1. Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false.

   (1) There is an integral domain with 6 elements.

   Let k be a positive integer. Let $\bar{\ }: \mathbb{Z} \to \mathbb{Z}_k$ be the mod function. Thus, *e.g.*, if $k = 7$, then $\overline{25} = 4$. This leads naturally to a homomorphism $\bar{\ }: \mathbb{Z}[x] \to \mathbb{Z}_k[x]$. Thus, *e.g.*, if $k = 7$, then $\overline{25x^2 + 12} = 4x^2 + 5 = -3x^2 - 2$. Consider the veracity or falsehood of each of the following statements. For those that are true give an argument, for those that are false, give a counterexample. Let $p(x) \in \mathbb{Z}[x]$ be monic.

   (2) If $p(x)$ has a root in $\mathbb{Z}$, then $\overline{p(x)}$ has a root in $\mathbb{Z}_k$.

   (3) If $\overline{p(x)}$ has a root in $\mathbb{Z}_k$, then $p(x)$ has a root in $\mathbb{Z}$.

   (4) If $p(x)$ is irreducible, then so is $\overline{p(x)}$.

   (5) If $\overline{p(x)}$ is irreducible, then so is $p(x)$.

   Solution.

   (1) a

2. Consider the integral domain $R = \mathbb{Z}[\sqrt{3}]$. Let $A = \begin{pmatrix} 5 & 3 \\ 9 & 5 \end{pmatrix}$.

   (1) Find a nontrivial unit, and show it has infinite order.

   (2) Compute $\dfrac{A}{\begin{pmatrix} 20 & 6 \\ 18 & 20 \end{pmatrix}}$ and its reciprocal $\dfrac{\begin{pmatrix} 20 & 6 \\ 18 & 20 \end{pmatrix}}{A}$. These elements may not be in the domain, but they are certainly in the field of quotients.

   (3) Decide if A and $\begin{pmatrix} 19 & 11 \\ 33 & 19 \end{pmatrix}$ are associates.

   (4) Is $\begin{pmatrix} 7789 & 4488 \\ 13464 & 7789 \end{pmatrix} \equiv \begin{pmatrix} 52 & 24 \\ 72 & 57 \end{pmatrix}$ mod A? Give reasons for your answer.

   Solution.

   (1) a

3. Consider the following element $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ of $GL(3, \mathbb{Z}_2)$.

   (1) Compute all of its powers.

(2) How many elements would you have to add for this set of powers to be closed under addition?

(3) Find the characteristic polynomial of each of the powers.

(4) Find the lowest degree polynomial that all of the powers satify.

(5) Have you constructed a field?

Bonus. Show that every irreducible cubic over $\mathbb{Z}_2$ has a root among these powers.

Solution.

(1) a

4. On $\mathbb{Z}_2[x]$. Consider the ring of polynomials $\mathbb{Z}_2[x]$ with coefficients in $\mathbb{Z}_2$,

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

(1) How many polynomials of degree n are there? **Hint.** Consider $n = 1, 2, 3, \ldots$.

(2) Consdier the function $E : \mathbb{Z}_2[x] \to \mathbb{Z}_2$ that sends any polynomial $p(x)$ to $p(1)$. Decide if it is a (ring) homomorphism or not. Decide if it is one-to-one and onto. Argue your case.

(3) Consider the function $S : \mathbb{Z}_2[x] \to \mathbb{Z}_2[x]$ that sends any polynomial $p(x)$ to $p^2(x)$, it square. Decide if it is a (ring) homomorphism or not. Decide if it is one-to-one and onto. Argue your case.

(4) Count the number of irreducible quadratics in $\mathbb{Z}_2[x]$.

(5) Count the number of irreducible cubics in $\mathbb{Z}_2[x]$.

(6) Count the number of irreducible quartics in $\mathbb{Z}_2[x]$.

Solution.

(1) a