7.14 Suppose $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in \mathbb{F}$, and $\epsilon > 0$. Prove that if there exists $v \in V$ such that $||v|| = 1$ and

$$||Tv - \lambda v|| < \epsilon,$$

then T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$.

Proof. By the Spectral Theorem, there exists an orthornormal basis $(e_1, \dots, e_n)$ of V consisting of eigenvectors of T with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. Given $v \in V$, with $||v|| = 1$, we thus have $v = a_1e_1 + \cdots + a_ne_n$ for some unique scalars. Since $||v|| = 1$, it follows that at least one of the a_is is nonzero. Thus

$$\begin{aligned}
||Tv - \lambda v||^2 &= ||T(a_1e_1 + \cdots + a_ne_n) - a_1\lambda e_1 - \cdots - a_n\lambda e_n||^2 \\
&= ||a_1\lambda_1e_1 + \cdots + a_n\lambda_ne_n - a_1\lambda e_1 - \cdots - a_n\lambda e_n||^2 \\
&= ||a_1(\lambda_1 - \lambda)e_1 + \cdots + a_n(\lambda_n - \lambda)e_n||^2 \\
&= |a_1(\lambda_1 - \lambda)|^2 + \cdots + |a_n(\lambda_n - \lambda)|^2 \\
&= |a_1|^2|\lambda_1 - \lambda|^2 + \cdots + |a_n|^2|\lambda_n - \lambda)|^2 \\
&= |a_1|^2|\lambda - \lambda_1|^2 + \cdots + |a_n|^2|\lambda - \lambda_n|^2.
\end{aligned}$$

Suppose to the contrary that $|\lambda - \lambda_i| \geq \epsilon$ for all $i = 1 \dots n$. Then we must have that

$$\begin{aligned}
||Tv - \lambda v||^2 &= |a_1|^2|\lambda - \lambda_1|^2 + \cdots + |a_n|^2|\lambda - \lambda_n|^2 \\
&\geq |a_1|^2\epsilon^2 + \cdots + |a_n|^2\epsilon^2 \\
&= (|a_1|^2 + \cdots + |a_n|^2)\epsilon^2 \\
&= ||v||\epsilon^2 = \epsilon^2,
\end{aligned}$$

so that $||Tv - \lambda v|| \geq \epsilon$, a contradiction. Thus, at least one of the eigenvalues of T, say λ_j, must be such that $|\lambda - \lambda_j| < \epsilon$. □

7.16 Give an example of an operator T on an inner product space such that T has an invariant subspace whose orthogonal complement is not invariant under T.

Answer. Consider $T \in \mathcal{L}(F^3)$, where $T((a, b, c)) = (b, c, 0)$. Let

$$U = \{(x, y, 0) : x, y \in \mathbb{F}\}.$$

Clearly U is a subspace of V and it is also invariant under T. Now we have $U^\perp = \{(0, 0, z) : z \in \mathbb{F}\}$. But $U^\perp$ is not invariant under T since $(0, 0, 10) \in U^\perp$ but $T((0, 0, 10) = (0, 10, 0) \notin U^\perp$.

7.17 Prove that the sum of any two positive operators on V is positive.

Proof. Suppose that S and T are positive operators on V. Since S and T are both self-adjoint, it follows immediately that $S + T$ is self-adjoint because

$$(S + T)^* = S^* + T^* = S + T.$$

Now let $v \in V$. Thus

$$\begin{aligned}
\langle (S+T)v, v\rangle &= \langle Sv + Tv, v\rangle \\
&= \langle Sv, v\rangle + \langle Tv, v\rangle \\
&\geq 0, && \text{[Since } Sv \geq 0, Tv \geq 0]
\end{aligned}$$

so that $S+T$ is a positive operator. □

7.19 Suppose that T is a positive operator on V. Prove that T is invertible if and only if

$$\langle Tv, v\rangle > 0$$

for every $v \in V \setminus \{0\}$.

Proof. Suppose first that T is invertible. Then it follows that T is injective, so that null $T = \{0\}$. Let $v \in V$. Now suppose that $\langle Tv, v\rangle = 0$. By the Spectral Theorem, there exists an orthonormal basis $(e_1, \ldots, e_n)$ of V consisting of eigenvectors of T. Let $\lambda_1, \ldots, \lambda_n$ denote the corresponding real (T is self-adjoint) and nonnegative (Theorem 7.27 (b)) eigenvalues. Since the eigenvectors are independent, they must be nonzero. Thus $e_i \notin$ null T, so that $0 \neq T(e_i) = \lambda_i e_i$. That is, all the eigenvalues are positive. Now we have $v = a_1 e_1 + \cdots + a_n e_n$, so that

$$\begin{aligned}
0 &= \langle Tv, v\rangle \\
&= \langle T(a_1e_1 + \cdots + a_ne_n), a_1e_1 + \cdots + a_ne_n\rangle \\
&= \langle T(a_1e_1) + \cdots + T(a_ne_n), a_1e_1 + \cdots + a_ne_n\rangle \\
&= \langle a_1\lambda_1e_1 + \cdots + a_n\lambda_ne_n, a_1e_1 + \cdots + a_ne_n\rangle \\
&= a_1\overline{a_1}\lambda_1\langle e_1, e_1\rangle + \cdots + a_n\overline{a_n}\lambda_n\langle e_n, e_n\rangle \\
&= |a_1|^2\lambda_1 + \cdots + |a_n|^2\lambda_n.
\end{aligned}$$

Since the eigenvalues are all positive, it must be the case $a_1 = \cdots = a_n = 0$, so that $v = 0$. So it follows that if $v \in V$ is nonzero, we must have that $\langle Tv, v\rangle > 0$.

Conversely suppose that $\langle Tv, v\rangle > 0$ for all nonzero $v \in V$. Let $x \in$ null T. Then it follows that x is not nonzero because

$$0 = \langle 0, x\rangle = \langle Tx, x\rangle.$$

Thus $x = 0$; that is null $T = \{0\}$. Thus T is injective (and surjective) and thus invertible. □

7.22 Prove that if $S \in \mathcal{L}(\mathbb{R}^3)$ is an isometry, then there exists a nonzero vector $x \in \mathbb{R}^3$ such that $S^2x = x$.

Proof. Assume that $S \in \mathcal{L}(\mathbb{R}^3)$ is an isometry. Since dim $\mathbb{R}^3$ is odd, it follows by Theorem 7.38 that S has an eigenvalue of 1 or -1. Suppose first that S has an eigenvalue of 1. Then there exists a nonzero vector x such that $Sx = x$. Thus we have that

$$S^2x = S(Sx) = Sx = x.$$

Now suppose that S has an eigenvalue of -1. Then there exists a nonzero vector c such that $Sc = -c$. Let $y = -c$. Then we have that

$$S^2 y = S(Sy) = S(S(-c)) = S(-S(c)) = S(c) = -c = y,$$

as desired. $\square$