Solutions to HW1

Note: These solutions were generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have added comments in italics where I thought more detail was appropriate.

Problem 1.2.1 •

A fax transmission can take place at any of three speeds depending on the condition of the phone connection between the two fax machines. The speeds are high (h) at $14,400$ b/s, medium (m) at 9600 b/s, and low (l) at 4800 b/s. In response to requests for information, a company sends either short faxes of two (t) pages, or long faxes of four (f) pages. Consider the experiment of monitoring a fax transmission and observing the transmission speed and length. An observation is a two-letter word, for example, a high-speed, two-page fax is ht.

(a) What is the sample space of the experiment?

(b) Let A_1 be the event "medium-speed fax." What are the outcomes in A_1?

(c) Let A_2 be the event "short (two-page) fax." What are the outcomes in A_2?

(d) Let A_3 be the event "high-speed fax or low-speed fax." What are the outcomes in A_3?

(e) Are A_1, A_2, and A_3 mutually exclusive?

(f) Are A_1, A_2, and A_3 collectively exhaustive?

Problem 1.2.1 Solution

(a) An outcome specifies whether the fax is high (h), medium (m), or low (l) speed, and whether the fax has two (t) pages or four (f) pages. The sample space is

$$S = \{ht, hf, mt, mf, lt, lf\}. \tag{1}$$

(b) The event that the fax is medium speed is $A_1 = \{mt, mf\}$.

(c) The event that a fax has two pages is $A_2 = \{ht, mt, lt\}$.

(d) The event that a fax is either high speed or low speed is $A_3 = \{ht, hf, lt, lf\}$.

(e) Since $A_1 \cap A_2 = \{mt\}$ and is not empty, A_1, A_2, and A_3 are not mutually exclusive. *(Equivalently, since $A_2 \cap A_3 \neq \emptyset$, A_1, A_2, and A_3 are not mutually exclusive.)*

(f) Since

$$A_1 \cup A_2 \cup A_3 = \{ht, hf, mt, mf, lt, lf\} = S, \tag{2}$$

the collection A_1, A_2, A_3 is collectively exhaustive.

Problem 1.2.2 •

An integrated circuit factory has three machines X, Y, and Z. Test one integrated circuit produced by each machine. Either a circuit is acceptable (a) or it fails (f). An observation is a sequence of three test results corresponding to the circuits from machines X, Y, and Z, respectively. For example, aaf is the observation that the circuits from X and Y pass the test and the circuit from Z fails the test.

(a) What are the elements of the sample space of this experiment?

(b) What are the elements of the sets

$$Z_F = \{\text{circuit from } Z \text{ fails}\},$$
$$X_A = \{\text{circuit from } X \text{ is acceptable}\}.$$

(c) Are Z_F and X_A mutually exclusive?

(d) Are Z_F and X_A collectively exhaustive?

(e) What are the elements of the sets

$$C = \{\text{more than one circuit acceptable}\},$$
$$D = \{\text{at least two circuits fail}\}.$$

(f) Are C and D mutually exclusive?

(g) Are C and D collectively exhaustive?

Problem 1.2.2 Solution

(a) The sample space of the experiment is

$$S = \{aaa, aaf, afa, faa, ffa, faf, aff, fff\}. \tag{1}$$

(b) The event that the circuit from Z fails is

$$Z_F = \{aaf, aff, faf, fff\}. \tag{2}$$

The event that the circuit from X is acceptable is

$$X_A = \{aaa, aaf, afa, aff\}. \tag{3}$$

(c) Since $Z_F \cap X_A = \{aaf, aff\} \neq \phi$, Z_F and X_A are not mutually exclusive.

(d) Since $Z_F \cup X_A = \{aaa, aaf, afa, aff, faf, fff\} \neq S$, Z_F and X_A are not collectively exhaustive.

(e) The event that more than one circuit is acceptable is

$$C = \{aaa, aaf, afa, faa\}. \tag{4}$$

The event that at least two circuits fail is

$$D = \{ffa, faf, aff, fff\}. \tag{5}$$

(f) Inspection shows that $C \cap D = \phi$ so C and D are mutually exclusive.

(g) Since $C \cup D = S$, C and D are collectively exhaustive.

Problem 1.2.3 •

Shuffle a deck of cards and turn over the first card. What is the sample space of this experiment? How many outcomes are in the event that the first card is a heart?

Problem 1.2.3 Solution

The sample space is

$$S = \{A\clubsuit, \ldots, K\clubsuit, A\diamondsuit, \ldots, K\diamondsuit, A\heartsuit, \ldots, K\heartsuit, A\spadesuit, \ldots, K\spadesuit\}. \tag{1}$$

The event H is the set

$$H = \{A\heartsuit, \ldots, K\heartsuit\}. \tag{2}$$

Problem 1.2.6 •

Let the sample space of the experiment consist of the measured resistances of two resistors. Give four examples of event spaces.

Problem 1.2.6 Solution

Let R_1 and R_2 denote the measured resistances. The pair (R_1, R_2) is an outcome of the experiment. Some event spaces include

1. If we need to check that neither resistance is too high, an event space is

$$A_1 = \{R_1 < 100, R_2 < 100\}, \qquad A_2 = \{\text{either } R_1 \geq 100 \text{ or } R_2 \geq 100\}. \tag{1}$$

2. If we need to check whether the first resistance exceeds the second resistance, an event space is

$$B_1 = \{R_1 > R_2\} \qquad B_2 = \{R_1 \leq R_2\}. \tag{2}$$

3. If we need to check whether each resistance doesn't fall below a minimum value (in this case 50 ohms for R_1 and 100 ohms for R_2), an event space is

$$C_1 = \{R_1 < 50, R_2 < 100\}, \qquad C_2 = \{R_1 < 50, R_2 \geq 100\}, \tag{3}$$

$$C_3 = \{R_1 \geq 50, R_2 < 100\}, \qquad C_4 = \{R_1 \geq 50, R_2 \geq 100\}. \tag{4}$$

4. If we want to check whether the resistors in parallel are within an acceptable range of 90 to 110 ohms, an event space is

$$D_1 = \left\{(1/R_1 + 1/R_2)^{-1} < 90\right\}, \tag{5}$$
$$D_2 = \left\{90 \le (1/R_1 + 1/R_2)^{-1} \le 110\right\}, \tag{6}$$
$$D_2 = \left\{110 < (1/R_1 + 1/R_2)^{-1}\right\}. \tag{7}$$

Problem 1.3.1 •

Computer programs are classified by the length of the source code and by the execution time. Programs with more than 150 lines in the source code are big (B). Programs with ≤ 150 lines are little (L). Fast programs (F) run in less than 0.1 seconds. Slow programs (W) require at least 0.1 seconds. Monitor a program executed by a computer. Observe the length of the source code and the run time. The probability model for this experiment contains the following information: $P[LF] = 0.5$, $P[BF] = 0.2$, and $P[BW] = 0.2$. What is the sample space of the experiment? Calculate the following probabilities:

(a) $P[W]$

(b) $P[B]$

(c) $P[W \cup B]$

Problem 1.3.1 Solution

The sample space of the experiment is

$$S = \{LF, BF, LW, BW\}. \tag{1}$$

From the problem statement, we know that $P[LF] = 0.5$, $P[BF] = 0.2$ and $P[BW] = 0.2$. This implies $P[LW] = 1 - 0.5 - 0.2 - 0.2 = 0.1$ *(because BF, BW, and LF are mutually exclusive)*. The questions can be answered using Theorem 1.5.

(a) *(Because $LW \cap BW = \emptyset$)* [t]he probability that a program is slow is

$$P[W] = P[LW] + P[BW] = 0.1 + 0.2 = 0.3. \tag{2}$$

(b) *(Because $BF \cap BW = \emptyset$)* [t]he probability that a program is big is

$$P[B] = P[BF] + P[BW] = 0.2 + 0.2 = 0.4. \tag{3}$$

(c) The probability that a program is slow or big is

$$P[W \cup B] = P[W] + P[B] - P[BW] = 0.3 + 0.4 - 0.2 = 0.5. \tag{4}$$

Problem 1.3.2 •

There are two types of cellular phones, handheld phones (H) that you carry and mobile phones (M) that are mounted in vehicles. Phone calls can be classified by the traveling speed of the user as fast (F) or slow (W). Monitor a cellular phone call and observe the type of telephone and the speed of the user. The probability model for this experiment has the following information: $P[F] = 0.5$, $P[HF] = 0.2$, $P[MW] = 0.1$. What is the sample space of the experiment? Calculate the following probabilities:

(a) $P[W]$

(b) $P[MF]$

(c) $P[H]$

Problem 1.3.2 Solution

A sample outcome indicates whether the cell phone is handheld (H) or mobile (M) and whether the speed is fast (F) or slow (W). The sample space is

$$S = \{HF, HW, MF, MW\}. \tag{1}$$

The problem statement tells us that $P[HF] = 0.2$, $P[MW] = 0.1$ and $P[F] = 0.5$. We can use these facts to find the probabilities of the other outcomes. In particular,

$$P[F] = P[HF] + P[MF], \tag{2}$$

(because $HF \cap MF = \emptyset$.)

This implies

$$P[MF] = P[F] - P[HF] = 0.5 - 0.2 = 0.3. \tag{3}$$

Also, since the probabilities must sum to 1,

$$P[HW] = 1 - P[HF] - P[MF] - P[MW] = 1 - 0.2 - 0.3 - 0.1 = 0.4. \tag{4}$$

Now that we have found the probabilities of the outcomes, finding any other probability is easy.

(a) The probability a cell phone is slow is

$$P[W] = P[HW] + P[MW] = 0.4 + 0.1 = 0.5. \tag{5}$$

(Equivalently, the probability that the cell phone is slow is one minus the probability that it is fast, since these are mutually exclusive, so

$$P[W] = 1 - P[F] = 1 - 0.5 = 0.5 \tag{6}$$

which gives the same answer.)

(b) The probability that a cell hpone is mobile and fast is $P[MF] = 0.3$.

(c) The probability that a cell phone is handheld is

$$P[H] = P[HF] + P[HW] = 0.2 + 0.4 = 0.6. \tag{7}$$

Problem 1.3.3 •
Shuffle a deck of cards and turn over the first card. What is the probability that the first card is a heart?

Problem 1.3.3 Solution
A reasonable probability model that is consistent with the notion of a shuffled deck is that each card in the deck is equally likely to be the first card. Let H_i denote the event that the first card drawn is the ith heart where the first heart is the ace, the second heart is the deuce and so on. In that case, $P[H_i] = 1/52$ for $1 \leq i \leq 13$. The event H that the first card is a heart can be written as the disjoint union

$$H = H_1 \cup H_2 \cup \cdots \cup H_{13}. \tag{1}$$

Using Theorem 1.1, we have

$$P[H] = \sum_{i=1}^{13} P[H_i] = 13/52. \tag{2}$$

This is the answer you would expect since 13 out of 52 cards are hearts. The point to keep in mind is that this is not just the common sense answer but is the result of a probability model for a shuffled deck and the axioms of probability.

Problem 1.3.4 •
You have a six-sided die that you roll once and observe the number of dots facing upwards. What is the sample space? What is the probability of each sample outcome? What is the probability of E, the event that the roll is even?

Problem 1.3.4 Solution
Let s_i denote the outcome that the down face has i dots. The sample space is $S = \{s_1, \ldots, s_6\}$. The probability of each sample outcome is $P[s_i] = 1/6$. From Theorem 1.1, the probability of the event E that the roll is even is

$$P[E] = P[s_2] + P[s_4] + P[s_6] = 3/6. \tag{1}$$

Problem 1.3.5 •
A student's score on a 10-point quiz is equally likely to be any integer between 0 and 10. What is the probability of an A, which requires the student to get a score of 9 or more? What is the probability the student gets an F by getting less than 4?

Problem 1.3.5 Solution
Let s_i equal the outcome of the student's quiz. The sample space is then composed of all the possible grades that she can receive.

$$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}. \tag{1}$$

Since each of the 11 possible outcomes is equally likely, the probability of receiving a grade of i, for each $i = 0, 1, \ldots, 10$ is $P[s_i] = 1/11$. The probability that the student gets an A is the probability that she gets a score of 9 or higher. That is

$$P[\text{Grade of A}] = P[9] + P[10] = 1/11 + 1/11 = 2/11. \tag{2}$$

The probability of failing requires the student to get a grade less than 4.

$$P[\text{Failing}] = P[3] + P[2] + P[1] + P[0] = 1/11 + 1/11 + 1/11 + 1/11 = 4/11. \tag{3}$$

Problem 1.4.1 •

Mobile telephones perform *handoffs* as they move from cell to cell. During a call, a telephone either performs zero handoffs (H_0), one handoff (H_1), or more than one handoff (H_2). In addition, each call is either long (L), if it lasts more than three minutes, or brief (B). The following table describes the probabilities of the possible types of calls.

	H_0	H_1	H_2
L	0.1	0.1	0.2
B	0.4	0.1	0.1

What is the probability $P[H_0]$ that a phone makes no handoffs? What is the probability a call is brief? What is the probability a call is long or there are at least two handoffs?

Problem 1.4.1 Solution

From the table we look to add all the disjoint events that contain H_0 to express the probability that a caller makes no hand-offs as

$$P[H_0] = P[LH_0] + P[BH_0] = 0.1 + 0.4 = 0.5. \tag{1}$$

In a similar fashion we can express the probability that a call is brief by

$$P[B] = P[BH_0] + P[BH_1] + P[BH_2] = 0.4 + 0.1 + 0.1 = 0.6. \tag{2}$$

The probability that a call is long or makes at least two hand-offs is

$$P[L \cup H_2] = P[LH_0] + P[LH_1] + P[LH_2] + P[BH_2] \tag{3}$$
$$= 0.1 + 0.1 + 0.2 + 0.1 = 0.5. \tag{4}$$

Problem 1.4.2 •

For the telephone usage model of Example 1.14, let B_m denote the event that a call is billed for m minutes. To generate a phone bill, observe the duration of the call in integer minutes (rounding up). Charge for M minutes $M = 1, 2, 3, \ldots$ if the exact duration T is $M - 1 < t \leq M$. A more complete probability model shows that for $m = 1, 2, \ldots$ the probability of each event B_m is

$$P[B_m] = \alpha(1 - \alpha)^{m-1}$$

where $\alpha = 1 - (0.57)^{1/3} = 0.171$.

(a) Classify a call as long, L, if the call lasts more than three minutes. What is $P[L]$?

(b) What is the probability that a call will be billed for nine minutes or less?

Problem 1.4.2 Solution

Note: (4) is not "obvious", at least to me, from (3). However, if you do the math, you'll find that it is the correct answer. Similarly, the final expression on the right-hand side of (5) is not "obvious" but can be obtained either by summing the terms directly or by using the fact that

$$1 + \beta + \beta^2 + \cdots + \beta^{p-1} = \frac{1-\beta^p}{1-\beta}.$$

(a) From the given probability distribution of billed minutes, M, the probability that a call is billed for more than 3 minutes is

$$\begin{aligned} P[L] &= 1 - P[\text{3 or fewer billed minutes}] && (1)\\ &= 1 - P[B_1] - P[B_2] - P[B_3] && (2)\\ &= 1 - \alpha - \alpha(1-\alpha) - \alpha(1-\alpha)^2 && (3)\\ &= (1-\alpha)^3 = 0.57. && (4) \end{aligned}$$

(b) The probability that a call will billed for 9 minutes or less is

$$P[\text{9 minutes or less}] = \sum_{i=1}^{9} \alpha(1-\alpha)^{i-1} = 1 - (0.57)^3. \qquad (5)$$

Problem 1.5.1 •

Given the model of handoffs and call lengths in Problem 1.4.1,

(a) What is the probability that a brief call will have no handoffs?

(b) What is the probability that a call with one handoff will be long?

(c) What is the probability that a long call will have one or more handoffs?

Problem 1.5.1 Solution

Each question requests a conditional probability.

(a) Note that the probability a call is brief is

$$P[B] = P[H_0B] + P[H_1B] + P[H_2B] = 0.6. \qquad (1)$$

The probability a brief call will have no handoffs is

$$P[H_0|B] = \frac{P[H_0B]}{P[B]} = \frac{0.4}{0.6} = \frac{2}{3}. \qquad (2)$$

(b) The probability of one handoff is $P[H_1] = P[H_1B] + P[H_1L] = 0.2$. The probability that a call with one handoff will be long is

$$P[L|H_1] = \frac{P[H_1L]}{P[H_1]} = \frac{0.1}{0.2} = \frac{1}{2}. \tag{3}$$

(c) The probability a call is long is $P[L] = 1 - P[B] = 0.4$. The probability that a long call will have one or more handoffs is

$$P[H_1 \cup H_2|L] = \frac{P[H_1L \cup H_2L]}{P[L]} = \frac{P[H_1L] + P[H_2L]}{P[L]} = \frac{0.1 + 0.2}{0.4} = \frac{3}{4}. \tag{4}$$

Problem 1.5.2 •

You have a six-sided die that you roll once. Let R_i denote the event that the roll is i. Let G_j denote the event that the roll is greater than j. Let E denote the event that the roll of the die is even-numbered.

(a) What is $P[R_3|G_1]$, the conditional probability that 3 is rolled given that the roll is greater than 1?

(b) What is the conditional probability that 6 is rolled given that the roll is greater than 3?

(c) What is $P[G_3|E]$, the conditional probability that the roll is greater than 3 given that the roll is even?

(d) Given that the roll is greater than 3, what is the conditional probability that the roll is even?

Problem 1.5.2 Solution

Let s_i denote the outcome that the roll is i. So, for $1 \le i \le 6$, $R_i = \{s_i\}$. Similarly, $G_j = \{s_{j+1}, \ldots, s_6\}$.

(a) Since $G_1 = \{s_2, s_3, s_4, s_5, s_6\}$ and all outcomes have probability 1/6, $P[G_1] = 5/6$. The event $R_3G_1 = \{s_3\}$ and $P[R_3G_1] = 1/6$ so that

$$P[R_3|G_1] = \frac{P[R_3G_1]}{P[G_1]} = \frac{1}{5}. \tag{1}$$

(b) The conditional probability that 6 is rolled given that the roll is greater than 3 is

$$P[R_6|G_3] = \frac{P[R_6G_3]}{P[G_3]} = \frac{P[s_6]}{P[s_4, s_5, s_6]} = \frac{1/6}{3/6}. \tag{2}$$

(c) The event E that the roll is even is $E = \{s_2, s_4, s_6\}$ and has probability 3/6. The joint probability of G_3 and E is

$$P[G_3E] = P[s_4, s_6] = 1/3. \tag{3}$$

The conditional probabilities of G_3 given E is

$$P[G_3|E] = \frac{P[G_3E]}{P[E]} = \frac{1/3}{1/2} = \frac{2}{3}. \tag{4}$$

(d) The conditional probability that the roll is even given that it's greater than 3 is

$$P[E|G_3] = \frac{P[EG_3]}{P[G_3]} = \frac{1/3}{1/2} = \frac{2}{3}. \tag{5}$$

Problem 1.5.3 •

You have a shuffled deck of three cards: 2, 3, and 4. You draw one card. Let C_i denote the event that card i is picked. Let E denote the event that card chosen is a even-numbered card.

(a) What is $P[C_2|E]$, the probability that the 2 is picked given that an even-numbered card is chosen?

(b) What is the conditional probability that an even-numbered card is picked given that the 2 is picked?

Problem 1.5.3 Solution

Since the 2 of clubs is an even numbered card, $C_2 \subset E$ so that $P[C_2E] = P[C_2] = 1/3$. Since $P[E] = 2/3$,

$$P[C_2|E] = \frac{P[C_2E]}{P[E]} = \frac{1/3}{2/3} = 1/2. \tag{1}$$

The probability that an even numbered card is picked given that the 2 is picked is

$$P[E|C_2] = \frac{P[C_2E]}{P[C_2]} = \frac{1/3}{1/3} = 1. \tag{2}$$

Problem 1.6.1 •

Is it possible for A and B to be independent events yet satisfy $A = B$?

Problem 1.6.1 Solution

This problems asks whether A and B can be independent events yet satisfy $A = B$? By definition, events A and B are independent if and only if $P[AB] = P[A]P[B]$. We can see that if $A = B$, that is they are the same set, then

$$P[AB] = P[AA] = P[A] = P[B]. \tag{1}$$

Thus, for A and B to be the same set and also independent,

$$P[A] = P[AB] = P[A]P[B] = (P[A])^2. \tag{2}$$

There are two ways that this requirement can be satisfied:

- $P[A] = 1$ implying $A = B = S$.
- $P[A] = 0$ implying $A = B = \phi$.

Problem 1.7.1 •

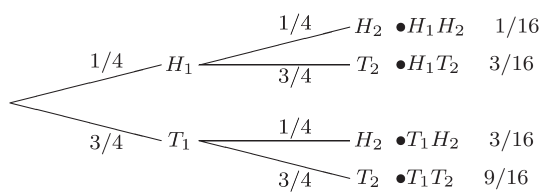
Suppose you flip a coin twice. On any flip, the coin comes up heads with probability 1/4. Use H_i and T_i to denote the result of flip i.

(a) What is the probability, $P[H_1|H_2]$, that the first flip is heads given that the second flip is heads?

(b) What is the probability that the first flip is heads and the second flip is tails?

Problem 1.7.1 Solution

A sequential sample space for this experiment is



(a) From the tree, we observe

$$P[H_2] = P[H_1H_2] + P[T_1H_2] = 1/4. \tag{1}$$

This implies

$$P[H_1|H_2] = \frac{P[H_1H_2]}{P[H_2]} = \frac{1/16}{1/4} = 1/4. \tag{2}$$

Note: If you have taken a course in probability before, you may also have used the fact that since the results of the two flips are independent, $P[H_1|H_2] = P[H_1] = 1/4$. However, while this is true, it hasn't been stated and proved a theorem yet in the text; so unless you prove it as part of your answer, you should not use it.

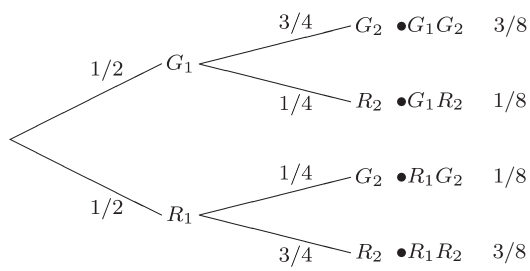
(b) The probability that the first flip is heads and the second flip is tails is $P[H_1T_2] = 3/16$ *because the results of the flips are independent.*

Problem 1.7.2 •

For Example 1.25, suppose $P[G_1] = 1/2$, $P[G_2|G_1] = 3/4$, and $P[G_2|R_1] = 1/4$. Find $P[G_2]$, $P[G_2|G_1]$, and $P[G_1|G_2]$.

Problem 1.7.2 Solution

The tree with adjusted probabilities is



From the tree, the probability the second light is green is

$$P[G_2] = P[G_1G_2] + P[R_1G_2] = 3/8 + 1/8 = 1/2. \tag{1}$$

The conditional probability that the first light was green given the second light was green is

$$P[G_1|G_2] = \frac{P[G_1G_2]}{P[G_2]} = \frac{P[G_2|G_1]\,P[G_1]}{P[G_2]} = 3/4. \tag{2}$$

Finally, from the tree diagram, we can directly read that $P[G_2|G_1] = 3/4$.

Alternatively, we can simply apply the Law of Total Probability (Thm 1.10, p. 19 of the textbook). First, since $\{R_1, G_1\}$ *is an event space,*

$$P[G_2] = P[G_2|G_1]\,P[G_1] + P[G_2|R_1]\,P[R_1] = \left(\frac{3}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{1}{2}. \tag{3}$$

Next, $P[G_2|G_1] = 3/4$ *is given. Finally, applying Bayes' Theorem (Thm 1.11, p. 20 of the textbook) yields*

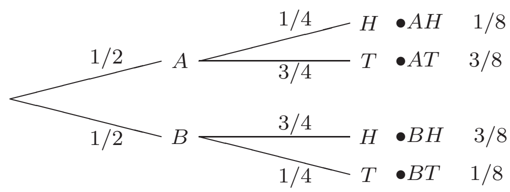
$$P[G_1|G_2] = \frac{P[G_2|G_1]\,P[G_1]}{P[G_2]} = \frac{\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)}{\frac{1}{2}} = \frac{3}{4}. \tag{4}$$

Problem 1.7.4 •

You have two biased coins. Coin A comes up heads with probability 1/4. Coin B comes up heads with probability 3/4. However, you are not sure which is which so you choose a coin randomly and you flip it. If the flip is heads, you guess that the flipped coin is B; otherwise, you guess that the flipped coin is A. Let events A and B designate which coin was picked. What is the probability $P[C]$ that your guess is correct?

Problem 1.7.4 Solution

The tree for this experiment is



The probability that you guess correctly is

$$P[C] = P[AT] + P[BH] = 3/8 + 3/8 = 3/4. \tag{1}$$

Alternatively, using the Law of Total Probability (Thm. 1.10, page 19 of the text),

$$P[C] = P[T|A]\,P[A] + P[H|B]\,P[B] = (3/4)(1/2) + (3/4)(1/2) = 3/4. \tag{2}$$

Problem 1.8.1 •

Consider a binary code with 5 bits (0 or 1) in each code word. An example of a code word is 01010. How many different code words are there? How many code words have exactly three 0's?

Problem 1.8.1 Solution

There are $2^5 = 32$ different binary codes with 5 bits. The number of codes with exactly 3 zeros equals the number of ways of choosing the bits in which those zeros occur. Therefore there are $\binom{5}{3} = 10$ codes with exactly 3 zeros.

Problem 1.8.2 •

Consider a language containing four letters: A, B, C, D. How many three-letter words can you form in this language? How many four-letter words can you form if each letter appears only once in each word?

Problem 1.8.2 Solution

Since each letter can take on any one of the 4 possible letters in the alphabet, the number of 3 letter words that can be formed is $4^3 = 64$. If we allow each letter to appear only once then we have 4 choices for the first letter and 3 choices for the second and two choices for the third letter. Therefore, there are a total of $4 \cdot 3 \cdot 2 = 24$ possible codes.

Problem 1.8.3 ■

Shuffle a deck of cards and pick two cards at random. Observe the sequence of the two cards in the order in which they were chosen.

(a) How many outcomes are in the sample space?

(b) How many outcomes are in the event that the two cards are the same type but different suits?

(c) What is the probability that the two cards are the same type but different suits?

(d) Suppose the experiment specifies observing the set of two cards without considering the order in which they are selected, and redo parts (a)–(c).

Problem 1.8.3 Solution

(a) The experiment of picking two cards and recording them in the order in which they were selected can be modeled by two sub-experiments. The first is to pick the first card and record it, the second sub-experiment is to pick the second card without replacing the first and recording it. For the first sub-experiment we can have any one of the possible 52 cards for a total of 52 possibilities. The second experiment consists of all the cards minus the one that was picked first(because we are sampling without replacement) for a total of 51 possible outcomes. So the total number of outcomes is the product of the number of outcomes for each sub-experiment.

$$52 \cdot 51 = 2652 \text{ outcomes.} \tag{1}$$

(b) To have the same card but different suit we can make the following sub-experiments. First we need to pick one of the 52 cards. Then we need to pick one of the 3 remaining cards that are of the same type but different suit out of the remaining 51 cards. So the total number outcomes is

$$52 \cdot 3 = 156 \text{ outcomes.} \tag{2}$$

(c) The probability that the two cards are of the same type but different suit is the number of outcomes that are of the same type but different suit divided by the total number of outcomes involved in picking two cards at random from a deck of 52 cards.

$$P\left[\text{same type, different suit}\right] = \frac{156}{2652} = \frac{1}{17}. \tag{3}$$

(d) Now we are not concerned with the ordering of the cards. So before, the outcomes $(K\heartsuit, 8\diamondsuit)$ and $(8\diamondsuit, K\heartsuit)$ were distinct. Now, those two outcomes are not distinct and are only considered to be the single outcome that a King of hearts and 8 of diamonds were selected. So every pair of outcomes before collapses to a single outcome when we disregard ordering. So we can redo parts (a) and (b) above by halving the corresponding values found in parts (a) and (b). The probability however, does not change because both the numerator and the denominator have been reduced by an equal factor of 2, which does not change their ratio.