5 Chapter 5

5.1 Section 1

5.01 If $f(x) = |x^3|$, find $f'(x)$.

Solution. By definition we have that

$$f(x) = \begin{cases} x^3 & \text{if } x \geq 0 \\ -x^3 & \text{if } x < 0 \end{cases}$$

so that $f'(x) = 3x^2$ if $x > 0$ and $f'(x) = -3x^2$ if $x < 0$. Now we have that

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h^3|}{h}.$$

And since

$$\lim_{h\to 0^+} \frac{|h^3|}{h} = \lim_{h\to 0^+} \frac{h^3}{h} = 0 = \lim_{h\to 0^-} \frac{-h^3}{h} = \lim_{h\to 0^-} \frac{|h^3|}{h},$$

it follows by Theorem 3.7 that $f'(0) = \lim_{h\to 0} \frac{|h^3|}{h} = 0$. Thus $f'(x) = |3x^2|$.

5.02 Let $f(x) = x|x|$; show that

$$f''(x) = \begin{cases} 2 & \text{if } x > 0 \\ -2 & \text{if } x < 0 \end{cases}$$

and that 0 is not in the domain of $f''(x)$.

Proof. From example 5.1, we know that $f'(x) = 2|x|$, so that $f''(x) = 2$ if $x > 0$ and $f''(x) = -2$ if $x < 0$. We now want to show that $f''(0)$ is undefined. To that end, we have that

$$f''(0) = \lim_{h\to 0} \frac{f'(0+h) - f'(0)}{h} = \lim_{h\to 0} \frac{2|h|}{h},$$

which does not exist since the one-sided limits are not equal. Thus 0 is not in the domain of f''.

5.03 Find $f'(x)$ if

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 2 \\ 4x - 4 & \text{if } x < 2 \end{cases}$$

Solution. It is clear that $f'(x) = 2x$ if $x > 2$ and that $f'(x) = 4$ if $x < 2$, so we only need to investigate if $f'(2)$ exists. To that end, we have that

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0} \frac{f(2+h) - 4}{h}. \end{aligned}$$

Since

$$\lim_{h\to 0^-}\frac{f(2+h)-4}{h}=\lim_{h\to 0^-}\frac{4(2+h)-4-4}{h}=4,$$

and

$$\lim_{h\to 0^+}\frac{f(2+h)-4}{h}=\lim_{h\to 0^+}\frac{(2+h)^2-4}{h}=\lim_{h\to 0^+}\frac{h^2+4h}{h}=4,$$

it follows that $f'(2)=4$. Thus

$$f'(x)=\begin{cases} 2x & \text{if } x\ge 2\\ 4 & \text{if } x<2\end{cases}$$

5.04 For what values of a and b is $f(x)$ differentiable at $x=1$ if

$$f(x)=\begin{cases} x^3 & \text{if } x<1\\ ax+b & \text{if } x\ge 1.\end{cases}$$

Solution. Suppose that f is differentiable at 1. Then by Theorem 5.1, it follows that f is continuous at 1. Particularly, f is left-continuous at 1 so that

$$1^3=1=\lim_{x\to 1^-}f(x)=f(1)=a+b.$$

Since f is differentiable at 1, it follows by Definition 5.1 that the limit

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

exists, so that

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}.$$

Thus

$$\begin{aligned}\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}&=\lim_{h\to 0^+}\frac{a(1+h)+b-a-b}{h}\\ &=a\\ &=\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}\\ &=\lim_{h\to 0^-}\frac{(1+h)^3-(a+b)}{h}\\ &=\lim_{h\to 0^-}\frac{(1+h)^3-1}{h}\\ &=3.\end{aligned}$$

Hence f is differentiable at 1 if and only if $a=3$ and $b=-2$.

5.11 (a) Define

$$f(x) = \begin{cases} \dfrac{1}{4^n} & \text{if } x = \dfrac{1}{2^n}\ (n = 1, 2, 3, \ldots) \\ 0 & \text{otherwise.} \end{cases}$$

Is f differentiable at $x = 0$? Verify.

(b) Define

$$g(x) = \begin{cases} \dfrac{1}{2^{n+1}} & \text{if } x = \dfrac{1}{2^n}\ (n = 1, 2, 3, \ldots) \\ 0 & \text{otherwise.} \end{cases}$$

Is g differentiable at $x = 0$? Verify.

Solution.

(a) By definition we have that

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{f(h) - 0}{h} \\ &= \lim_{h\to 0} \frac{f(h)}{h}. \end{aligned}$$

We now claim that f is differentiable at $x = 0$ because

$$\lim_{h\to 0} \frac{f(h)}{h} = 0.$$

Proof. Let $\varepsilon > 0$ be given. We want to find a corresponding $\delta > 0$ such that if $0 < |x| < \delta$, then $\left|\dfrac{f(x)}{x}\right| < \varepsilon$. Choose a large positive integer N such that $\dfrac{1}{2^N} < \varepsilon$. This suggests that we choose $\delta = \dfrac{1}{2^N}$. Now suppose $x \in N^*_\delta(0)$. We have the following possibilities:

Case 1. $x < 0$ or ($x > 0$ and $x \neq \dfrac{1}{2^n}$ $(n = 1, 2, 3, \ldots)$). Thus

$$\left|\frac{f(x)}{x}\right| = \left|\frac{0}{x}\right| = 0 < \varepsilon.$$

Case 2. $x > 0$ and $x = \dfrac{1}{2^n}$ for some positive integer n. Thus

$$\left|\frac{f(x)}{x}\right| = \left|\frac{\frac{1}{4^n}}{\frac{1}{2^n}}\right| = \left|\frac{1}{2^n}\right| = \frac{1}{2^n} < \delta = \frac{1}{2^N} < \varepsilon.$$

So it follows by definition that

$$f'(0) = \lim_{h \to 0} \frac{f(h)}{h} = 0.$$

□

That is, f is differentiable at $x = 0$.

(b) Similarly we have that

$$g'(0) = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{g(h)}{h}.$$

We claim that g is not differentiable at $x = 0$ because

$$\lim_{h \to 0} \frac{g(h)}{h}$$

does not exist.

Proof. Consider the sequences $x_n = \frac{1}{2^n}$ and $y_n = \frac{1}{3^n}$. We know from our previous discussions that $x_n \to 0$ and $y_n \to 0$. Now

$$\lim_{n \to \infty} \frac{g(x_n)}{x_n} = \lim_{n \to \infty} \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} = \lim_{n \to \infty} \frac{1}{2} = \frac{1}{2},$$

and

$$\lim_{n \to \infty} \frac{g(y_n)}{y_n} = \lim_{n \to \infty} 3^n g(y_n) = \lim_{n \to \infty} 3^n \cdot 0 = 0.$$

Since $\frac{1}{2} \neq 0$, it follows by Theorem 3.6 that

$$\lim_{h \to 0} \frac{g(h)}{h}$$

does not exist, so that g is not differentiable at $x = 0$. □