Probability Theory and Stochastic Processes

Fall 2012 - Exam 1

1. a) $E[Y/X] = \int y \, f_{Y/X}(y/x)\, dy$

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$$f_{Y/X}(y/x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

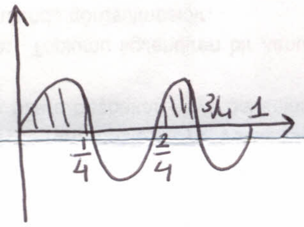
$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}\, dy = \int_{x}^{x+1} 3x^2\, dy = 3x^2\,(x+1-x) = 3x^2$$

$$\therefore\; f_{Y/X}(y/x) = \frac{3x^2}{3x^2} = 1$$

$$E[Y/X] = \int_{x}^{x+1} y \cdot 1\, dy = \left.\frac{y^2}{2}\right|_{x}^{x+1} = \frac{(x+1)^2}{2} - \frac{x^2}{2} = x + \frac{1}{2} \checkmark$$

b)

4



$$P[\sin(4\pi X) > 0]$$

$$= \int_{0}^{\frac{1}{4}} 3x^2\, dx + \int_{\frac{2}{4}}^{3/4} 3x^2\, dx$$

$$= \left. x^3 \right|_{0}^{\frac{1}{4}} + \left. x^3 \right|_{\frac{2}{4}}^{\frac{3}{4}} = \left(\frac{1}{4}\right)^3 + \left(\frac{3}{4}\right)^3 - \left(\frac{2}{4}\right)^3$$

$$= \frac{1+27-8}{64}$$

$$= \frac{20}{64} = \frac{5}{16} \checkmark$$

2. $$P(X>180 \,/\, 170<X<185) = \frac{P(X>180) \cap P(170<X<185)}{P(170<X<185)}$$

8

$$= \frac{P(180<X<185)}{P(170<X<185)} = \frac{\Phi(1.5) - \Phi(1)}{\Phi(1.5) - \Phi(0)} \checkmark$$

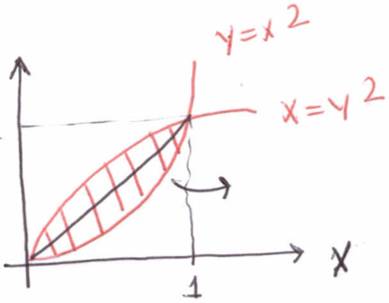
$$P(180<X<185) = \int_{180}^{185} \frac{1}{\sqrt{2\pi}\cdot 10}\, e^{-\frac{(x-170)^2}{2(10)^2}}\, dx \qquad \frac{x-170}{10} = y \qquad dx = 10\,dy$$

$$= \int_{1}^{1.5} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{y^2}{2}}\, dy = \Phi(1.5) - \Phi(0)$$

$$P(170<X<185) = \int_{170}^{185} \frac{1}{\sqrt{2\pi}\cdot 10}\, e^{-\frac{(x-170)^2}{2\cdot(10)^2}}\, dx = \Phi(1.5) - \Phi(0)$$

3. a)

6



$$P[A] = \int_0^1 \int_{y^2}^{\sqrt{y}} 4xy \, dx\, dy$$

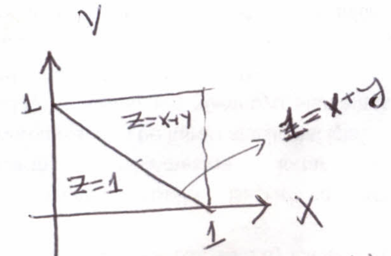
$$= \int_0^1 \Big|_{y^2}^{\sqrt{y}} 2x^2 y \quad dy$$

$$= \int_0^1 (2y^2 - 2y^5)\, dy$$

$$= \Big|_0^1 \left(\frac{2y^3}{3} - \frac{y^6}{3}\right) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$ ✓

b)

6



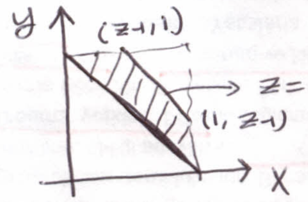
$$Z = \begin{cases} 1; & x+y \le 1 \\ x+y; & x+y > 1 \end{cases} \qquad R_Z = [1, 2]$$

for $z = 1$;
$$F_Z(z) = \int_{y=0}^{1} \int_{x=0}^{1-y} 4xy \, dx\, dy = \int_0^1 \Big|^{1-y} 2x^2 y \quad dy$$

$$= \int_0^1 2(1-y)^2 y \, dy = \int_0^1 2(1-2y+y^2) y \, dy$$

$$= \Big|_0^1 y^2 - \frac{4y^3}{3} + \frac{y^4}{2} = 1 - \frac{4}{3} + \frac{1}{2} = \frac{1}{6}$$ ✓

for $1 < z \le 2$;



$$F_Z(z) = P(Z \le z) = P(x+y \le z)$$

$$= 1 - P(x+y > z)$$

$$\Rightarrow F_Z(z) = \frac{5}{6}1 - \int_{y=z-1}^{1} \int_{x=z-y}^{1} 4xy \, dx\, dy$$

$$= \frac{5}{6}1 - \int_{z-1}^{1} \Big|_{z-y}^{1} 2x^2 y \quad dy = \frac{5}{6} - \int_{z-1}^{1} \left(2y - 2(z-y)^2 y\right) dy$$

$$= \frac{5}{6}1 - \int_{z-1}^{1} (2y - 2z^2 y + 4zy^2 - 2y^3)\, dy = \frac{5}{6} - \Big|_{z-1}^{1} y^2 - y^2 z^2 + \frac{4zy^3}{3} - \frac{y^4}{2}$$

$$= \frac{5}{6}1 - \Big[1 - z^2 + \frac{4z}{3} - \frac{1}{2} - \Big(z^2 - 2z + 1 - z^4 + 2z^3 - z^2 + \frac{4}{3} z (z^3 - 3z^2 + 3z - 1) - (z^2 - 2z + 1)(z^2 - 2z + 1)\Big)\Big]$$

$$= \cancel{\tfrac{5}{6}}1 - \left[\cancel{1} - \cancel{z^2} + \cancel{\tfrac{4z}{3}} - \cancel{\tfrac{1}{2}} - \cancel{z^2} + \cancel{2z} - \cancel{1} + \cancel{z^4} - \cancel{2z^3} + \cancel{z^2} - \tfrac{4}{3}z^4 + \cancel{4z^3} - \cancel{4z^2} + \cancel{\tfrac{4z}{3}}\right.$$

$$\left. + z^4 - \cancel{2z^3} + \cancel{z^2} - \cancel{2z^3} + \cancel{4z^2} - \cancel{2z} + \cancel{z^2} - \cancel{2z} + \cancel{1}\right]$$

$$= 1\frac{5}{6} - \left[\frac{3}{6} + \frac{2}{3}z^4 - 2z^3 + z^2 + \frac{2z}{3}\right] = \frac{1}{3} - \frac{2z^4}{3} + 2z^3 - z^2 - \frac{2z}{3}$$

$$\frac{1}{3} - \frac{2}{3} + 2 + 1 - \frac{2}{3}$$

$$\frac{4}{3} - \frac{2}{3} - \frac{2}{3} = 0$$

$$f_Z(z) = \frac{-8z^3}{3} + 6z^2 - 2z - \frac{2}{3} \quad 1 \leq z \leq 2$$

$$\frac{1}{3} - \frac{2}{3}\cdot 16 + 2\cdot 8 - 4 - \frac{4}{3}$$

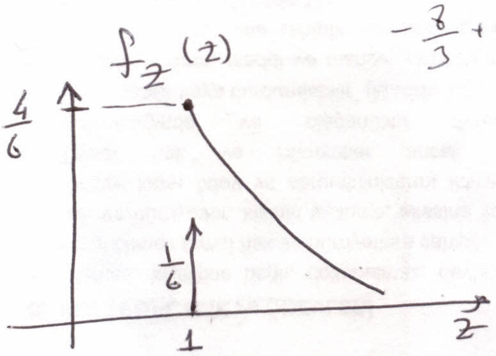
$$-\frac{8}{3} + 6 - 2 - \frac{2}{3}$$

$$\frac{1}{3} + \frac{16}{3} - \frac{16}{3} = \frac{1}{3} \; \frac{2}{6}$$

$$\frac{10}{3} - 6 - 2$$

$$-\frac{8}{3}\cdot 8 + 6\cdot 4 - 4 - \frac{2}{3}$$

$$\frac{8}{3} - \frac{64}{3} + \frac{72}{3} - \frac{14}{3}$$



$$f_Z(z) = \frac{1}{6}\delta(z-1) - \left(\frac{8}{3}z^3 - 6z^2 + 2z + \frac{2}{3}\right)u(z-1)$$

$$E[Z] = \frac{1}{6}\int_{-\infty}^{1} z\,\delta(z-1)\,dz + \int_{1}^{2} z\left(\quad\right)\,dz$$

$$= \frac{1}{6}\cdot 1 + \quad -$$

$$=$$

4-8

$Y=0 \rightarrow 0 \leq \frac{X}{2} < 1 \Rightarrow 0 \leq X < 2 \qquad P[Y=0] = \frac{2}{5}$

$Y=1 \rightarrow 1 \leq \frac{X}{2} < 2 \Rightarrow 2 \leq X < 4 \qquad P[Y=1] = \frac{2}{5}$

$Y=2 \rightarrow 2 \leq \frac{X}{2} < 3 \Rightarrow 4 \leq X < 5 \qquad P[Y=2] = \frac{1}{5}$

4 a) $\frac{5!}{2!3!}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^3 = \frac{\cancel{5}\cdot 4\cdot 2}{\cancel{2}}\cdot\frac{1}{25}\cdot\frac{64}{\cancel{125}\,25} = \frac{128}{625}$

4 b) $\dfrac{\frac{5!}{1!2!2!}\left(\frac{2}{5}\right)^1\left(\frac{1}{5}\right)^2\left(\frac{2}{5}\right)^2}{128/625} = \dfrac{\frac{5\cdot4\cdot3}{2}\cdot\frac{2}{25}\cdot\frac{4}{25}}{128/625} = \frac{128}{240} = \frac{16}{30} = \frac{8}{15}$