1	2	3	4	Total

Name:
Number:

KOM505E - Probability Theory and Stochastic Processes
Midterm #2

Dec 13, 2018

Rules:
- Closed book & notes.
- Write all answers within the frame given below the question.
- Each question is 25 pts.
- Duration: 90 min.

1. The covariance matrix of bivariate random variables X_1 and X_2 is:

$$C = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

Is this a valid covariance matrix? Explain.

For a valid covariance matrix
1) Diagonals should be nonnegative
2) Matrix should be symmetric
3) Matrix should be positive semidefinite

1 & 2 are satisfied. In order to show item 3, its eigenvalues should be nonnegative.

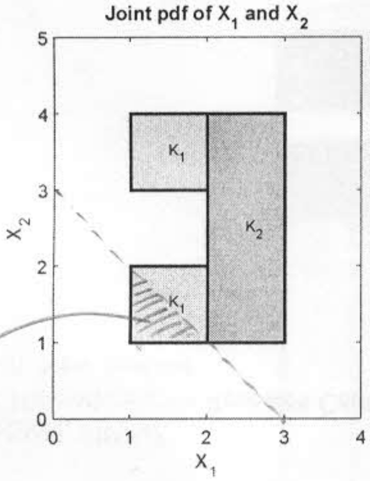
$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0$$

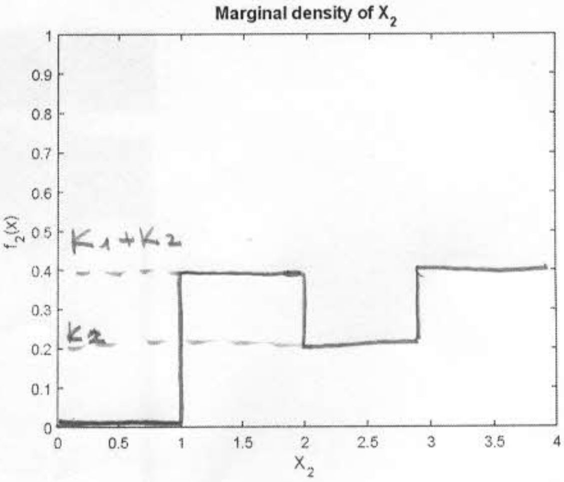
$$(1-\lambda)^2 - 9 = 0 \Rightarrow 1 - 2\lambda + \lambda^2 - 9 = 0$$
$$\lambda^2 - 2\lambda - 8 = 0$$
$$(\lambda+2)(\lambda-4) = 0$$
$$\lambda_1 = -2$$
$$\lambda_2 = 4$$

As one of the eigenvalues is negative, C is NOT a valid covariance matrix.

2. Consider two random variables X_1 and X_2 whose joint probability density function is given in the left figure below. In this figure, light and dark gray regions have flat values of K_1 and K_2 respectively.

$P(X_1+X_2 \leq 3)$





(a) Find the marginal density of X_2 in terms of K_1 and K_2, and draw it on the right figure above.

(b) Find the values of K_1 and K_2 if $F_2(1.5) = 0.35$, where F_2 is the cumulative distribution function of X_2.

The volume under $f(x_1, x_2) = 1$ and $F_2(1.5) = K_1 + K_2 = 0.35$

Hence: $2K_1 + 3K_2 = 1$

$K_1 + K_2 = 0.35$

$2K_1 + 2K_2 = 0.7 \Rightarrow K_2 = 0.3$

$K_1 = 0.05$

(c) Find probability of $X_1 + X_2 \leq 3$ in terms of K_1 and K_2. $P(X_1 + X_2 \leq 3)$=?

$P(X_1+X_2 \leq 3) = K_1/2$

$= 0.025$

(d) Find conditional probability of $X_1 \leq 2$ given that $X_2 \leq 2$ in terms of K_1 and K_2. $P(X_1 \leq 2 | X_2 \leq 2)$=?

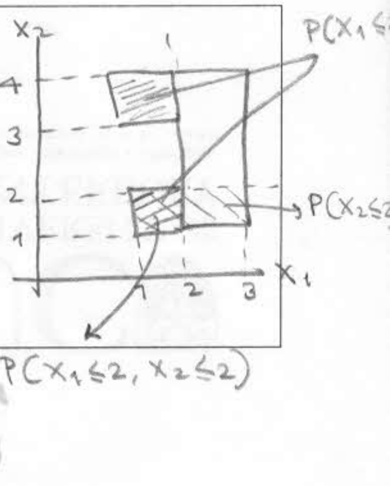
$$P(X_1 \leq 2 | X_2 \leq 2) = \frac{P(X_1 \leq 2, X_2 \leq 2)}{P(X_2 \leq 2)} = \frac{K_1}{K_1+K_2} = \frac{0.05}{0.35} = \frac{1}{7}$$

(e) Are X_1 and X_2 independent? Explain shortly.

$$\underbrace{P(X_1 \leq 2, X_2 \leq 2)}_{K_1} \stackrel{?}{=} \underbrace{P(X_1 \leq 2)}_{2K_1} \cdot \underbrace{P(X_2 \leq 2)}_{K_1+K_2}$$

$K_1 \neq 2K_1(K_1+K_2)$

Hence X_1 and X_2 are <u>not</u> independent.



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3. Let X and Y be random variables with joint probability density function of $f_{XY}(x, y)$. Consider another random variable Z that is obtained from X and Y as follows:

$$Z = X^2 + Y$$

Find the probability density function $f_Z(z)$.

Lets define $W = X$

$$J = \begin{vmatrix} \frac{\partial}{\partial x} z & \frac{\partial}{\partial y} \cancel{z} \\ \frac{\partial}{\partial x} w & \frac{\partial}{\partial y} w \end{vmatrix} = \begin{vmatrix} 2x & 1 \\ 1 & 0 \end{vmatrix} = 1$$

$X = W$

$Y = Z - X^2 = Z - W^2$

then

$$f_{Z,W}(z,w) = \frac{f_{XY}(w, z-w^2)}{|J|} = f_{XY}(w, z-w^2)$$

Lets find marginal density of z.

$$f_Z(z) = \int_w f_{Z,W}(z,w)\,dw$$

$$= \int_{x \in \mathcal{X}} f_{XY}(x, z-x^2)\,dx$$

where $\mathcal{X}$ is the support of X.

4. Consider two independent random variables X and Y which have both Gaussian probability density functions. Let $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$. Two new random variables are formed through the following linear transformation.

$$\begin{bmatrix} Z \\ W \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

Find the joint probability density function of the new random variables Z and W, $f_{ZW}(z,w) = ?$

Hint: Z and W will have bivariate Gaussian distribution as linear transformations do not change the type of distributions. Hence, it will suffice to find the mean and covariance matrix of the new random variables.

$$\begin{bmatrix} z \\ w \end{bmatrix} = G \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow G = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

From lectures

Mean $\begin{bmatrix} \mu_z \\ \mu_w \end{bmatrix} = G \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} = \begin{bmatrix} \mu_x - 2\mu_y \\ 2\mu_x + \mu_y \end{bmatrix}$

Covariance matrix

$$\Sigma_{z,w} = G^T \Sigma_{x,y} G$$

as X & Y are independent, they are also uncorrelated.

Hence $\Sigma_{xy} = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix}$

$$\Sigma_{z,w} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \sigma_x^2 & -2\sigma_x^2 \\ 2\sigma_y^2 & \sigma_y^2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_x^2 + 4\sigma_y^2 & -2\sigma_x^2 + 2\sigma_y^2 \\ -2\sigma_x^2 + 2\sigma_y^2 & 4\sigma_x^2 + \sigma_y^2 \end{bmatrix}$$

Then

$$f_{z,w}(z,w) = \frac{1}{2\pi |\Sigma_{zw}|} \exp\left\{ -\frac{1}{2} \left(\begin{bmatrix} z \\ w \end{bmatrix} - \begin{bmatrix} \mu_z \\ \mu_w \end{bmatrix} \right)^T \Sigma_{zw}^{-1} \left(\begin{bmatrix} z \\ w \end{bmatrix} - \begin{bmatrix} \mu_z \\ \mu_w \end{bmatrix} \right) \right\}$$