

1. If  $Z$  is a standard normal variable, find  
(a)  $P(Z^2 < 1)$       (b)  $P(Z^2 > 3.84146)$ .

**Solution.**

- (a) We have that

$$P(Z^2 < 1) = P(-1 < Z < 1) = 1 - 2 \cdot P(Z > 1) \approx 0.6826,$$

and

- (b)

$$P(Z^2 > 3.84146) = 2 \cdot P(Z > \sqrt{3.84146}) \approx 2 \cdot P(Z > 1.96) \approx 0.05.$$

2. If  $Y$  is a normal random variable with  $\mu = 20$  and variance  $\sigma^2 = 4$ , i.e.,  $Y \sim N(20, 4)$ , find  
(a)  $P(16 \leq Y \leq 22)$       (b)  $P(100 < 9Y - 80 < 145)$ .

**Solution.**

- (a) We have that

$$\begin{aligned} P(16 \leq Y \leq 22) &= P\left(\frac{16 - 20}{2} \leq Z \leq \frac{22 - 20}{2}\right) \\ &= P(-2 \leq Z \leq 1) \\ &= 1 - [P(Z < -2) + P(Z > 1)] \\ &= 1 - [P(Z > 2) + P(Z > 1)] \\ &\approx 0.8185, \end{aligned}$$

and

- (b)

$$\begin{aligned} P(100 < 9Y - 80 < 145) &= P(20 < Y < 25) \\ &= P\left(\frac{20 - 20}{2} < Z < \frac{25 - 20}{2}\right) \\ &= P(0 < Z < 2.5) \\ &= P(Z > 0) - P(Z > 2.5) \\ &\approx 0.4938. \end{aligned}$$

3. The scores of a pre-employment test are normally distributed with mean  $\mu = 70$  and standard deviation  $\sigma = 5$ . If only the top 1.5% of the applicants (based on their score on the pre-employment test) are to be considered, find the cut-off score (i.e., the value such that only 1.5% of the applicants score this value or higher).

**Solution.** Let  $y$  be the cut-off score. Then we have that

$$0.0015 = P(Y \geq y) = P\left(Z \geq \frac{y - 70}{5}\right),$$

so that  $(y - 70)/5 \approx 2.97$ ; i.e.,  $y \approx 84.85$ .

4. Using the fact that  $\int_0^\infty e^{-y^2/2} dy = \sqrt{\frac{\pi}{2}}$ , show that  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$  by making the transformation  $y = \frac{1}{2}x^2$ .

**Proof.** Using the substitution  $y = \frac{1}{2}x^2$  we have that

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty y^{-\frac{1}{2}} e^{-y} dy \\ &= \int_0^\infty \frac{\sqrt{2}}{x} e^{-\frac{1}{2}x^2} x dx \\ &= \sqrt{2} \int_0^\infty e^{-\frac{1}{2}x^2} dx \\ &= \sqrt{2} \sqrt{\frac{\pi}{2}} = \sqrt{\pi}.\end{aligned}$$

5. If  $Y$  has an exponential distribution with  $P(Y < 3) = 0.4512$ , find  
 (a)  $E[Y]$       (b)  $P(Y \geq 2)$ .

**Solution.**

- (a) We have that

$$\begin{aligned}0.4512 &= P(Y < 3) \\ &= P(Y \leq 3) \\ &= F(3) \\ &= \int_{-\infty}^3 \frac{1}{\beta} e^{-\frac{y}{\beta}} dy \\ &= \int_0^3 \frac{1}{\beta} e^{-\frac{y}{\beta}} dy \\ &= -e^{-\frac{3}{\beta}} + 1,\end{aligned}$$

so that  $e^{-\frac{3}{\beta}} = 0.5488$ ; i.e.,  $\beta \approx 5$ . Thus  $E[Y] \approx 5$ .

- (b)

$$\begin{aligned}P(Y \geq 2) &= 1 - P(Y < 2) \\ &= 1 - \int_0^2 \frac{1}{\beta} e^{-\frac{y}{\beta}} dy \\ &= e^{-\frac{2}{\beta}} \\ &\approx 0.6703.\end{aligned}$$

6. The length of time  $Y$  necessary to complete a key operation in the construction of houses has an exponential distribution with mean 10 hrs. The formula  $C = 100 + 40Y + 3Y^2$  gives the cost  $C$  of completing the operation. Find the mean and variance of  $C$ .

**Solution.** First we want to find  $E[Y^2]$ . So

$$\begin{aligned}
 E[Y^2] &= \frac{1}{10} \lim_{t \rightarrow \infty} \int_0^t y^2 e^{-\frac{y}{10}} dy \\
 &= \lim_{t \rightarrow \infty} \left[ -y^2 e^{-\frac{y}{10}} \Big|_0^t + 2 \int_0^t y e^{-\frac{y}{10}} dy \right] && \text{[Integration by parts]} \\
 &= 2 \lim_{t \rightarrow \infty} \left[ \int_0^t y e^{-\frac{y}{10}} dy \right] \\
 &= 2 \lim_{t \rightarrow \infty} \left[ -10y e^{-\frac{y}{10}} \Big|_0^t + 10 \int_0^t e^{-\frac{y}{10}} dy \right] && \text{[Integration by parts]} \\
 &= 200 \lim_{t \rightarrow \infty} \left[ \frac{1}{10} \int_0^t e^{-\frac{y}{10}} dy \right] \\
 &= 200 \cdot E[Y] = 2000.
 \end{aligned}$$

Now the mean of  $C$  is given by  $E[C]$  so that

$$\begin{aligned}
 E[C] &= E[100 + 40Y + 3Y^2] \\
 &= E[100] + 40E[Y] + 3E[Y^2] \\
 &= 100 + 40 \cdot 10 + 3 \cdot 2000 \\
 &= 6500,
 \end{aligned}$$

and the variance of  $C$ ,  $V[Y]$ , is  $E[Y^2] - E[Y]^2 = 2000 - 100 = 1900$ .