6.6 Prove that if $V$ is a real inner-product space, then

$$\langle u, v\rangle = \frac{||u+v||^2 - ||u-v||^2}{4}$$

for all $u.v \in V$.

**Proof.** Let $V$ be a real inner-product space and let $u, v \in V$. We have that

$$\begin{aligned}
\frac{||u+v||^2 - ||u-v||^2}{4} &= \frac{\langle u+v, u+v\rangle - \langle u-v, u-v\rangle}{4} \\
&= \frac{\langle u, u+v\rangle + \langle v, u+v\rangle - \langle u, u-v\rangle - \langle -v, u-v\rangle}{4} \\
&= \frac{\langle \overline{u+v, u}\rangle + \langle \overline{u+v, v}\rangle - \langle \overline{u-v, u}\rangle + \langle v, u-v\rangle}{4} \\
&= \frac{\langle u+v, u\rangle + \langle u+v, v\rangle - \langle u-v, u\rangle + \langle v, u-v\rangle}{4} \\
&= \frac{\langle u, u\rangle + \langle v, u\rangle + \langle u, v\rangle + \langle v, v\rangle - \langle u, u\rangle + \langle v, u\rangle + \langle \overline{u-v, v}\rangle}{4} \\
&= \frac{\langle v, u\rangle + \langle u, v\rangle + \langle v, v\rangle + \langle v, u\rangle + \langle u-v, v\rangle}{4} \\
&= \frac{\langle v, u\rangle + \langle u, v\rangle + \langle v, v\rangle + \langle v, u\rangle + \langle u, v\rangle - \langle v, v\rangle}{4} \\
&= \frac{\langle v, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle u, v\rangle}{4} \\
&= \frac{\langle \overline{u, v}\rangle + \langle u, v\rangle + \langle \overline{u, v}\rangle + \langle u, v\rangle}{4} \\
&= \frac{\langle u, v\rangle + \langle u, v\rangle + \langle u, v\rangle + \langle u, v\rangle}{4} \\
&= \frac{4\langle u, v\rangle}{4} = \langle u, v\rangle.
\end{aligned}$$

$\square$

6.10 On $\mathcal{P}_2(\mathbb{R})$, consider the inner product given by

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx.$$

Apply the Gram-Schmidt procedure to the basis $(1, x, x^2)$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$.

**Solution.** We want to construct an orthonormal basis $(e_1, e_2, e_3)$ for $\mathcal{P}_2(\mathbb{R})$; so applying the Gram-Schmidt procedure to the basis $(1, x, x^2)$, we have

$$\begin{aligned} e_1 &= \frac{1}{||1||} \\ e_2 &= \frac{x - \langle x, e_1 \rangle e_1}{||x - \langle x, e_1 \rangle e_1||} \\ e_3 &= \frac{x^2 - \langle x^2, e_1 \rangle e_1 - \langle x^2, e_2 \rangle e_2}{||x^2 - \langle x^2, e_1 \rangle e_1 - \langle x^2, e_2 \rangle e_2||}. \end{aligned}$$

So

$$||1|| = \sqrt{\langle 1, 1 \rangle} = \sqrt{\int_0^1 1 \, dx} = \sqrt{1} = 1,$$

so that $e_1 = 1$. Now

$$\langle x, e_1 \rangle = \int_0^1 x \, dx = \frac{1}{2},$$

and

$$\left|\left| x - \frac{1}{2} \right|\right| = \sqrt{\int_0^1 \left( x - \frac{1}{2} \right)^2 dx} = \frac{\sqrt{3}}{6}.$$

Thus $e_2 = \left( x - \frac{1}{2} \right) \cdot \frac{6}{\sqrt{3}} = 2x\sqrt{3} - \sqrt{3}$.

Similarly we find that

$$\langle x^2, e_1 \rangle e_1 = \int_0^1 x^2 \, dx = \frac{1}{3},$$

and

$$x^2 - \langle x^2, e_2 \rangle e_2 = x^2 - \left( \int_0^1 (2x^3\sqrt{3} - x^2\sqrt{3}) \, dx \right) (2x\sqrt{3} - \sqrt{3}) = x^2 - x + \frac{1}{6},$$

so that

$$\left|\left| x^2 - x + \frac{1}{6} \right|\right| = \sqrt{\int_0^1 \left( x^2 - x + \frac{1}{6} \right)^2 dx} = \frac{1}{6\sqrt{5}}.$$

Thus $e_3 = \left( x^2 - x + \frac{1}{6} \right) \cdot 6\sqrt{5} = 6x^2\sqrt{5} - 6x\sqrt{5} + \sqrt{5}$.

Thus an orthonormal basis for $\mathcal{P}_2(\mathbb{R})$ is

$$(1, 2x\sqrt{3} - \sqrt{3}, 6x^2\sqrt{5} - 6x\sqrt{5} + \sqrt{5}).$$

6.13 Suppose $(e_1, \ldots, e_m)$ is an orthonormal list of vectors in $V$. Let $v \in V$. Prove that

$$||v||^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2$$

if and only if $v \in \text{span}(e_1, \ldots, e_m)$.

**Proof.** Suppose $(e_1, \ldots, e_m)$ is an orthonormal list of vectors in $V$ and let $v \in V$.

($\Leftarrow$) Assume that $v \in \text{span}(e_1, \ldots, e_m)$. Therefore $v = a_1 e_1 + \cdots + a_m e_m$ for some scalars $a_1, \ldots, a_m$. By the orthonomality of $(e_1, \ldots, e_m)$, it follows that $\langle v, e_j \rangle = a_j$ for all $j \in \{1, 2, \ldots, m\}$, so we have that

$$\begin{aligned} ||v||^2 &= ||a_1 e_1 + \cdots + a_m e_m||^2 \\ &= ||\langle v, e_1 \rangle e_1 + \cdots + \langle v, e_m \rangle e_m||^2 \\ &= |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2 && \text{[Proposition 6.15]} \end{aligned}$$

($\Rightarrow$) Now assume that $||v||^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2$. Extend the orthonormal list $(e_1, \ldots, e_m)$ to an orthonormal basis $(e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+n})$ for $V$. Thus there exist scalars $b_1, \ldots, b_{m+n}$ such that $v = b_1 e_1 + \cdots + b_{m+n} e_{m+n}$. Thus

$$\begin{aligned} |b_1|^2 + \cdots + |b_m|^2 &= |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2 \\ &= ||v||^2 \\ &= ||b_1 e_1 + \cdots + b_{m+n} e_{m+n}||^2 \\ &= |b_1|^2 + \cdots + |b_{m+n}|^2 && \text{[Proposition 6.15]} \\ &= |b_1|^2 + \cdots + |b_m|^2 + |b_{m+1}|^2 + \cdots + |b_{m+n}|^2, \end{aligned}$$

so that $|b_{m+1}|^2 + \cdots + |b_{m+n}|^2 = 0$. Since $|b_{m+i}|$ is nonnegative for all $i \in \{1, \ldots, n\}$, it follows that $b_{m+1} = \cdots = b_{m+n} = 0$, so that $v = b_1 e_1 + \cdots + b_m$. That is

$$v \in \text{span}(e_1, \ldots, e_m).$$

□

6.17 Prove that if $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and every vector in null $P$ is orthogonal to every vector in range $P$, then $P$ is an orthogonal projection.

**Proof.** Suppose $P \in \mathcal{L}(V)$ such that $P^2 = P$. Also suppose that every vector in null $P$ is orthogonal to every vector in range $P$. First we want to show that $V =$ range $P \oplus$ null $P$. Let $v \in$ range $P \cap$ null $P$. By our hypothesis, we must have that $\langle v, v \rangle = 0$, so that $v = 0$; i.e., range $P \cap$ null $P = \{0\}$. Now let dim range $P = m$ and let dim null $P = n$. Then there exist a basis $(b_1, \ldots, b_m)$ for range $P$ and a basis $(e_1, \ldots, e_n)$ for null $P$. We claim that the list $(b_1, \ldots, b_m, e_1, \ldots, e_n)$ is linearly independent. So consider the equation

$$\alpha_1 b_1 + \cdots + \alpha_m b_m + \alpha_{m+1} e_1 + \cdots + \alpha_{m+n} e_n = 0.$$

It follows that

$$\alpha_1 b_1 + \cdots + \alpha_m b_m = \alpha_{m+1} e_1 + \cdots + \alpha_{m+n} e_n$$

and since range $P \cap$ null $P = \{0\}$, it must be the case that

$$\alpha_1 b_1 + \cdots + \alpha_m b_m = \alpha_{m+1} e_1 + \cdots + \alpha_{m+n} e_n = 0,$$

so that by the linear independence of $(b_1, \ldots, b_m)$ and $(e_1, \ldots, e_n)$, we must have that

$$\alpha_1 = \cdots = \alpha_m = \alpha_{m+1} = \cdots = \alpha_{m+n} = 0.$$

By the Rank-Nullity Theorem, we have that $\dim V = m + n$. Thus the list $(b_1, \ldots, b_m, e_1, \ldots, e_n)$ forms a basis for $V$, so that $V = \text{range } P + \text{ null } P$. We have thus shown that $V = \text{range } P \oplus \text{ null } P$. To complete the proof, we now want to show that $P = P_{\text{range}P}$. Let $v \in V$. Then we can write $v = r + n$ for unique $r \in \text{range } P$ and $n \in \text{null } P$. To show that $P = P_{\text{range}P}$, it suffices to show that $Pv = r$. Now we have that $r - P(r) \in \text{null } P$ because

$$P(r - P(r)) = P(r) - P(P(r)) = P(r) - P(r) = 0.$$

Notice that $r = P(r) + (r - P(r))$, where $P(r) \in \text{range } P$ and $r - P(r) \in \text{null } P$. But $r = r + 0$, so that $r = P(r)$ by the uniqueness of decomposition in direct sums. Thus

$$\begin{aligned} P(v) &= P(r + n) \\ &= P(r) + P(n) \\ &= P(r) + 0 \\ &= P(r) = r, \end{aligned}$$

which is what we wanted to show. □

6.29 Suppose $T \in \mathcal{L}(\mathcal{V})$ and $U$ is a subspace of $V$. Prove that $U$ is invariant under $T$ if and only if $U^\perp$ is invariant under $T^*$.

**Proof.** Suppose $T \in \mathcal{L}(\mathcal{V})$ and $U$ is a subspace of $V$.

($\Rightarrow$) Assume that $U$ is invariant under $T$. Let $v \in U^\perp$. In order to show that $U^\perp$ is invariant under $T^*$, it suffices to show that $T^*v \in U^\perp$. That is, we must show that $\langle u, T^*v \rangle = 0$ for all $u \in U$. So let $u \in U$. Thus

$$\begin{aligned} \langle u, T^*v \rangle &= \langle Tu, v \rangle && \text{[Definition]} \\ &= 0 && [Tu \in U \text{ and } v \in U^\perp], \end{aligned}$$

which is what we wanted to show.

($\Leftarrow$) Now assume that $U^\perp$ is invariant under $T^*$. Let $u \in U$. We want to show that $Tu \in U$. Consider any $v \in U^\perp$. We have that

$$\begin{aligned} \langle Tu, v \rangle = \langle u, T^*v \rangle & \text{[Definition]} \\ = 0 \qquad & [u \in U \text{ and } T^*v \in U^\perp], \end{aligned}$$

so that $Tu$ is orthogonal to every vector in $U^\perp$. That is, $Tu \in (U^\perp)^\perp$; but $(U^\perp)^\perp = U$. Thus $Tu \in U$, so that $U$ is invariant under $T$. □