1 Chapter 4

1.1 Section 1

Exam **12%**: Part of 3.3 (p.86)

$$\lim_{x\to\infty} f(x) = \infty, \lim_{x\to-\infty} f(x) = \infty, \lim_{x\to\infty} f(x) = -\infty, \lim_{x\to-\infty} f(x) = -\infty.$$

28%: 4.1 Continuous function at a point (both versions: $\lim_{x\to a} f(x) = f(a)$, $\varepsilon - \delta$ version)

Find and classify discontinuities: Removable discontinuity, jump discontinuity, discontinuity of 2$^{\text{nd}}$ kind.

Theorem 4.1, 4.2, 4.3, right continuous at $x = a$, left continuous at $x = a$. Continuity on an interval: $[a, b]$, (a, b).

28% 4.2 (p 102-106) Th 4.4, 4.5 (Extreme Value Theorem), 4.6 (Intermediate Value Theorem), Theorem 4,7 Fixed point.

32% 4.3 Uniform continuous on an interval. Be able to use definition to prove uniform continuity on I or to prove not uniform continuity on I. Theorem 4.12, 4.13, 4.14, 4.15 and its corollary, 4.16.

4.01 Prove that if f is continuous at x_0 then f is bounded at x_0.

Proof. Suppose that $f : A \to \mathbb{R}$ is continuous at x_0. To show that f is bounded at x_0, it suffices to show that f is bounded on $N_\delta(x_0) \cap A$ for some $\delta > 0$. Since f is continuous at x_0, it follows by definition that there exists $\delta_1 > 0$ such that $|f(x) - f(x_0)| < 1$ whenever $|x - x_0| < \delta_1$. Using the triangle inequality we have that

$$||f(x)| - |f(x_0)|| < |f(x) - f(x_0)| < 1,$$

so that $|f(x)| - |f(x_0)| < 1$, if $|x - x_0| < \delta_1$. Thus $|f(x)| < 1 + |f(x_0)|$, if $|x - x_0| < \delta_1$. We have thus shown that f is bounded on $N_{\delta_1}(x_0)$ by $1 + |f(x_0)|$, so that f is bounded at x_0. ☐

4.02 Find all points of discontinuity for the following functions, classify the discontinuities as removable, jump, or second kind, and determine where the function is right- and left-continuous.

(a) $$f(x) = \begin{cases} x^2 & \text{if } x < -1, \\ 2x + 3 & \text{if } -1 \le x \le 0, \\ |x - 1| & \text{if } 0 < x < 2, \\ x^3 - 7 & \text{if } 2 \le x < 3, \\ \dfrac{x-3}{x-4} & \text{if } 3 \le x < 4, \\ 0 & \text{if } 4 \le x. \end{cases}$$

(b) $f(x) = x + [\![-x]\!]$.

(c) $f(x) = x[\![x]\!]$.

(d) $f(x) = \operatorname{sgn}[\![|x|]\!]$.

(e) $f(x) = \begin{cases} [\![x+1]\!] \sin \dfrac{1}{x} & \text{if } x \in (-1,0) \cup (0,1) \\ 0 & \text{otherwise.} \end{cases}$

(f) $f(x) = \begin{cases} (1+x) \operatorname{sgn} x + \operatorname{sgn} |x| - 1 & \text{if } x \text{ is rational} \\ \operatorname{sgn} x & \text{if } x \text{ is irrational.} \end{cases}$

Solution.

(a) Since

$$3 = \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x) = 1 \text{ and } 27 = \lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x) = 0$$

it follows that f has jump discontinuities at 0 and 3. The function f has a discontinuity of the second kind at 4 because $\lim_{x\to 4^-} f(x) = -\infty$; f is not right-continuous at 0 because $1 = \lim_{x\to 0^+} f(x) \neq f(0) = 3$, but it is right-continuous at every other point; also, f is not left-continuous at 3 and 4 because $27 = \lim_{x\to 3^-} f(x) \neq f(3) = 0$ and $\lim_{x\to 4^-} f(x) = -\infty$, but it is left-continuous at every other point.

(b) Let z be an integer. Then it follows that

$$\begin{aligned} f(z) &= z + [\![-z]\!] \\ &= z + (-z) = 0, \end{aligned}$$

$$\begin{aligned} \lim_{x\to z^+} f(x) &= \lim_{x\to z^+} x + \lim_{x\to z^+} [\![-x]\!] \\ &= z + (-z-1) = -1, \text{ and} \end{aligned}$$

$$\begin{aligned} \lim_{x\to z^-} f(x) &= \lim_{x\to z^-} x + \lim_{x\to z^-} [\![-x]\!] \\ &= z + (-z) = 0, \end{aligned}$$

so that f has jump discontinuities at all integers; f is left-continuous at all points, and it is right-continuous at all points except the integers.

(c) It is clear that f is continuous at all non-integer points. So let z be an integer. Observe that $f(z) = z^2$, $\lim_{x\to z^-} f(x) = z(z-1)$, and $\lim_{x\to z^+} f(x) = z^2$, so that f is continuous at 0 and has jump discontinuities at all other integers; also f is left-continuous at all points except nonzero integers, and it is right continuous at all points.

(d) We have that

$$\operatorname{sgn}[\![|x|]\!] = \begin{cases} 1 & \text{if } x \leq -1 \text{ or } x \geq 1 \\ 0 & \text{if } -1 < x < 1, \end{cases}$$

so that f has jump discontinuites at -1 and 1, is left-continuous at all points except 1, and is right-continuous at all points except -1.

4.03 Prove that $f(x) = \cos x$ is continuous on $\mathbb{R}$.

4.04 Prove that if f is continuous at x_0 and g is discontinuous at x_0 then $f + g$ must have a discontinuity at x_0.

Proof. Assume that f is continuous at x_0 and g is discontinuous at x_0. Now suppose to the contrary that $f + g$ is continuous at x_0. By Theorem 4.2, it follows that $(f+g)+(-f) = g$ is also continuous at x_0, a contradiction. Thus $f+g$ has a discontinuity at x_0. $\square$

4.05 Show that $f + g$ can be continuous at x_0 even though both f and g have discontinuities at x_0.

Solution. For every nonzero x, let $f(x) = 1/x$; then define $f(0) = 0$ and $g(x) = -f(x)$ for all real x. It is clear that both functions are not continuous at 0, but $f + g = 0$ is continuous at 0.

4.06 Show that $f \cdot g$ can be continuous at x_0 even though both f and g have discontinuities at x_0.

Solution. Define

$$f(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ \dfrac{1}{x-1} & \text{if } x > 1, \end{cases}$$

$$g(x) = \begin{cases} \dfrac{1}{x-1} & \text{if } x < 1 \\ 0 & \text{if } x \geq 1. \end{cases}$$

The functions f and g have discontinuities at 1 because the former is not right-continuous at 1 and the latter is not left-continuous at 1; however $f \cdot g = 0$ is continuous at 1.

4.07 If f is continuous at x_0 and g is discontinuous at x_0, what can be said about continuity of the product $f \cdot g$ at x_0?

Answer. Nothing definite can be said about the continuity of the product at x_0 because if $f(x) = 1$ and $g(x) = 1/x$, then $f \cdot g = g$ is not continuous at 0, and if $f(x) = 0$ and $g(x) = 1/x$ with $g(0) = 0$ then $f \cdot g = 0$ is continuous at 0. Note that in either case f was continuous at 0 and g was not.

4.08 Show that the composition function $g \circ f$ can be continuous at x_0 even though f or g or both f and g are discontinuous at x_0.

Solution. Let $f(x) = 1/x$, with $f(0) = 0$, and let $g = f$. Then it follows that $(f \circ g)(x) = x$ so that $f \circ g$ is continuous at 0, but f (and thus g) is not continuous at 0.

4.09 Prove that the function

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

has a discontinuity of the second kind at each nonzero real number.

Proof. Let a be a nonzero real number. It suffices to show that $\lim_{x\to a^+} f(x)$ does not exist. By Theorems 1.9 and 1.10, we know that every interval in $\mathbb{R}$ contains a rational number and an irrational number. So let b_n be a sequence of rational numbers and c_n a sequence of irrational numbers such that

$$b_n \in (a, a+1/n) \cap \mathbb{Q}, \qquad c_n \in (a, a+1/n) \cap (\mathbb{R} - \mathbb{Q}).$$

It is clear that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = a$, but

$$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} b_n = a \neq 0 = \lim_{n\to\infty} 0 = \lim_{n\to\infty} c_n,$$

so that by Exercise 3.36, $\lim_{x\to a^+} f(x)$ does not exist. Thus f has a discontinuity of the second kind at each nonzero real number. □

4.10 Prove that if f is continuous at x_0 and f is nonnegative then $h(x) = \sqrt{f(x)}$ is continuous at x_0.

4.11 Find a function f which has a discontinuity of the second kind at every real number although $f \circ f$ is continuous on $\mathbb{R}$.

Solution. We know from Example 4.4 that

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational,} \end{cases}$$

has a discontinuity of the second kind at every real number. Now

$$(f \circ f)(x) = f(f(x)) = \begin{cases} f(1) = 1 & \text{if } x \text{ is rational,} \\ f(0) = 1 & \text{if } x \text{ is irrational,} \end{cases}$$

so that $f \circ f$ is identically equal to 1. Thus $f \circ f$ is continuous on $\mathbb{R}$.

4.12 If f is continuous on $(0,1)$ and $f(x) = 1 - x$ for every rational number $x \in (0,1)$, find $f(\pi/4)$. Explain your answer.

Solution. Since f is continuous on (0, 1) and $\pi/4 \in (0,1)$, it follows that

$$\lim_{x\to\pi/4} = f(\pi/4).$$

Consider the sequence of rationals a_n where $a_n \in \left(\frac{\pi}{4}, \frac{\pi}{4} + \frac{1}{10n}\right)$. Since each $a_n \in (0,1)$ and since a_n converges to $\pi/4$, it follows by Theorem 3.6 that $f(a_n)$ must converge to $f(\pi/4)$. Thus

$$\begin{aligned} f(\pi/4) &= \lim_{n\to\infty} f(a_n) \\ &= \lim_{n\to\infty} (1 - a_n) \\ &= \lim_{n\to\infty} 1 - \lim_{n\to\infty} a_n \\ &= 1 - \pi/4. \end{aligned}$$

4.13 Prove that if f and g are each continuous on (a, b) and $f(x) = g(x)$ for every rational $x \in (a, b)$ then $f(x) = g(x)$ for every $x \in (a, b)$.

4.14 Prove: f is right-continuous at x_0 if and only if $f(x_n) \to f(x_0)$ for every sequence $\{x_n\}$ in the domain of f with $x_n \to x_0$ and $x_n \geq x_0$ for $n = 1, 2, 3, ...$

4.15 Discuss one-sided continuity for the pie function.

4.16 Prove that if f is defined on $\mathbb{R}$ and continuous at $x_0 = 0$ and if $f(x_1+x_2) = f(x_1)+f(x_2)$ for each $x_1, x_2 \in \mathbb{R}$ then f is continuous on $\mathbb{R}$.

4.17 Find all functions f which are continuous on $\mathbb{R}$ and which satisfy the equation $f(x)^2 = x^2$ for each $x \in \mathbb{R}$. *Hint*: There are four possible solutions.

4.18 Prove that if g is continuous at $x_0 = 0$, $g(0) = 0$ and for some $\delta > 0$ $|f(x)| \leq |g(x)|$ for each $x \in N_\delta(0)$ then f is continuous at $x_0 = 0$.

4.19 Prove that if f is continuous on $[a, b]$ then there exists a function g continuous on $\mathbb{R}$ such that $g(x) = f(x)$ for each $x \in [a, b]$. The function g is called a *continuous extension* of f to $\mathbb{R}$.

4.20 The function $f(x) = \tan x$ defined on $(-\pi/2, \pi/2)$ clearly has no continuous extension to $\mathbb{R}$. Find a bounded continuous function on (a, b) which has no continuous extension to $\mathbb{R}$.

4.21 Assume that f is continuous on (a, b). Prove that f has a continuous extension to $\mathbb{R}$ if and only if both limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist.

4.22 Prove that if f is continuous on (a, b) and both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist then f is bounded on (a, b).

4.23 Suppose f is one-to-one on (a, b) and satisfies the following property: whenever $f(x_1) \neq f(x_2)$ for $x_1 < x_2$, x_1, $x_2 \in (a, b)$ and k is any number between $f(x_1)$ and $f(x_2)$, there exists a $c \in (x_1, x_2)$ with $f(c) = k$. Prove that f is continuous on (a, b).