

1. If  $Z$  is a standard normal variable, find  
(a)  $P(Z^2 < 1)$       (b)  $P(Z^2 > 3.84146)$ .

**Solution.**

- (a) We have that

$$P(Z^2 < 1) = P(-1 < Z < 1) = 1 - 2 \cdot P(Z > 1) \approx 0.6826,$$

and

- (b)

$$P(Z^2 > 3.84146) = 2 \cdot P(Z > \sqrt{3.84146}) \approx 2 \cdot P(Z > 1.96) \approx 0.05.$$

2. If  $Y$  is a normal random variable with  $\mu = 20$  and variance  $\sigma^2 = 4$ , i.e.,  $Y \sim N(20, 4)$ , find  
(a)  $P(16 \leq Y \leq 22)$       (b)  $P(100 < 9Y - 80 < 145)$ .

**Solution.**

- (a) We have that

$$\begin{aligned} P(16 \leq Y \leq 22) &= P\left(\frac{16 - 20}{2} \leq Z \leq \frac{22 - 20}{2}\right) \\ &= P(-2 \leq Z \leq 1) \\ &= 1 - [P(Z < -2) + P(Z > 1)] \\ &= 1 - [P(Z > 2) + P(Z > 1)] \\ &\approx 0.8185, \end{aligned}$$

and

- (b)

$$\begin{aligned} P(100 < 9Y - 80 < 145) &= P(20 < Y < 25) \\ &= P\left(\frac{20 - 20}{2} < Z < \frac{25 - 20}{2}\right) \\ &= P(0 < Z < 2.5) \\ &= P(Z > 0) - P(Z > 2.5) \\ &\approx 0.4938. \end{aligned}$$

3. The scores of a pre-employment test are normally distributed with mean  $\mu = 70$  and standard deviation  $\sigma = 5$ . If only the top 1.5% of the applicants (based on their score on the pre-employment test) are to be considered, find the cut-off score (i.e., the value such that only 1.5% of the applicants score this value or higher).

**Solution.** Let  $y$  be the cut-off score. Then we have that

$$0.0015 = P(Y \geq y) = P\left(Z \geq \frac{y - 70}{5}\right),$$

so that  $(y - 70)/5 \approx 2.97$ ; i.e.,  $y \approx 85$ .

4. Using the fact that  $\int_0^\infty e^{-y^2/2} dy = \sqrt{\frac{\pi}{2}}$ , show that  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$  by making the transformation  $y = \frac{1}{2}x^2$ .

**Proof.** Using the transformation  $y = \frac{1}{2}x^2$  we have that

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty y^{-\frac{1}{2}} e^{-y} dy \\ &= \int_0^\infty \frac{\sqrt{2}}{x} e^{-\frac{1}{2}x^2} x dx \\ &= \sqrt{2} \int_0^\infty e^{-\frac{1}{2}x^2} dx \\ &= \sqrt{2} \sqrt{\frac{\pi}{2}} = \sqrt{\pi},\end{aligned}$$

as desired. □

5. If  $Y$  has an exponential distribution with  $P(Y < 3) = 0.4512$ , find

(a)  $E[Y]$       (b)  $P(Y \geq 2)$ .

**Solution.**

(a) We have that

$$\begin{aligned}0.4512 &= P(Y < 3) \\ &= P(Y \leq 3) \\ &= F(3) \\ &= \int_{-\infty}^3 \frac{1}{\beta} e^{-\frac{y}{\beta}} dy \\ &= \int_0^3 \frac{1}{\beta} e^{-\frac{y}{\beta}} dy \\ &= -e^{-\frac{3}{\beta}} + 1,\end{aligned}$$

so that  $e^{-\frac{3}{\beta}} = 0.5488$ ; i.e.,  $\beta \approx 5$ . Thus  $E[Y] \approx 5$ .

(b)

$$\begin{aligned}P(Y \geq 2) &= 1 - P(Y < 2) \\ &= 1 - \int_0^2 \frac{1}{\beta} e^{-\frac{y}{\beta}} dy \\ &= e^{-\frac{2}{\beta}} \\ &\approx 0.6703.\end{aligned}$$

6. The length of time  $Y$  necessary to complete a key operation in the construction of houses has an exponential distribution with mean 10 hrs. The formula  $C = 100 + 40Y + 3Y^2$  gives the cost  $C$  of completing the operation. Find the mean and variance of  $C$ .

**Solution.** First we want to find  $E[Y^2]$ . So

$$\begin{aligned}
 E[Y^2] &= \frac{1}{10} \lim_{t \rightarrow \infty} \int_0^t y^2 e^{-\frac{y}{10}} dy \\
 &= \lim_{t \rightarrow \infty} \left[ -y^2 e^{-\frac{y}{10}} \Big|_0^t + 2 \int_0^t y e^{-\frac{y}{10}} dy \right] && \text{[Integration by parts]} \\
 &= 2 \lim_{t \rightarrow \infty} \left[ \int_0^t y e^{-\frac{y}{10}} dy \right] \\
 &= 2 \lim_{t \rightarrow \infty} \left[ -10y e^{-\frac{y}{10}} \Big|_0^t + 10 \int_0^t e^{-\frac{y}{10}} dy \right] && \text{[Integration by parts]} \\
 &= 200 \lim_{t \rightarrow \infty} \left[ \frac{1}{10} \int_0^t e^{-\frac{y}{10}} dy \right] \\
 &= 200 \cdot E[Y] = 2000.
 \end{aligned}$$

Now the mean of  $C$  is given by  $E[C]$  so that

$$\begin{aligned}
 E[C] &= E[100 + 40Y + 3Y^2] \\
 &= E[100] + 40E[Y] + 3E[Y^2] \\
 &= 100 + 40 \cdot 10 + 3 \cdot 2000 \\
 &= 6500,
 \end{aligned}$$

and the variance of  $C$ ,  $V[Y]$ , is  $E[Y^2] - E[Y]^2 = 2000 - 100 = 1900$ .

7. Suppose  $Y$  has density function  $f(y) = ky^9 e^{-y/2}$ ,  $y \geq 0$ . Find

- (a)  $k$ .
- (b)  $E[Y]$  and  $V(Y)$ .
- (c)  $P(Y > 34.1696)$ .
- (d) A value  $b$  such that  $P(Y < b) = 0.10$ .

**Solution.** By inspection we can see that  $f$  is the gamma distribution with  $\alpha = 10$ ,  $\beta = 2$ .

- (a)  $k = \frac{1}{2^{10} \cdot \Gamma(10)} = \frac{1}{2^{10} \cdot 9!}$ .
- (b)  $E[Y] = \alpha\beta = 20$  and  $V(Y) = \alpha\beta^2 = 40$ .

(c)

$$\begin{aligned}
 P(Y > 34.1696) &= \frac{1}{2^{10} \cdot 9!} \int_{34.1696}^{\infty} y^9 e^{-y/2} dy \\
 &= \frac{1}{2^{10} \cdot 9!} \int_{17.0848}^{\infty} 2^{10} z^9 e^{-z} dz \quad \left[ z = \frac{y}{2} \text{ substitution} \right] \\
 &= \frac{1}{9!} \int_{17.0848}^{\infty} z^9 e^{-z} dz \\
 &= \sum_{x=0}^9 \frac{17.0848^x e^{-17.0848}}{x!} \\
 &\approx 0.025.
 \end{aligned}$$

(d) Suppose there exists  $b$  with  $P(Y < b) = 0.10$ , then we must have that

$$0.90 = P(Y \geq b) = P(Z \geq b/2),$$

and from Appendix 3, Table 3, we get  $b/2 \approx 14$ , so that  $b \approx 28$ .

8. The function  $B(\alpha, \beta)$  is defined by  $B(\alpha, \beta) = \int_0^1 y^{\alpha-1} (1-y)^{\beta-1} dy$ .

(a) Letting  $y = \sin^2 \theta$ , show that  $B(\alpha, \beta) = 2 \int_0^{\pi/2} \sin^{2\alpha-1} \theta \cos^{2\beta-1} \theta d\theta$ .

(b) Write  $\Gamma(\alpha)\Gamma(\beta)$  as a double integral using variables of integration  $y$  and  $z$ , make the transformation  $y = r^2 \sin^2 \theta$  and  $z = r^2 \cos^2 \theta$ , and then show that

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

**Solution.**

(a) Let  $y = \sin^2 \theta$ , so that  $dy = 2 \sin \theta \cos \theta d\theta$ . Thus

$$\begin{aligned}
 B(\alpha, \beta) &= \int_0^1 y^{\alpha-1} (1-y)^{\beta-1} dy \\
 &= \int_0^{\pi/2} [(\sin \theta)^2]^{\alpha-1} (1 - \sin^2 \theta)^{\beta-1} 2 \sin \theta \cos \theta d\theta \\
 &= \int_0^{\pi/2} (\sin \theta)^{2\alpha-2} [(\cos \theta)^2]^{\beta-1} 2 \sin \theta \cos \theta d\theta \\
 &= 2 \int_0^{\pi/2} \sin^{2\alpha-1} \theta \cos^{2\beta-1} \theta d\theta.
 \end{aligned}$$

(b) By definition we have that

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^{\infty} y^{\alpha-1} e^{-y} dy \int_0^{\infty} z^{\beta-1} e^{-z} dz = \int_0^{\infty} \int_0^{\infty} y^{\alpha-1} e^{-(y+z)} z^{\beta-1} dy dz.$$

Consider the transformation  $y = r^2 \sin^2 \theta$  and  $z = r^2 \cos^2 \theta$ . The Jacobian of this transformation,  $\frac{\partial(y, z)}{\partial(r, \theta)}$ , is given by

$$\begin{vmatrix} \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{vmatrix} = -4r^3 \sin \theta \cos \theta.$$

Thus we have that

$$\begin{aligned} \Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty \int_0^\infty y^{\alpha-1} e^{-(y+z)} z^{\beta-1} dy dz \\ &= \int_0^{\pi/2} \int_0^\infty [(r \sin \theta)^2]^{\alpha-1} e^{-r^2} [(r \cos \theta)^2]^{\beta-1} \left| \frac{\partial(y, z)}{\partial(r, \theta)} \right| dr d\theta \\ &= \int_0^{\pi/2} \int_0^\infty r^{2\alpha-2} \sin^{2\alpha-2} \theta e^{-r^2} r^{2\beta-2} \cos^{2\beta-2} \theta \left| \frac{\partial(y, z)}{\partial(r, \theta)} \right| dr d\theta \\ &= \int_0^{\pi/2} \int_0^\infty r^{2\alpha+2\beta-4} \sin^{2\alpha-2} \theta e^{-r^2} \cos^{2\beta-2} \theta (4r^3 \sin \theta \cos \theta) dr d\theta \\ &= \int_0^{\pi/2} \int_0^\infty 4r^{2\alpha+2\beta-1} \sin^{2\alpha-1} \theta e^{-r^2} \cos^{2\beta-1} \theta dr d\theta \\ &= \left( 2 \int_0^{\pi/2} \sin^{2\alpha-1} \theta \cos^{2\beta-1} \theta d\theta \right) \left( \int_0^\infty r^{2(\alpha+\beta-1)} e^{-r^2} 2r dr \right) \\ &= B(\alpha, \beta) \int_0^\infty r^{2(\alpha+\beta-1)} e^{-r^2} 2r dr. \end{aligned}$$

Thus using the substitution  $x = r^2$  will give us

$$B(\alpha, \beta) \int_0^\infty r^{2(\alpha+\beta-1)} e^{-r^2} 2r dr = B(\alpha, \beta) \int_0^\infty x^{\alpha+\beta-1} e^{-x} dx = B(\alpha, \beta) \Gamma(\alpha + \beta),$$

so that  $\Gamma(\alpha)\Gamma(\beta) = B(\alpha, \beta)\Gamma(\alpha + \beta)$ , as desired.

9. Prove that the variance of a beta-distributed random variable with parameters  $\alpha$  and  $\beta$  are given by

$$\sigma^2 = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$

**Proof.** Let  $Y$  be a beta-distributed random variable with parameters  $\alpha$  and  $\beta$ . We

then have that

$$\begin{aligned}
 E[Y^2] &= \int_0^1 y^2 \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha, \beta)} dy \\
 &= \frac{1}{B(\alpha, \beta)} \int_0^1 y^{\alpha+1}(1-y)^{\beta-1} dy \\
 &= \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+2)\Gamma(\beta)} \\
 &= \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{(\alpha+1)\alpha\Gamma(\alpha)\Gamma(\beta)}{(\alpha+1)(\alpha+1)\Gamma(\alpha)\Gamma(\beta)} \\
 &= \frac{(\alpha+1)\alpha}{(\alpha+1)(\alpha+1)}.
 \end{aligned}$$

By the Proof on Pg 196 of the book we have that  $E[Y] = \frac{\alpha}{\alpha+\beta}$ . Thus the variance of  $Y$ ,  $V(Y)$ , is

$$\begin{aligned}
 E[Y^2] - E[Y]^2 &= \frac{(\alpha+1)\alpha}{(\alpha+1)(\alpha+1)} - \frac{\alpha^2}{(\alpha+\beta)^2} \\
 &= \frac{(\alpha+1)(\alpha+\beta)\alpha - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta+1)(\alpha+\beta)^2} \\
 &= \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2},
 \end{aligned}$$

which is what we wanted to prove.  $\square$

10. Suppose  $Y$  has the density function  $f(y) = k(y-2)^4(5-y)^6$ ,  $2 \leq y \leq 5$ . Find  
 (a)  $k$  (b)  $E[Y]$  and  $V(Y)$ .

**Solution.**

- (a) By definition we must have that

$$k \int_2^5 (y-2)^4(5-y)^6 dy = 1.$$

So make the substitution  $x = 5 - y$  to get

$$\begin{aligned}
 1 &= k \int_2^5 (y-2)^4(5-y)^6 dy \\
 &= -k \int_3^0 (3-x)^4 x^6 dx \\
 &= k \int_0^3 (3-x)^4 x^6 dx \\
 &= k \int_0^3 (x^4 - 12x^3 + 54x^2 - 108x + 81)x^6 dx \\
 &= k \int_0^3 (x^{10} - 12x^9 + 54x^8 - 108x^7 + 81x^6) dx \\
 &= k \left( \frac{1}{11}x^{11} - \frac{6}{5}x^{10} + 6x^9 - \frac{27}{2}x^8 + \frac{81}{7}x^7 \right) \Big|_0^3 \\
 &= \frac{59049}{770}k,
 \end{aligned}$$

so that  $k = \frac{770}{59049}$ .

(b) Using the same substitution  $x = 5 - y$ , it follows that

$$\begin{aligned}
 E[Y] &= k \int_2^5 y(y-2)^4(5-y)^6 dy \\
 &= k \int_0^3 (5-x)(x^{10} - 12x^9 + 54x^8 - 108x^7 + 81x^6) dx \\
 &= k \int_0^3 (-x^{11} + 17x^{10} - 114x^9 + 378x^8 - 621x^7 + 405x^6) dx \\
 &= k \left( -\frac{1}{12}x^{12} + \frac{17}{11}x^{11} - \frac{57}{5}x^{10} + 42x^9 - \frac{621}{8}x^8 + \frac{405}{7}x^7 \right) \Big|_0^3 \\
 &= \frac{770}{59049} \frac{767637}{3080} = \frac{13}{4},
 \end{aligned}$$

and

$$\begin{aligned}
 E[Y^2] &= k \int_2^5 y^2(y-2)^4(5-y)^6 dy \\
 &= k \int_0^3 (5-x)^2(x^{10} - 12x^9 + 54x^8 - 108x^7 + 81x^6) dx \\
 &= k \int_0^3 (x^2 - 10x + 25)(x^{10} - 12x^9 + 54x^8 - 108x^7 + 81x^6) dx \\
 &= k \int_0^3 (x^{12} - 22x^{11} + 199x^{10} - 948x^9 + 2511x^8 - 3510x^7 + 2025x^6) dx \\
 &= k \left( \frac{1}{13}x^{13} - \frac{11}{6}x^{12} + \frac{199}{11}x^{11} - \frac{474}{5}x^{10} + 279x^9 - \frac{1755}{4}x^8 + \frac{2025}{7}x^7 \right) \Big|_0^3 \\
 &= \frac{770}{59049} \frac{49424013}{60060} = \frac{279}{26}.
 \end{aligned}$$

We can then conclude that

$$V(Y) = E[Y^2] - E[Y]^2 = \frac{279^2}{26^2} - \frac{13^2}{4^2} = \frac{282803}{2704} \approx 104.59.$$