1. Suppose Y is a discrete random variable with probability function $p(y) = ky(1/4)^y$, $y = 0, 1, 2, 3, \ldots$. Find

   (a) k and (b) $E(Y)$ and $V(Y)$.

   Solution. Let $p = 1/4$.

   (a) We have that

$$\begin{aligned} 1 &= \sum_{y=0}^{\infty} kyp^y \\ &= \sum_{y=0}^{\infty} kp\left(\frac{d}{dp}p^y\right) \\ &= kp\frac{d}{dp}\sum_{y=0}^{\infty} p^y \\ &= kp\frac{d}{dp}\left(\frac{1}{1-p}\right) \\ &= \frac{kp}{(1-p)^2}. \end{aligned}$$

   It follows that $k = \dfrac{(1-p)^2}{p} = \dfrac{9}{4}$.

   (b) We have that

$$\begin{aligned} E(Y) &= \sum_{y=0}^{\infty} ky^2p^y \\ &= \sum_{y=0}^{\infty} ky^2p^y - \sum_{y=0}^{\infty} kyp^y + \sum_{y=0}^{\infty} kyp^y \\ &= \sum_{y=0}^{\infty} k(y^2-y)p^y + \sum_{y=0}^{\infty} kyp^y \\ &= \sum_{y=0}^{\infty} kp^2\left(\frac{d^2}{dp^2}p^y\right) + \sum_{y=0}^{\infty} kyp^y \\ &= kp^2\frac{d^2}{dp^2}\sum_{y=0}^{\infty} p^y + 1 \\ &= kp^2\frac{d^2}{dp^2}\left(\frac{1}{1-p}\right) + 1 \\ &= \frac{2kp^2}{(1-p)^3} + 1 \\ &= \frac{5}{3}, \end{aligned}$$

and

$$
\begin{aligned}
E(Y^2) &= \sum_{y=0}^{\infty} ky^3p^y \\
&= \sum_{y=0}^{\infty} ky^3p^y - 3\sum_{y=0}^{\infty} ky^2p^y + 2\sum_{y=0}^{\infty} kyp^y + 3\sum_{y=0}^{\infty} ky^2p^y - 2\sum_{y=0}^{\infty} kyp^y \\
&= \sum_{y=0}^{\infty} k(y^3 - 3y^2 + 2y)p^y + 3E(Y) - 2 \\
&= \sum_{y=0}^{\infty} ky(y-1)(y-2)p^y + 3 \\
&= \sum_{y=0}^{\infty} kp^3 \left(\frac{d^3}{dp^3}p^y\right) + 3 \\
&= kp^3 \frac{d^3}{dp^3} \sum_{y=0}^{\infty} p^y + 3 \\
&= \frac{6kp^3}{(1-p)^4} + 3.
\end{aligned}
$$

Since $V(Y) = E(Y^2) - E(Y)^2$, it follows that

$$
\begin{aligned}
V(Y) &= E(Y^2) - E(Y)^2 \\
&= \frac{6kp^3}{(1-p)^4} + 3 - \frac{25}{9} \\
&= \frac{8}{9}.
\end{aligned}
$$

2. Verify the identity $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ and use it to show that $E[Y^k] = npE[(X+1)^{k-1}]$ where Y is a binomial random variable with parameters n and p and X is a binomial random variable with parameters $n-1$ and p.

Proof. We have that

$$
\begin{aligned}
\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\
&= \frac{n(n-1)!}{k(k-1)!(n-k)!} \\
&= \frac{n}{k}\frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{n}{k}\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} \\
&= \frac{n}{k}\binom{n-1}{k-1}.
\end{aligned}
$$

Now

$$\begin{aligned}
E[Y^k] &= \sum_{y=0}^{n} y^k p(y) && \text{[Definition]}\\
&= \sum_{y=0}^{n} y^k \binom{n}{y} p^y (1-p)^{n-y}\\
&= \sum_{y=1}^{n} y^k \binom{n}{y} p^y (1-p)^{n-y}\\
&= \sum_{y=1}^{n} y^{k-1} n \binom{n-1}{y-1} p^y (1-p)^{n-y}\\
&= np \sum_{y=1}^{n} y^{k-1} \binom{n-1}{y-1} p^{y-1} (1-p)^{n-y}\\
&= np \sum_{x=0}^{n-1} (x+1)^{k-1} \binom{n-1}{x} p^x (1-p)^{(n-1)-x} && \text{[Let } y = x+1\text{]}\\
&= np \sum_{x=0}^{n-1} (x+1)^{k-1} p(x)\\
&= npE[(X+1)^{k-1}],
\end{aligned}$$

which is what we wanted to show. □

3. Using the recursion relation found in problem 2 for the binomial random variable with parameters n and p, find $E[Y^2]$ and then $V(Y)$.

   Solution. If we set $k = 2$ in the formula in problem 2, we get

$$\begin{aligned}
E[Y^2] &= npE[(X+1)]\\
&= np(E[X] + E[1])\\
&= np((n-1)p + 1)\\
&= (np)^2 - np^2 + np.
\end{aligned}$$

   Thus

$$\begin{aligned}
V(Y) &= E[Y^2] - E[Y]^2\\
&= (np)^2 - np^2 + np - (np)^2\\
&= np - np^2\\
&= np(1-p).
\end{aligned}$$

4. Using the identity from problem 2, show that

   (a) if Y is a negative binomial random variable with parameters r and p, then

$$E[Y^k] = \frac{r}{p} E[(X-1)^{k-1}],$$

where X is a negative binomial random variable with parameters $r+1$ and p.

(b) Use the relation in (a) to find $E[Y]$ and $V(Y)$.

Solution.

(a) From problem 2, we know that

$$\frac{1}{r}\binom{y-1}{r-1} = \frac{1}{y}\binom{y}{r};$$

thus

$$\begin{aligned}
E[Y^k] &= \sum_{y=r}^{\infty} y^k p(y) \\
&= \sum_{y=r}^{\infty} y^k \binom{y-1}{r-1} p^r (1-p)^{y-r} \\
&= \frac{r}{p}\frac{p}{r}\sum_{y=r}^{\infty} y^k \binom{y-1}{r-1} p^r (1-p)^{y-r} \\
&= \frac{r}{p}\sum_{y=r}^{\infty} y^k \frac{1}{r}\binom{y-1}{r-1} p^{r+1} (1-p)^{y-r} \\
&= \frac{r}{p}\sum_{y=r}^{\infty} y^k \frac{1}{y}\binom{y}{r} p^{r+1} (1-p)^{y-r} \\
&= \frac{r}{p}\sum_{y=r}^{\infty} y^{k-1} \binom{y}{r} p^{r+1} (1-p)^{y-r} \\
&= \frac{r}{p}\sum_{x=r+1}^{\infty} (x-1)^{k-1} \binom{x-1}{r} p^{r+1} (1-p)^{(x-1)-r} \\
&= \frac{r}{p}\sum_{x=r+1}^{\infty} (x-1)^{k-1} p(x) \\
&= \frac{r}{p} E[(X-1)^{k-1}].
\end{aligned}$$

(b) If we set $k = 1$ in (a), we immediately get that $E[Y] = \frac{r}{p}$. Now

$$\begin{aligned}
V(Y) &= E[Y^2] - E[Y]^2 \\
&= \frac{r}{p} E[(X-1)] - \frac{r^2}{p^2} \\
&= \frac{r}{p}(E[X] - E[1]) - \frac{r^2}{p^2} \\
&= \frac{r}{p}\left(\frac{r+1}{p} - 1\right) - \frac{r^2}{p^2} \\
&= \frac{r^2 + r}{p^2} - \frac{r}{p} - \frac{r^2}{p^2} \\
&= \frac{r - pr}{p^2} \\
&= \frac{r(1-p)}{p^2}.
\end{aligned}$$

5. Using the identity from problem 2, show that if Y is a hypergeometric random variable with parameters N, r, amd n, then

$$E[Y^k] = \frac{nr}{N} E[(X+1)^{k-1}],$$

where X is a hypergeometric random variable with parameters $N-1$, $r-1$, and $n-1$.

Proof. We have that

$$\begin{aligned}
E[Y^k] &= \sum_{y=0}^{n} y^k p(y) \\
&= \sum_{y=0}^{n} y^k \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}} \\
&= \sum_{y=1}^{n} y^k \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}} \\
&= \sum_{y=1}^{n} y^k \frac{\frac{r}{y}\binom{r-1}{y-1}\binom{N-r}{n-y}}{\frac{N}{n}\binom{N-1}{n-1}} \\
&= \frac{nr}{N} \sum_{y=1}^{n} y^{k-1} \frac{\binom{r-1}{y-1}\binom{N-r}{n-y}}{\binom{N-1}{n-1}} \\
&= \frac{nr}{N} \sum_{x=0}^{n-1} (x+1)^{k-1} \frac{\binom{r-1}{x}\binom{(N-1)-(r-1)}{(n-1)-x}}{\binom{N-1}{n-1}} \qquad [\text{Let } y = x+1] \\
&= \frac{nr}{N} \sum_{x=0}^{n-1} (x+1)^{k-1} p(x) \\
&= \frac{nr}{N} E[(X+1)^{k-1}],
\end{aligned}$$

which is what we wanted to prove. □