Lecture 2

- **Read:** Chapter 1.6-1.9
- Independence and Disjointness
- Pairwise Independence and Mutual Independence
- Counting
- Independent Trials
- Reliability Problems
- Multiple Outcomes

Randomness and Probability

- **Recall:** We call a phenomenon random if individual outcomes are uncertain but there is a regular distribution of outcomes in a large number of repetitions
- The probability of any outcome of a random phenomenon is the proportion of times the outcome would occur in a very long series of repetitions

Probabilities in a Finite Sample Space

- If the sample space is finite, each distinct event is assigned a probability
- The probability of an event is the sum of the probabilities of the distinct outcomes making up the event
- If a random phenomenon has k equally likely outcomes, each individual outcome has probability $1/k$
- For any event A,

$$P[A] = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } S}$$

Example

- Roll a fair die and look at the face value
- Sample space: $S = \{1, 2, 3, 4, 5, 6\}$
- This is a finite sample space, and each outcome is equally likely
- That is,
$$P[X = j] = 1/6 \ , \ \forall j \in S$$
where X is the face value of the die after rolling
- $P[X \geq 5] = P[X = 5] + P[X = 6] = 1/6 + 1/6 = 1/3$
- $P[X \leq 2] = ?$

Independent Events

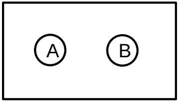
- Events A and B are independent if and only if

$$P[AB] = P[A]P[B]$$

- Always check this condition if you are asked about independence!

Are Disjoint Events Independent?

Example: Suppose $A \cap B = \emptyset$



- Then, $P[A \cap B] = P[\emptyset] = 0$
- For the independence condition ($P[A \cap B] = P[A] \cdot P[B]$) to hold, one of the events has to have probability 0

Definition: Two Independent Events

Definition 1.6: Events A and B are independent if and only if

$$P[AB] = P[A]P[B]$$

....

Equivalent definitions:

$$P[A|B] = P[A] \qquad\qquad P[B|A] = P[B]$$

Definition: Three Independent Events

Definition 1.7: A_1, A_2, and A_3 are independent if and only if

1. A_1 and A_2 are independent
2. A_2 and A_3 are independent
3. A_1 and A_3 are independent
4. $P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3]$

Independence and Disjointness

- Two events A and B are independent if knowing that one occurs does not change the probability that the other occurs. If A and B are independent, then $P[A \cap B] = P[A]P[B]$
- Events A and B are disjoint if they have no outcomes in common ($A \cap B = \emptyset$). If A and B are disjoint, then $P[A \cup B] = P[A] + P[B]$
- If A and B are disjoint, then the fact A occurs tells us that B cannot occur. So, disjoint events are not independent
- Independence cannot be shown in a Venn diagram because it involves the probabilities of the events rather than just the outcomes that make up the events
- If A and B are independent, so are A and B^c (Clearly, if B does not provide any information about A occurring, then neither does B^c)

Pairwise Independent Events

Definition: Two events A and B are independent if

$$P[A \cap B] = P[A] \cdot P[B]$$

- **Recall:** When $P[A] \neq 0$ and $P[B] \neq 0$,

$$P[A \cap B] = P[A|B] \cdot P[B] = P[B|A] \cdot P[A]$$

- So, equivalently, independence means:

$$P[A|B] = P[A] \text{ and } P[B|A] = P[B]$$

  $\Longrightarrow$ conditioning on an independent event does not change probability of the other

Remarks:

- $A \cap B = \emptyset$ does **not** mean A and B are independent (e.g., consider A and A^c)
- Independence is a property of events **and** of P

Pairwise Independent Events: Example

Example: Consider an experiment involving two rolls of a four-sided die, each outcome is equally likely with probability $1/16$

1. $A_i = \{\text{1st roll is i}\}$, $B_j = \{\text{2nd roll is j}\}$ independent? (Looks right. Yes!)
2. $A = \{\text{1st roll is 1}\}$, $B = \{\text{sum is 5}\}$ independent? (Does not look obvious, but calculate. Yes!)
3. $A = \{\text{max. of two rolls is 2}\}$, $B = \{\text{min. of two rolls is 2}\}$ independent? (Looks unlikely. No!)

Independence of a Collection of Events

- A collection $A_1, A_2, ..., A_n$ of events are mutually independent if for any finite subset S of $\{1, 2, ..., n\}$,

$$P[\cap_{i \in S} A_i] = \prod_{i \in S} P[A_i]$$

- Write out these conditions for three events: Three events are mutually independent if <u>all</u> these conditions hold:

$$\begin{aligned} P[A_1 \cap A_2 \cap A_3] &= P[A_1]P[A_2]P[A_3] \\ P[A_1 \cap A_2] &= P[A_1]P[A_2] \\ P[A_2 \cap A_3] &= P[A_2]P[A_3] \\ P[A_1 \cap A_3] &= P[A_1]P[A_3] \end{aligned}$$

- Warning: Pairwise independence $\not\Longrightarrow$ mutual independence

Independence of a Collection of Events: Example

Example: Two independent fair coin tosses

- $A = \{HT, HH\}$: First toss is H
- $B = \{TH, HH\}$: Second toss is H
- $C = \{TT, HH\}$: First toss $=$ second toss
- Verify they are pairwise independent, e.g., $P[C|A] = P[C]$
- Verify they are not mutually independent, $P[A \cap B \cap C] \neq P[A] \cdot P[B] \cdot P[C]$

Pairwise Independence and Mutual Independence

- Are there sets of random events which are pairwise independent but not mutually independent?
  Suppose a box contains 4 tickets labeled by: 112 121 211 222
  Choose one ticket at random and consider random events:
  $A_1 = \{1 \text{ occurs at the first place}\}$
  $A_2 = \{1 \text{ occurs at the second place}\}$
  $A_3 = \{1 \text{ occurs at the third place}\}$
  $P[A_1] = 1/2, P[A_2] = 1/2, P[A_3] = 1/2$
  $A_1A_2 = \{112\}, A_1A_3 = \{121\}, A_2A_3 = \{211\}$
  $P[A_1A_2] = P[A_1A_3] = P[A_2A_3] = 1/4$
  So, we conclude that the three events A_1, A_2, A_3 are pairwise independent
  However, $A_1A_2A_3 = \emptyset$
  $P[A_1A_2A_3] = 0 \neq P[A_1]P[A_2]P[A_3] = (1/2)^3$
  $\therefore$ Pairwise independence of a given set of random events does <u>not</u> imply that these events are mutually independent.

Pairwise Independence and Mutual Independence (cont.)

- So, we have seen that pairwise independence $\not\Longrightarrow$ mutual independence
- What about $\Longleftarrow$ (i.e., does mutual independence imply pairwise independence?)
  - Yes! The definition of mutual independence implies it!

Pairwise Independence and Mutual Independence (cont.)

- Suppose A, B, C are random events satisfying <u>just</u> the relation $P[ABC] = P[A]P[B]P[C]$. Does it follow that A, B, C are pairwise independent?

  Toss two different standard dice: white and black.

  The sample space S of the outcomes consists of all ordered pairs $(i, j), i, j = 1, ..., 6$, $S = \{(1,1), (1,2), ..., (6,6)\}$

  Each point in S has probability $1/36$

  $A_1 = \{\text{first die} = 1,2, \text{ or } 3\}$, $A_2 = \{\text{first die} = 3,4, \text{ or } 5\}$,

  $A_3 = \{\text{sum of faces is } 9\}$

  $A_1A_2 = \{31, 32, 33, 34, 35, 36\}$, $A_1A_3 = \{36\}$,

  $A_2A_3 = \{36, 45, 54\}$, $A_1A_2A_3 = \{36\}$

  $P[A_1] = 1/2, P[A_2] = 1/2, P[A_3] = 1/9$

  $P[A_1A_2A_3] = 1/36 = (1/2)(1/2)(1/9) = P[A_1]P[A_2]P[A_3]$

  $P[A_1A_2] = 1/6 \neq 1/4 = P[A_1]P[A_2]$

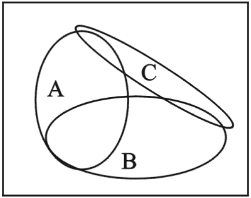
  $P[A_1A_3] = 1/36 \neq 1/18 = P[A_1]P[A_3]$

  $P[A_2A_3] = 1/12 \neq 1/18 = P[A_2]P[A_3]$

  $\therefore$ <u>Just</u> $P[ABC] = P[A]P[B]P[C]$ does not mean mutual independence. So, cannot imply pairwise independence, either.

Conditioning May Affect Independence

- Assume A and B are independent:

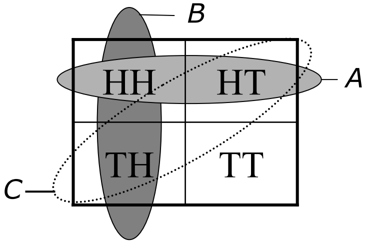


- If we are told that C occurred, are A and B independent?

Conditioning May Affect Independence

Example 1:

- Two independent fair ($p = 1/2$) coin tosses
- Event A: First toss is H
- Event B: Second toss is H
- $P[A] = P[B] = 1/2$

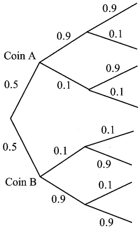


- Event C: The two outcomes are different.
- Conditioned on C, are A and B independent?

Conditioning May Affect Independence

Example 2: Choose at random among two unfair coins A and B

- $P[H|\text{coin } A] = 0.9$ and $P[H|\text{coin } B] = 0.1$
- Keep tossing the chosen coin



- Given coin A was selected, are future tosses independent? (Yes!)
- If we do not know which coin it is, are future tosses independent?
  - Compare $P[\text{toss } 11 = H]$ and $P[\text{toss } 11 = H \mid \text{first } 10 \text{ tosses are } H]$ ($\neq$!)

Conditionally Independent Events

Definition: Two events A and B are conditionally independent given an event C with probability $P[C] \geq 0$ if

$$P[A \cap B|C] = P[A|C] \cdot P[B|C]$$

- We can show this is equivalent to $P[A|B \cap C] = P[A|C]$, i.e., if C is known, the additional information that B has occurred does not change the probability of A
- **Remark:** Independence does not imply conditional independence and vice versa!

The King's Sibling

- The king comes from a family of two children
- What is the probability that his sibling is female?

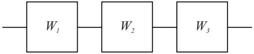
Using Independence in Modeling

Reliability: Complex systems are often simply modeled by assuming that they consists of several independent components

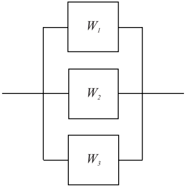
- Consider a network with n links
- For link i, event $W_i =$ link i is operational, and is independent of other components
- Probability that a link i is working well: $P[W_i] = p_i$

Using Independence in Modeling (cont.)

- What is probability that the network is reliable for communications?
  - Series connection: $P[\cap_i W_i] = p_1 p_2 \cdots p_n$
    Overall reliability is worse than any of the components

  - Parallel connection:
    $P[\text{system is up}] = 1 - P[\cap_i W_i^c] = 1-(1-p_1)(1-p_2)\cdots(1-p_n)$
    Reliability is better than any of the components, for small p_i, roughly sum

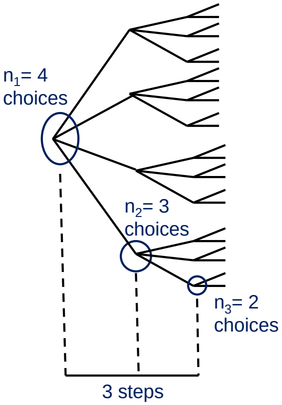
  - Mixture: Same story, can decompose the problem and reduce the series and parallel combinations

Counting

Motivation: Discrete Uniform Law: Recall that for such a law, all sample points are equally likely, and computing probabilities reduces to just counting, i.e.,

$$P[A] = \frac{\text{number of elements of } A}{\text{total number of sample points}}$$

Fundamental Principle of Counting



- r steps with n_i choices at each step i
- total number of choices$= n_1 \times n_2 \times ... \times n_r$

Fundamental Principle of Counting: Examples

- number of license plates with 3 letters followed by 4 digits$=$ $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10$
  - if repetition is prohibited: $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7$
- number of subsets of a set with n elements$= 2^n$

Permutations

- k-permutation: an ordered sequence of k distinguishable objects
- $(n)_k =$ no. of possible k-permutations of n distinguishable objects:

$$(n)_k = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

  This follows from the fundamental counting principle
- Example: number of words with 4 distinct letters$=$

$$\frac{26!}{22!} = 26 \times 25 \times 24 \times 23$$

Combinations

- k-combination:
  - Pick a subset of k out of n objects
  - Order of selection does not matter
  - Each subset is a k-combination
- **Remark:** In a combination, there is no ordering involved, e.g., 2-permutations of $\{$A, B, C$\}$ are AB, AC, BA, CA, BC, CB, while the combinations of 2 out of the 3 letters would be AB, AC, BC

How Many Combinations?

- $\binom{n}{k}$ = "n choose k" = no. of possible k-element **subsets** (i.e., order is not important) that can be obtained out of a set of n distinguishable objects
- To find $\binom{n}{k}$, we perform the following two subexperiments to assemble a *k-permutation* of n distinguishable objects:
  1. Choose a *k-combination* out of n objects.
  2. Choose a *k-permutation* of the k objects in the *k-combination*.

$$(n)_k = \binom{n}{k} k! \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Example: Independent Trials and Binomial Probabilities

- n independent coin tosses, $P[H] = p$
- What is the probability of obtaining an n-sequence with k heads?
- $P[HTTHHH] = p^4(1-p)^2$
- P[sequence]$=$ p$^{\#heads} \cdot (1-p)^{\#tails}$

$$
\begin{aligned}
P[k \text{ heads}] &= \sum_{k-\text{head seq.}} P[seq.] \\
&= p^k \cdot (1-p)^{n-k} \cdot (\# \text{ of k-head seqs.}) \\
&= \binom{n}{k} p^k \cdot (1-p)^{n-k}
\end{aligned}
$$

Partitions

How many ways are there to divide a set of n distinct elements into r disjoint sets of $n_1, n_2, ..., n_r$ elements each, with $n_1, n_2, ..., n_r \leq n$?

- A combination of k elements out of n breaks up into two disjoint sets of elements of size k and $n - k$
- Note: $\binom{n}{k} = \#$ of ways of breaking n elements into subsets of size k and $n - k$ each
- Using the counting principle, we can answer the above question as

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\dots\binom{n-n_1-n_2-\dots-n_{r-1}}{n_r}$$
$$= \frac{n!}{n_1!n_2!\dots n_r!} = \binom{n}{n_1, n_2, \dots n_r}$$

This is called the multinomial coefficient

Partitions: Example

- Anagrams: How many letter sequences can be obtained by rearranging the word TATTOO?
- Think of an anagram for this word as including 6 slots, which need to be filled with the letters T, A, or O
- Each solution corresponds to selecting 3 slots for T, 2 slots for O, and 1 slot for A
- How many ways are there to divide the 6 slots into such a partition?
- **Answer:** $\frac{6!}{1!2!3!} = 60$
- Note: The letters are not distinct, but the slots are!

Problem 1.8.6

- A basketball team has
  - 3 pure centers, 4 pure forwards, 4 pure guards
  - one swingman who can play either guard or forward
- A pure player can play only the designated position
- How many lineups are there (1 center, 2 forwards, 2 guards)?

Problem 1.8.6 Solution

Three possibilities:

1. swingman plays guard: N_1 lineups
2. swingman plays forward: N_2 lineups
3. swingman does not play: N_3 lineups

$$N = N_1 + N_2 + N_3$$

Problem 1.8.6 Solution (cont.)

$$N_1 = \binom{3}{1}\binom{4}{2}\binom{4}{1} = 72$$

$$N_2 = \binom{3}{1}\binom{4}{1}\binom{4}{2} = 72$$

$$N_3 = \binom{3}{1}\binom{4}{2}\binom{4}{2} = 108$$

Multiple Outcomes

- n independent trials
- r possible trial outcomes $(s_1, ..., s_r)$
- $P[\{s_k\}] = p_k$

Multiple Outcomes (2)

- Outcome is a sequence:
  - Example: $s_3 s_4 s_3 s_1$

$$
\begin{aligned}
P[s_3 s_4 s_3 s_1] = p_3 p_4 p_3 p_1 &= p_1 p_3^2 p_4 \\
&= p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}
\end{aligned}
$$

- Probability depends on how many times each outcome occurred

Multiple Outcomes (3)

$N_i =$ no. of times s_i occurs

$$P[N_1 = n_1, \ldots, N_r = n_r] = M p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}$$

$$\begin{aligned} M &= \text{Multinomial Coefficient} \\ &= \frac{n!}{n_1!n_2!\ldots n_r!} \end{aligned}$$

Service Facility Design

- c: service capacity of a facility
- n: # of customers it is assigned
- p: probability customer requires service
- $N =$ number of customers requiring service
- Given n and p, choose c
- Criterion?
  - We establish a probability p such that $P[N > c] < p$
  - We choose the smallest c such that $P[N > c]$ is still $< p$

Card Play

- 52-card deck, dealt to 4 players, i.e., 13 cards each
- Find P[each gets an ace]

- Count size of the sample space: Partition the 52 card deck into 4 sets of 13 cards each
$$\binom{52}{13,13,13,13} = \frac{52!}{13!13!13!13!}$$
- Count number of ways of distributing the four aces (One ace in each player's hand)
$$4 \times 3 \times 2$$
- Count number of ways of dealing the remaining 48 cards (Each hand has an additional 12 cards)
$$\binom{48}{12,12,12,12} = \frac{48!}{12!12!12!12!}$$
- So, P[each gets an ace] =
$$\frac{\#\text{ hands with one ace each}}{\#\text{ hands in sample space}} = \frac{4 \times 3 \times 2 \cdot \frac{48!}{12!12!12!12!}}{\frac{52!}{13!13!13!13!}}$$

N People

- N people in the class
- Each person can pick heads or tails
- A person will win if only $\underline{1}$ person selects heads

  $\rightarrow$ Ethernet network $\rightarrow$ what is the optimal strategy?

Three Doors

- Three doors, one has a prize behind it
- You pick one, what is the chance you win?
- After you pick your door, I open one of the others and show you there is no prize behind it
- Should you change your choice? (Yes, you should change!)
- If so, how? What is the probability you win?