1. Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false. Throughout G is a group.

   (1) If $g \in G$ is the only element of order 2, then $g \in Z(G)$, the center.

   (2) The intersection of two subgroups of G is also a subgroup.

   (3) The union of two subgroups of G is also a subgroup.

   (4) The largest order of an element in S_{12} is 60.

   (5) If an Abelian group has an element of order 10 and an element of order 12, then it has an element of order 30.

   Solution.

   (1) True.
   Proof. Assume that $g \in G$ is the only element of order 2. Let h be an arbitrary element in G. It suffices to show that $gh = hg$. We claim that $|hgh^{-1}| = 2$. So we have that $(hgh^{-1})^2 = hgh^{-1}hgh^{-1} = hg^2h^{-1} = hh^{-1} = e$. Now suppose that $hgh^{-1} = e$. Then it must be the case that $g = h^{-1}h = e$, a contradiction since $|g| = 2$. Thus we have that $|hgh^{-1}| = 2$. But since g is the only element of order 2, it follows that $hgh^{-1} = g$, so that $hg = gh$; since the choice of h was arbitrary, we can conclude that $g \in Z(G)$. $\square$

   (2) True.
   Proof. Let $H_1 \le G$, $H_2 \le G$, and $H' = H_1 \cap H_2$. Since e is in both H_1 and H_2, it follows that $e \in H'$. The set H' is also associative because it is a subset of G. Now let $a, b \in H'$. Thus we must have that $a, b \in H_1$ and $a, b \in H_2$. Since H_1 and H_2 are groups, it follows that they both contain ab and a^{-1} so that $ab, a^{-1} \in H'$. That is, H is closed under the operation of G and also closed under taking inverses. Thus $H' \le G$. $\square$

   (3) False.
   Counterexample: Consider $2\mathbb{Z}, 3\mathbb{Z} \le \mathbb{Z}$. We have that $2 \in 2\mathbb{Z}$ and $3 \in 3\mathbb{Z}$, but $2 + 3 = 5 \notin 2\mathbb{Z} \cup 3\mathbb{Z}$.

   (4) True.

   (5) True.
   Proof. Let g and h have orders 10 and 12 in some abelian group. The element g^2 has order 5 and the element h^2 has order 6. Since $\gcd(5, 6) = 1$, it follows that $|g^2h^2| = 5 \cdot 6 = 30$. $\square$

2. We have beads of four different colors.

   (1) How many distinct four-bead necklaces can we make?

   (2) How many distinct five-bead necklaces can we make?

(3) How many distinct six-bead necklaces can we make?

BONUS: Answer the same questions if we now have beads of five colors.

Solution.

(1) a

3. Consider the following two sets of matrices

$$S_1 = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\} \text{ and } S_2 = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}.$$

Do the following for both:

(1) Decide if they are rings or not—and give reasons.

(2) Decide if they are integral domains or not—and give reasons.

(3) Can you find a root for the polynomial $x^2 + 1$ in either place? If so find all the roots or give reasons.

Solution.

(1) a

4. Let R be a ring. An additive subgroup I is called an ideal if whenever $r \in R$ and $a \in I$, then $ra, ar \in I$.

(1) Find two ideals of $\mathbb{Z}$ that are neither 0 nor $\mathbb{Z}$.

(2) Let I be an ideal. Prove the following are true: if $I + x$ and $I + y$ are the same coset and $I + m$ and $I + n$ are the same coset, then $I + (x + m)$ and $I + (y + n)$ are the same coset, and so are $I + xm$ and $I + yn$.

(3) Let S be a ring, and let $\alpha : R \to S$ be a ring homomorphism—this means with respect to both operations. Show $I = \ker(\alpha) = \{a \in \mathbb{R} : \alpha(a) = 0\}$ is an ideal.

Solution.

(1) a