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Mathematics
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1. Prove that if f is continuous at x_0 and f is nonnegative then $h(x) = \sqrt{f(x)}$ is continuous at x_0.

Proof. Let $\varepsilon > 0$. We want to find $\delta > 0$ such that if $|x - x_0| < \delta$, then $|h(x) - h(x_0)| < \varepsilon$. We shall now investigate the following two cases:

Case 1. $h(x_0) = \sqrt{f(x_0)} = 0$. Now since f is continuous at x_0, it follows by definition that there exists a $\delta_1 > 0$ such that $|f(x) - f(x_0)| < \varepsilon^2$ whenever $|x - x_0| < \delta_1$. So let $x \in (x_0 - \delta_1, x_0 + \delta_1)$. Hence

$$f(x) = |f(x)| = |f(x) - 0| = |f(x) - f(x_0)| < \varepsilon^2$$

so that $\sqrt{f(x)} < \varepsilon$. Then it follows that

$$|h(x) - h(x_0)| = |h(x)| = |\sqrt{f(x)}| = \sqrt{f(x)} < \varepsilon,$$

if $|x - x_0| < \delta_1$.

Case 2. $h(x_0) = \sqrt{f(x_0)} > 0$. By the continuity of f at x_0, t follows by definition that there exists a $\delta_2 > 0$ such that $|f(x) - f(x_0)| < \varepsilon \cdot f(x_0)$ whenever $|x - x_0| < \delta_2$. Let $x \in (x_0 - \delta_2, x_0 + \delta_2)$. Then it follows that

$$\begin{aligned}
|h(x) - h(x_0)| &= |\sqrt{f(x)} - \sqrt{f(x_0)}| \\
&= |\sqrt{f(x)} - \sqrt{f(x_0)}| \cdot \frac{|\sqrt{f(x)} + \sqrt{f(x_0)}|}{|\sqrt{f(x)} + \sqrt{f(x_0)}|} \\
&= \frac{|f(x) - f(x_0)|}{|\sqrt{f(x)} + \sqrt{f(x_0)}|} \\
&= \frac{|f(x) - f(x_0)|}{\sqrt{f(x)} + \sqrt{f(x_0)}} && \text{[Since } f \text{ is nonnegative]} \\
&\le \frac{|f(x) - f(x_0)|}{\sqrt{f(x_0)}} \\
&< \varepsilon \cdot f(x_0) \cdot \frac{1}{\sqrt{f(x_0)}} \\
&= \varepsilon,
\end{aligned}$$

if $|x - x_0| < \delta_2$.

Thus it follows by definition that

$$\lim_{x \to x_0} h(x) = h(x_0),$$

so that h is continuous at x_0. $\square$