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MATHEMATICS
CAL STATE LONG BEACH

1. If Z is a standard normal variable, find
   (a) $P(Z^2 < 1)$ (b) $P(Z^2 > 3.84146)$.

   Solution.

   (a) We have that
   $$P(Z^2 < 1) = P(-1 < Z < 1) = 1 - 2 \cdot P(Z > 1) \approx 0.6826,$$
   and

   (b)
   $$P(Z^2 > 3.84146) = 2 \cdot P(Z > \sqrt{3.84146}) \approx 2 \cdot P(Z > 1.96) \approx 0.05.$$

2. If Y is a normal random variable with $\mu = 20$ and variance $\sigma^2 = 4$, i.e., $Y \sim N(20, 4)$, find
   (a) $P(16 \leq Y \leq 22)$ (b) $P(100 < 9Y - 80 < 145)$.

   Solution.

   (a) We have that
   $$\begin{aligned} P(16 \leq Y \leq 22) &= P\left(\frac{16-20}{2} \leq Z \leq \frac{22-20}{2}\right) \\ &= P(-2 \leq Z \leq 1) \\ &= 1 - [P(Z < -2) + P(Z > 1)] \\ &= 1 - [P(Z > 2) + P(Z > 1)] \\ &\approx 0.8185, \end{aligned}$$
   and

   (b)
   $$\begin{aligned} P(100 < 9Y - 80 < 145) &= P(20 < Y < 25) \\ &= P\left(\frac{20-20}{2} < Z < \frac{25-20}{2}\right) \\ &= P(0 < Z < 2.5) \\ &= P(Z > 0) - P(Z > 2.5) \\ &\approx 0.4938. \end{aligned}$$

3. The scores of a pre-employment test are normally distributed with mean $\mu = 70$ and standard deviation $\sigma = 5$. If only the top 1.5% of the applicants (based on their score on the pre-employment test) are to be considered, find the cut-off score (i.e., the value such that only 1.5% of the applicants score this value or higher).

   Solution. Let y be the cut-off score. Then we have that
   $$0.0015 = P(Y \geq y) = P\left(Z \geq \frac{y-70}{5}\right),$$
   so that $(y - 70)/5 \approx 2.97$; i.e., $y \approx 84.85$.

4. Using the fact that $\displaystyle\int_0^\infty e^{-y^2/2}dy = \sqrt{\frac{\pi}{2}}$, show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ by making the transformation $y = \frac{1}{2}x^2$.

   Proof. Using the substitution $y = \frac{1}{2}x^2$ we have that

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty y^{-\frac{1}{2}}e^{-y}dy \\ &= \int_0^\infty \frac{\sqrt{2}}{x}e^{-\frac{1}{2}x^2}x\,dx \\ &= \sqrt{2}\int_0^\infty e^{-\frac{1}{2}x^2}\,dx \\ &= \sqrt{2}\sqrt{\frac{\pi}{2}} = \sqrt{\pi}.\end{aligned}$$

5. If Y has an exponential distribution with $P(Y < 3) = 0.4512$, find

   (a) $E(Y)$ $\quad$ (b) $P(Y \geq 2)$.