Differential Equations

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Contents

Chapter 1

Introduction

1.3 Classification of Differential Equations

In each of Problems 1 through 6, determine the order of the given differential equation; also state whether the equation is linear or nonlinear.

1.3.1 $t^2 \dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + 2y = \sin(t).$

Answer. Linear of order 2.

1.3.2 $(1 + y^2)\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + y = e^t.$

Answer. Nonlinear of order 2.

1.3.3 $\dfrac{d^4y}{dt^4} + \dfrac{d^3y}{dt^3} + \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 1.$

Answer. Linear of order 4.

1.3.4 $\dfrac{dy}{dt} + ty^2 = 0.$

Answer. Nonlinear of order 1.

1.3.5 $\dfrac{d^2y}{dt^2} + \sin(t + y) = \sin(t).$

Answer. Nonlinear of order 2.

1.3.6 $\dfrac{d^3y}{dt^3} + t\dfrac{dy}{dt} + (\cos^2(t))y = t^3.$

Answer. Linear of order 3.

In problems 8 and 11, verify that each given function is a solution of the differential equation.

1.3.8 $y'' + 2y' - 3y = 0; \quad y_1(t) = e^{-3t}, \quad y_2(t) = e^t.$

Solution. We have

$$y_1{}' = -3e^{-3t} \text{ and } y_1{}'' = 9e^{-3t},$$

so that

$$y_1{}'' + 2y_1{}' - 3y_1 = 9e^{-3t} + 2(-3e^{-3t}) - 3e^{-3t} = 0.$$

Similarly,

$$y_2{}' = y_2{}'' = e^t,$$

so that

$$y_2{}'' + 2y_2{}' - 3y_2 = e^t + 2e^t - 3e^t = 0.$$

That is, y_1 and y_2 both satisfy the given differential equations.

1.3.11 $2t^2y'' + 3ty' - y = 0, \quad t > 0; \quad y_1(t) = t^{1/2}, \quad y_2(t) = t^{-1}.$

Solution. For $t > 0$, we have

$$y_1{}' = \frac{1}{2\sqrt{t}} \text{ and } y_1{}'' = -\frac{1}{4\sqrt{t^3}},$$

so that

$$\begin{aligned} 2t^2y_1{}'' + 3ty_1{}' - y_1 &= -2t^2\frac{1}{4\sqrt{t^3}} + 3t\frac{1}{2\sqrt{t}} - \sqrt{t} \\ &= -\frac{2}{4}t^{1/2} + \frac{3}{2}t^{1/2} - t^{1/2} \\ &= \left(-\frac{2}{4} + \frac{3}{2} - 1\right)t^{1/2} \\ &= 0 \cdot t^{1/2} = 0, \end{aligned}$$

and

$$y_2{}' = -t^{-2} \text{ and } y_2{}'' = 2t^{-3},$$

so that

$$\begin{aligned} 2t^2y_2{}'' + 3ty_2{}' - y_2 &= 2t^2(2t^{-3}) + 3t(-t^{-2}) - t^{-1} \\ &= 4t^{-1} - 3t^{-1} - t^{-1} \\ &= (4 - 3 - 1)t^{-1} = 0 \cdot t^{-1} = 0. \end{aligned}$$

Chapter 2

First Order Differential Equations

2.1 Linear Equations; Method of Integrating Factors

In Problems 1, 3, 7, 8, and 11, find the general solution of the differential equation, and use it to determine how solutions behave as $t \to \infty$.

2.1.1 $y' + 3y = t + e^{-2t}$.

Solution. Multiply the equation above by the integrating factor $e^{\int 3\, dt} = e^{3t}$ to get

$$e^{3t}y' + 3e^{3t}y = te^{3t} + e^{t},$$

so that

$$(ye^{3t})' = te^{3t} + e^{t}.$$

Differentiate the preceding equation to get

$$ye^{3t} = \int (te^{3t} + e^{t})\, dt = \frac{1}{9}e^{3t}(3t-1) + e^{t} + C.$$

That is, $y(t) = \dfrac{1}{9}(3t-1) + e^{-2t} + Ce^{-3t}$, and $y \to \infty$ as $t \to \infty$.

2.1.3 $y' + y = te^{-t} + 1$.

Solution. The integrating factor is $e^{\int dt} = e^{t}$. Now multiply the differential equation by e^{t} to get

$$e^{t}y' + e^{t}y = t + e^{t},$$

so that

$$(e^{t}y)' = t + e^{t}.$$

That is $e^{t}y = \displaystyle\int (t + e^{t}) = \frac{1}{2}t^2 + e^{t} + C$, and we conclude that $y(t) = \dfrac{1}{2}e^{-t}(t^2 + C') + 1$, and $y \to 1$ as $t \to \infty$.

2.1.7 $y' + 2ty = 2te^{-t^2}$.

Solution. The integrating factor is $e^{\int 2t\,dt} = e^{t^2}$. Multiply by the integrating factor to transform the equation into

$$(e^{t^2}y)' = 2t.$$

Integrate and simplify to get $y(t) = e^{-t^2}(t^2 + C)$, and thus, $y \to 0$ as $t \to \infty$.

2.1.8 $(1 + t^2)y' + 4ty = (1 + t^2)^{-2}$.

Solution. First we rewrite the equation in standard form to get:

$$y' + \frac{4t}{1 + t^2}y = (1 + t^2)^{-3}.$$

The integrating factor is thus $e^{\int((4t/1+t^2))\,dt} = (1 + t^2)^2$. So multiply the standard form by the integrating factor to get

$$(y(1 + t^2)^2)' = (1 + t^2)^{-1}.$$

Now integrate and simplify to get

$$y(t) = \frac{\arctan(t)}{(1 + t^2)^2} + \frac{C}{(1 + t^2)^2}.$$

Thus $y \to 0$ as $t \to \infty$.

2.1.11 $y' + y = 5\sin(2t)$.

Solution. The integrating factor is $e^{\int dt} = e^t$. So multiply the equation by e^t to arrive at

$$(e^t y)' = 5e^t \sin(2t).$$

Integrate (use by parts on the right) and divide the result by e^t to get

$$y(t) = \sin(2t) - 2\cos(2t) + Ce^{-t}.$$

It follows that y oscillates as t approaches infinity.

In Problems 13, 15, 16, 17, 18, and 20, find the solution of the given initial value problem.

2.1.13 $y' - y = 2te^{2t}, \quad y(0) = 1$

Solution. Using the integrating factor $e^{\int -t\,dt} = -e^{-t}$, it follows that

$$y(t) = 2te^{2t} - Ce^t - 2e^{2t}.$$

Now use the initial condition $y(0) = 1$ to get

$$y(t) = 2te^{2t} + 3e^t - 2e^{2t}$$

2.1.15 $ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}, \quad t > 0$

Solution. Put the equation in standard form:

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}.$$

Now use the integrating factor $e^{\int (2/t)\,dt} = t^2$ to solve; thus

$$y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{C}{t^2}.$$

Finally use the initial condition $y(1) = 1/2$ to get

$$y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{12t^2}$$

.

2.1.16 $y' + (2/t)y = (\cos(t))/t^2, \quad y(\pi) = 0, \quad t > 0$

Solution. The general solution using the integrating factor $e^{\int (2/t)\,dt} = t^2$ is

$$y(t) = \frac{\sin(t)}{t^2} + \frac{C}{t^2},$$

and, thus, $y(t) = \dfrac{\sin(t)}{t^2}$, after using the initial condition.

2.1.17 $y' - 2y = e^{2t}, \quad y(0) = 2$

Solution. Using the initial conditipn and the integrating factor $e^{\int -2\,dt} = e^{-2t}$, it follows that

$$y(t) = te^{2t} + Ce^{2t},$$

and thus $y(t) = te^{2t} + 2e^{2t}$.

2.1.18 $ty' + 2y = \sin(t), \quad y(\pi/2) = 1, \quad t > 0$

Solution. The standard form of the equation is

$$y' + \frac{2}{t}y = \frac{\sin(t)}{t}.$$

Now use the integrating factor $e^{\int (2/t)\,dt} = t^2$ to get

$$y(t) = -\frac{\cos(t)}{t} + \frac{\sin(t)}{t^2} + \frac{C}{t^2},$$

and thus

$$y(t) = -\frac{\cos(t)}{t} + \frac{\sin(t)}{t^2} + \frac{\pi^2 - 4}{4t^2}$$

by the initial conditon $y(\pi/2) = 1$.

2.1.20 $ty' + (t+1)y = t, \quad y(\ln(2)) = 1, \quad t > 0$

Solution. The standard form of the equation is

$$y' + \frac{t+1}{t}y = 1,$$

so that the integrating factor is $e^{\int (t+1/t)\,dt} = te^t$, and thus,

$$y(t) = \frac{C}{te^t} - \frac{1}{t} + 1,$$

so that

$$y(t) = \frac{2}{te^t} - \frac{1}{t} + 1$$

after using the initial condition.

2.2 Separable Equations

In Problems 2, 4, and 7, solve the given differential equation.

2.2.2 $y' = \dfrac{x^2}{y(1+x^3)}$.

Solution. Given the separable differential equation

$$\frac{dy}{dx} = \frac{x^2}{y(1+x^3)},$$

it follows that $y\,dy = \dfrac{x^2}{1+x^3}dx$. Integrate to get $\dfrac{y^2}{2} = \dfrac{1}{3}\ln|1+x^3| + C$. That is,

$$y(x) = \pm\sqrt{\frac{2}{3}\ln|1+x^3| + C'}.$$

2.2.4 $y' = \dfrac{3x^2-1}{3+2y}$.

Solution. We have

$$\frac{dy}{dx} = \frac{3x^2-1}{3+2y}.$$

That is, $(3+2y)\,dy = (3x^2-1)\,dx$. Integrate to get $3y + y^2 = x^3 - x + C$. Using the quadratic formula, it follows that

$$y(x) = \frac{-3 \pm \sqrt{4x^3 - 4x + C'}}{2}$$

2.2.7 $\dfrac{dy}{dx} = \dfrac{x - e^{-x}}{y + e^y}$.

Solution. Separate the equation so that $(y+e^y)\,dy = (x - e^{-x})\,dx$, and integrate to get $\dfrac{y^2}{2} + e^y = \dfrac{x^2}{2} + e^{-x} + C$. Multiply by 2 to get $y^2 + 2e^y = x^2 + 2e^{-x} + C'$. No further simplification is possible.

In Problems 9, 10, 12, 14, 15, 16, and 20, find the solution of the given initial value problem in explicit form and determine (at least approximately) the interval in which the solution is defined.

2.2.9 $y' = (1-2x)y^2, \qquad y(0) = -1/6.$

Solution. Separate the variables to get $\dfrac{1}{y^2}dy = (1-2x)\ dx \quad (y \neq 0)$. Integrate to get $-\dfrac{1}{y} = x - x^2 + C$. That is, $y(x) = \dfrac{1}{x^2 - x + C'}$. Use the initial condition to now conclude that $y(x) = \dfrac{1}{x^2 - x - 6}$. For y to be defined, its denominator must be nonzero, so this solution is defined on the intervals $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$. But since the initial condition (where $x = 0$) is contained only in the interval $(-2, 3)$, it follows that $y(x)$ is defined in the interval $(-2, 3)$.

2.2.10 $y' = \dfrac{1-2x}{y} \qquad y(1) = -2.$

Solution. Separate and integrate to obtain $y(x) = \pm\sqrt{2x - 2x^2 + C}$. Use initial condition to conclude that $y(x) = -\sqrt{2x - 2x^2 + 4} = -\sqrt{-2(x+1)(x-2)}$. By observing the differential equation, we see that $y \neq 0$. Thus the interval in which the solution is defined is $(-1, 2)$.

2.2.12 $\dfrac{dr}{d\theta} = \dfrac{r^2}{\theta}, \quad r(1) = 2.$

Solution. Separate the variables, integrate and use the initial condition to get

$$r(\theta) = \frac{-2}{2\ln|\theta| - 1}.$$

The interval on which the solution is defined is $(0, \sqrt{e})$.

2.2.14 $y' = xy^3(1+x^2)^{-1/2}, \qquad y(0) = 1.$

Solution. By separating the variables and integrating, we shall get

$$y = \pm\sqrt{\frac{1}{C - 2\sqrt{1+x^2}}}.$$

Use the initial condition to conclude that

$$y = \sqrt{\frac{1}{3 - 2\sqrt{1+x^2}}}.$$

The interval on which the solution is defined is thus $(-\sqrt{5}/2, \sqrt{5}/2)$.

2.2.15 $y' = \dfrac{2x}{1+2y}, \quad y(2) = 0.$

Solution. Separate the variable and solve to get $y^2+y=x^2+C$. Now use the quadratic formula to conclude that

$$y(x)=\frac{-1\pm\sqrt{4x^2+C'}}{2}$$

and that

$$y(x)=\frac{-1+\sqrt{4x^2-15}}{2}$$

by the initial condition. From the differential equation, we observe that $1+2y\neq 0$, so that $y\neq -1/2$. That is, for our solution, we must have that $\sqrt{4x-15}>0$. So the interval of our solution is $(\sqrt{15}/2,\infty)$.

2.2.16 $y'=\dfrac{x(x^2+1)}{4y^3}, \qquad y(0)=-\dfrac{1}{\sqrt{2}}.$

Solution. After solving we shall get

$$y(x)=\pm\left(\frac{x^4}{4}+\frac{x^2}{2}+C\right)^{1/4},$$

and, thus,

$$y(x)=-\left(\frac{x^4}{4}+\frac{x^2}{2}+\frac{1}{4}\right)^{1/4}=-\left[\left(\frac{x^2}{2}+\frac{1}{2}\right)^2\right]^{1/4}=-\left(\frac{x^2}{2}+\frac{1}{2}\right)^{1/2},$$

by the initial condition. The solution is defined on all the real numbers.

2.2.20 $y^2(1-x^2)^{1/2}\,dy=\arcsin(x)\,dx, \quad y(0)=1.$

Solution. Separate the variables and use the initial condition to get:

$$y(x)=\left[\frac{3}{2}(\arcsin(x))^2+1\right]^{1/3}.$$

The interval in which the solution is defined is thus $-1<x<1$.

2.3 Modeling with First Order Equations

2.3.1 Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 L/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

Solution. Let $Q(t)$ be the amount (in grams) of dye solution after t minutes. Thus it follows that

$$\begin{aligned} Q(0) &= \frac{200\ L}{1}\cdot\frac{1\ g}{L}=200\ g\\ \text{Rate of dye in} &= \frac{0\ g}{L}\cdot\frac{2\ L}{\text{min}}=0\\ \text{Rate of dye out} &= \frac{Q(t)}{200\ L}\cdot\frac{2\ L}{\text{min}}=\frac{Q(t)}{100}. \end{aligned}$$

Since

$$\frac{dQ}{dt} = \text{Rate of dye in} - \text{Rate of dye out},$$

it follows that

$$\frac{dQ}{dt} = 0 - \frac{Q(t)}{100} \Rightarrow$$
$$\frac{dQ}{dt} + \frac{Q(t)}{100} = 0.$$

Multiply the above equation by the integrating factor, $\mu(t) = e^{\int (1/100)\,dt} = e^{(1/100)t}$, and our differential equation becomes

$$\frac{d}{dt}\left(Q(t)e^{(1/100)t}\right) = 0.$$

Taking the integral of the above equation and using the initial condition $Q(0) = 200$ will give us

$$Q(t) = 200e^{-(1/100)t}.$$

Let t_2 be the time in minutes after which concentration of the dye reaches 1% of its original value. Thus $Q(t_2) = 2$. Solving this equation will give us

$$t_2 = 200\ln(10) \approx 461 \text{ minutes}.$$

2.3.2 A tank initially contains 120 L of pure water. A mixture containing a concentration of γ g/L of salt enters the tank at a rate of 2 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$.

Solution. Let $Q(t)$ be the amount (in grams) of salt in the tank after t minutes. Thus it follows that

$$Q(0) = 0$$
$$\text{Rate of dye in} = \frac{\gamma\ g}{L} \cdot \frac{2\ L}{\text{min}} = 2\gamma$$
$$\text{Rate of dye out} = \frac{Q(t)}{120\ L} \cdot \frac{2\ L}{\text{min}} = \frac{Q(t)}{60}.$$

Since

$$\frac{dQ}{dt} = \text{Rate of salt in} - \text{Rate of salt out},$$

it follows that

$$\frac{dQ}{dt} = 2\gamma - \frac{Q(t)}{60} \Rightarrow$$
$$\frac{dQ}{dt} + \frac{Q(t)}{60} = 2\gamma.$$

Solving with integrating factor $e^{(1/60)t}$ and the initial condition $Q(0) = 0$ will give us

$$120\gamma(1 - e^{-(1/60)t}).$$

That is, the limiting amount of salt is 120γ.

2.3.4 A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 9 gal/min, and the mixture is allowed to flow out of the tank at a rate of 6 gal/min. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow. Find the concentration (in pounds per gallon) of salt in the tank when it is on the point of overflowing. Compare this concentration with the theoretical limiting concentration if the tank had infinite capacity.

Solution. Let $Q(t)$ be the amount of salt after t minutes. It follows that

$$\begin{aligned}\frac{dQ}{dt} &= \text{rate of salt in} - \text{rate of salt out}\\ &= \frac{1 \text{ lb}}{1 \text{ gal}} \cdot \frac{9 \text{ gal}}{1 \text{ min}} - \frac{Q(t)}{200+3t} \cdot \frac{6 \text{ gal}}{1 \text{ min}}.\end{aligned}$$

That is,

$$\frac{dQ}{dt} + \frac{6Q(t)}{200+3t} = 9.$$

Multiplying the by the integrating factor $e^{\int 6dt/(200+3t)}$, solving, and then using the initial condition $Q(0) = 100$ will yield

$$Q(t) = 200 + 3t - \frac{4000000}{(200+3t)^2},$$

which is the amount of salt prior to the time before the tank starts to overflow. The volume of the solution becomes 500 at $t = 100$, so the concentration of the salt at this point is

$$\frac{Q(100)}{500} = 0.968 \text{ lb/gal}.$$

The theoretical limiting concentration if the tank had infinite capacity is

$$\lim_{t\to\infty} \frac{Q(t)}{200+3t} = 1 \text{ lb/gal}.$$

2.3.10 A home buyer can afford to spend no more than \$1500/month on mortgage payments. Suppose that the interest rate is 6%, that interest is compounded continuously, and that payments are also made continuously.

(a) Determine the maximum amount that this buyer can afford to borrow on a 20-year mortage; on a 30-year mortage.

(b) Determine the total interest paid during the term of the mortgage in each of the cases in part (a).

Solution.

(a) Let $S(t)$ be the amount of the loan that is left after t years. Mortage payments of \$1500/month = \$18000/year. So our differential equation is

$$\frac{dS}{dt} = rS - k = \frac{6S}{100} - 18000,$$

so that

$$\frac{dS}{dt} - \frac{6S}{100} = -18000.$$

Multiply by the integrating factor $e^{\int -(6/100)dt}$ and solve to get

$$S(t) = 300000 + Ce^{0.06t}.$$

The amount that the buyer can afford is $S(0)$, so we use the initial conditions $S(20) = 0$ and $S(30) = 0$ for the 20-year and 30-year mortgages respectively. Thus we get that \$209641.74 and \$250410.33 for the 20-year and 30-year mortgages.

(b) Interest for the 20-year mortgage is:

$$\$18000 \cdot 20 - \$209641.74 = \$150358.26$$

and interest for the 30-year mortgage is:

$$\$18000 \cdot 30 - \$250410.33 = \$289589.67.$$

2.3.21 Assume that the conditions are as in Problem 20 except there is a force due to air resistance of magnitude $|v|/30$ directed opposite to the velocity, where the velocity $|v|$ is measured in m/s.

(a) Find the maximum height above the ground that the ball reaches.

2.3.23 A skydiver weighing 180 lb (including equipment) falls vertically downward from an altitude of 5000 ft and opens the parachute after 10s of free fall. Assume that the force of air resistance, which is directed opposite to the velocity, is of magnitude $0.75|v|$ when the parachute is closed and is of magnitude $12|v|$ when the parachute is open, where the velocity v is measured in ft/s.

(a) Find the speed of the skydiver when the parachute opens.

(b) Find the distance fallen before the parachute opens.

(c) What is the limiting velocity v_L after the parachute opens?

2.3.26 A body of mass m is projected vertically upward with an initial velocity v_0 in a medium offering a resistance $k|v|$, where k is a constant. Assume that the gravitational attraction of the earth is constant.

(a) Find the velocity $v(t)$ of the body at any time.