1. Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false.

   (1) $\mathbb{Z}(\sqrt{2})$ is a UFD.

   (2) If $\alpha : R \to S$ is a ring homomorphism, then it is one-to-one if the only $r \in R$ satisfying $\alpha(r) = 0$ is $r = 0$.

   (3) Every integral domain is a field.

   (4) If every element of a ring is an idempotent, then the ring is commutative.

   (5) The group of units of $\mathcal{M}_2(\mathbb{Z}_3)$ has 56 elements.

   Solution.

   (1) a

2. Let $A = \begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$ be an element of $\mathbb{Z}(\sqrt{2})$. Suppose that $\det A$ is even. Show there is an element of $\mathbb{Z}(\sqrt{2})$ whose determinant is $\frac{1}{2}\det A$. **Hint.** Find an element of determinant 2 and show that you can divide A by it.

   Solution.

3. **On Factoring.**

   (1) Give all irreducible cubics over $\mathbb{Z}_2$.

   (2) Factor $p(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ completely in $\mathbb{Z}_2[x]$.

   (3) How many monic quadratics are there in $\mathbb{Z}_3[x]$?

   (4) Find all irreducible monic quadratics over $\mathbb{Z}_3$.

   (5) Factor $q(x) = x^6 + x^3 - x^2 - x$ completely in $\mathbb{Z}_3[x]$.

   (6) Consider the mod function from $\mathbb{Z}$ to $\mathbb{Z}_3$, and its natural extension to a homomorphism from $\mathbb{Z}[x]$ to $\mathbb{Z}_3[x]$ that mods out the coefficients, for example $5x$ goes to $2x$. Find the image under this homomorphism of the polynomial

   $$h(x) = x^6 + 9x^5 + 21x^4 + 37x^3 + 53x^2 + 29x + 15.$$

   (7) Do the same as in (6) except now one mods out 2.

   (8) Use parts (2) and (5) to discuss the possible factorization of $h(x)$. For example, does it have any integer roots? Or is it irreducible etc. Discuss as much as you can.

   Solution.

(1) a

4. **On Factorization.**

(1) Find the complete factorization of $x^5 - 1$ in $\mathbb{Q}[x]$.

(2) Find real numbers a and b such that

$$(x^2 + ax + 1)(x^2 + bx + 1) = x^4 + x^3 + x^2 + x + 1.$$

(3) Find the complete factorization of $x^5 - 1$ in $\mathbb{R}[x]$.

(4) Find the complete factorization of $x^5 - 1$ in $\mathbb{C}[x]$.

(5) Find the complete factorization of $x^5 - 1$ in $\mathbb{Z}_{11}[x]$.

(6) Find the complete factorization of $x^5 - 1$ in $\mathbb{Z}_{31}[x]$.

(7) Find the complete factorization of $x^{10} - 1$ in $\mathbb{Q}[x]$.

Solution.

(1) a