Solutions to HW6

Note: Most of these solutions were generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have added comments in italics where I thought more detail was appropriate.

Problem 3.2.5 ♦♦

For constants a and b, random variable X has PDF

$$f_X(x) = \begin{cases} ax^2 + bx & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

What conditions on a and b are necessary and sufficient to guarantee that $f_X(x)$ is a valid PDF?

Problem 3.2.5 Solution

$$f_X(x) = \begin{cases} ax^2 + bx & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

First, we note that a and b must be chosen such that the above PDF integrates to 1.

$$\int_0^1 (ax^2 + bx)\, dx = a/3 + b/2 = 1 \tag{2}$$

Hence, $b = 2 - 2a/3$ and our PDF becomes

$$f_X(x) = x(ax + 2 - 2a/3) \tag{3}$$

PDF's must take only nonnegative values. For the PDF to be non-negative for $x \in [0,1]$, we must have $ax + 2 - 2a/3 \ge 0$ for all $x \in [0,1]$. *We want to obtain constraints on a and b so first we must isolate a.* [The] requirement can be written as

$$a(2/3 - x) \le 2 \qquad (0 \le x \le 1) \tag{4}$$

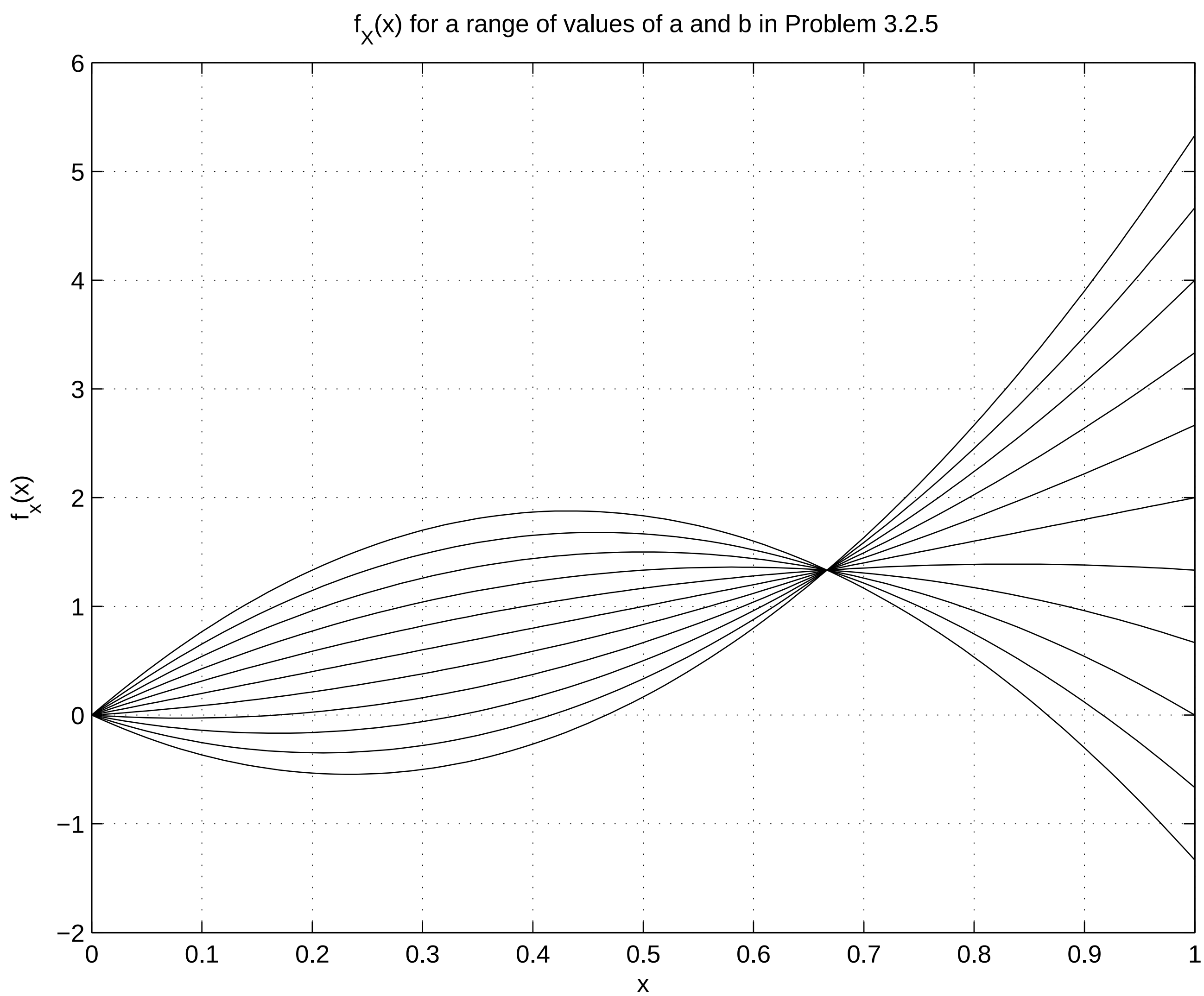
For $x = 2/3$, the requirement holds for all a. However, the problem is tricky because we must consider the cases $0 \le x < 2/3$ and $2/3 < x \le 1$ separately because of the sign change of the inequality. When $0 \le x < 2/3$, we have $2/3 - x > 0$ and the requirement is most stringent at $x = 0$ where we require $2a/3 \le 2$ or $a \le 3$. When $2/3 < x \le 1$, we can write the constraint as $a(x - 2/3) \ge -2$. In this case, the constraint is most stringent at $x = 1$, where we must have $a/3 \ge -2$ or $a \ge -6$. Thus a complete expression for our requirements are

$$-6 \le a \le 3 \qquad b = 2 - 2a/3 \tag{5}$$

Now, if the above reasoning was hard to follow, and even if it wasn't, I suggest that we use the tools we have available to help us gain insight into the behavior of the function $f_X(x)$ as a varies. All of the values of a that were of potential significance above were in the range $[-10, 10]$ so let's plot $f_X(x)$ for values of a in that range. A Matlab script that generates the values is given below.

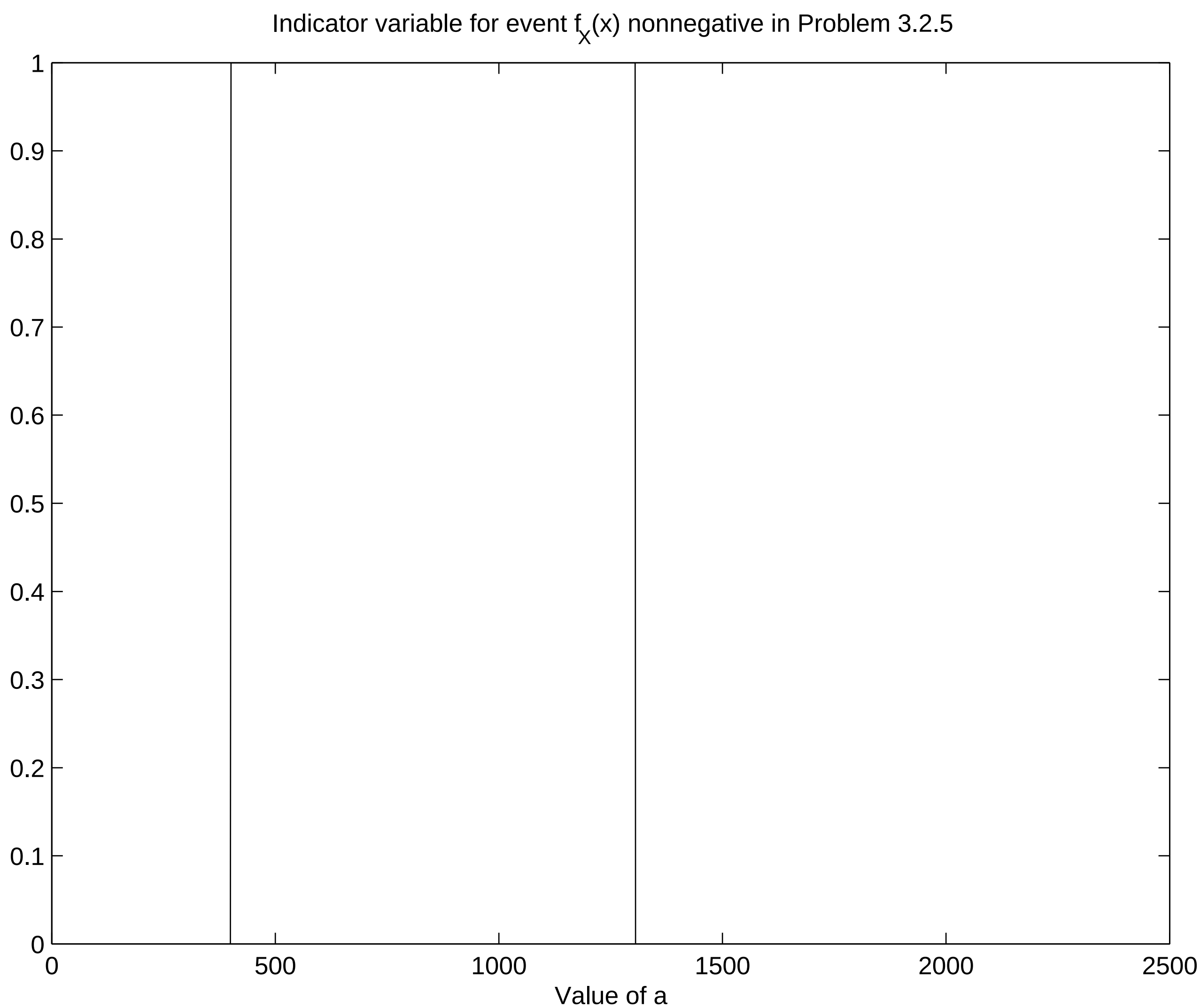
```
%%% 3.2.5                                                    2/19/06 sk
a = [-10:.01:10];b = 2*ones(size(a))-2*a/3;
x = [0:.01:10];
for index = 1:200:2001,plot(x,y(index,:)),hold on,end;
xlabel('x')
ylabel('f_X(x)')
title('f_X(x) for a range of values of a and b in Problem 3.2.5')
print -deps p3_2_5
```

The resulting figure clearly shows that there is a maximum and a minimum value of a.



Next, I defined an indicator random variable which takes the value 1 if the value of a results in an acceptable PDF and 0 otherwise. Here's the Matlab code and a plot of the value of the indicator random variable.

```
for index = 1:2001,if min(y(index,:))>=0, ok(index)=1;else ok(index)=0;end;end;
plot(ok)
figure(2)
title('Indicator variable for event f_X(x) nonnegative in Problem 3.2.5')
xlabel('Value of a')
print -deps p3_2_5b
```



Checking individual values to find the transitions in the plot of the indicator variable, we find values $a = -6$ and $a = 3 + \delta$ for a very small positive δ. We suspect that numerical error (the values plotted are generated for points at discrete intervals) is the source of the δ. Now that we see what is going on – for $a < -6$ the value of the function goes negative as x increases from zero and for $a > 3$, the value of the function goes negative at some x less than 1 – the analysis is more straightforward.

Problem 3.3.6 ■

The cumulative distribution function of random variable V is

$$F_V(v) = \begin{cases} 0 & v < -5, \\ (v+5)^2/144 & -5 \le v < 7, \\ 1 & v \ge 7. \end{cases}$$

(a) What is $E[V]$?

(b) What is $\mathrm{Var}[V]$?

(c) What is $E[V^3]$?

Problem 3.3.6 Solution

To evaluate the moments of V, we need the PDF $f_V(v)$, which we find by taking the derivative of the CDF $F_V(v)$. The CDF and corresponding PDF of V are

$$F_V(v) = \begin{cases} 0 & v < -5 \\ (v+5)^2/144 & -5 \le v < 7 \\ 1 & v \ge 7 \end{cases} \qquad f_V(v) = \begin{cases} 0 & v < -5 \\ (v+5)/72 & -5 \le v < 7 \\ 0 & v \ge 7 \end{cases} \tag{1}$$

We must check to see that there are no discontinuities in the CDF to determine whether we need any impulses in the PDF. We find that $F_V(-5) = 0$ and that if we let $(7+5)^2/144 = 1$ so there are no discontinuities and we have correctly determined the PDF.

(a) The expected value of V is

$$E[V] = \int_{-\infty}^{\infty} v f_V(v)\, dv = \frac{1}{72}\int_{-5}^{7} (v^2 + 5v)\, dv \tag{2}$$

$$= \frac{1}{72}\left(\frac{v^3}{3} + \frac{5v^2}{2}\right)\Bigg|_{-5}^{7} = \frac{1}{72}\left(\frac{343}{3} + \frac{245}{2} + \frac{125}{3} - \frac{125}{2}\right) = 3 \tag{3}$$

(b) To find the variance, we first find the second moment

$$E\left[V^2\right] = \int_{-\infty}^{\infty} v^2 f_V(v)\, dv = \frac{1}{72}\int_{-5}^{7} (v^3 + 5v^2)\, dv \tag{4}$$

$$= \frac{1}{72}\left(\frac{v^4}{4} + \frac{5v^3}{3}\right)\Bigg|_{-5}^{7} = 17 \tag{5}$$

The variance is $\mathrm{Var}[V] = E[V^2] - (E[V])^2 = 17 - 9 = 8$.

(c) The third moment of V is

$$E\left[V^3\right] = \int_{-\infty}^{\infty} v^3 f_V(v)\, dv = \frac{1}{72}\int_{-5}^{7} (v^4 + 5v^3)\, dv \tag{6}$$

$$= \frac{1}{72}\left(\frac{v^5}{5} + \frac{5v^4}{4}\right)\Bigg|_{-5}^{7} = 86.2 \tag{7}$$

Problem 3.3.7 ■

The cumulative distribution function of random variable U is

$$F_U(u) = \begin{cases} 0 & u < -5, \\ (u+5)/8 & -5 \leq u < -3m, \\ 1/4 & -3 \leq u < 3, \\ 1/4 + 3(u-3)/8 & 3 \leq u < 5, \\ 1 & u \geq 5. \end{cases}$$

(a) What is $E[U]$?

(b) What is $\mathrm{Var}[U]$?

(c) What is $E[2^U]$?

Problem 3.3.7 Solution

To find the moments, we first find the PDF of U by taking the derivative of $F_U(u)$. The CDF and corresponding PDF are

$$F_U(u) = \begin{cases} 0 & u < -5 \\ (u+5)/8 & -5 \leq u < -3 \\ 1/4 & -3 \leq u < 3 \\ 1/4 + 3(u-3)/8 & 3 \leq u < 5 \\ 1 & u \geq 5. \end{cases} \qquad f_U(u) = \begin{cases} 0 & u < -5 \\ 1/8 & -5 \leq u < -3 \\ 0 & -3 \leq u < 3 \\ 3/8 & 3 \leq u < 5 \\ 0 & u \geq 5. \end{cases} \tag{1}$$

Note that we should also verify that there are no jumps in the CDF. If there are jumps, we will need impulses of those magnitudes in the PDF. There are four values of u that we must check, as follows:

$$\left.\frac{u+5}{8}\right|_{u=-5} = 0 \qquad \left.\frac{u+5}{8}\right|_{u=-3} = \frac{1}{4} \qquad \frac{1}{4} + \left.\frac{3(u-8)}{8}\right|_{u=3} = \frac{1}{4} \qquad \frac{1}{4} + \left.\frac{3(u-8)}{8}\right|_{u=5} = 1 \tag{2}$$

So we have shown that there are no discontiuities (jumps) in the CDF.

(a) The expected value of U is

$$E[U] = \int_{-\infty}^{\infty} u f_U(u)\, du = \int_{-5}^{-3} \frac{u}{8}\, du + \int_{3}^{5} \frac{3u}{8}\, du \tag{3}$$

$$= \left.\frac{u^2}{16}\right|_{-5}^{-3} + \left.\frac{3u^2}{16}\right|_{3}^{5} = 2 \tag{4}$$

(b) The second moment of U is

$$E\left[U^2\right] \int_{-\infty}^{\infty} u^2 f_U(u)\, du = \int_{-5}^{-3} \frac{u^2}{8}\, du + \int_{3}^{5} \frac{3u^2}{8}\, du \tag{5}$$

$$= \left.\frac{u^3}{24}\right|_{-5}^{-3} + \left.\frac{u^3}{8}\right|_{3}^{5} = 49/3 \tag{6}$$

The variance of U is $\mathrm{Var}[U] = E[U^2] - (E[U])^2 = 37/3$.

(c) Note that $2^U = e^{(\ln 2)U}$. This implies that

$$\int 2^u \, du = \int e^{(\ln 2)u} \, du = \frac{1}{\ln 2} e^{(\ln 2)u} = \frac{2^u}{\ln 2} \tag{7}$$

The expected value of 2^U is then

$$E\left[2^U\right] = \int_{-\infty}^{\infty} 2^u f_U(u) \, du = \int_{-5}^{-3} \frac{2^u}{8} \, du + \int_3^5 \frac{3 \cdot 2^u}{8} \, du \tag{8}$$

$$= \left. \frac{2^u}{8 \ln 2} \right|_{-5}^{-3} + \left. \frac{3 \cdot 2^u}{8 \ln 2} \right|_3^5 = \frac{2307}{256 \ln 2} = 13.001 \tag{9}$$

Problem 3.4.5 ■

X is a continuous uniform $(-5, 5)$ random variable.

(a) What is the PDF $f_X(x)$?

(b) What is the CDF $F_X(x)$?

(c) What is $E[X]$?

(d) What is $E[X^5]$?

(e) What is $E[e^X]$?

Problem 3.4.5 Solution

(a) The PDF of a continuous uniform $(-5, 5)$ random variable is

$$f_X(x) = \begin{cases} 1/10 & -5 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) For $x < -5$, $F_X(x) = 0$. For $x \geq 5$, $F_X(x) = 1$. For $-5 \leq x \leq 5$, the CDF is

$$F_X(x) = \int_{-5}^{x} f_X(\tau) \, d\tau = \frac{x+5}{10} \tag{2}$$

The complete expression for the CDF of X is

$$F_X(x) = \begin{cases} 0 & x < -5 \\ (x+5)/10 & 5 \leq x \leq 5 \\ 1 & x > 5 \end{cases} \tag{3}$$

(c) The expected value of X is

$$\int_{-5}^{5} \frac{x}{10}\,dx = \left.\frac{x^2}{20}\right|_{-5}^{5} = 0 \tag{4}$$

Another way to obtain this answer is to use Theorem 3.6 which says the expected value of X is $E[X] = (5 + -5)/2 = 0$.

(d) The fifth moment of X is

$$\int_{-5}^{5} \frac{x^5}{10}\,dx = \left.\frac{x^6}{60}\right|_{-5}^{5} = 0 \tag{5}$$

(e) The expected value of e^X is

$$\int_{-5}^{5} \frac{e^x}{10}\,dx = \left.\frac{e^x}{10}\right|_{-5}^{5} = \frac{e^5 - e^{-5}}{10} = 14.84 \tag{6}$$

Problem 3.4.7 ■

The probability density function of random variable X is

$$f_X(x) = \begin{cases} (1/2)e^{-x/2} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is $P[1 \leq X \leq 2]$?

(b) What is $F_X(x)$, the cumulative distribution function of X?

(c) What is $E[X]$, the expected value of X?

(d) What is $\text{Var}[X]$, the variance of X?

Problem 3.4.7 Solution

Given that

$$f_X(x) = \begin{cases} (1/2)e^{-x/2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a)

$$P[1 \leq X \leq 2] = \int_1^2 (1/2)e^{-x/2}\,dx = e^{-1/2} - e^{-1} = 0.2387 \tag{2}$$

(b) The CDF of X may be be expressed as

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \int_0^x (1/2)e^{-\tau/2}\,d\tau & x \geq 0 \end{cases} = \begin{cases} 0 & x < 0 \\ 1 - e^{-x/2} & x \geq 0 \end{cases} \tag{3}$$

(c) X is an exponential random variable with parameter $a = 1/2$. By Theorem 3.8, the expected value of X is $E[X] = 1/a = 2$. *It is also simple to compute it directly.*

(d) By Theorem 3.8, the variance of X is $\text{Var}[X] = 1/a^2 = 4$.

Problem 3.5.5 ■

The peak temperature T, in degrees Fahrenheit, on a July day in Antarctica is a Gaussian random variable with a variance of 225. With probability 1/2, the temperature T exceeds 10 degrees. What is $P[T > 32]$, the probability the temperature is above freezing? What is $P[T < 0]$? What is $P[T > 60]$?

Problem 3.5.5 Solution

Moving to Antarctica, we find that the temperature, T is still Gaussian but with variance $\sigma^2 = 225$. We also know that with probability 1/2, T exceeds 10 degrees. First we would like to find the mean temperature, and we do so by looking at the second fact.

$$P[T > 10] = 1 - P[T \leq 10] = 1 - \Phi\left(\frac{10 - \mu_T}{15}\right) = 1/2 \tag{1}$$

By looking at the table we find that if $\Phi(\Gamma) = 1/2$, then $\Gamma = 0$. Therefore,

$$\Phi\left(\frac{10 - \mu_T}{15}\right) = 1/2 \tag{2}$$

implies that $(10 - \mu_T)/15 = 0$ or $\mu_T = 10$. Now we have a Gaussian T with mean 10 and standard deviation 15. So we are prepared to answer the following problems.

$$\begin{aligned}
P[T > 32] &= 1 - P[T \leq 32] = 1 - \Phi\left(\frac{32 - 10}{15}\right) && (3)\\
&= 1 - \Phi(1.47) = 1 - 0.929 = 0.071 && (4)\\
P[T < 0] &= F_T(0) = \Phi\left(\frac{0 - 10}{15}\right) && (5)\\
&= \Phi(-2/3) = 1 - \Phi(2/3) && (6)\\
&= 1 - \Phi(0.67) = 1 - 0.749 = 0.251 && (7)\\
P[T > 60] &= 1 - P[T \leq 60] = 1 - F_T(60) && (8)\\
&= 1 - \Phi\left(\frac{60 - 10}{15}\right) = 1 - \Phi(10/3) && (9)\\
&= Q(3.33) = 4.34 \cdot 10^{-4} && (10)
\end{aligned}$$

Problem 3.6.6 ■

When you make a phone call, the line is busy with probability 0.2 and no one answers with probability 0.3. The random variable X describes the conversation time (in minutes) of a phone call that is answered. X is an exponential random variable with $E[X] = 3$ minutes. Let the random variable W denote the conversation time (in seconds) of all calls ($W = 0$ when the line is busy or there is no answer.)

(a) What is $F_W(w)$?

(b) What is $f_W(w)$?

(c) What are $E[W]$ and $\text{Var}[W]$?

Problem 3.6.6 Solution

We are given that random variable X (conversation time in minutes of answered calls) has an exponential distribution with expected value $E = [X] = 3$. We define an event space $\{A, A^c\}$, where A is the event that the call is answered and, of course, A^c is the event that either the line was busy or the call was not answered. We define a new random variable W (conversation time in seconds of all calls) which is thus defined as

$$W = \begin{cases} 60X & \text{if the call is answered, } i.e. \text{ if } A \text{ occurs,} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

in terms of the random variable X. Now, we can work through the entire problem in terms of $F_X(x)$ and probably make a lot of mistakes in accounting for the scale factor, or we can let $V = 60X$, determine $F_V(v)$ from $F_X(x)$, and work with $F_V(v)$ instead. I recommend the latter approach, especially after having attempted the former a couple of times. X being exponential with parameter $1/3 = 1/E[X]$, we can apply Theorem 3.20 to show that

$$f_V(v) = \begin{cases} \frac{1}{180} e^{-v/180} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

so integrating we have

$$F_V(v) = \begin{cases} 1 - e^{-v/180} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and

$$W = \begin{cases} V & \text{if the call is answered, } i.e. \text{ if } A \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We will use Theorem 3.23 in determining the CDF. We note that this theorem gives a result for the PDF's not the CDF's so we will have to extend it as follows. Theorem 3.23 applied to our problem says that

$$f_W(w) = f_{W|A}(w)\, P[A] + f_{W|A^c}(w)\, P[A^c]. \tag{5}$$

To obtain the CDF from the PDF we integrate. This yields

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{v} f_W(y) dy = \int_{-\infty}^{v} \left(f_{W|A}(y)\, P[A] + f_{W|A^c}(y)\, P[A^c] \right) dy & (6) \\ &= \int_{-\infty}^{v} f_{W|A}(y)\, P[A] dy + \int_{-\infty}^{v} f_{W|A^c}(y)\, P[A^c] dy & (7) \\ &= P[A] \int_{-\infty}^{v} f_{W|A}(y)\, dy + P[A^c] \int_{-\infty}^{v} f_{W|A^c}(y)\, dy & (8) \\ &= P[A] F_{W|A}(y) + P[A^c] F_{W|A^c}(y) = F_{W|A}(w)\, P[A] + F_{W|A^c}(w)\, P[A^c]. & (9) \end{aligned}$$

Thus we have shown that a similar relation to that given for the PDF in Theorem 3.23 holds for the CDF.

(a) We determine the CDF of W separately on the intervals $(-\infty, 0)$ and $[0, \infty]$. Since the conversation time cannot be negative, we know that $F_W(w) = 0$ for $w < 0$. Also, the event A^c implies $W = 0$, whereas the event A implies $w > 0$. Using (6), we have, for $w \geq 0$,

$$F_W(w) = P[A^c] F_{W|A^c}(w) + P[A] F_{W|A}(w) = (1/2) + (1/2) F_V(w) \tag{10}$$

so

$$F_W(w) = \begin{cases} 0 & w < 0 \\ 1/2 + (1/2) F_V(w) & w \geq 0 \end{cases} \tag{11}$$

so

$$F_W(w) = \begin{cases} 0 & w < 0 \\ 1/2 + 1/2\left(1 - e^{-(w/180)}\right) & w \geq 0. \end{cases} \tag{12}$$

(b) Taking the derivative, we obtain that the PDF of W is

$$f_W(w) = \begin{cases} 0 & w < 0 \\ \frac{1}{2}\delta(w) + (1/360)e^{-w/180} & w \geq 0 \end{cases} \tag{13}$$

(c) From the PDF $f_W(w)$, calculating the moments is straightforward.

$$E[W] = \int_{-\infty}^{\infty} w f_W(w)\, dw = (1/2) \int_{-\infty}^{\infty} v f_V(v)\, dv = 1/2 E[V] = 90 \tag{14}$$

where we note that the contribution at zero is zero because we have $v f_V(v) = 0\delta(0)$. The second moment is

$$E\left[W^2\right] = \int_{-\infty}^{\infty} w^2 f_W(w)\, dw = (1/2) \int_{-\infty}^{\infty} v^2 f_V(v)\, dw = 1/2 E\left[V^2\right] = 16,200 \tag{15}$$

The variance of W is

$$\text{Var}[W] = E\left[W^2\right] - (E[W])^2 = 16,200 - 90^2 = 8100 \tag{16}$$

Problem 3.7.5 ■

U is a uniform $(0, 1)$ random variable and $X = -\ln(1 - U)$.

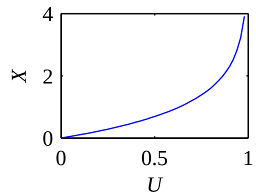
(a) What is $F_X(x)$?

(b) What is $f_X(x)$?

(c) What is $E[X]$?

Problem 3.7.5 Solution

Before solving for the PDF, it is helpful to have a sketch of the function $X = -\ln(1 - U)$.



(a) From the sketch, we observe that X will be nonnegative. Hence $F_X(x) = 0$ for $x < 0$. Since U has a uniform distribution on $[0, 1]$, for $0 \le u \le 1$, $P[U \le u] = u$. We use this fact to find the CDF of X. For $x \ge 0$,

$$F_X(x) = P[-\ln(1-U) \le x] = P\left[1 - U \ge e^{-x}\right] = P\left[U \le 1 - e^{-x}\right] \tag{1}$$

For $x \ge 0$, $0 \le 1 - e^{-x} \le 1$ and so

$$F_X(x) = F_U\left(1 - e^{-x}\right) = 1 - e^{-x} \tag{2}$$

The complete CDF can be written as

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \ge 0 \end{cases} \tag{3}$$

(b) By taking the derivative, the PDF is

$$f_X(x) = \begin{cases} e^{-x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Thus, X has an exponential PDF. In fact, since most computer languages provide uniform $[0, 1]$ random numbers, the procedure outlined in this problem provides a way to generate exponential random variables from uniform random variables.

(c) Since X is an exponential random variable with parameter $a = 1$, $E[X] = 1$ *by the table in Appendix A.*

Note that it is not necessary to have the textbook handy to determine the expected value. Letting $u = x$ and $dv = e^{-x}$ so that $du = dx$ and $v = -e^{-x}$ and performing integration by parts we obtain

$$\int_{-\infty}^{\infty} x f_X(x) dx = \int_0^{\infty} x e^{-x} dx = -x e^{-x}\Big|_0^1 - \int_0^{\infty} (-e^{-x}) dx = 0 - e^{-x}\Big|_0^{\infty} = 1 \tag{5}$$

Problem 3.7.9 ■

U is a uniform random variable with parameters 0 and 2. The random variable W is the output of the clipper:

$$W = g(U) = \begin{cases} U & U \le 1, \\ 1 & U > 1. \end{cases}$$

Find the CDF $F_W(w)$, the PDF $f_W(w)$, and the expected value $E[W]$.

Problem 3.7.9 Solution

The uniform $(0,2)$ random variable U has PDF and CDF

$$f_U(u) = \begin{cases} 1/2 & 0 \le u \le 2, \\ 0 & \text{otherwise}, \end{cases} \qquad F_U(u) = \begin{cases} 0 & u < 0, \\ u/2 & 0 \le u \le 2, \\ 1 & u > 2. \end{cases} \tag{1}$$

The uniform random variable U is subjected to the following clipper.

$$W = g(U) = \begin{cases} U & U \le 1 \\ 1 & U > 1 \end{cases} \tag{2}$$

To find the CDF of the output of the clipper, W, we remember that $W = U$ for $0 \le U \le 1$ while $W = 1$ for $1 \le U \le 2$. First, this implies W is nonnegative, i.e., $F_W(w) = 0$ *and* $f_W(w) = 0$ for $w < 0$. Furthermore, for $0 \le w \le 1$,

$$F_W(w) = P[W \le w] = P[U \le w] = F_U(w) = w/2 \tag{3}$$

Lastly, we observe that it is always true that $W \le 1$. This implies $F_W(w) = 1$ for $w \ge 1$. Therefore the CDF of W is

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w/2 & 0 \le w < 1 \\ 1 & w \ge 1 \end{cases} \tag{4}$$

From the jump in the CDF at $w = 1$, we see that $P[W = 1] = 1/2$. The corresponding PDF can be found by taking the derivative and using the delta function to model the discontinuity.

$$f_W(w) = \begin{cases} 1/2 + (1/2)\delta(w-1) & 0 \le w \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

The expected value of W is

$$E[W] = \int_{-\infty}^{\infty} w f_W(w)\, dw = \int_0^1 w[1/2 + (1/2)\delta(w-1)]\, dw \tag{6}$$

$$= 1/4 + 1/2 = 3/4. \tag{7}$$

where $1/4$ *is the integral of* w *over the interval* $[0,1]$ *and* $1/2$ *is obtained using the sifting property of the impulse function:*

$$\int_0^1 \frac{1}{2}\delta(w-1)dw = \frac{1}{2}. \tag{8}$$

(Recall that the sifting property holds because the impulse function makes the integrand zero except at the value $w = 1$*.)*

Problem 3.7.10 ■

X is a random variable with CDF $F_X(x)$. Let $Y = g(X)$ where

$$g(x) = \begin{cases} 10 & x < 0, \\ -10 & x \geq 0. \end{cases}$$

Express $F_Y(y)$ in terms of $F_X(x)$.

Problem 3.7.10 Solution

Given the following function of random variable X,

$$Y = g(X) = \begin{cases} 10 & X < 0 \\ -10 & X \geq 0 \end{cases} \tag{1}$$

we follow the same procedure as in Problem 3.7.4. We attempt to express the CDF of Y in terms of the CDF of X. We know that Y is *never* less than -10 *so for* $y < -10$, $F_Y(y) = 0$. We also know that $-10 \leq Y < 10$ when $X \geq 0$, and finally, that $Y = 10$ when $X < 0$. Therefore

$$F_Y(y) = P[Y \leq y] = \begin{cases} 0 & y < -10 \\ P[X \geq 0] = 1 - F_X(0) & -10 \leq y < 10 \\ 1 & y \geq 10 \end{cases} \tag{2}$$

Problem 3.7.13 ■

For a uniform $(0, 1)$ random variable U, find the CDF and PDF of $Y = a + (b - a)U$ with $a < b$. Show that Y is a uniform (a, b) random variable.

Problem 3.7.13 Solution

If X has a uniform distribution from 0 to 1 then the PDF and corresponding CDF of X are

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \tag{1}$$

We can determine the CDF interval by interval. First, let's consider the interval corresponding to $b - a > 0$*, i.e. when* y *is between* a *and* b*.* For $b - a > 0$, we can find the CDF of the function $Y = a + (b - a)X$

$$F_Y(y) = P[Y \leq y] = P[a + (b - a)X \leq y] \tag{2}$$

$$= P\left[X \leq \frac{y - a}{b - a}\right] \tag{3}$$

$$= F_X\left(\frac{y - a}{b - a}\right) = \frac{y - a}{b - a} \tag{4}$$

Now we note that since the value of the random variable Y never exceeds b, $F_Y(y) = 1$ for all $y > b$. Similarly, since the value of the random variable Y is never less than a, $F_Y(y) = 0$ for all $y < a$. Therefore the CDF of Y is

$$F_Y(y) = \begin{cases} 0 & y < a \\ \frac{y-a}{b-a} & a \le y \le b \\ 1 & y > b \end{cases} \tag{5}$$

By differentiating with respect to y we arrive at the PDF

$$f_Y(y) = \begin{cases} 1/(b-a) & a \le x \le b \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

which we recognize as the PDF of a uniform (a, b) random variable.

Problem 3.8.4 ■

W is a Gaussian random variable with expected value $\mu = 0$, and variance $\sigma^2 = 16$. Given the event $C = \{W > 0\}$,

(a) What is the conditional PDF, $f_{W|C}(w)$?

(b) Find the conditional expected value, $E[W|C]$.

(c) Find the conditional variance, $\mathrm{Var}[W|C]$.

Problem 3.8.4 Solution

From Definition 3.8, $W \sim \mathcal{N}(0, 4)$ *imlpies that* the PDF of W is

$$f_W(w) = \frac{1}{\sqrt{32\pi}} e^{-w^2/32} \tag{1}$$

(a) Since W has expected value $\mu = 0$, $f_W(w)$ is symmetric about $w = 0$. Hence $P[C] = P[W > 0] = 1/2$. From Definition 3.15, the conditional PDF of W given C is

$$f_{W|C}(w) = \begin{cases} f_W(w)/P[C] & w \in C \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2e^{-w^2/32}/\sqrt{32\pi} & w > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(b) The conditional expected value of W given C is

$$E[W|C] = \int_{-\infty}^{\infty} w f_{W|C}(w)\, dw = \frac{2}{4\sqrt{2\pi}} \int_0^{\infty} w e^{-w^2/32}\, dw \tag{3}$$

Making the substitution $v = w^2/32$, we obtain *($dv = 2w\,dw/32 = w\,dw/16$, so $w\,dw = 16dv$ and)*

$$E[W|C] = \frac{32}{\sqrt{32\pi}} \int_0^{\infty} e^{-v}\, dv = \frac{32}{\sqrt{32\pi}} \tag{4}$$

(c) The conditional second moment of W is

$$E\left[W^2|C\right] = \int_{-\infty}^{\infty} w^2 f_{W|C}(w)\,dw = \int_{0}^{\infty} w^2 2 f_W(w)\,dw \tag{5}$$

We observe that $w^2 f_W(w)$ is an even function. Hence

$$E\left[W^2|C\right] = 2\int_{0}^{\infty} w^2 f_W(w)\,dw \tag{6}$$

$$= \int_{-\infty}^{\infty} w^2 f_W(w)\,dw = E\left[W^2\right] = \sigma^2 = 16 \tag{7}$$

Lastly, the conditional variance of W given C is

$$\mathrm{Var}[W|C] = E\left[W^2|C\right] - (E[W|C])^2 = 16 - 32/\pi = 5.8141 \tag{8}$$

Problem 3.8.5 ■

The time between telephone calls at a telephone switch is an exponential random variable T with expected value 0.01. Given $T > 0.02$,

(a) What is $E[T|T > 0.02]$, the conditional expected value of T?

(b) What is $\mathrm{Var}[T|T > 0.02]$, the conditional variance of T?

Problem 3.8.5 Solution

(a) We first find the conditional PDF of T. The PDF of T is

$$f_T(t) = \begin{cases} 100e^{-100t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The conditioning event has probability

$$P[T > 0.02] = \int_{0.02}^{\infty} f_T(t)\,dt = -e^{-100t}\Big|_{0.02}^{\infty} = e^{-2} \tag{2}$$

From Definition 3.15, the conditional PDF of T is

$$f_{T|T>0.02}(t) = \begin{cases} \frac{f_T(t)}{P[T>0.02]} & t \geq 0.02 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 100e^{-100(t-0.02)} & t \geq 0.02 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The conditional expected value of T is

$$E[T|T > 0.02] = \int_{0.02}^{\infty} t(100)e^{-100(t-0.02)}\,dt \tag{4}$$

The substitution $\tau = t - 0.02$ yields $d\tau = dt$ *and*

$$E[T|T > 0.02] = \int_0^\infty (\tau + 0.02)(100)e^{-100\tau}\, d\tau \tag{5}$$

$$= \int_0^\infty (\tau + 0.02) f_T(\tau)\, d\tau = E[T + 0.02] = 0.03 \tag{6}$$

where the second to last equality is a result of both linearity of integration and linearity of the expected value.

(b) The conditional second moment of T is

$$E\left[T^2|T > 0.02\right] = \int_{0.02}^\infty t^2 (100)e^{-100(t-0.02)}\, dt \tag{7}$$

The substitution $\tau = t - 0.02$ yields

$$E\left[T^2|T > 0.02\right] = \int_0^\infty (\tau + 0.02)^2 (100)e^{-100\tau}\, d\tau \tag{8}$$

$$= \int_0^\infty (\tau + 0.02)^2 f_T(\tau)\, d\tau \tag{9}$$

$$= E\left[(T + 0.02)^2\right] \tag{10}$$

Now we can calculate the conditional variance.

$$\text{Var}[T|T > 0.02] = E\left[T^2|T > 0.02\right] - (E[T|T > 0.02])^2 \tag{11}$$

$$= E\left[(T + 0.02)^2\right] - (E[T + 0.02])^2 \tag{12}$$

$$= \text{Var}[T + 0.02] \tag{13}$$

$$= \text{Var}[T] = 0.01 \tag{14}$$

where the second to last equality can be justified by citing Theorem 3.5(d) or by direct calculation.

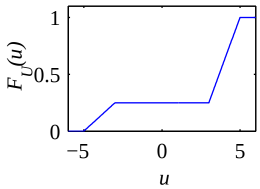
Problem 3.9.8 ■

Write a MATLAB function `u=urv(m)` that generates m samples of random variable U defined in Problem 3.3.7.

Problem 3.9.8 Solution

To solve this problem, we want to use Theorem 3.22. One complication is that in the theorem, U denotes the uniform random variable while X is the derived random variable. In this problem, we are using U for the random variable we want to derive. As a result, we will use Theorem 3.22 with the roles of X and U reversed. Given U with CDF $F_U(u) = F(u)$, we need to find the inverse functon $F^{-1}(x) = F_U^{-1}(x)$ so that for a uniform $(0, 1)$ random variable X, $U = F^{-1}(X)$.

Recall that random variable U defined in Problem 3.3.7 has CDF



$$F_U(u) = \begin{cases} 0 & u < -5 \\ (u+5)/8 & -5 \le u < -3 \\ 1/4 & -3 \le u < 3 \\ 1/4 + 3(u-3)/8 & 3 \le u < 5 \\ 1 & u \ge 5. \end{cases} \tag{1}$$

At $x = 1/4$, there are multiple values of u such that $F_U(u) = 1/4$. However, except for $x = 1/4$, the inverse $F_U^{-1}(x)$ is well defined over $0 < x < 1$. At $x = 1/4$, we can arbitrarily define a value for $F_U^{-1}(1/4)$ because when we produce sample values of $F_U^{-1}(X)$, the event $X = 1/4$ has probability zero. To generate the inverse CDF, given a value of x, $0 < x < 1$, we ave to find the value of u such that $x = F_U(u)$. From the CDF we see that

$$0 \le x \le \frac{1}{4} \qquad \Rightarrow x = \frac{u+5}{8} \tag{2}$$

$$\frac{1}{4} < x \le 1 \qquad \Rightarrow x = \frac{1}{4} + \frac{3}{8}(u-3) \tag{3}$$

$$\tag{4}$$

These conditions can be inverted to express u as a function of x.

$$u = F^{-1}(x) = \begin{cases} 8x - 5 & 0 \le x \le 1/4 \\ (8x+7)/3 & 1/4 < x \le 1 \end{cases} \tag{5}$$

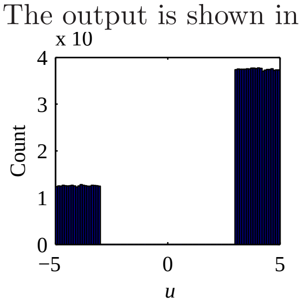
In particular, when X is a uniform $(0,1)$ random variable, $U = F^{-1}(X)$ will generate samples of the rndom variable U. A MATLAB program to implement this solution is now straightforward:

```
function u=urv(m)
%Usage: u=urv(m)
%Generates m samples of the random
%variable U defined in Problem 3.3.7
x=rand(m,1);
u=(x<=1/4).*(8*x-5);
u=u+(x>1/4).*(8*x+7)/3;
```

To see that this generates the correct output, we can generate a histogram of a million sample values of U using the commands

```
u=urv(1000000); hist(u,100);
```

The output is shown in the following graph, alongside the corresponding PDF of U.



$$f_U(u) = \begin{cases} 0 & u < -5 \\ 1/8 & -5 \le u < -3 \\ 0 & -3 \le u < 3 \\ 3/8 & 3 \le u < 5 \\ 0 & u \ge 5. \end{cases} \tag{6}$$

Note that the scaling constant 10^4 on the histogram plot comes from the fact that the histogram was generated using 10^6 sample points and 100 bins. The width of each bin is $\Delta = 10/100 = 0.1$. Consider a bin of idth Δ centered at u_0. A sample value of U would fall in that bin with probability $f_U(u_0)\Delta$. Given that we generate $m = 10^6$ samples, we would expect about $mf_U(u_0)\Delta = 10^5 f_U(u_0)$ samples in each bin. For $-5 < u_0 < -3$, we would expect to see about 1.25×10^4 samples in each bin. For $3 < u_0 < 5$, we would expect to see about 3.75×10^4 samples in each bin. As can be seen, these conclusions are consistent with the histogam data.

Finally, we comment that if you generate histograms for a range of values of m, the number of samples, you will see that the histograms will become more and more similar to a scaled version of the PDF. This gives the (false) impression that any bin centered on u_0 has a number of samples increasingly close to $mf_U(u_0)\Delta$. Because the histpgram is always the same height, what is actually happening is that the vertical axis is effectively scaled by $1/m$ and the height of a histogram bar is proportional to *the fraction* of m samples that land in that bin. We will see in Chapter 7 that the fraction of samples in a bin does converge to the probability of a sample being in that bin as the number of samples m goes to infinity.