1. Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false. Let $G = \langle g \rangle$ have order 300.

   1. There are exactly 80 generators of G.
   2. G has only one element of order 3.
   3. G can be embedded in S_{30}.
   4. G has a subgroup of order 20.
   5. G has a totality of 18 subgroups.

   Solution.

   1. True. Since $G = \langle g \rangle$ is cyclic and since $|g| = 300$, it follows that the number of generators of G is the number of positive integers relatively prime to 300, which is 80.
   2. False.
      Counterexample. We have that $g^{100} \neq g^{200}$ (since $|g| = 300$) and
      $$|g^{100}| = \frac{300}{\gcd(300, 100)} = 3 = \frac{300}{\gcd(300, 200)} = |g^{200}|.$$
   3. False.
      Proof. It suffices to show that S_{30} has no element of order 300. Suppose to the contrary that $\sigma \in S_{30}$ has order 300. Then we can write σ as a product of disjoint cycles (each of length greater than 1)
      $$\sigma = \alpha_1 \alpha_2 \cdots \alpha_n$$
      so that $|\sigma| = \operatorname{lcm}(|\alpha_1|, |\alpha_2|, \cdots, |\alpha_n|) = 300$. Now since $5^2 \mid 300$, it follows that 5^2 must divide the order of at least one of the cycles. We can assume without loss that $5^2 \mid |\alpha_1|$. Thus α_1 must be a 25-cycle. By a similar argument, it follows that 2^2 must divide the order of at least one of the cycles. Assume without loss that $2^2 \mid |\alpha_2|$. Since there are 25 elements in α_1, there can be at most 5 elements in α_2, so that α_2 is a 4-cycle. Thus
      $$\sigma = \alpha_1 \alpha_2,$$
      a contradiction since $\operatorname{lcm}(|\alpha_1|, |\alpha_2|) = 100 \neq 300$. $\square$
   4. True. This subgroup of G, $\langle g^{15} \rangle$, has 20 elements.
   5. True. Since the number of positive divisors of 300 is 18, it follows that G has exactly 18 subgroups.

2. Let G be an abelian group and let $a, b \in G$ be of order 120 and 72 respectively. Do the following:

(1) Find an element of order 15.

(2) What is the order of b^{10}?

(3) Find an element of as large an order as you can.

Solution.

(1) The element a^8 has order 15.

(2) $|b^{10}| = \dfrac{72}{\gcd(72,10)} = 36.$

(3) The element $a^{24}b$ has order 360.

3. Consider the non-abelian group of order 55 from **Homework #4**. View this group as acting on all column vectors of size 2 (with entries in $\mathbb{Z}_{11}$).

(1) Find the number of fixed points of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(2) Find the number of fixed points of $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$.

(3) Decide on the number of fixed elements each of the elements of the group has.

(4) Use Burnside's Lemma to count the orbits.

Solution.

(1) Suppose $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ for some $a, b \in \mathbb{Z}_{11}$. Then it follows that

$$\begin{pmatrix} a+b \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix},$$

so that $b = 0$. Thus the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ fixes 11 elements and they are

$$\left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} : a \in \mathbb{Z}_{11} \right\}.$$

(2) Suppose $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ for some $a, b \in \mathbb{Z}_{11}$. Then it follows that

$$\begin{pmatrix} 3a \\ 4b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix},$$

so that $2a = 0$ and $3b = 0$. Multiply the former equality by 6 and the latter by 4 to get $a = b = 0$.Thus the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ fixes only 1 element, the zero vector.

(3) There are 44 matrices of the form $\begin{pmatrix} b & x \\ 0 & b^{-1} \end{pmatrix}$, with $b \neq 1$, and each only fixes the zero vector. The identity matrix fixes all the vectors (121 of them), while each of the remaining 10 matrices fixes exactly 11 vectors.

(4) Let n be the number of orbits. Using our results from (3) and Burnside's Lemma, it follows that that

$$n \cdot 55 = 44 \cdot 1 + 1 \cdot 121 + 10 \cdot 11 = 275,$$

so that $n = 5$.

4. Let the vertices of the cube be given as follows:

$$1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, 2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, 3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, 4 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, 5 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, 6 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, 7 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$

and $8 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$.

(1) Label the faces A, A', B, B', C, and C' (where the prime means opposite), and give each as a set of four vertices. Let A be the intersection with the plane $x = 1$, B with the plane $y = 1$ and C with $z = 1$.

(2) Find 24 3×3 matrices of determinant 1 that are isometries of the cube, and write each as a permutation in S_8 (of the eight vertices) and also as a permutation of the faces. **Hint:** Start with the six permutation matrices of size 3.

Assume these 24 matrices form a group G. **Bonus.** Prove this.
Assume these $G \simeq S_4$. **Bonus.** Prove this.

(3) Find the number of ways to color a cube with two colors.

(4) Find the number of ways to color a cube with three colors.

Bonus. Find the number of ways to color the cube with n colors.

Solution.

(1) s