1. Suppose $Y$ is a discrete random variable with probability function $p(y) = ky(1/4)^y$, $y = 0, 1, 2, 3, \ldots$. Find

   $(a)$ $k$ and $(b)$ $E(Y)$ and $V(Y)$.

   **Solution.** Let $p = 1/4$.

   (a) We have that

$$\begin{aligned} 1 &= \sum_{y=0}^{\infty} kyp^y \\ &= \sum_{y=0}^{\infty} kp\left(\frac{d}{dp}p^y\right) \\ &= kp\frac{d}{dp}\sum_{y=0}^{\infty} p^y \\ &= kp\frac{d}{dp}\left(\frac{1}{1-p}\right) \\ &= \frac{kp}{(1-p)^2}. \end{aligned}$$

   It follows that $k = \dfrac{(1-p)^2}{p} = \dfrac{9}{4}$.

   (b) We have that

$$\begin{aligned} E(Y) &= \sum_{y=0}^{\infty} ky^2p^y \\ &= \sum_{y=0}^{\infty} ky^2p^y - \sum_{y=0}^{\infty} kyp^y + \sum_{y=0}^{\infty} kyp^y \\ &= \sum_{y=0}^{\infty} k(y^2-y)p^y + \sum_{y=0}^{\infty} kyp^y \\ &= \sum_{y=0}^{\infty} kp^2\left(\frac{d^2}{dp^2}p^y\right) + \sum_{y=0}^{\infty} kyp^y \\ &= kp^2\frac{d^2}{dp^2}\sum_{y=0}^{\infty} p^y + 1 \\ &= kp^2\frac{d^2}{dp^2}\left(\frac{1}{1-p}\right) + 1 \\ &= \frac{2kp^2}{(1-p)^3} + 1 \\ &= \frac{5}{3}, \end{aligned}$$

and

$$\begin{aligned}
E(Y^2) &= \sum_{y=0}^{\infty} ky^3p^y \\
&= \sum_{y=0}^{\infty} ky^3p^y - 3\sum_{y=0}^{\infty} ky^2p^y + 2\sum_{y=0}^{\infty} kyp^y + 3\sum_{y=0}^{\infty} ky^2p^y - 2\sum_{y=0}^{\infty} kyp^y \\
&= \sum_{y=0}^{\infty} k(y^3 - 3y^2 + 2y)p^y + 3E(Y) - 2 \\
&= \sum_{y=0}^{\infty} ky(y-1)(y-2)p^y + 3 \\
&= \sum_{y=0}^{\infty} kp^3\left(\frac{d^3}{dp^3}p^y\right) + 3 \\
&= kp^3\frac{d^3}{dp^3}\sum_{y=0}^{\infty} p^y + 3 \\
&= \frac{6kp^3}{(1-p)^4} + 3.
\end{aligned}$$

Since $V(Y) = E(Y^2) - E(Y)^2$, it follows that

$$\begin{aligned}
V(Y) &= E(Y^2) - E(Y)^2 \\
&= \frac{6kp^3}{(1-p)^4} + 3 - \frac{25}{9} \\
&= \frac{8}{9}.
\end{aligned}$$