1. **Quickie Queries. It is essential you put down reasons for your answers and show your work. 30 points.**

Throughout assume $g, h \in G$, an abelian group, and that the order of g is 1000.

1 The order of g^{2120}.

2 The smallest n such that S_n has an element of the same order as g.

3 The number of generators of $\langle g \rangle$.

4 The number of subgroups of $\langle g \rangle$.

5 The number of subgroups of $\langle g \rangle$ of order 100.

6 The number of elements of $\langle g \rangle$ of order 100.

7 Given that h is of order 2400, the largest possible order of an element in G (as far as you know).

8 An element of that largest order (as in 7).

Solution.

1 The order of g^{2120} is

$$\frac{1000}{\gcd(2120, 1000)} = 25.$$

2 Since $1000 = 2^3 5^3$, it follows that $n = 2^3 + 5^3 = 133$.

3 Let $\varphi(n)$ be the number of positive integers relatively prime to a positive integer n. Then the number of generators of $\langle g \rangle$ is $\varphi(1000) = \varphi(2^3 5^3) = \varphi(2^3)\varphi(5^3) = 400$.

4 The number of subgroups of $\langle g \rangle$ is the number of positive divisors of 1000; since $1000 = 2^3 5^3$, it follows that we have $4 \cdot 4 = 16$ subgroups of $\langle g \rangle$.

5 There is 1 subgroup of $\langle g \rangle$ of order 100.

6 There are $\varphi(100) = \varphi(2^2 5^2) = \varphi(2^2)\varphi(5^2) = 40$ elements of $\langle g \rangle$ of order 100.

7 The largest possible order of an element as far we know is

$$\frac{1000 \cdot 2400}{\gcd(1000, 2400)} = 12000.$$

8 The order of h^{25} is 96 and the order of g^8 is 125. Since $\gcd(96, 125) = 1$, it follows that the order of $g^8 h^{25}$ is $96 \cdot 125 = 12000$.

2. **15 points.** Recall that the centralizer of an element $a \in G$ (a group) is given by

$$C(a) = \{g \in G : ag = ga\}.$$

Do the following:

(1) Show that $gag^{-1} = hah^{-1}$ if and only if $h^{-1}g \in C(a)$.

(2) Assume G is finite. Show that $|C(a)| \times \# = |G|$ where $\#$ is the number of conjugates of a.

Solution.

(1) Suppose $h^{-1}g \in C(a)$. Then

$$\begin{aligned} h^{-1}ga &= ah^{-1}g && \Longleftrightarrow \\ ga &= hah^{-1}g && \Longleftrightarrow \\ gag^{-1} &= hah^{-1}. \end{aligned}$$

Now suppose $gag^{-1} = hah^{-1}$. Then

$$\begin{aligned} gag^{-1} &= hah^{-1} && \Longleftrightarrow \\ ga &= hah^{-1}g && \Longleftrightarrow \\ h^{-1}ga &= ah^{-1}g && \Longleftrightarrow \\ h^{-1}g &\in C(a). \end{aligned}$$

(2) **Proof.** Let $a \in G$. We know that

$$|G_a| \cdot |Ga| = |G|,$$

where G_a is the stablilizer of a and Ga is the orbit of a (note that $\# = |Ga|$). It suffices to show that $C(a) = G_a$. Now

$$\begin{aligned} x &\in C(a) && \Longleftrightarrow \\ xa &= ax && \Longleftrightarrow \\ xax^{-1} &= a && \Longleftrightarrow \\ x &\in Ga, \end{aligned}$$

so that $C(a) = Ga$, and we have that $|C(a)| \cdot |Ga| = |G_a| \cdot \# = |G|$.

3. Let A be an abelian group with identity e. **15 points.**

(1) Show that $\{a \in A : a^3 = e\}$ is a subgroup.

(2) Find the elements of this subgroup when A is the multiplicative group of nozero elements of $\mathbb{Z}_{19}$.

(3) Give necessary and sufficient conditions on the size of A in order for this subgroup to have other elements besides e, and give reasons.

Solution. Let $G = \{a \in A : a^3 = e\}$.

(1) G is clearly associative under the operation of A since it is a subset of A, so in order to show that G is a subgroup, we need to show that it contains the e and that it is closed under the operation of A and taking inverses.

Identity. Clearly $e \in G$ since $e^3 = e$.

Closure. Suppose $g, h \in G$. Then since G is abelian, it follows that $(gh)^3 = g^3h^3 = ee = e$, so that $gh \in G$.

Inverse. Suppose $g \in G$. Then it follows that $ggg = g^3 = e$. Now

$$ggg = e \Rightarrow gg = g^{-1} \Rightarrow g = (g^{-1})^2 \Rightarrow e = (g^{-1})^3 \Rightarrow g^{-1} \in G,$$

so that G is closed under taking inverses.

Thus we can conclude that G is a subgroup of A.

(2) We want the elements a of $\mathbb{Z}_{19}$ such that $a^3 = 1$. By computation we find that the subgroup of A that satisfies this condition is $\{1, 7, 13\}$.

(3) If $a^3 = e$, then the order of a divides 3 so that the order of a is 1 or 3. So we want the order of a to be 3. Thus we must require that 3 divides $|A|$, so that by Cauchy's Theorem, an element of order 3 will be in G.