Practice Questions

KOMS05

1) Let X & Y be jointly normal rv with

$\mu_X = 10 \quad \sigma_X^2 = 4 \quad \rho_{XY} = 0.5$

$\mu_Y = 0 \quad \sigma_Y^2 = 1$

Find the joint density function of Z & W

$$Z = X + Y$$
$$W = X - Y$$

Answer

As linear transformations do not change the type of distributions Z&W will be jointly normal.

$$\begin{bmatrix} Z \\ W \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{G} \begin{bmatrix} X \\ Y \end{bmatrix} \Rightarrow \begin{bmatrix} \mu_Z \\ \mu_W \end{bmatrix} = G \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$\rho_{XY} \cdot \sigma_X \cdot \sigma_Y = Cov(X, Y)$

$$\Sigma_{ZW} = G \Sigma_{XY} G^T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

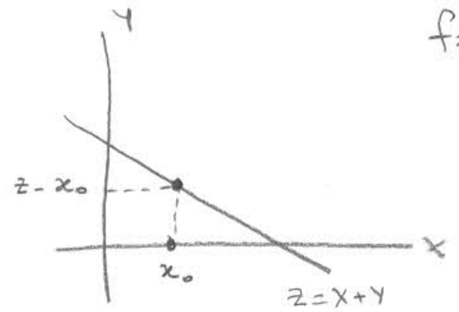
$$= \begin{bmatrix} 5 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 3 & 3 \end{bmatrix} \Rightarrow \begin{array}{l} \sigma_Z^2 = 7 \\ \sigma_W^2 = 3 \\ \rho_{ZW} = 3/\sqrt{3.7} = \frac{\sqrt{3}}{\sqrt{7}} \end{array}$$

Hence

$$f(z, w) = \frac{1}{2\pi \, |\Sigma_{ZW}|} \exp\left\{ -\frac{1}{2} \begin{bmatrix} z-10 \\ w-10 \end{bmatrix}^T \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix}^{-1} \begin{bmatrix} z-10 \\ w-10 \end{bmatrix} \right\}$$

2) Let X & Y be independent discrete r.v. and $Z = X + Y$.

Find pmf of Z.



$$f_Z(z) = P(Z = z) = \sum_{x+y=z} f_{XY}(x, y)$$

$$= \sum_{x=-\infty}^{\infty} f_{XY}(x, z-x)$$

or

$$= \sum_{y=-\infty}^{\infty} f_{XY}(z-y, y)$$

$$= \sum_{x=-\infty}^{\infty} f_X(x) f_Y(z-y) \quad \text{as } X \text{ \& } Y \text{ are indep.}$$

3) Let X & Y be independent discrete rv with λ_x and λ_y.

Find pdf of $Z = X + Y$

Answer

$$f_x(k) = P(X = k) = \frac{\lambda_x^k e^{-\lambda_x}}{k!} \quad \text{when } k \geq 0$$

As X & Y are independent $f_Z(z) = f_x(x) * f_y(y)$

For integer z values

$$P(Z = z) = \sum_{x=0}^{\infty} f_x(x) f_y(z-x)$$

$$= \sum_{x=0}^{\infty} \frac{\lambda_x^x e^{-\lambda_x}}{x!} \cdot \frac{\lambda_y^{z-x} e^{-\lambda_y}}{(z-x)!}$$

$$= \underbrace{\left[\sum_{x=0}^{\infty} \frac{z!}{x!(z-x)!} \lambda_x^x \lambda_y^{z-x} \right]}_{(\lambda_x + \lambda_y)^z} \frac{e^{-(\lambda_x + \lambda_y)}}{z!}$$

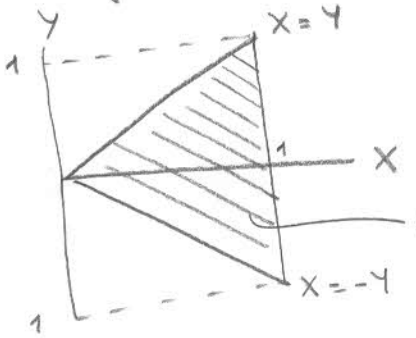
$$= (\lambda_x + \lambda_y)^z \frac{e^{-(\lambda_x + \lambda_y)}}{z!}$$

Hence z has also Poisson distribution with $\lambda_z = \lambda_x + \lambda_y$

4) Let X, Y have the following joint pdf

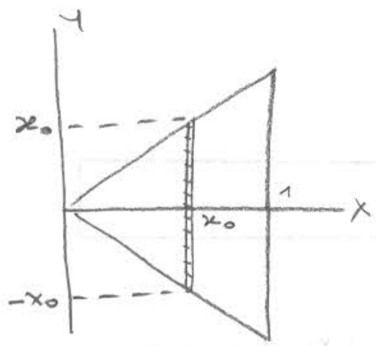
$$f_{xy}(x,y) = \begin{cases} 1 & 0 < |y| < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

a) Find marginal densities of X & Y

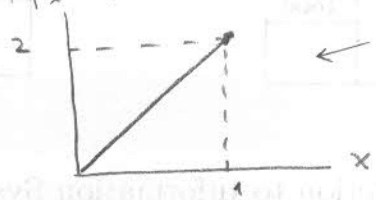


$|y| < x$ means $-y < x < y$

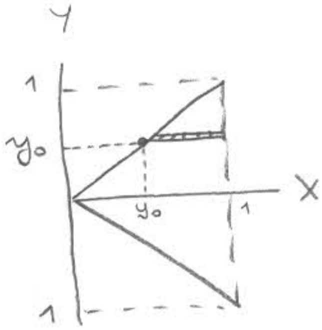
$f_{xy}(x,y) = 1$ in the shaded area



$$f_X(x_0) = \int_{-x_0}^{x_0} f_{XY}(x,y)\,dx = 2x_0 \quad \text{when } 0 \le x_0 \le 1$$



← marginal density ~~of~~ X



$$f_Y(y_0) = \int_{|y_0|}^{1} f_{XY}(x,y)\,dy = 1 - |y_0| \quad \text{when } |y_0| \le 1$$

b) Find $E(X|Y)$

$$E(X|Y) = \int_0^1 x \cdot f_{X|Y}(x|y)\,dx$$

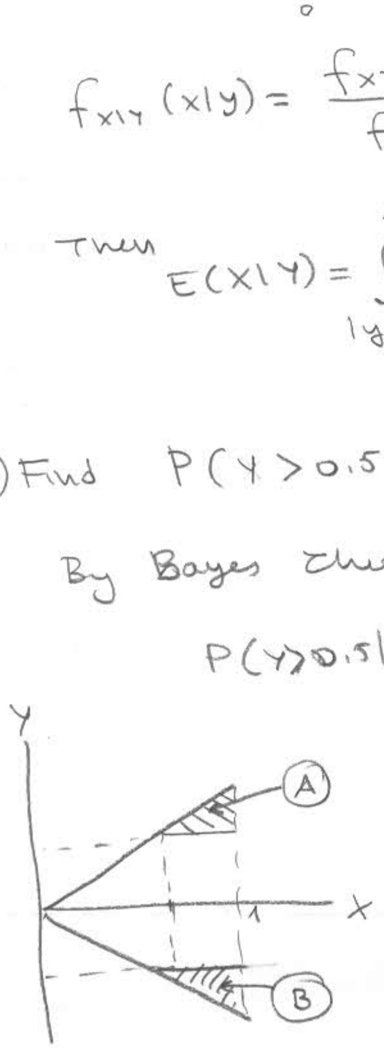
$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1}{1-|y|} \quad \text{when } 0 < |y| < x < 1$$

Then
$$E(X|Y) = \int_{|y|}^{1} x \cdot \frac{1}{1-|y|}\,dx = \frac{1-|y|^2}{2} \cdot \frac{1}{1-|y|} = \frac{1+|y|}{2} \quad |y| < 1$$

c) Find $P(Y > 0.5 \mid X > 0.5)$

By Bayes theorem

$$P(Y>0.5|X>0.5) = \frac{P(X>0.5,\, Y>0.5)}{P(X>0.5)} = \frac{A}{A+B} = \frac{1/8}{1/4} = \frac{1}{2}$$



Area A = Area B = 1/8

5) If X & Y are independent rv and $Y \sim Uni(0,1)$

Find pdf of $Z = X + Y$ in terms of $f_X(x)$ and $F_X(x)$

Answer: As they are independent

$$f_Z(z) = f_X(x) * f_Y(y) = \int_{-\infty}^{z} f_X(x) \cdot \underbrace{f_Y(z-x)}_{= \begin{cases} 1 & \text{if } 0 < z-x < 1 \\ 0 & \text{else} \end{cases}} dx \qquad z-1 < x < z$$

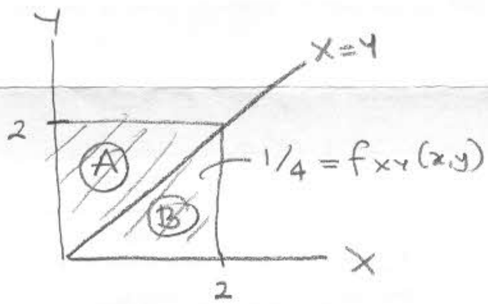
$$= \int_{z-1}^{z} f_X(x)\,dx = F_X(x)\Big|_{z-1}^{z}$$

$$f_Z(z) = F_X(z) - F_X(z-1)$$

6) Let X & Y iid $\sim Uni(0,2)$

Find pdf of $Z = |X - Y|$

Answer:

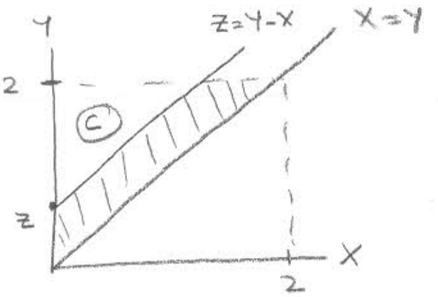


In region A $Z = Y - X$

In region B $Z = X - Y$

for $Z < 0$ $f_Z(z) = 0$

for $z > 0$ in region A



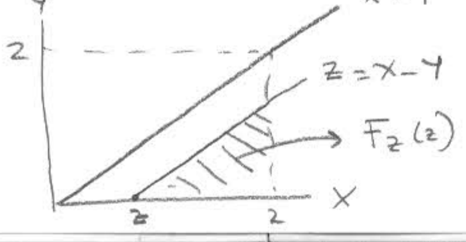
$F_Z(z) = P(Y - X \leq z)$ is the shaded area on the left figure

$$= \left(\frac{1}{2} - \text{area of region } C\right) \times \frac{1}{4}$$

$$= \frac{1}{2} - \frac{(2-z)^2}{2} = \frac{1 - 4 + 4z + z^2}{2} \cdot \frac{1}{4}$$

$$\Rightarrow f_Z(z) = \frac{z+2}{4} \quad \text{when} \quad 0 \leq z \leq 2$$

for $z > 0$ in region B



$$F_Z(z) = \frac{(2-z)^2}{2 \times 4} \Rightarrow f_Z(z) = \frac{z-2}{4} \quad 0 \leq z \leq 2$$

Then in total

$$f_Z(z) = \frac{2z}{4} = \frac{z}{2} \quad 0 \leq z \leq 2$$