Find the group of units in each of the following integral domains:

1. $\mathbb{Z}[x]$.
2. $\mathbb{Q}[x]$.
3. $\mathbb{Z}_5[x]$.
4. $\mathbb{Z}_8[x]$.
5. $S = \left\{ \begin{pmatrix} a & b \\ -2b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}$.

Solution. The group of units simply consists of all the elements that have a multiplicative inverse. Let $\mathbb{I}(G)$ denote the group of units of a domain G. Recall that the units in $G[x]$ are exactly the units in G. Thus

1. $\mathbb{I}(\mathbb{Z}[x]) = \{-1, 1\}$.
2. $\mathbb{I}(\mathbb{Q}[x]) = \mathbb{Q} - \{0\}$.
3. $\mathbb{I}(\mathbb{Z}_5[x]) = \{1, 2, 3, 4\}$.
4. $\mathbb{I}(\mathbb{Z}_8[x]) = \{1, 3, 5, 7\}$.
5. An matrix in S is a unit if and only if its determinant is ± 1. So we require $a^2 + 2b^2 = \pm 1$. The only possibilities are $a = \pm 1$ and $b = 0$. Thus

$$\mathbb{I}(S) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$