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MATHEMATICS
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MATH 380, SPRING 2015
SECTION 4 (10519)
HW #5, DUE: 2015, MARCH 23

1. Suppose Y is a discrete random variable with probability function $p(y) = ky(1/4)^y$, $y = 0, 1, 2, 3, \ldots$. Find

(a) k and (b) $E(Y)$ and $V(Y)$.

Solution. Let $p = 1/4$.

(a) We have that

$$1 = \sum_{y=0}^{\infty} kyp^y$$
$$= \sum_{y=0}^{\infty} kp \left(\frac{d}{dp} p^y \right)$$
$$= kp \frac{d}{dp} \sum_{y=0}^{\infty} p^y$$
$$= kp \frac{d}{dp} \left(\frac{1}{1-p} \right)$$
$$= \frac{kp}{(1-p)^2}.$$

It follows that $k = \dfrac{(1-p)^2}{p} = \dfrac{9}{4}$.

(b) We have that

$$E(Y) = \sum_{y=0}^{\infty} ky^2 p^y$$
$$= \sum_{y=0}^{\infty} ky^2 p^y - \sum_{y=0}^{\infty} kyp^y + \sum_{y=0}^{\infty} kyp^y$$
$$= \sum_{y=0}^{\infty} k(y^2 - y)p^y + \sum_{y=0}^{\infty} kyp^y$$
$$= \sum_{y=0}^{\infty} kp^2 \left(\frac{d^2}{dp^2} p^y \right) + \sum_{y=0}^{\infty} kyp^y$$
$$= kp^2 \frac{d^2}{dp^2} \sum_{y=0}^{\infty} p^y + 1$$
$$= kp^2 \frac{d^2}{dp^2} \left(\frac{1}{1-p} \right) + 1$$
$$= \frac{2kp^2}{(1-p)^3} + 1$$
$$= \frac{5}{3},$$

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and

$$E(Y^2) = \sum_{y=0}^{\infty} ky^3 p^y$$

$$= \sum_{y=0}^{\infty} ky^3 p^y - 3 \sum_{y=0}^{\infty} ky^2 p^y + 2 \sum_{y=0}^{\infty} kyp^y + 3 \sum_{y=0}^{\infty} ky^2 p^y - 2 \sum_{y=0}^{\infty} kyp^y$$

$$= \sum_{y=0}^{\infty} k(y^3 - 3y^2 + 2y)p^y + 3E(Y) - 2$$

$$= \sum_{y=0}^{\infty} ky(y-1)(y-2)p^y + 3$$

$$= \sum_{y=0}^{\infty} kp^3 \left(\frac{d^3}{dp^3} p^y \right) + 3$$

$$= kp^3 \frac{d^3}{dp^3} \sum_{y=0}^{\infty} p^y + 3$$

$$= \frac{6kp^3}{(1-p)^4} + 3.$$

Since $V(Y) = E(Y^2) - E(Y)^2$, it follows that

$$V(Y) = E(Y^2) - E(Y)^2$$

$$= \frac{6kp^3}{(1-p)^4} + 3 - \frac{25}{9}$$

$$= \frac{8}{9}.$$

2. Verify the identity $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ and use it to show that $E[Y^k] = npE[(X+1)^{k-1}]$ where Y is a binomial random variable with parameters n and p and X is a binomial random variable with parameters $n-1$ and p.

Proof. We have that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \frac{n(n-1)!}{k(k-1)!(n-k)!}$$

$$= \frac{n}{k} \frac{(n-1)!}{(k-1)!(n-k)!}$$

$$= \frac{n}{k} \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!}$$

$$= \frac{n}{k}\binom{n-1}{k-1}.$$

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Now

$$E[Y^k] = \sum_{y=0}^{n} y^k p(y) \qquad \text{[Definition]}$$

$$= \sum_{y=0}^{n} y^k \binom{n}{y} p^y (1-p)^{n-y}$$

$$= \sum_{y=1}^{n} y^k \binom{n}{y} p^y (1-p)^{n-y}$$

$$= \sum_{y=1}^{n} y^{k-1} n \binom{n-1}{y-1} p^y (1-p)^{n-y}$$

$$= np \sum_{y=1}^{n} y^{k-1} \binom{n-1}{y-1} p^{y-1} (1-p)^{n-y}$$

$$= np \sum_{x=0}^{n-1} (x+1)^{k-1} \binom{n-1}{x} p^x (1-p)^{(n-1)-x} \qquad \text{[Let } y = x+1\text{]}$$

$$= np \sum_{x=0}^{n-1} (x+1)^{k-1} p(x)$$

$$= npE[(X+1)^{k-1}],$$

which is what we wanted to show. $\qquad\square$

3. Using the recursion relation found in problem 2 for the binomial random variable with parameters n and p, find $E[Y^2]$ and then $V(Y)$.

Solution. If we set $k = 2$ in the formula in problem 2, we get

$$\begin{aligned} E[Y^2] &= npE[X+1] \\ &= np(E[X] + E[1]) \\ &= np((n-1)p + 1) \\ &= (np)^2 - np^2 + np. \end{aligned}$$

Thus

$$\begin{aligned} V(Y) &= E[Y^2] - E[Y]^2 \\ &= (np)^2 - np^2 + np - (np)^2 \\ &= np - np^2 \\ &= np(1-p). \end{aligned}$$

4. Using the identity from problem 2, show that

(a) if Y is a negative binomial random variable with parameters r and p, then

$$E[Y^k] = \frac{r}{p} E[(X-1)^{k-1}],$$

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where X is a negative binomial random variable with parameters $r + 1$ and p.

(b) Use the relation in (a) to find $E[Y]$ and $V(Y)$.

Solution.

(a) From problem 2, we know that

$$\frac{1}{r}\binom{y-1}{r-1} = \frac{1}{y}\binom{y}{r};$$

thus

$$
\begin{aligned}
E[Y^k] &= \sum_{y=r}^{\infty} y^k p(y) \\
&= \sum_{y=r}^{\infty} y^k \binom{y-1}{r-1} p^r (1-p)^{y-r} \\
&= \frac{r}{p}\frac{p}{r}\sum_{y=r}^{\infty} y^k \binom{y-1}{r-1} p^r (1-p)^{y-r} \\
&= \frac{r}{p}\sum_{y=r}^{\infty} y^k \frac{1}{r}\binom{y-1}{r-1} p^{r+1} (1-p)^{y-r} \\
&= \frac{r}{p}\sum_{y=r}^{\infty} y^k \frac{1}{y}\binom{y}{r} p^{r+1} (1-p)^{y-r} \\
&= \frac{r}{p}\sum_{y=r}^{\infty} y^{k-1} \binom{y}{r} p^{r+1} (1-p)^{y-r} \\
&= \frac{r}{p}\sum_{x=r+1}^{\infty} (x-1)^{k-1} \binom{x-1}{r} p^{r+1} (1-p)^{(x-1)-r} \\
&= \frac{r}{p}\sum_{x=r+1}^{\infty} (x-1)^{k-1} p(x) \\
&= \frac{r}{p} E[(X-1)^{k-1}].
\end{aligned}
$$

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(b) If we set $k = 1$ in (a), we immediately get that $E[Y] = \dfrac{r}{p}$. Now

$$
\begin{aligned}
V(Y) &= E[Y^2] - E[Y]^2 \\
&= \frac{r}{p} E[X - 1] - \frac{r^2}{p^2} \\
&= \frac{r}{p} (E[X] - E[1]) - \frac{r^2}{p^2} \\
&= \frac{r}{p} \left(\frac{r+1}{p} - 1 \right) - \frac{r^2}{p^2} \\
&= \frac{r^2 + r}{p^2} - \frac{r}{p} - \frac{r^2}{p^2} \\
&= \frac{r - pr}{p^2} \\
&= \frac{r(1 - p)}{p^2}.
\end{aligned}
$$

5. Using the identity from problem 2, show that if Y is a hypergeometric random variable with parameters N, r, amd n, then

$$
E[Y^k] = \frac{nr}{N} E[(X + 1)^{k-1}],
$$

where X is a hypergeometric random variable with parameters $N - 1$, $r - 1$, and $n - 1$.

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Proof. We have that

$$E[Y^k] = \sum_{y=0}^{n} y^k p(y)$$

$$= \sum_{y=0}^{n} y^k \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}$$

$$= \sum_{y=1}^{n} y^k \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}$$

$$= \sum_{y=1}^{n} y^k \frac{\frac{r}{y}\binom{r-1}{y-1}\binom{N-r}{n-y}}{\frac{N}{n}\binom{N-1}{n-1}}$$

$$= \frac{nr}{N} \sum_{y=1}^{n} y^{k-1} \frac{\binom{r-1}{y-1}\binom{N-r}{n-y}}{\binom{N-1}{n-1}}$$

$$= \frac{nr}{N} \sum_{x=0}^{n-1} (x+1)^{k-1} \frac{\binom{r-1}{x}\binom{(N-1)-(r-1)}{(n-1)-x}}{\binom{N-1}{n-1}} \qquad [\text{Let } y = x+1]$$

$$= \frac{nr}{N} \sum_{x=0}^{n-1} (x+1)^{k-1} p(x)$$

$$= \frac{nr}{N} E[(X+1)^{k-1}],$$

which is what we wanted to prove. □

6. If Y is a hypergeometric random variable with parameters N, r, and n, use the recursion relation found in problem 5 to find $E[Y]$ and $V(Y)$.

   Solution. Plugging in $k = 1$ in the formula from problem 5 immediately shows us that

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$E[Y] = \dfrac{nr}{N}$. Using this same formula, we have that

$$
\begin{aligned}
V(Y) &= E[Y^2] - E[Y]^2 \\
&= \frac{nr}{N}E[X+1] - \left(\frac{nr}{N}\right)^2 \\
&= \frac{nr}{N}(E[X] + E[1]) - \left(\frac{nr}{N}\right)^2 \\
&= \frac{nr}{N}\left(\frac{(n-1)(r-1)}{N-1} + 1\right) - \left(\frac{nr}{N}\right)^2 \\
&= \frac{nr}{N}\left(\frac{(n-1)(r-1)+N-1}{N-1}\right) - \left(\frac{nr}{N}\right)^2 \\
&= \frac{nr}{N}\left(\frac{(n-1)(r-1)+N-1}{N-1} - \frac{nr}{N}\right) \\
&= \frac{nr}{N}\left(\frac{N(n-1)(r-1)+N^2-N-nr(N-1)}{N(N-1)}\right) \\
&= \frac{nr}{N}\left(\frac{Nnr-Nn-Nr+N+N^2-N-Nnr+nr}{N(N-1)}\right) \\
&= \frac{nr}{N}\left(\frac{-Nn-Nr+N^2+nr}{N(N-1)}\right) \\
&= \frac{nr}{N}\left(\frac{N^2-Nr-Nn+nr}{N(N-1)}\right) \\
&= \frac{nr}{N}\left(\frac{N(N-r)-n(N-r)}{N(N-1)}\right) \\
&= \frac{nr}{N}\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right).
\end{aligned}
$$