MAT 271E Probability and Statistics

Homework 2 Solutions

Assigned: February 19, 2012

Due: February 22, 2012 (in class, before class starts)

No late homework will be accepted!

Do not copy from solutions from your classmates. All work must be your own!

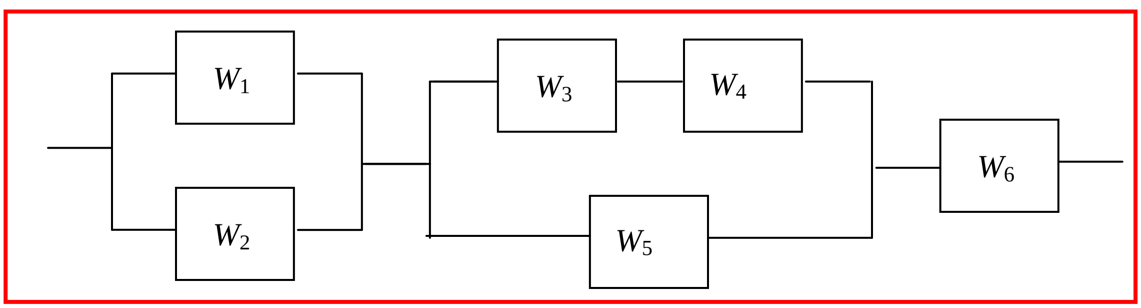
Show all your work!

Read: "Probability and Stochastic Processes", Yates and Goodman, Ch. 2

1) A particular operation has six components. Each component has a failure probability q, independent of any other component. The operation is successful if and only if all three hold:

- Component 1 or component 2 works.
- Components 3 and 4 both work or component 5 works.
- Component 6 works.

a) Sketch a block diagram for this operation.



b) What is the probability $P[W]$ that the operation is successful?

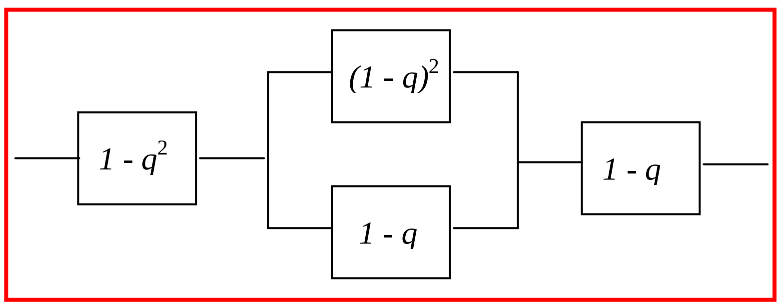
We replace parallel devices 1 and 2 with a single device labeled with the probability that it works. In particular,

$$P[W_1 \cup W_2] = 1 - P[W_1^c W_2^c] = 1 - q^2$$

We replace series devices 3 and 4 with a single device labeled with the probability that it works. In particular,

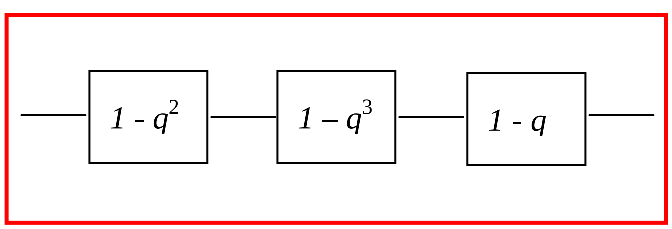
$$P[W_3 W_4] = (1 - q)^2$$

This yields a composite device of the form:



We replace parallel devices in the above figure with a single device labeled with the probability that it works. The probability P[*parallel*], that the two devices in parallel work is 1 minus the probability that neither works. In particular,

$$P[\textit{parallel}] = 1 - [(1 - (1 - q^2))q] = 1 - q^3$$



Finally, for the device to work, all three composite devices in series must work. Thus, the probability the device works is:

$$P[W] = [1 - q^2]\,[1 - q^3]\,[1 - q]$$

2) Assume that people try to access your website five times and then give up. You want to make sure that you are able to serve at least 98% of the people who want to access the site. Let *p* be probability that a person accesses your site. What should be the smallest value for *p*?

The probability that a person fails to access your website in five tries is $(1 - p)^5$. To be sure that at least 98% of people who want to access your website can do so, we need $(1 - p)^5 \leq 0.02$. Taking the natural logarithm of both sides,

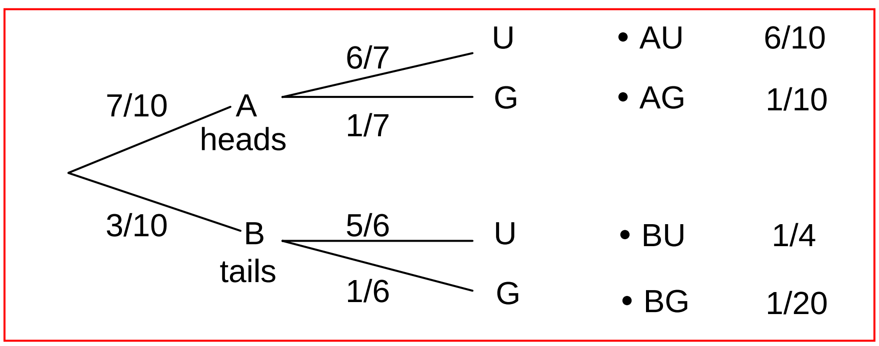
$5 \ln (1 - p) \leq \ln 0.02$

$1 - p \leq e^{(\ln 0.02)/5}$

This implies $p = 0.54$.

3) Assume that we have two classes that both undergraduate and graduate students are allowed to take. Class A contains 30 undergraduate students and 5 graduate students, while Class B contains 40 undergraduate and 8 graduate students. We toss a biased coin that has a 0.7 probability of turning up heads. If the result of the toss is heads, we pick a student from Class A; if it is tails, we pick a student from Class B. Find the probability that a graduate student is chosen.

The tree diagram for this experiment is shown below:

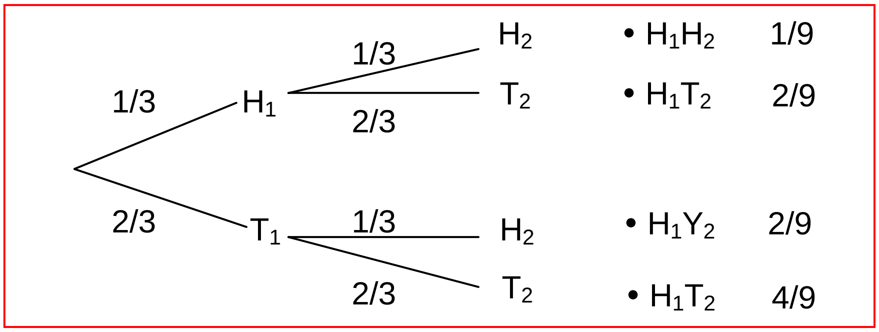


Let A represent the event of "class A being chosen" (as a result of heads coming up). Let B represent the event of "class B being chosen" (as a result of tails coming up). Let U and G represent the events of an undergraduate student and a graduate student being chosen respectively. Since a graduate student can be chosen from class A or class B, applying the law of total probability, the probability of a graduate student being chosen is

$$P[G] = P[G|A]P[A] + P[G|B]P[B] = \left(\frac{5}{30+5}\right)\left(\frac{7}{10}\right) + \left(\frac{8}{40+8}\right)\left(\frac{3}{10}\right) = \frac{3}{20}$$

4) Suppose you flip a coin twice. On any flip, the coin comes up heads with probability 1/3. Use H_i and T_i to denote the result of flip i.

A sequential sample space for this experiment is:



a) What is the probability, P[T_1 | T_2], that the first flip is tails given that the second flip is tails?

From the tree diagram, we observe

$$P[T_2] = P[H_1T_2] + P[T_1T_2] = 2/9 + 4/9 = 6/9 = 2/3$$

This implies

$$P[T_1|T_2] = \frac{P[T_1T_2]}{P[T_2]} = \frac{4/9}{2/3} = \frac{2}{3}$$

b) What is the probability that the first flip is tails and the second flip is heads?

From the tree diagram, the probability that the first flip is tails and the second flip is heads is

$$P[T_1H_2] = 2/9.$$

5) Assume you are allowed to use only six numbers to create your PIN for the student registration system: 1, 2, 3, 4, 5, and 6.
How many five-digit PINs are possible?

Each digit can take on any one of the 6 possible numbers.

Number of choices

6 6 6 6 6

The number of five-digit PINs that can be formed is

$6 \times 6 \times 6 \times 6 \times 6 = 6^5 = 7776$.

How many six-digit PINs are possible if each number appears only once in each PIN?

6 5 4 3 2 1

If we allow each number to appear only once, then we have 6 choices for the first digit, 5 choices for the second, 4 choices for the third, etc.
Therefore, we can form $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720$ six-digit PINs.

6) Consider 6 tosses of a coin. A possible sequence of outcomes is HTHTHT. Assume that the probability of getting heads is 0.6 and each toss is independent.

a) What is the probability of the sequence THTTHT?

Since the probability of a toss turning up heads is 0.6, $p_H = 0.6$ and $p_T = 0.4$. We can express the sequence THTTHT as the appearance of H twice and the appearance of T four times:

$$\underbrace{\text{THTTHT}}_{\text{Two Hs, four Ts}}$$

Thus,

$$P[\text{THTTHT}] = (0.6)^2(0.4)^4 = 0.009216$$

b) What is the probability that a sequence contains exactly three tails?

$$P[\text{three Ts}] = \binom{6}{3}(0.6)^3(0.4)^3 = \frac{6!}{3!(6-3)!}(0.6)^3(0.4)^3 = 0.276$$