7.1 Make $\mathcal{P}_2(\mathbb{R})$ into an inner-product space by defining

$$\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx.$$

Define $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ by $T(a_0 + a_1x + a_2x^2) = a_1x$.

(a) Show that T is not self-adjoint.

(b) The matrix of T with respect to the basis $(1, x, x^2)$ is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix equals its conjugate transpose, even though T is not self-adjoint. Explain why this is not a contradiction.

Solution.

(a) Suppose to the contrary that T is self-adjoint. Consider $x, 1 \in \mathcal{P}_2(\mathbb{R})$. Then we must have that $\langle T(1), x\rangle = \langle T^*(1), x\rangle = \langle 1, T(x)\rangle$. But

$$\begin{aligned} \langle T^*(1), x\rangle &= \langle T(1), x\rangle \\ &= \langle 0, x\rangle \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \langle 1, T(x)\rangle &= \langle 1, x\rangle \\ &= \int_0^1 x\,dx \\ &= \frac{1}{2}, \end{aligned}$$

so that $\langle T(1), x\rangle \neq \langle T(x), 1\rangle$; i.e., T is not self-adjoint.

7.4 Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that P is an orthogonal projection if and only if P is self-adjoint.

7.6 Prove that if $T \in \mathcal{L}(V)$ is normal, then

$$\text{range } T = \text{range } T^*.$$

7.7 Prove that if $T \in \mathcal{L}(V)$ is normal, then

$$\text{null } T^k = \text{null } T \text{ and range } T^k = \text{range } T$$

for every positive integer k.

7.9 Prove that a normal operator on a complex inner-product space is self-adjoint if and only if all its eigenvalues are real.