JOSEPH OKONOBOH
Mathematics
Cal State Long Beach

1. Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false. Let $G = \langle g \rangle$ have order 300.

   ① There are exactly 80 generators of G.

   ② G has only one element of order 3.

   ③ G can be embedded in S_{30}.

   ④ G has a subgroup of order 20.

   ⑤ G has a totality of 18 subgroups.

   Solution.

   ① True. Since $G = \langle g \rangle$ is cyclic and since $|g| = 300$, it follows that the number of generators of G is the number of positive integers relatively prime to 300, which is 80.

   ② False. We have that $g^{100} \neq g^{200}$ (since $|g| = 300$) and

$$|g^{100}| = \frac{300}{\gcd(300, 100)} = 3 = \frac{300}{\gcd(300, 200)} = |g^{200}|.$$

2. Let G be an abelian group and let $a, b \in G$ be of order 120 and 72 respectively. Do the following:

   ① Find an element of order 15.

   ② What is the order of b^{10}?

   ③ Find an element of as large an order as you can.

   Solution.

   ① w

3. Consider the non-abelian group of order 55 from **Homework #4**. View this group as acting on all column vectors of size 2 (with entries in $\mathbb{Z}_{11}$).

   ① Find the number of fixed points of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

   ② Find the number of fixed points of $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$.

   ③ Decide on the number of fixed elements each of the elements of the group has.

   ④ Use Burnside's Lemma to count the orbits.

4. Let the vertices of the cube be given as follows:

$$1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, 2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, 3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, 4 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, 5 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, 6 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, 7 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$

and $8 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$.

(1) Label the faces A, A', B, B', C, and C' (where the prime means opposite), and give each as a set of four vertices. Let A be the intersection with the plane $x = 1$, B with the plane $y = 1$ and C with $z = 1$.

(2) Find 24 3×3 matrices of determinant 1 that are isometries of the cube, and write each as a permutation in S_8 (of the eight vertices) and also as a permutation of the faces. **Hint:** Start with the six permutation matrices of size 3.

Assume these 24 matrices form a group G. **Bonus.** Prove this.
Assume these $G \simeq S_4$. **Bonus.** Prove this.

(3) Find the number of ways to color a cube with two colors.

(4) Find the number of ways to color a cube with three colors.

Bonus. Find the number of ways to color the cube with n colors.

Solution.

(1) s