Practice Questions for Final Exam

14/05/07

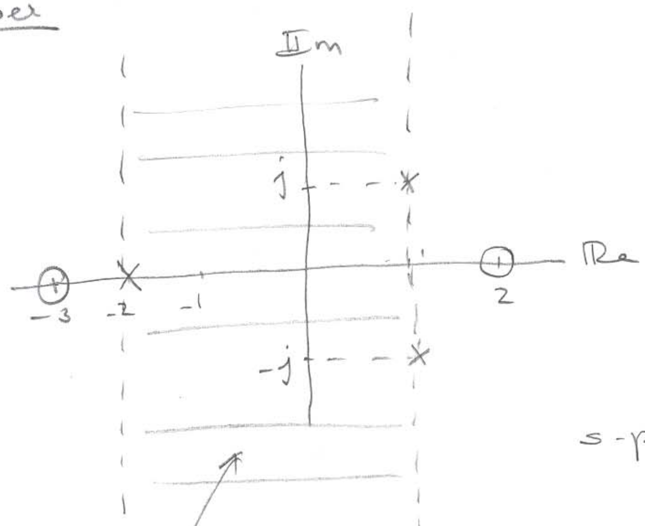
1) Consider the Laplace transform $X(s)$ has two zeros located at $s=-3$ and $s=2$. It has three poles at $s=-2$, $s=1-j$, and $s=1+j$.

a) Plot pole-zeros of $X(s)$ and show its ROC such that Fourier transform of $x(t)$ exists.

b) Write $X(s)$

Answer

a)



Pole-zero plot of $X(s)$

The ROC should contain $Re\{s\}=0$ (vertical axis). There therefore $-2<Re\{s\}<1$ is the ROC.

b)

$$X(s) = \frac{K(s-2)(s-3)}{(s+2)(s-1-j)(s-1+j)} = \frac{K(s^2-5s+6)}{(s+2)(s^2-2s+2)}$$

2. Assume $$X(s) = \frac{s-3}{(s-1)^2(s+2)}$$

Find all possible ROC and compute the corresponding $x(t)$ for each ROC.

Answer: $$X(s) = \frac{A}{s+2} + \frac{B}{(s-1)^2} + \frac{C}{s-1}$$ partial fraction expansion

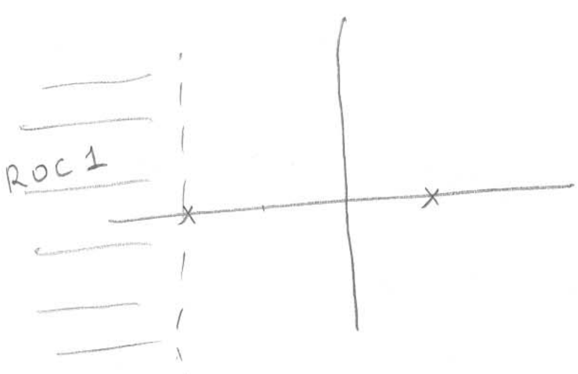
$$A = \left.\frac{s-3}{(s-1)^2}\right|_{s=-2} = -5/9$$

$$B = \left.\frac{s-3}{s+2}\right|_{s=1} = -2/3$$

$$C = \left.\frac{d}{ds}\frac{s-3}{s+2}\right|_{s=1} = 5/9$$

$$X(s) = \frac{-5/9}{s+2} + \frac{-2/3}{(s-1)^2} + \frac{5/9}{(s-1)}$$

Possibility 1

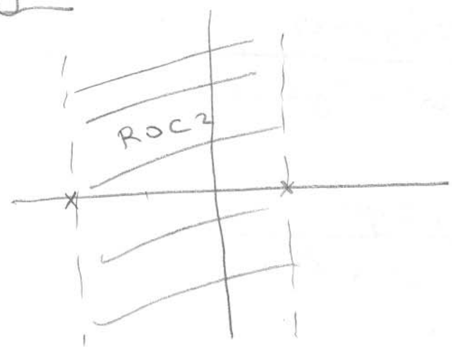


$x(t)$ is a left-sided signal

then

$$x(t) = \frac{5}{9}e^{-2t}u(-t) + \frac{2t}{3}e^{t}u(-t) - \frac{5}{9}e^{t}u(-t)$$

Possibility 2

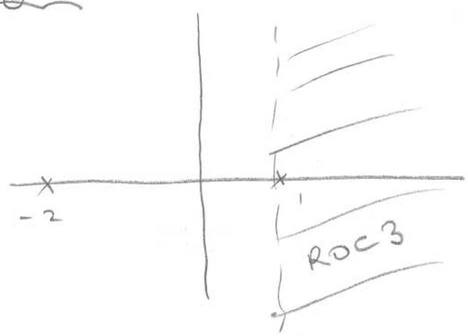


then $x(t)$ is a two-sided signal

$$x(t) = -\frac{5}{9} e^{-2t} u(t) + \frac{2t}{3} e^{t} u(-t)$$

$$-\frac{5}{9} e^{t} u(-t)$$

Possibility 3



$x(t)$ is a right sided signal

$$x(t) = -\frac{5}{9} e^{-2t} u(t) - \frac{2t}{3} e^{t} u(t)$$

$$+ \frac{5}{9} e^{t} u(t)$$

(3) $x(t) \xleftrightarrow{\mathcal{L}} X(s) = \dfrac{2(s+2)}{s^2 + 7s + 12}$

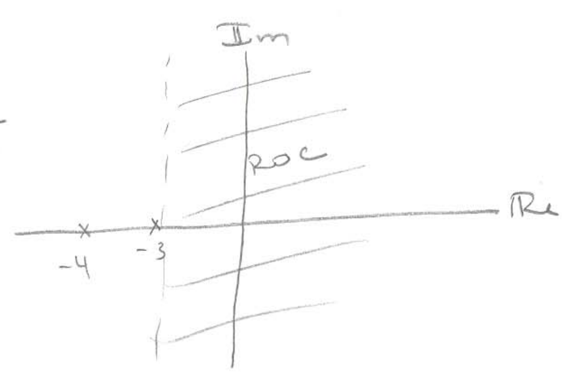
a) Find the ROC of $X(s)$ such that the Fourier transform of $x(t)$ exists

b) Write the Fourier transform of $x(t)$

c) According to the ROC of part (a), compute $x(t)$

Answers

a) $X(s) = \dfrac{2(s+2)}{(s+4)(s+3)}$



* $\Re\{s\} > -3$ includes the $\Re\{s\} = 0$ axis

b). $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$

$$X(j\omega) = X(s)\Big|_{s=j\omega} = \frac{2(j\omega+2)}{-\omega^2+7j\omega+12}$$

$$= \frac{4+2j\omega}{(12-\omega^2)+j7\omega}$$

c) Using partial fraction expansion

$$X(s) = \frac{A}{s+4} + \frac{B}{s+3}$$

$$\left.\begin{aligned} A &= \frac{2(s+2)}{s+3}\Big|_{s=-4} = 4 \\ B &= \frac{2(s+2)}{s+4}\Big|_{s=-3} = -2 \end{aligned}\right\} \quad X(s) = \frac{4}{s+4} - \frac{2}{s+3}$$

ROC $\Re e\{s\} > -3$ (from part-a)

then

$$x(t) = 4e^{-4t}u(t) - 2e^{-3t}u(t)$$

4 The system function of a causal LTI system is

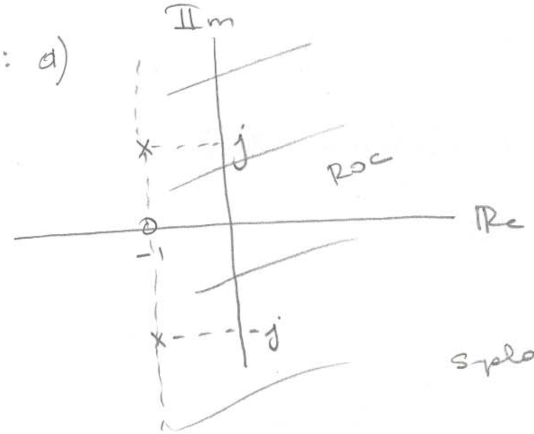
$$H(s) = \frac{s+1}{s^2+2s+2}$$

a) Plot pole-zero graph and show ROC.

b) If the input to this system is $x(t) = e^{-|t|}$ $\forall t$, then

find the output of the system.

Answer: a)



* The zero of the system is at $s=-1$, the poles are at $s=-1 \pm j$

* Since the system is causal ROC is $\mathrm{Re}\{s\} > -1$

b) $$e^{-|t|} = e^{t}u(-t) + e^{-t}u(t)$$

$$= \frac{-1}{s-1} + \frac{1}{s+1} = \frac{-2}{(s+1)(s-1)}$$

ROC $-1 < \mathrm{Re}\{s\} < 1$

$$Y(s) = H(s)\,X(s)$$

$$= \frac{\cancel{(s+1)}}{s^2+2s+2} \cdot \frac{-2}{\cancel{(s+1)}(s-1)}$$

Using partial fraction expansion

$$Y(s) = -\frac{2/5}{s-1} + \frac{2s/5 + 6/5}{s^2+2s+2}$$

This can be rewritten as

$$Y(s) = \frac{-2/5}{s-1} + \frac{2}{5}\frac{s+1}{(s+1)^2+1} + \frac{4}{5}\frac{1}{(s+1)^2+1}$$

ROC $-1 < \mathrm{Re}\{s\} < 1$

Using table 9.2 from textbook

$$y(t) = \frac{2}{5}e^{t}u(-t) + \frac{2}{5}e^{-t}\cos t\; u(t) + \frac{4}{5}e^{-t}\sin t\; u(t)$$

(5) $$x(t) = \sum_{n=0}^{\infty} e^{-nT}\,\delta(t-nT)$$

a) Compute its Laplace transform, $X(s)$

b) Find the poles of $X(s)$

Answer: a) $$X(s) \triangleq \int_{-\infty}^{\infty} \left(\sum_{n=0}^{\infty} e^{-nT}\,\delta(t-nT) \right) e^{-st}\,dt$$

Let's rearrange this by changing the order of the integral and summation.

$$X(s) = \sum_{n=0}^{\infty} e^{-nT} \int_{-\infty}^{\infty} \delta(t-nT)\,e^{-st}\,dt$$

Recall that
$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

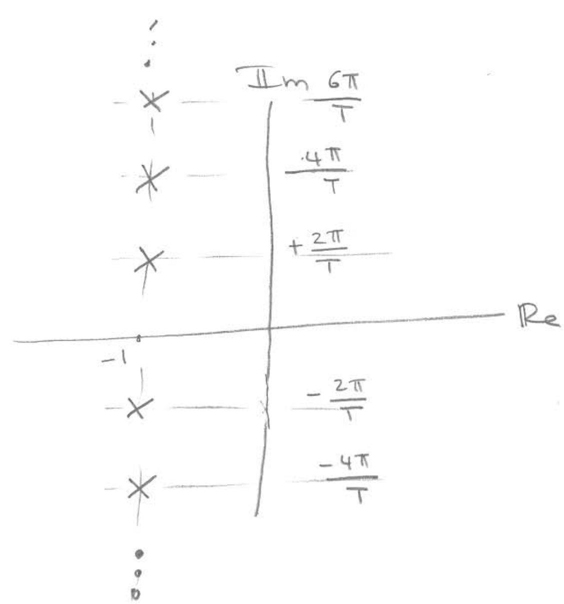
$$= \sum_{n=0}^{\infty} e^{-nT}\,e^{-snT} = \frac{1}{1-e^{-T(1+s)}}$$

b) The poles are located at

$$e^{-T(1+s)} = 1$$

then $$-T(1+s) = j\,2\pi k$$

$$s = -1 - j\,\frac{2\pi k}{T}$$



(6) Find the unilateral Laplace transform of

$$x(t) = \delta(t+1) + \delta(t) + e^{-2(t+3)}\, u(t+1)$$

Answer

$$\mathcal{X}(s) = \int_{0^-}^{\infty} \left\{ \delta(t+1) + \delta(t) + e^{-2(t+3)}\, u(t+1) \right\} e^{-st}\, dt$$

$$= \underbrace{\cancel{\int_{0^-}^{\infty} \delta(t+1)\, e^{-st}\, dt}}_{0} + \underbrace{\cancel{\int_{0^-}^{\infty} \delta(t)\, e^{-st}\, dt}}_{1} + \underbrace{\int_{0^-}^{\infty} e^{-2(t+3)}\, e^{-st}\, dt}_{\frac{e^{-6}}{s+2}}$$

$$= 1 + \frac{e^{-6}}{s+2}$$

(7) Let $X(z) = \dfrac{1 + z^{-1}}{1 + \frac{1}{3} z^{-1}}$

a) If ROC is $|z| > 1/3$, find $x[0]$, $x[1]$ and $x[2]$

b) " $|z| < 1/3$, " "

Answer:

a)

$$\begin{array}{l|l}
1 + z^{-1} & 1 + \frac{1}{3} z^{-1} \\ \hline
1 + \frac{1}{3} z^{-1} & 1 + \frac{2}{3} z^{-1} + \frac{-2}{9} z^{-2} \\ \hline
\frac{2}{3} z^{-1} & \\
\frac{2}{3} z^{-1} + \frac{2}{9} z^{-2} & \\ \hline
-\frac{2}{9} z^{-2} & \\
-\frac{2}{9} z^{-2} - \frac{2}{27} z^{-3} & \\ \hline
2/27\, z^{-3} &
\end{array}$$

$$\Rightarrow X(z) = \frac{1 + z^{-1}}{1 + \frac{1}{3} z^{-1}} = 1 + \frac{2}{3} z^{-1} - \frac{2}{9} z^{-2} + \cdots$$

then $x[0] = 1$

$x[1] = 2/3$

$x[2] = -2/9$

b) Since ROC is $|z| < 1/3$, the signal is left sided

$$
\begin{array}{r|l}
z^{-1} + 1 & \frac{1}{3} z^{-1} + 1 \\
\underline{z^{-1} + 3} & \overline{3 - 6z + 18z^2} \\
-2 & \\
\underline{-2 - 6z} & \\
6z & \\
\underline{6z + 18z^2} & \\
-18z^2 &
\end{array}
$$

then

$$X(z) = \frac{1 + z^{-1}}{1 + \frac{1}{3} z^{-1}} = 3 - 6z + 18z^2 + \cdots$$

$x[0] = 3$

$x[1] = -6$

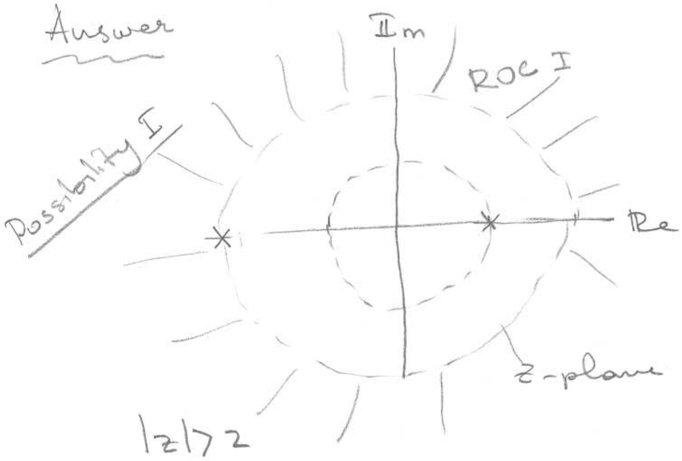
$x[2] = 18$

(8) Find the inverse z-transform of

$$X(z) = \frac{1 - \frac{1}{3} z^{-1}}{(1 - z^{-1})(1 + 2z^{-1})}$$

for all possible ROC. Does the DTFT exist for $x[n]$ for any of these ROC possibilities?

Answer



Right sided signal

Partial fraction expansion

$$X(z) = \frac{A}{1 - z^{-1}} + \frac{B}{1 + 2z^{-1}}$$

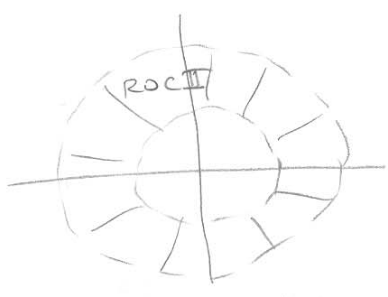
$$A = \left.\frac{1 - \frac{1}{3} z^{-1}}{1 + 2z^{-1}}\right|_{z^{-1}=1} = \frac{2/3}{3} = 2/9$$

$$B = \left.\frac{1 - \frac{1}{3} z^{-1}}{1 - z^{-1}}\right|_{z^{-1}=-1/2} = \frac{1 + 1/6}{3/2} = 7/9$$

For ROC-I the signal is right sided

$$x[n] = \frac{2}{9} u[n] + \frac{7}{9} (-2)^n u[n]$$

Possibility II

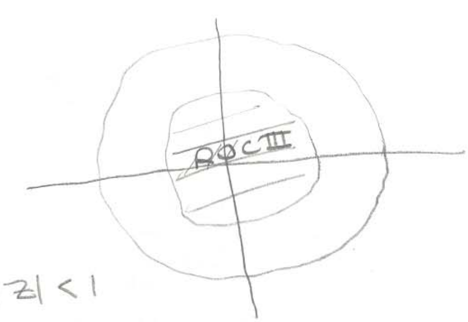


$1 < |z| < 2$

The signal is two-sided

$$x[n] = \frac{-7}{9} (-2)^n u[-n-1] + \frac{2}{9} u[n]$$

Possibility III



$|z| < 1$

The signal is left sided

$$x[n] = \frac{-2}{9} u[-n-1] - \frac{7}{9} (-2)^n u[-n-1]$$

Note that none of these ROC include unit circle,

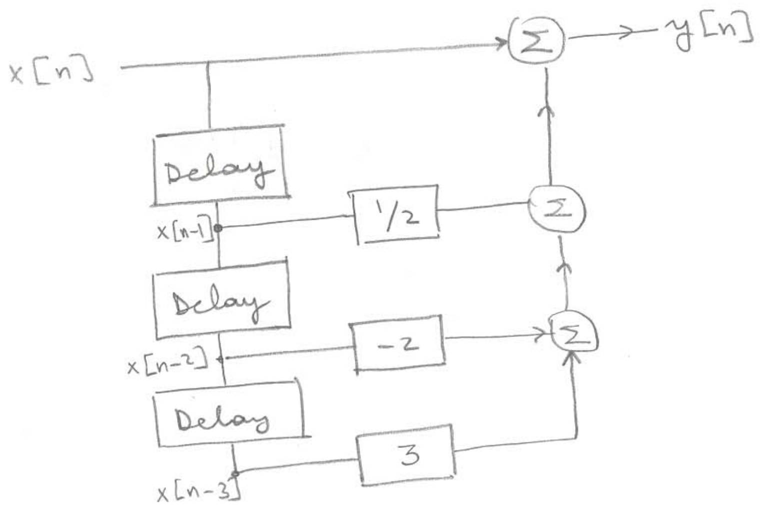
because there is a pole on the unit circle $|z|=1$.

Therefore, DTFT of $x[n]$ DOES NOT exist for any

of the ROC.

(9) Find the z-transform of the following systems.

a)



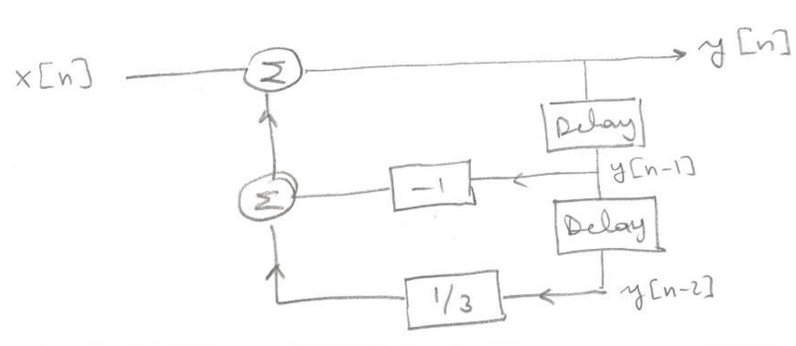
Answer

$$y[n] = x[n] + \frac{1}{2} x[n-1] - 2x[n-2] + 3x[n-3]$$

$$Y(z) = X(z) + \frac{1}{2} z^{-1} X(z) - 2z^{-2} X(z) + 3z^{-3} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 + \frac{1}{2} z^{-1} - 2z^{-2} + 3z^{-3}$$

b)

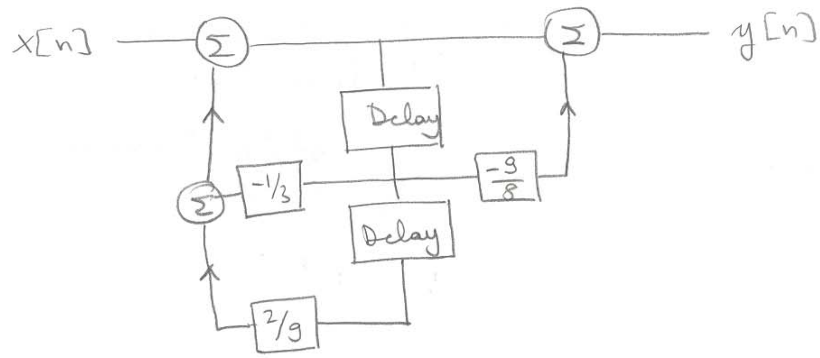


$$y[n] = x[n] - y[n-1] + \frac{1}{3} y[n-2]$$

$$Y(z) = X(z) - z^{-1} Y(z) + \frac{1}{3} z^{-2} Y(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + z^{-1} - \frac{1}{3} z^{-2}}$$

c)



is this system stable?

Answer

$$y[n] = x[n] - \frac{9}{8} x[n-1] - \frac{1}{3} y[n-1] + \frac{2}{9} y[n-2]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{9}{8} z^{-1}}{1 - \frac{1}{3} z^{-1} + \frac{2}{9} z^{-2}}$$

$$= \frac{1 - \frac{9}{8} z^{-1}}{\left(1 + \frac{2}{3} z^{-1}\right)\left(1 - \frac{1}{3} z^{-1}\right)}$$

The poles of $H(z)$ are at $\frac{1}{3}$ and $\frac{-2}{3}$, which are inside the unit circle. Therefore its ROC includes unit circle. Therefore the system is STABLE.

(10) Let

$$X(z) = \frac{1 - \frac{1}{3} z^{-1}}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 - z^{-1}\right)} \quad \text{and ROC} \quad |z| > 1$$

a) Rewrite $X(z)$ in terms of z^n

b) Use partial fraction expansion of $X(z)$ of part (a)

c) find $x[n]$

Fenerbahçe şampiyon oldu. Tebrikler ☺

Answer

a) $$X(z) = \frac{z^2 - \frac{1}{3} z}{\left(z - \frac{1}{2}\right)(z-1)}$$

b) Instead of expanding $X(z)$, we'll expand $\frac{X(z)}{z}$

$$\frac{X(z)}{z} = \frac{z - \frac{1}{3}}{\left(z - \frac{1}{2}\right)(z-1)} = \frac{A}{z - 1/2} + \frac{B}{z-1}$$

$$A = \left.\frac{z - 1/3}{z-1}\right|_{z=1/2} = -1/3$$

$$B = \left.\frac{z - 1/3}{z - 1/2}\right|_{z=1} = 4/3$$

$$X(z) = \frac{-\frac{1}{3} z}{z - 1/2} + \frac{4/3\, z}{z-1}$$

Since we are trying to expand into $\frac{z}{z-a}$ format, we expand $\frac{X(z)}{z}$ into fractions of $\frac{1}{z-a}$ and then multiply by z to obtain $\frac{z}{z-a}$

c) Since ROC is $|z| > 1$, the signal is right sided.

$$x[n] = \frac{-1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{4}{3} u[n]$$

(11) Let $$X(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}$$ ROC . $|z| < 1/2$

a) Rewrite $X(z)$ in terms of z^n

b) Use partial fraction expansion to find $x[n]$

Answer:

a) $$X(z) = \frac{z^2}{(z-1/2)(z-1)}$$

b) $$\frac{X(z)}{z} = \frac{z}{(z-1/2)(z-1)} = \frac{A}{z-1/2} + \frac{B}{z-1}$$

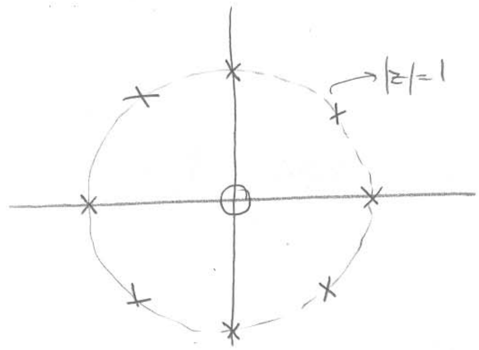
$$\left.\begin{aligned} A &= \left.\frac{z}{z-1}\right|_{z=1/2} = -1 \\ B &= \left.\frac{z}{z-1/2}\right|_{z=1} = 2 \end{aligned}\right\} \quad X(z) = \frac{-z}{z-1/2} + \frac{2z}{z-1}$$

Since the ROC is $|z| < 1/2$, this is a left-sided signal

then

$$x[n] = \left(\frac{1}{2}\right)^n u[-n-1] - 2u[-n-1]$$

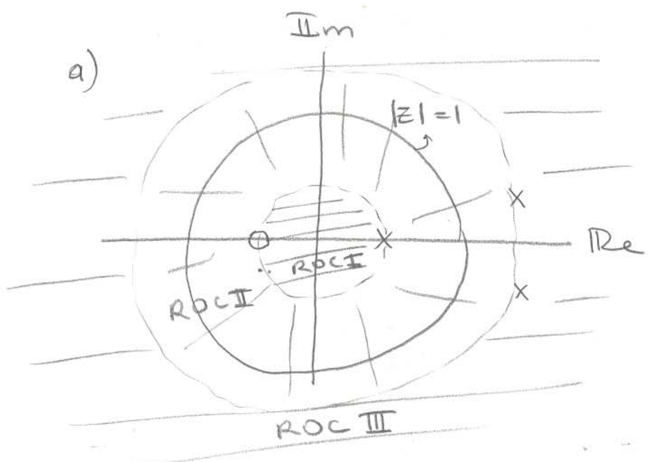
(12) Find the corresponding z-transform



Answer: $X(z) = \dfrac{z}{z^8 - 1}$

(13) The pole-zero plots of the LTI systems are given. State if the system is causal and stable.

a)



Answer

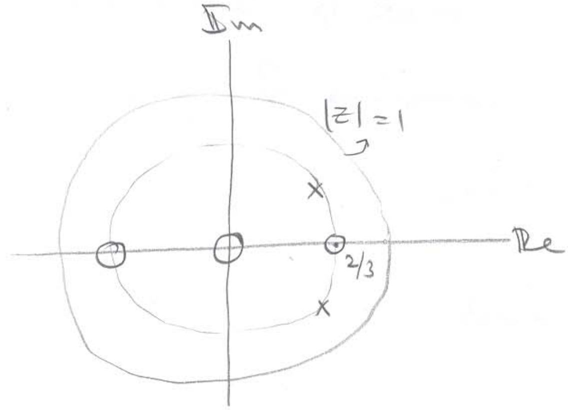
This system cannot be causal and stable at the same time.

ROC II corresponds to a stable but noncausal system

ROC III " to instable but causal "

ROC I " to instable and noncausal system.

b)



Answer:

This system can be stable, but it cannot be causal. Because it has more zeros than its poles. This means as $z \to \infty$ $X(z) \to \infty$. Therefore the system has another pole at the infinity. Therefore, it cannot be causal.

(14) Consider the signal

$$x(t) = \cos^2(\omega_0 t)$$

We want to sample $x(t)$ using impulse sequence

$$x(t) \longrightarrow \otimes \longrightarrow x_p(t)$$

$\uparrow$

$p(t)$: impulse sequence

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

a) Compute the minimum sampling frequency such that no aliasing occurs.

b) Draw $X(j\omega)$

c) Assume that we used sampling frequency of $2\omega_0$.

Draw $P(j\omega)$ and, $X_p(j\omega)$.

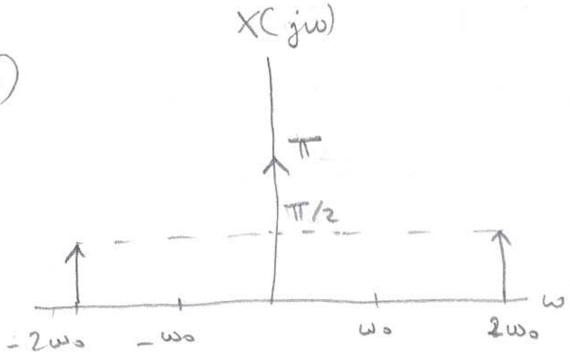
d) Using the sampling frequency of $16\omega_0$, draw $X_p(j\omega)$

e) What is the difference between $X_p(j\omega)$ found in part(c) and (d)

Answer: a) Recall that if the signal is bandlimited to ω_c, then the minimum sampling frequency (aka Nyquist frequency) is $2\omega_c$.

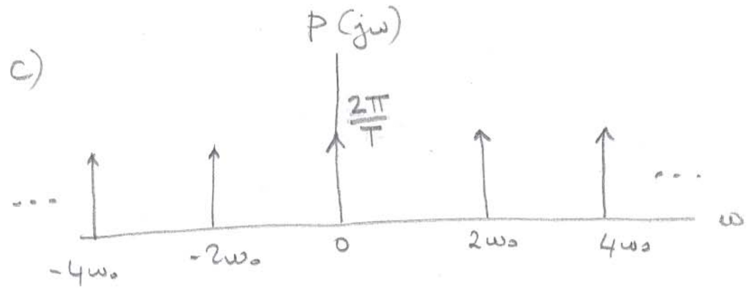
$$x(t) = \cos^2 \omega_0 t = \frac{1}{2} + \frac{1}{2}\cos 2\omega_0 t$$

b)



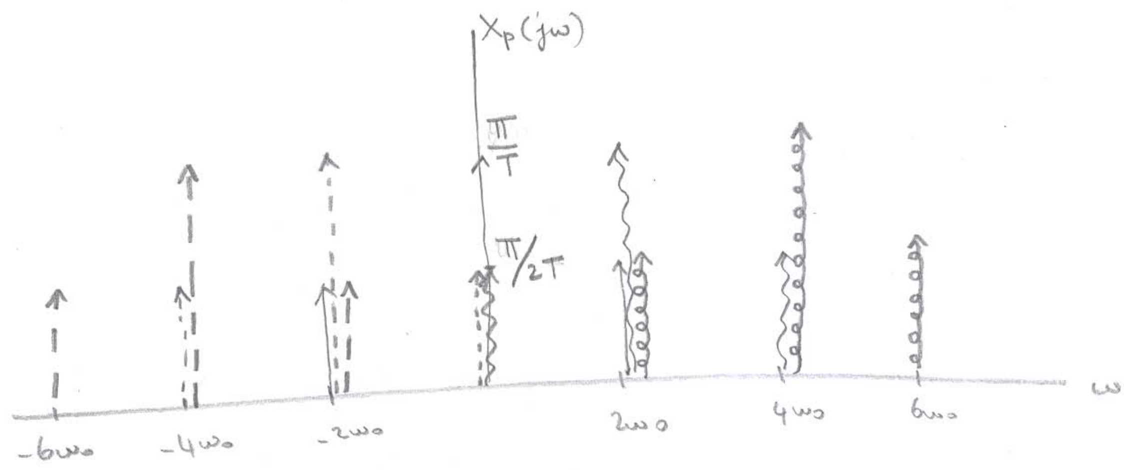
Therefore $x(t)$ is bandlimited to $2\omega_0$ and the minimum sampling frequency is $\underline{4\omega_0}$.

c)

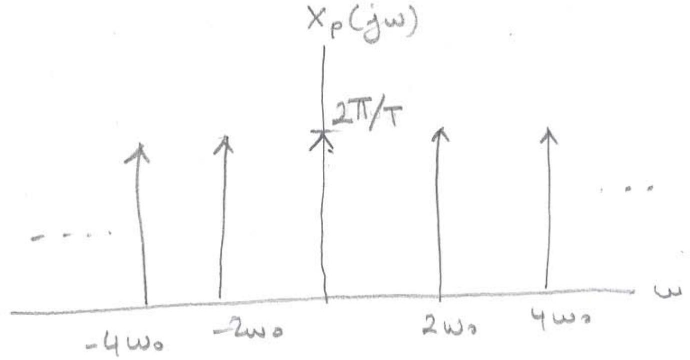


Remember that multiplication in time domain corresponds to convolution in frequency domain. Then

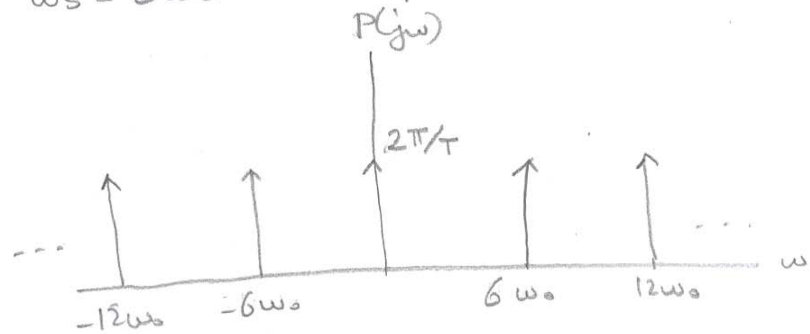
$$X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)\,P(j\omega - j\theta)\,d\theta$$

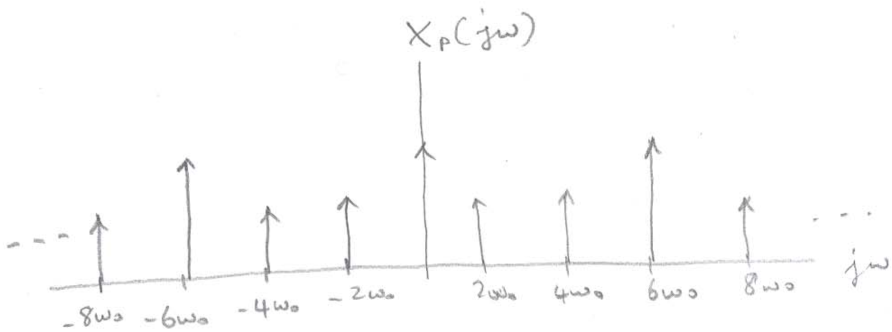


As you may see from this figure each $2\omega_0 k$ has π/T contribution from $X(j\omega - j2\omega_0 k)$, $\pi/2T$ contribution from $X(j\omega - j2\omega_0(k-1))$ and $\pi/2T$ contribution from $X(j\omega - j2\omega_0(k+1))$. They add up to $2\pi/T$.



d) $\omega_s = 6\omega_0$ ← greater than Nyquist frequency





e) There is aliasing in part (c) and no aliasing in part (d). Therefore, we can recover the original signal $x(t)$ from $X_p(j\omega)$ of part (d), but we cannot recover $x(t)$ from $X_p(j\omega)$ of part (c).