4 Chapter 4

4.1 Section 1

Exam **12%**: Part of 3.3 (p.86)

$$\lim_{x\to\infty} f(x) = \infty, \lim_{x\to-\infty} f(x) = \infty, \lim_{x\to\infty} f(x) = -\infty, \lim_{x\to-\infty} f(x) = -\infty.$$

28%: 4.1 Continuous function at a point (both versions: $\lim_{x\to a} f(x) = f(a)$, $\varepsilon - \delta$ version)

Find and classify discontinuities: Removable discontinuity, jump discontinuity, discontinuity of 2$^{\text{nd}}$ kind.

Theorem 4.1, 4.2, 4.3, right continuous at $x = a$, left continuous at $x = a$. Continuity on an interval: $[a, b]$, (a, b).

28% 4.2 (p 102-106) Th 4.4, 4.5 (Extreme Value Theorem), 4.6 (Intermediate Value Theorem), Theorem 4,7 Fixed point.

32% 4.3 Uniform continuous on an interval. Be able to use definition to prove uniform continuity on I or to prove not uniform continuity on I. Theorem 4.12, 4.13, 4.14, 4.15 and its corollary, 4.16.

4.01 Prove that if f is continuous at x_0 then f is bounded at x_0.

Proof. Suppose that $f : A \to \mathbb{R}$ is continuous at x_0. To show that f is bounded at x_0, it suffices to show that f is bounded on $N_\delta(x_0) \cap A$ for some $\delta > 0$. Since f is continuous at x_0, it follows by definition that there exists $\delta_1 > 0$ such that $|f(x) - f(x_0)| < 1$ whenever $|x - x_0| < \delta_1$. Using the triangle inequality we have that

$$||f(x)| - |f(x_0)|| < |f(x) - f(x_0)| < 1,$$

so that $|f(x)| - |f(x_0)| < 1$, if $|x - x_0| < \delta_1$. Thus $|f(x)| < 1 + |f(x_0)|$, if $|x - x_0| < \delta_1$. We have thus shown that f is bounded on $N_{\delta_1}(x_0)$ by $1 + |f(x_0)|$, so that f is bounded at x_0. □

4.02 Find all points of discontinuity for the following functions, classify the discontinuities as removable, jump, or second kind, and determine where the function is right- and left-continuous.

(a) $$f(x) = \begin{cases} x^2 & \text{if } x < -1, \\ 2x + 3 & \text{if } -1 \le x \le 0, \\ |x - 1| & \text{if } 0 < x < 2, \\ x^3 - 7 & \text{if } 2 \le x < 3, \\ \dfrac{x-3}{x-4} & \text{if } 3 \le x < 4, \\ 0 & \text{if } 4 \le x. \end{cases}$$

(b) $f(x) = x + [\![-x]\!]$.

(c) $f(x) = x[\![x]\!]$.

(d) $f(x) = \operatorname{sgn}[\![|x|]\!]$.

(e) $f(x) = \begin{cases} [\![x+1]\!] \sin \dfrac{1}{x} & \text{if } x \in (-1,0) \cup (0,1) \\ 0 & \text{otherwise.} \end{cases}$

(f) $f(x) = \begin{cases} (1+x) \operatorname{sgn} x + \operatorname{sgn} |x| - 1 & \text{if } x \text{ is rational} \\ \operatorname{sgn} x & \text{if } x \text{ is irrational.} \end{cases}$

Solution.

(a) Since

$$3 = \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x) = 1 \text{ and } 27 = \lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x) = 0$$

it follows that f has jump discontinuities at 0 and 3. The function f has a discontinuity of the second kind at 4 because $\lim_{x \to 4^-} f(x) = -\infty$; f is not right-continuous at 0 because $1 = \lim_{x \to 0^+} f(x) \neq f(0) = 3$, but it is right-continuous at every other point; also, f is not left-continuous at 3 and 4 because $27 = \lim_{x \to 3^-} f(x) \neq f(3) = 0$ and $\lim_{x \to 4^-} f(x) = -\infty$, but it is left-continuous at every other point.

(b) Let z be an integer. Then it follows that

$$\begin{aligned} f(z) &= z + [\![-z]\!] \\ &= z + (-z) = 0, \end{aligned}$$

$$\begin{aligned} \lim_{x \to z^+} f(x) &= \lim_{x \to z^+} x + \lim_{x \to z^+} [\![-x]\!] \\ &= z + (-z-1) = -1, \text{ and} \end{aligned}$$

$$\begin{aligned} \lim_{x \to z^-} f(x) &= \lim_{x \to z^-} x + \lim_{x \to z^-} [\![-x]\!] \\ &= z + (-z) = 0, \end{aligned}$$

so that f has jump discontinuities at all integers; f is left-continuous at all points, and it is right-continuous at all points except the integers.

(c) It is clear that f is continuous at all non-integer points. So let z be an integer. Observe that $f(z) = z^2$, $\lim_{x \to z^-} f(x) = z(z-1)$, and $\lim_{x \to z^+} f(x) = z^2$, so that f is continuous at 0 and has jump discontinuities at all other integers; also f is left-continuous at all points except nonzero integers, and it is right continuous at all points.

(d) We have that

$$\operatorname{sgn}[\![|x|]\!] = \begin{cases} 1 & \text{if } x \leq -1 \text{ or } x \geq 1 \\ 0 & \text{if } -1 < x < 1, \end{cases}$$

so that f has jump discontinuites at -1 and 1, is left-continuous at all points except 1, and is right-continuous at all points except -1.

4.03 Prove that $f(x) = \cos x$ is continuous on $\mathbb{R}$.

4.04 Prove that if f is continuous at x_0 and g is discontinuous at x_0 then $f + g$ must have a discontinuity at x_0.

Proof. Assume that f is continuous at x_0 and g is discontinuous at x_0. Now suppose to the contrary that $f + g$ is continuous at x_0. By Theorem 4.2, it follows that $(f+g)+(-f) = g$ is also continuous at x_0, a contradiction. Thus $f+g$ has a discontinuity at x_0. □

4.05 Show that $f + g$ can be continuous at x_0 even though both f and g have discontinuities at x_0.

Solution. For every nonzero x, let $f(x) = 1/x$; then define $f(0) = 0$ and $g(x) = -f(x)$ for all real x. It is clear that both functions are not continuous at 0, but $f + g = 0$ is continuous at 0.

4.06 Show that $f \cdot g$ can be continuous at x_0 even though both f and g have discontinuities at x_0.

Solution. Define

$$f(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ \dfrac{1}{x-1} & \text{if } x > 1, \end{cases}$$

$$g(x) = \begin{cases} \dfrac{1}{x-1} & \text{if } x < 1 \\ 0 & \text{if } x \geq 1. \end{cases}$$

The functions f and g have discontinuities at 1 because the former is not right-continuous at 1 and the latter is not left-continuous at 1; however $f \cdot g = 0$ is continuous at 1.

4.07 If f is continuous at x_0 and g is discontinuous at x_0, what can be said about continuity of the product $f \cdot g$ at x_0?

Answer. Nothing definite can be said about the continuity of the product at x_0 because if $f(x) = 1$ and $g(x) = 1/x$, then $f \cdot g = g$ is not continuous at 0, and if $f(x) = 0$ and $g(x) = 1/x$ with $g(0) = 0$ then $f \cdot g = 0$ is continuous at 0. Note that in either case f was continuous at 0 and g was not.

4.08 Show that the composition function $g \circ f$ can be continuous at x_0 even though f or g or both f and g are discontinuous at x_0.

Solution. Let $f(x) = 1/x$, with $f(0) = 0$, and let $g = f$. Then it follows that $(f \circ g)(x) = x$ so that $f \circ g$ is continuous at 0, but f (and thus g) is not continuous at 0.

4.09 Prove that the function

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational}, \\ 0 & \text{if } x \text{ is irrational}. \end{cases}$$

has a discontinuity of the second kind at each nonzero real number.

Proof. Let a be a nonzero real number. It suffices to show that $\lim_{x\to a^+} f(x)$ does not exist. By Theorems 1.9 and 1.10, we know that every interval in $\mathbb{R}$ contains a rational number and an irrational number. So let b_n be a sequence of rational numbers and c_n a sequence of irrational numbers such that

$$b_n \in (a, a+1/n) \cap \mathbb{Q}, \qquad c_n \in (a, a+1/n) \cap (\mathbb{R} - \mathbb{Q}).$$

It is clear that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = a$, but

$$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} b_n = a \neq 0 = \lim_{n\to\infty} 0 = \lim_{n\to\infty} c_n,$$

so that by Exercise 3.36, $\lim_{x\to a^+} f(x)$ does not exist. Thus f has a discontinuity of the second kind at each nonzero real number. □

4.10 Prove that if f is continuous at x_0 and f is nonnegative then $h(x) = \sqrt{f(x)}$ is continuous at x_0.

4.11 Find a function f which has a discontinuity of the second kind at every real number although $f \circ f$ is continuous on $\mathbb{R}$.

Solution. We know from Example 4.4 that

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational,} \end{cases}$$

has a discontinuity of the second kind at every real number. Now

$$(f \circ f)(x) = f(f(x)) = \begin{cases} f(1) = 1 & \text{if } x \text{ is rational,} \\ f(0) = 1 & \text{if } x \text{ is irrational,} \end{cases}$$

so that $f \circ f$ is identically equal to 1. Thus $f \circ f$ is continuous on $\mathbb{R}$.

4.12 If f is continuous on $(0, 1)$ and $f(x) = 1 - x$ for every rational number $x \in (0, 1)$, find $f(\pi/4)$. Explain your answer.

Solution. Since f is continuous on (0, 1) and $\pi/4 \in (0, 1)$, it follows that

$$\lim_{x\to\pi/4} = f(\pi/4).$$

Consider the sequence of rationals a_n where $a_n \in \left(\frac{\pi}{4}, \frac{\pi}{4} + \frac{1}{10n}\right)$. Since each $a_n \in (0, 1)$ and since a_n converges to $\pi/4$, it follows by Theorem 3.6 that $f(a_n)$ must converge to $f(\pi/4)$. Thus

$$\begin{aligned} f(\pi/4) &= \lim_{n\to\infty} f(a_n) \\ &= \lim_{n\to\infty} (1 - a_n) \\ &= \lim_{n\to\infty} 1 - \lim_{n\to\infty} a_n \\ &= 1 - \pi/4. \end{aligned}$$

4.13 Prove that if f and g are each continuous on (a, b) and $f(x) = g(x)$ for every rational $x \in (a, b)$ then $f(x) = g(x)$ for every $x \in (a, b)$.

Proof. Assume the hypothesis holds. It suffices to show that $f(x) = g(x)$ for all $x \in (a, b) \cap (\mathbb{R} - \mathbb{Q})$. So let c be an irrational number in (a, b). Since f and g are continuous at c, we must have that $\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$. Let $\{t_n\}$ be a sequence of rationals in (a, b) that converges to c. By Theorem 3.6, it follows that $\{f(t_n)\}$ must converge to $f(c)$ and $\{g(t_n)\}$ must converge to $g(c)$. But $f(t_n) = g(t_n)$ for each n, so that by the uniqueness of the limit of a sequence, we must have that $f(c) = g(c)$, which is what we wanted to show. □

4.14 Prove: f is right-continuous at x_0 if and only if $f(x_n) \to f(x_0)$ for every sequence $\{x_n\}$ in the domain of f with $x_n \to x_0$ and $x_n \geq x_0$ for $n = 1, 2, 3, ...$

Proof. Use Exercise 3.36. □

4.15 Discuss one-sided continuity for the pie function.

4.16 Prove that if f is defined on $\mathbb{R}$ and continuous at $x_0 = 0$ and if $f(x_1+x_2) = f(x_1)+f(x_2)$ for each $x_1, x_2 \in \mathbb{R}$ then f is continuous on $\mathbb{R}$.

Proof.

4.17 Find all functions f which are continuous on $\mathbb{R}$ and which satisfy the equation $f(x)^2 = x^2$ for each $x \in \mathbb{R}$. *Hint*: There are four possible solutions.

4.18 Prove that if g is continuous at $x_0 = 0$, $g(0) = 0$ and for some $\delta > 0$ $|f(x)| \leq |g(x)|$ for each $x \in N_\delta(0)$ then f is continuous at $x_0 = 0$.

4.19 Prove that if f is continuous on $[a, b]$ then there exists a function g continuous on $\mathbb{R}$ such that $g(x) = f(x)$ for each $x \in [a, b]$. The function g is called a *continuous extension* of f to $\mathbb{R}$.

4.20 The function $f(x) = \tan x$ defined on $(-\pi/2, \pi/2)$ clearly has no continuous extension to $\mathbb{R}$. Find a bounded continuous function on (a, b) which has no continuous extension to $\mathbb{R}$.

4.21 Assume that f is continuous on (a, b). Prove that f has a continuous extension to $\mathbb{R}$ if and only if both limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist.

4.22 Prove that if f is continuous on (a, b) and both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist then f is bounded on (a, b).

4.23 Suppose f is one-to-one on (a, b) and satisfies the following property: whenever $f(x_1) \neq f(x_2)$ for $x_1 < x_2$, x_1, $x_2 \in (a, b)$ and k is any number between $f(x_1)$ and $f(x_2)$, there exists a $c \in (x_1, x_2)$ with $f(c) = k$. Prove that f is continuous on (a, b).

4.2 Section 2

4.24 Prove Lemma 2.

4.25 Prove Lemma 4.

4.26 Find $M = \sup_{x\in A} f(x)$ and $m = \inf_{x\in A} f(x)$ for the following bounded functions f defined on the indicated domain A and then find points $x_1, x_2 \in A$ (if they exist) such that $f(x_1) = M$ and $f(x_2) = m$.

(a) $f(x) = 3 + 2x - x^2$ on $[0, 4]$.
(b) $f(x) = 2 - |x - 1|$ on $[-2, 2)$.
(c) $f(x) = e^{-1/x}$ on $(0, \infty)$.
(d) $f(x) = 1 - x^2$ on $(-2, 1)$.

4.27 Find a function $f : A \to B$ such that f is bounded on $C \subset A$ but there exists an $x_0 \in C$ such that f is not bounded at x_0.

4.28 Suppose f is continuous on (a, b). Prove that if $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ both exist then f is bounded on (a, b) but the converse does not hold.

4.29 Suppose f is continuous on $\mathbb{R}$. Prove that if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ both exist then f is bounded on $\mathbb{R}$ but the converse does not hold.

4.30 Verify that the function in Example 4.5 satisfies the intermediate-value property on $[-1, 1]$.

4.31 Prove that if f is continuous on any interval then f satisfies the intermediate-value property on that interval.

4.32 Prove that if f is continuous on any interval then the range of of f is again an interval.

4.33 Prove that the polynomial $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, where $a_0 \neq 0$ and n is an odd natural number, has at least one real root; that is, there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = 0$.

4.34 Prove that there exists $x_0 \in (0, 1)$ such that $f(x_0) = 0$, where $f(x) = e^x - 3x - \sin x$.

4.35 Suppose that f is continuous on $[0, 1]$, $f(x)$ is rational for every $x \in [0, 1]$, and $f(0) = 0$. Find $f(\sqrt{2}/2)$.

4.36 Suppose that f is continuous on $(0, \infty)$, $\lim_{x\to 0^+} f(x) = 0$, and $\lim_{x\to\infty} f(x) = 1$. Prove that there exists $x_0 > 0$ such that $f(x_0) = \sqrt{3}/2$.

4.37 Prove that the function $f(x) = x^3 + x^2 - 3x - 3$ has a root between 1 and 2, between 1.5 and 2, between 1.5 and 1.75, between 1.625 and 1.75, etc. Note that if we continue this procedure we shall be able to approximate the root as closely as we wish.

4.38 Suppose that f is continuous on $[a, b]$ except at $x_0 \in (a, b)$, where f has a discontinuity. Suppose further that $f(x)$ is rational for every $x \in [a, b]$. Prove that f has a simple discontinuity at x_0.

4.39 Suppose that f is continuous on $[-1, 1]$ and $|f(x)| \leq 1$ for every $x \in [-1, 1]$. Suppose that g is continuous on $[-1, 1]$ with $g(-1) = -1$ and $g(1) = 1$. Prove that there exists an $x_0 \in [-1, 1]$ with $f(x_0) = g(x_0)$.

4.40 Prove that if f is monotone on $[a, b]$ and f satisfies the intermediate-value property on $[a, b]$ then f is continuous on $[a, b]$.

4.41 Prove that there is no continuous function f on $\mathbb{R}$ such that for every real number c, $f(x) = c$ has exactly has two solutions.

4.42 Find a continuous function f on $\mathbb{R}$ such that for every real number c, $f(x) = c$ has exactly three solutions.

4.43 Use the fact that $f(x) = \tan x$ is continuous and one-to-one on $(-\pi/2, \pi/2)$ to establish that $\tan^{-1} x$ is continuous on $\mathbb{R}$.