ (b) $P(100 < 9Y - 80 < 145)$.

   Solution.

   (a) We have that

   $$\begin{aligned} P(16 \leq Y \leq 22) &= P\left(\frac{16-20}{2} \leq Z \leq \frac{22-20}{2}\right) \\ &= P(-2 \leq Z \leq 1) \\ &= 1 - [P(Z < -2) + P(Z > 1)] \\ &= 1 - [P(Z > 2) + P(Z > 1)] \\ &\approx 0.8185, \end{aligned}$$

   and

   (b)

   $$\begin{aligned} P(100 < 9Y - 80 < 145) &= P(20 < Y < 25) \\ &= P\left(\frac{20-20}{2} < Z < \frac{25-20}{2}\right) \\ &= P(0 < Z < 2.5) \\ &= P(Z > 0) - P(Z > 2.5) \\ &\approx 0.4938. \end{aligned}$$

3. The scores of a pre-employment test are normally distributed with mean $\mu = 70$ and standard deviation $\sigma = 5$. If only the top 1.5% of the applicants (based on their score on the pre-employment test) are to be considered, find the cut-off score (i.e., the value such that only 1.5% of the applicants score this value or higher).

   Solution. Let y be the cut-off score. Then we have that

   $$0.0015 = P(Y \geq y) = P\left(Z \geq \frac{y-70}{5}\right),$$

   so that $(y - 70)/5 \approx 2.97$; i.e., $y \approx 85$.

4. Using the fact that $\int_0^\infty e^{-y^2/2}dy = \sqrt{\frac{\pi}{2}}$, show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ by making the transformation $y = \frac{1}{2}x^2$.

Proof. Using the transformation $y = \frac{1}{2}x^2$ we have that

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty y^{-\frac{1}{2}}e^{-y}dy \\ &= \int_0^\infty \frac{\sqrt{2}}{x}e^{-\frac{1}{2}x^2}x\,dx \\ &= \sqrt{2}\int_0^\infty e^{-\frac{1}{2}x^2}\,dx \\ &= \sqrt{2}\sqrt{\frac{\pi}{2}} = \sqrt{\pi},\end{aligned}$$

as desired. $\square$

5. If Y has an exponential distribution with $P(Y < 3) = 0.4512$, find
(a) $E[Y]$ (b) $P(Y \geq 2)$.

Solution.

(a) We have that

$$\begin{aligned}0.4512 &= P(Y < 3) \\ &= P(Y \leq 3) \\ &= F(3) \\ &= \int_{-\infty}^3 \frac{1}{\beta}e^{-\frac{y}{\beta}}dy \\ &= \int_0^3 \frac{1}{\beta}e^{-\frac{y}{\beta}}dy \\ &= -e^{-\frac{3}{\beta}} + 1,\end{aligned}$$

so that $e^{-\frac{3}{\beta}} = 0.5488$; i.e., $\beta \approx 5$. Thus $E[Y] \approx 5$.

(b)

$$\begin{aligned}P(Y \geq 2) &= 1 - P(Y < 2) \\ &= 1 - \int_0^2 \frac{1}{\beta}e^{-\frac{y}{\beta}}dy \\ &= e^{-\frac{2}{\beta}} \\ &\approx 0.6703.\end{aligned}$$

6. The length of time Y necessary to complete a key operation in the construction of houses has an exponential distribution with mean 10 hrs. The formula $C = 100 + 40Y + 3Y^2$ gives the cost C of completing the operation. Find the mean and variance of C.

Solution. First we want to find $E[Y^2]$. So

$$\begin{aligned}
E[Y^2] &= \frac{1}{10}\lim_{t\to\infty}\int_0^t y^2 e^{-\frac{y}{10}}\,dy \\
&= \lim_{t\to\infty}\left[-y^2 e^{-\frac{y}{10}}\Big|_0^t + 2\int_0^t y e^{-\frac{y}{10}}\,dy\right] && \text{[Integration by parts]} \\
&= 2\lim_{t\to\infty}\left[\int_0^t y e^{-\frac{y}{10}}\,dy\right] \\
&= 2\lim_{t\to\infty}\left[-10y e^{-\frac{y}{10}}\Big|_0^t + 10\int_0^t e^{-\frac{y}{10}}\,dy\right] && \text{[Integration by parts]} \\
&= 20\lim_{t\to\infty}\left[\frac{1}{10}\int_0^t e^{-\frac{y}{10}}\,dy\right] \\
&= 20\cdot E[Y] = 200.
\end{aligned}$$

Now the mean of C is given by $E[C]$ so that

$$\begin{aligned}
E[C] &= E[100 + 40Y + 3Y^2] \\
&= E[100] + 40E[Y] + 3E[Y^2] \\
&= 100 + 40\cdot 10 + 3\cdot 200 \\
&= 1100,
\end{aligned}$$

and the variance of C, $V[Y]$, is $E[Y^2] - E[Y]^2 = 2000 - 100 = 1900$.

7. Suppose Y has density function $f(y) = ky^9 e^{-y/2}$, $y \geq 0$. Find
   (a) k.
   (b) $E[Y]$ and $V(Y)$.
   (c) $P(Y > 34.1696)$.
   (d) A value b such that $P(Y < b) = 0.10$.

Solution. By inspection we can see that f is the gamma distribution with $\alpha = 10$, $\beta = 2$.

(a) $k = \dfrac{1}{2^{10}\cdot\Gamma(10)} = \dfrac{1}{2^{10}\cdot 9!}$.

(b) $E[Y] = \alpha\beta = 20$ and $V(Y) = \alpha\beta^2 = 40$.

(c)

$$\begin{aligned}
P(Y > 34.1696) &= \frac{1}{2^{10}\cdot 9!}\int_{34.1696}^{\infty} y^9 e^{-y/2}dy \\
&= \frac{1}{2^{10}\cdot 9!}\int_{17.0848}^{\infty} 2^{10}z^9 e^{-z}dz \qquad \left[z = \frac{y}{2} \text{ substitution}\right] \\
&= \frac{1}{9!}\int_{17.0848}^{\infty} z^9 e^{-z}dz \\
&= \sum_{x=0}^{9} \frac{17.0848^x e^{-17.0848}}{x!} \\
&\approx 0.025.
\end{aligned}$$

(d) Suppose there exists b with $P(Y < b) = 0.10$, then we must have that

$$0.90 = P(Y \geq b) = P(Z \geq b/2),$$

and from Appendix 3, Table 3, we get $b/2 \approx 14$, so that $b \approx 28$.

8. The function $B(\alpha,\beta)$ is defined by $B(\alpha,\beta) = \displaystyle\int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy$.

(a) Letting $y = \sin^2\theta$, show that $B(\alpha,\beta) = 2\displaystyle\int_0^{\pi/2} \sin^{2\alpha-1}\theta\cos^{2\beta-1}\theta\, d\theta$.

(b) Write $\Gamma(\alpha)\Gamma(\beta)$ as a double integral using variables of integration y and z, make the transformation $y = r^2\sin^2\theta$ and $z = r^2\cos^2\theta$, and then show that

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Solution.

(a) Let $y = \sin^2\theta$, so that $dy = 2\sin\theta\cos\theta\, d\theta$. Thus

$$\begin{aligned}
B(\alpha,\beta) &= \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy \\
&= \int_0^{\pi/2} [(\sin\theta)^2]^{\alpha-1}(1-\sin^2\theta)^{\beta-1} 2\sin\theta\cos\theta\, d\theta \\
&= \int_0^{\pi/2} (\sin\theta)^{2\alpha-2}[(\cos\theta)^2]^{\beta-1} 2\sin\theta\cos\theta\, d\theta \\
&= 2\int_0^{\pi/2} \sin^{2\alpha-1}\theta\cos^{2\beta-1}\theta\, d\theta.
\end{aligned}$$

(b) By definition we have that

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty y^{\alpha-1}e^{-y}\, dy \int_0^\infty z^{\beta-1}e^{-z}\, dz = \int_0^\infty\int_0^\infty y^{\alpha-1}e^{-(y+z)}z^{\beta-1}\, dydz.$$

Consider the transformation $y = r^2 \sin^2\theta$ and $z = r^2\cos^2\theta$. The Jacobian of this transformation, $\dfrac{\partial(y,z)}{\partial(r,\theta)}$, is given by

$$\begin{vmatrix} \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = -4r^3 \sin\theta\cos\theta.$$

Thus we have that

$$\begin{aligned}
\Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty \int_0^\infty y^{\alpha-1} e^{-(y+z)} z^{\beta-1}\, dydz \\
&= \int_0^{\pi/2}\int_0^\infty [(r\sin\theta)^2]^{\alpha-1} e^{-r^2} [(r\cos\theta)^2]^{\beta-1} \left|\frac{\partial(y,z)}{\partial(r,\theta)}\right| drd\theta \\
&= \int_0^{\pi/2}\int_0^\infty r^{2\alpha-2}\sin^{2\alpha-2}\theta e^{-r^2} r^{2\beta-2}\cos^{2\beta-2}\theta \left|\frac{\partial(y,z)}{\partial(r,\theta)}\right| drd\theta \\
&= \int_0^{\pi/2}\int_0^\infty r^{2\alpha+2\beta-4}\sin^{2\alpha-2}\theta e^{-r^2}\cos^{2\beta-2}\theta\,(4r^3\sin\theta\cos\theta)\,drd\theta \\
&= \int_0^{\pi/2}\int_0^\infty 4r^{2\alpha+2\beta-1}\sin^{2\alpha-1}\theta e^{-r^2}\cos^{2\beta-1}\theta\,drd\theta \\
&= \left(2\int_0^{\pi/2}\sin^{2\alpha-1}\theta\cos^{2\beta-1}\theta\,d\theta\right)\left(\int_0^\infty r^{2(\alpha+\beta-1)}e^{-r^2}2r\,dr\right) \\
&= B(\alpha,\beta)\int_0^\infty r^{2(\alpha+\beta-1)}e^{-r^2}2r\,dr.
\end{aligned}$$

Thus using the substitution $x = r^2$ will give us

$$B(\alpha,\beta)\int_0^\infty r^{2(\alpha+\beta-1)}e^{-r^2}2r\,dr = B(\alpha,\beta)\int_0^\infty x^{\alpha+\beta-1}e^{-x}\,dx = B(\alpha,\beta)\Gamma(\alpha+\beta),$$

so that $\Gamma(\alpha)\Gamma(\beta) = B(\alpha,\beta)\Gamma(\alpha+\beta)$, as desired.

9. Prove that the variance of a beta-distributed random variable with parameters α and β are given by
$$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

Proof. Let Y be a beta-distributed random variable with parameters α and β. We then have that

$$\begin{aligned}
E[Y^2] &= \int_0^1 y^2 \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} \\
&= \frac{1}{B(\alpha,\beta)} \int_0^1 y^{\alpha+1}(1-y)^{\beta-1} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{(\alpha+1)\alpha\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)} \\
&= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}.
\end{aligned}$$

By the Proof on Pg 196 of the book we have that $E[Y] = \dfrac{\alpha}{\alpha+\beta}$. Thus the variance of Y, $V(Y)$, is

$$\begin{aligned}
E[Y^2] - E[Y]^2 &= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2} \\
&= \frac{(\alpha+1)(\alpha+\beta)\alpha - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta+1)(\alpha+\beta)^2} \\
&= \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2},
\end{aligned}$$

which is what we wanted to prove. $\square$

10. Suppose Y has the density function $f(y) = k(y-2)^4(5-y)^6$, $2 \le y \le 5$. Find (a) k (b) $E[Y]$ and $V(Y)$.

Solution.

(a) By definition we must have that

$$k \int_2^5 (y-2)^4(5-y)^6 \, dy = 1.$$

So make the substitution $x = 5 - y$ to get

$$\begin{aligned}
1 &= k \int_2^5 (y-2)^4 (5-y)^6 \, dy \\
&= -k \int_3^0 (3-x)^4 x^6 \, dx \\
&= k \int_0^3 (3-x)^4 x^6 \, dx \\
&= k \int_0^3 (x^4 - 12x^3 + 54x^2 - 108x + 81) x^6 \, dx \\
&= k \int_0^3 (x^{10} - 12x^9 + 54x^8 - 108x^7 + 81x^6) \, dx \\
&= k \left(\frac{1}{11} x^{11} - \frac{6}{5} x^{10} + 6x^9 - \frac{27}{2} x^8 + \frac{81}{7} x^7 \right) \Big|_0^3 \\
&= \frac{59049}{770} k,
\end{aligned}$$

so that $k = \dfrac{770}{59049}$.

(b) Using the same substitution $x = 5 - y$, it follows that

$$\begin{aligned}
E[Y] &= k \int_2^5 y(y-2)^4 (5-y)^6 \, dy \\
&= k \int_0^3 (5-x)(x^{10} - 12x^9 + 54x^8 - 108x^7 + 81x^6) \, dx \\
&= k \int_0^3 (-x^{11} + 17x^{10} - 114x^9 + 378x^8 - 621x^7 + 405x^6) \, dx \\
&= k \left(-\frac{1}{12} x^{12} + \frac{17}{11} x^{11} - \frac{57}{5} x^{10} + 42x^9 - \frac{621}{8} x^8 + \frac{405}{7} x^7 \right) \Big|_0^3 \\
&= \frac{770}{59049} \frac{767637}{3080} = \frac{13}{4},
\end{aligned}$$

and

$$\begin{aligned}
E[Y^2] &= k\int_2^5 y^2(y-2)^4(5-y)^6\,dy \\
&= k\int_0^3 (5-x)^2(x^{10}-12x^9+54x^8-108x^7+81x^6)\,dx \\
&= k\int_0^3 (x^2-10x+25)(x^{10}-12x^9+54x^8-108x^7+81x^6)\,dx \\
&= k\int_0^3 (x^{12}-22x^{11}+199x^{10}-948x^9+2511^8-3510x^7+2025x^6)\,dx \\
&= k\left(\frac{1}{13}x^{13}-\frac{11}{6}x^{12}+\frac{199}{11}x^{11}-\frac{474}{5}x^{10}+279x^9-\frac{1755}{4}x^8+\frac{2025}{7}x^7\right)\Big|_0^3 \\
&= \frac{770}{59049}\frac{49424013}{60060} = \frac{279}{26}.
\end{aligned}$$

We can then conclude that

$$V(Y) = E[Y^2]-E[Y]^2 = \frac{279}{26}-\frac{13^2}{4^2} = \frac{35}{208}.$$