1	2	3	4	Total

Name:
Number:

KOM505E - Probability Theory and Stochastic Processes
Midterm #2

Dec. 10, 2015

Rules:

- Closed book & notes.
- Cell phones are not allowed.
- This exam will count for 20% of your final grade.
- Duration: 110 min.

1. (25 pts) Consider two random variables X and Y. Prove that if $Y = aX + b$ where a and b are constants, then the correlation coefficient ρ is given as

$$\rho_{X,Y} = \begin{cases} 1 & \text{if } a > 0 \\ -1 & \text{if } a < 0 \end{cases}$$

$$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}$$

$$Cov(X,Y) = E\{(X-\mu_X)(Y-\mu_Y)\}$$

$$\mu_Y = E\{aX+b\} = aE\{X\}+b$$
$$= a\mu_X + b$$

then

$$Cov(X,Y) = E\{(X-\mu_X)\underbrace{(aX+b-a\mu_X-b)}_{a(X-\mu_X)}\}$$

$$= E\{a(X-\mu_X)^2\} = a\sigma_X^2$$

In addition:

$$\sigma_Y^2 = a^2\sigma_X^2 \Rightarrow \sigma_Y = |a|\sigma_X$$

Then

$$\rho_{XY} = \frac{a\sigma_X^2}{\sigma_X \cdot |a| \cdot \sigma_X} = \frac{a}{|a|} = sgn(a) = \begin{cases} +1 & \text{if } a>0 \\ -1 & \text{if } a<0 \end{cases}$$

2. (25 pts) If a joint pdf is given as $p_{X,Y}(x,y) = (\frac{1}{4})^2 \exp\{-\frac{1}{2}(|x|+|y|)\}$ for $-\infty < x, y < \infty$. New random variables W and Z are obtained from X and Y as follows:

$$\begin{bmatrix} W \\ Z \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

(a) What is the range of W and Z?

(b) Find the joint pdf of W and Z.

a) As $\left.\begin{matrix} -\infty < x < \infty \\ -\infty < y < \infty \end{matrix}\right\}$ $\begin{matrix} W = 2X + 1Y \Rightarrow & -\infty < w < \infty \\ Z = 3X + 2Y \Rightarrow & -\infty < z < \infty \end{matrix}$

b) $G = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \Rightarrow |G| = |G^{-1}| = 1$

$G^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} W \\ Z \end{bmatrix}$

Hence

$$f_{W,Z}(w,z) = \left(\frac{1}{4}\right)^2 \exp\left\{-\frac{1}{2}\left(|2W - Z| + |-3W + 2Z|\right)\right\} \qquad -\infty < w, z < \infty$$

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3. (25 pts) Consider two discrete random variables X and Y, whose joint PMF are given below:

	$Y=-8$	$Y=0$	$Y=8$
$X=-6$	1/12	1/6	1/12
$X=0$	1/6	1/12	1/12
$X=6$	1/12	1/12	K

(a) Find the value of K.

(b) Find the conditional probability of $P(X=0|Y=8)$.

(c) Are X and Y independent? Show your work.

(d) Find the expected value of X and Y.

(e) Find the covariance matrix C of (X, Y).

(f) Now, assuming that the covariance matrix of X and Y are given by (for the sake of easy computations):

$$C = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix},$$

find a matrix A that transforms X and Y to uncorrelated random variables W and Z:

$$\begin{bmatrix} W \\ Z \end{bmatrix} = A \begin{bmatrix} X \\ Y \end{bmatrix}$$

a) The values in the table should add upto 1.

Hence $K = 1/6$

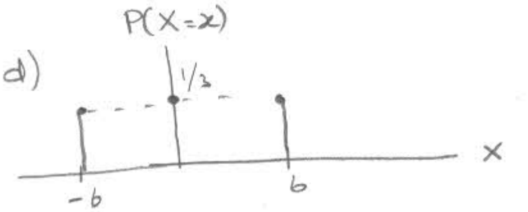
b) $P(X=0|Y=8) = \dfrac{P(X=0, Y=8)}{P(Y=8)} = \dfrac{1/12}{1/3} = 1/4$

$P(Y=8) = \sum_{k\in\{-6,0,6\}} P(X=k, Y=8) = 1/12 + 1/12 + 1/6 = 1/3$

c) $P(X=0) = \sum_{k\in\{-8,0,8\}} P(X=0, Y=k) = 1/6 + 1/12 + 1/12 = 1/3$

$P(X=0|Y=8) \neq P(X=0) \Rightarrow X$ & Y are NOT independent.

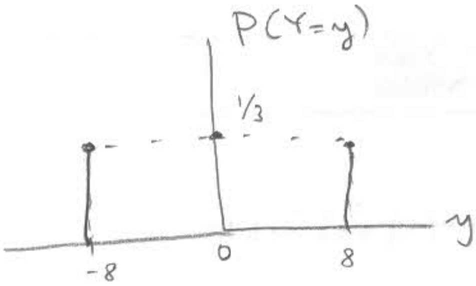
d)



$\Rightarrow E[X] = \frac{1}{3}(-6) + \frac{1}{3}(0) + \frac{1}{3}6 = 0$

$\mu_X = 0$

Similarly



$$E[Y] = \frac{1}{3}(-8) + \frac{1}{3}(0) + \frac{1}{3}(8)$$

$$\mu_Y = 0$$

e) $\mathrm{Cov}(X,Y) = E((X-\mu_X)(Y-\mu_Y)) = E[XY]$

$$= \frac{1}{12}(-6.-8) + \frac{1}{12}(-6.8) + \frac{1}{12}(6.-8) + \frac{1}{12}(6.8)$$

$$= +4 - 4 - 4 + 4 = 0$$

$$\sigma_X^2 = E[X^2] = \frac{1}{3}(-6)^2 + \frac{1}{3}0^2 + \frac{1}{3}6^2 = 24$$

$$\sigma_Y^2 = E[Y^2] = \frac{1}{3}(-8)^2 + \frac{1}{3}0^2 + \frac{1}{3}8^2 = 128/3$$

Then

$$C = \begin{bmatrix} 24 & 0 \\ 0 & 128/3 \end{bmatrix}$$

f) $C = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$ $\quad \det(\lambda I - C) = \begin{vmatrix} 6-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = (6-\lambda)(3-\lambda) - 4 = 0$

$$= 18 - 9\lambda + \lambda^2 - 4 = 0$$

$$(\lambda - 2)(\lambda - 7) = 0$$

$\left.\begin{matrix}\lambda = 2 \\ \lambda = 7\end{matrix}\right\}$ eigenvalues of C

For $\lambda = 2$ $\quad \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 2\begin{bmatrix} a \\ b \end{bmatrix}$ $\quad 6a + 2b = 2a$, $\; b = -2a$ $\quad v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda = 7$ $\quad \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 7\begin{bmatrix} a \\ b \end{bmatrix}$ $\quad 6a + 2b = 7a$, $\; 2b = a$ $\quad v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Hence $\quad A = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

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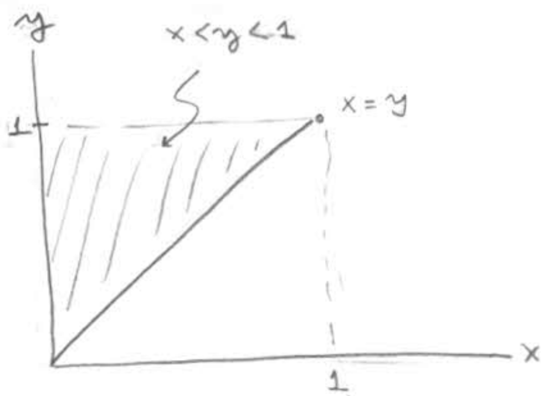
4. (25 pts) The joint pdf $p_{X,Y}$ of two random variables X and Y are given as uniform over the region $0 < y < 1$ and $0 < x < y$, and zero otherwise.

   (a) Find the conditional PDF $p_{Y|X}(y|x)$.

   (b) Find the minimum mean square error (MMSE) estimate Y given $X = x$, assuming a nonlinear prediction model.

   (c) Plot the estimated value of $\hat{Y}$ versus x, by also indicating the region in the $x - y$ plane for which the joint PDF is nonzero.

   (d) Based on the plot of the predictor, comment on the intuition behind this predictor.



$$f_{XY}(x,y) = \begin{cases} 2 & \text{in the shaded area} \\ 0 & \text{otherwise} \end{cases}$$

a) $$f_X(x) = \int_x^1 f_{XY}(x,y)\,dy = 2y\Big|_x^1 = 2(1-x) \qquad 0<x<y$$

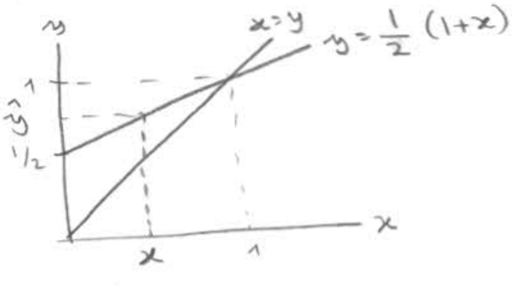
$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{2}{2(1-x)} = \frac{1}{1-x} \qquad \begin{matrix} 0<x<y \\ x<y<1 \end{matrix}$$

b) $$\hat{Y} = E_{Y|X}[Y|X] = \int_x^1 y \cdot f_{Y|X}(y|x)\,dy$$

$$= \int_x^1 \frac{y}{1-x}\,dy = \frac{1}{1-x} \frac{y^2}{2}\Big|_x^1 = \frac{1}{2}\left(\frac{1}{1-x} \cdot (1-x^2)\right)$$

$$= \frac{1}{2}(1+x)$$

c) Hence $$\hat{Y} = \frac{1}{2}(1+x)$$



d) Given an x value, the predictor y picks the <u>average of all possible y values.</u>

$$\hat{Y} = E_{Y|X}(y|x)$$