KOMSOSE

3/1/2019

Stoch. Proc. Sample Questions

1) Assume $X(t)$ & $Y(t)$ are random proc. that are jointly WSS.

Show that

$$R_{XY}(\tau) \leq \frac{1}{2}\left[R_{XX}(0) + R_{YY}(0)\right]$$

Answer

Consider

$$E\left(\left[X(t+\tau) - Y(t)\right]^2\right) \geq 0$$

$$= E\left(X(t+\tau)^2 - 2X(t+\tau)Y(t) + Y(t)^2\right)$$

$$= \underbrace{E\left(X(t+\tau)^2\right)}_{R_{XX}(0)} - 2\underbrace{E\left(X(t+\tau)Y(t)\right)}_{R_{XY}(\tau)} + \underbrace{E\left(Y(t)^2\right)}_{R_{YY}(0)} \geq 0$$

Hence $\quad 2R_{XY}(\tau) \leq R_{XX}(0) + R_{YY}(0)$

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2) A biased random walk process is defined as

$$X[n] = \sum_{i=0}^{n} \underbrace{U[i]}_{\text{Bernoulli process}} \quad \text{where} \quad p_U(k) = \begin{cases} 1/4 & \text{for } k=-1 \\ 3/4 & \text{for } k=1 \end{cases}$$

a) Find $E[X(n])$ and $Var(X[n])$

b) Is $X[n]$ stationary?

c) Does $X[n]$ have stationary & independent increments.

Answer:

a) $E(X[n]) = E\left(\sum_{i=0}^{n} U[i]\right) = \sum_{i=0}^{n} \underbrace{E(U[i])}_{\frac{1}{4}\times -1 + \frac{3}{4}\times 1 = \frac{1}{2}} = \frac{n+1}{2}$

$Var(X[n]) = Var\left(\sum_{i=0}^{n} U[i]\right) = \sum_{i=0}^{n} \underbrace{Var(U[i])}_{E(U^2[i]) - \underbrace{E(U[i])^2}_{\left(\frac{1}{2}\right)^2}}$ as $U[i]$ are iid

$$E\left(u^2[i]\right) = \frac{1}{4}(-1)^2 + \frac{3}{4}(1)^2 = 1$$

Hence

$$\mathrm{Var}\left(u[i]\right) = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow \mathrm{Var}\left[x[n]\right) = \frac{3}{4}(n+1)$$

b) As $E(x[n])$ is a function of n (time) it is not stationary.

c) $x[n] - x[n-1] = u[n] \leftarrow$ increments are Bernoulli r.p. & iid

iid processes are independent & stationary.

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3) A symmetric Bernoulli r.p. $Y[n]$ takes values of 0 and 1 with prob. 1/2. A new rp $Z[n]$ is defined as

$$Z[n] = (-1)^n Y[n]$$

Is $Z[n]$ iid ?

Answer

$Z[n]$ samples are independent as $Y[n]$ are independent

However $Z[n]$ are not identically distributed

For example

$$Z[1] = \begin{cases} 1/2 & k=0 \\ 1/2 & k=-1 \end{cases}$$

$P_{Z1} \neq P_{Z2} \rightarrow$ not identical

$\Downarrow$

not stationary

$$Z[2] = \begin{cases} 1/2 & k=0 \\ 1/2 & k=1 \end{cases}$$

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4) Let $X[n]$ be an iid Gaussian rp. with mean μ and $\sigma^2 = 1$.

Consider $Y[n] = X[n] - X[n-1]$

a) Find joint pdf of $Y[1]$ and $Y[2]$

b) Is $Y[n]$ iid?

c) Find $R_{YY}[k]$ in terms of R_{XX}

d) Is $Y[n]$ wss ?

a) $Y[1] = X[1] - X[0]$
$Y[2] = X[2] - X[1]$

as $X[n]$ is Gaussian r.p. $X[2], X[1], X[0]$ have joint Gaussian distr. In addition $Y[n]$ is a linear transform. Hence, its samples will also have joint Gaussian distr.

$$\underbrace{\begin{bmatrix} Y[1] \\ Y[0] \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} X[0] \\ X[1] \\ X[2] \end{bmatrix}}_{X}$$

$X \sim N(\mu_x, \Sigma_x)$

$\Rightarrow Y \sim N(G\mu_x, G\Sigma_x G^T)$

$\mu_x = \begin{bmatrix} \mu \\ \mu \\ \mu \end{bmatrix}$ and $\underbrace{\Sigma_x = \sigma_x^2 I = I}_{\text{as } X[n] \text{ is iid.}}$

Hence $\mu_Y = G\mu_x = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \mu \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Sigma_Y = G\Sigma_x G^T = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

b) As diagonals of Σ_Y are not zero $Y[n]$ are correlated

$\Rightarrow$ not independent

c) $R_{YY}[k] = E(Y[n]Y[n+k])$

$= E((X[n] - X[n-1])(X[n+k] - X[n+k-1]))$

$= 2R_{XX}[k] - R_{XX}[k-1] - R_{XX}[k+1]$

d) $E(Y[n]) = E(X[n] - X[n-1]) = \mu - \mu = 0$ (time independent)

From part (c), R_{YY} is a function of time difference

$\Rightarrow$ Hence $Y[n]$ is WSS.

⑤ A symmetric Bernoulli random walk is defined as

$$X[n] = \sum_{i=0}^{n} U[i] \quad \text{where} \quad U[i] \text{ is iid with}$$

$$p_U[k] = \begin{cases} 1/2 & k = 1 \\ 1/2 & k = -1 \end{cases}$$

Show that $Z[n] = X[n]^2 - n$ is a Martingale process

Answer:

Show $E(Z(n)) < \infty \quad \forall n$

$$E[X[n]^2 - n] = E\left[\sum_{i=0}^{n}\sum_{j=0}^{n} U[i]U[j]\right] - n$$

$$= \underbrace{E\left[\sum_{i=0}^{n} U[i]^2\right]}_{} + \underbrace{E\left[\sum_{i=0}^{n}\sum_{\substack{j=0 \\ i\neq j}}^{n} U[i]U[j]\right]}_{} - n$$

$$= \sum_{i=0}^{n}\Big[\underbrace{E(U^2(i))}_{=1} - \underbrace{E(U(i))^2}_{=0}\Big] + \sum_{i=0}^{n}\sum_{\substack{j=0 \\ i\neq j}}^{n} \underbrace{E(U[i])}_{=0}\underbrace{E(U[j])}_{=0} - n$$

$E(U[i]) = \frac{1}{2}(-1) + \frac{1}{2}(1) = 0$

$E(U^2[i]) = \frac{1}{2}(-1)^2 + \frac{1}{2}(1)^2 = 1$

$$= \Big(\underbrace{\sum_{i=0}^{n} 1}_{n}\Big) - n = 0 < \infty \quad \forall n$$

Show

$$E(Z[n_2] \mid Z[n_1] = z_1) = z_1 \quad \forall n_2 \geq n_1$$

$$Z[n_2] = X[n_2]^2 - n_2 = \Big[X(n_1) + \sum_{i=n_1+1}^{n_2} U[i]\Big]^2 - n_2$$

$$= \underbrace{X[n_1]^2 - n_1}_{Z[n_1]} + 2X[n_1]\sum_{i=n_1+1}^{n_2} U[i]$$

$$+ \sum_{i=n_1+1}^{n_2}\sum_{j=n_1+1}^{n_2} U[i]U[j] - [n_2 - n_1]$$

$$E\left(Z[n_2] \mid Z[n_1] = z_1\right) = z_1 + 2E\left(X[n_1]\right) \sum_{i=n_1+1}^{n_2} \overbrace{E\left(U[i]\right)}^{=0}$$

$$+ \sum_{i=n_1+1}^{n_2} \underbrace{E\left(U^2[i]\right)}_{=1} + \sum_{\substack{i=n_1+1 \\ i \neq j}}^{n_2} \sum_{j=n_1+1}^{n_2} \underbrace{E\left(U[i]\right)}_{=0} \underbrace{E\left(U[j]\right)}_{=0}$$

$$- [n_2 - n_1]$$

$$= z_1 + \underbrace{\left(\sum_{i=n_1+1}^{n_2} 1 \right)}_{n_2 - n_1} - [n_2 - n_1] = z_1$$

Hence $Z[n]$ is a Martingale process.