

1. If Z is a standard normal variable, find
(a) $P(Z^2 < 1)$ (b) $P(Z^2 > 3.84146)$.

Solution.

- (a) We have that

$$P(Z^2 < 1) = P(-1 < Z < 1) = 1 - 2 \cdot P(Z > 1) \approx 0.6826,$$

and

- (b)

$$P(Z^2 > 3.84146) = 2 \cdot P(Z > \sqrt{3.84146}) \approx 2 \cdot P(Z > 1.96) \approx 0.05.$$

2. If Y is a normal random variable with $\mu = 20$ and variance $\sigma^2 = 4$, i.e., $Y \sim N(20, 4)$, find
(a) $P(16 \leq Y \leq 22)$ (b) $P(100 < 9Y - 80 < 145)$.

Solution.

- (a) We have that

$$\begin{aligned} P(16 \leq Y \leq 22) &= P\left(\frac{16 - 20}{2} \leq Z \leq \frac{22 - 20}{2}\right) \\ &= P(-2 \leq Z \leq 1) \\ &= 1 - [P(Z < -2) + P(Z > 1)] \\ &= 1 - [P(Z > 2) + P(Z > 1)] \\ &\approx 0.8185, \end{aligned}$$

and

- (b)

$$\begin{aligned} P(100 < 9Y - 80 < 145) &= P(20 < Y < 25) \\ &= P\left(\frac{20 - 20}{2} < Z < \frac{25 - 20}{2}\right) \\ &= P(0 < Z < 2.5) \\ &= P(Z > 0) - P(Z > 2.5) \\ &\approx 0.4938. \end{aligned}$$

3. The scores of a pre-employment test are normally distributed with mean $\mu = 70$ and standard deviation $\sigma = 5$. If only the top 1.5% of the applicants (based on their score on the pre-employment test) are to be considered, find the cut-off score (i.e., the value such that only 1.5% of the applicants score this value or higher).

Solution. Let y be the cut-off score. Then we have that

$$0.0015 = P(Y \geq y) = P\left(Z \geq \frac{y - 70}{5}\right),$$

so that $(y - 70)/5 \approx 2.97$; i.e., $y \approx 84.85$.

4. Using the fact that $\int_0^\infty e^{-y^2/2} dy = \sqrt{\frac{\pi}{2}}$, show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ by making the transformation $y = \frac{1}{2}x^2$.

Proof. Using the substitution $y = \frac{1}{2}x^2$ we have that

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty y^{-\frac{1}{2}} e^{-y} dy \\ &= \int_0^\infty \frac{\sqrt{2}}{x} e^{-\frac{1}{2}x^2} x dx \\ &= \sqrt{2} \int_0^\infty e^{-\frac{1}{2}x^2} dx \\ &= \sqrt{2} \sqrt{\frac{\pi}{2}} = \sqrt{\pi}.\end{aligned}$$

5. If Y has an exponential distribution with $P(Y < 3) = 0.4512$, find
 (a) $E[Y]$ (b) $P(Y \geq 2)$.

Solution.

- (a) We have that

$$\begin{aligned}0.4512 &= P(Y < 3) \\ &= P(Y \leq 3) \\ &= F(3) \\ &= \int_{-\infty}^3 \frac{1}{\beta} e^{-\frac{y}{\beta}} dy \\ &= \int_0^3 \frac{1}{\beta} e^{-\frac{y}{\beta}} dy \\ &= -e^{-\frac{3}{\beta}} + 1,\end{aligned}$$

so that $e^{-\frac{3}{\beta}} = 0.5488$; i.e., $\beta \approx 5$. Thus $E[Y] \approx 5$.

- (b)

$$\begin{aligned}P(Y \geq 2) &= 1 - P(Y < 2) \\ &= 1 - \int_0^2 \frac{1}{\beta} e^{-\frac{y}{\beta}} dy \\ &= e^{-\frac{2}{\beta}} \\ &\approx 0.6703.\end{aligned}$$

6. The length of time Y necessary to complete a key operation in the construction of houses has an exponential distribution with mean 10 hrs. The formula $C = 100 + 40Y + 3Y^2$ gives the cost C of completing the operation. Find the mean and variance of C .

Solution. First we want to find $E[Y^2]$. So

$$\begin{aligned}
 E[Y^2] &= \frac{1}{10} \lim_{t \rightarrow \infty} \int_0^t y^2 e^{-\frac{y}{10}} dy \\
 &= \lim_{t \rightarrow \infty} \left[-y^2 e^{-\frac{y}{10}} \Big|_0^t + 2 \int_0^t y e^{-\frac{y}{10}} dy \right] && \text{[Integration by parts]} \\
 &= 2 \lim_{t \rightarrow \infty} \left[\int_0^t y e^{-\frac{y}{10}} dy \right] \\
 &= 2 \lim_{t \rightarrow \infty} \left[-10y e^{-\frac{y}{10}} \Big|_0^t + 10 \int_0^t e^{-\frac{y}{10}} dy \right] && \text{[Integration by parts]} \\
 &= 200 \lim_{t \rightarrow \infty} \left[\frac{1}{10} \int_0^t e^{-\frac{y}{10}} dy \right] \\
 &= 200 \cdot E[Y] = 2000.
 \end{aligned}$$

Now the mean of C is given by $E[C]$ so that

$$\begin{aligned}
 E[C] &= E[100 + 40Y + 3Y^2] \\
 &= E[100] + 40E[Y] + 3E[Y^2] \\
 &= 100 + 40 \cdot 10 + 3 \cdot 2000 \\
 &= 6500,
 \end{aligned}$$

and the variance of C , $V[Y]$, is $E[Y^2] - E[Y]^2 = 2000 - 100 = 1900$.

7. Suppose Y has density function $f(y) = ky^9 e^{-y/2}$, $y \geq 0$. Find

- (a) k .
- (b) $E[Y]$ and $V(Y)$.
- (c) $P(Y > 34.1696)$.
- (d) A value b such that $P(Y < b) = 0.10$.

Solution. By inspection we can see that f is the gamma distribution with $\alpha = 10$, $\beta = 2$.

- (a) $k = \frac{1}{2^{10} \cdot \Gamma(10)} = \frac{1}{2^{10} \cdot 9!}$.
- (b) $E[Y] = \alpha\beta = 20$ and $V(Y) = \alpha\beta^2 = 40$.

(c)

$$\begin{aligned}
 P(Y > 34.1696) &= \frac{1}{2^{10} \cdot 9!} \int_{34.1696}^{\infty} y^9 e^{-y/2} dy \\
 &= \frac{1}{2^{10} \cdot 9!} \int_{17.0848}^{\infty} 2^{10} z^9 e^{-z} dz \quad \left[z = \frac{y}{2} \text{ substitution} \right] \\
 &= \frac{1}{9!} \int_{17.0848}^{\infty} z^9 e^{-z} dz \\
 &= \sum_{x=0}^9 \frac{17.0848^x e^{-17.0848}}{x!} \\
 &\approx 0.025.
 \end{aligned}$$

(d) Suppose there exists b with $P(Y < b) = 0.10$, then we must have that

$$0.90 = P(Y \geq b) = P(Z \geq b/2),$$

and from Appendix 3, Table 3, we get $b/2 \approx 14$, so that $b \approx 28$.