Solutions to HW8

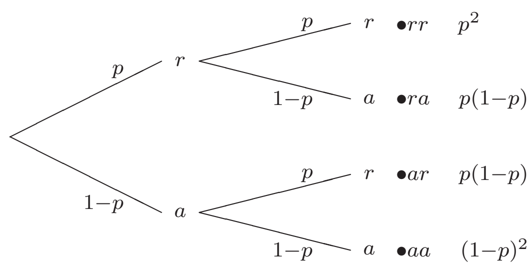
Note: Most of these solutions were generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have added comments in italics where I thought more detail was appropriate.

Problem 4.2.3 •

Test two integrated circuits. In each test, the probability of rejecting the circuit is p. Let X be the number of rejects (either 0 or 1) in the first test and let Y be the number of rejects in the second test. Find the joint PMF $P_{X,Y}(x,y)$.

Problem 4.2.3 Solution

Let r (reject) and a (accept) denote the result of each test. There are four possible outcomes: rr, ra, ar, aa. The sample tree is



Now we construct a table that maps the sample outcomes to values of X and Y.

outcome	$P[\cdot]$	X	Y
rr	p^2	1	1
ra	$p(1-p)$	1	0
ar	$p(1-p)$	0	1
aa	$(1-p)^2$	0	0

(1)

This table is esentially the joint PMF $P_{X,Y}(x,y)$.

$$P_{X,Y}(x,y) = \begin{cases} p^2 & x=1, y=1 \\ p(1-p) & x=0, y=1 \\ p(1-p) & x=1, y=0 \\ (1-p)^2 & x=0, y=0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Problem 4.4.3 ■

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P[X > Y]$ and $P[X + Y \leq 1]$.

(b) Find $P[\min(X, Y) \geq 1]$.

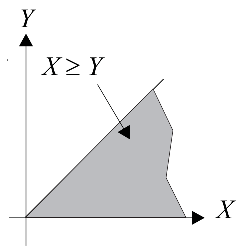
(c) Find $P[\max(X, Y) \leq 1]$.

Problem 4.4.3 Solution

The joint PDF of X and Y is

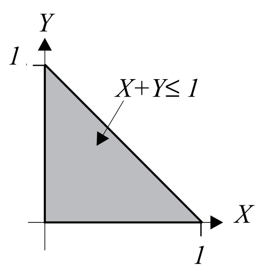
$$f_{X,Y}(x, y) = \begin{cases} 6e^{-(2x+3y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The probability that $X \geq Y$ is:



$$P[X \geq Y] = \int_0^\infty \int_0^x 6e^{-(2x+3y)}\, dy\, dx \tag{2}$$
$$= \int_0^\infty 2e^{-2x} \left(-e^{-3y}\Big|_{y=0}^{y=x} \right) dx \tag{3}$$
$$= \int_0^\infty [2e^{-2x} - 2e^{-5x}]\, dx = 3/5 \tag{4}$$

The $P[X + Y \leq 1]$ is found by integrating over the region where $X + Y \leq 1$



$$P[X + Y \leq 1] = \int_0^1 \int_0^{1-x} 6e^{-(2x+3y)}\, dy\, dx \tag{5}$$
$$= \int_0^1 2e^{-2x} \left[-e^{-3y}\Big|_{y=0}^{y=1-x} \right] dx \tag{6}$$
$$= \int_0^1 2e^{-2x} \left[1 - e^{-3(1-x)} \right] dx \tag{7}$$
$$= -e^{-2x} - 2e^{x-3}\Big|_0^1 \tag{8}$$
$$= 1 + 2e^{-3} - 3e^{-2} \tag{9}$$

We should check here to make sure that our answer is not obviously invalid. A probability must not exceed one so, not knowing offhand whether $2e^{-3}$ *is less than or greater than* $3e^{-2}$*, we consult Matlab. Matlab says that* $1 + 2e^{-3} - 3e^{-2} = 0.6936$ *which is less than one so we have no cause for alarm.*

(b) The event $\min(X, Y) \geq 1$ is the same as the event $\{X \geq 1, Y \geq 1\}$. Thus,

$$P[\min(X, Y) \geq 1] = \int_1^\infty \int_1^\infty 6e^{-(2x+3y)}\, dy\, dx = e^{-(2+3)} \tag{10}$$

(c) The event $\max(X, Y) \leq 1$ is the same as the event $\{X \leq 1, Y \leq 1\}$ so that

$$P[\max(X, Y) \leq 1] = \int_0^1 \int_0^1 6e^{-(2x+3y)}\, dy\, dx = (1 - e^{-2})(1 - e^{-3}) \tag{11}$$

Problem 5.1.3 •

The random variables $X_1, \ldots, X_n$ have the joint PDF

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \begin{cases} 1 & 0 \le x_i \le 1; \\ & i = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the joint CDF, $F_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$?

(b) For $n = 3$, what is the probability that $\min_i X_i \le 3/4$?

Problem 5.1.3 Solution

(a) In terms of the joint PDF, we can write joint CDF as

$$F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{X_1,\ldots,X_n}(y_1, \ldots, y_n)\, dy_1 \cdots dy_n \tag{1}$$

However, simplifying the above integral depends on the values of each x_i. In particular, $f_{X_1,\ldots,X_n}(y_1, \ldots, y_n) = 1$ if and only if $0 \le y_i \le 1$ for each i. Since $F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = 0$ if any $x_i < 0$, we limit, for the moment, our attention to the case where $x_i \ge 0$ for all i. In this case, some thought will show that we can write the limits in the following way:

$$\begin{aligned} F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) &= \int_0^{\max(1,x_1)} \cdots \int_0^{\min(1,x_n)} dy_1 \cdots dy_n && (2) \\ &= \min(1, x_1) \min(1, x_2) \cdots \min(1, x_n) && (3) \end{aligned}$$

A complete expression for the CDF of $X_1, \ldots, X_n$ is

$$F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \begin{cases} \prod_{i=1}^n \min(1, x_i) & 0 \le x_i, i = 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(b) For $n = 3$,

$$\begin{aligned} 1 - P\left[\min_i X_i \le 3/4\right] &= P\left[\min_i X_i > 3/4\right] && (5) \\ &= P\left[X_1 > 3/4, X_2 > 3/4, X_3 > 3/4\right] && (6) \\ &= \int_{3/4}^1 \int_{3/4}^1 \int_{3/4}^1 dx_1\, dx_2\, dx_3 && (7) \\ &= (1 - 3/4)^3 = 1/64 && (8) \end{aligned}$$

Thus $P[\min_i X_i \le 3/4] = 63/64$.

Problem 5.2.1 •
For random variables $X_1, \ldots, X_n$ in Problem 5.1.3, let $\mathbf{X} = \begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix}'$. What is $f_{\mathbf{X}}(\mathbf{x})$?

Problem 5.2.1 Solution
This problem is very simple. In terms of the vector $\mathbf{X}$, the PDF is

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} 1 & \mathbf{0} \leq \mathbf{x} \leq \mathbf{1} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

However, just keep in mind that the inequalities $\mathbf{0} \leq \mathbf{x}$ and $\mathbf{x} \leq \mathbf{1}$ are vector inequalities that must hold for every component x_i. *As noted in class, we should say* $x_i \in [0,1]$, $\forall i \in \{1, 2, \ldots, n\}$ *rather than* $\mathbf{0} \leq \mathbf{x} \leq \mathbf{1}$.

Problem 4.3.4 ■
Random variables N and K have the joint PMF

$$P_{N,K}(n,k) = \begin{cases} (1-p)^{n-1}p/n & k = 1, \ldots, n; \\ & n = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal PMFs $P_N(n)$ and $P_K(k)$.

Problem 4.3.4 Solution
The joint PMF of N, K is

$$P_{N,K}(n,k) = \begin{cases} (1-p)^{n-1}p/n & k = 1, 2, \ldots, n \\ & n = 1, 2 \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

For $n \geq 1$, the marginal PMF of N is

$$P_N(n) = \sum_{k=1}^{n} P_{N,K}(n,k) = \sum_{k=1}^{n} (1-p)^{n-1}p/n = (1-p)^{n-1}p \tag{2}$$

The marginal PMF of K is found by summing $P_{N,K}(n,k)$ over all possible N. Note that if $K = k$, then $N \geq k$. Thus,

$$P_K(k) = \sum_{n=k}^{\infty} \frac{1}{n}(1-p)^{n-1}p \tag{3}$$

Unfortunately, this sum cannot be simplified.

Problem 4.5.3 ■

Over the circle $X^2 + Y^2 \le r^2$, random variables X and Y have the uniform PDF

$$f_{X,Y}(x, y) = \begin{cases} 1/(\pi r^2) & x^2 + y^2 \le r^2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the marginal PDF $f_X(x)$?

(b) What is the marginal PDF $f_Y(y)$?

Problem 4.5.3 Solution

Random variables X and Y have joint PDF

$$f_{X,Y}(x, y) = \begin{cases} 1/(\pi r^2) & 0 \le x^2 + y^2 \le r^2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The marginal PDF of X is

$$f_X(x) = 2\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{1}{\pi r^2}\, dy = \begin{cases} \frac{2\sqrt{r^2-x^2}}{\pi r^2} & -r \le x \le r, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(b) Similarly, for random variable Y,

$$f_Y(y) = 2\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\, dx = \begin{cases} \frac{2\sqrt{r^2-y^2}}{\pi r^2} & -r \le y \le r, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Problem 5.3.7 ■

For Example 5.2, we derived the joint PMF of three types of fax transmissions:

$$P_{X,Y,Z}(x, y, z) = \binom{4}{x, y, z} \frac{1}{3^x} \frac{1}{2^y} \frac{1}{6^z}.$$

(a) In a group of four faxes, what is the PMF of the number of 3-page faxes?

(b) In a group of four faxes, what is the expected number of 3-page faxes?

(c) Given that there are two 3-page faxes in a group of four, what is the joint PMF of the number of 1-page faxes and the number of 2-page faxes?

(d) Given that there are two 3-page faxes in a group of four, what is the expected number of 1-page faxes?

(e) In a group of four faxes, what is the joint PMF of the number of 1-page faxes and the number of 2-page faxes?

Problem 5.3.7 Solution

(a) Note that Z is the number of three-page faxes. In principle, we can sum the joint PMF $P_{X,Y,Z}(x,y,z)$ over all x, y to find $P_Z(z)$. However, it is better to realize that each fax has 3 pages with probability 1/6, independent of any other fax. Thus, Z has the binomial PMF

$$P_Z(z) = \begin{cases} \binom{4}{z}(1/6)^z(5/6)^{4-z} & z = 0,1,\ldots,4 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) From the properties of the binomial distribution given in Appendix A, we know that $E[Z] = 4(1/6)$.

(c) We want to find the conditional PMF of the number X of 1-page faxes and number Y of 2-page faxes given $Z = 2$ 3-page faxes. Note that given $Z = 2$, $X + Y = 2$. Hence for non-negative integers x, y satisfying $x + y = 2$,

$$P_{X,Y|Z}(x,y|2) = \frac{P_{X,Y,Z}(x,y,2)}{P_Z(2)} = \frac{\frac{4!}{x!y!2!}(1/3)^x(1/2)^y(1/6)^2}{\binom{4}{2}(1/6)^2(5/6)^2} \tag{2}$$

With some algebra, the complete expression of the conditional PMF is

$$P_{X,Y|Z}(x,y|2) = \begin{cases} \frac{2!}{x!y!}(2/5)^x(3/5)^y & x+y=2, x \geq 0, y \geq 0; x,y \text{ integer} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

In the above expression, we note that if $Z = 2$, then $Y = 2 - X$ and

$$P_{X|Z}(x|2) = P_{X,Y|Z}(x, 2-x|2) = \begin{cases} \binom{2}{x}(2/5)^x(3/5)^{2-x} & x = 0,1,2,3 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

That is, given $Z = 2$, there are 2 faxes left, each of which independently could be a 1-page fax. The conditonal PMF of the number of 1-page faxes is binomial where 2/5 is the conditional probability that a fax has 1 page given that it either has 1 page or 2 pages. Moreover given $X = x$ and $Z = 2$ we must have $Y = 2 - x$.

(d) Given $Z = 2$, the conditional PMF of X is binomial for 3 trials and success probability 2/5. The conditional expectation of X given $Z = 2$ is $E[X|Z = 2] = 2(2/5) = 4/5$.

(e) There are several ways to solve this problem. The most straightforward approach is to realize that for integers $0 \leq x \leq 4$ and $0 \leq y \leq 4$, the event $\{X = x, Y = y\}$ occurs iff $\{X = x, Y = y, Z = 4 - (x + y)\}$. For the rest of this problem, we assume x and y are non-negative integers so that

$$\begin{aligned} P_{X,Y}(x,y) &= P_{X,Y,Z}(x,y,4-(x+y)) && (5) \\ &= \begin{cases} \frac{4!}{x!y!(4-x-y)!}\left(\frac{1}{3}\right)^x\left(\frac{1}{2}\right)^y\left(\frac{1}{6}\right)^{4-x-y} & 0 \leq x+y \leq 4, x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases} && (6) \end{aligned}$$

The above expression may seem unwieldy and it isn't even clear that it will sum to 1. To simplify the expression, we observe that

$$P_{X,Y}(x,y) = P_{X,Y,Z}(x,y,4-x-y) = P_{X,Y|Z}(x,y|4-x+y)\,P_Z(4-x-y) \quad (7)$$

Using $P_Z(z)$ found in part (c), we can calculate $P_{X,Y|Z}(x,y|4-x-y)$ for $0 \leq x+y \leq 4$, integer valued.

$$\begin{aligned} P_{X,Y|Z}(x,y|4-x+y) &= \frac{P_{X,Y,Z}(x,y,4-x-y)}{P_Z(4-x-y)} && (8) \\ &= \binom{x+y}{x}\left(\frac{1/3}{1/2+1/3}\right)^x\left(\frac{1/2}{1/2+1/3}\right)^y && (9) \\ &= \binom{x+y}{x}\left(\frac{2}{5}\right)^x\left(\frac{3}{5}\right)^{(x+y)-x} && (10) \end{aligned}$$

In the above expression, it is wise to think of $x+y$ as some fixed value. In that case, we see that given $x+y$ is a fixed value, X and Y have a joint PMF given by a binomial distribution in x. This should not be surprising since it is just a generalization of the case when $Z=2$. That is, given that there were a fixed number of faxes that had either one or two pages, each of those faxes is a one page fax with probability $(1/3)/(1/2+1/3)$ and so the number of one page faxes should have a binomial distribution, Moreover, given the number X of one page faxes, the number Y of two page faxes is completely specified.

Finally, by rewriting $P_{X,Y}(x,y)$ given above, the complete expression for the joint PMF of X and Y is

$$P_{X,Y}(x,y) = \begin{cases} \binom{4}{4-x-y}\left(\frac{1}{6}\right)^{4-x-y}\left(\frac{5}{6}\right)^{x+y}\binom{x+y}{x}\left(\frac{2}{5}\right)^x\left(\frac{3}{5}\right)^y & x,y \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

Problem 4.10.10 ■

An ice cream company orders supplies by fax. Depending on the size of the order, a fax can be either

1 page for a short order,
2 pages for a long order.

The company has three different suppliers:

The vanilla supplier is 20 miles away.
The chocolate supplier is 100 miles away.
The strawberry supplier is 300 miles away.

An experiment consists of monitoring an order and observing N, the number of pages, and D, the distance the order is transmitted. The following probability model describes the experiment:

	van.	choc.	straw.
short	0.2	0.2	0.2
long	0.1	0.2	0.1

(a) What is the joint PMF $P_{N,D}(n,d)$ of the number of pages and the distance?

(b) What is $E[D]$, the expected distance of an order?

(c) Find $P_{D|N}(d|2)$, the conditional PMF of the distance when the order requires 2 pages.

(d) Write $E[D|N=2]$, the expected distance given that the order requires 2 pages.

(e) Are the random variables D and N independent?

(f) The price per page of sending a fax is one cent per mile transmitted. C cents is the price of one fax. What is $E[C]$, the expected price of one fax?

Problem 4.10.10 Solution

The key to this problem is understanding that "short order" and "long order" are synonyms for $N = 1$ and $N = 2$. Similarly, "vanilla", "chocolate", and "strawberry" correspond to the events $D = 20$, $D = 100$ and $D = 300$.

(a) The following table is given in the problem statement.

	vanilla	choc.	strawberry
short order	0.2	0.2	0.2
long order	0.1	0.2	0.1

This table can be translated directly into the joint PMF of N and D.

$$\begin{array}{c|ccc} P_{N,D}(n,d) & d=20 & d=100 & d=300 \\ \hline n=1 & 0.2 & 0.2 & 0.2 \\ \hline n=2 & 0.1 & 0.2 & 0.1 \end{array} \tag{1}$$

(b) We find the marginal PMF $P_D(d)$ by summing the columns of the joint PMF. This yields

$$P_D(d) = \begin{cases} 0.3 & d = 20, \\ 0.4 & d = 100, \\ 0.3 & d = 300, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(c) To find the conditional PMF $P_{D|N}(d|2)$, we first need to find the probability of the conditioning event

$$P_N(2) = P_{N,D}(2,20) + P_{N,D}(2,100) + P_{N,D}(2,300) = 0.4 \tag{3}$$

The conditional PMF of N D given $N = 2$ is

$$P_{D|N}(d|2) = \frac{P_{N,D}(2,d)}{P_N(2)} = \begin{cases} 1/4 & d = 20 \\ 1/2 & d = 100 \\ 1/4 & d = 300 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(d) The conditional expectation of D given $N = 2$ is

$$E[D|N=2] = \sum_d dP_{D|N}(d|2) = 20(1/4) + 100(1/2) + 300(1/4) = 130 \tag{5}$$

(e) To check independence, we could calculate the marginal PMFs of N and D. In this case, however, it is simpler to observe that $P_D(d) \neq P_{D|N}(d|2)$. Hence N and D are dependent.

(f) In terms of N and D, the cost (in cents) of a fax is $C = ND$. The expected value of C is

$$E[C] = \sum_{n,d} ndP_{N,D}(n,d) \tag{6}$$

$$= 1(20)(0.2) + 1(100)(0.2) + 1(300)(0.2) \tag{7}$$

$$+ 2(20)(0.1) + 2(100)(0.2) + 2(300)(0.1) = 188 \tag{8}$$

Problem 5.4.6 ■

The random vector $\mathbf{X}$ has PDF

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} e^{-x_3} & 0 \leq x_1 \leq x_2 \leq x_3, \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal PDFs $f_{X_1}(x_1)$, $f_{X_2}(x_2)$, and $f_{X_3}(x_3)$.

Problem 5.4.6 Solution

This problem was not assigned but contains the straightforward solution to problem 5.4.5, hence is included here in the solutions.

We find the marginal PDFs using Theorem 5.5. First we note that for $x < 0$, $f_{X_i}(x) = 0$. For $x_1 \geq 0$,

$$f_{X_1}(x_1) = \int_{x_1}^{\infty} \left(\int_{x_2}^{\infty} e^{-x_3}\, dx_3 \right) dx_2 = \int_{x_1}^{\infty} e^{-x_2}\, dx_2 = e^{-x_1} \tag{1}$$

Similarly, for $x_2 \geq 0$, X_2 has marginal PDF

$$f_{X_2}(x_2) = \int_0^{x_2} \left(\int_{x_2}^{\infty} e^{-x_3}\, dx_3 \right) dx_1 = \int_0^{x_2} e^{-x_2}\, dx_1 = x_2 e^{-x_2} \tag{2}$$

Lastly,

$$f_{X_3}(x_3) = \int_0^{x_3} \left(\int_{x_1}^{x_3} e^{-x_3}\, dx_2 \right) dx_1 = \int_0^{x_3} (x_3 - x_1) e^{-x_3}\, dx_1 \tag{3}$$

$$= -\frac{1}{2}(x_3 - x_1)^2 e^{-x_3} \Big|_{x_1=0}^{x_1=x_3} = \frac{1}{2} x_3^2 e^{-x_3} \tag{4}$$

The complete expressions for the three marginal PDFs are

$$f_{X_1}(x_1) = \begin{cases} e^{-x_1} & x_1 \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$f_{X_2}(x_2) = \begin{cases} x_2 e^{-x_2} & x_2 \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$f_{X_3}(x_3) = \begin{cases} (1/2) x_3^2 e^{-x_3} & x_3 \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

In fact, each X_i is an Erlang $(n, \lambda) = (i, 1)$ random variable.

Problem 4.6.6 ■

Random variables X and Y have joint PDF

$$f_{X,Y}(x, y) = \begin{cases} x + y & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $W = \max(X, Y)$.

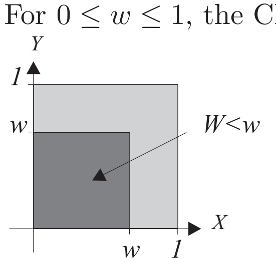
(a) What is S_W, the range of W?

(b) Find $F_W(w)$ and $f_W(w)$.

Problem 4.6.6 Solution

(a) The minimum value of W is $W = 0$, which occurs when $X = 0$ and $Y = 0$. The maximum value of W is $W = 1$, which occurs when $X = 1$ or $Y = 1$. The range of W is $S_W = \{w | 0 \leq w \leq 1\}$.

(b) For $0 \leq w \leq 1$, the CDF of W is



$$F_W(w) = P[\max(X, Y) \leq w] \tag{1}$$

$$= P[X \leq w, Y \leq w] \tag{2}$$

$$= \int_0^w \int_0^w f_{X,Y}(x, y)\, dy\, dx \tag{3}$$

Substituting $f_{X,Y}(x,y) = x + y$ yields

$$F_W(w) = \int_0^w \int_0^w (x+y)\,dy\,dx \tag{4}$$

$$= \int_0^w \left(xy + \frac{y^2}{2}\Big|_{y=0}^{y=w} \right) dx = \int_0^w (wx + w^2/2)\,dx = w^3 \tag{5}$$

The complete expression for the CDF is

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w^3 & 0 \le w \le 1 \\ 1 & \text{otherwise} \end{cases} \tag{6}$$

The PDF of W is found by differentiating the CDF.

$$f_W(w) = \frac{dF_W(w)}{dw} = \begin{cases} 3w^2 & 0 \le w \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Problem 5.5.3 ■

In an automatic geolocation system, a dispatcher sends a message to six trucks in a fleet asking their locations. The waiting times for responses from the six trucks are iid exponential random variables, each with expected value 2 seconds.

(a) What is the probability that all six responses will arrive within 5 seconds?

(b) If the system has to locate all six vehicles within 3 seconds, it has to reduce the expected response time of each vehicle. What is the maximum expected response time that will produce a location time for all six vehicles of 3 seconds or less with probability of at least 0.9?

Problem 5.5.3 Solution

The response time X_i of the ith truck has PDF $f_{X_i}(x_i)$ and CDF $F_{X_i}(x_i)$ given by

$$f_{X_i}(x_i) = \begin{cases} \frac{1}{2}e^{-x/2} & x \ge 0, \\ 0 & \text{otherwise}, \end{cases} \qquad F_{X_i}(x_i) = F_X(x_i) = \begin{cases} 1 - e^{-x/2} & x \ge 0 \\ 0 & \text{otherwise}. \end{cases} \tag{1}$$

Let $R = \max(X_1, X_2, \ldots, X_6)$ denote the maximum response time. From Theorem 5.7, R has PDF

$$F_R(r) = (F_X(r))^6. \tag{2}$$

(a) The probability that all six responses arrive within five seconds is

$$P[R \le 5] = F_R(5) = (F_X(5))^6 = (1 - e^{-5/2})^6 = 0.5982. \tag{3}$$

(b) This question is worded in a somewhat confusing way. The "expected response time" refers to $E[X_i]$, the response time of an individual truck, rather than $E[R]$. If the expected response time of a truck is τ, then each X_i has CDF

$$F_{X_i}(x) = F_X(x) = \begin{cases} 1 - e^{-x/\tau} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

The goal of this problem is to find the maximum permissible value of τ. When each truck has expected response time τ, the CDF of R is

$$F_R(r) = (F_X(x)\, r)^6 = \begin{cases} (1 - e^{-r/\tau})^6 & r \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

We need to find τ such that

$$P[R \leq 3] = (1 - e^{-3/\tau})^6 = 0.9. \quad (6)$$

This implies

$$\tau = \frac{-3}{\ln\left(1 - (0.9)^{1/6}\right)} = 0.7406 \text{ s.} \quad (7)$$

Problem 4.7.9 ■

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 4xy & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What are $E[X]$ and $\text{Var}[X]$?

(b) What are $E[Y]$ and $\text{Var}[Y]$?

(c) What is $\text{Cov}[X,Y]$?

(d) What is $E[X+Y]$?

(e) What is $\text{Var}[X+Y]$?

Problem 4.7.9 Solution

(a) The first moment of X is

$$E[X] = \int_0^1 \int_0^1 4x^2 y \, dy \, dx = \int_0^1 2x^2 \, dx = \frac{2}{3} \quad (1)$$

The second moment of X is

$$E\left[X^2\right] = \int_0^1 \int_0^1 4x^3 y \, dy \, dx = \int_0^1 2x^3 \, dx = \frac{1}{2} \quad (2)$$

The variance of X is $\text{Var}[X] = E[X^2] - (E[X])^2 = 1/2 - (2/3)^2 = 1/18$.

(b) The mean of Y is

$$E[Y] = \int_0^1 \int_0^1 4xy^2\, dy\, dx = \int_0^1 \frac{4x}{3}\, dx = \frac{2}{3} \tag{3}$$

The second moment of Y is

$$E\left[Y^2\right] = \int_0^1 \int_0^1 4xy^3\, dy\, dx = \int_0^1 x\, dx = \frac{1}{2} \tag{4}$$

The variance of Y is $\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 1/2 - (2/3)^2 = 1/18$.

(c) To find the covariance, we first find the correlation

$$E[XY] = \int_0^1 \int_0^1 4x^2y^2\, dy\, dx = \int_0^1 \frac{4x^2}{3}\, dx = \frac{4}{9} \tag{5}$$

The covariance is thus

$$\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = \frac{4}{9} - \left(\frac{2}{3}\right)^2 = 0 \tag{6}$$

(d) $E[X+Y] = E[X] + E[Y] = 2/3 + 2/3 = 4/3$.

(e) By Theorem 4.15, the variance of $X+Y$ is

$$\text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X,Y] = 1/18 + 1/18 + 0 = 1/9 \tag{7}$$

Problem 5.6.5 •

In the message transmission system in Problem 5.3.2, the solution to Problem 5.5.2 is a formula for the PMF of $\mathbf{J}$, the number of transmissions of individual messages. For $p = 0.8$, find the expected value vector $E[\mathbf{J}]$, the correlation matrix $\mathbf{R_J}$, and the covariance matrix $\mathbf{C_J}$.

Problem 5.6.5 Solution

The random variable J_m is the number of times that message m is transmitted. Since each transmission is a success with probability p, independent of any other transmission, J_1, J_2 and J_3 are iid geometric (p) random variables with

$$E[J_m] = \frac{1}{p}, \qquad \text{Var}[J_m] = \frac{1-p}{p^2}. \tag{1}$$

Thus the vector $\mathbf{J} = \begin{bmatrix} J_1 & J_2 & J_3 \end{bmatrix}'$ has expected value

$$E[\mathbf{J}] = \begin{bmatrix} E[J_1] & E[J_2] & EJ_3 \end{bmatrix}' = \begin{bmatrix} 1/p & 1/p & 1/p \end{bmatrix}'. \tag{2}$$

For $m \neq n$, the correlation matrix $\mathbf{R_J}$ has m,nth entry

$$R_{\mathbf{J}}(m,n) = E[J_mJ_n] = E[J_m]\,J_n = 1/p^2 \tag{3}$$

For $m = n$,

$$R_{\mathbf{J}}(m,m) = E\left[J_m^2\right] = \text{Var}[J_m] + (E\left[J_m^2\right])^2 = \frac{1-p}{p^2} + \frac{1}{p^2} = \frac{2-p}{p^2}. \tag{4}$$

Thus

$$\mathbf{R_J} = \frac{1}{p^2}\begin{bmatrix} 2-p & 1 & 1 \\ 1 & 2-p & 1 \\ 1 & 1 & 2-p \end{bmatrix}. \tag{5}$$

Because J_m and J_n are independent, off-diagonal terms in the covariance matrix are

$$C_{\mathbf{J}}(m,n) = \text{Cov}\left[J_m, J_n\right] = 0 \tag{6}$$

Since $C_{\mathbf{J}}(m,m) = \text{Var}[J_m]$, we have that

$$\mathbf{C_J} = \frac{1-p}{p^2}\mathbf{I} = \frac{1-p}{p^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

Problem 5.6.6 ■

In the message transmission system in Problem 5.3.2,

$$P_{\mathbf{K}}(\mathbf{k}) = \begin{cases} p^3(1-p)^{k_3-3}; & k_1 < k_2 < k_3; \\ & k_i \in \{1, 2, \ldots\} \\ 0 & \text{otherwise.} \end{cases}$$

For $p = 0.8$, find the expected value vector $E[\mathbf{K}]$, the covariance matrix $\mathbf{C_K}$, and the correlation matrix $\mathbf{R_K}$.

Problem 5.6.6 Solution

This problem is quite difficult unless one uses the observation that the vector $\mathbf{K}$ can be expressed in terms of the vector $\mathbf{J} = \begin{bmatrix} J_1 & J_2 & J_3 \end{bmatrix}'$ where J_i is the number of transmissions of message i. Note that we can write

$$\mathbf{K} = \mathbf{AJ} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{J} \tag{1}$$

We also observe that since each transmission is an independent Bernoulli trial with success probability p, the components of $\mathbf{J}$ are iid geometric (p) random variables. Thus $E[J_i] = 1/p$ and $\text{Var}[J_i] = (1-p)/p^2$. Thus $\mathbf{J}$ has expected value

$$E\left[\mathbf{J}\right] = \boldsymbol{\mu}_J = \begin{bmatrix} E\left[J_1\right] & E\left[J_2\right] & E\left[J_3\right] \end{bmatrix}' = \begin{bmatrix} 1/p & 1/p & 1/p \end{bmatrix}'. \tag{2}$$

Since the components of $\mathbf{J}$ are independent, it has the diagonal covariance matrix

$$\mathbf{C}_J = \begin{bmatrix} \text{Var}[J_1] & 0 & 0 \\ 0 & \text{Var}[J_2] & 0 \\ 0 & 0 & \text{Var}[J_3] \end{bmatrix} = \frac{1-p}{p^2}\mathbf{I} \tag{3}$$

Given these properties of $\mathbf{J}$, finding the same properties of $\mathbf{K} = \mathbf{AJ}$ is simple.

(a) The expected value of $\mathbf{K}$ is

$$E[\mathbf{K}] = \mathbf{A}\boldsymbol{\mu}_J = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/p \\ 1/p \\ 1/p \end{bmatrix} = \begin{bmatrix} 1/p \\ 2/p \\ 3/p \end{bmatrix} \tag{4}$$

(b) From Theorem 5.13, the covariance matrix of $\mathbf{K}$ is

$$\mathbf{C}_K = \mathbf{A}\mathbf{C}_J\mathbf{A}' \tag{5}$$

$$= \frac{1-p}{p^2}\mathbf{A}\mathbf{I}\mathbf{A}' \tag{6}$$

$$= \frac{1-p}{p^2} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1-p}{p^2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \tag{7}$$

(c) Given the expected value vector $\boldsymbol{\mu}_K$ and the covariance matrix $\mathbf{C}_K$, we can use Theorem 5.12 to find the correlation matrix

$$\mathbf{R}_K = \mathbf{C}_K + \boldsymbol{\mu}_K\boldsymbol{\mu}_K' \tag{8}$$

$$= \frac{1-p}{p^2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1/p \\ 2/p \\ 3/p \end{bmatrix} \begin{bmatrix} 1/p & 2/p & 3/p \end{bmatrix} \tag{9}$$

$$= \frac{1-p}{p^2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} + \frac{1}{p^2} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \tag{10}$$

$$= \frac{1}{p^2} \begin{bmatrix} 2-p & 3-p & 4-p \\ 3-p & 6-2p & 8-2p \\ 4-p & 8-2p & 12-3p \end{bmatrix} \tag{11}$$

Problem 4.11.4 ■

An archer shoots an arrow at a circular target of radius 50cm. The arrow pierces the target at a random position (X, Y), measured in centimeters from the center of the disk at position $(X, Y) = (0, 0)$. The "bullseye" is a solid black circle of radius 2cm, at the center of the target. Calculate the probability $P[B]$ of the event that the archer hits the bullseye under each of the following models:

(a) X and Y are iid continuous uniform $(-50, 50)$ random variables.

(b) The PDF $f_{X,Y}(x, y)$ is uniform over the 50cm circular target.

(c) X and Y are iid Gaussian $(\mu = 0, \sigma = 10)$ random variables.

Problem 4.11.4 Solution

The event B is the set of outcomes satisfying $X^2 + Y^2 \leq 2^2$. Of ocurse, the calculation of $P[B]$ depends on the probability model for X and Y.

(a) In this instance, X and Y have the same PDF

$$f_X(x) = f_Y(x) = \begin{cases} 0.01 & -50 \le x \le 50 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Since X and Y are independent, their joint PDF is

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \begin{cases} 10^{-4} & -50 \le x \le 50, -50 \le y \le 50 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Because X and Y have a uniform PDF over the bullseye area, $P[B]$ is just the value of the joint PDF over the area times the area of the bullseye.

$$P[B] = P\left[X^2 + Y^2 \le 2^2\right] = 10^{-4} \cdot \pi 2^2 = 4\pi \cdot 10^{-4} \approx 0.0013 \tag{3}$$

(b) In this case, the joint PDF of X and Y is inversely proportional to the area of the target.

$$f_{X,Y}(x,y) = \begin{cases} 1/[\pi 50^2] & x^2 + y^2 \le 50^2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

The probability of a bullseye is

$$P[B] = P\left[X^2 + Y^2 \le 2^2\right] = \frac{\pi 2^2}{\pi 50^2} = \left(\frac{1}{25}\right)^2 \approx 0.0016. \tag{5}$$

(c) In this instance, X and Y have the identical Gaussian $(0, \sigma)$ PDF with $\sigma^2 = 100$; i.e.,

$$f_X(x) = f_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \tag{6}$$

Since X and Y are independent, their joint PDF is

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2} \tag{7}$$

To find $P[B]$, we write

$$P[B] = P\left[X^2 + Y^2 \le 2^2\right] = \iint_{x^2+y^2 \le 2^2} f_{X,Y}(x,y)\, dx\, dy \tag{8}$$

$$= \frac{1}{2\pi\sigma^2} \iint_{x^2+y^2 \le 2^2} e^{-(x^2+y^2)/2\sigma^2}\, dx\, dy \tag{9}$$

This integral is easy using polar coordinates. With the substitutions $x^2 + y^2 = r^2$, and $dx\, dy = r\, dr\, d\theta$

$$P[B] = \frac{1}{2\pi\sigma^2} \int_0^2 \int_0^{2\pi} e^{-r^2/2\sigma^2} r\, d\theta\, dr \tag{10}$$

$$= \frac{1}{\sigma^2} \int_0^2 r e^{-r^2/2\sigma^2}\, dr \tag{11}$$

$$= -e^{-r^2/2\sigma^2} \Big|_0^2 = 1 - e^{-4/200} \approx 0.0198. \tag{12}$$

Problem 5.7.2 •

Given the Gaussian random vector $\mathbf{X}$ in Problem 5.7.1, $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1 & -1/2 & 2/3 \end{bmatrix}$$

and $\mathbf{b} = \begin{bmatrix} -4 & -4 \end{bmatrix}'$. Calculate

(a) the expected value $\boldsymbol{\mu}_\mathbf{Y}$,

(b) the covariance $\mathbf{C}_\mathbf{Y}$,

(c) the correlation $\mathbf{R}_\mathbf{Y}$,

(d) the probability that $-1 \leq Y_2 \leq 1$.

Problem 5.7.2 Solution

We are given that $\mathbf{X}$ is a Gaussian random vector with

$$\boldsymbol{\mu}_\mathbf{X} = \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix} \qquad \mathbf{C}_\mathbf{X} = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix}. \tag{1}$$

We are also given that $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1 & -1/2 & 2/3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}. \tag{2}$$

Since the two rows of $\mathbf{A}$ are linearly independent row vectors, $\mathbf{A}$ has rank 2. By Theorem 5.16, $\mathbf{Y}$ is a Gaussian random vector. Given these facts, the various parts of this problem are just straightforward calculations using Theorem 5.16.

(a) The expected value of $\mathbf{Y}$ is

$$\boldsymbol{\mu}_\mathbf{Y} = \mathbf{A}\boldsymbol{\mu}_\mathbf{X} + \mathbf{b} = \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1 & -1/2 & 2/3 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix} + \begin{bmatrix} -4 \\ -4 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}. \tag{3}$$

(b) The covariance matrix of $\mathbf{Y}$ is

$$\mathbf{C}_\mathbf{Y} = \mathbf{A}\mathbf{C}_\mathbf{X}\mathbf{A} \tag{4}$$

$$= \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1 & -1/2 & 2/3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1/2 & -1/2 \\ 2/3 & 2/3 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 43 & 55 \\ 55 & 103 \end{bmatrix}. \tag{5}$$

(c) $\mathbf{Y}$ has correlation matrix

$$\mathbf{R_Y} = \mathbf{C_Y} + \boldsymbol{\mu}_\mathbf{Y}\boldsymbol{\mu}'_\mathbf{Y} = \frac{1}{9}\begin{bmatrix} 43 & 55 \\ 55 & 103 \end{bmatrix} + \begin{bmatrix} 8 \\ 0 \end{bmatrix}\begin{bmatrix} 8 & 0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 619 & 55 \\ 55 & 103 \end{bmatrix} \tag{6}$$

(d) From $\boldsymbol{\mu}_\mathbf{Y}$, we see that $E[Y_2] = 0$. From the covariance matrix $\mathbf{C_Y}$, we learn that Y_2 has variance $\sigma_2^2 = C_\mathbf{Y}(2,2) = 103/9$. Since Y_2 is a Gaussian random variable,

$$P[-1 \le Y_2 \le 1] = P\left[-\frac{1}{\sigma_2} \le \frac{Y_2}{\sigma_2} \le \frac{1}{\sigma_2}\right] \tag{7}$$

$$= \Phi\left(\frac{1}{\sigma_2}\right) - \Phi\left(\frac{-1}{\sigma_2}\right) \tag{8}$$

$$= 2\Phi\left(\frac{1}{\sigma_2}\right) - 1 \tag{9}$$

$$= 2\Phi\left(\frac{3}{\sqrt{103}}\right) - 1 = 0.2325. \tag{10}$$

Problem 4.12.2 •

For random variables X and Y in Example 4.27, use MATLAB to calculate $E[X]$, $E[Y]$, the correlation $E[XY]$, and the covariance $\text{Cov}[X,Y]$.

Problem 4.12.2 Solution

In this problem, we need to calculate $E[X]$, $E[Y]$, the correlation $E[XY]$, and the covariance $\text{Cov}[X,Y]$ for random variables X and Y in Example 4.27. In this case, we can use the script `imagepmf.m` in Example 4.27 to generate the grid variables `SX`, `SY` and `PXY` that describe the joint PMF $P_{X,Y}(x,y)$.

However, for the rest of the problem, a general solution is better than a specific solution. The general problem is that given a pair of finite random variables described by the grid variables `SX`, `SY` and `PXY`, we want MATLAB to calculate an expected value $E[g(X,Y)]$. This problem is solved in a few simple steps. First we write a function that calculates the expected value of any finite random variable.

```
function ex=finiteexp(sx,px);
%Usage: ex=finiteexp(sx,px)
%returns the expected value E[X]
%of finite random variable X described
%by samples sx and probabilities px
ex=sum((sx(:)).*(px(:)));
```

Note that `finiteexp` performs its calculations on the sample values `sx` and probabilities `px` using the column vectors `sx(:)` and `px(:)`. As a result, we can use the same `finiteexp` function when the random variable is represented by grid variables. For example, we can calculate the correlation $r = E[XY]$ as

```
r=finiteexp(SX.*SY,PXY)
```

It is also convenient to define a function that returns the covariance:

```
function covxy=finitecov(SX,SY,PXY);
%Usage: cxy=finitecov(SX,SY,PXY)
%returns the covariance of
%finite random variables X and Y
%given by grids SX, SY, and PXY
ex=finiteexp(SX,PXY);
ey=finiteexp(SY,PXY);
R=finiteexp(SX.*SY,PXY);
covxy=R-ex*ey;
```

The following script calculates the desired quantities:

```
%imageavg.m
%Solution for Problem 4.12.2
imagepmf; %defines SX, SY, PXY
ex=finiteexp(SX,PXY)
ey=finiteexp(SY,PXY)
rxy=finiteexp(SX.*SY,PXY)
cxy=finitecov(SX,SY,PXY)
```

```
>> imageavg
ex =
        1180
ey =
   860
rxy =
     1064000
cxy =
       49200
>>
```

The careful reader will observe that `imagepmf` is inefficiently coded in that the correlation $E[XY]$ is calculated twice, once directly and once inside of `finitecov`. For more complex problems, it would be worthwhile to avoid this duplication.

Problem 4.12.3 •

Write a script `trianglecdfplot.m` that generates the graph of $F_{X,Y}(x,y)$ of Figure 4.4.

Problem 4.12.3 Solution

The script is just a MATLAB calculation of $F_{X,Y}(x,y)$ in Equation (4.29).

```
%trianglecdfplot.m
[X,Y]=meshgrid(0:0.05:1.5);
R=(0<=Y).*(Y<=X).*(X<=1).*(2*(X.*Y)-(Y.^2));
R=R+((0<=X).*(X<Y).*(X<=1).*(X.^2));
R=R+((0<=Y).*(Y<=1).*(1<X).*((2*Y)-(Y.^2)));
R=R+((X>1).*(Y>1));
mesh(X,Y,R);
xlabel('\it x');
ylabel('\it y');
```

For functions like $F_{X,Y}(x,y)$ that have multiple cases, we calculate the function for each case and multiply by the corresponding boolean condition so as to have a zero contribution when that case doesn't apply. Using this technique, its important to define the boundary conditions carefully to make sure that no point is included in two different boundary conditions.