March 25, 2007
MK

TEL252E Problems

Problem 1 *CT System Properties.*
Consider a CT system with input $x(t)$ and output $y(t)$. For each of the following systems, i) prove that it is linear or give a counter example, ii) prove that it is time-invariant or give a counter example, iii) determine whether it is causal or noncausal, and iv) determine if it is a memoryless or memory system.

(a) $y(t) = u(t)x(t)$

(b) $y(t) = x(\sin(t))$

(c) $y(t) = \sin(x(t))$

(d) $y(t) = \frac{dx(t)}{dt}$

(e) $y(t) = x(2t) - x(t-1)$

(f) $y(t) = x(0)$

(g) $y(t) = \int_0^t x(\tau)d\tau$

Problem 2 *DT System Properties.*
Consider a system with input $x[n]$ and output $y[n]$. For each of the following systems, i) prove that it is linear or give a counter example, ii) prove that it is time-invariant or give a counter example.

(a) $y[n] = x[n] + 1$

(b) $y[n] = x[2n]$ (This operation is known as *decimation*.)

(c) $y[n] = \begin{cases} x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

(d) $y[n] = \begin{cases} x[n], & x[n] < 4 \\ 4, & \text{else} \end{cases}$

Problem 3 *Determining the impulse response of DT LTI systems.*
For each of the following systems with input x and output y, i) prove that the system is linear; ii) prove that the system is time invariant; iii) compute the system's impulse response. Simplify your answer as much as possible.

(a) $y_n = \sum_{k=0}^{\infty} b_k x_{n-k}$

(b) $y_n = \frac{1}{3}\left(x_n - \frac{1}{2}(x_{n-1} + x_{n+1})\right)$

(c) $y_n = \frac{1}{2}y_{n-1} + x_n$

Problem 4 *Determining the impulse response of CT LTI systems.*
Determine the impulse response for the following system under the assumption that the system is initially at rest.

$$y(t) = ay(t) + x(t)$$

Problem 5 *Determining the impulse response of CT LTI systems.*
For each of the following systems with input x and output y, i) prove that the system is linear; ii) prove that the system is time invariant; iii) compute the system's impulse response. Simplify your answer as much as possible.

(a) $y(t) = \int_{-\infty}^{\infty} r(\tau - t)x(\tau)d\tau$

(b) $y(t) = x(t) + 2x(t+1) + 3x(t-1)$

(c) $\frac{dy(t)}{dt} = -x(t)$

Problem 6 *Properties of LTI systems.*
A time invariant system $T[\cdot]$ is observed to have the following input/output relationships.

$$\begin{aligned} \delta_{n-1} + 2\delta_{n-2} &= T\left[\,\delta_n + 2\delta_{n-2}\,\right] \\ \delta_{n-1} + 2\delta_{n-3} &= T\left[\,3\delta_{n-2}\,\right] \\ \delta_{n+1} + 2\delta_n + \delta_{n-1} &= T\left[\,\delta_{n-3}\,\right] \end{aligned}$$

(a) Prove that the system is linear or nonlinear.

(b) Compute the response to an input of δ_n, that is compute $T\left[\delta_n\right]$.

Problem 7 *Properties of LTI systems.*
Prove the following properties.

(a) The commutative property of DT convolution, that is, $x_n * y_n = y_n * x_n$

(b) The associative property of DT convolution, that is, $(x_n * y_n) * z_n = x_n * (y_n * z_n)$

(c) The distributive property of DT convolution, that is, $x_n * (y_n + z_n) = x_n * y_n + x_n * z_n$

(d) Let h_n be the impulse response of a DT system. Then the system is causal if and only if $h_n = 0$ for $n < 0$.

Problem 8 *Responses of LTI systems.*
Find the outputs of the following LTI systems with the following inputs.

(a) Impulse response of $h(t) = u(t+1) - u(t-1)$; input of $x(t) = u(t) - u(t-2)$

(b) Impulse response of $h_n = a^n u_n$; input of $x_n = u_n$ for $|a| < 1$.

(c) Impulse response of $h_n = (-a)^n u_n$; input of $x_n = u_n$ for $|a| < 1$.

Problem 9 *Convolutions.*
Calculate the output of a LTI system with impulse response $h(n)$, input $x(n)$, and output $y(n)$.

(a) $h(n) = a^n u(n)$ and $x(n) = b^n u(n)$ where $a \neq b$.

(b) $h(n) = a^n u(n)$ and $x(n) = a^n u(n)$

(c) $h(n) = a^n u(n)$ and $x(n) = \cos(\omega n)$ where $|a| < 1$.

(d) $h(n) = u(n) - u(n-N)$ and $x(n) = u(n) - u(n-P)$ for $P > N$.

Problem 10 *Causal and Stable LTI systems.*
For the following discrete-time and continuous-time LTI systems, determine whether each system is causal and/or stable. Justify your answers.

(a) $h_n = (\frac{1}{2})^n u_{-n}$

(b) $h_n = (-\frac{1}{2})^n u_n + (1.01)^n u_{n-1}$

(c) $h(t) = e^{2t}u(-1-t)$

(d) $h(t) = te^{-t}u(t)$

Problem 11 *Properties of convolution.*

(a) Consider a CT LTI system $y(t) = x(t) * h(t)$. Show the input $\frac{dx(t)}{d(t)}$ results in the output $\frac{dy(t)}{d(t)}$.

(b) Consider the DT LTI system $y_n = x_n * h_n$. Prove that

$$\sum_{n=-\infty}^{\infty} y_n = \left(\sum_{n=-\infty}^{\infty} x_n\right)\left(\sum_{n=-\infty}^{\infty} h_n\right)$$

(c) Consider a CT LTI system $y(t) = x(t) * h(t)$. Prove that if $x(t)$ is periodic with period T, then $y(t)$ is also periodic with period T.

Problem 12 *Properties of convolution.*
Let x_n be a signal which is nonzero only in the interval $0 \leq n < M$ and h_n be a signal which is nonzero only in the interval $0 \leq n < N$.

(a) Determine the interval $L_1 \leq n \leq L_2$ over which $y_n = x_n * h_n$ is nonzero. Express L_1 and L_2 in terms of M and N.

(b) Verify the result in the the previous part by analytically computing the convolution of the signals $x_n = u_n - u_{n-5}$ and $h_n = 2(u_n - u_{n-3})$.

(c) Verify the result in the the previous part by graphically computing the convolution of the signals $x_n = u_n - u_{n-5}$ and $h_n = 2(u_n - u_{n-2})$.

Problem 13 *Discrete-time Impulse Response*
Consider the discrete-time LTI system described by the equation

$$y_n = x_n - 3x_{n-1} + 2x_{n-2}$$

(a) Compute the impulse response of the system.

(b) Express the system in the form $y_n = x_n * h_n$.

(c) Find the output when the input is given by $x_n = u_n$.

(d) Find the output when the input is given by $x_n = 1$.

Problem 14 *System response to a complex exponential input.*
For the following continuous-time and discrete-time systems with the given input and output, determine whether the system is definitely *not* LTI.

(a) $S_1[e^{j7t}] = te^{j7t}$

(b) $S_2[e^{j7t}] = e^{j7(t-2)}$

(c) $S_3[e^{j7t}] = \sin(7t)$

(d) $S_4[e^{j\pi n/4}] = e^{j\pi n/4}u_n$

(e) $S_5[e^{j\pi n/4}] = e^{j3\pi n/4}$

(f) $S_6[e^{j\pi n/4}] = 2e^{3\pi/4}e^{j\pi n/4}$

Problem 15 *DT Impulse Response*
Consider the DT LTI system described by the equation

$$y[n] = \frac{1}{2}y[n-1] + x[n]$$

where $\lim_{n\to-\infty} y[n] = 0$.

(a) Compute the impulse response of the system.

(b) Express the system in the form $y[n] = x[n] * h[n]$.

(c) Find the output when the input is given by $x[n] = u[n]$.

(d) Find the output when the input is given by $x[n] = 1$.

Problem 16 *Parseval's formula.*
For a signal expressed using Equation 1 show that

$$\int_a^b |x(t)|^2\, dt = \sum_{m=-\infty}^{\infty} |a_m|^2$$

This important result is known as the Parseval's formula. Note that the left side is the energy in $x(t)$.
Problem 17 *Determining Fourier series coefficients.*
Each of the following functions is periodic with period T. For each function sketch the real and imaginary parts of the function on the interval $[0, 2T]$ and calculate the Fourier series coeffcients.

(a) $x(t) = e^{j2\pi t/3}$ with period $T = 3$.

(b) $x(t) = \sin(2\pi t/3) + 3\cos(\pi t/6)$ with period $T = 12$.

(c) $x(t) = \text{rect}(t)$ for $|t| < T/2$ with period $T = 2$. (put in simplest form)

(d) $x(t) = \Lambda(t)$ for $|t| < T/2$ with period $T = 2$. (put in simplest form)

Problem 18 *Properties of Fourier series.*
Suppose that the Fourier series coefficients for the function $x(t)$ with period T are given as a_k, and the Fourier series coefficients for the function $y(t)$ with period T are given as b_k. Prove the following relationships.

(a) If $y(t) = \frac{d\,x(t)}{dt}$ then $b_k = jk\frac{2\pi}{T}a_k$.

(b) If $y(t) = x(-t)$ then $b_k = a_{-k}$.

(c) If $x(t)$ is real, then $a_k = a^*_{-k}$.

(d) If $x(t)$ is real and $x(t) = x(-t)$, then a_k are real and $a_k = a_{-k}$.

Problem 19 *Reconstructing signals from Fourier series coefficients.*
In each of the following, the Fourier series coefficients and the period of a signal are specified. Determine the signal $x(t)$ in each case.

(a) $a_k = (\frac{1}{2})^{|k|}$ and $T = 2$.

(b) $a_k = \begin{cases} jk & |k| < 3 \\ 0 & \text{otherwise} \end{cases}$ and $T = 4$.

(c) $a_k = \cos(\pi k/4)$ and $T = 4$.

Problem 20 *Fourier series and LTI systems.*
Suppose that the signal $x(t)$ is periodic with period T and Fourier series coefficients a_k. Let $y(t) = h(t) * x(t)$ where $h(t)$ is the impulse response of an LTI system.

(a) Show that $y(t)$ is also periodic with period T.

(b) Show that the Fourier series coefficients of $y(t)$ have the form $b_k = c_k\, a_k$ where c_k are multiplicative constants.

(c) Derive an expression for the multiplicative constants c_k.

Problem 21 *Evaluating CTFTs.*
Calculate the continuous-time Fourier transform for the following signals:

a) $x(t) = e^{-at}u(t)$ for $a > 0$

b) $x(t) = te^{-at}u(t)$ for $a > 0$

c) $x(t) = \text{rect}(t)$

d) $x(t) = \text{rect}\left(\frac{t-a}{b}\right)$ for any two real numbers a and b.

e) $x(t) = \delta(t)$

f) $x(t) = a\delta(t - b)$ for any two real numbers a and b.

Problem 22 *Properties of CTFTs.*
For the following problems, let $X(\omega)$ and $Y(\omega)$ be the CTFT's of $x(t)$ and $y(t)$, respectively. Calculate the CTFT of each function in terms of the functions $x(t)$, $y(t)$, $X(\omega)$, and $Y(\omega)$.

(a) $5x(t - a)$

(b) $X(t)$

(c) $x(t) * y(t)$

(d) $x(t)y(t)$

(e) $x(-t)$

(f) $x(t)e^{j\omega_0 t}$

(g) $\frac{1}{|a|}X\left(\frac{\omega}{a}\right)$

Problem 23 *Evaluating inverse CTFTs.*
Calculate the **inverse** CTFT for the following signals.

a) $X(\omega) = \delta(\omega)$

b) $X(\omega) = \delta(\omega - \omega_0)$

c) $X(\omega) = \text{rect}(\omega)$

Problem 24 *Evaluating CTFTs.*
Use answers to Problems 1 and 2 above to compute the CTFT for the following signals.

a) $x(t) = \text{sinc}(t)$.

b) $x(t) = \text{sinc}\left(\frac{t-a}{b}\right)$ for any two real numbers a and b.

c) $x(t) = 1$

d) $x(t) = e^{j\omega_0 t}$

e) $x(t) = \cos(\omega_0 t)$

f) $x(t) = \sin(\omega_0 t)$

Problem 25 *Transfer functions for LTI systems.*
For an LTI system T we have

$$T[e^{-2t}u(t)] = te^{-t}u(t) + 2e^{-2t}u(t)$$

Determine the transfer function, $H(\omega) = \frac{Y(\omega)}{X(\omega)}$, for this system.

Problem 26 *Deriving CTFT Properties*
Derive each of the following CTFT properties. Assume that in each case the CTFT of $x(t)$ and $y(t)$ are $X(\omega)$ and $Y(\omega)$ respectively.

b) $x(-t) \stackrel{CTFT}{\Leftrightarrow} X(-\omega)$

c) $x(t-t_0) \stackrel{CTFT}{\Leftrightarrow} X(\omega)e^{-j\omega t_0}$

d) $x(at) \stackrel{CTFT}{\Leftrightarrow} \frac{1}{|a|}X(\frac{\omega}{a})$

e) $X(\omega) = X^*(-\omega)$ if $x(t)$ is real

h) $x(t)y(t) \stackrel{CTFT}{\Leftrightarrow} \frac{1}{2\pi}X(\omega) * Y(\omega)$

j) $\frac{dx(t)}{dt} \stackrel{CTFT}{\Leftrightarrow} j\omega X(\omega)$

Problem 27 *Computing CTFT Transforms*
For each of the following functions, compute the CTFT then sketch the function $x(t)$ and its Fourier transform $X(\omega)$. (Hint: Use CTFT property 12 from notes.)

a) $x(t) = \displaystyle\sum_{k=-\infty}^{\infty} \delta(t - k/2)$

b) $x(t) = \text{sinc}(t) \displaystyle\sum_{k=-\infty}^{\infty} \delta(t - k/2)$

c) $x(t) = \text{sinc}(t) \displaystyle\sum_{k=-\infty}^{\infty} \delta(t - k)$

Problem 28 *Frequency analysis of linear differential equations*
Consider the system with input $x(t)$ and output $y(t)$ described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt} + x(t)$$

where the system is assumed to be initially at rest.

a) Calculate the frequency response of the system $H(\omega)$.

b) Express $H(\omega)$ as the ratio of factored polynomials.

Problem 29 *Duality property of the CTFT*
Use the duality property to determine the CTFT of the following signals

(a) $x(t) = \frac{1}{5+j2\pi t}$

(b) $x(t) = \frac{t}{(1+t^2)^2}$ (Hint: see question 4.12 in the textbook.)

Problem 30 *Symmetry properties of the CTFT*
For each of the following transforms, determine whether the corresponding time-domain signal is (i) real, purely imaginary, or complex, and (ii) even, odd, or neither even nor odd. Do this without evaluating the inverse CTFT.

(a) $X(\omega) = \sin(2\omega)\cos(3\omega)$

(b) $X(\omega) = \sin(\omega)\, e^{j(2\omega+\pi/2)}$

(c) $X(\omega) = u(\omega) - u(\omega - 4\pi)$

Problem 31 *Frequency analysis of LTI systems*
Consider a LTI system with frequency response $H(\omega)$, input $x(t)$, and output $y(t)$.

(a) Derive an expression for

$$\int_{-\infty}^{\infty} h(t)dt$$

in terms of the function $H(\omega)$.

(b) Derive an expression for $h(0)$ in terms of $H(\omega)$.

(c) If the input is $x(t) = a$, then express the output $y(t)$ in terms of a and $H(\omega)$.

(d) If the input is $x(t) = a$, then express the output $y(t)$ in terms of a and $h(t)$.

(e) You are asked to design a LTI system with a DC gain of A. What do you know about the impulse response of the system?

(f) You are asked to design a LTI system with a DC gain of A. What do you know about the frequency response of the system?

Problem 32 *Frequency analysis of linear differential equations*
Consider the system with input $x(t)$ and output $y(t)$ described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 10y(t) = \frac{dx(t)}{dt} - x(t)$$

where the system is assumed to be initially at rest.

(a) Determine the frequency response of the system $H(\omega)$.

(b) Determine the impulse response of the system $h(t)$.

(c) If the input to the system is $x(t) = e^{-t}u(t)$ find the corresponding output.

Problem 33 *Inverse CTFT's*
Calculate the inverse CTFT's of the following transforms.

(a) $X(\omega) = \frac{1}{j\omega+5}$

(b) $X(\omega) = \frac{1}{(j\omega+5)^2}$

(c) $X(\omega) = \frac{1}{(j\omega+5)(j\omega+2)}$

Problem 34 *Convolution and CTFT's*
For each of the following, calculate $X(\omega)$, $Y(\omega)$, $Z(\omega) = X(\omega)Y(\omega)$, and $z(t)$.

(a) $x(t) = e^{-t}u(t)$ and $y(t) = e^{-t}u(t)$

(b) $x(t) = (e^{-t}u(t)) * (e^{-t}u(t))$ and $y(t) = e^{-t}u(t)$

(c) $x(t) = \frac{t^{n-1}}{(n-1)!}e^{-t}u(t)$ and $y(t) = e^{-t}u(t)$

(d) $x(t) = e^{-t}u(t)$ and $y(t) = e^{-2t}u(t)$

(e) $x(t) = e^{-t}u(t)$ and $y(t) = te^{-2t}u(t)$

Problem 35 *DFT*
For each of the following discrete-time signals $x(n)$, calculate the DFT X_k for $0 \leq k < N$. In each case, assume that m is an integer.

(a) $x(n) = \delta(n)$ for $0 \leq n < N$.

(b) $x(n) = \delta(n-m)$ for $0 \leq n, m < N$.

(c) $x(n) = e^{j\frac{2\pi nm}{N}}$ for $0 \leq n, m < N$.

(d) $x(n) = \cos\left(\frac{2\pi nm}{N}\right)$ for $0 \leq n, m < N$.

(e) $x(n) = \sin\left(\frac{2\pi nm}{N}\right)$ for $0 \leq n, m < N$.

Problem 36 *DFT of a Sine Wave*
Let $x(n) = e^{j\omega n}$ for $0 \leq n < N$ and let X_k be its DFT.

(a) Calculate an explicit expresion for X_k that is correct for any value of ω.

(b) Sketch a plot of $|X_k|$ for $\omega = \pi/N$ and $N = 20$.

(c) Calculate a simplified expression for X_k when $\omega = 2\pi m/N$ where m is an integer

(d) Sketch a plot of $|X_k|$ for $\omega = 2\pi/N$ and $N = 20$.

Problem 37 *Parseval's Theorem for the DFT*

(a) Let the functions $\phi_k(n)$ for $0 \leq n, k < N$ have the property that

$$< \phi_k, \phi_l >= \alpha \delta(k-l)$$

and let

$$x(n) = \sum_{k=0}^{N-1} X_k \phi_k(n)$$

then prove that

$$\sum_{n=0}^{N-1} |x(n)|^2 = \alpha \sum_{k=0}^{N-1} |X_k|^2$$

(b) Specify the functions $\phi_k(n)$, the constant α, and the form of the innerproduct $< \phi_k, \phi_l >$ so that the transform described in part a) is a DFT as described in lecture.

Problem 38 *DTFT Transforms*
Compute the DTFT, $X(\omega)$, for the following signals.

(a) $x(n) = u(n) - u(n-m)$ for $m \geq 0$.

(b) $x(n) = \delta(n-m)$ for m an integer.

(c) $x(n) = e^{(j\omega_0 - a)n} u(n)$

(d) $x(n) = \cos(\omega_0 n + \phi)$

(e) $x(n) = \sin(\omega_0 n + \phi)$

(f) $x(n) = a^n u(n)$ where $|a| < 1$

(g) $x(n) = a^{|n|}$ where $|a| < 1$

(h) $x(n) = n a^n u(n)$ where $|a| < 1$

(i) $x(n) = a^{n-1} u(n-1)$ where $|a| < 1$

Problem 39 *Difference Equations*
Consider the discrete time system $y(n) = T[x(n)]$ with input $x(n)$ and output $y(n)$ which obeys the following difference equation

$$y(n) = 2r\cos(\theta) y(n-1) - r^2 y(n-2) + x(n)$$

where $|r| < 1$ and θ are real valued constants.

(a) Prove the system $T[\cdot]$ is linear.

(b) Prove the system $T[\cdot]$ is time invariant.

(c) Calculate the frequency response $H(\omega)$ of the system.

(d) Calculate the impulse response $h(n)$ of the system.

Problem 40 *Sampling and DTFT's*
Consider the functions

$$y(n) = x(nT)$$

For each example, i) sketch $x(t)$, ii) calculate $X(\omega)$ the CTFT of $x(t)$, iii) sketch $|X(\omega)|$, iv) sketch $y(n)$, v) calculate $Y(\omega)$ the DTFT of $y(n)$, vi) sketch $|Y(\omega)|$, vii) indicate if there is aliasing.

(a) $x(t) = (\text{sinc}(t))^2$ and $T = 3/8$.

(b) $x(t) = (\text{sinc}(t))^2$ and $T = 1/2$.

(c) $x(t) = (\text{sinc}(t))^2$ and $T = 5/8$.

Problem 41 *Sampling and Reconstruction*
A signal $x(t)$ is sampled at period T to form $y(n)$.

$$y(n) = x(nT)$$

The signal $y(n)$ is then used as the input to an impulse generator to form $s(t)$.

$$s(t) = \sum_{k=-\infty}^{\infty} y(n)\delta(t - kT)$$

The signal $s(t)$ is then filtered to form the final output $z(t)$ using the filter $H(\omega)$.

(a) Sketch a general function $|X(\omega)|$ which is bandlimited to $|\omega| < \frac{\pi}{T}$.

(b) Calculate $Y(\omega)$ in terms of $X(\omega)$.

(c) Sketch $|Y(\omega)|$ for a typical function $X(\omega)$.

(d) Calculate $S(\omega)$ in terms of $X(\omega)$.

(e) Sketch $|S(\omega)|$.

(f) Calculate $Z(\omega)$ in terms of $X(\omega)$.

(g) Calculate $Z(\omega)$ in terms of $X(\omega)$ assuming that $H(\omega) = T\text{rect}\,(T\omega/(2\pi))$

(h) Sketch $|Z(\omega)|$ assuming that $H(\omega) = T\text{rect}\,(T\omega/(2\pi))$

Problem 42 *Sampling and reconstruction*
Consider a sampling system

$$y(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

where $T = 1$ and $x(t)$ is a function that is band-limited to $|\omega| < \pi$. Then, consider the signal

$$z(t) = y(t) * h(t)$$

where $h(t) = \text{sinc}(t)$.

(a) Determine $Y(\omega)$ in terms of $X(\omega)$.

(b) Sketch $Y(\omega)$ for a typical function $X(\omega)$.

(c) Determine and sketch $H(\omega)$.

(d) Determine $Z(\omega)$ in terms of $X(\omega)$.

(e) Sketch $Z(\omega)$ for a typical function $X(\omega)$.

(f) Determine $z(t)$ in terms of $x(t)$.