Lecture 1

- **Read:** Chapter 1.1-1.5.

- Set Theory
- Elements of Probability
- Conditional Probability
- Sequential Calculation of Probability
- Total Probability and Bayes' Theorem

Probability

- Life and our environment are uncertain
- Common analysis method for uncertain situations
  - Use “long-term averages,” i.e., probabilities
- Common approach for decision under uncertainty
  - Optimizing the “average value” of the result of the decision
- Probability theory deals with phenomena whose outcome is not fully predictable
  - but exhibit some regularity when observed many times

History

- Cardano, Galileo (16th Century)
  - Emergence - gambling in Italy...
- Pascal, Fermat (17th Century)
  - Rational thought
- Bernoulli, Laplace, Poisson, Gauss (18th-19th Century)
  - Mathematical organization
- Kolmogorov (20th Century)
  - Axiomization

Course Objective

1. Develop your ability to describe uncertain events in terms of probabilistic models
2. Develop your skill in probabilistic reasoning

Motivation - Applications

- Engineering
  - Communications, information theory
  - Signal processing and systems control
  - Queuing theory and modeling computer systems
  - Decision and resource allocation under uncertainty
  - Reliability, estimation, and detection
- Statistics: collection and organization of data so that useful inferences can be drawn from them
- Physics, statistical mechanics, thermodynamics
- Computer science: randomized algorithms, random search
- Economics and finance: investment/insurance risk assessment

Two Interpretations for Probabilities

- **Frequency of occurrence:** probability = % of successes in a moderately large number of situations (*Reality may or may not involve repetition!*)
  - When is this appropriate? For example, 50% probability that a coin comes up heads versus 90% probability that Homer wrote the Odyssey and the Illiad?
- **Subjective belief:** probability = an expert's opinion that an event occurred/will occur
  - For example, likelihood that a medication will work when it is used for the first time

Role of Math

- "Probability is common sense reduced to calculation." (Laplace)
- "The book of the universe is written in the language of mathematics." (Galileo)
- Probabilistic analysis is mathematical, but intuition dominates and guides the math. (Our goal!)
- Problem formulation in terms of probabilities is typically more challenging than the calculations. (Need to work lots of problems!)

Getting Started

- But, thinking probabilistically is fairly unnatural, unless you are used to it! So, let us get to work.
- Basic idea is to assign probabilities to collections (sets) of possible outcomes, so we start by briefly reviewing set theory.

Set Theory Preliminaries

- Venn Diagrams
- Universal Set/Empty Set
- Union/Intersection
- Complement
- Mutually Exclusive/Collectively Exhaustive

Set Theory Review: Sets

A set A is a collection of objects which are *elements* of the set.

- If x is an element of A, we write $x \in A$
- If x is *not* an element of A, we write $x \notin A$
- A set with no elements is the **empty set** $\emptyset$
- The set with *all* the elements relevant to a particular context is called the **universal set**, say S

Set Theory Review: Describing Sets

"Make a list" versus "describe its elements"

- **list of elements:** $A = \{x_1, x_2, ..., x_n\}$, e.g., possible outcomes of the roll of a die, $\{1, 2, 3, 4, 5, 6\}$ or coin toss, $\{H, T\}$
- **properties of elements:** $A = \{x | x \text{ satisfies } P\}$, e.g., $\{x | x \text{ is an even integer}\}$ or $\{x : 0 \leq x \leq 1\}$. Note that "|" and ":" both mean "such that"

Set Theory Review: Describing Sets

- **countable vs. uncountable:** A set is **countable** if it can be written down as a list, otherwise it is **uncountable.**
- **ordered pair** of two objects (x,y): e.g., set of scalars
  - Note that order is indicated by the use of (x,y) versus {x,y}
- **subset:** $A \subset B$ if every element in A is in B
- **equality of sets:** $A = B$ if and only if $A \subset B$ and $B \subset A$

Set Theory Review: Set Operations and Venn Diagrams

- **union:** (logical OR) of two sets $A \cup B$, i.e., in A or B **or both** (e.g., round and/or blue elements)
  - One can also define the union of a finite or even infinite number of sets, e.g.:

$$\cup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \ldots = \{x : x \in A_i \text{ for some } i\}$$
$$\cup_{\alpha \in R}^{\infty} B_\alpha = \{x : x \in B_\alpha \text{ for some } \alpha\}$$

- **intersection**: (logical AND) of two sets $A \cap B$ (e.g., round and blue elements)
  - Similarly, one can also define intersections of a finite or infinite number of sets, e.g.:

$$\cap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \ldots = \{x : x \in A_i \text{ for all } i\}$$

Set Theory Review: Set Operations and Venn Diagrams

- **complement**: of A in S is A^c, the set of elements which are not in A, e.g., the complement of $\emptyset$ is S
- **difference**: of two sets $A \backslash B = A \cap B^c$
- **disjoint or mutually exclusive sets**: have no common elements, i.e., $A \cap B = \emptyset$ iff A and B are disjoint
- **collectively exhaustive**: a (possibly infinite) collection of sets $A_1, ..., A_n$ is said to be **collectively exhaustive** iff $\cup_{i=1}^n A_i = S$
- **partition**: a (possibly infinite) collection of sets $A_1, ..., A_n$ is said to be a **partition** of S iff $\cup_{i=1}^n A_i = S$ (i.e., they are collectively exhaustive) and the sets are disjoint (i.e., mutually exclusive), e.g., A and A^c are a partition of S

Set Theory Review: Algebra of Sets

Elementary properties follow from the definitions:

- **Associative**: $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive**: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- **de Morgan's Law**: $(A \cup B)^c = A^c \cap B^c$, similarly $(A \cap B)^c = A^c \cup B^c$
    - **Proof:** We can show that $(A \cup B)^c \subset A^c \cap B^c$ as follows. If $x \in (A \cup B)^c$, then x is not in A and not in B, thus x must be in both A^c and B^c. Similarly, one can establish that $A^c \cap B^c = (A \cup B)^c$

What is Probability?

- a number between 0 and 1.
- a physical property (like mass or volume) that can be measured?
- measure of our knowledge?

Probabilistic Models

Going from *experiments* in the physical world to *probabilistic models*

- Experiment = Procedure + Observation, e.g., flip a coin and see if it landed heads or tails or transmit a waveform over a channel and observe what is received
- Real Experiments are TOO complicated
- Instead we analyze/develop models of experiments
  - A coin flip is equally likely to be H or T
- Probabilistic model is (usually) a simplified mathematical description used to *study* the situation

Example 1.1

An experiment consists of the following procedure, observation, and model:

- Procedure: Flip a coin and let it land on a table.
- Observation: Observe which side (head or tail) faces you after the coin lands.
- Model: Heads and tails are equally likely. The result of each flip is unrelated to the results of previous flips.

Components of Probabilistic Models

- An *outcome* s, s_1 or x of an experiment is one of the possible observations of that experiment.
- The *sample space* S of an experiment is the finest-grain, mutually exclusive, collectively exhaustive set of all possible outcomes.
- An *event* A is a set of outcomes of an experiment, i.e., $A \in S$, e.g. $\{s_1\}$.
- $B_1, B_2, ..., B_n$ make up an *event space* or *partition* S iff
  - $B_i \cap B_j = \emptyset, i \neq j$
  - $\cup_{i=1}^{n} B_i = S$

Representing Sample Spaces

- Sequential models: sample space vs. tree-based sequential description, e.g., two rolls of a tetrahedral (four-sided) die. Draw the two sample spaces, show the event that the second roll is 4, and exhibit a partition, e.g., the sets first roll is 1, 2, 3, or 4.
- A continuous sample space: throw a dart at a square target with area 1, e.g., $S = \{(x, y) | 0 \leq x, y \leq 1\}$

Correspondences

Set Algebra	Probability
set	event
universal set	sample space
element	outcome

Example 1.9

Flip four coins, a penny, a nickel, a dime, and a quarter. Examine the coins in order (penny, then nickel, then quarter) and observe whether each coin shows a head (h) or a tail (t). What is the sample space? How many elements are in the sample space?

. . . .

The sample space consists of 16 four-letter words:

$$\{\text{tttt, ttth, ttht, ..., hhhh}\}$$

Event Spaces

- An *event space* is a collectively exhaustive, mutually exclusive set of events.
- **Example 1.10:** For i= 0, 1, 2, 3, 4,

$$B_i = \{\text{outcomes with } i \text{ heads}\}$$

- Each B_i is an event containing one or more outcomes.
- The set $B = \{B_0, B_1, B_2, B_3, B_4, \}$ is an event space.

Theorem 1.2

- For an event space $B = \{B_1, B_2, ...\}$ and any event A, let $C_i = A \cap B_i$.
- For $i \neq j$, the events $C_i \cap C_j = \emptyset$

$$A = C_1 \cup C_2 \cup \cdots$$

Probability Measure (or Law) and Axioms of Probability

A **probability measure (or law)** $P[\cdot]$ is a function that maps events in the sample space to real numbers ($P[A] \mapsto [0,1]$) such that

- **Axiom 1** (nonnegativity) For any event A, $P[A] \geq 0$.
- **Axiom 2** (normalization) $P[S] = 1$.
- **Axiom 3** (additivity) For any countable collection $A_1, A_2, ...$ of mutually exclusive (i.e., disjoint) events
$$P[A_1 \cup A_2 \cup ...] = P[A_1] + P[A_2] + ...$$

Consequences of the Axioms

Theorem 1.7: The probability measure $P[\cdot]$ satisfies

- $P[\emptyset] = 0$.
- $P[A^c] = 1 - P[A]$.
- For any A and B,

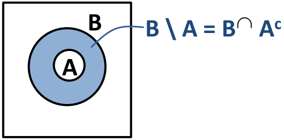
$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

- If $A \subset B$, then $P[A] \leq P[B]$.

Consequences of the Axioms (cont.)

- $P[A^c] = 1 - P[A]$
  Proof:
$$A \cap A^c = \emptyset$$
$$A \cup A^c = S$$
$$P[A \cup A^c] = P[A] + P[A^c] = 1$$

- If $A \subset B$, then $P[A] \leq P[B]$.
  Proof: $B = A \cup [B \backslash A]$, A and $B \backslash A$ are disjoint
$$P[B] = P[A] + P[B \backslash A]$$
$$\Rightarrow P[B] \geq P[A]$$



- $P[A \cup B] = P[A] + P[B] - P[A \cap B]$

Consequences of the Axioms (cont.)

- **Theorem 1.4:** If

$$B = B_1 \cup B_2 \cup \ldots \cup B_m$$

and for $i \neq j$,

$$B_i \cap B_j = \emptyset$$

then

$$P[B] = \sum_{i=1}^{m} P[B_i]$$

Problem 1.3.5

A student's score on a 10-point quiz is equally likely to be any integer between 0 and 10. What is the probability of an A, which requires the student to get a score of 9 or more? What is the probability the student gets an F by getting less than 4?

Problem 1.4.5

A cellphone is equally likely to make zero handoffs (H_0), one handoff (H_1), or more than one handoff (H_2). Also, a caller is on foot (F) with probability $5/12$ or in a vehicle (V).

- Find three ways to fill in the following probability table:

	H_0	H_1	H_2
F			
V			

Problem 1.4.5 (cont.)

- If $1/4$ of all callers are on foot making calls with no handoffs and that $1/6$ of all callers are vehicle users making calls with a single handoff, what is the table?

	H_0	H_1	H_2
F	$1/4$	$1/6$	0
V	$1/12$	$1/6$	$1/3$

Discrete Models

Building models means

1. defining sample/event space and
2. specifying a suitable probability law, i.e., consistent with the axioms

Examples:

1. fair coin toss $S = \{H, T\}$, P[H]= P[T]= 1/2
2. three fair coin tosses $S = \{HHH, HHT, HTH, ...\}$, each outcome with probability 1/8, suppose $A = \{\text{exactly 2 heads occur}\} = \{HHT, HTH, THH\}$

$$
\begin{aligned}
P[A] &= P[\{HHT, HTH, THH\}] \\
&= P[\{HHT\}] + P[\{HTH\}] + P[\{THH\}]
\end{aligned}
$$

Discrete Probability Law

If S consists of a countable set of outcomes, then for any event $A = \{s_1, s_2, ..., s_n\}$,

$$P[A] = P[\{s_1\}] + P[\{s_2\}] + P[\{s_n\}]$$

Equally Likely Outcomes: Discrete Uniform Probability Law

If $S = \{s_1, s_2, ..., s_n\}$, i.e., consists of n possible outcomes, and they are *equally likely*, then for any event A, we have

$$P[A] = \frac{(\#\text{ of elements in A})}{n}$$

$$\text{e.g.,} P[\{s_i\}] = \frac{1}{n}$$

Note: For such laws, computing probabilities boils down to counting events! Here arises a basic link between combinatorics and probability. Later, we will go over counting methods, e.g., permutations, combinations, etc.

Equally Likely Outcomes: Example

- Two rolls of a fair four-sided die
- 16 outcomes with the same probability $1/16$
- Draw figure
- Find probability that at least one roll is 4, or probability that first roll is equal to the second

Continuous Models

- Sample space does need not be finite; probability of single outcomes is not enough to describe what is going on, so we use continuous models
- **Example:** Romeo and Juliet have a date at 12 pm, but each will arrive at the meeting place with a delay between 0 and 1 hour, with all pairs of delays "equally likely." The first to arrive will wait for 15 mins and leave if the other does not arrive. What is the probability that they will meet?
- **Model and solution:** $S = [0,1] \times [0,1]$. Equally likely means the probability of a subset of S is equal to its area. This probability law satisfies our axioms. Draw and explain the event that they meet $M = \{(x, y) : |x - y| \leq 1/4, (x, y) \in S\}$. The area of this event is $7/16$, i.e., 1 minus the area of the two triangles, i.e., square with area $(3/4)^2$.

Conditional Probability

Reasoning about events when we have partial information: Key to dealing with many real world problems, e.g., signal detection, stock market, etc.

1. A spot shows up on the radar screen. How likely is it that it corresponds to an aircraft?
2. A fair die is rolled and you are told that the outcome was even. How likely is it that the outcome was a 6?

Example (2) above: fair die $\rightarrow$ all outcomes equally likely; we know the outcome is even, and expect each even outcome to be equally likely, i.e., $P[\text{outcome } 6 \mid \text{outcome even}] = 1/3$

Conditioning

- Consider two events A and B
- $P[A]$ = our knowledge of the likelihood of A
- $P[A]$ = "a priori" probability
- Suppose we cannot completely observe an experiment
  - We learn that event B occurred
  - We do not learn the precise outcome
- Learning B occurred changes $P[A]$
- $P[A|B]$ = probability of A given B occurred (B is our new universe)

Conditional Probability Definition

- Assuming $P[B] \neq 0$, the **conditional probability** A given the occurrence of B is

$$P[A|B] = \frac{P[AB]}{P[B]} = \frac{P[A \cap B]}{P[B]}$$

- **Note:** $A \cap B = A \cdot B = AB$
- It follows that $P[A \cap B] = P[B] \cdot P[A|B] = P[A] \cdot P[B|A]$

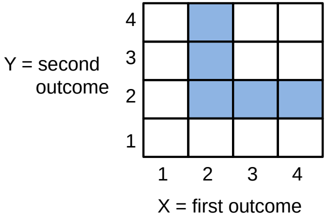
Conditional Probability: Subsets

- Suppose that $B \subset A$, then

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[B]}{P[B]} = 1$$

Die Roll Example

Consider rolling a four-sided die twice; denote the outcomes X and Y.



- Let B be the event: $min(X, Y) = 2$
- Let $M = max(X, Y)$
- Compute $\mathrm{P}[\mathrm{M} = \mathrm{m}|\mathrm{B}]$ for $m = 1, 2, 3, 4$

Die Roll Example (cont.)

- $P[B] = 5/16$

$$P[M = 1|B] = \frac{P[A \cap B]}{P[B]} = 0$$
$$P[M = 2|B] = \frac{P[A \cap B]}{P[B]} = \frac{1/16}{5/16} = \frac{1}{5}$$
$$P[M = 3|B] = \frac{P[A \cap B]}{P[B]} = \frac{2/16}{5/16} = \frac{2}{5}$$
$$P[M = 4|B] = \frac{P[A \cap B]}{P[B]} = \frac{2/16}{5/16} = \frac{2}{5}$$

Models Based on Conditional Probabilities: Estimation Problems

Radar Example

- Event $A =$ airplane is flying above
- Event $B =$ something registers on radar screen
- What are A^c and B^c?

$$P[\text{false alarm}] = P[A^c \cap B] = P[A^c]P[B|A^c] = 0.95 \times 0.10 = 0.095$$

Models Based on Conditional Probabilities: Tree-Based Sequential Description

1. Set up tree with event of interest as a leaf
2. Record the conditional probabilities associated with branches
3. Multiply the probability to get to the leaf

Models Based on Conditional Probabilities: Multiplication Rule

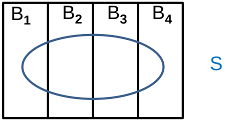
Assuming all the conditioning events have positive probability, we have that

$$P[\cap_{i=1}^{n} A_i] = P[A_1]P[A_2|A_1]P[A_3|A_1 \cap A_2] \ldots P[A_n| \cap_{i=1}^{n} A_i]$$

Law of Total Probability (divide and conquer approach to computing the probability of an event)

- If $B_1, B_2, ..., B_m$ is an event space (in other words, given a partition $B_1, B_2, ..., B_m$) and $P[B_i] > 0$ for $i = 1, ..., m$, then

$$P[A] = \sum_{i=1}^{m} P[A|B_i]P[B_i]$$



Bayes' Theorem

- In many situations, we have advance information about $P[A|B]$ and need to calculate $P[B|A]$
- To do so, we have **Bayes' Theorem**:

$$P[B|A] = \frac{P[A|B]P[B]}{P[A]}$$

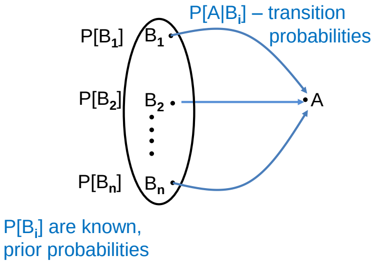
- **Proof:**

$$\frac{P[A|B]P[B]}{P[A]} = \frac{\frac{P[A \cap B]}{P[B]} \times P[B]}{P[A]} = \frac{P[A \cap B]}{P[A]}$$

Bayes' Theorem: Rules for Combining Evidence

- Suppose we are given
  1. "Prior" probabilities: $P[B_i]$
  2. "Transition" probabilities: $P[A|B_i]$ for each i
- Then
- For an event space (or partition) $B_1, B_2, \ldots, B_n$,

$$P[B_i|A] = \frac{P[A|B_i]P[B_i]}{P[A]} = \frac{P[A|B_i]P[B_i]}{\sum_{i=1}^{n} P[A|B_i]P[B_i]}$$



Radar Example Revisited

- Event $A=$ airplane is flying above
- Event $B=$ something registers on radar screen

$$\begin{aligned} P[airplane|register] &= P[A|B] \\ &= \frac{P[A]P[B|A]}{P[B]} \\ &= 0.34 \end{aligned}$$

$$\begin{aligned} P[B] = P[B|A]P[A] + P[B|A^c]P[A^c] &= 0.99 \times 0.05 + 0.10 \times 0.95 \\ &= 0.1445 \end{aligned}$$

Why so small when P[B|A]= 0.99? (probability of correct detection) *Probability of A^c and false alarm are quite high!*

Example: Decoding of Noise Corrupted Messages

- Prior probabilities: P[a]$=1/3$, P[b]$=2/3$
- Received 0001. What was transmitted?

$$P[a|0001] = \frac{P[a]P[0001|a]}{P[a]P[0001|a] + P[b]P[0001|b]}$$
$$P[b|0001] = \frac{P[b]P[0001|b]}{P[a]P[0001|a] + P[b]P[0001|b]}$$

- Take the most likely event

Summary: One Picture, Two Formulas

- Total probability theorem
- Bayes' theorem
  - "Reverse the direction" of conditioning (contrast with logical statements)
  - Often used to make inferences "most likely cause for a given effect", commonly applied in drugs testing, detection, ...

Sequential Experiments - Example

- Two coins, one biased, one fair, but you do not know which is which.
- Coin 1: P[H]$= 3/4$. Coin 2: P[H]$= 1/2$
- Pick a coin at random and flip it. Let C_i denote the event that coin i is picked. What is $P[C_1|H]$?

Solution: Tree Diagram

$$P[C_1|H] = \frac{P[C_1H]}{P[C_1H] + P[C_2H]} = \frac{3/8}{3/8 + 1/4} = 3/5$$