SEN301 OPERATIONS RESEARCH I
LECTURE NOTES
(2013-2014)

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We retain responsibility for all errors and would love to hear from visitors of this site!
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1. INTRODUCTION TO OPERATONS RESEARH

1.1 TERMINOLOGY

The British/Europeans refer to "operational research", the Americans to "operations research" - but both are often shortened to just "OR" (which is the term we will use).

Another term which is used for this field is "management science" ("MS"). The Americans sometimes combine the terms OR and MS together and say "***OR/MS***" or "ORMS".

Yet other terms sometimes used are "industrial engineering" ("IE"), "decision science" ("DS"), and “problem solving”.

In recent years there has been a move towards a standardization upon a single term for the field, namely the term "OR".

“Operations Research (Management Science) is a scientific approach to decision making that seeks to best design and operate a system, usually under conditions requiring the allocation of scarce resources.”

A system is an organization of interdependent components that work together to accomplish the goal of the system.

1.2 THE METHODOLOGY OF OR

When OR is used to solve a problem of an organization, the following seven step procedure should be followed:

Step 1. Formulate the Problem

OR analyst first defines the organization's problem. Defining the problem includes specifying the organization's objectives and the parts of the organization (or system) that must be studied before the problem can be solved.

Step 2. Observe the System

Next, the analyst collects data to estimate the values of parameters that affect the organization's problem. These estimates are used to develop (in Step 3) and evaluate (in Step 4) a mathematical model of the organization's problem.

Step 3. Formulate a Mathematical Model of the Problem

The analyst, then, develops a mathematical model (in other words an idealized representation) of the problem. In this class, we describe many mathematical techniques that can be used to model systems.

Step 4. Verify the Model and Use the Model for Prediction

The analyst now tries to determine if the mathematical model developed in Step 3 is an accurate representation of reality. To determine how well the model fits reality, one determines how valid the model is for the current situation.

Step 5. Select a Suitable Alternative

Given a model and a set of alternatives, the analyst chooses the alternative (if there is one) that best meets the organization's objectives.

Sometimes the set of alternatives is subject to certain restrictions and constraints. In many situations, the best alternative may be impossible or too costly to determine.

Step 6. Present the Results and Conclusions of the Study

In this step, the analyst presents the model and the recommendations from Step 5 to the decision making individual or group. In some situations, one might present several alternatives and let the organization choose the decision maker(s) choose the one that best meets her/his/their needs.

After presenting the results of the OR study to the decision maker(s), the analyst may find that s/he does not (or they do not) approve of the recommendations. This may result from incorrect definition of the problem on hand or from failure to involve decision maker(s) from the start of the project. In this case, the analyst should return to Step 1, 2, or 3.

Step 7. Implement and Evaluate Recommendation

If the decision maker(s) has accepted the study, the analyst aids in implementing the recommendations. The system must be constantly monitored (and updated dynamically as the environment changes) to ensure that the recommendations are enabling decision maker(s) to meet her/his/their objectives.

1.3 HISTORY OF OR

(Prof. Beasley’s lecture notes)

OR is a relatively new discipline. Whereas 70 years ago it would have been possible to study mathematics, physics or engineering (for example) at university it would not have been possible to study OR, indeed the term OR did not exist then. It was only really in the late 1930's that operational research began in a systematic fashion, and it started in the UK.

Early in 1936 the British Air Ministry established Bawdsey Research Station, on the east coast, near Felixstowe, Suffolk, as the centre where all pre-war radar experiments for both the Air Force and the Army would be carried out. Experimental radar equipment was brought up to a high state of reliability and ranges of over 100 miles on aircraft were obtained.

It was also in 1936 that Royal Air Force (RAF) Fighter Command, charged specifically with the air defense of Britain, was first created. It lacked however any effective fighter aircraft - no Hurricanes or Spitfires had come into service - and no radar data was yet fed into its very elementary warning and control system.

It had become clear that radar would create a whole new series of problems in fighter direction and control so in late 1936 some experiments started at Biggin Hill in Kent into the effective use of such data. This early work, attempting to integrate radar data with ground based observer data for fighter interception, was the start of OR.

The first of three major pre-war air-defense exercises was carried out in the summer of 1937. The experimental radar station at Bawdsey Research Station was brought into operation and the information derived from it was fed into the general air-defense warning and control system. From the early warning point of view this exercise was encouraging, but the tracking information obtained from radar, after filtering and transmission through the control and display network, was not very satisfactory.

In July 1938 a second major air-defense exercise was carried out. Four additional radar stations had been installed along the coast and it was hoped that Britain now had an aircraft location and control system greatly improved both in coverage and effectiveness. Not so! The exercise revealed, rather, that a new and serious problem had arisen. This was the need to coordinate and correlate the additional, and often conflicting, information received from the additional radar stations. With the out-break of war apparently imminent, it was obvious that something new - drastic if necessary - had to be attempted. Some new approach was needed.

Accordingly, on the termination of the exercise, the Superintendent of Bawdsey Research Station, A.P. Rowe, announced that although the exercise had again demonstrated the technical feasibility of the radar system for detecting aircraft, its operational achievements still fell far short of requirements. He therefore proposed that a crash program of research into the operational - as opposed to the technical -

aspects of the system should begin immediately. The term "operational research" [RESEARCH into (military) OPERATIONS] was coined as a suitable description of this new branch of applied science. The first team was selected from amongst the scientists of the radar research group the same day.

In the summer of 1939 Britain held what was to be its last pre-war air defense exercise. It involved some 33,000 men, 1,300 aircraft, 110 antiaircraft guns, 700 searchlights, and 100 barrage balloons. This exercise showed a great improvement in the operation of the air defense warning and control system. The contribution made by the OR teams was so apparent that the Air Officer Commander-in-Chief RAF Fighter Command (Air Chief Marshal Sir Hugh Dowding) requested that, on the outbreak of war, they should be attached to his headquarters at Stanmore.

On May 15th 1940, with German forces advancing rapidly in France, Stanmore Research Section was asked to analyze a French request for ten additional fighter squadrons (12 aircraft a squadron) when losses were running at some three squadrons every two days. They prepared graphs for Winston Churchill (the British Prime Minister of the time), based upon a study of current daily losses and replacement rates, indicating how rapidly such a move would deplete fighter strength. No aircraft were sent and most of those currently in France were recalled.

This is held by some to be the most strategic contribution to the course of the war made by OR (as the aircraft and pilots saved were consequently available for the successful air defense of Britain, the Battle of Britain).

In 1941 an Operational Research Section (ORS) was established in Coastal Command which was to carry out some of the most well-known OR work in World War II.

Although scientists had (plainly) been involved in the hardware side of warfare (designing better planes, bombs, tanks, etc) scientific analysis of the operational use of military resources had never taken place in a systematic fashion before the Second World War. Military personnel, often by no means stupid, were simply not trained to undertake such analysis.

These early OR workers came from many different disciplines, one group consisted of a physicist, two physiologists, two mathematical physicists and a surveyor. What such people brought to their work were "scientifically trained" minds, used to querying assumptions, logic, exploring hypotheses, devising experiments, collecting data, analyzing numbers, etc. Many too were of high intellectual caliber (at least four wartime OR personnel were later to win Nobel prizes when they returned to their peacetime disciplines).

By the end of the war OR was well established in the armed services both in the UK and in the USA.

OR started just before World War II in Britain with the establishment of teams of scientists to study the strategic and tactical problems involved in military operations. The objective was to find the most effective utilization of limited military resources by the use of quantitative techniques.

Following the end of the war OR spread, although it spread in different ways in the UK and USA.

You should be clear that the growth of OR since it began (and especially in the last 30 years) is, to a large extent, the result of the increasing power and widespread availability of computers. Most (though not all) OR involves carrying out a large number of numeric calculations. Without computers this would simply not be possible.

2. BASIC OR CONCEPTS

"OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing."

We can also define a mathematical model as consisting of:

- *Decision variables*, which are the unknowns to be determined by the solution to the model.
- *Constraints* to represent the physical limitations of the system
- An *objective* function
- An *optimal solution* to the model is the identification of a set of variable values which are feasible (satisfy all the constraints) and which lead to the optimal value of the objective function.

An optimization model seeks to find values of the decision variables that optimize (maximize or minimize) an objective function among the set of all values for the decision variables that satisfy the given constraints.

Two Mines Example

The Two Mines Company own two different mines that produce an ore which, after being crushed, is graded into three classes: high, medium and low-grade. The company has contracted to provide a smelting plant with 12 tons of high-grade, 8 tons of medium-grade and 24 tons of low-grade ore per week. The two mines have different operating characteristics as detailed below.

Mine	**Cost per day (£'000)**	**Production (tons/day)**		
		Hgh	**Medium**	**Low**
X	180	6	3	4
Y	160	1	1	6

Consider that mines cannot be operated in the weekend. How many days per week should each mine be operated to fulfill the smelting plant contract?

Guessing

To explore the Two Mines problem further we might simply guess (i.e. use our judgment) how many days per week to work and see how they turn out.

- work one day a week on *X*, one day a week on *Y*

This does not seem like a good guess as it results in only 7 tones a day of high-grade, insufficient to meet the contract requirement for 12 tones of high-grade a day. We say that such a solution is *infeasible*.

- work 4 days a week on *X*, 3 days a week on *Y*

This seems like a better guess as it results in sufficient ore to meet the contract. We say that such a solution is *feasible*. However it is quite expensive (costly).

We would like a solution which supplies what is necessary under the contract at minimum cost. Logically such a minimum cost solution to this decision problem must exist. However even if we keep guessing we can never be sure whether we have found this minimum cost solution or not. Fortunately our structured approach will enable us to find the minimum cost solution.

Solution

What we have is a verbal description of the Two Mines problem. What we need to do is to translate that verbal description into an *equivalent* mathematical description.

In dealing with problems of this kind we often do best to consider them in the order:

- Variables
- Constraints
- Objective

This process is often called *formulating* the problem (or more strictly formulating a mathematical representation of the problem).

Variables

These represent the "decisions that have to be made" or the "unknowns".

We have two decision variables in this problem:

x = number of days per week mine *X* is operated

y = number of days per week mine *Y* is operated

Note here that $x \geq 0$ and $y \geq 0$.

Constraints

It is best to first put each constraint into words and then express it in a mathematical form.

ore production constraints - balance the amount produced with the quantity required under the smelting plant contract

Ore	
High	$6x + 1y \geq 12$
Medium	$3x + 1y \geq 8$
Low	$4x + 6y \geq 24$

days per week constraint - we cannot work more than a certain maximum number of days a week e.g. for a 5 day week we have

$x \leq 5$

$y \leq 5$

Inequality constraints

Note we have an inequality here rather than an equality. This implies that we may produce more of some grade of ore than we need. In fact we have the general rule: given a choice between an equality and an inequality choose the inequality

For example - if we choose an equality for the ore production constraints we have the three equations $6x+y=12$, $3x+y=8$ and $4x+6y=24$ and there are no values of x and y which satisfy all three equations (the problem is therefore said to be "over-constrained"). For example the values of x and y which satisfy $6x+y=12$ and $3x+y=8$ are $x=4/3$ and $y=4$, but these values do not satisfy $4x+6y=24$.

The reason for this general rule is that choosing an inequality rather than an equality gives us more flexibility in optimizing (maximizing or minimizing) the objective (deciding values for the decision variables that optimize the objective).

Implicit constraints

Constraints such as days per week constraint are often called implicit constraints because they are implicit in the definition of the variables.

Objective

Again in words our objective is (presumably) to minimize cost which is given by

$180x + 160y$

Hence we have the ***complete mathematical representation*** of the problem:

minimize

$$180x + 160y$$

subject to

$$6x + y \geq 12$$
$$3x + y \geq 8$$
$$4x + 6y \geq 24$$
$$x \leq 5$$
$$y \leq 5$$
$$x, y \geq 0$$

Some notes

The mathematical problem given above has the form

- all variables continuous (i.e. can take fractional values)
- a single objective (maximize or minimize)
- the objective and constraints are linear i.e. any term is either a constant or a constant multiplied by an unknown (e.g. 24, $4x$, $6y$ are linear terms but xy or x^2 is a non-linear term)

Any formulation which satisfies these three conditions is called a *linear program* (LP).

We have (implicitly) assumed that it is permissible to work in fractions of days - problems where this is not permissible and variables must take integer values will be dealt with under *Integer Programming* (IP).

Discussion

This problem was a *decision problem.*

We have taken a real-world situation and constructed an equivalent mathematical representation - such a representation is often called a mathematical *model* of the real-world situation (and the process by which the model is obtained is called *formulating* the model).

Just to confuse things the mathematical model of the problem is sometimes called the *formulation* of the problem.

Having obtained our mathematical model we (hopefully) have some quantitative method which will enable us to numerically solve the model (i.e. obtain a numerical solution) - such a quantitative method is often called an *algorithm* for solving the model.

Essentially an algorithm (for a particular model) is a set of instructions which, when followed in a step-by-step fashion, will produce a numerical solution to that model.

Our model has an *objective*, that is something which we are trying to *optimize*.

Having obtained the numerical solution of our model we have to translate that solution back into the real-world situation.

"OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing."

3. FORMULATING LINEAR PROGRAMS

It can be recalled from the Two Mines example that the conditions for a mathematical model to be a linear program (LP) were:

- all variables continuous (i.e. can take fractional values)
- a single objective (minimize or maximize)
- the objective and constraints are linear i.e. any term is either a constant or a constant multiplied by an unknown.

LP's are important - this is because:

- many practical problems can be formulated as LP's
- there exists an algorithm (called the *simplex* algorithm) which enables us to solve LP's numerically relatively easily

We will return later to the simplex algorithm for solving LP's but for the moment we will concentrate upon formulating LP's.

Some of the major application areas to which LP can be applied are:

- Work scheduling
- Production planning & Production process
- Capital budgeting
- Financial planning
- Blending (e.g. Oil refinery management)
- Farm planning
- Distribution
- Multi-period decision problems
  - Inventory model
  - Financial models
  - Work scheduling

Note that the key to formulating LP's is practice. However a useful hint is that common objectives for LP's are maximize profit/minimize cost.

There are four basic assumptions in LP:

- Proportionality
    - The contribution to the objective function from each decision variable is proportional to the value of the decision variable (The contribution to the objective function from making four soldiers (4×\$3=\$12) is exactly four times the contribution to the objective function from making one soldier (\$3))
    - The contribution of each decision variable to the LHS of each constraint is proportional to the value of the decision variable (It takes exactly three times as many finishing hours (2hrs×3=6hrs) to manufacture three soldiers as it takes to manufacture one soldier (2 hrs))
- Additivity
    - The contribution to the objective function for any decision variable is independent of the values of the other decision variables (No matter what the value of train (x_2), the manufacture of soldier (x_1) will always contribute $3x_1$ dollars to the objective function)
- The contribution of a decision variable to LHS of each constraint is independent of the values of other decision variables (No matter what the value of x_1, the manufacture of x_2 uses x_2 finishing hours and x2 carpentry hours)
    - *1st implication*: The value of objective function is the sum of the contributions from each decision variables.
    - *2nd implication*: LHS of each constraint is the sum of the contributions from each decision variables.
- Divisibility
    - Each decision variable is allowed to assume fractional values. If we actually can not produce a fractional number of decision variables, we use IP (It is acceptable to produce 1.69 trains)
- Certainty
    - Each parameter is known with certainty

3.1 LINEAR PROGRAMMING EXAMPLES

3.1.1 Giapetto Example

(Winston 3.1, p. 49)

Giapetto's wooden soldiers and trains. Each soldier sells for $27, uses $10 of raw materials and takes $14 of labor & overhead costs. Each train sells for $21, uses $9 of raw materials, and takes $10 of overhead costs. Each soldier needs 2 hours finishing and 1 hour carpentry; each train needs 1 hour finishing and 1 hour carpentry. Raw materials are unlimited, but only 100 hours of finishing and 80 hours of carpentry are available each week. Demand for trains is unlimited; but at most 40 soldiers can be sold each week. How many of each toy should be made each week to maximize profits?

Answer

Decision variables completely describe the decisions to be made (in this case, by Giapetto). Giapetto must decide how many soldiers and trains should be manufactured each week. With this in mind, we define:

x_1 = the number of soldiers produced per week

x_2 = the number of trains produced per week

Objective function is the function of the decision variables that the decision maker wants to maximize (revenue or profit) or minimize (costs). Giapetto can concentrate on maximizing the total weekly profit (z).

Here profit equals to (weekly revenues) – (raw material purchase cost) – (other variable costs). Hence Giapetto’s objective function is:

Maximize $z = 3x_1 + 2x_2$

Constraints show the restrictions on the values of the decision variables. Without constraints Giapetto could make a large profit by choosing decision variables to be very large. Here there are three constraints:

Finishing time per week

Carpentry time per week

Weekly demand for soldiers

Sign restrictions are added if the decision variables can only assume nonnegative values (Giapetto can not manufacture negative number of soldiers or trains!)

All these characteristics explored above give the following ***Linear Programming*** (LP) model

$\max z = 3x_1 + 2x_2$ (The Objective function)

s.t. $2x_1 + x_2 \leq 100$ (Finishing constraint)

$x_1 + x_2 \leq 80$ (Carpentry constraint)

$x_1 \leq 40$ (Constraint on demand for soldiers)

$x_1, x_2 \geq 0$ (Sign restrictions)

A value of (x_1, x_2) is in the ***feasible region*** if it satisfies all the constraints and sign restrictions.

Graphically and computationally we see the solution is (x_1, x_2) = (20, 60) at which z = 180. (***Optimal solution***)

Report

The maximum profit is $180 by making 20 soldiers and 60 trains each week. Profit is limited by the carpentry and finishing labor available. Profit could be increased by buying more labor.

3.1.2 Advertisement Example

(Winston 3.2, p.61)

Dorian makes luxury cars and jeeps for high-income men and women. It wishes to advertise with 1 minute spots in comedy shows and football games. Each comedy spot costs $50K and is seen by 7M high-income women and 2M high-income men. Each football spot costs $100K and is seen by 2M high-income women and 12M high-income men. How can Dorian reach 28M high-income women and 24M high-income men at the least cost?

Answer

The decision variables are

x_1 = the number of comedy spots

x_2 = the number of football spots

The model of the problem:

$$\begin{aligned} &\min z = 50x_1 + 100x_2 \\ &\text{st} \quad 7x_1 + 2x_2 \geq 28 \\ &\quad\quad 2x_1 + 12x_2 \geq 24 \\ &\quad x_1, x_2 \geq 0 \end{aligned}$$

The graphical solution is z = 320 when (x_1, x_2) = (3.6, 1.4). From the graph, in this problem rounding up to (x_1, x_2) = (4, 2) gives the best *integer* solution.

Report

The minimum cost of reaching the target audience is $400K, with 4 comedy spots and 2 football slots. The model is dubious as it does not allow for saturation after repeated viewings.

3.1.3 Diet Example

(Winston 3.4., p. 70)

Ms. Fidan's diet requires that all the food she eats come from one of the four "basic food groups". At present, the following four foods are available for consumption: brownies, chocolate ice cream, cola, and pineapple cheesecake. Each brownie costs 0.5$, each scoop of chocolate ice cream costs 0.2$, each bottle of cola costs 0.3$, and each pineapple cheesecake costs 0.8$. Each day, she must ingest at least 500 calories, 6 oz of chocolate, 10 oz of sugar, and 8 oz of fat. The nutritional content per unit of each food is shown in Table. Formulate an LP model that can be used to satisfy her daily nutritional requirements at minimum cost.

	Calories	Chocolate (ounces)	Sugar (ounces)	Fat (ounces)
Brownie	400	3	2	2
Choc. ice cream (1 scoop)	200	2	2	4
Cola (1 bottle)	150	0	4	1
Pineapple cheesecake (1 piece)	500	0	4	5

Answer

The decision variables:

x_1: number of brownies eaten daily

x_2: number of scoops of chocolate ice cream eaten daily

x_3: bottles of cola drunk daily

x_4: pieces of pineapple cheesecake eaten daily

The objective function (the total cost of the diet in cents):

$$\min w = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

Constraints:

$$400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 \quad \text{(daily calorie intake)}$$

$$3x_1 + 2x_2 \geq 6 \quad \text{(daily chocolate intake)}$$

$$2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10 \quad \text{(daily sugar intake)}$$

$$
\begin{array}{llllll}
2x_1 + & 4x_2 + & x_3 + & 5x_4 & \geq & 8 \qquad \text{(daily fat intake)} \\
\multicolumn{6}{l}{x_i \geq 0, \; i = 1, 2, 3, 4 \qquad \text{(Sign restrictions!)}}
\end{array}
$$

Report

The minimum cost diet incurs a daily cost of 90 cents by eating 3 scoops of chocolate and drinking 1 bottle of cola ($w = 90$, $x_2 = 3$, $x_3 = 1$)

3.1.4 Post Office Example

(Winston 3.5, p.74)

A PO requires different numbers of employees on different days of the week. Union rules state each employee must work 5 consecutive days and then receive two days off. Find the minimum number of employees needed.

	Mon	Tue	Wed	Thur	Fri	Sat	Sun
Staff Needed	17	13	15	19	14	16	11

Answer

The decision variables are x_i (# of employees starting on day i)

Mathematically we must

$$
\begin{array}{lllllllllll}
\min z = & x_1 & + x_2 & + x_3 & + x_4 & + x_5 & + x_6 & + x_7 & & \\
 & x_1 & & & + x_4 & + x_5 & + x_6 & + x_7 & \geq & 17 \\
 & x_1 & + x_2 & & & + x_5 & + x_6 & + x_7 & \geq & 13 \\
 & x_1 & + x_2 & + x_3 & & & + x_6 & + x_7 & \geq & 15 \\
 & x_1 & + x_2 & + x_3 & + x_4 & & & + x_7 & \geq & 19 \\
 & x_1 & + x_2 & + x_3 & + x_4 & + x_5 & & & \geq & 14 \\
 & & + x_2 & + x_3 & + x_4 & + x_5 & + x_6 & & \geq & 16 \\
 & & & + x_3 & + x_4 & + x_5 & + x_6 & + x_7 & \geq & 11 \\
\multicolumn{3}{l}{x_t \geq 0, \forall t}
\end{array}
$$

The solution is (x_i) = (4/3, 10/3, 2, 22/3, 0, 10/3, 5) giving $z = 67/3$.

We could round this up to (x_i) = (2, 4, 2, 8, 0, 4, 5) giving $z = 25$ (may be wrong!).

However restricting the decision var.s to be integers and using Lindo again gives (x_i) = (4, 4, 2, 6, 0, 4, 3) giving $z = 23$.

3.1.5 Sailco Example

(Winston 3.10, p. 99)

Sailco must determine how many sailboats to produce in the next 4 quarters. The demand is known to be 40, 60, 75, and 25 boats. Sailco must meet its demands. At the beginning of the 1st quarter Sailco starts with 10 boats in inventory. Sailco can produce up to 40 boats with regular time labor at \$400 per boat, or additional boats at

$450 with overtime labor. Boats made in a quarter can be used to meet that quarter's demand or held in inventory for the next quarter at an extra cost of $20.00 per boat.

Answer

The decision variables are for $t = 1,2,3,4$

x_t = # of boats in quarter *t* built in regular time

y_t = # of boats in quarter *t* built in overtime

For convenience, introduce variables:

i_t = # of boats in inventory at the end quarter *t*

d_t = demand in quarter *t*

We are given that $d_1 = 40, d_2 = 60, d_3 = 75, d_4 = 25, i_0 = 10$

$x_t \leq 40, \forall t$

By logic $i_t = i_{t-1} + x_t + y_t - d_t, \forall t.$

Demand is met iff $i_t \geq 0, \forall t$

(Sign restrictions $x_t, y_t \geq 0, \forall t$)

We need to minimize total cost z subject to these three sets of conditions where

$$z = 400\,(x_1 + x_2 + x_3 + x_4) + 450\,(y_1 + y_2 + y_3 + y_4) + 20\,(i_1 + i_2 + i_3 + i_4)$$

Report:

Lindo reveals the solution to be $(x_1, x_2, x_3, x_4) = (40, 40, 40, 25)$ and $(y_1, y_2, y_3, y_4) = (0, 10, 35, 0)$ and the minimum cost of $78450.00 is achieved by the schedule

		Q_1	Q_2	Q_3	Q_4
Regular time (x_t)		40	40	40	25
Overtime (y_t)		0	10	35	0
Inventory (i_t)	10	10	0	0	0
Demand (d_t)		40	60	75	25

3.1.6 Customer Service Level Example

(Winston 3.12, p. 108)

CSL services computers. Its demand (hours) for the time of skilled technicians in the next 5 months is

t	Jan	Feb	Mar	Apr	May
d_t	6000	7000	8000	9500	11000

It starts with 50 skilled technicians at the beginning of January. Each technician can work 160 hrs/month. To train a new technician they must be supervised for 50 hrs by an experienced technician for a period of one month time. Each experienced technician is paid \$2K/mth and a trainee is paid \$1K/mth. Each month 5% of the skilled technicians leave. CSL needs to meet demand and minimize costs.

Answer

The decision variable is

x_t = # to be trained in month t

We must minimize the total cost. For convenience let

y_t = # experienced tech. at start of t^{th} month

d_t = demand during month t

Then we must

min $z = 2000\,(y_1+...+ y_5) + 1000\,(x_1 +...+ x_5)$

subject to

$160y_t - 50x_t \geq d_t$ for $t = 1,..., 5$

$y_1 = 50$, $d_1 = 6000$, $d_2 = 7000$, $d_3 = 8000$, $d_4 = 9500$, $d_5 = 11000$

$y_t = .95y_{t-1} + x_{t-1}$ for $t = 2,3,4,5$

$x_t, y_t \geq 0$

3.1.7 Oil Blending Example

(Based on Winston 3.8)

Sunco manufactures three types of gasoline (G1, G2, G3). Each type is produced by blending three types of crude oil (C1, C2, C3). Octane rating and sulfur content should met certain standards:

- G1: average octane rating of at least 10, at most 2% sulfur
- G2: average octane rating of at least 8, at most 4% sulfur
- G3: average octane rating of at least 6, at most 3% sulfur

Sunco's customers require the following amounts of each gasoline, respectively: 3000, 2000, and 1000 barrels (daily demand). Each dollar spent daily in advertising a particular type of gas increases the daily demand for that type of gas by 10 barrels.

The sales price per barrel of gasoline and the purchase price per barrel of crude oil as well as the octane rating and the sulfur content of the three types of oil are given in the following table. Formulate an LP that would maximize the profit of Sunco.

Crude oil	Octane	Sulfur (%)	Purch. price (\$/barrel)
1	12	1	45
2	6	3	35
3	8	5	25

Gasoline	Selling price (\$/barrel)
1	70
2	60
3	50

Answer

Decision variables

x_{ij} : barrels of crude oil *i* used daily to produce gas *j*, *i*=1,2,3; *j*=1,2,3.

a_j : dollars spent daily on advertising gas *j* ($), *j*=1,2,3.

Objective function (maximizing profit)

Max Z = Profit = revenue – cost

Max Z = $(70\sum_i x_{i1} + 60\sum_i x_{i2} + 50\sum_i x_{i3}) - \left(45\sum_j x_{1j} + 35\sum_j x_{2j} + 25\sum_j x_{3j}\right) - \sum_j a_j$

Constraints

Octane rating

$(12x_{11} + 6x_{21} + 8x_{31})/(x_{11} + x_{21} + x_{31}) \geq 10$ →

$12x_{11} + 6x_{21} + 8x_{31} \geq 10(x_{11} + x_{21} + x_{31})$ octane rating for G1

$12x_{12} + 6x_{22} + 8x_{32} \geq 8(x_{12} + x_{22} + x_{32})$ octane rating for G2

$12x_{13} + 6x_{23} + 8x_{33} \geq 6\ (x_{13} + x_{23} + x_{33})$ octane rating for G3

Sulfur content

$(.01x_{11} + .03x_{21} + .05x_{31})/(x_{11} + x_{21} + x_{31}) \leq .02$ →

$x_{11} + 3x_{21} + 5x_{31} \leq 2\ (x_{11} + x_{21} + x_{31})$ sulfur content for G1

$x_{12} + 3x_{22} + 5x_{32} \leq 4\ (x_{12} + x_{22} + x_{32})$ sulfur content for G2

$x_{13} + 3x_{23} + 5x_{33} \leq 3\ (x_{13} + x_{23} + x_{33})$ sulfur content for G3

Demands

$\sum_i x_{ij} \leq D_j + 10a_j \quad \forall j.$ (D_j : demand for gas *j* without advertisement)

Sign restrictions

x_{ij} , $a_j \geq 0$, $\forall i,j$.

3.2 ADDING ABSOLUTE VALUES TO LP FORMULATION

3.2.1 Formulation

Functions involving absolute values are not linear but can be formulated in an LP. Consider a model that includes the absolute value of the function $f(x_1, x_2,\ldots, x_n)$ (i.e., $|f(x_1, x_2,\ldots, x_n)|$). In order to formulate this function in the LP, a decision variable λ is defined and following constraints are added to the LP:

$\lambda \geq f(x_1, x_2,\ldots, x_n)$

$\lambda \geq -f(x_1, x_2,\ldots, x_n)$

The related absolute value function $|f(x_1, x_2,\ldots, x_n)|$ is replaced by λ in the LP. This approach works only if the model tends to minimize λ. Otherwise this formulation does not reflect the desired aim as λ does not have an upper bound.

Similar approach can be used to add Min-Max and Max-Min functions to an LP. In order to add { Min (Max $[f_1(\boldsymbol{x}), f_2(\boldsymbol{x}),\ldots, f_k(\boldsymbol{x})]$) } function to an LP, a decision variable λ is defined and following constraints are added to the LP:

$\lambda \geq f_1(\boldsymbol{x}), \lambda \geq f_2(\boldsymbol{x}), \ldots., \lambda \geq f_k(\boldsymbol{x})$

In order to add { Max (Min $[f_1(\boldsymbol{x}), f_2(\boldsymbol{x}),\ldots, f_k(\boldsymbol{x})]$) } a decision variable λ is defined and following constraints are added to the LP:

$\lambda \leq f_1(\boldsymbol{x}), \lambda \leq f_2(\boldsymbol{x}), \ldots., \lambda \leq f_k(\boldsymbol{x})$

3.2.2 Plant Layout Example

(Bazaraa, 2010; p.30.)

Consider the problem of locating a new machine to an existing layout consisting of four machines. These machines are located at the following coordinates in two-dimensional space: $\binom{3}{1}, \binom{0}{-3}, \binom{-2}{2}, \binom{1}{4}$. Use Manhattan distance to calculate the distances between machines. For instance the distance between $\binom{x_1}{x_2}$ and $\binom{3}{1}$: $|x_1 - 3| + |x_2 - 1|$. Formulate an LP that would minimize the sum of the distances from the new machine to the four machines.

Answer

Decision variables

x_1 and x_2: coordinates of the new machine

λ_{ij}: distance between new machine and machine i at coordinate j, i=1,2,3,4; j=1,2.

Objective function

Min $\sum_{i=1}^{4}\sum_{j=1}^{2}\lambda_{ij}$

Constraints (*Distance calculation)*

$\lambda_{ij} \geq k_{ij} - x_j, \quad \lambda_{ij} \geq -k_{ij} + x_j \quad \forall i, j.$ k_{ij}: value for coordinate j of machine i

For instance; for i =1 and j=1,2;

$\lambda_{11} \geq 3 - x_1$ $\quad$ $\lambda_{11} \geq -3 + x_1$

$\lambda_{12} \geq 1 - x_2$ $\quad$ $\lambda_{12} \geq -1 + x_2$

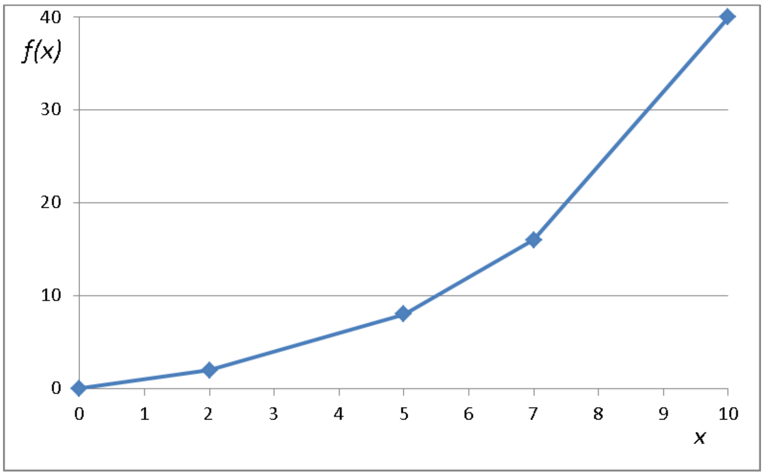
Sign restrictions

x_1, x_2 urs; $\lambda_{ij} \geq 0$, $\forall i,j$.

3.3 PIECEWISE LINEAR FUNCTIONS

3.3.1 Representing Piecewise Linear Convex Functions in an LP

A piecewise linear function consists of several straight line segments. For example the function given below consists of four straight line segments.



The function in the figure can be expressed as follows:

$$f(x) = \begin{cases} x & 0 \leq x < 2 \\ 2 + 2(x-2) & 2 \leq x < 5 \\ 8 + 4(x-5) & 5 \leq x < 7 \\ 16 + 8(x-7) & 7 \leq x < 10 \end{cases}$$

The points where the slope of the function changes are regarded as break points. In the figure, breakpoints are 0, 2, 5, 7, and 10. If the slopes of straight line segments increase when value of *x* increases, that function is a piecewise linear convex function. If a minimization objective function of a mathematical model *f*(*x*) is a piecewise linear convex function, two different methods can be used to represent this objective in an LP.

Let *f*(*x*) is a piecewise linear convex function and d_1, d_2, … ,d_n are break points.

Method 1.

In the model replace *f*(*x*) by $\sum_{i=1}^{n-1} c_i y_i$,

replace *x* by $\sum_{i=1}^{n-1} y_i$,

add $y_i \leq d_{i+1} - d_i, i = 1, \dots, n-1$ to constraints.

Here y_i's are the decision variables $(i = 1, \dots, n-1)$

c_i is the slope of the *i*th piecewise function $(i = 1, \dots, n-1)$.

For the function given in the example, LP formulation is as follows:

$$f(x) = y_1 + 2y_2 + 4y_3 + 8y_4$$

$$x = y_1 + y_2 + y_3 + y_4$$

$$y_1 \leq 2$$

$$y_2 \leq 3$$

$$y_3 \leq 2$$

$$y_4 \leq 3$$

Method 2.

In the model replace *f*(*x*) by $\sum_{i=1}^{n} z_i f(d_i)$,

replace *x* by $\sum_{i=1}^{n} z_i d_i$,

add $\sum_{i=1}^{n} z_i = 1$ to constraints.

Here z_i's are decision variables ($i = 1, \ldots, n$),

$f(d_i)$ is the function value of the *i*th break point.

For the function given in the example, LP formulation is as follows:

$$f(x) = 0z_1 + 2z_2 + 8z_3 + 16z_4 + 40z_5$$

$$x = 0z_1 + 2z_2 + 5z_3 + 7z_4 + 10z_5$$

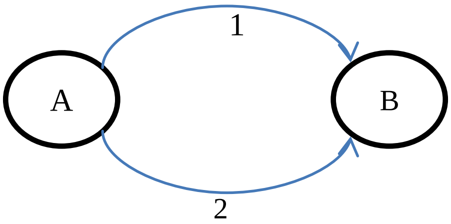
$$z_1 + z_2 + z_3 + z_4 + z_5 = 1$$

3.3.2 Transformation of Nonlinear Convex Functions

By transforming nonlinear convex objective function into piecewise linear convex function, nonlinear function can be approximately represented in linear form. For this purpose, non-linear function is divided into n-1 pieces and by assuming each piece to be linear. The piecewise linear convex function is then represented in an LP model by utilizing one of the aforementioned methods.

3.3.3 Oil Shipment Example

10,000 barrels of oil will be sent from point A to point B through two pipelines. Shipment duration is related with shipment quantity. If x_1 thousand barrels of oil is sent through the first pipeline, shipment duration will be x_1^2 hours; if x_2 thousand barrels of oil is sent through the second pipeline, shipment duration will be $x_2^{1,5}$ hours. Formulate an LP model that will minimize the shipment duration if two pipelines are used simultaneously for the oil shipment.



Answer

First of all, shipment duration functions are transformed into piecewise functions. As x_1 and x_2 can take values between 0 and 10, the functions can be separated to 4 equal pieces:

x	$f(x_1) = \boldsymbol{x_1^2}$	$f(x_2) = \boldsymbol{x_2^{1,5}}$
0.0	0.000	0.000
2.5	6.250	3.953
5.0	25.000	11.180
7.5	56.250	20.540
10.0	100.000	31.623

In this case, LP formulation is as follows:

Decision variables

x_i : the quantity of oil sent through pipeline i (*1000 barrels),

f_i : shipment duration through pipeline i (hours),

λ : longest shipment duration (hours),

z_{ij} : additional variables for piecewise functions, i =1,2, j=1,…,5.

Objective functions

Min λ

Constraints

Longest shipment duration should not be less than duration through pipelines

$\lambda \geq f_1$

$\lambda \geq f_2$

Piecewise linear function representation for the 1st pipeline (Method 2)

$x_1 = 0z_{11} + 2,5\ z_{12} + 5\ z_{13} + 7,5\ z_{14} + 10\ z_{15}$

$f_1 = 0z_{11} + 6,25\ z_{12} + 25\ z_{13} + 56,25\ z_{14} + 100\ z_{15}$

$z_{11} + z_{12} + z_{13} + z_{14} + z_{15} = 1$

Piecewise linear function representation for the 2nd pipeline

$x_2 = 0z_{21} + 2,5\ z_{22} + 5\ z_{23} + 7,5\ z_{24} + 10\ z_{25}$

$f_2 = 0z_{21} + 3,953\ z_{22} + 11,18\ z_{23} + 20,54\ z_{24} + 31,623\ z_{25}$

$z_{21} + z_{22} + z_{23} + z_{24} + z_{25} = 1$

Total quantity is 10,000 barrels

$x_1 + x_2 = 10$

Sign restrictions

all variables ≥ 0.

Report

Solution for LP model:

$\lambda = f_1 = f_2 = 15{,}781$; $x_1 = 3{,}771$; $x_2 = 6{,}229$

When x_1 and x_2 values are replaced in functions:

$f_1 = 14{,}22$; $f_2 = 15{,}546$

If the same problem is solved with Nonlinear Programming

$f_1 = f_2 = 15{,}112$; $x_1 = 3{,}887$; $x_2 = 6{,}113$

Solution to “transformation to piecewise function” and solution to “NLP” are very close to each other. Solving LP is easier than solving NLP. To find more accurate results, functions can be divided into more pieces.

4. SOLVING LP

4.1 LP SOLUTIONS: FOUR CASES

When an LP is solved, one of the following four cases will occur:

1. The LP has a **unique optimal solution**.
2. The LP has **alternative (multiple) optimal solutions**. It has more than one (actually an infinite number of) optimal solutions
3. The LP is **infeasible**. It has no feasible solutions (The feasible region contains no points).
4. The LP is **unbounded**. In the feasible region there are points with arbitrarily large (in a max problem) objective function values.

4.2 THE GRAPHICAL SOLUTION

Any LP with only two variables can be solved graphically

Example 1. Giapetto

(*Winston 3.1, p. 49*)

Since the Giapetto LP has two variables, it may be solved graphically.

Answer

The feasible region is the set of all points satisfying the constraints.

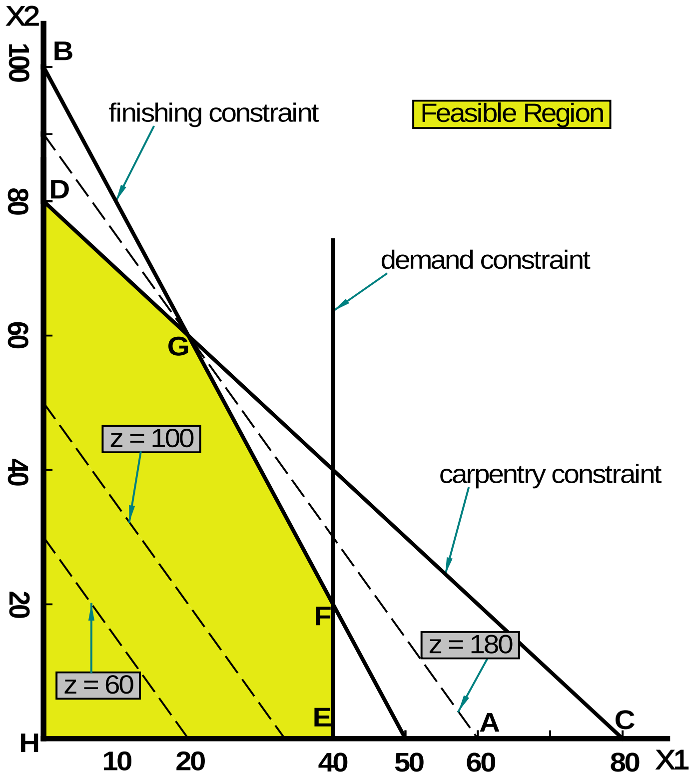
$$\max z = 3x_1 + 2x_2$$

s.t.	$2x_1 + x_2 \leq 100$	(Finishing constraint)
	$x_1 + x_2 \leq 80$	(Carpentry constraint)
	$x_1 \leq 40$	(Demand constraint)
	$x_1, x_2 \geq 0$	(Sign restrictions)

The set of points satisfying the LP is bounded by the five sided polygon DGFEH. Any point ***on*** or ***in*** the interior of this polygon (the shade area) is in the ***feasible region***.

Having identified the feasible region for the LP, a search can begin for the ***optimal solution*** which will be the point in the feasible region with the *largest* z-value (maximization problem).

To find the optimal solution, a line on which the points have the same *z*-value is graphed. In a max problem, such a line is called an ***isoprofit*** line while in a min problem, this is called the ***isocost*** line. (*The figure shows the isoprofit lines for z = 60, z = 100, and z = 180*).



In the unique optimal solution case, isoprofit line last hits a point (vertex - corner) before leaving the feasible region.

The optimal solution of this LP is point G where (x_1, x_2) = (20, 60) giving z = 180.

A constraint is ***binding*** (active, tight) if the left-hand and right-hand side of the constraint are equal when the optimal values of the decision variables are substituted into the constraint.

A constraint is ***nonbinding*** (inactive) if the left-hand side and the right-hand side of the constraint are unequal when the optimal values of the decision variables are substituted into the constraint.

In Giapetto LP, the finishing and carpentry constraints are binding. On the other hand the demand constraint for wooden soldiers is nonbinding since at the optimal solution $x_1 < 40$ ($x_1 = 20$).

Example 2. Advertisement

(*Winston 3.2, p. 61*)

Since the Advertisement LP has two variables, it may be solved graphically.

Answer

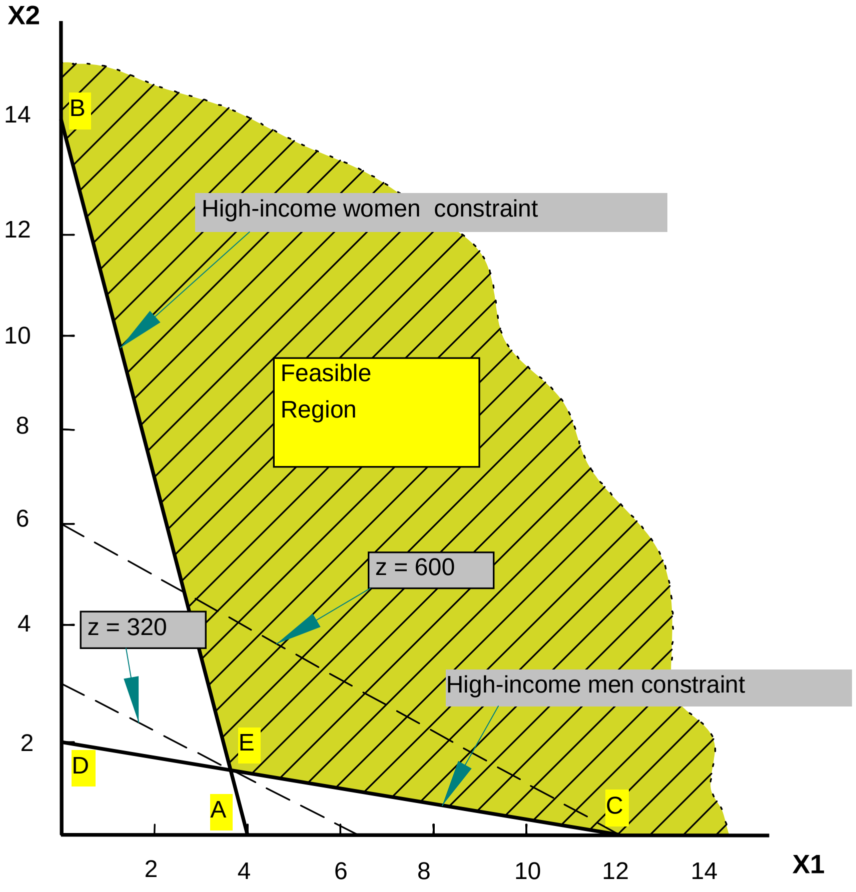
The feasible region is the set of all points satisfying the constraints.

min $z = 50x_1 + 100x_2$

s.t. $7x_1 + 2x_2 \geq 28$ (high income women)

$2x_1 + 12x_2 \geq 24$ (high income men)

$x_1, x_2 \geq 0$



Since Dorian wants to minimize total advertising costs, the optimal solution to the problem is the point in the feasible region with the smallest z value.

An isocost line with the smallest z value passes through point E and is the optimal solution at $x_1 = 3.6$ and $x_2 = 1.4$ giving $z = 320$.

Both the high-income women and high-income men constraints are satisfied, both constraints are binding.

Example 3. Two Mines

min $180x + 160y$

st $6x + y \geq 12$

$3x + y \geq 8$

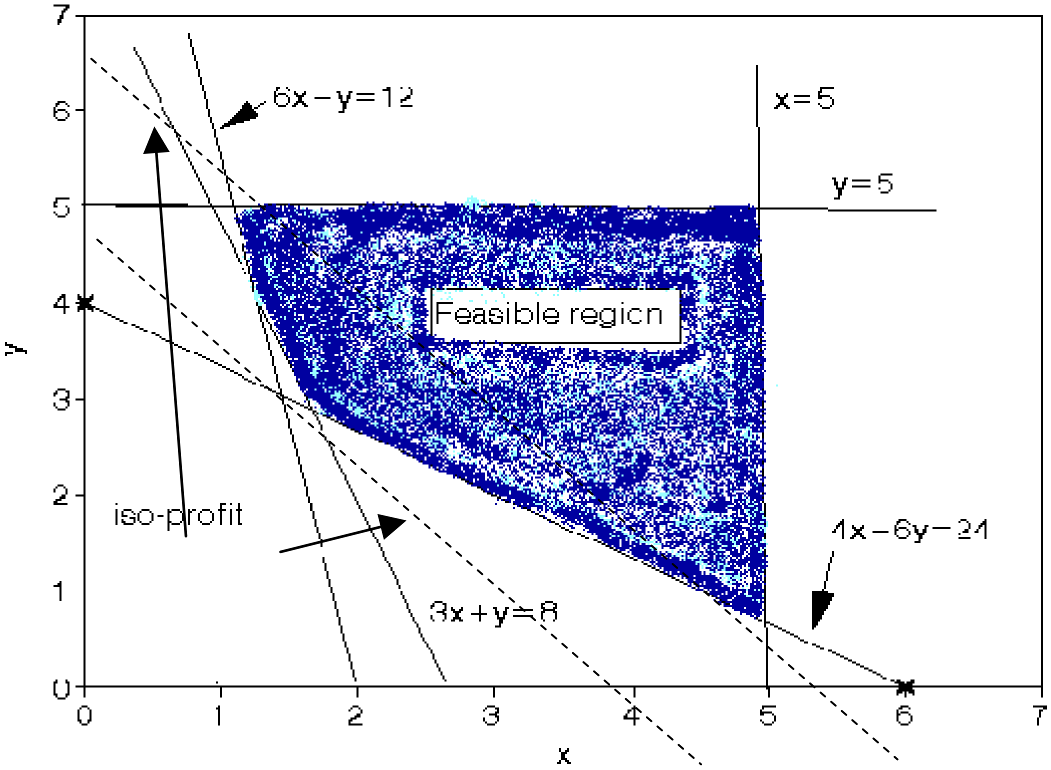
$4x + 6y \geq 24$

$x \leq 5$

$y \leq 5$

$x, y \geq 0$

Answer



Optimal sol'n is 765.71. 1.71 days mine *X* and 2.86 days mine *Y* are operated.

Example 4. Modified Giapetto

max $z = 4x_1 + 2x_2$

s.t.	$2x_1 + x_2 \leq 100$	(Finishing constraint)
	$x_1 + x_2 \leq 80$	(Carpentry constraint)
	$x_1 \leq 40$	(Demand constraint)
	$x_1, x_2 \geq 0$	(Sign restrictions)

Answer

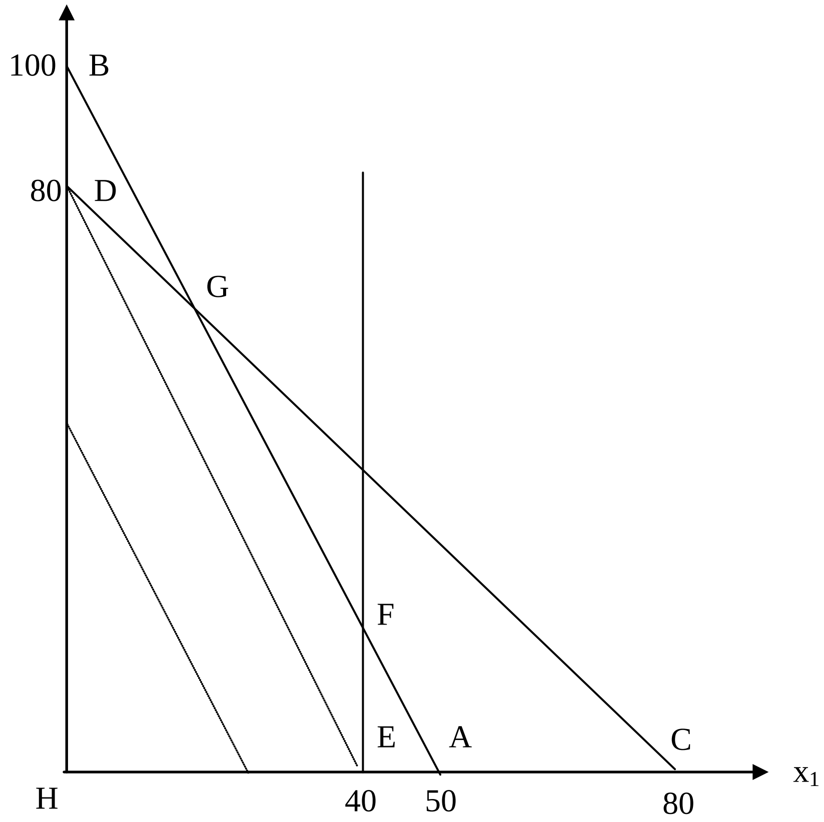
Points on the line between points G (20, 60) and F (40, 20) are the ***alternative optimal solutions*** (see figure below).

Thus, for $0 \le c \le 1$,

c [20 60] + (1 - c) [40 20] = **[40 - 20*c*, 20 + 40*c*]**

will be optimal

For all optimal solutions, the optimal objective function value is 200.



Example 5. Modified Giapetto (v. 2)

Add constraint $x_2 \ge 90$ (Constraint on demand for trains).

Answer

No feasible region: ***Infeasible LP***

Example 6. Modified Giapetto (v. 3)

Only use constraint $x_2 \ge 90$

Answer

Isoprofit line never lose contact with the feasible region: ***Unbounded LP***

4.3 THE SIMPLEX ALGORITHM

Note that in the examples considered at the graphical solution, the unique optimal solution to the LP occurred at a vertex (corner) of the feasible region. In fact it is true that for *any* LP the optimal solution occurs at a vertex of the feasible region. This fact is the key to the simplex algorithm for solving LP's.

Essentially the simplex algorithm starts at one vertex of the feasible region and moves (at each iteration) to another (adjacent) vertex, improving (or leaving unchanged) the objective function as it does so, until it reaches the vertex corresponding to the optimal LP solution.

The simplex algorithm for solving linear programs (LP's) was developed by Dantzig in the late 1940's and since then a number of different versions of the algorithm have been developed. One of these later versions, called the *revised simplex* algorithm (sometimes known as the "product form of the inverse" simplex algorithm) forms the basis of most modern computer packages for solving LP's.

Steps

1. Convert the LP to standard form
2. Obtain a basic feasible solution (bfs) from the standard form
3. Determine whether the current bfs is optimal. If it is optimal, stop.
4. If the current bfs is not optimal, determine which nonbasic variable should become a basic variable and which basic variable should become a nonbasic variable to find a new bfs with a better objective function value
5. Go back to Step 3.

Related concepts:

- Standard form: all constraints are equations and all variables are nonnegative
- bfs: any basic solution where all variables are nonnegative
- Nonbasic variable: a chosen set of variables where variables equal to 0
- Basic variable: the remaining variables that satisfy the system of equations at the standard form

Example 1. Dakota Furniture

(Winston 4.3, p. 134)

Dakota Furniture makes desks, tables, and chairs. Each product needs the limited resources of lumber, carpentry and finishing; as described in the table. At most 5 tables can be sold per week. Maximize weekly revenue.

Resource	Desk	Table	Chair	Max Avail.
Lumber (board ft.)	8	6	1	48
Finishing hours	4	2	1.5	20
Carpentry hours	2	1.5	.5	8
Max Demand	unlimited	5	unlimited	
Price ($)	60	30	20	

LP Model:

Let x_1, x_2, x_3 be the number of desks, tables and chairs produced.

Let the weekly profit be $z. Then, we must

$$\begin{aligned}
\max z = {} & 60x_1 + 30x_2 + 20x_3 \\
\text{s.t.} \quad & 8x_1 + 6x_2 + x_3 \le 48 \\
& 4x_1 + 2x_2 + 1.5x_3 \le 20 \\
& 2x_1 + 1.5x_2 + .5x_3 \le 8 \\
& x_2 \le 5 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}$$

Solution with Simplex Algorithm

First introduce slack variables and convert the LP to the standard form and write a canonical form

$$\begin{array}{llllllllll}
R_0 & z & -60x_1 & -30x_2 & -20x_3 & & & & & = 0 \\
R_1 & & 8x_1 & +6x_2 & + x_3 & +s_1 & & & & = 48 \\
R_2 & & 4x_1 & +2x_2 & +1.5x_3 & & +s_2 & & & = 20 \\
R_3 & & 2x_1 & +1.5x_2 & +.5x_3 & & & +s_3 & & = 8 \\
R_4 & & & x_2 & & & & & +s_4 & = 5
\end{array}$$

$x_1, x_2, x_3, s_1, s_2, s_3, s_4 \ge 0$

Obtain a starting bfs.

As (x_1, x_2, x_3) = 0 is feasible for the original problem, the below given point where three of the variables equal 0 (the ***non-basic variables***) and the four other variables (the ***basic variables***) are determined by the four equalities is an obvious bfs:

$x_1 = x_2 = x_3 = 0$, $s_1 = 48$, $s_2 = 20$, $s_3 = 8$, $s_4 = 5$.

Determine whether the current bfs is optimal.

Determine whether there is any way that z can be increased by increasing some nonbasic variable.

If each nonbasic variable has a nonnegative coefficient in the objective function row (***row 0***), current bfs is optimal.

However, here all nonbasic variables have negative coefficients: It is not optimal.

Find a new bfs

- *z* increases most rapidly when x_1 is made non-zero; i.e. x_1 is the ***entering variable***.
- Examining R_1, x_1 can be increased only to 6. More than 6 makes $s_1 < 0$. Similarly R_2, R_3, and R_4, give limits of 5, 4, and no limit for x_1 (***ratio test***). The smallest ratio is the largest value of the entering variable that will keep all the current basic variables nonnegative. Thus by R_3, x_1 can only increase to $x_1 = 4$ when s_3 becomes 0. We say s_3 is the ***leaving variable*** and R_3 is the ***pivot equation***.
- Now we must rewrite the system so the values of the basic variables can be read off.

The new *pivot equation* ($R_3/2$) is

$$R_3': x_1+.75x_2+.25x_3+ \quad .5s_3 \quad = 4$$

Then use R_3' to eliminate x_1 in all the other rows.

$$R0'=R0+60R3', \quad R1'=R1-8R3', \quad R2'=R2-4R3', \quad R4'=R4$$

$$\begin{array}{lccccccccrl}
R_0' & z & & +15x_2 & -5x_3 & & & +30s_3 & & = 240 & z = 240 \\
R_1' & & & & -x_3 & +s_1 & & -4s_3 & & = 16 & s_1 = 16 \\
R_2' & & & -x_2 & +.5x_3 & & +s_2 & -2s_3 & & = 4 & s_2 = 4 \\
R_3' & & x_1 & +.75x_2 & +.25x_3 & & & +.5s_3 & & = 4 & x_1 = 4 \\
R_4 & & & x_2 & & & & & +s_4 & = 5 & s_4 = 5
\end{array}$$

The new bfs is $x_2 = x_3 = s_3 = 0$, $x_1 = 4$, $s_1 = 16$, $s_2 = 4$, $s_4 = 5$ making $z = 240$.

Check optimality of current bfs. Repeat steps until an optimal solution is reached

- We increase *z* fastest by making x_3 non-zero (i.e. x_3 enters).
- x_3 can be increased to at most $x_3 = 8$, when $s_2 = 0$ (i.e. s_2 leaves.)

Rearranging the pivot equation gives

$$R_2'' \quad -2x_2 + x_3 + 2s_2 - 4s_3 = 8 \quad (R_2' \times 2).$$

Row operations with R_2'' eliminate x_3 to give the new system

$$R_0''= R_0' + 5R_2'', \quad R_1'' = R_1' + R_2'', \quad R_3'' = R_3' - .5R_2'', \quad R_4'' = R_4'$$

The bfs is now $x_2 = s_2 = s_3 = 0$, $x_1 = 2$, $x_3 = 8$, $s_1 = 24$, $s_4 = 5$ making $z = 280$.

Each nonbasic variable has a nonnegative coefficient in row 0 ($5x_2$, $10s_2$, $10s_3$).

THE CURRENT SOLUTION IS OPTIMAL

Report: Dakota furniture's optimum weekly profit would be 280$ if they produce 2 desks and 8 chairs.

This was once written as a tableau.

(Use tableau format for each operation in all HW and exams!!!)

$$\begin{aligned}
&\max z = 60x_1 + 30x_2 + 20x_3 \\
&\text{s.t.} \quad 8x_1 + 6x_2 + x_3 \le 48 \\
&\qquad\; 4x_1 + 2x_2 + 1.5x_3 \le 20 \\
&\qquad\; 2x_1 + 1.5x_2 + .5x_3 \le 8 \\
&\qquad\qquad x_2 \le 5 \\
&\quad x_1, x_2, x_3 \geq 0
\end{aligned}$$

Initial tableau:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	-60	-30	-20	0	0	0	0	0	$z = 0$	
0	8	6	1	1	0	0	0	48	$s_1 = 48$	6
0	4	2	1.5	0	1	0	0	20	$s_2 = 20$	5
0	2	1.5	0.5	0	0	1	0	8	$s_3 = 8$	4
0	0	1	0	0	0	0	1	5	$s_4 = 5$	-

First tableau:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	0	15	-5	0	0	30	0	240	$z = 240$	
0	0	0	-1	1	0	-4	0	16	$s_1 = 16$	-
0	0	-1	0.5	0	1	-2	0	4	$s_2 = 4$	8
0	1	0.75	0.25	0	0	0.5	0	4	$x_1 = 4$	16
0	0	1	0	0	0	0	1	5	$s_4 = 5$	-

Second and optimal tableau:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	0	5	0	0	10	10	0	280	$z = 280$	
0	0	-2	0	1	2	-8	0	24	$s_1 = 24$	
0	0	-2	1	0	2	-4	0	8	$x_3 = 8$	
0	1	1.25	0	0	-0.5	1.5	0	2	$x_1 = 2$	
0	0	1	0	0	0	0	1	5	$s_4 = 5$	

Example 2. Modified Dakota Furniture

Dakota example is modified: $35/table

$$\text{new } z = 60\, x_1 + 35\, x_2 + 20\, x_3$$

Second and optimal tableau for the modified problem:

z	x_1	⇓ x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	0	**0**	0	0	10	10	0	280	z=280	
0	0	-2	0	1	2	-8	0	24	s_1=24	-
0	0	-2	1	0	2	-4	0	8	x_3=8	-
0	1	1.25	0	0	-0.5	1.5	0	2	x_1=2	2/1.25 ⇒
0	0	1	0	0	0	0	1	5	s_4=5	5/1

Another optimal tableau for the modified problem:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV
1	0	0	0	0	10	10	0	280	z=280
0	1.6	0	0	1	1.2	-5.6	0	27.2	s_1=27.2
0	1.6	0	1	0	1.2	-1.6	0	11.2	x_3=11.2
0	0.8	1	0	0	-0.4	1.2	0	1.6	x_2=1.6
0	-0.8	0	0	0	0.4	-1.2	1	3.4	s_4=3.4

Therefore the optimal solution is as follows:

z = 280 and for $0 \le c \le 1$

$$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = c \begin{vmatrix} 2 \\ 0 \\ 8 \end{vmatrix} + (1 - c) \begin{vmatrix} 0 \\ 1.6 \\ 11.2 \end{vmatrix} = \begin{vmatrix} 2c \\ 1.6 - 1.6c \\ 11.2 - 3.2c \end{vmatrix}$$

Example 3. Unbounded LPs

z	x_1	x_2	⇓ x_3	s_1	s_2	z	RHS	BV	Ratio
1	0	2	**-9**	0	12	4	100	z=100	
0	0	1	-6	1	6	-1	20	x_4=20	None
0	1	1	-1	0	1	0	5	x_1=5	None

Since ratio test fails, the LP under consideration is an unbounded LP.

4.4 THE BIG M METHOD

If an LP has any ≥ or = constraints, a starting bfs may not be readily apparent.

When a bfs is not readily apparent, the Big M method or the two-phase simplex method may be used to solve the problem.

The Big M method is a version of the Simplex Algorithm that first finds a bfs by adding "artificial" variables to the problem. The objective function of the original LP must, of course, be modified to ensure that the artificial variables are all equal to 0 at the conclusion of the simplex algorithm.

Steps

1. Modify the constraints so that the RHS of each constraint is nonnegative (This requires that each constraint with a negative RHS be multiplied by -1. Remember that if you multiply an inequality by any negative number, the direction of the inequality is reversed!). After modification, identify each constraint as a ≤, ≥ or = constraint.
2. Convert each inequality constraint to standard form (If constraint i is a ≤ constraint, we add a slack variable s_i; and if constraint i is a ≥ constraint, we subtract an excess variable e_i).
3. Add an artificial variable a_i to the constraints identified as ≥ or = constraints at the end of Step 1. Also add the sign restriction $a_i \geq 0$.
4. Let M denote a very large positive number. If the LP is a min problem, add (for each artificial variable) Ma_i to the objective function. If the LP is a max problem, add (for each artificial variable) -Ma_i to the objective function.
5. Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. Now solve the transformed problem by the simplex (In choosing the entering variable, remember that M is a very large positive number!).

If all artificial variables are equal to zero in the optimal solution, we have found the **optimal solution** to the original problem.

If any artificial variables are positive in the optimal solution, the original problem is **infeasible**!!!

Example 1. Oranj Juice

(Winston 4.10, p. 164)

Bevco manufactures an orange flavored soft drink called Oranj by combining orange soda and orange juice. Each ounce of orange soda contains 0.5 oz of sugar and 1 mg of vitamin C. Each ounce of orange juice contains 0.25 oz of sugar and 3 mg of vitamin C. It costs Bevco 2¢ to produce an ounce of orange soda and 3¢ to produce an ounce of orange juice. Marketing department has decided that each 10 oz bottle of Oranj must contain at least 20 mg of vitamin C and at most 4 oz of sugar. Use LP to determine how Bevco can meet marketing dept.'s requirements at minimum cost.

LP Model:

Let x_1 and x_2 be the quantity of ounces of orange soda and orange juice (respectively) in a bottle of Oranj.

$$\begin{array}{lll} \min z = 2x_1 + 3x_2 & & \\ \text{s.t.} \quad 0.5\,x_1 + 0.25\,x_2 \leq 4 & & \text{(sugar const.)} \\ \quad x_1 + 3\,x_2 \geq 20 & & \text{(vit. C const.)} \\ \quad x_1 + x_2 = 10 & & \text{(10 oz in bottle)} \\ x_1, x_2 \geq 0 & & \end{array}$$

Solving with Big M Method:

1. Modify the constraints so that the RHS of each constraint is nonnegative

The RHS of each constraint is nonnegative

2. Convert each inequality constraint to standard form

$$\begin{array}{rcl} z - 2x_1 - 3x_2 & = & 0 \\ 0.5x_1 + 0.25x_2 + s_1 & = & 4 \\ x_1 + 3x_2 - e_2 & = & 20 \\ x_1 + x_2 & = & 10 \end{array}$$

all variables nonnegative

3. Add a_i to the constraints identified as > or = const.s

$$\begin{array}{rcll} z - 2x_1 - 3x_2 & = & 0 & \text{Row 0} \\ 0.5x_1 + 0.25x_2 + s_1 & = & 4 & \text{Row 1} \\ x_1 + 3x_2 - e_2 + a_2 & = & 20 & \text{Row 2} \\ x_1 + x_2 + a_3 & = & 10 & \text{Row 3} \end{array}$$

all variables nonnegative

4. Add $\mathrm{M}a_\mathrm{i}$ to the objective function (min problem)

$$\min z = 2x_1 + 3x_2 + \mathrm{M}a_2 + \mathrm{M}a_3$$

Row 0 will change to

$$z - 2x_1 - 3x_2 - Ma_2 - Ma_3 = 0$$

5. Since each artificial variable are in our starting bfs, they must be eliminated from row 0

New Row 0 = Row 0 + M * Row 2 + M * Row 3 ⇒

$$z + (2M-2)\,x_1 + (4M-3)\,x_2 - M\,e_2 = 30M \qquad \text{New Row 0}$$

Initial tableau:

z	x_1	⇓ x_2	s_1	e_2	a_2	a_3	RHS	BV	Ratio
1	2M-2	**4M-3**	0	-M	0	0	30M	z=30M	
0	0.5	0.25	1	0	0	0	4	s_1=4	16
0	1	$\boxed{\mathbf{3}}$	0	-1	1	0	20	a_2=20	**20/3** ⇒
0	1	1	0	0	0	1	10	a_3=10	10

In a min problem, entering variable is the variable that has the "most positive" coefficient in row 0!

First tableau:

z	⇓ x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV	Ratio
1	**(2M-3)/3**	0	0	(M-3)/3	(3-4M)/3	0	20+3.3M	z	
0	5/12	0	1	1/12	-1/12	0	7/3	s_1	28/5
0	1/3	1	0	-1/3	1/3	0	20/3	x_2	20
0	$\boxed{\mathbf{2/3}}$	0	0	1/3	-1/3	1	10/3	a_3	**5** ⇒

Optimal tableau:

z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV
1	0	0	0	-1/2	(1-2M)/2	(3-2M)/2	25	z=25
0	0	0	1	-1/8	1/8	-5/8	1/4	s_1=1/4
0	0	1	0	-1/2	1/2	-1/2	5	x_2=5
0	1	0	0	1/2	-1/2	3/2	5	x_1=5

Report:

In a bottle of Oranj, there should be 5 oz orange soda and 5 oz orange juice.

In this case the cost would be 25¢.

Example 2. Modified Oranj Juice

Consider Bevco's problem. It is modified so that 36 mg of vitamin C are required. Related LP model is given as follows:

Let x_1 and x_2 be the quantity of ounces of orange soda and orange juice (respectively) in a bottle of Oranj.

$$\begin{array}{llr}
\min z = 2x_1 + 3x_2 & & \\
\text{s.t.} \quad 0.5\,x_1 + 0.25\,x_2 \le 4 & & \text{(sugar const.)} \\
x_1 + 3\,x_2 \ge 36 & & \text{(vit. C const.)} \\
x_1 + x_2 = 10 & & \text{(10 oz in bottle)} \\
x_1, x_2 \ge 0 & &
\end{array}$$

Solving with Big M method:

Initial tableau:

z	x_1	⇓ x_2	s_1	e_2	a_2	a_3	RHS	BV	Ratio
1	2M-2	**4M-3**	0	-M	0	0	46M	z=46M	
0	0.5	0.25	1	0	0	0	4	s_1=4	16
0	1	3	0	-1	1	0	36	a_2=36	36/3
0	1	$\boxed{\mathbf{1}}$	0	0	0	1	10	a_3=10	**10** ⇒

Optimal tableau:

z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV
1	1-2M	0	0	-M	0	3-4M	30+6M	z=30+6M
0	1/4	0	1	0	0	-1/4	3/2	s_1=3/2
0	-2	0	0	-1	1	-3	6	a_2=6
0	1	1	0	0	0	1	10	x_2=10

An artificial variable (a_2) is BV so the original LP has no feasible solution

Report:

It is impossible to produce Oranj under these conditions.

4.5 TWO-PHASE SIMPLEX METHOD

When a basic feasible solution is not readily available, the two-phase simplex method may be used as an alternative to the Big M method. In the two-phase simplex method, we add artificial variables to the same constraints as we did in the Big M method. Then we find a bfs to the original LP by solving the Phase I LP. In the Phase I LP, the objective function is to minimize the sum of all artificial variables. At the completion of Phase I, we reintroduce the original LP's objective function and determine the optimal solution to the original LP.

Steps

1. Modify the constraints so that the RHS of each constraint is nonnegative (This requires that each constraint with a negative RHS be multiplied by -1. Remember that if you multiply an inequality by any negative number, the direction of the inequality is reversed!). After modification, identify each constraint as a ≤, ≥ or = constraint.

2. Convert each inequality constraint to standard form (If constraint *i* is a ≤ constraint, we add a slack variable s_i; and if constraint *i* is a ≥ constraint, we subtract an excess variable e_i).
3. Add an artificial variable a_i to the constraints identified as ≥ or = constraints at the end of Step 1. Also add the sign restriction $a_i \geq 0$.
4. In the phase I, ignore the original LP's objective function, instead solve an LP whose objective function is minimizing $w = \Sigma a_i$ (sum of all the artificial variables). The act of solving the Phase I LP will force the artificial variables to be zero.
5. Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. Now solve the transformed problem by the simplex.

Solving the Phase I LP will result in one of the following three cases:

I. Case 1. If w > 0 then the original LP has no feasible solution (stop here).

II. Case 2. If w = 0, and no artificial variables are in the optimal Phase I basis:
   i. Drop all columns in the optimal Phase I tableau that correspond to the artificial variables. Drop Phase I row 0.
   ii. Combine the original objective function with the constraints from the optimal Phase I tableau (Phase II LP). If original objective function coefficients of BVs are nonzero row operations are done.
   iii. Solve Phase II LP using the simplex method. The optimal solution to the Phase II LP is the optimal solution to the original LP.

III. Case 3. If w = 0, and at least one artificial variable is in the optimal Phase I basis:
   i. Drop all columns in the optimal Phase I tableau that correspond to the nonbasic artificial variables and any variable from the original problem that has a negative coefficient in row 0 of the optimal Phase I tableau. Drop Phase I row 0.
   ii. Combine the original objective function with the constraints from the optimal Phase I tableau (Phase II LP). If original objective function coefficients of BVs are nonzero row operations are done.
   iii. Solve Phase II LP using the simplex method. The optimal solution to the Phase II LP is the optimal solution to the original LP.

Example 1. Oranj Juice

Let x_1 and x_2 be the quantity of ounces of orange soda and orange juice (respectively) in a bottle of Oranj.

$$\min z = 2x_1 + 3x_2$$

$$\begin{array}{lrl}
\text{s.t.} & 0.5\,x_1 + 0.25\,x_2 \le 4 & \text{(sugar const.)} \\
 & x_1 + 3\,x_2 \ge 20 & \text{(vit. C const.)} \\
 & x_1 + x_2 = 10 & \text{(10 oz in bottle)} \\
 & x_1, x_2 \ge 0 &
\end{array}$$

Solving with two-phase simplex method:

1. Modify the constraints so that the RHS of each constraint is nonnegative

The RHS of each constraint is nonnegative

2. Convert each inequality constraint to standard form

$$\begin{array}{rcl}
z - 2x_1 - 3x_2 & = & 0 \\
0.5x_1 + 0.25x_2 + s_1 & = & 4 \\
x_1 + 3x_2 - e_2 & = & 20 \\
x_1 + x_2 & = & 10
\end{array}$$

all variables nonnegative

3. Add a_i to the constraints identified as > or = constraints

$$\begin{array}{rcll}
z - 2x_1 - 3x_2 & = & 0 & \text{Row 0} \\
0.5x_1 + 0.25x_2 + s_1 & = & 4 & \text{Row 1} \\
x_1 + 3x_2 - e_2 + a_2 & = & 20 & \text{Row 2} \\
x_1 + x_2 + a_3 & = & 10 & \text{Row 3}
\end{array}$$

all variables nonnegative

4. Set objective function is mininimization of sum of all the artificial variables)

$$\text{Min } w = a_2 + a_3$$

Row 0 is as follows:

$$w - a_2 - a_3 = 0$$

5. Eliminate all artificial variables from row 0

$$\text{New } R_0 = R_0 + R_2 + R_3 \Rightarrow$$

$$w + (1+1)\,x_1 + (3+1)\,x_2 - e_2 = 30 \quad \text{New } R_0$$

Phase I LP - Initial Tableau:

w	x_1	⇓ x_2	s_1	e_2	a_2	a_3	RHS	BV	Ratio
1	2	**4**	0	-1	0	0	30	w=30	
0	1/2	1/4	1	0	0	0	4	s_1=4	16
0	1	$\boxed{\mathbf{3}}$	0	-1	1	0	20	a_2=20	**20/3**⇒
0	1	1	0	0	0	1	10	a_3=10	10

Phase I LP - First Tableau:

w	⇓ x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV	Ratio
1	**2/3**	0	0	1/3	-4/3	0	10/3	w=10/3	
0	5/12	0	1	1/12	-1/12	0	7/3	s_1=7/3	28/5
0	1/3	1	0	-1/3	1/3	0	20/3	x_2=20/3	20
0	$\boxed{\mathbf{2/3}}$	0	0	1/3	-1/3	1	10/3	a_3=10/3	**5**⇒

Phase I LP - Optimal Tableau:

w	x_1	**x_2**	s_1	e_2	a_2	a_3	RHS	BV
1	0	0	0	0	-1	-1	0	w=0
0	0	0	1	-1/8	1/8	-5/8	1/4	s_1=1/4
0	0	1	0	-1/2	1/2	-1/2	5	x_2=5
0	1	0	0	1/2	-1/2	3/2	5	x_1=5

Solving the Phase I LP will result in one of the three cases

According to Phase I optimal tableau; *w* = 0 and no artificial variables are in the optimal Phase I basis, therefore the problem is an example of Case 2.

i. Drop all columns in the optimal Phase I tableau that correspond to the artificial variables. Drop Phase I row 0.

w	**x_1**	x_2	s_1	e_2	a_2	a_3	RHS	BV
1	0	0	0	0	-1	-1	0	w=0
0	0	0	1	-1/8	1/8	-5/8	1/4	s_1=1/4
0	0	1	0	-1/2	1/2	-1/2	5	x_2=5
0	1	0	0	1/2	-1/2	3/2	5	x_1=5

ii. Combine the original objective function with the constraints from the optimal Phase I tableau

$\min z = 2x_1 + 3x_2$

z	x_1	x_2	s_1	e_2	RHS	BV
1	-2	-3	0	0	0	z=0
0	0	0	1	-1/8	1/4	s_1=1/4
0	0	1	0	-1/2	5	x_2=5
0	1	0	0	1/2	5	x_1=5

Since x_1 and x_2 are basic variables, they are eliminated from row 0 using elementary row operations: New R0 = R0 + 2R3 + 3R2

Phase II LP - Initial Tableau:

z	x_1	x_2	s_1	e_2	RHS	BV
1	0	0	0	-1/2	25	z=25
0	0	0	1	-1/8	1/4	s_1=1/4
0	0	1	0	-1/2	5	x_2=5
0	1	0	0	1/2	5	x_1=5

iii. Phase II LP is solved with simplex algorithm. As R0 values for nonbasic variables are non-positive, the initial tableau is the optimal. It is solution the original problem.

The solution is $x_1 = x_2 = 5$; $z = 25$.

Report:

In a bottle of Oranj, there should be 5 oz orange soda and 5 oz orange juice.

In this case the cost would be 25¢.

Example 2. Modified Oranj Juice

Let x_1 and x_2 be the quantity of ounces of orange soda and orange juice (respectively) in a bottle of Oranj.

$$\begin{array}{llr}
\min z = 2x_1 + 3x_2 & & \\
\text{s.t.} \quad 0.5\,x_1 + 0.25\,x_2 \leq 4 & & \text{(sugar const.)} \\
\quad x_1 + 3\,x_2 \geq 36 & & \text{(vit. C const.)} \\
\quad x_1 + x_2 = 10 & & \text{(10 oz in bottle)} \\
\quad x_1, x_2 \geq 0 & &
\end{array}$$

Solving with two-phase simplex method:

1. Modify the constraints so that the RHS of each constraint is nonnegative

The RHS of each constraint is nonnegative

2. Convert each inequality constraint to standard form

$$\begin{array}{lr}
z - 2x_1 - 3x_2 & = 0 \\
0.5x_1 + 0.25x_2 + s_1 & = 4 \\
x_1 + 3x_2 - e_2 & = 36 \\
x_1 + x_2 & = 10 \\
\text{all variables nonnegative} &
\end{array}$$

3. Add a_i to the constraints identified as > or = constraints

$$\begin{array}{lrl}
z - 2x_1 - 3x_2 & = 0 & \text{Row 0} \\
0.5x_1 + 0.25x_2 + s_1 & = 4 & \text{Row 1} \\
x_1 + 3x_2 - e_2 + a_2 & = 36 & \text{Row 2} \\
x_1 + x_2 + a_3 & = 10 & \text{Row 3} \\
\text{all variables nonnegative} & &
\end{array}$$

4. Set objective function is mininimization of sum of all the artificial variables)

Min w = $a_2 + a_3$

Row 0 is as follows:

$$w - a_2 - a_3 = 0$$

5. Eliminate all artificial variables from row 0

New $R_0 = R_0 + R_2 + R_3 \Rightarrow$

$w + (1+1)\,x_1 + (3+1)\,x_2 - e_2 = 46$ New R_0

Phase I LP - Initial Tableau:

w	x_1	⇓ x_2	s_1	e_2	a_2	a_3	RHS	BV	Ratio
1	2	**4**	0	-1	0	0	46	w=46	
0	1/2	1/4	1	0	0	0	4	s_1=4	16
0	1	3	0	-1	1	0	36	a_2=36	12
0	1	$\boxed{\mathbf{1}}$	0	0	0	1	10	a_3=10	**10**⇒

Phase I LP - Optimal Tableau:

w	x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV
1	-2	0	0	-1	0	-4	6	w=6
0	1/4	0	1	0	0	-1/4	3/2	s_1=3/2
0	-2	0	0	-1	1	-3	6	a_2=6
0	1	1	0	0	0	1	10	x_2=10

Solving the Phase I LP will result in one of the three cases

According to Phase I optimal tableau; $w = 6 > 0$, therefore the problem is an example of Case 1. So the original LP must have no feasible solution.

This is reasonable, because if the original LP had a feasible solution, it would have been feasible in the Phase I LP (after setting $a_2 = a_3 = 0$). This feasible solution would have yielded $w^* = 0$. Because the simplex could not find a Phase I solution with $w = 0$, the original LP must have no feasible solution.

Report:

It is impossible to produce Oranj under these conditions.

Example 3. (*Winston*, 4.13)

Solve following LP using two-phase simplex method.

$$
\begin{array}{ll}
\max z = 40x_1 + 10x_2 + 7x_5 + 14x_6 & \\
\text{s.t.} \quad x_1 - x_2 + 2x_5 = 0 & \\
\quad -2x_1 + x_2 - 2x_5 = 0 & \\
\quad x_1 + x_3 + x_5 - x_6 = 3 & \\
\quad 2x_2 + x_3 + x_4 + 2x_5 + x_6 = 4 & \\
\quad \text{All } x_i \geq 0 &
\end{array}
$$

Solving with two-phase simplex method:

1. Modify the constraints so that the RHS of each constraint is nonnegative

The RHS of each constraint is nonnegative.

2. Convert each inequality constraint to standard form

All variables nonnegative and all constraints are equal to constraints so it is in standard form.

3. Add a_j to the constraints identified as > or = constraints

$$\begin{array}{l} z - 40x_1 - 10x_2 \qquad -7x_5 - 14x_6 = 0 \\ x_1 - x_2 \qquad +2x_5 \qquad + a_1 = 0 \\ -2x_1 + x_2 \qquad -2x_5 \qquad + a_2 = 0 \\ x_1 \qquad + x_3 \qquad + x_5 - x_6 \qquad + a_3 = 3 \\ 2x_2 + x_3 + x_4 + 2x_5 + x_6 = 4 \end{array}$$

(Notice that x_4 can be stated as the basic variable, therefore no artificial variable is needed.)

4. Set objective function is mininimization of sum of all the artificial variables)

$$\text{Min } w = a_1 + a_2 + a_3$$

Row 0 is as follows:

$$w - a_1 - a_2 - a_3 = 0$$

5. Eliminate all artificial variables from row 0

$$\text{New } R_0 = R_0 + R1 + R_2 + R_3 \Rightarrow$$

$$w + x_3 + x_5 - x_6 = 3 \quad \text{New } R_0$$

Phase I LP - Initial Tableau:

⇓

w	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	a_3	RHS	BV	Ratio
1	0	0	**1**	0	1	-1	0	0	0	3	w=3	
0	1	-1	0	0	2	0	1	0	0	0	a_1=0	-
0	-2	1	0	0	-2	0	0	1	0	0	a_2=0	-
0	1	0	$\boxed{\mathbf{1}}$	0	1	-1	0	0	1	3	a_3=3	**3** ⇒
0	0	2	1	1	2	1	0	0	0	4	x_4=4	4

Phase I LP - Optimal Tableau:

w	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	a_3	RHS	BV
1	-1	0	0	0	0	0	0	0	-1	0	w=0
0	1	-1	0	0	2	0	1	0	0	0	a_1=0
0	-2	1	0	0	-2	0	0	1	0	0	a_2=0
0	1	0	1	0	1	-1	0	0	1	3	x_3=3
0	-1	2	0	1	1	2	0	0	-1	1	x_4=1

Solving the Phase I LP will result in one of the three cases

According to Phase I optimal tableau; $w = 0$ and two artificial variables (a_1 and a_2) are in the optimal Phase I basis, therefore the problem is an example of Case 3.

i. Drop from the optimal Phase I tableau all nonbasic artificial variables and any variable from the original problem that has a negative coefficient in row 0 of the optimal Phase I tableau. Drop Phase I row 0.

w	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	a_3	RHS	BV
1	-1	0	0	0	0	0	0	0	-1	0	w=0
0	1	-1	0	0	2	0	1	0	0	0	a_1=0
0	-2	1	0	0	-2	0	0	1	0	0	a_2=0
0	1	0	1	0	1	-1	0	0	1	3	x_3=3
0	-1	2	0	1	1	2	0	0	-1	1	x_4=1

ii. Combine the original objective function (*z*) with the constraints from the optimal Phase I tableau.

$$z - 40x_1 - 10x_2 \qquad -7x_5 - 14x_6 = 0$$

All basic variables has zero coefficient in the objective function, so there is no need for any row operation.

Phase II – Initial Tableau

z	x_2	x_3	x_4	x_5	⇓ x_6	a_1	a_2	RHS	BV	Ratio
1	-10	0	0	-7	**-14**	0	0	0	z=0	
0	-1	0	0	2	0	1	0	0	a_1=0	-
0	1	0	0	-2	0	0	1	0	a_2=0	-
0	0	1	0	1	-1	0	0	3	x_3=3	-
0	2	0	1	1	$\boxed{\mathbf{2}}$	0	0	1	x_4=1	**1/2** ⇒

iii. Solve Phase II LP using the simplex method. The optimal solution to the Phase II LP is the optimal solution to the original LP.

Phase II – Optimal Tableau

z	x_2	x_3	x_4	x_5	x_6	a_1	a_2	RHS	BV
1	4	0	7	0	0	0	0	7	z=7
0	-1	0	0	2	0	1	0	0	a_1=0
0	1	0	0	-2	0	0	1	0	a_2=0
0	1	1	1/2	3/2	0	0	0	7/2	x_3=7/2
0	1	0	1/2	1/2	1	0	0	1/2	x_6=1/2

Report:

$z = 7$, $x_3 = 3.5$; $x_6 = 0.5$; $x_1 = x_2 = x_5 = x_4 = 0$

4.6 UNRESTRICTED IN SIGN VARIABLES

Some variables are allowed to be unrestricted in sign (urs).

This topic will be covered at the class.

5. SENSITIVITY ANALYSIS AND DUALITY

5.1 SENSITIVITY ANALYSIS

5.1.1 Reduced Cost

For any nonbasic variable, the reduced cost for the variable is the amount by which the nonbasic variable's objective function coefficient must be improved before that variable will become a basic variable in some optimal solution to the LP.

If the objective function coefficient of a nonbasic variable x_k is improved by its reduced cost, then the LP will have alternative optimal solutions at least one in which x_k is a basic variable, and at least one in which x_k is not a basic variable.

If the objective function coefficient of a nonbasic variable x_k is improved by more than its reduced cost, then any optimal solution to the LP will have x_k as a basic variable and $x_k > 0$.

Reduced cost of a basic variable is zero (see definition)!

5.1.2 Shadow Price

We define the shadow price for the *i*th constraint of an LP to be the amount by which the optimal z value is "improved" (increased in a max problem and decreased in a min problem) if the RHS of the *i*th constraint is increased by 1.

This definition applies only if the change in the RHS of the constraint leaves the current basis optimal!

A ≥ constraint will always have a nonpositive shadow price; a ≤ constraint will always have a nonnegative shadow price.

5.1.3 Conceptualization

$$\max z = 6\,x_1 + x_2 + 10x_3$$

$$x_1 + x_3 \leq 100$$

$$x_2 \leq 1$$

All variables ≥ 0

This is a very easy LP model and can be solved manually without utilizing Simplex.

x_2 = 1 (This variable does not exist in the first constraint. In this case, as the problem is a maximization problem, the optimum value of the variable equals the RHS value of the second constraint).

x_1 = 0, x_3 = 100 (These two variables do exist only in the first constraint and as the objective function coefficient of x_3 is greater than that of x_1, the optimum value of x_3 equals the RHS value of the first constraint).

Hence, the optimal solution is as follows:

z = 1001, [x_1, x_2, x_3] = [0, 1, 100]

Similarly, sensitivity analysis can be executed manually.

Reduced Cost

As x_2 and x_3 are in the basis, their reduced costs are 0.

In order to have x_1 enter in the basis, we should make its objective function coefficient as great as that of x_3. In other words, improve the coefficient by 4 (10-6). New objective function would be (max $z = 10x_1 + x_2 + 10x_3$) and there would be at least two optimal solutions for [x_1, x_2, x_3]: [0, 1, 100] and [100, 1, 0].

Therefore reduced cost of x_1 equals to 4.

If we improve the objective function coefficient of x_1 more than its reduced cost, there would be a unique optimal solution: [100, 1, 0].

Shadow Price

If the RHS of the first constraint is increased by 1, new optimal solution of x_3 would be 101 instead of 100. In this case, new z value would be 1011.

If we use the definition: 1011 - 1001 = 10 is the shadow price of the first constraint.

Similarly the shadow price of the second constraint can be calculated as 1 (please find it).

5.1.4 Utilizing Lindo Output for Sensitivity

NOTICE: The objective function which is regarded as Row 0 in Simplex is accepted as Row 1 in Lindo.

Therefore the first constraint of the model is always second row in Lindo!!!

```
MAX     6 X1 + X2 + 10 X3
SUBJECT TO
2)   X1 + X3 <=   100
3)      X2   <=   1
  END


 LP OPTIMUM FOUND AT STEP      2
```

```
        OBJECTIVE FUNCTION VALUE

        1)      1001.000

  VARIABLE        VALUE          REDUCED COST
        X1         0.000000          4.000000
        X2         1.000000          0.000000
        X3       100.000000          0.000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         0.000000         10.000000
        3)         0.000000          1.000000

 NO. ITERATIONS=       2

 RANGES IN WHICH THE BASIS IS UNCHANGED:

                           OBJ COEFFICIENT RANGES
 VARIABLE          CURRENT        ALLOWABLE        ALLOWABLE
                    COEF          INCREASE         DECREASE
       X1        6.000000         4.000000         INFINITY
       X2        1.000000         INFINITY         1.000000
       X3       10.000000         INFINITY         4.000000

                           RIGHTHAND SIDE RANGES
      ROW          CURRENT        ALLOWABLE        ALLOWABLE
                     RHS          INCREASE         DECREASE
        2      100.000000         INFINITY       100.000000
        3        1.000000         INFINITY         1.000000
```

Lindo output reveals the reduced costs of x_1, x_2, and x_3 as 4, 0, and 0 respectively.

In the maximization problems, the reduced cost of a non-basic variable can also be read from the allowable increase value of that variable at obj. coefficient ranges. Here, the corresponding value of x_1 is 4.

In the minimization problems, the reduced cost of a non-basic variable can also be read from the allowable decrease value of that variable at obj. coefficient ranges.

The same Lindo output reveals the shadow prices of the constraints in the "dual price" section:

Here, the shadow price of the first constraint (Row 2) equals 10.

The shadow price of the second constraint (Row 3) equals 1.

If the change in the RHS of the constraint leaves the current basis optimal (within the allowable RHS range), the following equations can be used to calculate new objective function value:

for maximization problems

- new obj. fn. value = old obj. fn. value + (new RHS – old RHS) × shadow price

for minimization problems

- new obj. fn. value = old obj. fn. value – (new RHS – old RHS) × shadow price

For Lindo example, as the allowable increases in RHS ranges are infinity for each constraint, we can increase RHS of them as much as we want. But according to allowable decreases, RHS of the first constraint can be decreased by 100 and that of second constraint by 1.

Lets assume that new RHS value of the first constraint is 60.

As the change is within allowable range, we can use the first equation (max. problem):

$$z_{\text{new}} = 1001 + (60 - 100)\ 10 = 601.$$

5.1.5 Utilizing Graphical Solution for Sensitivity

Will be treated at the class.

5.1.6 The 100% Rule

Will be treated at the class.

5.2 DUALITY

5.2.1 Primal – Dual

Associated with any LP is another LP called the ***dual***. Knowledge of the dual provides interesting economic and sensitivity analysis insights. When taking the dual of any LP, the given LP is referred to as the ***primal***. If the primal is a max problem, the dual will be a min problem and vice versa.

5.2.2 Finding the Dual of an LP

The dual of a ***normal max*** problem is a ***normal min*** problem.

> *Normal max problem is a problem in which all the variables are required to be nonnegative and all the constraints are ≤ constraints.*
>
> *Normal min problem is a problem in which all the variables are required to be nonnegative and all the constraints are ≥ constraints.*

Similarly, the dual of a normal min problem is a normal max problem.

Finding the Dual of a Normal Max Problem

PRIMAL

$$\begin{array}{ll} \max z = & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{s.t.} & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \\ & \ldots \quad \ldots \quad \ldots \quad \ldots \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \\ & x_j \geq 0 \ (j = 1, 2, \ldots, n) \end{array}$$

DUAL

$$\begin{array}{ll} \min w = & b_1y_1 + b_2y_2 + \ldots + b_my_m \\ \text{s.t.} & a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \geq c_1 \\ & a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \geq c_2 \\ & \ldots \quad \ldots \quad \ldots \quad \ldots \\ & a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \geq c_n \\ & y_i \geq 0 \ (i = 1, 2, \ldots, m) \end{array}$$

Finding the Dual of a Normal Min Problem

PRIMAL

$$\begin{array}{ll} \min w = & b_1y_1 + b_2y_2 + \ldots + b_my_m \\ \text{s.t.} & a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \geq c_1 \\ & a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \geq c_2 \\ & \ldots \quad \ldots \quad \ldots \quad \ldots \\ & a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \geq c_n \\ & y_i \geq 0 \ (i = 1, 2, \ldots, m) \end{array}$$

DUAL

$$\begin{array}{ll} \max z = & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{s.t.} & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \\ & \ldots \quad \ldots \quad \ldots \quad \ldots \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \\ & x_j \geq 0 \ (j = 1, 2, \ldots, n) \end{array}$$

Finding the Dual of a Nonnormal Max Problem

- If the *i*th primal constraint is a ≥ constraint, the corresponding dual variable y_i must satisfy $y_i \leq 0$
- If the *i*th primal constraint is an equality constraint, the dual variable y_i is now unrestricted in sign (urs).
- If the *i*th primal variable is urs, the *i*th dual constraint will be an equality constraint

Finding the Dual of a Nonnormal Min Problem

- If the *i*th primal constraint is a ≤ constraint, the corresponding dual variable x_i must satisfy $x_i \leq 0$
- If the *i*th primal constraint is an equality constraint, the dual variable x_i is now urs.
- If the *i*th primal variable is urs, the *i*th dual constraint will be an equality constraint

5.2.3 The Dual Theorem

The primal and dual have equal optimal objective function values (if the problems have optimal solutions).

Weak duality implies that if for any feasible solution to the primal and an feasible solution to the dual, the *w*-value for the feasible dual solution will be at least as large as the *z*-value for the feasible primal solution → $z \leq w$.

Consequences

- Any feasible solution to the dual can be used to develop a bound on the optimal value of the primal objective function.
- If the primal is unbounded, then the dual problem is infeasible.
- If the dual is unbounded, then the primal is infeasible.
- How to read the optimal dual solution from Row 0 of the optimal tableau if the primal is a max problem:

  'optimal value of dual variable y_i'
  = 'coefficient of s_i in optimal row 0' (if const. *i* is a ≤ const.)
  = –'coefficient of e_i in optimal row 0' (if const. *i* is a ≥ const.)
  = 'coefficient of a_i in optimal row 0' – M (if const. *i* is a = const.)
- How to read the optimal dual solution from Row 0 of the optimal tableau if the primal is a min problem:

  'optimal value of dual variable *xi*'
  = 'coefficient of s_i in optimal row 0' (if const. *i* is a ≤ const.)
  = –' coefficient of e_i in optimal row 0' (if const. *i* is a ≥ const.)
  = 'coefficient of a_i in optimal row 0' + M (if const. *i* is a = const.)

5.2.4 Economic Interpretation

When the primal is a normal max problem, the dual variables are related to the value of resources available to the decision maker. For this reason, dual variables are often referred to as ***resource shadow prices***.

Example 1.

PRIMAL

Let $x_{1,}$ $x_{2,}$ x_3 be the number of desks, tables and chairs produced. Let the weekly profit be $\$z$. Then, we must

$$\max z = 60x_1 + 30x_2 + 20x_3$$

$$\begin{aligned} \text{s.t.} \quad & 8x_1 + 6x_2 + x_3 \le 48 \text{ (Lumber constraint)} \\ & 4x_1 + 2x_2 + 1.5x_3 \le 20 \text{ (Finishing hour constraint)} \\ & 2x_1 + 1.5x_2 + 0.5x_3 \le 8 \text{ (Carpentry hour constraint)} \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

DUAL

Suppose an entrepreneur wants to purchase all of Dakota's resources.

In the dual problem $y_{1,}$ $y_{2,}$ y_3 are the resource prices (price paid for one board ft of lumber, one finishing hour, and one carpentry hour).

$w is the cost of purchasing the resources.

Resource prices must be set high enough to induce Dakota to sell. i.e. total purchasing cost equals total profit.

$$\min w = 48y_1 + 20y_2 + 8y_3$$

$$\begin{aligned} \text{s.t.} \quad & 8y_1 + 4y_2 + 2y_3 \ge 60 \text{ (Desk constraint)} \\ & 6y_1 + 2y_2 + 1.5y_3 \ge 30 \text{ (Table constraint)} \\ & y_1 + 1.5y_2 + 0.5y_3 \ge 20 \text{ (Chair constraint)} \\ & y_1, y_2, y_3 \ge 0 \end{aligned}$$

5.3 DUALITY AND SENSITIVITY

Will be treated at the class.

5.4 COMPLEMENTARY SLACKNESS THEOREM

The Theorem of Complementary Slackness is an important result that relates the optimal primal and dual solutions. This is a very important theorem relating the primal and dual problems. It provides optimality conditions for any feasible solution of primal model and dual model.

Assume a PRIMAL LP, a normal maximization problem, consists of *n* decision variables x_1, x_2,…, x_n and *m* less than or equal to (≤) constraints with corresponding slack variables s_1, s_2,…, s_m. The related DUAL LP is a normal minimization problem that consists of *m* decision variables y_1, y_2,…, y_m , and *n* greater than or equal to (≥) constraints with excess variables e_1, e_2,…, e_n.

Suppose **x** = [x_1, x_2, … x_n] and **y** = [y_1, y_2, … y_m] are any feasible solutions to the primal and dual problems. Then they are respectively optimal if and only if

$s_i\, y_i = 0$ (i = 1,2,…,m)
$e_j\, x_j = 0$ (j = 1,2,…,n)

Hence, at optimality, if a variable (y_i or x_j) in one problem is positive, then the corresponding constraint in the other problem must be binding/tight (s_i or e_j = 0). If a constraint in one problem is not binding/tight (s_i or e_j > 0), then the corresponding variable in the other problem must be zero.

If solution to the dual problem is given the solution to the primal problem can be calculated using the complementary slackness theorem.

Example:

Consider the following LP,

$$\begin{aligned} &\min z = 3x_1 + 2x_2 + 4x_3 \\ &\text{S.t.} \quad 2x_1 + x_2 + 3x_3 = 60 \\ &\qquad 3x_1 + 3x_2 + 5x_3 \geq 120 \\ &\qquad x_1 + x_2 - 3x_3 \leq 150 \\ &x_1, x_2, x_3 \geq 0 \end{aligned}$$

The optimal solution of this LP is z= 90, $x_1 = 0$, $x_2 = 15$, $x_3 = 15$. Find solution of the dual model, and shadow prices and reduced costs of the primal model.

Answer:

Standard form of PRIMAL LP

$$\begin{array}{lll} \min z = 3x_1 + 2x_2 + 4x_3 & & \\ \text{s.t.} & 2x_1 + x_2 + 3x_3 & = 60 \\ & 3x_1 + 3x_2 + 5x_3 - e_2 & = 120 \\ & x_1 + x_2 - 3x_3 + s_3 & = 150 \\ \text{all variables} \geq 0 & & \end{array}$$

Standard form of DUAL LP

$$\begin{array}{lll} \max w = 60\,y_1 + 120\,y_2 + 150\,y_3 & & \\ \text{s.t.} & 2\,y_1 + 3\,y_2 + y_3 + sd_1 & = 3 \\ & y_1 + 3\,y_2 + y_3 + sd_2 & = 2 \\ & 3\,y_1 + 5\,y_2 - 3\,y_3 + sd_3 & = 4 \\ y_1 \text{ urs}, y_2 \geq 0, y_3 \leq 0\ ;\ sd_1, sd_2, sd_3 \geq 0. & & \end{array}$$

Out of the variables in Dual LP; y_1, y_2, y_3 are shadow prices of the primal model and sd_1, sd_2, sd_3 are reduced costs of the primal model.

Following conditions should hold at the optimal solution according to complementary slackness theorem:

$$x_1 * sd_1 = 0\ ;\ x_2 * sd_2 = 0;\ \ x_3 * sd_3 = 0$$

$$y_2 * e_2 = 0\ ;\ y_3 * s_3 = 0$$

As $z= 90$, $x_1 = 0$, $x_2 = 15$, $x_3 = 15$; $e_2 = 0$ and $s_3 = 180$ are found.

Utilizing the theorem of complementary slackness, we further know that $sd_2 = 0$, $sd_3 = 0$; $y_3 = 0$ since none of the corresponding complementary dual constraints are tight. When these values are utilized to dual LP, we come up with three equations with three unknowns

$$\begin{array}{ll} 2\,y_1 + 3\,y_2 + sd_1 & = 3 \\ y_1 + 3\,y_2 & = 2 \\ 3\,y_1 + 5\,y_2 & = 4 \end{array}$$

Here $y_1 = 1/2$; $y_2 = 1/2$; $sd_1 = 1/2$

Report: Solution of the dual LP model: $w = 90$; $y_1 = 1/2$; $y_2 = 1/2$; $y_3 = 0$; $sd_1 = 1/2$; $sd_2 = 0$; $sd_3 = 0$. Thus, for the primal LP, shadow prices of first and second constraints are 1/2, and shadow price of the third constraint is 0. Reduced costs of second and third decision variables are 0, and reduced cost of the first decision variable is 1/2.

5.5 DUAL SIMPLEX ALGORITHM

5.5.1 Three uses of the dual simplex

- Finding the new optimal solution after a constraint is added to an LP
- Finding the new optimal solution after changing a RHS of an LP
- Solving a normal minimization problem

5.5.2 Steps

1. Select the most negative RHS.
2. Basic variable of this pivot row leaves the solution.
3. Ratios are calculated for the variables with negative coefficients in the pivot row (coefficient at Row Zero / coefficient at pivot row)
4. The variable with the smallest absolute ratio enters the solution.

- If each variable in the pivot row has a nonnegative coefficient, then the LP has no feasible solution

Example:

z	x_1	x_2	**s_1**	s_2	s_3	RHS	BV
1	0	0	1.25	0.75	0	41.25	z
0	0	1	2.25	-0.25	0	2.25	x_2
0	1	0	-1.25	0.25	0	3.75	x_1
0	0	0	-0.75	-0.25	1	**-0.75**	**s_3**

s_3 leaves the solution since it has a negative RHS.

The variable with the smallest absolute ratio is s_1 among variables that have a negative coefficient in the pivot row ($|1.25 / -0.75|$ and $|0.75 / -0.25|$), therefore s_1 enters the solution.

Row operations are made.

z	x_1	x_2	s_1	s_2	s_3	RHS	BV
1	0	0	0	0.333	1.667	40	z
0	0	1	0	-1	3	0	x_2
0	1	0	0	0.667	-1.667	5	x_1
0	0	0	1	0.333	-1.333	1	s_1

Optimal solution: $z = 40$, $x_1 = 5$, $x_2 = 0$

5.5.3 Adding a Constraint

Supplementary example 1

Suppose that in the Dakota problem, marketing considerations dictate that at least 1 table be manufactured.

Answer

Add $x_2 \geq 1$

Because the current optimal solution ($z = 280$, $x_1 = 2$, $x_2 = 0$, $x_3 = 8$) does not satisfy the new constraint; it is no longer feasible.

To find the new optimal solution, we add a new row to the optimal tableau:

$$x_2 - e_5 = 1$$

To have e_5 as a BV, we should multiply this equation through by –1:

$$-x_2 + e_5 = -1$$

New tableau:

z	x_1	**x_2**	x_3	s_1	s_2	s_3	s_4	e_5	RHS	BV
1	0	5	0	0	10	10	0	0	280	z = 280
0	0	-2	0	1	2	-8	0	0	24	$s_1 = 24$
0	0	-2	1	0	2	-4	0	0	8	$x_3 = 8$
0	1	1.25	0	0	-0.5	1.5	0	0	2	$x_1 = 2$
0	0	1	0	0	0	0	1	0	5	$s_4 = 5$
0	0	**-1**	0	0	0	0	0	1	-1	**$e_5 = -1$**

e_5 leaves the solution and x_2 enters the solution.

The optimal solution:

$$z = 275,\ s_1 = 26,\ x_3 = 10,\ x_1 = 3/4,\ s_4 = 4,\ x_2 = 1$$

Supplementary example 2

Suppose that we add $x_1 + x_2 \geq 12$

Answer

Because the current optimal solution ($z = 280$, $x_1 = 2$, $x_2 = 0$, $x_3 = 8$) does not satisfy the new constraint; it is no longer feasible.

To find the new optimal solution, we add a new row to the optimal tableau:

$$x_1 + x_2 - e_5 = 12$$

To have e_5 as a BV, we should multiply this equation through by –1:

$$-x_1 - x_2 + e_5 = -12$$

New tableau:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	e_5	RHS	BV
1	0	5	0	0	10	10	0	0	280	z
0	0	-2	0	1	2	-8	0	0	24	s_1
0	0	-2	1	0	2	-4	0	0	8	x_3
0	1	1.25	0	0	-0.5	1.5	0	0	2	
0	0	1	0	0	0	0	1	0	5	s_4
0	-1	-1	0	0	0	0	0	1	-12	e_5

To have x_1 as a BV, row operations are made:

z	x_1	x_2	x_3	s_1	**s_2**	s_3	s_4	e_5	RHS	BV
1	0	5	0	0	10	10	0	0	280	z
0	0	-2	0	1	2	-8	0	0	24	s_1
0	0	-2	1	0	2	-4	0	0	8	x_3
0	1	1.25	0	0	-0.5	1.5	0	0	2	x_1
0	0	1	0	0	0	0	1	0	5	s_4
0	0	0.25	0	0	**-0.5**	1.5	0	1	**-10**	**e_5**

Iterations:

z	x_1	**x_2**	x_3	s_1	s_2	s_3	s_4	e_5	RHS	BV
1	0	10	0	0	0	40	0	20	80	z
0	0	-1	0	1	0	-2	0	4	-16	s_1
0	0	**-1**	1	0	0	2	0	4	**-32**	**x_3**
0	1	1	0	0	0	0	0	-1	12	x_1
0	0	1	0	0	0	0	1	0	5	s_4
0	0	-0.5	0	0	1	-3	0	-2	20	s_2

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	e_5	RHS	BV
1	0	0	10	0	0	60	0	60	-240	z
0	0	0	-1	1	0	-4	0	0	16	s_1
0	0	1	-1	0	0	-2	0	-4	32	x_2
0	1	0	1	0	0	2	0	3	-20	x_1
0	0	0	1	0	0	2	1	4	**-27**	**s_4**
0	0	0	-0.5	0	1	-4	0	-4	36	s_2

Each variable in the pivot row has a nonnegative coefficient:

LP has <u>no feasible solution</u>

5.5.4 Solving a normal minimization problem

Supplementary example 3

Solve the following LP

$$\min z = x_1 + 2x_2$$

$$\text{s.t.} \quad x_1 - 2x_2 + x_3 \geq 4$$

$$2x_1 + x_2 - x_3 \geq 6$$

$$x_1, x_2, x_3 \geq 0$$

Answer

z	**x_1**	x_2	x_3	e_1	e_2	RHS	BV
1	-1	-2	0	0	0	0	z
0	-1	2	-1	1	0	-4	e_1
0	**-2**	-1	1	0	1	**-6**	**e_2**

z	x_1	x_2	**x_3**	e_1	e_2	RHS	BV
1	0	-1.5	-0.5	0	-0.5	3	z
0	0	2.5	**-1.5**	1	-0.5	**-1**	**e_1**
0	1	0.5	-0.5	0	-0.5	3	x_1

z	x_1	x_2	x_3	e_1	e_2	RHS	BV
1	0	-2.333	0	-0.333	-0.333	3.333	z
0	0	-1.667	1	-0.667	0.333	0.667	x_3
0	1	-0.333	0	-0.333	-0.333	3.333	x_1

Optimal solution: $z = 10/3$, $x_1 = 10/3$, $x_2 = 0$, $x_3 = 2/3$

6. ADVANCED TOPICS IN LP

6.1 REVISED SIMPLEX ALGORITHM

Classical Simplex algorithm is not an efficient method especially for solving big size problems using computers. In simplex algorithm unnecessary data is computed and stored. The revised simplex method is a systematic procedure for implementing the steps of the simplex method using a smaller array, thus saving storage space.

The only pieces of information relevant at each iteration are

- the coefficients of the nonbasic variables in Row0,
- the coefficients of the entering basic variable in the other equations,
- and the right-hand sides of the equations.

The revised simplex method is useful to have a procedure that obtain this information efficiently without computing and storing all the other coefficients.

6.1.1 Representation of the Simplex Method in matrix form

Suppose number of variables=*n*, number of constraints =*m*,

Max Z = **cx**

Subject to **Ax** = **b** , **x** ≥ 0.

where **x** vector of decision variables, **c** vector of objective function coefficients, **A** matrix of technology coefficients, **b** vector of rhs values.

For example, consider Dakota Furniture LP;

$$\begin{aligned} \max z &= 60x_1+30x_2+20x_3 \\ \text{S.t.} \quad 8x_1+6x_2+x_3 &\le 48 \\ 4x_1+2x_2+1.5x_3 &\le 20 \\ 2x_1+1.5x_2+.5x_3 &\le 8 \\ x_2 &\le 5 \\ x_1,x_2,x_3 &\ge 0 \end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{c} = [60,\ 30, 20\ ], \mathbf{A} = \begin{bmatrix} 8 & 6 & 1 \\ 4 & 2 & 1.5 \\ 2 & 1.5 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 48 \\ 20 \\ 8 \\ 5 \end{bmatrix}.$$

Table of notations

c	$1 \times n$ row vector, objective function coefficients
x	$n \times 1$ column, decision variables
A	$m \times n$ matrix; technology coefficients
b	right-hand-side vector of the original tableau's constraints
BV	any set of basic variables (the first element of BV is the basic variable in the first constraint, the second variable in BV is the basic variable in the second constraint, and so on)
BV_j	the basic variable for constraint *j* in the desired tableau
NBV	the set of nonbasic variables
$\mathbf{a}_j$	column for x_j in the constraints of the original problem
B	$m \times m$ matrix whose *j*th column is the column for BV_j in the original constraints
N	$m \times (n - m)$ matrix whose columns are the columns for the nonbasic variables
c_j	coefficients of x_j in the objective function
$\mathbf{c}_B$	$1 \times m$ row vector whose *j*th element is the objective function coefficient for BV_j
$\mathbf{c}_N$	$1 \times (n\text{-}m)$ row vector whose *j*th element is the objective function coefficient for *j*th element of NBV
$\mathbf{x}_B$	$m \times 1$ column vector, basic variables
$\mathbf{x}_N$	$n\text{-}m \times 1$ column vector, nonbasic variables

An basic feasible solution (bfs) in simplex method can be defined with the basic variables in that bfs. For this, set of basic variables BV is defined.

For an BV **A**, **x** and **c** are divided into two parts corresponding to basic and nonbasic variables:

$\mathbf{A} = [\mathbf{B}, \mathbf{N}]$

$\mathbf{x} = [\mathbf{x}_B, \mathbf{x}_N]$

$\mathbf{c} = [\mathbf{c_B}, \mathbf{c_N}]$

Using these definitions the generic LP is formulated as follows:

$$\text{Max } Z = \mathbf{c}_B\mathbf{x}_B + \mathbf{c}_N\mathbf{x}_N$$

$$\text{S.t. } \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_{NB} = \mathbf{b}$$

$$\mathbf{x}_{BV}, \mathbf{x}_{NBV} \geq 0$$

where the definitions of the abbreviations are given in the table of notations.

B is an invertible matrix as it is composed of linear independent vectors. If the constraint of the given LP is multiplied by $\mathbf{B}^{-1}$ from two sides:

$$\mathbf{B}^{-1}\mathbf{B}\mathbf{x}_{\mathrm{B}} + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_{\mathrm{N}} = \mathbf{B}^{-1}\mathbf{b}$$

$$\mathbf{x}_{\mathrm{B}} + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_{\mathrm{N}} = \mathbf{B}^{-1}\mathbf{b}$$

Where $\mathbf{B}^{-1}\mathbf{N}$ gives the coefficients of nonbasic basic variables in the simplex tableau and $\mathbf{B}^{-1}\mathbf{b}$ gives the rhs values.

To find Row 0 of the simplex tableau we replace $\mathbf{x}_{\mathrm{B}}$ with $\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_{\mathrm{N}}$ in $\mathrm{Z} = \mathbf{c}_{\mathrm{B}}\mathbf{x}_{\mathrm{B}} + \mathbf{c}_{\mathrm{N}}\mathbf{x}_{\mathrm{N}}$ as follows:

$$\mathrm{Z} = \mathbf{c}_{\mathrm{B}}(\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_{\mathrm{N}}) + \mathbf{c}_{\mathrm{N}}\mathbf{x}_{\mathrm{N}}$$

$$\mathrm{Z} + (\mathbf{c}_{\mathrm{B}}B^{-1}N - \mathbf{c}_{\mathrm{N}})\mathbf{x}_{\mathrm{N}} = \mathbf{c}_{\mathrm{B}}B^{-1}\mathbf{b}$$

Where $(\mathbf{c}_{\mathrm{B}}B^{-1}N - \mathbf{c}_{\mathrm{N}})$ gives the coefficient of nonbasic variables in Row 0 and $\mathbf{c}_{\mathrm{B}}B^{-1}\mathbf{b}$ gives the rhs of Row 0. The coefficient of nonbasic variable in Row 0 is called reduced cost and shown as $z_j - c_j = \mathbf{c}_{\mathrm{BV}}\mathbf{B}^{-1}\mathbf{a}_j - c_j$.

The simplex tableau of a BV can be constructed with the given formulas as follows.

	z	$\mathbf{x}_B$	$\mathbf{x}_N$	RHS	
z	1	**0**	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$	Row 0 (R_0)
$\mathbf{x}_B$	**0**	$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{B}^{-1}\mathbf{b}$	Row 1 – m (R_1-R_m)

The optimality condition for this bfs is $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N \geq 0$ (for a max problem). If $z_j - c_j = \mathbf{c}_{\mathrm{BV}}\mathbf{B}^{-1}\mathbf{a}_j - c_j < 0$ for any *j* nonbasic variable, the current tableau is not optimal. We find a new bf after determining the entering and leaving variables.

For the above given table, $\mathbf{B}^{-1}\mathbf{N}$ values for the nonbasic variables except the entering variable are unnecessary but calculated in simplex method. To improve the efficiency of the simplex method the revised simplex method is developed.

6.1.2 Steps of Revised Simplex Method

(for Max problem)

Step 0: Note the columns from which the current $\mathbf{B}^{-1}$ will be read. Initially, $\mathbf{B}^{-1} = I$.

Step 1: For the current tableau, compute $\mathbf{w} = \mathbf{c}_{\mathrm{BV}}B^{-1}$. (**w** is called as simplex multipliers or shadow prices (dual prices))

Step 2: Price out all nonbasic variables $(z_j - c_j = \mathrm{c}_{\mathrm{BV}}\mathrm{B}^{-1}\mathrm{a}_j - c_j = \mathrm{w}\mathrm{a}_j - c_j)$ in the current tableau.

- If each nonbasic variable prices out to be nonnegative, then the current basis is optimal.
- If the current basis is not optimal, then enter into the basis the nonbasic variable with the most negative coefficient in row 0. Call this variable x_k.

Step 3: To determine the row in which x_k enters the basis,

- compute x_k's column in the current tableau ($\mathbf{y}_j = \mathbf{B}^{-1}\mathbf{a}_j$)
- compute the right-hand side of the current tableau ($\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}$)
- Use the ratio test to determine the row in which x_k should enter the basis.
- We now know the set of basic variables (BV) for the new tableau.

Step 4: Use the column for x_k in the current tableau to determine the EROs needed to enter x_k into the basis. Perform these EROs on the current B^{-1}. This will yield the new B^{-1}. Return to step 1.

Formulas used in Revised Simplex method

Formula	**Definition**
$\mathbf{y}_j = \mathbf{B}^{-1}\mathbf{a}_j$	Column for x_j in BV tableau
$\mathbf{w} = \mathbf{c_B}\mathbf{B}^{-1}$	Simplex multipliers – shadow prices (dual price)
$z_j - c_j = \mathbf{c_B}\mathbf{B}^{-1}\mathbf{a}_j - c_j$ $= \mathbf{w}\mathbf{a}_j - c_j$	Coefficient of x_j in row 0
$\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}$	Right-hand side of constraints in BV tableau – values of basic variables
$Z = \mathbf{c_B}\mathbf{B}^{-1}\mathbf{b} = \mathbf{c_B}\bar{\mathbf{b}} = \mathbf{w}\mathbf{b}$	Right-hand side of BV row 0 – objective function value

Example 1. Solve the below-given LP using revised simplex method.

$$\begin{aligned} \text{Max } Z = \ & x_1 + 2x_2 - x_3 + x_4 + 4x_5 - 2x_6 \\ \text{S.t. } \ & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \le 6 \\ & 2x_1 - x_2 - 2x_3 + x_4 \le 4 \\ & x_3 + x_4 + 2x_5 + x_6 \le 4 \end{aligned}$$

All variables ≥ 0

Convert each inequality constraint to standard form:

$$\begin{aligned} \text{Max } Z = \ & x_1 + 2x_2 - x_3 + x_4 + 4x_5 - 2x_6 \\ \text{S.t. } \ & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + s_1 = 6 \\ & 2x_1 - x_2 - 2x_3 + x_4 + s_2 = 4 \\ & x_3 + x_4 + 2x_5 + x_6 + s_3 = 4 \end{aligned}$$

All variables ≥ 0

Initially, slack variables are basic variables. BV = {s_1, s_2 , s_3 }

Step 0: **B** = [a$_7$, a$_8$, a$_9$] = I, **B**$^{-1}$ = **B** = I = $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

Iteration 1

Step 1: BV = {s_1, s_2 , s_3 }, **B**$^{-1}$ = I, $\mathbf{c}_{\mathrm{B}} = [0,0,0]$

$\mathbf{w} = \mathbf{c}_{\mathrm{B}}\mathbf{B}^{-1}$; $\mathbf{w} = [0,0,0]\mathbf{I} = [0,0,0]$

Step 2: calculate $(z_j - c_j = \mathrm{c_{BV}B^{-1}a}_j - c_j = \mathrm{wa}_j - c_j)$

$z_1 - c_1 = [0,0,0]\begin{bmatrix}1\\2\\0\end{bmatrix} - 1 = -1$; $\quad z_2 - c_2 = [0,0,0]\begin{bmatrix}1\\-1\\0\end{bmatrix} - 2 = -2$

$z_3 - c_3 = [0,0,0]\begin{bmatrix}1\\-2\\1\end{bmatrix} - (-1) = 1$; $\quad z_4 - c_4 = [0,0,0]\begin{bmatrix}1\\1\\1\end{bmatrix} - 1 = -1$

$z_5 - c_5 = [0,0,0]\begin{bmatrix}1\\0\\2\end{bmatrix} - 4 = -4$; $\quad z_6 - c_6 = [0,0,0]\begin{bmatrix}1\\0\\1\end{bmatrix} - (-2) = 2$

The current solution is not optimal because some of reduced costs are negative. x_5 is entering the basis as it has the most negative reduced cost.

Step 3: determining the leaving variable:

Column of x_5 in the current tableau: $\mathbf{y}_5 = \mathbf{B}^{-1}\mathbf{a}_5$=**I**$\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}1\\0\\2\end{bmatrix}$;

Rhs of the current tableau: $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} = \mathbf{I}\begin{bmatrix}6\\4\\4\end{bmatrix} = \begin{bmatrix}6\\4\\4\end{bmatrix}$

Ratio test: $\begin{bmatrix}6\\4\\4\end{bmatrix} / \begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{matrix}6\\-\\2*\end{matrix}$ $\quad s_3$ is the leaving variable

The new BV= {s_1, s_2 , x_5 }

Step 4: Calculate B^{-1} for the new BV. Apply EROs to convert $\mathbf{y}_5$ to $\begin{bmatrix}0\\0\\1\end{bmatrix}$:

To convert $\begin{bmatrix}1\\0\\2\end{bmatrix} \rightarrow \begin{bmatrix}0\\0\\1\end{bmatrix}$ required EROs: R3' = R3 / 2; R1' = R1 – R3'; R2' = R2.

If we apply the same EROs to $\boldsymbol{B}^{-1} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ we get the new $\boldsymbol{B}^{-1} = \begin{bmatrix}1&0&-1/2\\0&1&0\\0&0&1/2\end{bmatrix}$.

Iteration 2

Step 1: BV= {s_1, s_2 , x_5 } , $\boldsymbol{B}^{-1} = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$, $\mathbf{c}_{\mathrm{B}} = [0,0,4]$

$$\mathbf{w} = \mathbf{c}_{\mathrm{B}}\mathbf{B}^{-1} \text{ ; } \mathbf{w} = [0,0,4]\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} = [0,0,2]$$

Step 2: Calculate $(z_j - c_j = \mathrm{c}_{\mathrm{BV}}\mathrm{B}^{-1}\mathrm{a}_j - c_j = \mathrm{wa}_j - c_j)$.

$$z_1 - c_1 = [0,0,2]\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - 1 = -1 \text{ ;} \qquad z_2 - c_2 = [0,0,2]\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - 2 = -2$$

$$z_3 - c_3 = [0,0,2]\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} - (-1) = 3 \text{ ;} \qquad z_4 - c_4 = [0,0,2]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1 = 1$$

$$z_6 - c_6 = [0,0,2]\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - (-2) = 4; \qquad z_9 - c_9 = [0,0,2]\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 0 = 2 \text{ ;}$$

The current solution is not optimal because some of reduced costs are negative. x_2 is entering the basis as it has the most negative reduced cost.

Step 3: determining the leaving variable:

Column of x_2 in the current tableau: $\mathbf{y}_2 = \mathbf{B}^{-1}\mathbf{a}_2 = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$;

Rhs of the current tableau: $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}$

Ratio test: $\begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} / \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{matrix} 4 \; * \\ - \\ - \end{matrix}$ $\quad s_1$ is the leaving variable

The new BV= {x_2, s_2 , x_5 }

Step 4: Calculate B^{-1} for the new BV. Apply EROs to convert $\mathbf{y}_2$ to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$:

To convert $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ required EROs: R1' = R1; R2' = R2 + R1'; R3' = R3.

If we apply the same EROs to $\boldsymbol{B}^{-1} = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$ we get new $\boldsymbol{B}^{-1} = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 1 & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$.

Iteration 3

Step 1: BV= {x_2, s_2 , x_5 }, $\boldsymbol{B}^{-1} = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 1 & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$, $\mathbf{c}_{\mathrm{B}} = [2,0,4]$,

$$\mathbf{w} = \mathbf{c}_{\mathrm{B}}\mathbf{B}^{-1}\ ;\ \mathbf{w} = [2,0,4]\begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 1 & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix} = [2,0,1]$$

Step 2: Calculate $(z_j - c_j = \mathrm{c_{BV}B^{-1}a}_j - c_j = \mathrm{wa}_j - c_j)$

$$z_1 - c_1 = [2,0,1]\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - 1 = 1\ ; \qquad z_3 - c_3 = [2,0,1]\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} - (-1) = 4\ ;$$

$$z_4 - c_4 = [2,0,1]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1 = 2; \qquad z_6 - c_6 = [2,0,1]\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - (-2) = 5;$$

$$z_7 - c_7 = [2,0,1]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 0 = 2\ ; \qquad z_9 - c_9 = [2,0,1]\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 0 = 1\ ;$$

All reduced costs are nonnegative, therefore the current solution is optimal.

Calculate values of basic variables using the formula $x_B = \bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}$**:**

$\begin{bmatrix} x_2 \\ s_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 1 & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix}$. All nonbasic variables are equal to zero.

Calculate objective function value using the formula $\boldsymbol{Z} = \mathbf{c_B B^{-1} b} = \mathbf{c_{BV}\bar{b}} = \mathbf{wb};$

$$\mathbf{Z} = \mathbf{wb} = [2,0,1]\begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix} = 16.$$

6.1.3 The Revised Simplex Method in Tableau Format

The revised simplex method can be applied in tableau format especially while manually solving. The Revised simplex tableau includes right hand side values, simplex multipliers, and invers of the basic matrix. The column of the entering variable is added to the table when it is required.

Initialization Step

Find an initial basic feasible solution with basis inverse $\mathbf{B}^{-1}$. Calculate $\mathbf{w} = \mathbf{c}_B\mathbf{B}^{-1}$, $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}$, and form the following array (revised simplex tableau):

Basis Inverse	Rhs
w	$\mathbf{c}_B\bar{\mathbf{b}}$
$\mathbf{B}^{-1}$	$\bar{\mathbf{b}}$

Main Step

For each nonbasic variable, calculate $z_j - c_j = \mathbf{w}\mathbf{a}_j - \mathbf{c}_j$.

Let $z_k - c_k = \min_{j \in J}\{z_j - c_j\}$. If $z_k - c_k \geq 0$ stop! The current feasible solution is optimal.

Otherwise, calculate $\mathbf{y}_k = \mathbf{B}^{-1}\mathbf{a}_k$. If $\mathbf{y}_k \leq 0$ stop; the optimal objective value is unbounded. If $\mathbf{y}_k \nleq 0$ insert the column $\begin{bmatrix} z_k - c_k \\ \mathbf{y}_k \end{bmatrix}$ to the right of the tableau as follows:

Basis Inverse	Rhs
$\mathbf{w}$	$\mathbf{c}_B\bar{\mathbf{b}}$
$\mathbf{B}^{-1}$	$\bar{\mathbf{b}}$

x$_k$
$z_k - c_k$
$\mathbf{y}_k$

Determine the index *r* via the minimum ratio test: $\frac{\bar{b}_r}{y_{rk}} = \min_{1 \leq i \leq m}\left\{\frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0\right\}$

Pivot at *y*$_{rk}$. This updates the tableau. Repeat the main step.

Example 2. Solve the following LP using revised simplex method.

$$\begin{aligned}
\max z &= 60x_1 + 30x_2 + 20x_3 \\
\text{s.t.}\quad & 8x_1 + 6x_2 + x_3 \leq 48 \\
& 4x_1 + 2x_2 + 1{,}5x_3 \leq 20 \\
& 2x_1 + 1{,}5x_2 + 0{,}5x_3 \leq 8 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

Convert the problem to standard form:

$$\begin{aligned}
\max z &= 60x_1+30x_2+20x_3 \\
\text{s.t.}\quad & 8x_1 + 6x_2 + x_3 + s_1 = 48 \\
& 4x_1 + 2x_2 + 1{,}5x_3 + s_2 = 20 \\
& 2x_1 + 1{,}5x_2 + 0{,}5x_3 + s_3 = 8 \\
& \text{All variables} \geq 0
\end{aligned}$$

<u>Initialization Step</u>

Initially, slacks are basic variables: BV = {s_1, s_2 , s_3 },

$$\mathbf{B}^{-1} = \mathbf{B} = \mathrm{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad \mathbf{w} = \mathbf{c}_B\mathbf{B}^{-1} = [0,0,0]\mathbf{I} = [0,0,0]$$

$$\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} = \mathbf{I}\begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix} \qquad \mathbf{c}_B\bar{\mathbf{b}} = [0,0,0]\begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix} = 0$$

Form the revised simplex tableau:

	Basis Inverse			Rhs
z	0	0	0	0
s_1	1	0	0	48
s_2	0	1	0	20
s_3	0	0	1	8

Main Step – Iteration 1

Calculate $z_j - c_j = \mathbf{wa}_j - \mathbf{c}_j$ for each non basic variable:

$$z_1 - c_1 = \mathbf{wa}_1 - \mathbf{c}_1 = [0,0,0]\begin{bmatrix}8\\4\\2\end{bmatrix} - 60 = -60$$

$$z_2 - c_2 = \mathbf{wa}_2 - \mathbf{c}_2 = [0,0,0]\begin{bmatrix}6\\2\\1.5\end{bmatrix} - 30 = -30$$

$$z_3 - c_3 = \mathbf{wa}_3 - \mathbf{c}_3 = [0,0,0]\begin{bmatrix}1\\1.5\\0.5\end{bmatrix} - 20 = -20$$

$\boldsymbol{z_k - c_k} = \mathbf{min}_{j\in J}\{\boldsymbol{z_j - c_j}\} = \boldsymbol{min}\{-60,-30,-20\} = -60 < 0$; the current bfs is not optimal.

x_1 is entering variable; $k = 1$. $\mathbf{y}_1 = \mathbf{B}^{-1}\mathbf{a}_1 = \mathbf{I}\begin{bmatrix}8\\4\\2\end{bmatrix} = \begin{bmatrix}8\\4\\2\end{bmatrix}$: Column $\begin{bmatrix}\frac{z_k - c_k}{\mathbf{y}_k}\end{bmatrix}$ is inserted to the right of the revised simplex tableau.

	Basis Inverse			rhs	x_1	Ratio
z	0	0	0	0	-60	
s_1	1	0	0	48	8	48/8 = 6
s_2	0	1	0	20	4	20/4=5
s_3	0	0	1	8	2	8/2=4**

s_3 is specified as leaving variable through the ratio test.

EROs are applied to the tableau according to the column $\begin{bmatrix}-60\\8\\4\\1\end{bmatrix} \rightarrow \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

R3' = R3 / 2; R2' = R2 – 4R3', R1' = R1 – 8R3', R0' = R0 + 60R3'

	Basis Inverse			rhs
z	0	0	30	240
s_1	1	0	-4	16
s_2	0	1	-2	4
x_1	0	0	0.5	4

Main Step – Iteration 2

Calculate $z_j - c_j = \mathbf{wa}_j - \mathbf{c}_j$ for each non basic variable. Get $\mathbf{w}$ from revised simplex tableau!

$$z_2 - c_2 = \mathbf{wa}_2 - \mathbf{c}_2 = [0,0,30]\begin{bmatrix}6\\2\\1.5\end{bmatrix} - 30 = 15$$

$$z_3 - c_3 = \mathbf{wa_3} - \mathbf{c_3} = [0,0,30]\begin{bmatrix}1\\1.5\\0.5\end{bmatrix} - \mathbf{20} = -5$$

$$z_6 - c_6 = \mathbf{wa}_6 - \mathbf{c}_6 = [0,0,30]\begin{bmatrix}0\\0\\1\end{bmatrix} - 0 = 30$$

$\boldsymbol{z_k - c_k} = \mathbf{min}_{j\in J}\{\boldsymbol{z_j - c_j}\} = \boldsymbol{min}\{15, -5, 30\} = -5 < 0$; the current bfs is not optimal.

x_3 is the entering variable; k = 3. $\mathbf{y}_3 = \mathbf{B}^{-1}\mathbf{a}_3 = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 1.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} -1 \\ 0.5 \\ 0.25 \end{bmatrix}$. Column $\begin{bmatrix} \frac{z_k - c_k}{\mathbf{y}_k} \end{bmatrix}$ is inserted to the right of the revised simplex tableau.

	Basis Inverse			rhs	x_3	Ratio
z	0	0	30	240	-5	
s_1	1	0	-4	16	-1	--
s_2	0	1	-2	4	$\boxed{0.5}$	4/0.5=8**
x_1	0	0	0.5	4	0.25	4/0.25=16

s_2 is specified as leaving variable through the ratio test.

EROs are applied to the tableau according to the column $\begin{bmatrix} -5 \\ -1 \\ 0.5 \\ 0.25 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$:

R2' = R2 / 0.5; R1' = R1 + R2', R1' = R1 – 0.25 R2', R0' = R0 + 5R2'

	Basis Inverse			rhs
z	0	10	10	280
s_1	1	2	-8	24
x_3	0	2	-4	8
x_1	0	-0.5	1.5	2

Main Step – Iteration 3

Calculate $z_j - c_j = \mathbf{w}\mathbf{a}_j - \mathbf{c}_j$ for each non basic variable. Get $\mathbf{w}$ from revised simplex tableau!

$$z_2 - c_2 = \mathbf{w}\mathbf{a}_2 - \mathbf{c}_2 = [0, 10, 10]\begin{bmatrix} 6 \\ 2 \\ 1.5 \end{bmatrix} - 30 = 5$$

$$z_5 - c_5 = \mathbf{w}\mathbf{a}_5 - \mathbf{c}_5 = [0, 10, 10]\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \mathbf{0} = 10$$

$$z_6 - c_6 = \mathbf{w}\mathbf{a}_6 - \mathbf{c}_6 = [0, 10, 10]\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 0 = 10$$

$\boldsymbol{z_k - c_k} = \mathbf{min}_{j\in J}\{\boldsymbol{z_j - c_j}\} = \boldsymbol{min}\{5, 10, 10\} = 5 \geq 0$; the current bfs is optimal.

Get the values of the decision variables and objective function from the tableau:

x_1 = 2, x_2 = 0, x_3 = 8, z = 280.

6.2 UTILIZING SIMPLEX FOR SENSITIVITY

This topic is explained through the Dakota furniture example. Consider that x_1, x_2, and x_3 are representing the number of desks, tables, and chairs produced.

The LP formulated for profit maximization is given as follows:

$$\begin{array}{llllllllll}
\max z = & 60\,x_1 & 30\,x_2 & 20x_3 & & & & & & \\
& 8\,x_1 & +\ 6\,x_2 & +\ \ x_3 & +\,s_1 & & & & = 48 & \text{Lumber} \\
& 4\,x_1 & +\ 2\,x_2 & +1.5\,x_3 & & +\,s_2 & & & = 20 & \text{Finishing} \\
& 2\,x_1 & +1.5\,x_2 & +\ .5\,x_3 & & & +\,s_3 & & = \ 8 & \text{Carpentry} \\
& & x_2 & & & & & +\,s_4 & = \ 5 & \text{Demand}
\end{array}$$

The optimal solution is:

$$\begin{array}{lllllllll}
z & & +5\,x_2 & & & +10\,s_2 & +10\,s_3 & & = 280 \\
& & -2\,x_2 & & +s_1 & +2\,s_2 & -8\,s_3 & & = \ \ 24 \\
& & -2\,x_2 & +\,x_3 & & +2\,s_2 & -4\,s_3 & & = \ \ \ 8 \\
& +\,x_1 & +\,1.25\,x_2 & & & -.5\,s_2 & +1.5\,s_3 & & = \ \ \ 2 \\
& & x_2 & & & & & +\,s_4 & = \ \ \ 5
\end{array}$$

In the optimal solution BV: {s_1, x_3, x_1} , NBV:{x_2,s_2,s_3}, $\mathbf{w} = \mathbf{c}_B\mathbf{B}^{-1}$= [0,10,10] and

$$B^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0{,}5 & 1{,}5 \end{bmatrix}.$$

Analysis 1: Changing the Objective Function Coefficient of a Nonbasic Variable

If objective function coefficient of a nonbasic variable x_j is changed to c'_j , we check the reduced cost of that variable [$z_j - c_j = \mathbf{c}_B\mathbf{B}^{-1}a_j$ - c'_j]:

If $z_j - c_j \geq 0$ (for a max problem), the current basis remain optimal and current solution does not change.

If $z_j - c_j < 0$ (for a max problem), the current basis does not remain optimal. The simplex algorithm is used to find the new optimal solution: x_j is the entering variable and the ratio test gives the leaving variable.

Suppose δ represents the amount by which x_j is changed from its current value, i.e., $c'_j = c_j + \delta$. To find the ranges for the objective function coefficient of x_j in which the current basis remain optimal, we find values of δ which satisfies $z_j - c_j \geq 0$.

Example 1. In the Dakota problem, find the range for the objective function coefficient of x_2 in which the current basis does not change.

Answer: $c'_2 = 30 + \delta \Rightarrow z_2 - c_2 = [0,10,10]\begin{bmatrix}6\\2\\1,5\end{bmatrix} - (30 + \delta) \geq 0$

$$5 - \delta \geq 0 \qquad \Rightarrow \qquad \delta \leq 5$$

Or If $c'_2 \leq 35$ then the current basis does not change.

Example 2. In the Dakota problem, what would be the optimal solution if the sales price of the table is changed to 40?

Answer: Sales price of a table is related to the objective function coefficient of x_2. According to the ranges found in Example 1, If it is changed to 40 (δ =10), the current basis changes. x_1'in will enter the new basis, and leaving variables is found by using ratio test (Please find the new optimal solution by yourself).

Analysis 2: Changing the Objective Function Coefficient of a Basic Variable

If objective function coefficient of a basic variable x_k is changed to c'_k, we check the reduced costs of all nonbasic variables [$z_j - c_j = \mathbf{c}_B\mathbf{B}^{-1}\mathrm{a}_j - c'_j$]:

If $z_j - c_j \geq 0$ for all nonbasic variables (for a max problem), the current basis remain optimal. The current solution may change. We utilize $Z = \mathbf{c_B B^{-1} b} = \mathbf{c_B \bar{b}}$ to find the new optimal solution.

If $z_j - c_j < 0$ for at least one nonbasic variable (for a max problem), the current basis does not remain optimal. The simplex algorithm is used to find the new optimal solution.

Suppose δ represents the amount by which x_k is changed from its current value, i.e., $c'_k = c_k + \delta$. To find the ranges for the objective function coefficient of x_k in which the current basis remain optimal, we find values of δ which satisfies $z_j - c_j \geq 0$ for all nonbasic varaibles.

Example 3. In the Dakota problem, find the range for the sales price of the desk (the objective function coefficient of x_1) in which the current basis does not change.

Answer: $c'_1 = 60 + \delta$

$$z_2 - c_2 = [0, 20, 60 + \delta]\begin{bmatrix}1 & 2 & -8\\0 & 2 & -4\\0 & -0,5 & 1,5\end{bmatrix}\begin{bmatrix}6\\2\\1,5\end{bmatrix} - 30 \geq 0 \Rightarrow 5 + 1.25\delta \geq 0 \Rightarrow \delta \geq -4$$

$$z_5 - c_5 = [0, 20, 60 + \delta]\begin{bmatrix}1 & 2 & -8\\0 & 2 & -4\\0 & -0,5 & 1,5\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} - 0 \geq 0 \Rightarrow 10 - 0.5\delta \geq 0 \Rightarrow \delta \leq 20$$

$$z_6 - c_6 = [0, 20, 60 + \delta] \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0{,}5 & 1{,}5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 0 \geq 0 \Rightarrow 10 + 1.5\delta \geq 0 \Rightarrow \delta \geq -20/3$$

Thus, the current basis will remain optimal if and only if $-4 \leq \delta \leq 20$ or $56 \leq c'_1 \leq 80$.

Example 4. In the Dakota problem, what would be the optimal solution if the sales price of the desk (the objective function coefficient of x_1) is changed to 50?

Answer: According to the ranges found in Example 3, If the sales price of the desk (the objective function coefficient of x_1) is changed to 50, the current basis does not remain optimal. The revised simplex tableau is built to find the new solution:

	Basis Inverse			RHS
z	0	15	-5	260
s_1	1	2	-8	24
x_3	0	2	-4	8
x_1	0	-0.5	1.5	2

Calculate reduced costs for nonbasic variables:

$z_2 - c_2 = -7.5;\quad z_5 - c_5 = 15;\quad z_6 - c_6 = -5$

x_2 is entering variable with its most negative reduced cost. Add x_2 column:

	Basis Inverse			RHS	x_2	Ratio
z	0	15	-5	260	-7,5	
s_1	1	2	-8	24	-2	-
x_3	0	2	-4	8	-2	-
x_1	0	-0.5	1.5	2	1.25	1.6*

x_1 is leaving variable. The new tableau:

	Basis Inverse			RHS
z	0	12	4	272
s_1	1	1,2	-5,6	27,2
x_3	0	1,2	-1,6	11,2
x_2	0	-0,4	1,2	1,6

It is required to calculate reduced costs for all nonbasic variables to check its optimality (Please calculate yourself). All reduced costs are nonnegative, therefore it is optimal.

Consequently, if the sales price of the desk is decreased to 50, Dakato is better to produce table instead of desk. Solution is $x_2 = 1.6$; $x_3 = 11.2$, and the profit is 272.

Analysis 3: Changing the Right-Hand Side of a Constraint

If rhs of constraint *i* is changed to b'_i , we check the rhs of the optimal tableau $[\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}]$.

If $\bar{\mathbf{b}} \geq \mathbf{0}$ the current basis remain optimal and feasible. The current solution and the values of the decision variables may change. We utilize $[\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}]$ and $Z = \mathbf{c_B}\mathbf{B}^{-1}\mathbf{b} = \mathbf{c_B}\bar{\mathbf{b}}$ to find the new values of decision variables and the optimal solution.

If $\bar{\mathbf{b}} \ngeq \mathbf{0}$ (at least one of the rhs values is negative), the current basis is not feasible. The dual simplex method is used to find the new optimal solution.

Suppose δ represents the amount by which b_i is changed from its current value, i.e., $b'_i = b_i + \delta$. To find the ranges for the rhs of that constraint in which the current basis remain optimal, we find values of δ which satisfies $\bar{\mathbf{b}} \geq \mathbf{0}$.

Example 5. In the Dakota problem, find the range for the available finishing hours (the rhs of the second constraint) in which the current basis remain optimal.

Answer: $b'_2 = 20+\delta$

$$\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0{,}5 & 1{,}5 \end{bmatrix} \begin{bmatrix} 48 \\ 20+\delta \\ 8 \end{bmatrix} = \begin{bmatrix} 24+2\delta \\ 8+2\delta \\ 2-0.5\delta \end{bmatrix} \geq 0 \quad \Rightarrow \quad \begin{matrix} \delta \geq -12 \\ \delta \geq -4 \\ \delta \leq 4 \end{matrix}$$

The current basis remain optimal if and only if $-4 \leq \delta \leq 4$ or $16 \leq b'_2 \leq 24$.

Example 6. In the Dakota problem, what would be the optimal solution if available finishing hours (the rhs of the second constraint) is decreased to 18?

Answer: According to the ranges found in example 5, if available finishing hours is decreased to 18, the current basis remain optimal. The values of the decision variables are $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0{,}5 & 1{,}5 \end{bmatrix} \begin{bmatrix} 48 \\ 18 \\ 8 \end{bmatrix} = \begin{bmatrix} 20 \\ 4 \\ 3 \end{bmatrix}$, which yields $x_1 = 3$, $x_2 = 0$, $x_3 = 4$. The new objective function value is $= \mathbf{c_B}\bar{\mathbf{b}} = [0, 20, 60] \begin{bmatrix} 20 \\ 4 \\ 3 \end{bmatrix} = 240.$

Analysis 4: Adding a new decision variable

If a new decision variable x_j is added to a problem, we check the reduced cost of that variable $[z_j - c_j = \mathbf{c_B}\mathbf{B}^{-1}\mathrm{a}_j - c'_j]$:

If $z_j - c_j \geq 0$ (for a max problem), the current basis remain optimal and current solution does not change.

If $z_j - c_j < 0$ (for a max problem), the current basis does not remain optimal. The simplex algorithm is used to find the new optimal solution: x_j is the entering variable and the ratio test gives the leaving variable.

Example 7. In the Dakota problem, the company plans to produce a new product: a coffee table. The coffee table will consume one units of lumber, finishing and carpentry and will be sold for 15$. Find out if the production of this new product is feasible or not.

Answer: Suppose x_7 is amount of coffee table produced, c_7 = 15, $a_7 = \begin{bmatrix}1\\1\\1\end{bmatrix}$. Calculate reduced cost for x_7: $z_8 - c_8 =$ [0, 10, 10] $\begin{bmatrix}1\\1\\1\end{bmatrix} - 15 = 5$. It is positive, therefore current basis remain optimal. it is not feasible (will not increase the profit) to produce coffee table.

Analysis 5: Adding a new constraint

If a new constraint is added to a problem, we check the rhs of the optimal tableau [$\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}$]. When a new constraint is introduced, the number of basic variables increases by one. The new constraint's slack or excess variable become basic variable in the current basis. Therefore **B** and $\mathbf{B}^{-1}$ are revised before calculating the new rhs.

If $\bar{\mathbf{b}} \geq \mathbf{0}$ the current basis does not violate the new constraint. The current basis remain optimal, and the current solution does not change.

If $\bar{\mathbf{b}} \ngeq \mathbf{0}$ (at least one of the rhs values is negative), the current basis violets the new constraint. The current basis is not feasible. The dual simplex method is used to find the new optimal solution.

Example 8. In the Dakota problem, It is required to make a final quality control for all products. Each product is controlled at 0.5 hours and 7 hours per week are available for quality control. Find the solution for the new situation.

Answer:

The new constraint: $0{,}5x_1 + 0{,}5x_2 + 0{,}5x_3 \leq 8 \quad \Rightarrow \quad 0{,}5x_1 + 0{,}5x_2 + 0{,}5x_3 + s_4 = 8$

The new basis: BV= $\{s_1, x_3, x_1, s_4\}$, NBV=$\{x_2, s_2, s_3\}$

$B = \begin{bmatrix} 1 & 1 & 8 & 0 \\ 0 & 1.5 & 4 & 0 \\ 0 & 0.5 & 2 & 0 \\ 0 & 0.5 & 0.5 & 1 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 2 & -8 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & -0.5 & 1.5 & 0 \\ 0 & -0.75 & 1.25 & 1 \end{bmatrix}$.

$\bar{\mathbf{b}} = \mathbf{B^{-1}b} = \begin{bmatrix} 1 & 2 & -8 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & -0.5 & 1.5 & 0 \\ 0 & -0.75 & 1.25 & 1 \end{bmatrix} \begin{bmatrix} 48 \\ 20 \\ 8 \\ 7 \end{bmatrix} = \begin{bmatrix} 24 \\ 8 \\ 2 \\ 2 \end{bmatrix} \geq 0$, the current basis remain optimal.

7. TRANSPORTATION PROBLEMS

7.1 FORMULATING TRANSPORTATION PROBLEMS

In general, a transportation problem is specified by the following information:

- A set of *m* ***supply points*** form which a good is shipped. Supply point *i* can supply at most s_i units.
- A set of *n* ***demand points*** to which the good is shipped. Demand point *j* must receive at least d_j units of the shipped good.
- Each unit produced at supply point *i* and shipped to demand point *j* incurs a variable cost of c_{ij}.

The relevant data can be formulated in a ***transportation tableau***:

	Demand point 1	Demand point 2	.....	Demand point *n*	SUPPLY
Supply point 1	c_{11}	c_{12}		c_{1n}	s_1
Supply point 2	c_{21}	c_{22}		c_{2n}	s_2
.....					
Supply point *m*	c_{m1}	c_{m2}		c_{mn}	s_m
DEMAND	d_1	d_2		d_n	

If total supply equals total demand then the problem is said to be a ***balanced transportation problem***.

Let x_{ij} = number of units shipped from supply point i to demand point j

then the general LP representation of a transportation problem is

min $\Sigma_i \Sigma_j c_{ij} x_{ij}$

s.t. $\Sigma_j x_{ij} \leq s_i$ (i=1,2, ..., m) Supply constraints

$\Sigma_i x_{ij} \geq d_j$ (j=1,2, ..., n) Demand constraints

$x_{ij} \geq 0$

If a problem has the constraints given above and is a *maximization* problem, it is still a transportation problem.

7.1.1 Formulating a Balanced Transportation Problem

Example 1. Powerco

Powerco has three electric power plants that supply the needs of four cities. Each power plant can supply the following numbers of kwh of electricity: plant 1, 35 million; plant 2, 50 million; and plant 3, 40 million. The peak power demands in these cities as follows (in kwh): city 1, 45 million; city 2, 20 million; city 3, 30 million; city 4, 30 million. The costs of sending 1 million kwh of electricity from plant to city is given in the table below. To minimize the cost of meeting each city's peak power demand, formulate a balanced transportation problem in a transportation tableau and represent the problem as an LP model.

	To			
From	City 1	City 2	City 3	City 4
Plant 1	\$8	\$6	\$10	\$9
Plant 2	\$9	\$12	\$13	\$7
Plant 3	\$14	\$9	\$16	\$5

Answer:

Representation of the problem as a LP model

x_{ij}: number of (million) kwh produced at plant *i* and sent to city *j*.

$$\min z = 8\,x11 + 6\,x12 + 10\,x13 + 9\,x14 + 9\,x21 + 12\,x22 + 13\,x23 + 7\,x24 + 14\,x31 + 9\,x32 + 16\,x33 + 5\,x34$$

s.t. $x11 + x12 + x13 + x14 \leq 35$ (supply constraints)

$x21 + x22 + x23 + x24 \leq 50$

$x31 + x32 + x33 + x34 \leq 40$

$x11 + x21 + x31 \geq 45$ (demand constraints)

$x12 + x22 + x32 \geq 20$

$x13 + x23 + x33 \geq 30$

$x14 + x24 + x34 \geq 30$

$x_{ij} \geq 0$ (i = 1, 2, 3; j = 1, 2, 3, 4)

Formulation of the transportation problem

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8	6	10	9	35
Plant 2	9	12	13	7	50
Plant 3	14	9	16	5	40
DEMAND	45	20	30	30	125

Total supply & total demand both equal 125: "balanced transportation problem".

7.1.2 Balancing an Unbalanced Transportation Problem

Excess Supply

If total supply exceeds total demand, we can balance a transportation problem by creating a ***dummy demand point*** that has a demand equal to the amount of excess supply. Since shipments to the dummy demand point are not real shipments, they are assigned a cost of zero. These shipments indicate unused supply capacity.

Unmet Demand

If total supply is less than total demand, actually the problem has no feasible solution. To solve the problem it is sometimes desirable to allow the possibility of leaving some demand unmet. In such a situation, *a penalty is often associated with unmet demand.* This means that a ***dummy supply point*** should be introduced.

Example 2. Modified Powerco for Excess Supply

Suppose that demand for city 1 is 40 million kwh. Formulate a balanced transportation problem.

Answer: Total demand is 120, total supply is 125.

To balance the problem, we would add a dummy demand point with a demand of 125 – 120 = 5 million kwh.

From each plant, the cost of shipping 1 million kwh to the dummy is 0.

For details see the following table.

	City 1	City 2	City 3	City 4	Dummy	SUPPLY
Plant 1	8	6	10	9	0	35
Plant 2	9	12	13	7	0	50
Plant 3	14	9	16	5	0	40
DEMAND	40	20	30	30	5	125

Example 3. Modified Powerco for Unmet Demand

Suppose that demand for city 1 is 50 million kwh. For each million kwh of unmet demand, there is a penalty of 80$. Formulate a balanced transportation problem.

Answer:

We would add a dummy supply point having a supply of 5 million kwh representing shortage.

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8	6	10	9	35
Plant 2	9	12	13	7	50
Plant 3	14	9	16	5	40
Dummy (Shortage)	80	80	80	80	5
DEMAND	50	20	30	30	130

7.2 FINDING A BFS FOR A TRANSPORTION PROBLEM

For a balanced transportation problem, general LP representation may be written as:

$$\min \Sigma_i \Sigma_j c_{ij} x_{ij}$$

s.t. $\Sigma_j x_{ij} = s_i$ (i=1,2, ..., m) Supply constraints

$\Sigma_i x_{ij} = d_j$ (j=1,2, ..., n) Demand constraints

$x_{ij} \geq 0$

To find a bfs to a balanced transportation problem, we need to make the following important observation:

If a set of values for the x_{ij}'s satisfies all but one of the constraints of a balanced transportation problem, the values for the x_{ij}'s will automatically satisfy the other constraint.

This observation shows that when we solve a balanced transportation, we may omit from consideration any one of the problem's constraints and solve an LP having m+n-1 constraints. We arbitrarily assume that the first supply constraint is omitted from consideration. In trying to find a bfs to the remaining m+n-1 constraints, you might think that any collection of m+n-1 variables would yield a basic solution. But this is not the case: If the m+n-1 variables yield a basic solution, the cells corresponding to this set contain ***no loop***.

An ordered sequence of at least four different cells is called a loop if

- Any two consecutives cells lie in either the same row or same column
- No three consecutive cells lie in the same row or column
- The last cell in the sequence has a row or column in common with the first cell in the sequence

The following three methods can be used to find a bfs for a balanced transportation problem:

1. Northwest Corner method
2. Minimum cost method
3. Vogel's method

7.2.1 Northwest Corner Method

We begin in the upper left corner of the transportation tableau and set x_{11} as large as possible (clearly, x_{11} can be no larger than the smaller of s_1 and d_1).

- If $x_{11}=s_1$, cross out the first row of the tableau. Also change d_1 to d_1-s_1.
- If $x_{11}=d_1$, cross out the first column of the tableau. Change s_1 to s_1-d_1.
- If $x_{11}=s_1=d_1$, cross out either row 1 or column 1 (but not both!).
    - If you cross out row, change d_1 to 0.
    - If you cross out column, change s_1 to 0.

Continue applying this procedure to the most northwest cell in the tableau that does not lie in a crossed out row or column.

Eventually, you will come to a point where there is only one cell that can be assigned a value. Assign this cell a value equal to its row or column demand, and cross out both the cell's row or column. A bfs has now been obtained.

Example 4. Consider a balanced transportation problem (costs are not needed to find a bfs!):

				5
				1
				3
2	4	2	1	

2				3
▨				1
▨				3
X	4	2	1	

2	3	▨	▨	X
▨				1
▨				3
X	1	2	1	

2	3	▨	▨	X
▨	1	▨	▨	X
▨				3
X	0	2	1	

2	3	▨	▨	X
▨	1	▨	▨	X
▨	0	2	1	3
X	0	2	1	

NWC method assigned values to $m+n-1$ (3+4-1 = 6) variables. The variables chosen by NWC method can not form a loop, so a bfs is obtained.

7.2.2 Minimum Cost Method

Northwest Corner method does not utilize shipping costs, so it can yield an initial bfs that has a very high shipping cost. Then determining an optimal solution may require several pivots.

To begin the minimum cost method, find the variable with the smallest shipping cost (call it x_{ij}). Then assign x_{ij} its largest possible value, min $\{s_i, d_j\}$.

As in the NWC method, cross out row *i* or column *j* and reduce the supply or demand of the noncrossed-out of row or column by the value of x_{ij}.

Continue like NWC method (instead of assigning upper left corner, the cell with the minimum cost is assigned). See Northwest Corner Method for the details!

Example 5.

2	3	5	6	5
2	1	3	5	10
3	8	4	6	15
12	8	4	6	

2	3	5	6	5
2	**1** 8	3	5	2
3	8	4	6	15
12	X	4	6	

2	3	5	6	5
2 2	1 8	3	5	X
3	8	4	6	15
10	X	4	6	

2 5	3	5	6	X
2 2	1 8	3	5	X
3	8	4	6	15
5	X	4	6	

2 5	3	5	6	X
2 2	1 8	3	5	X
3 5	8	4 4	6 6	10
5	X	4	6	

7.2.3 Vogel's Method

Begin by computing for each row and column a penalty equal to the difference between the two smallest costs in the row and column. Next find the row or column with the largest penalty. Choose as the first basic variable the variable in this row or column that has the smallest cost. As described in the NWC method, make this variable as large as possible, cross out row or column, and change the supply or demand associated with the basic variable (See Northwest Corner Method for the details!). Now recomputed new penalties (using only cells that do not lie in a crossed out row or column), and repeat the procedure until only one uncrossed cell remains. Set this variable equal to the supply or demand associated with the variable, and cross out the variable's row and column.

Example 6.

				Supply	Row penalty
	6	7	8	10	7-6=1
	15	80	78	15	78-15=63
Demand	15	5	5		
Column penalty	15-6=9	80-7=**73**	78-8=70		

				Supply	Row penalty
	6	**7** 5	8	5	8-6=2
	15	80	78	15	78-15=63
Demand	15	X	5		
Column penalty	15-6=9	-	78-8=**70**		

				Supply	Row penalty
	6	7 5	**8** 5	X	-
	15	80	78	15	-
Demand	15	X	0		
Column penalty	15-6=9	-	-		

	6	7 5	8 5	X
	15 15	80	78 0	15
Demand	15	X	0	

7.3 THE TRANSPORTATION SIMPLEX METHOD

Steps of the Method

1. If the problem is unbalanced, balance it
2. Use one of the methods to find a bfs for the problem
3. Use the fact that $u_1 = 0$ and $u_i + v_j = c_{ij}$ for all basic variables to find the u's and v's for the current bfs.
4. If $u_i + v_j - c_{ij} \leq 0$ for all nonbasic variables, then the current bfs is optimal. If this is not the case, we enter the variable with the most positive $u_i + v_j - c_{ij}$ into the basis using the *pivoting procedure*. This yields a new bfs. Return to Step 3.

For a maximization problem, proceed as stated, but replace Step 4 by the following step:

> If $u_i + v_j - c_{ij} \geq 0$ for all nonbasic variables, then the current bfs is optimal. Otherwise, enter the variable with the most negative $u_i + v_j - c_{ij}$ into the basis using the *pivoting procedure*. This yields a new bfs. Return to Step 3.

Pivoting procedure

1. Find the loop (there is only one possible loop!) involving the entering variable (determined at step 4 of the transport'n simplex method) and some or all of the basic variables.
2. Counting *only cells in the loop*, label those that are an even number (0, 2, 4, and so on) of cells away from the entering variable as *even cells*. Also label those that are an odd number of cells away from the entering variable as *odd cells*.
3. Find the odd cell whose variable assumes the smallest value. Call this value Φ. The variable corresponding to this odd cell will leave the basis. To perform the pivot, decrease the value of each odd cell by Φ and increase the value of each even cell by Φ. The values of variables not in the loop remain unchanged. The pivot is now complete. If Φ = 0, the entering variable will equal 0, and odd variable that has a current value of 0 will leave the basis.

Example 7. Powerco

The problem is balanced (total supply equals total demand).

When the NWC method is applied to the Powerco example, the bfs in the following table is obtained (check: there exist $m+n-1=6$ basic variables).

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8 35	6	10	9	35
Plant 2	9 10	12 20	13 20	7	50
Plant 3	14	9	16 10	5 30	40
DEMAND	45	20	30	30	125

u1 = 0

u1 + v1 = 8 yields v1 = 8

u2 + v1 = 9 yields u2 = 1

u2 + v2 = 12 yields v2 = 11

u2 + v3 = 13 yields v3 = 12

u3 + v3 = 16 yields u3 = 4

u3 + v4 = 5 yields v4 = 1

For each nonbasic variable, we now compute $\hat{c}_{ij} = u_i + v_j - c_{ij}$

$\hat{c}_{12} = 0 + 11 - 6 = 5$

$\hat{c}_{13} = 0 + 12 – 10 = 2$

$\hat{c}_{14} = 0 + 1 – 9 = -8$

$\hat{c}_{24} = 1 + 1 – 7 = -5$

$\hat{c}_{31} = 4 + 8 – 14 = -2$

$\hat{c}_{32} = 4 + 11 – 9 = 6$

Since $\hat{c}_{32}$ is the most positive one, we would next enter x_{32} into the basis: Each unit of x_{32} that is entered into the basis will decrease Powerco’s cost by $6.

The loop involving x_{32} is (3,2)-(3,3)-(2,3)-(2,2). Φ = 10 (see table)

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8 35	6	10	9	35
Plant 2	9 10	12 **20–Φ**	13 **20+Φ**	7	50
Plant 3	14	9 **Φ**	16 **10–Φ**	5 30	40
DEMAND	45	20	30	30	125

x_{33} would leave the basis. New bfs is shown at the following table:

u_i/v_j	8	11	12	7	SUPPLY
0	8 35	6	10	9	35
1	9 10	12 10	13 30	7	50
-2	14	9 10	16	5 30	40
DEMAND	45	20	30	30	125

$\hat{c}_{12} = 5$, $\hat{c}_{13} = 2$, $\hat{c}_{14} = -2$, $\hat{c}_{24} = 1$, $\hat{c}_{31} = -8$, $\hat{c}_{33} = -6$

Since $\hat{c}_{12}$ is the most positive one, we would next enter x_{12} into the basis.

The loop involving x_{12} is (1,2)-(2,2)-(2,1)-(1,1). Φ = 10 (see table)

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8 **35–Φ**	6 **Φ**	10	9	35
Plant 2	9 **10+Φ**	12 **10–Φ**	13 30	7	50
Plant 3	14	9 10	16	5 30	40
DEMAND	45	20	30	30	125

x_{22} would leave the basis. New bfs is shown at the following table:

u_i/v_j	8	6	12	2	SUPPLY
0	8 25	6 10	10	9	35
1	9 20	12	13 30	7	50
3	14	9 10	16	5 30	40
DEMAND	45	20	30	30	125

$\hat{c}_{13} = 2$, $\hat{c}_{14} = -7$, $\hat{c}_{22} = -5$, $\hat{c}_{24} = -4$, $\hat{c}_{31} = -3$, $\hat{c}_{33} = -1$

Since $\hat{c}_{13}$ is the most positive one, we would next enter x_{13} into the basis.

The loop involving x_{13} is (1,3)-(2,3)-(2,1)-(1,1). $\Phi = 25$ (see table)

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8 **25–Φ**	6 10	10 **Φ**	9	35
Plant 2	9 **20**+**Φ**	12	13 **30–Φ**	7	50
Plant 3	14	9 10	16	5 30	40
DEMAND	45	20	30	30	125

x_{11} would leave the basis. New bfs is shown at the following table:

u_i/v_j	6	6	10	2	SUPPLY
0	8	6 10	10 25	9	35
3	9 45	12	13 5	7	50
3	14	9 10	16	5 30	40
DEMAND	45	20	30	30	125

$\hat{c}_{11} = -2$, $\hat{c}_{14} = -7$, $\hat{c}_{22} = -3$, $\hat{c}_{24} = -2$, $\hat{c}_{31} = -5$, $\hat{c}_{33} = -3$

Since all $\hat{c}_{ij}$'s are negative, an optimal solution has been obtained.

Report: 45 million kwh of electricity would be sent from plant 2 to city 1.

10 million kwh of electricity would be sent from plant 1 to city 2. Similarly, 10 million kwh of electricity would be sent from plant 3 to city 2.

25 million kwh of electricity would be sent from plant 1 to city 3. 5 million kwh of electricity would be sent from plant 2 to city 3.

30 million kwh of electricity would be sent from plant 3 to city 4 and

Total shipping cost is:

$$z = .9\ (45) + 6\ (10) + 9\ (10) + 10\ (25) + 13\ (5) + 5\ (30) = \$\ 1020$$

7.4 SENSITIVITY ANALYSIS FOR TRANSPORTATION PROBLEMS

In this section, we discuss the following three aspects of sensitivity analysis for the transportation problem:

- Changing the objective function coefficient of a nonbasic variable.
- Changing the objective function coefficient of a basic variable.
- Increasing a single supply by Δ and a single demand by Δ.

We illustrate three changes using the Powerco problem. Recall that the optimal solution for the Powerco problem was z=$1,020 and the optimal tableau was:

		City 1	City 2	City 3	City 4	Supply
	u_i/v_j	6	6	10	2	
Plant 1	0	8	6 10	10 25	9	35
Plant 2	3	9 45	12	13 5	7	50
Plant 3	3	14	9 10	16	5 30	40
Demand		45	20	30	30	

Changing the Objective Function Coefficient of a Nonbasic Variable

Changing the objective function coefficient of a nonbasic variable x_{ij} will leave the right hand side of the optimal tableau unchanged. Thus, the current basis will still be feasible. We are not changing $\mathbf{c}_{BV}\mathbf{B}^{-1}$, so the u_i's and v_j's remain unchanged. In row 0, only the coefficient of x_{ij} will change. Thus, as long as the coefficient of x_{ij} in the optimal row 0 is nonpositive, the current basis remains optimal.

To illustrate the method, we answer the following question: For what range of values of the cost of shipping 1 million kwh of electricity from plant 1 to city 1 will the current basis remain optimal? Suppose we change c_{11} from 8 to 8+Δ. For what values of Δ will the current basis remain optimal? Now $\bar{c}_{11} = u_1 + v_1 - c_{11} = 0 + 6 - (8 + \Delta) = -2 - \Delta$. Thus, the current basis remains optimal for $-2 - \Delta \le 0$, or $\Delta \ge -2$, and $c_{11} \ge 8 - 2 = 6$.

Changing the Objective Function Coefficient of a Basic Variable

Because we are changing $\mathbf{c}_{BV}\mathbf{B}^{-1}$, the coefficient of each nonbasic variable in row 0 may change, and to determine whether the current basis remains optimal, we must find the new u_i's and v_j's and use these values to price out all nonbasic variables. The current basis remains optimal as long as all nonbasic variables price out nonpositive.

To illustrate the idea, we determine for the Powerco problem the range of values of the cost of shipping 1 million kwh from plant 1 to city 3, for which the current basis remains optimal.

Suppose we change c_{13} from 10 to 10+Δ. Then the equation $\bar{c}_{13} = 0$ changes from $u_1 + v_3 = 10$ to $u_1 + v_3 = 10 + \Delta$. Thus, to find the u_i's and v_j's, we must solve the following equations:

u_1=0
$u_2 + v_1 = 9$
$u_1 + v_2 = 6$
$u_2 + v_3 = 13$
$u_3 + v_2 = 9$
$u_1 + v_3 = 10 + \Delta$
$u_3 + v_4 = 5$

Solving these equations, we obtain $u_1 = 0$, $v_2 = 6$, $v_3 = 10 + \Delta$, $v_1 = 6 + \Delta$, $u_2 = 3 - \Delta$, $u_3 = 3$, and $v_4 = 2$.

We now price out each nonbasic variable. The current basis will remain optimal as long as each nonbasic variable has a nonpositive coefficient in row 0.

$$\bar{c}_{11} = u_1 + v_1 - 8 = \Delta - 2 \leq 0 \qquad for \qquad \Delta \leq 2$$
$$\bar{c}_{14} = u_1 + v_4 - 9 = -7$$
$$\bar{c}_{22} = u_2 + v_2 - 12 = -3 - \Delta \leq 0 \qquad for \qquad \Delta \geq -3$$
$$\bar{c}_{24} = u_2 + v_4 - 7 = -2 - \Delta \leq 0 \qquad for \qquad \Delta \geq -2$$
$$\bar{c}_{31} = u_3 + v_1 - 14 = -5 + \Delta \leq 0 \qquad for \qquad \Delta \leq 5$$
$$\bar{c}_{33} = u_3 + v_3 - 16 = \Delta - 3 \leq 0 \qquad for \qquad \Delta \leq 3$$

Thus, the current basis remains optimal for $-2 \leq \Delta \leq 2$,

or $8 = 10 - 2 \leq c_{13} \leq 10 + 2 = 12$.

Increasing Both Supply s_i and Demand d_j by Δ

Observe that this change maintains a balanced transportation problem. Because the u_i's and v_j's may be thought of as the negative of each constraints shadow prices, we know that if the current basis remains optimal,

New z-value = old z-value + + Δ (u_i) + Δ (v_j)

For example, if we increase plant 1's supply and city 2's demand by 1 unit, then

new cost = 1,020 + 1 (0) + 1 (6) = $ 1,026.

We may also find the new values of the decision variables as follows:

1. If x_{ij} is a basic variable in the optimal solution, then increase x_{ij} by Δ.

2. If x_{ij} is a nonbasic variable in the optimal solution, then find the loop involving x_{ij} and some of the basic variables. Find an odd cell in the loop that is in row *i*. Increase the value of this odd cell by Δ and go around the loop, alternately increasing and then decreasing current basic variables in the loop by Δ.

To illustrate the first situation, suppose we increase s_1 and d_2 by 2. Because x_{12} is a basic variable in the optimal solution, the new optimal solution will be as follows:

		City 1	City 2	City 3	City 4	Supply
	u_i/v_j	6	6	10	2	
Plant 1	0	8	6 12	10 25	9	37
Plant 2	3	9 45	12	13 5	7	50
Plant 3	3	14	9 10	16	5 30	40
Demand		45	22	30	30	

New *z*-value is 1,020 + $2u_1 + 2v_2$ = \$ 1,032.

To illustrate the second situation, suppose we increase both s_1 and d_1 by 1. Because x_{11} is a nonbasic variable in the current optimal solution, we must find the loop involving x_{11} and some of the basic variables. The loop is (1, 1) – (1, 3) – (2, 3) – (2, 1). The odd cell in the loop and row 1 is x_{13}. Thus, the new optimal solution will be obtained by increasing both x_{13} and x_{21} by 1 and decreasing x_{23} by 1. This yields the optimal solution shown as:

		City 1	City 2	City 3	City 4	Supply
	u_i/v_j	6	6	10	2	
Plant 1	0	8	6 10	10 26	9	36
Plant 2	3	9 46	12	13 4	7	50
Plant 3	3	14	9 10	16	5 30	40
Demand		46	20	30	30	

New *z*-value is found from (new z-value) = 1,020 + u_i + v_j = \$ 1,026.

Observe that if both s_1 and d_1 were increased by more than 5, the current basis would be infeasible.

7.5 TRANSSHIPMENT PROBLEMS

Sometimes a point in the shipment process can both receive goods from other points and send goods to other points. This point is called as **transshipment point** through which goods can be transshipped on their journey from a supply point to demand point. Shipping problem with this characteristic is a transshipment problem.

The optimal solution to a transshipment problem can be found by converting this transshipment problem to a transportation problem and then solving this transportation problem.

Remark

As stated in “Formulating Transportation Problems”, we define a **supply point** to be a point that can send goods to another point but cannot receive goods from any other point.

Similarly, a **demand point** is a point that can receive goods from other points but cannot send goods to any other point.

Steps

1. If the problem is unbalanced, balance it

   Let *s* = total available supply (or demand) for balanced problem
2. Construct a transportation tableau as follows

   A row in the tableau will be needed for each supply point and transshipment point

   A column will be needed for each demand point and transshipment point

   Each supply point will have a supply equal to its original supply

   Each demand point will have a demand equal to its original demand

   Each transshipment point will have a supply equal to “that point’s original supply + *s*”

   Each transshipment point will have a demand equal to “that point’s original demand + *s*”
3. Solve the transportation problem

Example 8. Bosphorus *(Based on Winston 7.6.)*

Bosphorus manufactures LCD TVs at two factories, one in Istanbul and one in Bruges. The Istanbul factory can produce up to 150 TVs per day, and the Bruges factory can produce up to 200 TVs per day. TVs are shipped by air to customers in London and Paris. The customers in each city require 130 TVs per day. Because of the deregulation of air fares, Bosphorus believes that it may be cheaper to first fly some TVs to Amsterdam or Munchen and then fly them to their final destinations. The costs of flying a TV are shown at the table below. Bosphorus wants to minimize the total cost of shipping the required TVs to its customers.

€	To					
From	Istanbul	Bruges	Amsterdam	Munchen	London	Paris
Istanbul	0	-	8	13	25	28
Bruges	-	0	15	12	26	25
Amsterdam	-	-	0	6	16	17
Munchen	-	-	6	0	14	16
London	-	-	-	-	0	-
Paris	-	-	-	-	-	0

Answer: In this problem Amsterdam and Munchen are *transshipment points*.

Step 1. Balancing the problem

Total supply = 150 + 200 = 350

Total demand = 130 + 130 = 260

Dummy's demand = 350 – 260 = 90

s = 350 (total available supply or demand for balanced problem)

Step 2. Constructing a transportation tableau

Transshipment point's demand = Its original demand + *s* = 0 + 350 = 350

Transshipment point's supply = Its original supply + *s* = 0 + 350 = 350

	Amsterdam	Munchen	London	Paris	Dummy	Supply
Istanbul	8	13	25	28	0	150
Bruges	15	12	26	25	0	200
Amsterdam	0	6	16	17	0	350
Munchen	6	0	14	16	0	350
Demand	350	350	130	130	90	

Step 3. Solving the transportation problem

	Amsterdam	Munchen	London	Paris	Dummy	Supply
Istanbul	8 130	13	25	28	0 20	150
Bruges	15	12	26	25 130	0 70	200
Amsterdam	0 220	6	16 130	17	0	350
Munchen	6	0 350	14	16	0	350
Demand	350	350	130	130	90	1050

Report: Bosphorus should produce 130 TVs at Istanbul, ship them to Amsterdam, and transship them from Amsterdam to London.

The 130 TVs produced at Bruges should be shipped directly to Paris.

The total shipment is 6370 Euros.

7.6 ASSIGNMENT PROBLEMS

There is a special case of transportation problems where each supply point should be assigned to a demand point and each demand should be met. This certain class of problems is called as “assignment problems”. For example determining which employee or machine should be assigned to which job is an assignment problem.

7.6.1 LP Formulation

An assignment problem is characterized by knowledge of the cost of assigning each supply point to each demand point: c_{ij}

On the other hand, a 0-1 integer variable x_{ij} is defined as follows

x_{ij} = 1 if supply point i is assigned to meet the demands of demand point j

x_{ij} = 0 if supply point i is not assigned to meet the demands of point j

In this case, the general LP representation of an assignment problem is

$$\min \Sigma_i \Sigma_j c_{ij} x_{ij}$$

$$\text{s.t.} \quad \Sigma_j x_{ij} = 1 \; (i=1,2,\ldots, m) \qquad \text{Supply constraints}$$

$$\Sigma_i x_{ij} = 1 \; (j=1,2,\ldots, n) \qquad \text{Demand constraints}$$

$$x_{ij} = 0 \text{ or } x_{ij} = 1$$

7.6.2 Hungarian Method

Since all the supplies and demands for any assignment problem are integers, all variables in optimal solution of the problem must be integers. Since the RHS of each constraint is equal to 1, each x_{ij} must be a nonnegative integer that is no larger than 1, so each x_{ij} must equal 0 or 1.

Ignoring the x_{ij} = 0 or x_{ij} = 1 restrictions at the LP representation of the assignment problem, we see that we confront with a balanced transportation problem in which each supply point has a supply of 1 and each demand point has a demand of 1.

However, the high degree of degeneracy in an assignment problem may cause the Transportation Simplex to be an inefficient way of solving assignment problems.

For this reason and the fact that the algorithm is even simpler than the Transportation Simplex, the Hungarian method is usually used to solve assignment problems.

Remarks

1. To solve an assignment problem in which the goal is to maximize the objective function, multiply the profits matrix through by –1 and solve the problem as a **minimization** problem.
2. If the number of rows and columns in the cost matrix are unequal, the assignment problem is **unbalanced**. Any assignment problem should be balanced by the addition of one or more dummy points before it is solved by the Hungarian method.

Steps

1. Find the minimum cost each row of the $m*m$ cost matrix.
2. Construct a new matrix by subtracting from each cost the minimum cost in its row
3. For this new matrix, find the minimum cost in each column
4. Construct a new matrix (reduced cost matrix) by subtracting from each cost the minimum cost in its column
5. Draw the minimum number of lines (horizontal and/or vertical) that are needed to cover all the zeros in the reduced cost matrix. If m lines are required, an optimal solution is available among the covered zeros in the matrix. If fewer than m lines are needed, proceed to next step
6. Find the smallest cost (k) in the reduced cost matrix that is uncovered by the lines drawn in Step 5
7. Subtract k from each uncovered element of the reduced cost matrix and add k to each element that is covered by two lines. Return to Step 5

Example 9. Flight Crew *(Based on Winston 7.5.)*

Four captain pilots (CP1, CP2, CP3, CP4) has evaluated four flight officers (FO1, FO2, FO3, FO4) according to perfection, adaptation, morale motivation in a 1-20 scale (1: very good, 20: very bad). Evaluation grades are given in the table. Flight Company wants to assign each flight officer to a captain pilot according to these evaluations. Determine possible flight crews.

	FO1	FO2	FO3	FO4
CP1	2	4	6	10
CP2	2	12	6	5
CP3	7	8	3	9
CP4	14	5	8	7

Answer:

Step 1. For each row in the table we find the minimum cost: 2, 2, 3, and 5 respectively

Step 2 & 3. We subtract the row minimum from each cost in the row. For this new matrix, we find the minimum cost in each column

	0	2	4	8
	0	10	4	3
	4	5	0	6
	9	0	3	2
Column minimum	0	0	0	2

Step 4. We now subtract the column minimum from each cost in the column obtaining reduced cost matrix.

0	2	4	6
0	10	4	1
4	5	0	4
9	0	3	0

Step 5. As shown, lines through row 3, row 4, and column 1 cover all the zeros in the reduced cost matrix. The minimum number of lines for this operation is 3. Since fewer than four lines are required to cover all the zeros, solution is not optimal: we proceed to next step.

0	2	4	6
0	10	4	1
4	5	0	4
9	0	3	0

Step 6 & 7. The smallest uncovered cost equals 1. We now subtract 1 from each uncovered cost, add 1 to each twice-covered cost, and obtain

0	1	3	5
0	9	3	0
5	5	0	4
10	0	3	0

Four lines are now required to cover all the zeros: An optimal s9olution is available. Observe that the only covered 0 in column 3 is x_{33}, and in column 2 is x_{42}. As row 5 cannot be used again, for column 4 the remaining zero is x_{24}. Finally we choose x_{11}.

CP1 should fly with FO1; CP2 should fly with FO4; CP3 should fly with FO3; and CP4 should fly with FO4.

Example 10. Maximization problem

	F	G	H	I	J
A	6	3	5	8	10
B	2	7	6	3	2
C	5	8	3	4	6
D	6	9	3	1	7
E	2	2	2	2	8

Report:

Optimal profit = 36

Assigments: A-I, B-H, C-G, D-F, E-J

Alternative optimal sol'n: A-I, B-H, C-F, D-G, E-J

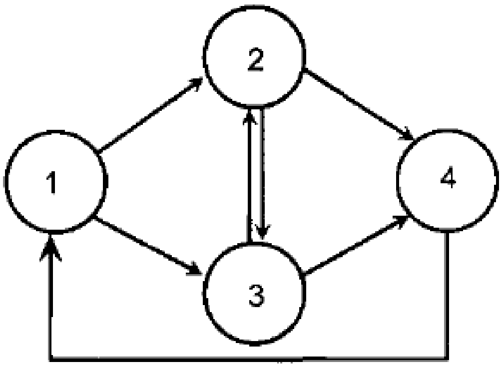
8. INTRODUCTION TO NETWORK MODELS

Many physical networks such as telephone lines, internet, highways, electric systems, water delivery systems, etc. are very familiar to us. In each of these settings, there are similar problems to be solved: to send some goods from one point to another in a shortest route or via some minimum cost flow pattern. Similar to these physical systems, many optimization problems can be analyzed by means of a network representation.

The topic of network optimization has its origins in the 1940's with the development of linear programming. Since then it has grown rapidly as a result of many theoretical and applied researches as well as applications in a wide range of practical situations.

We will make an introduction to some important network models during the course: Shortest path problem, maximum flow problem, minimum cost network flow problem, and project management.

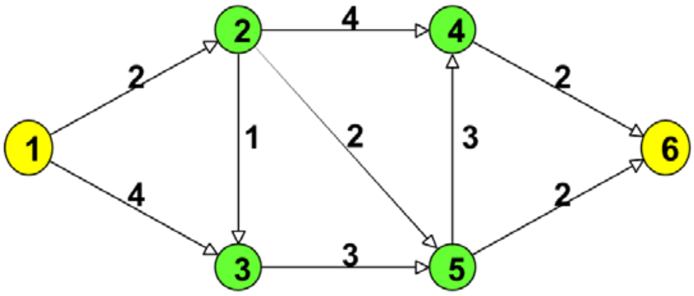
A graph, or network, is defined by two types of elements. A directed network or a digraph $G(N,S)$ consists of a finite set of nodes (vertices or points) N = {1,2,..., m} and a set of directed arcs (links, branches, edges, or lines) S = {(i,j), (k,l),...,(s, t)} joining pairs of nodes in N. Arc (i,j) is said to be incident at nodes i and j and it is directed from node i to node j. We shall assume that the network has m nodes and n arcs. Following figure presents a network having four nodes and seven arcs, in which N = {1,2,3,4} and S = {(1,2), (1,3), (2,4), (2,3), (3,4), (4,1)}.



8.1 SHORTEST-PATH PROBLEM

Consider a network $G(N,S)$ that has m nodes and n arcs. Given a cost c_{ij} associated to arc (i,j) for all arcs in S. The shortest path problem is defined as finding the shortest path or route from a starting point (node 1), to another point (node m).

Example 1. ATK-brown is transportation company that ought to deliver the goods of one of its customers from the distribution center (point 1) to customer's warehouse (point 6). Possible paths to follow and their lengths in kms are given in the following figure. The problem is to find the shortest route from point 1 to point 6.



8.1.1 LP formulation of shortest path problem

Suppose x_{ij} = 0 or 1 indicate that the arc (i,j) is either in the shortest path or not.

Minimize $$\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}x_{ij}$$

Subject to $\sum_{j=1}^{m} x_{1j} = 1$

$\sum_{j=1}^{m} x_{ij} - \sum_{k=1}^{m} x_{ki} = 0$ $\quad i=2,\ldots,m-1$

$\sum_{i=1}^{m} x_{im} = 1$

x_{ij} = 0 or 1, $i,j=1,2,\ldots,m$.

where the sums are taken over existing arcs in the network.

8.1.2 Dijkstra's Algorithm

Consider the case when all $c_{ij} \geq 0$. In this case, a very simple and efficient procedure, known as Dijkstra's algorithm, exists for finding a shortest path from a node (node 1) to all other nodes. Dijkstra's algorithm is a labeling method, in which the nodes are given temporary and permanent labels.

INITIALIZATION STEP

The starting point (node 1) is given a permanent label 0.

All other nodes have a temporary label of ∞.

MAIN STEP

1) Update all temporary labels:

node *j*'s temporary label =

$$min\begin{cases} \text{node } j'\text{s current temporary label} \\ \text{node } i'\text{s permanent label} + \text{ length of arc } (i,j), \text{for } (i,j) \in S \end{cases}$$

2) Make the smallest temporary label a permanent label.

3) Repeat the main step until the destination node has a permanent label, then stop.

To find the shortest path from node 1 to node *m*, work backward from node *m* by finding nodes having labels differing by exactly the length of the connecting arc.

Example 2. Find the shortest path for the problem in Example 1.

Answer: consider P(*i*): permanent label of *i*; T(*i*): temporary label of *I*;

INITIALIZATION STEP
P(1) = 0, T(i) = ∞, i = 2,…,6.

MAIN STEP – 1st run
T(2) = min (∞, P(1) + c_{12}) = min (∞, 2) = 2
T(3) = min (∞, P(1) + c_{13}) = min (∞, 4) = 4
T(4) = T(5) = T(6) = ∞
We make temporary label of node 2 permanent; P(2) = 2.

MAIN STEP – 2nd run
T(3) = min (4, P(2) + c_{23}) = min (4, 2+1) = 3
T(4) = 6, T(5) = 4, T(6) = ∞
We make temporary label of node 3 permanent; P(3) = 3.

MAIN STEP – 3rd run
T(4) = 6
T(5) = min (4, P(3) + c_{35}) = min (4, 6) = 4
T(6) = ∞
We make temporary label of node 5 permanent; P(5) = 4.

MAIN STEP – 4th run
T(4) = min (6, P(5) + c_{54}) = min (6, 7) = 6
T(6) = min (∞, P(5) + c_{56}) = (∞, 6) = 6
We make temporary label of node 6 permanent; P(6) = 6. We stop.

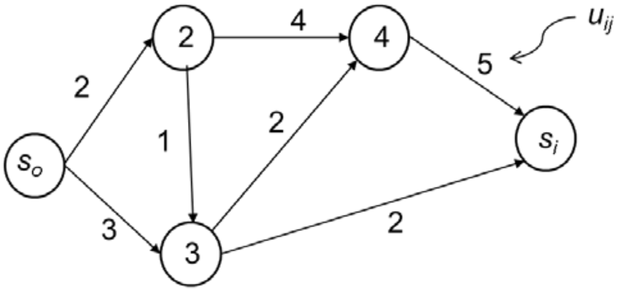
Shortest path is 1-2-5-6, the total cost (total length) is 6.

8.2 MAXIMUM-FLOW PROBLEM

Consider a network G(N,S) that has *m* nodes and *n* arcs. A single commodity will flow through the network. We associate with each arc (*i*,*j*) an upper bound on flow of u_{ij} . There are no costs involved in the maximal flow problem. Maximum-flow problem is defined as finding the maximum amount of flow from a starting point (node 1) to another point (node *m*).

Example 3. (based on Winston 8.3)

ATK-Oil wants to ship the maximum possible amount of oil (per hour) via pipeline from node s_o to node s_i in the following figure. On its way from node s_o to node s_i, oil must pass through some or all of stations 2, 3, and 4. The various arcs represent pipelines of different diameters. The maximum number of barrels of oil (millions of barrels per hour) that can be pumped through each arc is given on each arc. Each number is called an arc capacity. The problem is to determine the maximum number of barrels of oil per hour that can be sent from s_0 to s_i.



8.2.1 LP formulation of maximum flow problem

Suppose *f* represent the amount of flow in the network from node 1 to node *m* and x_{ij} represent the amount of flow through the arc (*i*,*j*).

Maximize f

Subject to $\sum_{j=1}^{m} x_{1j} = f$

$\sum_{j=1}^{m} x_{ij} - \sum_{k=1}^{m} x_{ki} = 0$ i=2,…,*m*-1

$\sum_{i=1}^{m} x_{im} = f$

$x_{ij} \leq u_{ij} \;\; i,j = 1,2,\ldots,m.$

$x_{ij} \geq 0$, *i*,*j*=1,2,…,*m*.

where the sums and inequalities are taken over existing arcs in the network.

Example 4. Formulate an LP that can be used for the problem given in Example 3.

Answer:

Maximize f

Subject to

$$\begin{aligned}
&x_{12}+x_{13}=f\\
&x_{23}+x_{24}-x_{12}=0\\
&x_{34}+x_{35}-x_{13}-x_{23}=0\\
&x_{45}-x_{24}-x_{34}=0\\
&x_{45}+x_{35}=f\\
&x_{12}\leq 2\\
&x_{13}\leq 3\\
&x_{23}\leq 1\\
&x_{24}\leq 4\\
&x_{34}\leq 2\\
&x_{35}\leq 2\\
&x_{45}\leq 7
\end{aligned}$$

All variables ≥ 0

(notice that s_0 to s_i are considered as nodes 1 and 5, respectively.)

8.3 MINIMUM-COST NETWORK FLOW PROBLEM

The minimum-cost network flow problem is general form of the most of the network problems including the transportation, assignment, transshipment, shortest-path, maximum flow, and CPM problems. Demands and supplies of each nodes and costs as well as the bounds of the flows on each arc can be included in a minimum-cost network flow problem. Formally, it is defined as follows.

Consider a network *G*(N,S) that has *m* nodes and *n* arcs. With each node *i* in *N* we associate a number b_i that represents the available supply of an item (if $b_i > 0$) or the required demand for the item (if $b_i < 0$). Nodes with $b_i > 0$ are called sources, and nodes with $b_i < 0$ are called sinks. If $b_i = 0$, then none of the item is available at node *i* and none is required; in this case, node *i* is called a transshipment (or intermediate) node. Associated with each arc (*i*,*j*) we let x_{ij} be the amount of flow on the arc, and we let c_{ij} be the unit shipping cost along this arc. We associate with each arc (*i*,*j*) an upper bound on flow of u_{ij} and a lower bound on flow of l_{ij}.

The minimal-cost network flow problem is stated as shipping the available supply through the network to satisfy demand at minimal cost.

Mathematically, this problem becomes the following (where summations are taken over existing arcs):

$$\text{Minimize} \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}x_{ij}$$

Subject to $\sum_{j=1}^{m} x_{ij} - \sum_{k=1}^{m} x_{ki} = b_i$ i=1,…,m

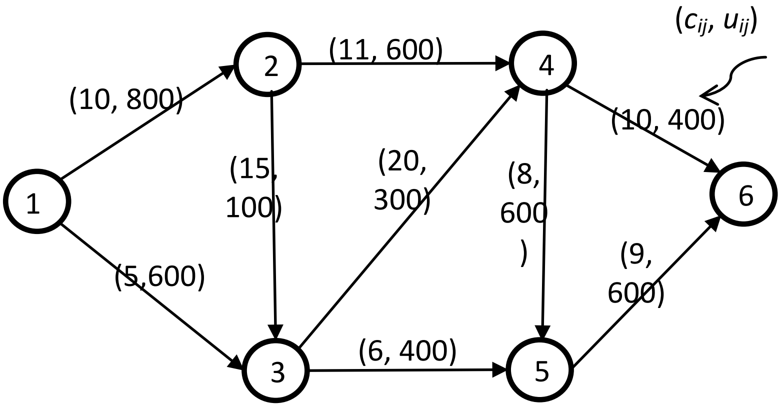
$l_{ij} \le x_{ij} \le u_{ij}$ $i, j = 1,2, \dots, m.$

$x_{ij} \ge 0$, i,j=1,2,…,m.

where the first constraint, which usually refereed as the flow balance equations for the network, are stipulate that the net flow out of node i must equal b_i. the second constraint ensure that the flow through each arc satisfies the arc capacity restrictions.

Example 5. (based on Winston 8.5)

Each hour, an average of 900 cars enter the network in the following figure at node 1. 300 of the cars seek to travel to node 4, 500 cars will travel to node 6, and 100 cars will go to node 5. In the figure, the time it takes a car to traverse each arc and the maximum number of cars that can pass by any point on the arc during a one-hour period are given. Formulate an minimum-cost network flow problem that minimizes the total time required for all cars to travel from node 1 to nodes 4, 5, and 6.



9. PROJECT MANAGEMENT

9.1 CONCEPTS

Organizations perform work – either operations or projects.

Shared characteristics of projects and operations:

- Performed by people
- Constrained by limited resources
- Planned, executed and controlled

Operations and projects differ:

- Operations are ongoing and repetitive
- Projects are temporary and unique

“A ***project*** is a temporary and intensely serious attempt undertaken to create a unique product or service.” Here, *temporary* means “definite beginning and end” and *unique* means “different in some distinguishing characteristic”.

Resources used in projects are time, finance, labor, materials, tools & machinery, and personnel.

Project Examples:

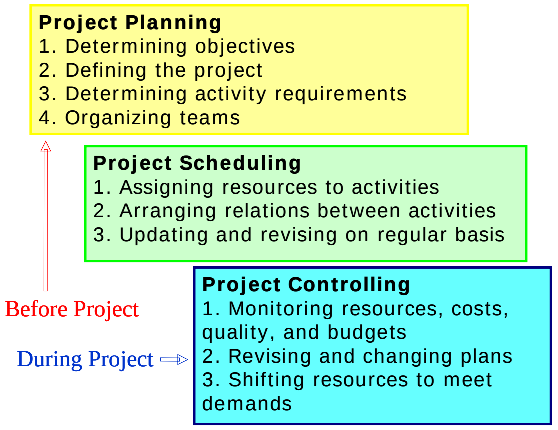
- Developing a new product or service
- Effecting a change in structure, staffing, or style of an organization
- Designing a new transportation vehicle
- Constructing a building or facility
- Running a campaign for political office
- Implementing a new business procedure or process

Management is generally perceived to be concerned with planning, organization, and control of an ongoing process or activity.

Project Management reflects a commitment of resources and people to a typically important activity for a relatively short time frame, after which the management effort is dissolved. Project management is the application of knowledge, skills, tools, and techniques to project activities in order to meet or exceed stakeholder needs and expectations from a project.

Meeting or exceeding stakeholder needs and expectations invariably involves balancing competing demands among:

- Scope, time, cost, and quality
- Stakeholders with differing needs and expectations
- Identified needs and unidentified expectations - “client relations challenge”



Project Planning
1. Determining objectives
2. Defining the project
3. Determining activity requirements
4. Organizing teams

Project Scheduling
1. Assigning resources to activities
2. Arranging relations between activities
3. Updating and revising on regular basis

Project Controlling
1. Monitoring resources, costs, quality, and budgets
2. Revising and changing plans
3. Shifting resources to meet demands

9.2 THE PROJECT NETWORK

The project network consisted of nodes and directed arcs and shows the relation between activities. It has two types:

- Arc Diagrams (Activity on Arc – ***AOA***): Arcs represent the activities, Nodes represent the beginning and termination of activities (events).
- Block Diagrams (Activity on Node – ***AON***): Nodes represent activities, Arcs represent precedence relations between activities.

Example 1. Network

Assume a project of 5 activities.

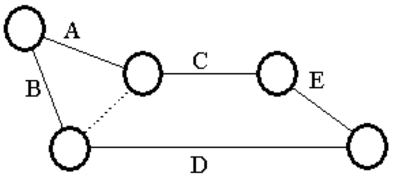
Activities A and B are predecessors of activity C.

B is predecessor of D.

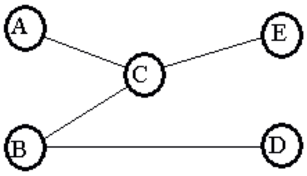
C is predecessors of E.

Answer:

ARROW DIAGRAM



BLOCK DIAGRAM



9.3 CPM/PERT

Network models can be used as an aid in the scheduling of large complex projects that consist of many activities.

If the duration of each activity is known with certainty, the ***Critical Path Method*** (***CPM***) can be used to determine the length of time required to complete a project. CPM can also be used to determine how long each activity in the project can be delayed without delaying the completion of the project. It was developed in the late 1950s by researchers at DuPont and Sperry Rand.

If the duration of activities is not known with certainty, the ***Program Evaluation and Review Technique*** (***PERT***) can be used to estimate the probability that the project will be completed by a given deadline. PERT was developed in the late 1950s by consultants working on the development of the Polaris missile.

Application Examples for CPM/PERT

- Scheduling construction projects such as buildings, highways, and airports...
- Installing new computer systems
- Designing and marketing new products
- Completing corporate mergers
- Building ships
- Developing countdown and hold procedure for the launching of space crafts

Six Steps Common to CPM/PERT

1. Define the project and all significant activities.
2. Develop relationships among the activities. Identify precedence relationships.
3. Draw the network.
4. Assign time and/or cost estimates to each activity.

5. Compute the longest time path (***critical path***) through the network.
6. Use the network to help plan, schedule, monitor, and control the project.

Questions Addressed by CPM/PERT

- When will the project be completed?
- What are the critical activities or tasks in the project?
- Which are the non-critical activities?
- What is the probability that the project will be completed by a specific date?
- Is the project on schedule, ahead of schedule, or behind schedule?
- Is the project over or under the budgeted amount?
- Are there enough resources available to finish the project on time?
- If the project must be finished in less than the scheduled amount of time, what is the best way to accomplish this at least cost?

Advantages of CPM/PERT

- Useful at several stages of project management,
- Straightforward in concept, not mathematically complex,
- Uses graphical displays employing networks to help user perceive relationships among project activities,
- Critical path and slack time analyses help pinpoint activities that need to be closely watched,
- Networks generated provide valuable project documentation and graphically point out who is responsible for various project activities,
- Applicable to a wide variety of projects and industries,
- Useful in monitoring not only schedules, but costs as well.

Limitations of CPM/PERT

- Project activities must be clearly defined, independent, and stable in their relationships,
- Precedence relationships must be specified and networked together,
- Time activities in PERT are assumed to follow the beta probability distribution -- must be verified,
- Time estimates tend to be subjective, and are subject to fudging by managers,
- There is inherent danger in too much emphasis being placed on the critical path.

Utilization of CPM/PERT

To apply CPM or PERT, we need a list of activities that make up the project.

The project is considered to be completed when all activities have been completed.

For each activity there is a set of activities (called the ***predecessors*** of the activity) that must be completed before the activity begins.

A ***project network*** (***project diagram***) is used to represent the precedence relationships between activities → ***AOA representation*** of a project

Given a list of activities and predecessors, an AOA representation of a project can be constructed by using the following rules.

- Node 1 represents the ***start*** of the project. An arc should lead from node 1 to represent each activity that has no predecessors.
- A node (called the ***finish node***) representing the completion of the project should be included in the network.
- Number the nodes in the network so that the node representing the completion time of an activity always has a larger number than the node representing the beginning of an activity.
- An activity should not be represented by more than one arc in the network
- Two nodes can be connected by at most one arc.

To avoid violating last two rules, it can be sometimes necessary to utilize a ***dummy*** activity that takes zero time.

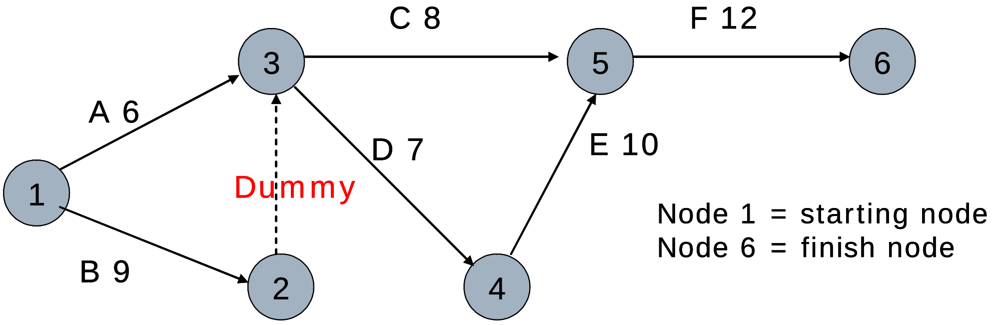
Example 2. Widgetco (*Winston 8.4., p. 433*)

Widgetco is about to introduce a new product (product 3). A list of activities and their predecessors and of the duration of each activity is given.

Draw a project network for this project.

Activity	**Predecessors**	**Duration(days)**
A:train workers	-	6
B:purchase raw materials	-	9
C:produce product 1	A, B	8
D:produce product 2	A, B	7
E:test product 2	D	10
F:assemble products 1&2	C, E	12

Answer:



9.3.1 CPM

Two key building blocks in CPM:

The ***early event time*** for node *i*, represented by ***ET*(*i*)**, is the earliest time at which the event corresponding to node *i* can occur.

The ***late event time*** for node *i*, represented by ***LT*(*i*)**, is the latest time at which the event corresponding to node *i* can occur without delaying the completion of the project.

EARLY EVENT TIME

Note that *ET*(1) = 0

Then compute *ET*(2), *ET*(3), and so on...

Stop when *ET*(*n*) has been calculated (*n: finish node*)

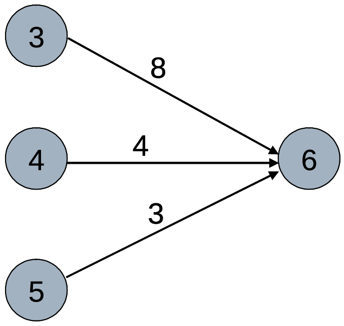
Computation of *ET*(*i*):

- Find each prior event to node *i* that is connected by an arc to node *i*. These are immediate predecessors.
- To the *ET* for each immediate predecessor of the node *i,* add the duration of the activity connecting the immediate predecessor to node *i*.
- *ET*(*i*) equals the maximum of the sums computed in previous step.

Example 3. ET

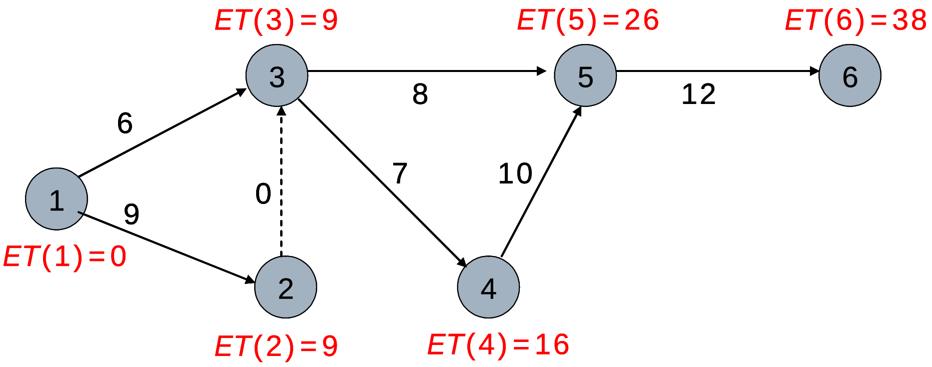
Suppose that for the segment of the project network given below we have already determined

ET(3)=6, *ET*(4)=8, and *ET*(5)=10



Answer: $ET(6)$ = max {$ET(3)$+8, $ET(4)$+4, $ET(5)$+3} = max {14, 12, 13} = 14

Example 4. *ET*(*i*)s for Widgetco



LATE EVENT TIME

Work backward, begin with the finish node

Note that $LT(n) = ET(n)$

Then compute $LT(n\text{-}1)$, $LT(n\text{-}2)$, ... $LT(1)$.

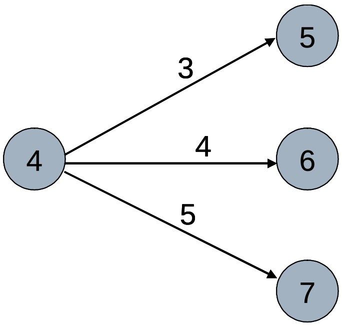
Computation of $LT(i)$:

- Find each node that occurs after node *i* and is connected to node *i* by an arc. These events are immediate successors of node *i*.
- From the *LT* for each immediate successor to node *i,* subtract the duration of the activity.
- $LT(i)$ is the smallest of the differences determined in previous step.

Example 5. LT

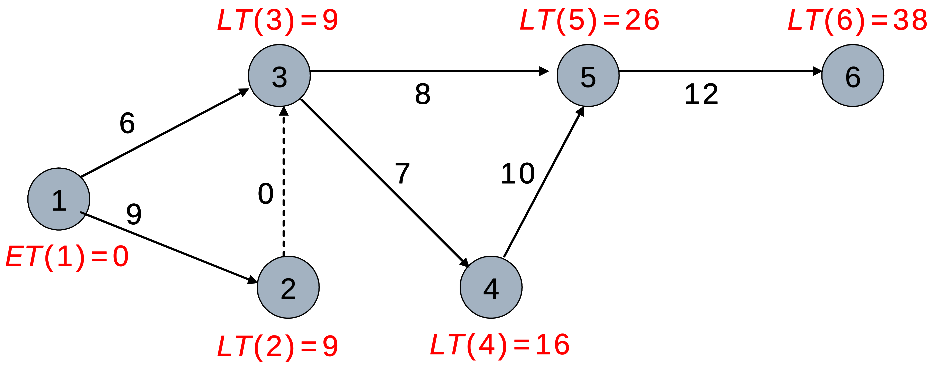
Suppose that for the segment of the project network given below we have already determined

$LT(5)$=24, $LT(6)$=26, and $LT(7)$=28



Answer: *LT*(4) = min {*LT*(5)–3, *LT*(6)–4, *LT*(7)–5} = min {21, 22, 23} = 21

Example 6. *LT*(*i*)s for Widgetco



TOTAL FLOAT

Before the project is begun, the duration of an activity is unknown, and the duration of each activity is used to construct the project network is just an estimate of the activity's actual completion time. The concept of ***total float*** of an activity can be used as a measure of how important it is to keep each activity's duration from greatly exceeding our estimate of its completion time.

For an arbitrary arc representing activity (*i*,*j*), the total float, represented by ***TF*(*i*,*j*)**, of the activity is the amount by which the starting time of activity (*i*,*j*) could be delayed beyond its earliest possible starting time without delaying the completion of the project (assuming no other activities are delayed).

Equivalently, *TF*(*i*,*j*) is the amount by which the duration of the activity can be increased without delaying the completion of the project.

$$TF(i,j) = LT(j) – ET(i) – t_{ij}$$

Example 7. *TF*(*i*,*j*)s for Widgetco

Activity B: *TF*(1,2) = *LT*(2) – *ET*(1) – 9 = 0

Activity A: *TF*(1,3) = *LT*(3) – *ET*(1) – 6 = 3

Activity D: $TF(3,4) = LT(4) – ET(3) – 7 = 0$

Activity C: $TF(3,5) = LT(5) – ET(3) – 8 = 9$

Activity E: $TF(4,5) = LT(5) – ET(4) – 10 = 0$

Activity F: $TF(5,6) = LT(6) – ET(5) – 12 = 0$

Dummy activity: $TF(2,3) = LT(3) – ET(2) – 0 = 0$

FINDING A CRITICAL PATH

If an activity has a total float of zero, then any delay in the start of the activity will delay the completion of the project:

An activity with a total float of zero is a ***critical activity***. A path from node 1 to the finish node that consists entirely of critical activities is called a ***critical path***.

Example 8. Critical Path for Widgetco

$TF(1,2) = 0$
$TF(1,3) = 3$
$TF(2,3) = 0$
$TF(3,4) = 0$
$TF(3,5) = 9$
$TF(4,5) = 0$
$TF(5,6) = 0$
Widgetco critical path is 1-2-3-4-5-6

FREE FLOAT

The ***Free Float*** of the activity corresponding to arc(i,j), denoted by FF(i,j) is the amount by which the starting time of the activity corresponding to arc(i,j) can be delayed without delaying the start of any later activity beyond the earliest possible starting time.

$$FF(i,j) = ET(j) – ET(i) – t_{ij}$$

Example 9. *FF*(*i*,*j*)s for Widgetco

Activity B: FF(1,2) = 9 – 0 – 9 = 0

Activity A: FF(1,3) = 9 – 0 – 6 = 3

Activity D: FF(3,4) = 16 – 9 – 7 = 0

Activity C: FF(3,5) = 26 – 9 – 8 = 9

Activity E: FF(4,5) = 26 – 16 – 10 = 0

Activity F: FF(5,6) = 38 – 26 – 12 = 0

For example, because FF for activity C is 9 days, a delay in the start of this activity (occurrence of node 3) more than 9 days will delay the start of some later activity (in this case activity F)

USING LP FOR FINDING A CRITICAL PATH

LP can also be used to determine the length of the critical path.

Decision variable (x_j): the time that the event corresponding to node *j* occurs

Note that for each activity (*i*,*j*), before node *j* occurs, node *i* must occur and activity (*i*,*j*) must be completed →

$$x_j \geq x_i + t_{ij}$$

Goal is to minimize the time required to complete the project: objective function →

$$\min z = x_n - x_1$$

A critical path for a project network consists of a path from the start of the project to the finish in which each arc in the path corresponds to a constraint having a dual price of -1.

For each constraint with a dual price of -1, increasing the duration of the activity corresponding to that constraint by Δ will increase the duration of the project by Δ.

Example 10. Using LP approach to Widgetco

$$\begin{array}{lll} \min z = & x_6 - x_1 & \\ \text{s.t.} & x_3 \geq x_1 + 6 & \text{(Arc (1,3) constraint)} \\ & x_2 \geq x_1 + 9 & \text{(Arc (1,2) constraint)} \\ & x_5 \geq x_3 + 8 & \text{(Arc (3,5) constraint)} \\ & x_4 \geq x_3 + 7 & \text{(Arc (3,4) constraint)} \\ & x_5 \geq x_4 + 10 & \text{(Arc (4,5) constraint)} \\ & x_6 \geq x_5 + 12 & \text{(Arc (5,6) constraint)} \\ & x_3 \geq x_2 & \text{(Arc (2,3) constraint)} \end{array}$$

All variables urs

Optimal Solution & Report

```
OBJECTIVE FUNCTION VALUE
        1)      38.00000
  VARIABLE        VALUE          REDUCED COST
        X6        38.000000          0.000000
        X1         0.000000          0.000000
        X3         9.000000          0.000000
        X2         9.000000          0.000000
        X5        26.000000          0.000000
        X4        16.000000          0.000000
       ROW   SLACK OR SURPLUS     DUAL PRICES
 ARC (1,3)         3.000000          0.000000
 ARC (1,2)         0.000000         -1.000000
 ARC (3,5)         9.000000          0.000000
 ARC (3,4)         0.000000         -1.000000
 ARC (4,5)         0.000000         -1.000000
```

```
 ARC (5,6)          0.000000          -1.000000
 ARC (2,3)          0.000000          -1.000000
```

The project can be completed in 38 days

Critical path is 1-2-3-4-5-6

9.3.2 Crashing the Project

In many situations, the project manager must complete the project in a time that is less than the length of the critical path. LP can often be used to determine the allocation of resources that minimizes the cost of meeting the project deadline. This process is called ***crashing a project***.

Example 11. Crashing Widgetco Project

Widgetco believes that to have any chance of being a success, product 3 must be available for sale before the competitor's product hits the market.

Widgetco knows that the competitor's product is scheduled to hit the market 26 days from now, so Widgetco must introduce product 3 within 25 days.

Because the project can be completed in 38 days, Widgetco will have to expend additional resources to meet 25 day project deadline.

Suppose that by allocating additional resources to an activity, Widgetco can reduce the duration of any activity by as many as 5 days.

The cost per day of reducing the duration of an activity is shown below:

Activity A	$10
Activity B	$20
Activity C	$3
Activity D	$30
Activity E	$40
Activity F	$50

Find the minimum cost of completing the project by the 25-day deadline

Answer:

Decision variables

A: # of days by which duration of activity A is reduced

...

F: # of days by which duration of activity F is reduced

x_j: time that the event corresponding to node j occurs

LP

$$\begin{aligned}
&\min\ 10A + 20B + 3C + 30D + 40E + 50F\\
&\text{s.t.}\quad A \le 5\\
&\qquad B \le 5\\
&\qquad C \le 5\\
&\qquad D \le 5\\
&\qquad E \le 5\\
&\qquad F \le 5\\
&\qquad x_3 \ge x_1 + 6 - A\\
&\qquad x_2 \ge x_1 + 9 - B\\
&\qquad x_5 \ge x_3 + 8 - C\\
&\qquad x_4 \ge x_3 + 7 - D\\
&\qquad x_5 \ge x_4 + 10 - E\\
&\qquad x_6 \ge x_5 + 12 - F\\
&\qquad x_3 \ge x_2\\
&\qquad x_6 - x_1 \le 25\\
&\ A, B, C, D, E, F \ge 0;\ x_j \text{ urs}
\end{aligned}$$

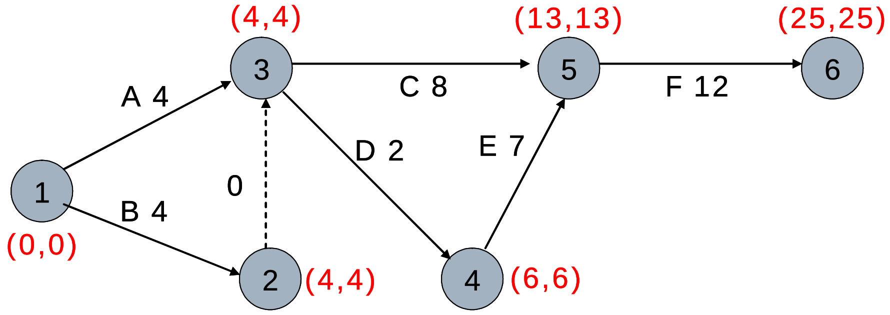
Optimal Solution & Report

$z = 390$, $A = 2$, $B = 5$, $C = 0$, $D = 5$, $E = 3$, $F = 0$

$x_1 = 0, x_2 = 4, x_3 = 4, x_4 = 6, x_5 = 13, x_6 = 25$

After reducing the durations of project A, B, D, and E by the given amounts in the optimal solution, the project deadline of 25 days can be met for a cost of $390.

Project Network & Critical Path



Critical path is 1-2-3-4-5-6 or 1-3-4-5-6

9.3.3 PERT

CPM assumes that the duration of each activity is known with certainty. For many projects, this is clearly not applicable.

PERT is an attempt to correct this shortcoming of CPM by modeling the duration of each activity as a random variable.

For each activity, PERT requires that the project manager estimate three quantities:

- optimistic duration (a)
- pessimistic duration (b)
- the most likely value for duration (m)

Let $\mathbf{T}_{ij}$ be the duration of activity (i,j).

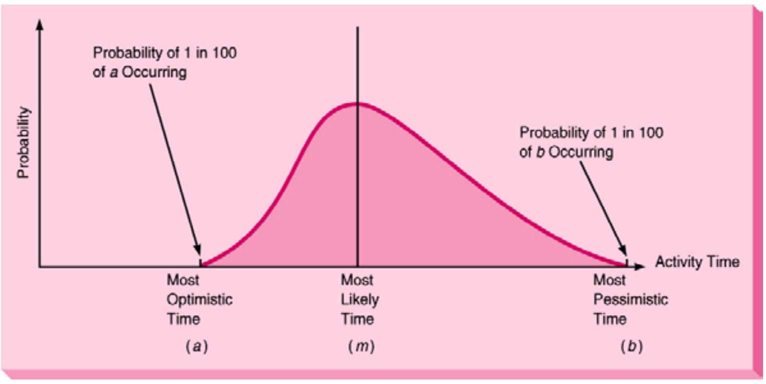
PERT requires the assumption that $\mathbf{T}_{ij}$ follows a beta distribution.

According to this assumption, it can be shown that the mean and variance of $\mathbf{T}_{ij}$ may be approximated by :

$$E(\mathbf{T}_{ij}) = (a + 4m + b) / 6$$

$$\text{var } \mathbf{T}_{ij} = (b - a)^2 / 36$$

Beta Probability Distribution:



PERT requires the assumption that the durations of all activities are independent.

In this case, the mean and variance of the time required to complete the activities on any path are given by

Mean of time required to complete the activities: $\sum_{(i,j)\in \text{path}} E(T_{ij})$

Variance of time required to complete the activities: $\sum_{(i,j)\in \text{path}} var(T_{ij})$

Let **CP** be the random variable denoting the total duration of the activities on a critical path found by CPM. PERT assumes that the critical path found by CPM contains enough activities to allow us to invoke the Central Limit Theorem and conclude that the following is normally distributed:

$$\mathbf{CP} = \sum_{(i,j)\in \text{critical path}} T_{ij}$$

Example 12. Modified Widgetco

Suppose that for Widgetco example a, b, and m for each activity are given as follows.

Activity	a	b	m
(1,2)	5	13	9
(1,3)	2	10	6
(3,5)	3	13	8
(3,4)	1	13	7
(4,5)	8	12	10
(5,6)	9	15	12

Calculate the expected completion time and the variance of the project.

Answer: $E(\mathbf{T}_{12}) = (5+13+9\times4)/6 = 9$, $\mathrm{var}\mathbf{T}_{12} = (13-5)^2/36 = 1.78$

$E(\mathbf{T}_{13}) = 6$ $\mathrm{var}\mathbf{T}_{13} = 1.78$

$E(\mathbf{T}_{35}) = 8$ $\mathrm{var}\mathbf{T}_{35} = 2.78$

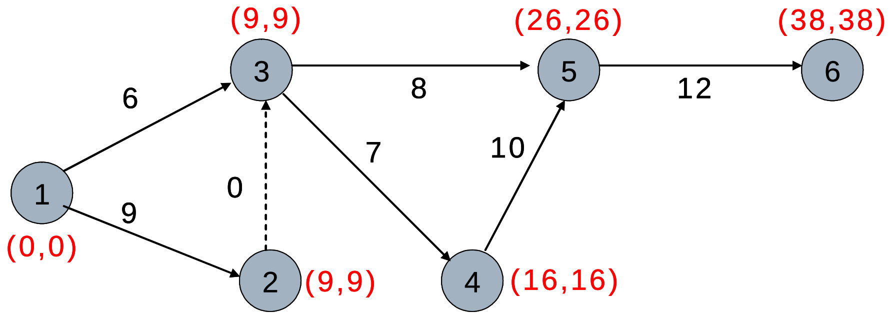
$E(\mathbf{T}_{34}) = 7$ $\mathrm{var}\mathbf{T}_{34} = 4$

$E(\mathbf{T}_{45}) = 10$ $\mathrm{var}\mathbf{T}_{45} = 0.44$

$E(\mathbf{T}_{56}) = 12$ $\mathrm{var}\mathbf{T}_{56} = 1$

$E(\mathbf{T}_{23}) = 0$ $\mathrm{var}\mathbf{T}_{23} = 0$

Project Network & Critical Path



Critical path: 1-2-3-4-5-6

$E(\mathbf{CP}) = 9 + 0 + 7 + 10 + 12 = 38$

$\mathrm{var}\mathbf{CP} = 1.78 + 0 + 4 + 0.44 + 1 = 7.22$

standard deviation for **CP** = $(7.22)^{1/2} = 2.69$

9.3.4 Probability Analysis For CP

Example 13. CP Analysis for Widgetco

What is the probability that Modified Widgetco project will be completed within 35 days?

Answer: Standardizing and applying the assumption that **CP** is normally distributed, we find that **Z** is a standardized normal random variable with mean 0 and variance 1.

Using standard normal cumulative probabilities (*Winston 12.6, p. 724-725*):

$P(\mathbf{CP}\leq 35) = P[(\mathbf{CP}-38)/2.69 \leq (35-38)/2.69)] = P(\mathbf{Z}\leq -1.12) = 0.1314$

Thus, the probability that the project will be completed within 35 days is 13.14%.