8.02 Define $T \in \mathcal{L}(\mathbb{C}^2)$ by

$$T(w,z) = (-z, w).$$

Find all generalized eigenvectors.

Solution. First we find the eigenvalues of T. Suppose $T(w,z) = \lambda(w,z)$. Then it follows that $(-z,w) = (\lambda w, \lambda z)$, so that $\lambda w = -z$ and $w = \lambda z$. Solving these equations will gives $\lambda = \pm i$. Thus the eigenvalues of T are i and $-i$. Since $\dim \mathbb{C}^2 = 2$, it follows that the generalized eignevectors corresponding to i is null $(T - iI)^2$ and the generalized eignevectors corresponding to $-i$ is null $(T+iI)^2$. Suppose $(a,b) \in$ null $(T-iI)^2$. Then we have that

$$\begin{aligned}
0 &= (T - iI)^2(a,b) \\
&= (T - iI)[(T - iI)(a,b)] \\
&= (T - iI)[T(a,b) - (iI)(a,b)] \\
&= (T - iI)[(-b,a) + (-ai, -bi)] \\
&= (T - iI)(-b - ai, a - bi) \\
&= T(-b - ai, a - bi) - (iI)(-b - ai, a - bi) \\
&= (-a + bi, -b - ai) + (-a + bi, -b - ai) \\
&= (-2a + 2bi, -2b - 2ai),
\end{aligned}$$

so that $-2a + 2bi = 0$ and $-2b - 2ai = 0$. That is $a = bi$. Similarly if $(c,d) \in$ null $(T + iI)^2$. Then we have that

$$\begin{aligned}
0 &= (T + iI)^2(c,d) \\
&= (T + iI)[(T + iI)(c,d)] \\
&= (T + iI)[T(c,d) + (iI)(c,d)] \\
&= (T + iI)[(-d,c) + (ci, di)] \\
&= (T + iI)(-d + ci, c + di) \\
&= T(-d + ci, c + di) + (iI)(-d + ci, c + di) \\
&= (-c - di, -d + ci) + (-c - di, -d + ci) \\
&= (-2c - 2di, -2d + 2ci),
\end{aligned}$$

so that $-2c - 2di = 0$ and $-2d + 2ci = 0$. That is $c = -di$. Thus

$$\text{null } (T - iI)^2 = \{(xi, x) : x \in \mathbb{C}\} \text{ and null } (T + iI)^2 = \{(-yi, y) : y \in \mathbb{C}\}.$$

8.03 Suppose $T \in \mathcal{L}(V)$, m is a positive integer, and $v \in V$ is such that $T^{m-1}v \neq 0$ but $T^m v = 0$. Prove that

$$(v, Tv, T^2v, \dots, T^{m-1}v)$$

is linearly independent.

Proof. Consider the equation

$$a_0 v + a_1 Tv + a_2 T^2 v + \cdots + a_{m-1} T^{m-1} v = 0. \tag{1}$$

Applying T^{m-1} to equation (**??**) above will result in

$$a_0T^{m-1}v + a_1T^mv + a_2T^{m+1}v + \cdots + a_{m-1}T^{2m-2}v = 0. \tag{2}$$

Notice since $T^mv = 0$, we must have $T^{m+i}v = 0$ for all $i \geq 0$. Thus equation (2) reduces to $a_0T^{m-1}v = 0$, so that $a_0 = 0$ since $T^{m-1}v \neq 0$. Now equation (1) reduces to

$$a_1Tv + a_2T^2v + \cdots + a_{m-1}T^{m-1}v = 0. \tag{3}$$

Now apply T^{m-2} to equation (3) to get $a_1 = 0$. Then apply T^{m-3} to the simplified equation to get $a_2 = 0$, $\ldots$, and so on to get $a_{m-1} = 0$. Thus

$$(v, Tv, T^2v, \ldots, T^{m-1}v)$$

is linearly independent.

8.06 Suppose $N \in \mathcal{L}(V)$ is nilpotent. Prove (without using 8.26) that 0 is the only eigenvalue of N.

Proof.

8.07 Suppose V is an inner-product space. Prove that if $N \in \mathcal{L}(V)$ is self-adjoint and nilpotent, then $N = 0$.

Proof.

8.11 Prove that if $T \in \mathcal{L}(V)$, then

$$V = \text{null } T^n \oplus \text{range } T^n,$$

where $n = \dim V$.

Proof.