

# 1 Chapter 4

## 1.1 Section 1

4.01 Prove that if  $f$  is continuous at  $x_0$  then  $f$  is bounded at  $x_0$ .

**Proof.** Suppose that  $f : A \rightarrow \mathbb{R}$  is continuous at  $x_0$ . To show that  $f$  is bounded at  $x_0$ , it suffices to show that  $f$  is bounded on  $N_{x_0}(\delta) \cap A$  for some  $\delta > 0$ . Since  $f$  is continuous at  $x_0$ , it follows by definition that there exists  $\delta_1 > 0$  such that  $|f(x) - f(x_0)| < 1$  whenever  $|x - x_0| < \delta_1$ . Using the triangle inequality we have that

$$||f(x)| - |f(x_0)|| < |f(x) - f(x_0)| < 1,$$

so that  $|f(x)| - |f(x_0)| < 1$ , if  $|x - x_0| < \delta_1$ . Thus  $|f(x)| < 1 + |f(x_0)|$ , if  $|x - x_0| < \delta_1$ . We have thus shown that  $f$  is bounded on  $N_{x_0}(\delta_1)$  by  $1 + |f(x_0)|$ , so that  $f$  is bounded at  $x_0$ .  $\square$

4.02 Find all points of discontinuity for the following functions, classify the discontinuities as removable, jump, or second kind, and determine where the function is right- and left-continuous.

$$(a) \ f(x) = \begin{cases} x^2 & \text{if } x < -1, \\ 2x + 3 & \text{if } -1 \leq x \leq 0, \\ |x - 1| & \text{if } 0 < x < 2, \\ x^3 - 7 & \text{if } 2 \leq x < 3, \\ \frac{x-3}{x-4} & \text{if } 3 \leq x < 4, \\ 0 & \text{if } 4 \leq x. \end{cases}$$

$$(b) \ f(x) = x + \llbracket -x \rrbracket.$$

$$(c) \ f(x) = x \llbracket x \rrbracket.$$

$$(d) \ f(x) = \operatorname{sgn} \llbracket |x| \rrbracket.$$

$$(e) \ f(x) = \begin{cases} \llbracket x + 1 \rrbracket \sin \frac{1}{x} & \text{if } x \in (-1, 0) \cup (0, 1) \\ 0 & \text{otherwise.} \end{cases}$$

$$(f) \ f(x) = \begin{cases} (1+x) \operatorname{sgn} x + \operatorname{sgn} |x| - 1 & \text{if } x \text{ is rational} \\ \operatorname{sgn} x & \text{if } x \text{ is irrational.} \end{cases}$$

4.03 Prove that  $f(x) = \cos x$  is continuous on  $\mathbb{R}$ .

4.04 Prove that if  $f$  is continuous at  $x_0$  and  $g$  is discontinuous at  $x_0$  then  $f + g$  must have a discontinuity at  $x_0$ .

4.05 Show that  $f + g$  can be continuous at  $x_0$  even though both  $f$  and  $g$  have discontinuities at  $x_0$ .

4.06 Show that  $f \cdot g$  can be continuous at  $x_0$  even though both  $f$  and  $g$  have discontinuities at  $x_0$ .

- 4.07 If  $f$  is continuous at  $x_0$  and  $g$  is discontinuous at  $x_0$ , what can be said about continuity of the product  $f \cdot g$  at  $x_0$ ?
- 4.08 Show that the composition function  $g \circ f$  can be continuous at  $x_0$  even though  $f$  or  $g$  or both  $f$  and  $g$  are discontinuous at  $x_0$ .
- 4.09 Prove that the function
- $$f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$
- has a discontinuity of the second kind at each nonzero real number.
- 4.10 Prove that if  $f$  is continuous at  $x_0$  and  $f$  is nonnegative then  $h(x) = \sqrt{f(x)}$  is continuous at  $x_0$ .
- 4.11 Find a function  $f$  which has a discontinuity of the second kind at every real number although  $f \circ f$  is continuous on  $\mathbb{R}$ .
- 4.12 If  $f$  is continuous on  $(0, 1)$  and  $f(x) = 1 - x$  for every rational number  $x \in (0, 1)$ , find  $f(\pi/4)$ . Explain your answer.
- 4.13 Prove that if  $f$  and  $g$  are each continuous on  $(a, b)$  and  $f(x) = g(x)$  for every rational  $x \in (a, b)$  then  $f(x) = g(x)$  for every  $x \in (a, b)$ .
- 4.14 Prove:  $f$  is right-continuous at  $x_0$  if and only if  $f(x_n) \rightarrow f(x_0)$  for every sequence  $\{x_n\}$  in the domain of  $f$  with  $x_n \rightarrow x_0$  and  $x_n \geq x_0$  for  $n = 1, 2, 3, \dots$
- 4.15 Discuss one-sided continuity for the pie function.
- 4.16 Prove that if  $f$  is defined on  $\mathbb{R}$  and continuous at  $x_0 = 0$  and if  $f(x_1 + x_2) = f(x_1) + f(x_2)$  for each  $x_1, x_2 \in \mathbb{R}$  then  $f$  is continuous on  $\mathbb{R}$ .
- 4.17 Find all functions  $f$  which are continuous on  $\mathbb{R}$  and which satisfy the equation  $f(x)^2 = x^2$  for each  $x \in \mathbb{R}$ . *Hint:* There are four possible solutions.
- 4.18 Prove that if  $g$  is continuous at  $x_0 = 0$ ,  $g(0) = 0$  and for some  $\delta > 0$   $|f(x)| \leq |g(x)|$  for each  $x \in N_\delta(0)$  then  $f$  is continuous at  $x_0 = 0$ .
- 4.19 Prove that if  $f$  is continuous on  $[a, b]$  then there exists a function  $g$  continuous on  $\mathbb{R}$  such that  $g(x) = f(x)$  for each  $x \in [a, b]$ . The function  $g$  is called a *continuous extension* of  $f$  to  $\mathbb{R}$ .
- 4.20 The function  $f(x) = \tan x$  defined on  $(-\pi/2, \pi/2)$  clearly has no continuous extension to  $\mathbb{R}$ . Find a bounded continuous function on  $(a, b)$  which has no continuous extension to  $\mathbb{R}$ .
- 4.21 Assume that  $f$  is continuous on  $(a, b)$ . Prove that  $f$  has a continuous extension to  $\mathbb{R}$  if and only if both limits  $\lim_{x \rightarrow a^+} f(x)$  and  $\lim_{x \rightarrow b^-} f(x)$  exist.
- 4.22 Prove that if  $f$  is continuous on  $(a, b)$  and both  $\lim_{x \rightarrow a^+} f(x)$  and  $\lim_{x \rightarrow b^-} f(x)$  exist then  $f$  is bounded on  $(a, b)$ .

- 4.23 Suppose  $f$  is one-to-one on  $(a, b)$  and satisfies the following property: whenever  $f(x_1) \neq f(x_2)$  for  $x_1 < x_2$ ,  $x_1, x_2 \in (a, b)$  and  $k$  is any number between  $f(x_1)$  and  $f(x_2)$ , there exists a  $c \in (x_1, x_2)$  with  $f(c) = k$ . Prove that  $f$  is continuous on  $(a, b)$ .