JOSEPH OKONOBOH   MATH 444, SPRING 2015
MATHEMATICS   SECTION 1 (5562)
CAL STATE LONG BEACH   HW #7, DUE: 2015, MARCH 18

1. Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false. Let $G = \langle g \rangle$ have order 300.

   (1) There are exactly 80 generators of G.

   (2) G has only one element of order 3.

   (3) G can be embedded in S_{30}.

   (4) G has a subgroup of order 20.

   (5) G has a totality of 18 subgroups.

   Solution.

   (1) True. Since $G = \langle g \rangle$ is cyclic and since $|g| = 300$, it follows that the number of generators of G is the number of positive integers relatively prime to 300, which is 80.

   (2) False.
   Counterexample. We have that $g^{100} \neq g^{200}$ (since $|g| = 300$) and

   $$|g^{100}| = \frac{300}{\gcd(300, 100)} = 3 = \frac{300}{\gcd(300, 200)} = |g^{200}|.$$

   (3) False.
   Proof. It suffices to show that S_{30} has no element of order 300. Suppose to the contrary that $\sigma \in S_{30}$ has order 300. Then we can write σ as a product of disjoint cycles (each of length greater than 1)

   $$\sigma = \alpha_1 \alpha_2 \cdots \alpha_n$$

   so that $|\sigma| = \mathrm{lcm}(|\alpha_1|, |\alpha_2|, \cdots, |\alpha_n|) = 300$. Now since $5^2 \mid 300$, it follows that 5^2 must divide the order of at least one of the cycles. We can assume without loss that $5^2 \mid |\alpha_1|$. Thus α_1 must be a 25-cycle. By a similar argument, it follows that 2^2 must divide the order of at least one of the cycles. Assume without loss that $2^2 \mid |\alpha_2|$. Since there are 25 elements in α_1, there can be at most 5 elements in α_2, so that α_2 is a 4-cycle. Thus

   $$\sigma = \alpha_1 \alpha_2,$$

   a contradiction since $\mathrm{lcm}(|\alpha_1|, |\alpha_2|) = 100 \neq 300$.   □

   (4) True. This subgroup of G, $\langle g^{15} \rangle$, has 20 elements.

   (5) True. Since the number of positive divisors of 300 is 18, it follows that G has exactly 18 subgroups.

2. Let G be an abelian group and let $a, b \in G$ be of order 120 and 72 respectively. Do the following:

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① Find an element of order 15.

② What is the order of b^{10}?

③ Find an element of as large an order as you can.

Solution.

① The element a^8 has order 15.

② $|b^{10}| = \dfrac{72}{\gcd(72, 10)} = 36.$

③ The element $a^{24}b$ has order 360.

3. Consider the non-abelian group of order 55 from **Homework #4**. View this group as acting on all column vectors of size 2 (with entries in $\mathbb{Z}_{11}$).

① Find the number of fixed points of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

② Find the number of fixed points of $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$.

③ Decide on the number of fixed elements each of the elements of the group has.

④ Use Burnside's Lemma to count the orbits.

4. Let the vertices of the cube be given as follows:

$$1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, 2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, 3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, 4 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, 5 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, 6 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, 7 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$

and $8 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$.

① Label the faces $A, A', B, B', C,$ and C' (where the prime means opposite), and give each as a set of four vertices. Let A be the intersection with the plane $x = 1$, B with the plane $y = 1$ and C with $z = 1$.

② Find 24 3×3 matrices of determinant 1 that are isometries of the cube, and write each as a permutation in S_8 (of the eight vertices) and also as a permutation of the faces. **Hint:** Start with the six permutation matrices of size 3.

Assume these 24 matrices form a group G. **Bonus.** Prove this.
Assume these $G \simeq S_4$. **Bonus.** Prove this.

③ Find the number of ways to color a cube with two colors.

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④ Find the number of ways to color a cube with three colors.

Bonus. Find the number of ways to color the cube with n colors.

Solution.

① s