1. Suppose Y is a discrete random variable with probability function $p(y) = ky(1/4)^y$, $y = 0, 1, 2, 3, \ldots$. Find

   (a) k and (b) $E(Y)$ and $V(Y)$.

   Solution. Let $p = 1/4$.

   (a) We have that

$$\begin{aligned}
1 &= \sum_{y=0}^{\infty} kyp^y \\
&= \sum_{y=0}^{\infty} kp\left(\frac{d}{dp}p^y\right) \\
&= kp\frac{d}{dp}\sum_{y=0}^{\infty} p^y \\
&= kp\frac{d}{dp}\left(\frac{1}{1-p}\right) \\
&= \frac{kp}{(1-p)^2}.
\end{aligned}$$

   It follows that $k = \dfrac{(1-p)^2}{p} = \dfrac{9}{4}$.

   (b) We have that

$$\begin{aligned}
E(Y) &= \sum_{y=0}^{\infty} ky^2p^y \\
&= \sum_{y=0}^{\infty} ky^2p^y - \sum_{y=0}^{\infty} kyp^y + \sum_{y=0}^{\infty} kyp^y \\
&= \sum_{y=0}^{\infty} k(y^2-y)p^y + \sum_{y=0}^{\infty} kyp^y \\
&= \sum_{y=0}^{\infty} kp^2\left(\frac{d^2}{dp^2}p^y\right) + \sum_{y=0}^{\infty} kyp^y \\
&= kp^2\frac{d^2}{dp^2}\sum_{y=0}^{\infty} p^y + 1 \\
&= kp^2\frac{d^2}{dp^2}\left(\frac{1}{1-p}\right) + 1 \\
&= \frac{2kp^2}{(1-p)^3} + 1 \\
&= \frac{5}{3},
\end{aligned}$$

and

$$\begin{aligned}
E(Y^2) &= \sum_{y=0}^{\infty} ky^3p^y \\
&= \sum_{y=0}^{\infty} ky^3p^y - 3\sum_{y=0}^{\infty} ky^2p^y + 2\sum_{y=0}^{\infty} kyp^y + 3\sum_{y=0}^{\infty} ky^2p^y - 2\sum_{y=0}^{\infty} kyp^y \\
&= \sum_{y=0}^{\infty} k(y^3 - 3y^2 + 2y)p^y + 3E(Y) - 2 \\
&= \sum_{y=0}^{\infty} ky(y-1)(y-2)p^y + 3 \\
&= \sum_{y=0}^{\infty} kp^3 \left(\frac{d^3}{dp^3}p^y\right) + 3 \\
&= kp^3 \frac{d^3}{dp^3} \sum_{y=0}^{\infty} p^y + 3 \\
&= \frac{6kp^3}{(1-p)^4} + 3.
\end{aligned}$$

Since $V(Y) = E(Y^2) - E(Y)^2$, it follows that

$$\begin{aligned}
V(Y) &= E(Y^2) - E(Y)^2 \\
&= \frac{6kp^3}{(1-p)^4} + 3 - \frac{25}{9} \\
&= \frac{8}{9}.
\end{aligned}$$

2. Verify the identity $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ and use it to show that $E[Y^k] = npE[(X+1)^{k-1}]$ where Y is a binomial random variable with parameters n and p and X is a binomial random variable with parameters $n-1$ and p.

Proof. We have that

$$\begin{aligned}
\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\
&= \frac{n(n-1)!}{k(k-1)!(n-k)!} \\
&= \frac{n}{k}\frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{n}{k}\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} \\
&= \frac{n}{k}\binom{n-1}{k-1}.
\end{aligned}$$

Now

$$\begin{aligned}
E[Y^k] &= \sum_{y=0}^{n} y^k p(y) && \text{[Definition]}\\
&= \sum_{y=0}^{n} y^k \binom{n}{y} p^y (1-p)^{n-y}\\
&= \sum_{y=1}^{n} y^k \binom{n}{y} p^y (1-p)^{n-y}\\
&= \sum_{y=1}^{n} y^{k-1} n \binom{n-1}{y-1} p^y (1-p)^{n-y}\\
&= np \sum_{y=1}^{n} y^{k-1} \binom{n-1}{y-1} p^{y-1} (1-p)^{n-y}\\
&= np \sum_{x=0}^{n-1} (x+1)^{k-1} \binom{n-1}{x} p^x (1-p)^{(n-1)-x} && \text{[Let } y = x+1\text{]}\\
&= np \sum_{x=0}^{n-1} (x+1)^{k-1} p(x)\\
&= npE[(X+1)^{k-1}],
\end{aligned}$$

which is what we wanted to show. $\square$