MAT 271E Probability and Statistics

Homework 4 Solutions

Assigned: March 1, 2012

Due: March 7, 2011 (in class, before class starts)

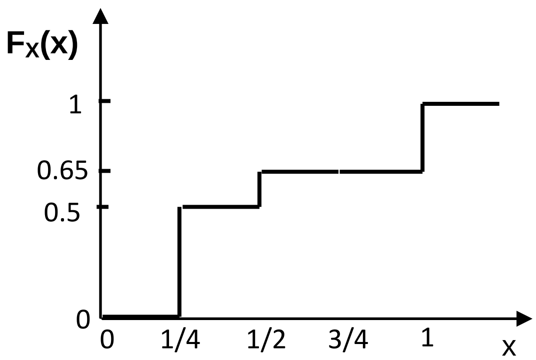
No late homework will be accepted!

Do not copy from solutions from your classmates. All work must be your own!

Show all your steps! Just writing a number as a result is not enough. Make sure you answer everything that is asked (subquestions, etc.). This homework includes **6 problems** all of which must be answered!

Read: "Probability and Stochastic Processes", Yates and Goodman, Ch. 2 and Ch. 3

1) Discrete random variable X has the *CDF* $F_X(x)$ as shown:



a) Find $p_{X|A}(x)$, where $A = \{X < 7/8\}$.

From the CDF, we can find the PMF

$$p_X(x) = \begin{cases} 0.5 & , x = 1/4 \\ 0.15 & , x = 1/2 \\ 0.35 & , x = 1 \\ 0 & , \text{otherwise} \end{cases}$$

The probability of event A is

$$P[A] = P[X < 7/8] = p_X(1/4) + p_X(1/2) = 0.5 + 0.15 = 0.65$$

Given the event A, the conditional probability of X is

$$p_{X|A}(x) = \begin{cases} \dfrac{p_X(x)}{P[A]} & , x \in A \\ 0 & , \text{otherwise} \end{cases}$$

$$p_{X|A}(x) = \begin{cases} 0.5/0.65 & , x = 1/4 \\ 0.15/0.65 & , x = 1/2 \\ 0 & , \text{otherwise} \end{cases}$$

$$p_{X|A}(x) = \begin{cases} 10/13 & , x = 1/4 \\ 3/13 & , x = 1/2 \\ 0 & , \text{otherwise} \end{cases}$$

b) Calculate $E[X|A]$.

The conditional expected value of X given A is

$$E[X|A] = \sum_{x=\frac{1}{4},\frac{1}{2}} x p_{X|A}(x) = \frac{1}{4}\left(\frac{10}{13}\right) + \frac{1}{2}\left(\frac{3}{13}\right) = \frac{5}{26} + \frac{3}{26} = \frac{8}{26} = \frac{4}{13}$$

c) Calculate $Var[X|A]$.

The conditional expected value of X^2 given A is

$$E[X^2|A] = \sum_{x=\frac{1}{4},\frac{1}{2}} x^2 p_{X|A}(x) = \left(\frac{1}{4}\right)^2\left(\frac{10}{13}\right) + \left(\frac{1}{2}\right)^2\left(\frac{3}{13}\right) = \frac{11}{8 \times 13} = \frac{11}{104}$$

The conditional variance is

$$Var[X|A] = E[X^2|A] - (E[X|A])^2 = \frac{11}{104} - \left(\frac{4}{13}\right)^2 = \frac{15}{1352}$$

2) Suppose that we have a box of toys. Each toy is suitable for either

- children who are aged between 3 and 8, or
- children who are over 8 years old.

Assume that there is a 0.3 probability of a toy being suitable for ages 3-8, and we need toys for 5 such children. Let T represent the number of toys we have to check to find these 5 toys. Consider the condition $A = \{T \geq 14\}$.

a) Find the PMF $p_T(t)$.

T is a Pascal RV. So, the PMF is

$$p_T(t) = \begin{cases} \binom{t-1}{4}(0.3)^5(0.7)^{t-5} & , t = 5, 6, 7, \dots \\ 0 & , \text{otherwise} \end{cases}$$

b) Find the conditional PMF of T given that we have checked 14 toys and still not found all of the needed 5 toys, $p_{T|A}(t)$.

$$P[A] = P[T \geq 14] = 1 - P[T < 14]$$

$$P[T < 14] = \sum_{5<t<14} tp_T(t) = \sum_{5<t<14} t\binom{t-1}{4}(0.3)^5(0.7)^{t-5}$$

$$P[A] = 1 - \sum_{5<t<14} t\binom{t-1}{4}(0.3)^5(0.7)^{t-5}$$

$$p_{T|A}(t) = \begin{cases} \frac{1}{P[A]}\cdot\binom{t-1}{4}(0.3)^5(0.7)^{t-5} & , t = 14, 15, \dots \\ 0 & , \text{otherwise} \end{cases}$$

c) Calculate the expected number of toys we have to check given that we have checked 14 toys and still not found all of the needed toys for the 5 children, $E[T|A]$.

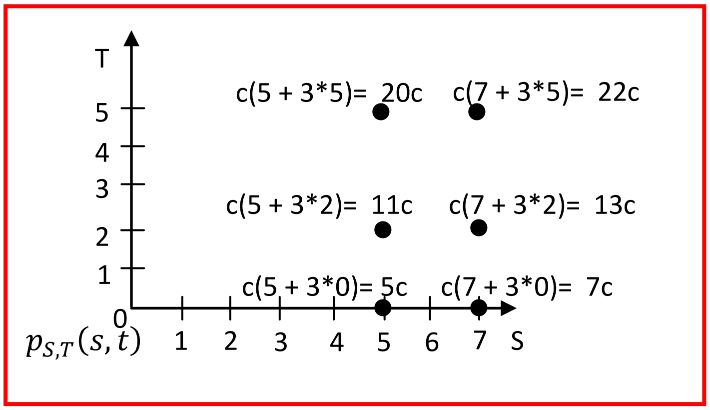
$$E[T|A] = \sum_{t\geq=14} tp_{T|A}(t) = \sum_{t\geq=14} t\frac{1}{P[A]}\cdot\binom{t-1}{4}(0.3)^5(0.7)^{t-5}$$
$$= \frac{1}{P[A]}\cdot\sum_{t\geq=14} t\binom{t-1}{4}(0.3)^5(0.7)^{t-5}$$

3) Random variables S and T have the joint PMF

$$p_{S,T}(s,t) = \begin{cases} c(s+3t) & ,s = 5,7;\ t = 0,2,5 \\ 0 & ,\text{otherwise} \end{cases}$$

a) What is the value of the constant c?

In this problem, it is helpful to label points with nonzero probability on the S-T plane:



We must choose c so that the PMF sums to 1.

$$\sum_{s=5,7}\sum_{t=0,2,5} p_{S,T}(s,t) = c\sum_{s=5,7}\sum_{t=0,2,5}(s+3t)$$

$$= c[(5+3\cdot 0)+(5+3\cdot 2)+(5+3\cdot 5)+(7+3\cdot 0)+(7+3\cdot 2) + (7+3\cdot 5)] = (5+11+20+7+13+22)c = 78c$$

Thus, $c = 1/78$.

b) What is $P[T < S]$?

The event $\{T < S\}$ has probability

$$P[T<S] = \sum_{s=5,7}\sum_{t<s} p_{S,T}(s,t)$$

$$= \frac{(5+3\cdot 0)+(5+3\cdot 2)+(7+3\cdot 0)+(7+3\cdot 2)+(7+3\cdot 5)}{78}$$

$$= \frac{58}{78} = \frac{29}{39}$$

c) What is $P[T > S]$?

Since T never takes on a value higher than that of S, the event {T > S} has probability

$$P[T > S] = \sum_{s=5,7} \sum_{t>s} p_{S,T}(s,t) = 0$$

d) What is $P[T = S]$?

There are two ways to solve this part:

(1) $$P[T = S] = \sum_{s=5,7} \sum_{t=s} p_{S,T}(s,t) = \frac{(5 + 3 \cdot 5)}{78} = \frac{20}{78} = \frac{10}{39}$$

(2) We observe that P[T = S] = 1 - P[T < S] - P[T > S] and use the values we found in the previous parts of the problem. So, P[T = S] = 1 – 29/39 - 0 = 10/39

e) What is $P[T = 5]$?

$$P[T = 5] = \sum_{s=5,7} p_{S,T}(s,5) = \frac{(5 + 3 \cdot 5) + (7 + 3 \cdot 5)}{78} = \frac{42}{78} = \frac{7}{13}$$

4) The random variables S and T have the joint PMF given above in **Problem 3.**

a) Find the marginal PMFs $p_S(s)$ and $p_T(t)$.

We can sum along the S and T axes of the graph for the joint PMF in Problem 3(a). The marginal PMFs of S and T are

$$p_S(s) = \sum_{t=0,2,5} p_{S,T}(s,t) = \begin{cases} \frac{36}{78} = \frac{6}{13} & , s = 5 \\ \frac{42}{78} = \frac{7}{13} & , s = 7 \\ 0 & , \text{otherwise} \end{cases}$$

$$p_T(t) = \sum_{s=5,7} p_{S,T}(s,t) = \begin{cases} \frac{12}{78} = \frac{2}{13} & ,t = 0 \\ \frac{24}{78} = \frac{4}{13} & ,t = 2 \\ \frac{42}{78} = \frac{7}{13} & ,t = 5 \\ 0 & ,\text{otherwise} \end{cases}$$

b) Find the expected values $E[S]$ and $E[T]$.

The expected values of S and T are

$$E[S] = \sum_{s=5,7} s p_S(s) = 5(6/13) + 7(7/13) = 79/13$$

$$E[T] = \sum_{t=0,2,5} t p_T(t) = 0(2/13) + 2(4/13) + 5(7/13) = 43/13$$

c) Find the standard deviations σ_S and σ_T.

The expectations of S^2 and T^2 are

$$E[S^2] = \sum_{s=5,7} s^2 p_S(s) = 5^2(6/13) + 7^2(7/13) = 493/13$$

$$E[T^2] = \sum_{t=0,2,5} t^2 p_T(t) = 0^2(2/13) + 2^2(4/13) + 5^2(7/13) = 191/13$$

The variances are

$Var[S] = E[S^2] - (E[S])^2 = 168/169$ $\qquad$ $Var[T] = E[T^2] - (E[T])^2 = 634/169$

The standard deviations are

$$\sigma_S = \sqrt{Var[S]} = \sqrt{168/169} \qquad \sigma_T = \sqrt{Var[T]} = \sqrt{634/169}$$

5) The random variables S and T have the joint PMF given above in **Problem 3.** We define another random variable $U = 5S + T$.

a) Find the probability mass function $p_U(u)$.

To find the PMF of U, we simply add the probabilities associated with each possible value of U.

$$p_U(u) = \sum_{(s,t):g(s,t)=u} p_{S,T}(s,t)$$

S_U = {25, 27, 30, 35, 37, 40}

$$p_U(25) = p_{S,T}(5,0) = \frac{1}{78}(5 + 3\cdot 0) = 5/78$$
$$p_U(27) = p_{S,T}(5,2) = \frac{1}{78}(5 + 3\cdot 2) = 11/78$$
$$p_U(30) = p_{S,T}(5,5) = \frac{1}{78}(5 + 3\cdot 5) = 20/78$$
$$p_U(35) = p_{S,T}(7,0) = \frac{1}{78}(7 + 3\cdot 0) = 7/78$$
$$p_U(37) = p_{S,T}(7,2) = \frac{1}{78}(7 + 3\cdot 2) = 13/78$$
$$p_U(40) = p_{S,T}(7,5) = \frac{1}{78}(7 + 3\cdot 5) = 22/78$$

$$p_U(u) = \begin{cases} 5/78 & , \quad u = 25 \\ 11/78 & , \quad u = 27 \\ 20/78 & , \quad u = 30 \\ 7/78 & , \quad u = 35 \\ 13/78 & , \quad u = 37 \\ 22/78 & , \quad u = 40 \\ 0 & , \quad \text{otherwise} \end{cases}$$

b) Find the expected value E[U].

$$E[U] = \sum_u u p_U(u) = 25\left(\frac{5}{78}\right) + 27\left(\frac{11}{78}\right) + 30\left(\frac{20}{78}\right) + 35\left(\frac{7}{78}\right) + 37\left(\frac{13}{78}\right) + 40\left(\frac{22}{78}\right)$$

$$= \frac{125 + 297 + 600 + 245 + 481 + 880}{78}$$

$$= \frac{2628}{78} = \frac{438}{13}$$

c) Find $P[U < 28]$.

$P[U < 28] = P[U = 25] + P[U = 27] = p_U(25) + p_U(27)$

$$= \frac{5}{78} + \frac{11}{78} = \frac{16}{78} = \frac{8}{39}$$

6) The random variables S and T have the joint PMF given above in **Problem 3.** (You may refer to the results you find in **Problem 4** above to answer some of these questions).

a) Find the expected value of $U = T \times S$.

The possible values of U are:

5 x 0 = 0
7 x 0 = 0
5 x 2 = 10
7 x 2 = 14
5 x 5 = 25
7 x 5 = 35

$$p_U(u) = \begin{cases} (1/78)(5+3\cdot 0) + (1/78)(7+3\cdot 0) & , \quad u = 0 \\ (1/78)(5+3\cdot 2) & , \quad u = 10 \\ (1/78)(7+3\cdot 2) & , \quad u = 14 \\ (1/78)(5+3\cdot 5) & , \quad u = 25 \\ (1/78)(7+3\cdot 5) & , \quad u = 35 \\ 0 & , \quad \text{otherwise} \end{cases}$$

$$p_U(u) = \begin{cases} 12/78 & , \quad u = 0 \\ 11/78 & , \quad u = 10 \\ 13/78 & , \quad u = 14 \\ 20/78 & , \quad u = 25 \\ 22/78 & , \quad u = 35 \\ 0 & , \quad \text{otherwise} \end{cases}$$

$$E[U] = 0\left(\frac{12}{78}\right) + 10\left(\frac{11}{78}\right) + 14\left(\frac{13}{78}\right) + 25\left(\frac{20}{78}\right) + 35\left(\frac{22}{78}\right)$$
$$= 0 + \frac{110}{78} + \frac{182}{78} + \frac{500}{78} + \frac{770}{78}$$

$$E[U] = \frac{1562}{78} = \frac{781}{39}$$

b) Find the correlation $E[ST]$.

$$E[ST] = \sum_{s\in S_S}\sum_{t\in S_T} st p_{S,T}(s,t) = \sum_{s\in S_S}\sum_{t\in S_T} st\left(\frac{1}{78}(s+3t)\right) = \frac{1}{78}\sum_{s\in S_S}\sum_{t\in S_T} st(s+3t)$$

$$E[ST] = \frac{1}{78}[(5\cdot 0)(5+3\cdot 0) + (5\cdot 2)(5+3\cdot 2) + (5\cdot 5)(5+3\cdot 5) + (7\cdot 0)(7+3\cdot 0) + (7\cdot 2)(7+3\cdot 2) + (7\cdot 5)(7+3\cdot 5)]$$

$$E[ST] = \frac{1}{78}(0+110+500+0+182+770)$$

$$E[ST] = \frac{1562}{78} = \frac{781}{39}$$

c) Find the covariance $Cov[S,T]$.

$$Cov[S,T] = E[ST] - \mu_S\mu_T$$

In Problem 4(b) above, we found that $\mu_S = 79/13$ and $\mu_T = 43/13$.

In Part (b) above, we found that $E[ST] = 781/39$.

$$Cov[S,T] = \frac{781}{39} - \left(\frac{79}{13}\right)\left(\frac{43}{13}\right) = \frac{10153}{507} - \frac{10191}{507} = -\frac{38}{507}$$

d) Find the correlation coefficient $\rho_{S,T}$.

$$\rho_{S,T} = \frac{Cov[S,T]}{\sqrt{Var[S]Var[T]}}$$

In Problem 4(c) above, we found that $Var[S] = 168/169$ and $Var[T] = 634/169$.

In Part (c) above, $Cov[S,T] = -38/507$.

$$\rho_{S,T} = \frac{-38/507}{\sqrt{\frac{168}{169}\cdot\frac{634}{169}}}$$

$$\rho_{S,T} = -\frac{38}{507} \cdot \frac{169}{\sqrt{106512}}$$

$$\rho_{S,T} = -0.0388$$

e) Find the variance of *S* - *T*, Var[*S* - *T*].

$$Var[S-T] = Var[S] + Var[-T] + 2Cov[S,-T]$$

$$= Var[S] + Var[T] - 2Cov[S,T]$$

$$= \frac{168}{169} + \frac{634}{169} - 2\left(-\frac{38}{507}\right)$$

$$= \frac{802}{169} - 2\left(-\frac{38}{507}\right)$$

$$= \frac{2406}{507} + \frac{76}{507}$$

$$Var[S-T] = \frac{2482}{507}$$