1 Chapter 3

1.1 Section 1

1. Give a geometrical argument to verify that $|\sin x| \leq |x|$ for every real number x.

2. Prove that if f is bounded on A and f is also bounded on B then f is bounded on $A \cup B$.

   Proof. Assume that f is bounded on A and that f is also bounded on A. Then it follows by definition that there exist positive real numbers M_1 and M_2 such that

   $$|f(x)| \leq M_1 \text{ for all } x \in A \text{ and } |f(y)| \leq M_2 \text{ for all } y \in B.$$

   Now let $M = \max\{M_1, M_2\}$, and consider $z \in A \cup B$. If z is in A, then we must have that $|f(z)| \leq M_1 \leq M$. Otherwise z must be in B, so that $|f(z)| \leq M_2 \leq M$. In either case, we have that $|f(z)| \leq M$. Thus, by definition, f is bounded on $A \cup B$. □

3. Prove that if f and g are each bounded above (below) on A then $f + g$ is bounded above (below) on A.

   Proof. Assume that f and g are both bounded above on A. Then it follows by definition that there exist real numbers M_1 and M_2 such that

   $$f(x) \leq M_1 \text{ and } g(x) \leq M_2 \text{ for all } x \in A.$$

   Let $y \in A$. Then it follows that

   $$\begin{aligned}(f+g)(y) &= f(y) + g(y) \\ &\leq M_1 + M_2,\end{aligned}$$

   so that $f + g$ is bounded above by $M_1 + M_2$. The proof is similar if f and g are bounded below. □

4. Prove: If f is bounded above (below) on A and $k > 0$ then $k \cdot f$ is bounded above (below) on A; if f is bounded above (below) on A and $k < 0$ then $k \cdot f$ is bounded below (above) on A.

   Proof. Assume that f is bounded above on A and that $k > 0$. Then it follows by definition that there exists a real number M such that $f(x) \leq M$, for every $x \in A$. Let $y \in A$, so that $f(y) \leq M$. Multiply the inequality $f(y) \leq M$ by the positive number k to get $kf(y) \leq kM$. The preceding inequality tells us that kf is bounded above by kM on A. Now assume that $k < 0$. Then we must have that $-k > 0$, so that $-kf(y) \leq -kM$. Hence multiplying the inequality $-kf(y) \leq -kM$ by -1 will lead us to conclude that $kf(y) \geq kM$. That is $k \cdot f$ is bounded below by kM on A. The proof is similar if f is bounded below. □

5. Show that if f and g are both bounded above on A the product $f \cdot g$ may fail to be bounded above on A.

   Proof. Let $f = -x$, $g = -1$, and $A = \mathbb{R}$. The function f is clearly bounded above by 0 and g is bounded above by -1. But $f \cdot g = x$ is not bounded on $\mathbb{R}$. □

6. Prove that each polynomial function

$$p : \mathbb{R} \to \mathbb{R}, \quad x \mapsto a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

is bounded on every bounded interval I.

Proof. Let I be a bounded interval and consider the polynomial

$$p : \mathbb{R} \to \mathbb{R}, \quad x \mapsto a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n.$$

We know from the discussion in the notes that the identity function $f(x) = x$ is bounded on I. Thus there exists a postive real number M such that $|f(x)| \leq M$ for all $x \in I$. Let $y \in I$. Then we have that

$$\begin{aligned} |p(y)| &= |a_0 y^n + a_1 y^{n-1} + \cdots + a_{n-1} y + a_n| \\ &\leq |a_0 y^n| + |a_1 y^{n-1}| + \cdots + |a_{n-1} y| + |a_n| && \text{[Lemma 3.1]} \\ &= |a_0||y^n| + |a_1||y^{n-1}| + \cdots + |a_{n-1}||y| + |a_n| \\ &= |a_0||y|^n + |a_1||y|^{n-1} + \cdots + |a_{n-1}||y| + |a_n| \\ &\leq |a_0| M^n + |a_1| M^{n-1} + \cdots + |a_{n-1}| M + |a_n|. \end{aligned}$$

Let $M' = |a_0| M^n + |a_1| M^{n-1} + \cdots + |a_{n-1}| M + |a_n| + 1$. Clearly $M' > 0$, and since

$$|p(y)| \leq |a_0| M^n + |a_1| M^{n-1} + \cdots + |a_{n-1}| M + |a_n| < M',$$

it follows by definition that the function p is bounded on I. □

7. Prove that each of the following functions is bounded on the indicated interval:

(a) $f(x) = \dfrac{\sin x}{1 + x^2}$ on $(-\infty, \infty)$ $\quad$ (b) $f(x) = \dfrac{\sin 1/x}{x + 2}$ on $(0, 2)$

(c) $f(x) = \dfrac{x^4 - 3x^2 + 2}{2 - \cos x}$ on $[0, 2\pi]$ $\quad$ (d) $f(x) = \dfrac{\sin x}{3x - 2x \sin x}$ on $(0, \infty)$

(e) $f(x) = \dfrac{\cos x}{x^2 - 2x + 2}$ on $(-\infty, \infty)$ $\quad$ (f) $f(x) = \dfrac{5x^2 + 3x + 1}{x^2 - 2}$ on $[-1, 1]$

(g) $f(x) = \dfrac{\sin x}{\sqrt{x}}$ on $(0, \infty)$ $\quad$ (h) $f(x) = \dfrac{1 - \cos x}{x^2}$ on $(0, \infty)$

(i) $f(x) = \dfrac{1 + x^2}{1 + x^3}$ on $[0, \infty)$ $\quad$ (j) $f(x) = \dfrac{1 - x^2}{1 + x^3}$ on $(-1, 1)$

Solutions.

(a) Let $y \in \mathbb{R}$. Then $|1 + y^2| \geq 1$, so that $\dfrac{1}{|1 + y^2|} \leq 1$. And since $|\sin(y)| \leq 1$, it follows that $\left|\dfrac{\sin(y)}{1 + y^2}\right| = |\sin(y)| \cdot \dfrac{1}{|1 + y^2|} \leq 1$, so that f is bounded on $\mathbb{R}$.

(b) Let $y \in (0,2)$. Then $2 \le |y+2| \le 4$, so that $.25 \le \dfrac{1}{|y+2|} \le .5$ And since $|\sin(1/y)| \le 1$, it follows that $\left|\dfrac{\sin(1/y)}{y+2}\right| = |\sin(1/y)| \cdot \dfrac{1}{|y+2|} \le .5$, so that f is bounded on $(0,2)$.

(c) By (6), we know there exists $M_1 > 0$ and $M_2 > 0$ such that $|5x^2+3x+1| \le M_1$ and $|x^2-2| \le M_2$ for all $x \in [-1,1]$.

8. Prove that a nonconstant polynomial cannot be bounded on an unbounded interval.

9. Suppose that f is bounded on A and g is unbounded on A. Prove that $f+g$ fails to be bounded on A.

   Proof. Suppose that f is bounded on A and that g is unbounded on A. Now suppose to the contrary that $f+g$ is bounded on A. Let h be the constant function that is identically equal to -1. It is clear that h is bounded on A. Thus $h \cdot f = -f$ is bounded on A by Theorem 3.1. Thus $g = (f+g)+(-f)$ is also bounded on A by Theorem 3.1, a contradiction since we g is unbounded on A by hypothesis. Thus $f+g$ is not bounded on A. □

10. Find functions f and g neither of which is bounded on A but such that the product $f \cdot g$ is bounded on A.

    Answer. Let $A = \mathbb{R}$, $g(x) = x$, and

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

so that

$$(f \cdot g)(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Both f and g are unbounded on A, but their product, $f \cdot g = 1$, is bounded on A.

11. If f is bounded on A and g is unbounded on A, what can be said regarding the boundedness of $f \cdot g$? Explain.

    Answer. Suppose that f is bounded on A and g is unbounded on A, then nothing can be said about the boundedness of $f \cdot g$. Let $A = \mathbb{R}$, $f(x) = 1$, and $g(x) = x$ for all $x \in \mathbb{R}$, then $f \cdot g$ is unbounded on $\mathbb{R}$. However if we let $A = (1,\infty)$, $f(x) = 1/x$, and $g(x) = x$ for all $x \in (1,\infty)$, then $f \cdot g$ is bounded on $(0,\infty)$.

12. Find $\sup f(x)$ and $\inf f(x)$ for each of the following functions on the indicated domain:

(a) $f(x) = 3 + 2x - x^2$ on $(0, 4)$

(b) $f(x) = 2 - |x - 1|$ on $(-2, 2)$

(c) $f(x) = -e^{-|x|}$ on $(-\infty, \infty)$

(d) $f(x) = \dfrac{x}{x-2}$ on $(-\infty, 2) \cup (2, \infty)$

(e) $f(x) = e^{-1/x}$ on $(-\infty, 0) \cup (0, \infty)$

(f) $f(x) = x \sin \dfrac{1}{x}$ on $(0, \infty)$

(g) $f(x) = \dfrac{1 - x^2}{1 + x^2}$ on $(-\infty, \infty)$

(h) $f(x) = x \sin \dfrac{1}{\sqrt{x}}$ on $(0, \infty)$

13. Prove results 1 to 3 in the paragraph following Example 3.4.

14. Show that any function $f : \mathbb{R} \to \mathbb{R}$ can be decomposed into the sum of an even function and an odd function. *Hint*: Consider

$$e(x) = \frac{1}{2}(f(x) + f(-x)) \text{ and } o(x) = \frac{1}{2}(f(x) - f(-x)).$$

15. Prove that if f is an even function and $f(x) \neq 0$ for all $x \in \mathbb{R}$ then $1/f$ is an even function. Why isn't a similar statement for odd functions meaningful?

16. A rational number $r = p/q$, where $p, q \in \mathbb{Z}$ and $q \neq 0$, is said to be *properly reduced* if p and q ($q > 0$) have no common integral factor other than ± 1. Define the function f as follows:

$$f(x) = \begin{cases} q & \text{if } x = p/q, \text{ properly reduced} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Prove that for every real number x_0, f fails to be bounded at x_0.