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MATHEMATICS
CAL STATE LONG BEACH

1. Suppose f is continuous on (a, b). Prove that if $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ both exist then f is bounded on (a, b) but the converse does not hold.

   Proof. Assume f is continuous on (a, b). Now suppose that $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ both exist. Let

   $$\lim_{x\to a^+} f(x) = R \text{ and } \lim_{x\to b^-} f(x) = L.$$

   It follows by definition that there exists a $\delta_1 > 0$ such that if $x \in (a, a + \delta_1)$ then $|f(x) - R| < 1$, so that $|f(x)| < 1 + |R|$. Similarly, there exists a $\delta_2 > 0$ such that if $x \in (b - \delta_2, b)$ then $|f(x) - L| < 1$, so that $|f(x)| < 1 + |L|$. Let

   $$\delta = \min\left\{\frac{b-a}{3}, \delta_1, \delta_2\right\}.$$

   Now since f is continuous on (a, b) it must also be continuous on $[a + \delta, b - \delta]$, so that by Theorem 4.2, there exists $N > 0$ such that if $x \in [a + \delta, b - \delta]$, then $|f(x)| \leq N$. Let

   $$M = \max\{N, 1 + |R|, 1 + |L|\}.$$

   Finally consider any $x \in (a, b)$. It must be the case that x is in exactly one of the following intervals: $(a, a + \delta)$, $[a + \delta, b - \delta]$, and $(b - \delta, b)$. The first interval is bounded by $1 + |R|$, the second by $1 + |L|$, and the third by N, so that $|f(x)| \leq M$. Thus by definition f is bounded on (a, b).

   Converse. Suppose that f is bounded on (a, b). Fix some positive integer N such that $N > \frac{1}{b-a}$, so that $\frac{1}{N} < b - a$. Let

   $$X_1 = \left\{\left[a + \frac{1}{n+1},\ a + \frac{1}{n}\right) : n \in \mathbb{N}, n \geq N\right\} \text{ and } X_2 = \left[a + \frac{1}{N},\ b\right),$$

   so that $(a, b) = X_1 \cup X_2$. Let $P \in X_1$. Then

   $$P = \left[a + \frac{1}{N+i+1},\ a + \frac{1}{N+i}\right)$$

   for some nonnegative integer i. We want to map P to the interval $[0, 1)$ for even i and $[1, 0)$ for odd i such that $f(P)$ is a straight line. If i is even, for every $x \in P$, let

   $$f(x) = \left(\frac{1}{\frac{1}{N+i} - \frac{1}{N+i+1}}\right)\left(x - a - \frac{1}{N+i+1}\right),$$

   and if i is odd, for every $x \in P$, let

   $$f(x) = \left(\frac{-1}{\frac{1}{N+i} - \frac{1}{N+i+1}}\right)\left(x - a - \frac{1}{N+i+1}\right) + 1.$$

   Now if $x \in X_2$, let $f(x) = 1$. Notice that f oscillates between 0 and 1 for x close to a, so that $\lim_{x\to a^+}$ does not exist. $\square$