Math 444 Review

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Chapters 1 and 2.

Equivalence Relations.

Let X be a nonempy set. A *relation* on X is a subset of $X \times X$, where $X \times X$ is the set of ordered pairs $\{(x, y) : x, y \in X\}$. We write $x \sim y$ if and only if (x, y) is a member of this relation. A relation on X that satisfies the following properties:

- *Reflexivity.* For each $x \in X$, we have that $x \sim x$.
- *Symmetry.* If $x \sim y$ for some $x, y \in X$, then it follows that $y \sim x$.
- *Transitivity.* If $x \sim y$ and $y \sim z$ for some $x, y, z \in X$, then it follows that $x \sim z$.

is said to be an *equivalence relation*.

Exercise 1.

Prove that if n is a natural number, then

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{1}$$

Proof. We shall proceed by induction.

Base Case. $n = 1$. Since $1 = \dfrac{1(1+1)(2 \cdot 1 + 1)}{6}$, it follows that (1) holds whenever n is 1.

Inductive Hypothesis. Suppose that (1) holds for some positive integer k. That is,

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}. \tag{2}$$

To complete the proof, we must now show that (1) holds for $k+1$. Thus

$$\begin{aligned}
1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad \text{[From (2)]} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6},
\end{aligned}$$

so that (1) holds for $k+1$; thus it follows by Mathematical Induction that (1) for all natural numbers. $\square$

Functions.

Let $f : X \to Y$ be a function. Then this function is

- *injective (or one-to-one)* if $f(x_1) = f(x_2)$, with $x_1, x_2 \in X$, then $x_1 = x_2$.
- *surjective (or onto)* if for every $y \in Y$, there exists an $x \in X$ such that $f(x) = y$.
- *well defined* if $x_1 = x_2$, with $x_1, x_2 \in X$, then $f(x_1) = f(x_2)$.

A function $h : S \to T$ is called an *identity* on S if $h(s) = s$ for all $s \in S$. Now consider the functions $f : X \to Y$ and $g : Y \to X$. The function g is an inverse function of f if $f \circ g$ is an identity on Y and if $g \circ f$ is an identity on X.