MAT 271E Probability and Statistics

Homework 3 Solutions

Assigned: February 23, 2012

Due: February 29, 2012 (in class, before class starts)

No late homework will be accepted!

Do not copy from solutions from your classmates. All work must be your own!

Show all your steps! Just writing a number as a result is not enough. Make sure you answer everything that is asked (subquestions, etc.).

Read: "Probability and Stochastic Processes", Yates and Goodman, Ch. 2 and Ch. 3

1) The random variable H has PMF

$$p_H(h) = \begin{cases} 2c/h & ,h = 1,3,4 \\ 0 & ,\text{otherwise} \end{cases}$$

a) What is the value of the constant c?

We wish to find the value of c that makes the PMF sum up to one.

$$\sum_{h \in S_H} p_H(h) = c(2/1) + c(2/3) + c(2/4) = 2c + c(2/3) + c(1/2) = 1$$

$$\frac{19}{6}c = 1$$

$$c = 6/19$$

b) What is $P[H > 1]$?

The probability that H > 1 is

P[N > 1] = P[H = 3] + P[H = 4] = 6/19 x (2/3 + 1/2) = 6/19 x 7/6 = 7/19

2) Suppose we observe three pea seeds, and green seeds (g) and yellow seeds (y) are equally likely. Let W denote the number of yellow peas, X the number of green peas, and let $Y = W + 2X$. The sample space and the corresponding values of the random variables W, X, and Y are given in the table below.

	Outcomes	ggg	ggy	gyg	gyy	ygg	ygy	yyg	yyy
	$P[\cdot]$	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8
Random Variables	W	0	1	1	2	1	2	2	3
	X	3	2	2	1	2	1	1	0
	Y	6	5	5	4	5	4	4	0

a) For random variables W and X, calculate $p_W(w)$ ve $p_X(x)$.

$$p_W(w) = \begin{cases} 1/8 \quad , & w = 0 \\ 3/8 \quad , & w = 1 \\ 3/8 \quad , & w = 2 \\ 1/8 \quad , & w = 3 \\ 0 \quad , & \text{otherwise} \end{cases}$$

$$p_X(x) = \begin{cases} 1/8 \quad , & x = 0 \\ 3/8 \quad , & x = 1 \\ 3/8 \quad , & x = 2 \\ 1/8 \quad , & x = 3 \\ 0 \quad , & \text{otherwise} \end{cases}$$

b) Find the probability $P[W = 1]$.

$P[W = 1] = p_W(1) = 3/8$

c) Find the probability $P[W > 2]$.

$P[W > 2] = p_w(3) = 1/8$

d) Find the probability $P[X < 2]$.

$P[X < 2] = p_X(0) + p_X(1) = 1/8 + 3/8 = 4/8 = 1/2$

3) When a conventional paging system transmits a message, the probability that the message will be received by the pager it is sent to is p. To be confident that a message is received at least once, a system transmits the message n times.

a) Assuming that all transmissions are independent, what is the PMF of K, the number of times a pager receives the same message?

$$p_K(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & , k = 0,1,\ldots,n \\ 0 & , \text{otherwise} \end{cases}$$

b) Assume $p = 0.9$. What is the minimum value of n that produces a probability of 0.93 of receiving the message at least once?

Let R denote the event that the paging message was received at least once. The event R has probability

$$P[R] = P[K > 0] = 1 - P[K = 0] = 1 - (1 - p)^n$$

To ensure $P[R] \geq 0.93$ requires

$$1 - (1 - p)^n \geq 0.93$$
$$0.07 \geq (1 - 0.9)^n$$
$$0.1^n \leq 0.07$$
$$\log_{0.1}(0.07) \geq n$$

If we do a logarithm base change,

$$n \geq \ln(0.07)/\ln(0.1)$$
$$n \geq 1.1549$$

Thus, n = 2 pages would be necessary (because we can only send a whole number of pages, i.e., we cannot send 1.15 pages!).

4) The number of orders that arrive at an online retailer in T minutes is a Poisson random variable, O, with expected value $\alpha = T/8$.

a) What is the PMF of O (the number of orders that arrive in T minutes)?

Since $E[O] = \alpha = T/8$, the PMF of O, the number of orders in T minutes, is

$$P_O(o) = \begin{cases} \dfrac{(T/8)^o e^{-(T/8)}}{o!} & , o = 0, 1, 2, \ldots \\ 0 & , \text{otherwise} \end{cases}$$

b) Calculate the probability that in a three minute interval, fifteen orders will arrive.

Since T = 3,

$P[O = 15] = P_O(15) = (3/8)^{15} e^{-(3/8)} / 15! \approx 2.143 \times 10^{-19}$

c) Calculate the probability of no orders arriving in a 1-minute interval.

Since T = 1,

$P[O = 0] = P_O(0) = (1/8)^0 e^{-(1/8)} / 0! = e^{-1/8} \approx 0.882$

d) Calculate how much time you should allow so that with probability 0.97 at least one order arrives.

We want $P[O \geq 1] \geq 0.97$. We can calculate $P[O \geq 1]$ as:

$$\begin{aligned} P[O \geq 1] &= 1 - P[O = 0] \\ &= 1 - P_O(0) \\ &= 1 - (T/8)^0 e^{-(T/8)} / 0! \\ &= 1 - e^{-T/8} \end{aligned}$$

This means that the condition $1 - e^{-T/8} \geq 0.97$ must hold. We can rewrite this condition as $0.03 \geq e^{-T/8}$ or $3 \geq 100\, e^{-T/8}$.

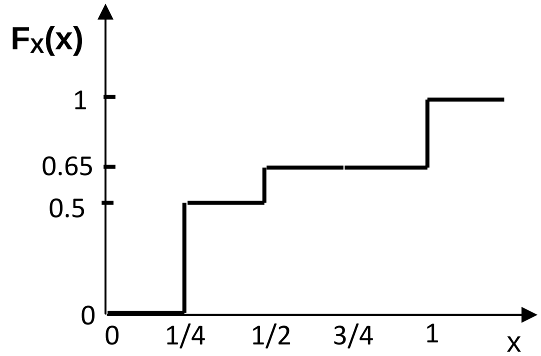
This means that $e^{T/8} \geq 100/3$.

Taking the natural logarithm of both sides,

$$T/8 \geq \ln(100/3)$$

$$T \geq 8 \ln(100/3) \approx 28 \text{ minutes}$$

5) Discrete random variable *X* has the *CDF* $F_X(x)$ as shown:



Use the CDF to find the following probabilities:

a) $P[X < 1/4]$

$P[X < 1/4] = F_X(1/4^-) = 0$

b) $P[X \leq 1/4]$

$P[X \leq 1/4] = F_X(1/4) = 0.5$

c) $P[X > 1/2]$

$P[X > 1/2] = 1 - P[X \leq 1/2] = 1 - F_X(1/2) = 1 - 0.65 = 0.35$

d) $P[X \geq 1/2]$

$P[X \geq 1/2] = 1 - P[X < 1/2] = 1 - F_X(1/2^-) = 1 - 0.5 = 0.5$

e) $P[X = 1/4]$

$P[X = 1/4] = P[X \leq 1/4] - P[X < 1/4] = F_X(1/4^+) - F_X(1/4^-) = 0.5 - 0 = 0.5$

f) $P[X = 1]$

$P[X = 1] = P[X \leq 1] - P[X < 1] = F_X(1^+) - F_X(1^-) = 1 - 0.65 = 0.35$

g) $p_X(x)$

The jumps in the CDF occur at the values that X can take on. The height of each jump equals the probability of that value. The PMF of X is

$$p_X(x) = \begin{cases} 0.5 & , \quad x = 1/4 \\ 0.15 & , \quad x = 1/2 \\ 0.35 & , \quad x = 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

6) At a pizza restaurant, a pizza sold has mushrooms with probability $p = 3/7$. On a day in which 150 pizzas are sold, let N equal the number of pizzas sold before the first pizza with mushrooms is sold.

a) Find the PMF of N.

Mushrooms occur with probability 3/7. After n pizzas without mushrooms have been sold, if the $(n + 1)$th pizza sold has mushrooms, the number of pizzas sold before this first pizza with mushroms is N = n < 150. Also, it is possible that N = 150 if all 150 pizzas are sold without mushrooms. So, the resulting PMF is

$$p_N(n) = \begin{cases} (4/7)^n(3/7) & , n = 0, 1, \dots, 149 \\ (4/7)^{150} & , n = 150 \\ 0 & , \quad \text{otherwise} \end{cases}$$

b) Find the CDF of N.

For integers n < 150, the CDF of N follows

$$F_N(n) = \sum_{i=0}^{n} p_N(i) = \sum_{i=0}^{n} (4/7)^i(3/7) = 1 - (4/7)^{n+1}$$

A complete expression for $F_N(n)$ must give a valid answer for every value of n, including non-integer values. We can write the CDF using the floor function $\lfloor x \rfloor$ which denotes the largest integer less than or equal to x. The complete expression for the CDF is

$$F_N(x) = \begin{cases} 0 & , x < 150 \\ 1 - (4/7)^{\lfloor x \rfloor + 1} & , 0 \leq x < 150 \\ 1 & , \mathrm{x} \geq 150 \end{cases}$$

7) Let K have the uniform PMF

$$p_K(k) = \begin{cases} 1/301 & , k = 1, 2, \dots, 301 \\ 0 & , \text{otherwise} \end{cases}$$

For this problem, we just need to pay careful attention to the definitions of mode and median.

a) Find a mode, x_{mode} of X. If the mode is not unique, find X_{mode}, the set of all modes of X.

The mode must satisfy $p_X(x_{mode}) \geq p_X(x)$ for all x. In the case of the uniform PMF, any integer x' between 1 and 301 is a mode of the random variable X. Hence, the set of all modes is

$$X_{mode} = \{1, 2, \dots, 301\}$$

b) Find a median, x_{median} of X. If the median is not unique, find X_{median}, the set of all numbers x that are medians of X.

The median must satisfy $P[X < x_{median}] = P[X > x_{median}]$. Since

$$P[X < 151] = P[X > 151] = 1/2$$

we observe that $x_{median} = 151$ is the median since it satisfies

$$P[X < x_{median}] = P[X > x_{median}] = 1/2$$

8) Voice calls cost 0.20 TL each and data calls cost 0.30 TL each. *C* is the cost of one telephone call. The probability that a call is a voice call is P[*V*] = 0.6. The probability of a data call is *P*[*D*] = 0.4.

a) Find $P_C(c)$, the PMF of *C*.

Since each call is either a voice or data call, the cost of one call can only take the two values associated with the cost of each type of call. Therefore, the PMF of *C* is

$$p_C(c) = \begin{cases} 0.6 & , \ c = 0.2 \\ 0.4 & , \ c = 0.3 \\ 0 & , \ \text{otherwise} \end{cases}$$

b) Find E[*C*], the expected value of *C*.

The expected cost, E[C], is simply the sum of each type of call multiplied by the probability of such a call occurring.

E[C] = 0.2(0.6) + 0.3(0.4) = 0.24 TL

9) Suppose that a cellular phone costs 20 TL per month with 30 minutes of use included and that each additional minute of use costs 0.50 TL. If the number of minutes you use the phone in a month is a geometric random variable *M* with expected value E[*M*] = 1/*p* = 30 minutes, what is the PMF of *C*, the cost of the phone for one month?

The cellular plan charges a flat rate of 20 TL per month up to and including the 30th minute, and an additional 0.50 TL for each minute over 30 minutes. Knowing that the time you spend on the phone is a geometric random variable *M* with mean 1/p = 30, the PMF of *M* is

$$p_M(m) = \begin{cases} (1-p)^{m-1}p & , m = 1, 2, \ldots \\ 0 & , \text{otherwise} \end{cases}$$

The monthly cost *C* obeys

$$p_C(20) = \mathrm{P}[M \le 30] = \sum_{m=1}^{30} (1-p)^{m-1}p = 1 - (1-p)^{30}$$

When M ≥ 30, C = 20 + 0.5(M – 30) or rearranging, M = 2C - 10. Thus,

$$p_M(m) = p_M(2c - 10), \qquad c = 20.5, 21, 21.5, \ldots$$

The complete PMF of *C* is

$$p_C(c) = \begin{cases} 1 - (1-p)^{30} & , c = 20 \\ (1-p)^{2c-10-1}p & , c = 20.5, 21, 21.5, \ldots \end{cases}$$

10) Let X have the binomial PMF

$$p_x(x) = \begin{cases} \binom{3}{x}(1/2)^3 & , x = 0, 1, 2, 3 \\ 0 & , \text{otherwise} \end{cases}$$

a) Find the standard deviation of the random variable X.

The expected value of X is

$$E[X] = \sum_{x=0}^{3} x p_X(x) = 0\binom{3}{0}\frac{1}{2^3} + 1\binom{3}{1}\frac{1}{2^3} + 2\binom{3}{2}\frac{1}{2^3} + 3\binom{3}{3}\frac{1}{2^3}$$

$$= \frac{[0+3+6+3]}{2^3} = \frac{12}{8} = \frac{3}{2}$$

The expected value of X^2 is

$$E[X^2] = \sum_{x=0}^{3} x p_X(x) = 0^2\binom{3}{0}\frac{1}{2^3} + 1^2\binom{3}{1}\frac{1}{2^3} + 2^2\binom{3}{2}\frac{1}{2^3} + 3^2\binom{3}{3}\frac{1}{2^3}$$

$$= \frac{[0+3+12+9]}{2^3} = \frac{24}{8} = 3$$

The variance of X is

$Var[X] = E[X^2] - (E[X])^2 = 3 - (3/2)^2 = 3/4$

Thus, X has standard deviation

$$\sigma_X = \sqrt{Var[X]} = \frac{\sqrt{3}}{2}$$

b) What is $P[\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X]$, the probability that X is within one standard deviation of the expected value?

The probability that X is within one standard deviation of its expected value is

$$P[\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X] = P[3/2 - \sqrt{3}/2 \leq X \leq 3/2 + \sqrt{3}/2]$$
$$= P[0.634 \leq X \leq 2.366]$$

We can use the PMF of X to calculate this probability:

$P[0.634 \leq X \leq 2.366] = p_X(1) + p_X(2) = 3/8 + 3/8 = 3/4$