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Mathematics
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1. Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false. Throughout G is a group.

   1. If $g \in G$ is the only element of order 2, then $g \in Z(G)$, the center.
   2. The intersection of two subgroups of G is also a subgroup.
   3. The union of two subgroups of G is also a subgroup.
   4. The largest order of an element in S_{12} is 60.
   5. If an Abelian group has an element of order 10 and an element of order 12, then it has an element of order 30.

   Solution.

   1. a

2. We have beads of four different colors.

   1. How many distinct four-bead necklaces can we make?
   2. How many distinct five-bead necklaces can we make?
   3. How many distinct six-bead necklaces can we make?

BONUS: Answer the same questions if we now have beads of five colors.

   Solution.

   1. a

3. Consider the following two sets of matrices

$$S_1 = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\} \text{ and } S_2 = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}.$$

   Do the following for both:

   1. Decide if they are rings or not—and give reasons.
   2. Decide if they are integral domains or not—and give reasons.
   3. Can you find a root for the polynomial $x^2 + 1$ in either place? If so find all the roots or give reasons.

   Solution.

   1. a

4. Let R be a ring. An additive subgroup I is called an ideal if whenever $r \in R$ and $a \in I$, then $ra, ar \in I$.

1. Find two ideals of $\mathbb{Z}$ that are neither 0 nor $\mathbb{Z}$.
2. Let I be an ideal. Prove the following are true: if $I+x$ and $I+y$ are the same coset and $I+m$ and $I+n$ are the same coset, then $I+(x+m)$ and $I+(y+n)$ are the same coset, and so are $I+xm$ and $I+yn$.
3. Let S be a ring, and let $\alpha : R \to S$ be a ring homomorphism—this means with respect to both operations. Show $I = \ker(\alpha) = \{a \in \mathbb{R} : \alpha(a) = 0\}$ is an ideal.

Solution.

1. a