

1. If Z is a standard normal variable, find

(a) $P(Z^2 < 1)$ (b) $P(Z^2 > 3.84146)$.

Solution.

- (a) We have that

$$P(Z^2 < 1) = P(-1 < Z < 1) = 1 - 2 \cdot P(Z > 1) \approx 0.6826,$$

and

- (b)

$$P(Z^2 > 3.84146) = 2 \cdot P(Z > \sqrt{3.84146}) \approx 2 \cdot P(Z > 1.96) \approx 0.05.$$

2. If Y is a normal random variable with $\mu = 20$ and variance $\sigma^2 = 4$, i.e., $Y \sim N(20, 4)$, find

(a) $P(16 \leq Y \leq 22)$ (b) $P(100 < 9Y - 80 < 145)$.

Solution.

- (a) We have that

$$\begin{aligned} P(16 \leq Y \leq 22) &= P\left(\frac{16-20}{2} \leq Z \leq \frac{22-20}{2}\right) \\ &= P(-2 \leq Z \leq 1) \\ &= 1 - [P(Z < -2) + P(Z > 1)] \\ &= 1 - [P(Z > 2) + P(Z > 1)] \\ &\approx 0.8185, \end{aligned}$$

and

- (b)

$$\begin{aligned} P(100 < 9Y - 80 < 145) &= P(20 < Y < 25) \\ &= P\left(\frac{20-20}{2} < Z < \frac{25-20}{2}\right) \\ &= P(0 < Z < 2.5) \\ &= P(Z > 0) - P(Z > 2.5) \\ &\approx 0.4938. \end{aligned}$$

3. The scores of a pre-employment test are normally distributed with mean $\mu = 70$ and standard deviation $\sigma = 5$. If only the top 1.5% of the applicants (based on their score on the pre-employment test) are to be considered, find the cut-off score (i.e., the value such that only 1.5% of the applicants score this value or higher).

Solution. Let y be the cut-off score. Then we have that

$$0.0015 = P(Y \geq y) = P\left(Z \geq \frac{y-70}{5}\right),$$

so that $(y - 70)/5 \approx 2.97$; i.e., $y \approx 85$.

4. Using the fact that $\int_0^\infty e^{-y^2/2} dy = \sqrt{\frac{\pi}{2}}$, show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ by making the transformation $y = \frac{1}{2}x^2$.

Proof. Using the transformation $y = \frac{1}{2}x^2$ we have that

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty y^{-\frac{1}{2}} e^{-y} dy \\ &= \int_0^\infty \frac{\sqrt{2}}{x} e^{-\frac{1}{2}x^2} x dx \\ &= \sqrt{2} \int_0^\infty e^{-\frac{1}{2}x^2} dx \\ &= \sqrt{2} \sqrt{\frac{\pi}{2}} = \sqrt{\pi},\end{aligned}$$

as desired. □

5. If Y has an exponential distribution with $P(Y < 3) = 0.4512$, find
 (a) $E[Y]$ (b) $P(Y \geq 2)$.

Solution.

- (a) We have that

$$\begin{aligned}0.4512 &= P(Y < 3) \\ &= P(Y \leq 3) \\ &= F(3) \\ &= \int_{-\infty}^3 \frac{1}{\beta} e^{-\frac{y}{\beta}} dy \\ &= \int_0^3 \frac{1}{\beta} e^{-\frac{y}{\beta}} dy \\ &= -e^{-\frac{3}{\beta}} + 1,\end{aligned}$$

so that $e^{-\frac{3}{\beta}} = 0.5488$; i.e., $\beta \approx 5$. Thus $E[Y] \approx 5$.

- (b)

$$\begin{aligned}P(Y \geq 2) &= 1 - P(Y < 2) \\ &= 1 - \int_0^2 \frac{1}{\beta} e^{-\frac{y}{\beta}} dy \\ &= e^{-\frac{2}{\beta}} \\ &\approx 0.6703.\end{aligned}$$

6. The length of time Y necessary to complete a key operation in the construction of houses has an exponential distribution with mean 10 hrs. The formula $C = 100 + 40Y + 3Y^2$ gives the cost C of completing the operation. Find the mean and variance of C .

Solution. First we want to find $E[Y^2]$. So

$$\begin{aligned} E[Y^2] &= \frac{1}{10} \lim_{t \rightarrow \infty} \int_0^t y^2 e^{-\frac{y}{10}} dy \\ &= \lim_{t \rightarrow \infty} \left[-y^2 e^{-\frac{y}{10}} \Big|_0^t + 2 \int_0^t y e^{-\frac{y}{10}} dy \right] && \text{[Integration by parts]} \\ &= 2 \lim_{t \rightarrow \infty} \left[\int_0^t y e^{-\frac{y}{10}} dy \right] \\ &= 2 \lim_{t \rightarrow \infty} \left[-10y e^{-\frac{y}{10}} \Big|_0^t + 10 \int_0^t e^{-\frac{y}{10}} dy \right] && \text{[Integration by parts]} \\ &= 200 \lim_{t \rightarrow \infty} \left[\frac{1}{10} \int_0^t e^{-\frac{y}{10}} dy \right] \\ &= 200 \cdot E[Y] = 2000. \end{aligned}$$

Now the mean of C is given by $E[C]$ so that

$$\begin{aligned} E[C] &= E[100 + 40Y + 3Y^2] \\ &= E[100] + 40E[Y] + 3E[Y^2] \\ &= 100 + 40 \cdot 10 + 3 \cdot 2000 \\ &= 6500, \end{aligned}$$

and the variance of C , $V[Y]$, is $E[Y^2] - E[Y]^2 = 2000 - 100 = 1900$.

7. Suppose Y has density function $f(y) = ky^9 e^{-y/2}$, $y \geq 0$. Find

- (a) k .
- (b) $E[Y]$ and $V(Y)$.
- (c) $P(Y > 34.1696)$.
- (d) A value b such that $P(Y < b) = 0.10$.

Solution. By inspection we can see that f is the gamma distribution with $\alpha = 10$, $\beta = 2$.

- (a) $k = \frac{1}{2^{10} \cdot \Gamma(10)} = \frac{1}{2^{10} \cdot 9!}$.
- (b) $E[Y] = \alpha\beta = 20$ and $V(Y) = \alpha\beta^2 = 40$.

(c)

$$\begin{aligned}
 P(Y > 34.1696) &= \frac{1}{2^{10} \cdot 9!} \int_{34.1696}^{\infty} y^9 e^{-y/2} dy \\
 &= \frac{1}{2^{10} \cdot 9!} \int_{17.0848}^{\infty} 2^{10} z^9 e^{-z} dz \quad \left[z = \frac{y}{2} \text{ substitution} \right] \\
 &= \frac{1}{9!} \int_{17.0848}^{\infty} z^9 e^{-z} dz \\
 &= \sum_{x=0}^9 \frac{17.0848^x e^{-17.0848}}{x!} \\
 &\approx 0.025.
 \end{aligned}$$

(d) Suppose there exists b with $P(Y < b) = 0.10$, then we must have that

$$0.90 = P(Y \geq b) = P(Z \geq b/2),$$

and from Appendix 3, Table 3, we get $b/2 \approx 14$, so that $b \approx 28$.

8. The function $B(\alpha, \beta)$ is defined by $B(\alpha, \beta) = \int_0^1 y^{\alpha-1} (1-y)^{\beta-1} dy$.

(a) Letting $y = \sin^2 \theta$, show that $B(\alpha, \beta) = 2 \int_0^{\pi/2} \sin^{2\alpha-1} \theta \cos^{2\beta-1} \theta d\theta$.

(b) Write $\Gamma(\alpha)\Gamma(\beta)$ as a double integral using variables of integration y and z , make the transformation $y = r^2 \sin^2 \theta$ and $z = r^2 \cos^2 \theta$, and then show that

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

Solution.

(a) Let $y = \sin^2 \theta$, so that $dy = 2 \sin \theta \cos \theta d\theta$. Thus

$$\begin{aligned}
 B(\alpha, \beta) &= \int_0^1 y^{\alpha-1} (1-y)^{\beta-1} dy \\
 &= \int_0^{\pi/2} [(\sin \theta)^2]^{\alpha-1} (1 - \sin^2 \theta)^{\beta-1} 2 \sin \theta \cos \theta d\theta \\
 &= \int_0^{\pi/2} (\sin \theta)^{2\alpha-2} [(\cos \theta)^2]^{\beta-1} 2 \sin \theta \cos \theta d\theta \\
 &= 2 \int_0^{\pi/2} \sin^{2\alpha-1} \theta \cos^{2\beta-1} \theta d\theta.
 \end{aligned}$$

(b) By definition we have that

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^{\infty} y^{\alpha-1} e^{-y} dy \int_0^{\infty} z^{\beta-1} e^{-z} dz = \int_0^{\infty} \int_0^{\infty} y^{\alpha-1} e^{-(y+z)} z^{\beta-1} dy dz.$$

Consider the transformation $y = r^2 \sin^2 \theta$ and $z = r^2 \cos^2 \theta$. The Jacobian of this transformation, $\frac{\partial(y, z)}{\partial(r, \theta)}$, is given by

$$\begin{vmatrix} \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{vmatrix} = -4r^3 \sin \theta \cos \theta.$$

Thus we have that

$$\begin{aligned} \Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty \int_0^\infty y^{\alpha-1} e^{-(y+z)} z^{\beta-1} dy dz \\ &= \int_0^{\pi/2} \int_0^\infty [(r \sin \theta)^2]^{\alpha-1} e^{-r^2} [(r \cos \theta)^2]^{\beta-1} \left| \frac{\partial(y, z)}{\partial(r, \theta)} \right| dr d\theta \\ &= \int_0^{\pi/2} \int_0^\infty r^{2\alpha-2} \sin^{2\alpha-2} \theta e^{-r^2} r^{2\beta-2} \cos^{2\beta-2} \theta \left| \frac{\partial(y, z)}{\partial(r, \theta)} \right| dr d\theta \\ &= \int_0^{\pi/2} \int_0^\infty r^{2\alpha+2\beta-4} \sin^{2\alpha-2} \theta e^{-r^2} \cos^{2\beta-2} \theta (4r^3 \sin \theta \cos \theta) dr d\theta \\ &= \int_0^{\pi/2} \int_0^\infty 4r^{2\alpha+2\beta-1} \sin^{2\alpha-1} \theta e^{-r^2} \cos^{2\beta-1} \theta dr d\theta \\ &= \left(2 \int_0^{\pi/2} \sin^{2\alpha-1} \theta \cos^{2\beta-1} \theta d\theta \right) \left(\int_0^\infty r^{2(\alpha+\beta-1)} e^{-r^2} 2r dr \right) \\ &= B(\alpha, \beta) \int_0^\infty r^{2(\alpha+\beta-1)} e^{-r^2} 2r dr. \end{aligned}$$

Thus using the substitution $x = r^2$ will give us

$$B(\alpha, \beta) \int_0^\infty r^{2(\alpha+\beta-1)} e^{-r^2} 2r dr = B(\alpha, \beta) \int_0^\infty x^{\alpha+\beta-1} e^{-x} dx = B(\alpha, \beta) \Gamma(\alpha + \beta),$$

so that $\Gamma(\alpha)\Gamma(\beta) = B(\alpha, \beta)\Gamma(\alpha + \beta)$, as desired.

9. Prove that the variance of a beta-distributed random variable with parameters α and β are given by

$$\sigma^2 = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$

Proof. Let Y be a beta-distributed random variable with parameters α and β . We

then have that

$$\begin{aligned}
 E[Y^2] &= \int_0^1 y^2 \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha, \beta)} \\
 &= \frac{1}{B(\alpha, \beta)} \int_0^1 y^{\alpha+1}(1-y)^{\beta-1} \\
 &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha + 2)\Gamma(\beta)}{\Gamma(\alpha + \beta + 2)} \\
 &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{(\alpha + 1)\alpha\Gamma(\alpha)\Gamma(\beta)}{(\alpha + \beta + 1)(\alpha + \beta)\Gamma(\alpha + \beta)} \\
 &= \frac{(\alpha + 1)\alpha}{(\alpha + \beta + 1)(\alpha + \beta)}.
 \end{aligned}$$

By the Proof on Pg 196 of the book we have that $E[Y] = \frac{\alpha}{\alpha + \beta}$. Thus the variance of Y , $V(Y)$, is

$$\begin{aligned}
 E[Y^2] - E[Y]^2 &= \frac{(\alpha + 1)\alpha}{(\alpha + \beta + 1)(\alpha + \beta)} - \frac{\alpha^2}{(\alpha + \beta)^2} \\
 &= \frac{(\alpha + 1)(\alpha + \beta)\alpha - \alpha^2(\alpha + \beta + 1)}{(\alpha + \beta + 1)(\alpha + \beta)^2} \\
 &= \frac{\alpha\beta}{(\alpha + \beta + 1)(\alpha + \beta)^2},
 \end{aligned}$$

which is what we wanted to prove. \square

10. Suppose Y has the density function $f(y) = k(y - 2)^4(5 - y)^6$, $2 \leq y \leq 5$. Find
 (a) k (b) $E[Y]$ and $V(Y)$.

Solution.

- (a) By definition we must have that

$$k \int_2^5 (y - 2)^4(5 - y)^6 dy = 1.$$

So make the substitution $x = 5 - y$ to get

$$\begin{aligned}
 1 &= k \int_2^5 (y-2)^4 (5-y)^6 dy \\
 &= -k \int_3^0 (3-x)^4 x^6 dx \\
 &= k \int_0^3 (3-x)^4 x^6 dx \\
 &= k \int_0^3 (x^4 - 12x^3 + 54x^2 - 108x + 81)x^6 dx \\
 &= k \int_0^3 (x^{10} - 12x^9 + 54x^8 - 108x^7 + 81x^6) dx \\
 &= k \left(\frac{1}{11}x^{11} - \frac{6}{5}x^{10} + 6x^9 - \frac{27}{2}x^8 + \frac{81}{7}x^7 \right) \Big|_0^3 \\
 &= \frac{59049}{770}k,
 \end{aligned}$$

so that $k = \frac{770}{59049}$.

(b) Using the same substitution $x = 5 - y$, it follows that

$$\begin{aligned}
 E[Y] &= k \int_2^5 y(y-2)^4 (5-y)^6 dy \\
 &= k \int_0^3 (5-x)(x^{10} - 12x^9 + 54x^8 - 108x^7 + 81x^6) dx \\
 &= k \int_0^3 (-x^{11} + 17x^{10} - 114x^9 + 378x^8 - 621x^7 + 405x^6) dx \\
 &= k \left(-\frac{1}{12}x^{12} + \frac{17}{11}x^{11} - \frac{57}{5}x^{10} + 42x^9 - \frac{621}{8}x^8 + \frac{405}{7}x^7 \right) \Big|_0^3 \\
 &= \frac{770}{59049} \frac{767637}{3080} = \frac{13}{4},
 \end{aligned}$$

and

$$\begin{aligned} E[Y^2] &= k \int_2^5 y^2(y-2)^4(5-y)^6 dy \\ &= k \int_0^3 (5-x)^2(x^{10} - 12x^9 + 54x^8 - 108x^7 + 81x^6) dx \\ &= k \int_0^3 (x^2 - 10x + 25)(x^{10} - 12x^9 + 54x^8 - 108x^7 + 81x^6) dx \\ &= k \int_0^3 (x^{12} - 22x^{11} + 199x^{10} - 948x^9 + 2511x^8 - 3510x^7 + 2025x^6) dx \\ &= k \left(\frac{1}{13}x^{13} - \frac{11}{6}x^{12} + \frac{199}{11}x^{11} - \frac{474}{5}x^{10} + 279x^9 - \frac{1755}{4}x^8 + \frac{2025}{7}x^7 \right) \Big|_0^3 \\ &= \frac{770}{59049} \frac{49424013}{60060} = \frac{279}{26}. \end{aligned}$$

We can then conclude that

$$V(Y) = E[Y^2] - E[Y]^2 = \frac{279^2}{26^2} - \frac{13^2}{4^2} = \frac{282803}{2704} \approx 104.59.$$