

1. Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false. Let $G = \langle g \rangle$ have order 300.

- ① There are exactly 80 generators of G .
- ② G has only one element of order 3.
- ③ G can be embedded in S_{30} .
- ④ G has a subgroup of order 20.
- ⑤ G has a totality of 18 subgroups.

Solution.

- ① True. Since $G = \langle g \rangle$ is cyclic and since $|g| = 300$, it follows that the number of generators of G is the number of positive integers relatively prime to 300, which is 80.
- ② False. We have that $g^{100} \neq g^{200}$ (since $|g| = 300$) and

$$|g^{100}| = \frac{300}{\gcd(300, 100)} = 3 = |g^{200}| = \frac{300}{\gcd(300, 200)}.$$

2. Let G be an abelian group and let $a, b \in G$ be of order 120 and 72 respectively. Do the following:

- ① Find an element of order 15.
- ② What is the order of b^{10} ?
- ③ Find an element of as large an order as you can.

Solution.

- ① w

3. Consider the non-abelian group of order 55 from **Homework #4**. View this group as acting on all column vectors of size 2 (with entries in \mathbb{Z}_{11}).

- ① Find the number of fixed points of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
- ② Find the number of fixed points of $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$.
- ③ Decide on the number of fixed elements each of the elements of the group has.
- ④ Use Burnside's Lemma to count the orbits.

4. Let the vertices of the cube be given as follows:

$$1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, 2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, 3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, 4 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, 5 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, 6 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, 7 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$

$$\text{and } 8 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

- ① Label the faces A, A', B, B', C , and C' (where the prime means opposite), and give each as a set of four vertices. Let A be the intersection with the plane $x = 1$, B with the plane $y = 1$ and C with $z = 1$.
- ② Find 24 3×3 matrices of determinant 1 that are isometries of the cube, and write each as a permutation in S_8 (of the eight vertices) and also as a permutation of the faces. **Hint:** Start with the six permutation matrices of size 3.

Assume these 24 matrices form a group G . **Bonus.** Prove this.
 Assume these $G \simeq S_4$. **Bonus.** Prove this.

- ③ Find the number of ways to color a cube with two colors.
- ④ Find the number of ways to color a cube with three colors.

Bonus. Find the number of ways to color the cube with n colors.

Solution.

- ① s