Question 1)

We are given that $h_2[n] = \delta[n] + \delta[n-1]$. Therefore,

$$h_2[n] * h_2[n] = \delta[n] + 2\delta[n-1] + \delta[n-2].$$

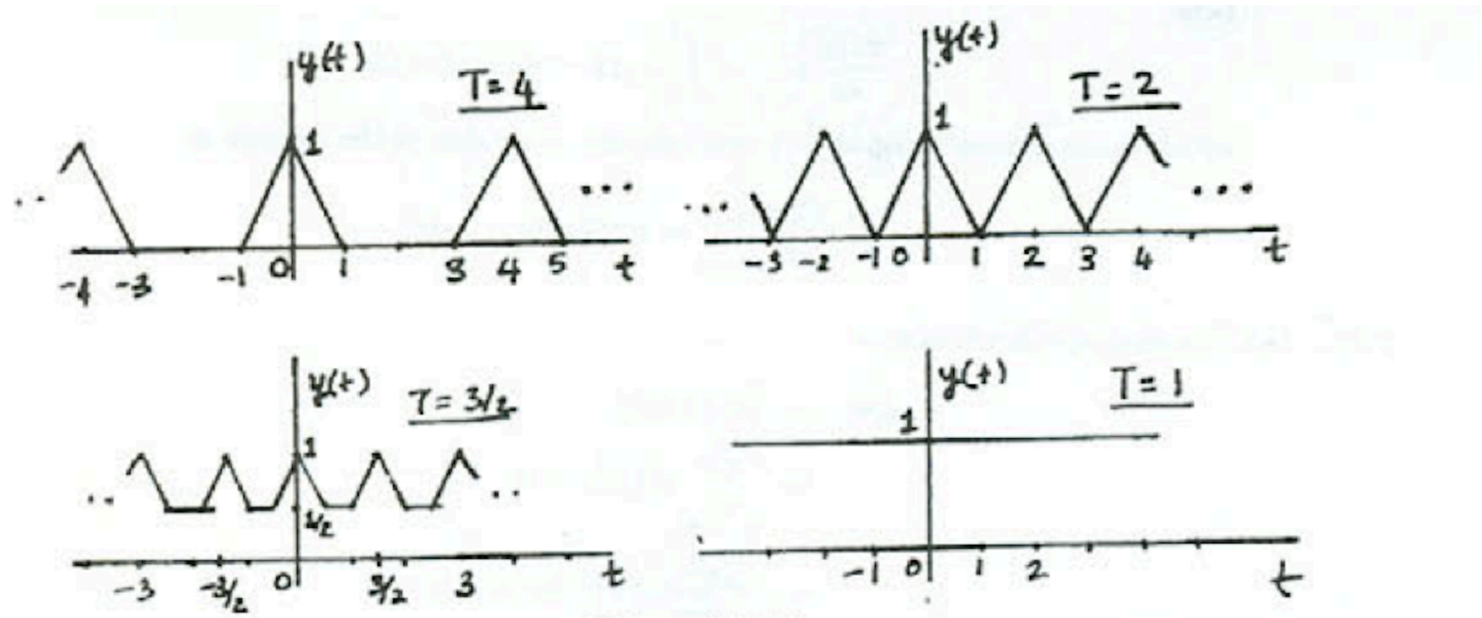

Figure S2.23

Since

$$h[n] = h_1[n] * [h_2[n] * h_2[n]],$$

we get

$$h[n] = h_1[n] + 2h_1[n-1] + h_1[n-2].$$

Therefore,

$$\begin{aligned}
h[0] &= h_1[0] &\Rightarrow\quad& h_1[0] = 1,\\
h[1] &= h_1[1] + 2h_1[0] &\Rightarrow\quad& h_1[1] = 3,\\
h[2] &= h_1[2] + 2h_1[1] + h_1[0] &\Rightarrow\quad& h_1[2] = 3,\\
h[3] &= h_1[3] + 2h_1[2] + h_1[1] &\Rightarrow\quad& h_1[3] = 2,\\
h[4] &= h_1[4] + 2h_1[3] + h_1[2] &\Rightarrow\quad& h_1[4] = 1\\
h[5] &= h_1[5] + 2h_1[4] + h_1[3] &\Rightarrow\quad& h_1[5] = 0.
\end{aligned}$$

$h_1[n] = 0$ for $n < 0$ and $n \geq 5$.

Question 2)

Let

$$x(t) = 1 + \sin \omega_0 t + 2\cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

which has fundamental frequency ω_0. As with Example 3.3, we can again expand $x(t)$ directly in terms of complex exponentials, so that

$$x(t) = 1 + \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}] + [e^{j\omega_0 t} + e^{-j\omega_0 t}] + \frac{1}{2}[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}].$$

Collecting terms, we obtain

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j(\pi/4)}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j(\pi/4)}\right)e^{-j2\omega_0 t}.$$

Thus, the Fourier series coefficients for this example are

$$a_0 = 1,$$

$$a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j,$$

$$a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j,$$

$$a_2 = \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1 + j),$$

$$a_{-2} = \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1 - j),$$

$$a_k = 0, \ |k| > 2.$$

In Figure 3.5, we show a bar graph of the magnitude and phase of a_k.

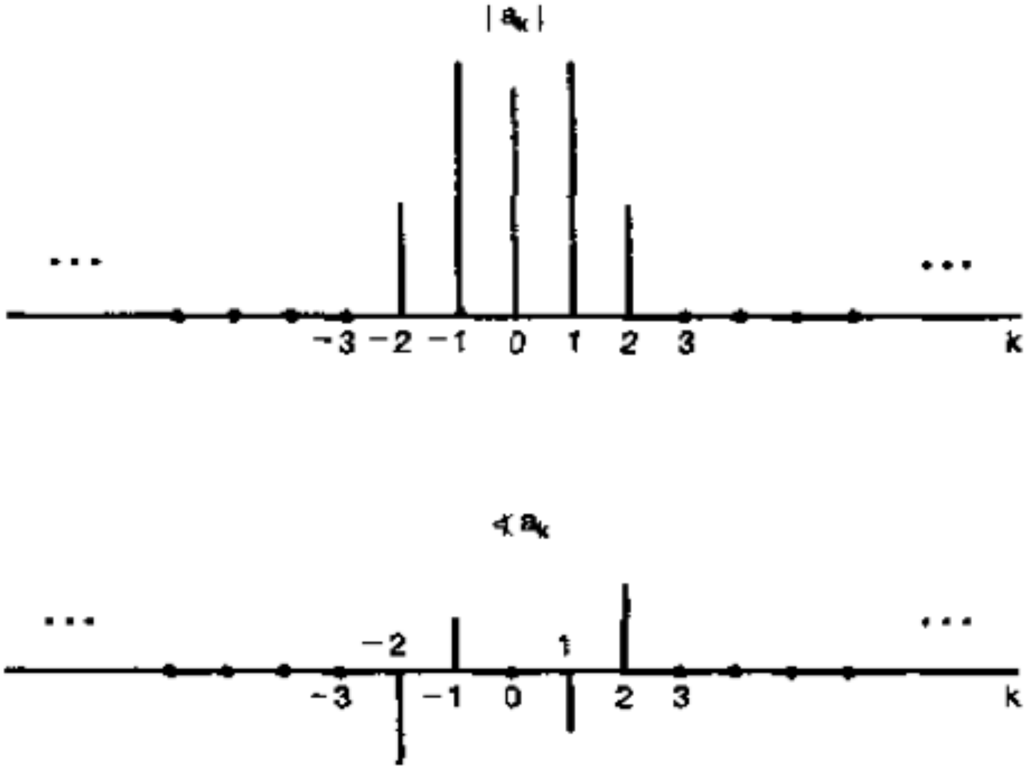

Figure 3.5 Plots of the magnitude and phase of the Fourier coefficients of the signal considered in Example 3.4.

Question 3)

Consider the response of an LTI system with impulse response

$$h(t) = e^{-at}u(t), \quad a > 0,$$

to the input signal

$$x(t) = e^{-bt}u(t), \quad b > 0.$$

Rather than computing $y(t) = x(t) * h(t)$ directly, let us transform the problem into the frequency domain. From Example 4.1, the Fourier transforms of $x(t)$ and $h(t)$ are

$$X(j\omega) = \frac{1}{b + j\omega}$$

and

$$H(j\omega) = \frac{1}{a + j\omega}.$$

Therefore,

$$Y(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)}. \tag{4.67}$$

To determine the output $y(t)$, we wish to obtain the inverse transform of $Y(j\omega)$. This is most simply done by expanding $Y(j\omega)$ in a partial-fraction expansion. Such expansions are extremely useful in evaluating inverse transforms, and the general method for performing a partial-fraction expansion is developed in the appendix. For this example, assuming that $b \neq a$, the partial fraction expansion for $Y(j\omega)$ takes the form

$$Y(j\omega) = \frac{A}{a + j\omega} + \frac{B}{b + j\omega}, \tag{4.68}$$

where A and B are constants to be determined. One way to find A and B is to equate the right-hand sides of eqs. (4.67) and (4.68), multiply both sides by $(a + j\omega)(b + j\omega)$, and solve for A and B. Alternatively, in the appendix we present a more general and efficient method for computing the coefficients in partial-fraction expansions such as eq. (4.68) Using either of these approaches, we find that

$$A = \frac{1}{b - a} = -B,$$

and therefore,

$$Y(j\omega) = \frac{1}{b - a}\left[\frac{1}{a + j\omega} - \frac{1}{b + j\omega}\right]. \tag{4.69}$$

The inverse transform for each of the two terms in eq. (4.69) can be recognized by inspection. Using the linearity property of Section 4.3.1, we have

$$y(t) = \frac{1}{b - a}[e^{-at}u(t) - e^{-bt}u(t)].$$

When $b = a$, the partial fraction expansion of eq. (4.69) is not valid. However, with $b = a$, eq. (4.67) becomes

$$Y(j\omega) = \frac{1}{(a + j\omega)^2}.$$

Recognizing this as

$$\frac{1}{(a + j\omega)^2} = j\frac{d}{d\omega}\left[\frac{1}{a + j\omega}\right],$$

we can use the dual of the differentiation property, as given in eq. (4.40). Thus,

$$e^{-at}u(t) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{a + j\omega}$$

$$te^{-at}u(t) \overset{\mathfrak{F}}{\longleftrightarrow} j\frac{d}{d\omega}\left[\frac{1}{a + j\omega}\right] = \frac{1}{(a + j\omega)^2},$$

and consequently,

Question 4)

$$h_1(t) = \frac{\sin 4t}{\pi t}.$$

The Fourier transform $H_1(j\omega)$ of $h_1(t)$ is as shown in Figure S4.32.

From the above figure it is clear that $h_1(t)$ is the impulse response of an ideal lowpass filter whose passband is in the range $|\omega| < 4$. Therefore, $h(t)$ is the impulse response of an ideal lowpass filter shifted by one to the right. Using the shift property,

$$H(j\omega) = \begin{cases} e^{-j\omega}, & |\omega| < 4 \\ 0, & \text{otherwise} \end{cases}.$$

(a) We have

$$X_1(j\omega) = \pi e^{j\frac{\pi}{12}}\delta(\omega - 6) + \pi e^{j\frac{\pi}{12}}\delta(\omega + 6).$$

It is clear that

$$Y_1(j\omega) = X_1(j\omega)H(j\omega) = 0 \Rightarrow y_1(t) = 0.$$

Question 5)

2. From the Nyquist theorem, we know that the sampling frequency in this case must be at least $\omega_s = 2000\pi$. In other words, the sampling period should be at most $T = 2\pi/(\omega_s) = 1 \times 10^{-3}$. Clearly, only (a) and (c) satisfy this condition.

Question 6)

For comparison with Example 9.1, let us consider as a second example the signal

$$x(t) = -e^{-at}u(-t). \tag{9.16}$$

Then

$$\begin{aligned} X(s) &= -\int_{-\infty}^{\infty} e^{-at}e^{-st}u(-t)\,dt \\ &= -\int_{-\infty}^{0} e^{-(s+a)t}\,dt, \end{aligned} \tag{9.17}$$

or

$$X(s) = \frac{1}{s+a}. \tag{9.18}$$

For convergence in this example, we require that $\Re e\{s+a\} < 0$, or $\Re e\{s\} < -a$; that is,

$$-e^{-at}u(-t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \qquad \Re e\{s\} < -a. \tag{9.19}$$

Comparing eqs. (9.14) and (9.19), we see that the algebraic expression for the Laplace transform is identical for both of the signals considered in Examples 9.1 and 9.2. However, from the same equations, we also see that the set of values of s for which the expression is valid is very different in the two examples. This serves to illustrate the fact that, in specifying the Laplace transform of a signal, both the algebraic expression and the range of values of s for which this expression is valid are required. In general, the range of values of s for which the integral in eq.(9.3) converges is referred to as the *region of convergence* (which we abbreviate as ROC) of the Laplace transform. That is, the ROC consists of those values of $s = \sigma + j\omega$ for which the Fourier transform of $x(t)e^{-\sigma t}$ converges. We will have more to say about the ROC as we develop some insight into the properties of the Laplace transform.

A convenient way to display the ROC is shown in Figure 9.1. The variable s is a complex number, and in the figure we display the complex plane, generally referred to as the s-plane, associated with this complex variable. The coordinate axes are $\Re e\{s\}$ along the horizontal axis and $\Im m\{s\}$ along the vertical axis. The horizontal and vertical axes are sometimes referred to as the σ-axis and the $j\omega$-axis, respectively. The shaded region in Figure 9.1(a) represents the set of points in the s-plane corresponding to the region of convergence for Example 9.1. The shaded region in Figure 9.1(b) indicates the region of convergence for Example 9.2.

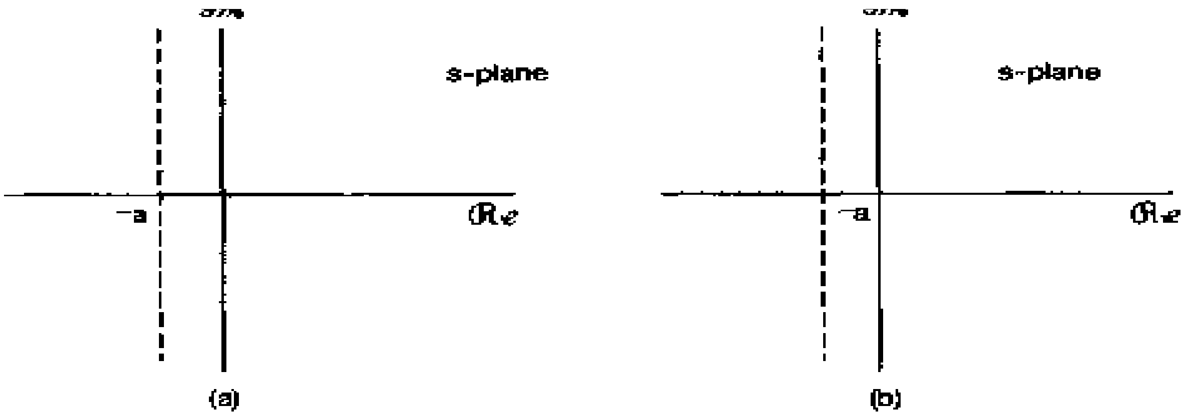

Figure 9.1 (a) ROC for Example 9 1; (b) ROC for Example 9.2.

Question 7)

Consider the z-transform

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}, \quad |z| > \frac{1}{3}. \tag{10.42}$$

There are two poles, one at $z = 1/3$ and one at $z = 1/4$, and the ROC lies outside the outermost pole. That is, the ROC consists of all points with magnitude greater than that of the pole with the larger magnitude, namely the pole at $z = 1/3$. From Property 4 in Section 10.2, we then know that the inverse transform is a right-sided sequence. As described in the appendix, $X(z)$ can be expanded by the method of partial fractions. For this example, the partial-fraction expansion, expressed in polynomials in z^{-1}, is

$$X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}}. \tag{10.43}$$

Thus, $x[n]$ is the sum of two terms, one with z-transform $1/[1 - (1/4)z^{-1}]$ and the other with z-transform $2/[1 - (1/3)z^{-1}]$. In order to determine the inverse z-transform of each of these individual terms, we must specify the ROC associated with each. Since the ROC for $X(z)$ is outside the outermost pole, the ROC for each individual term in eq. (10.43) must also be outside the pole associated with that term. That is, the ROC for each term consists of all points with magnitude greater than the magnitude of the corresponding pole. Thus,

$$x[n] = x_1[n] + x_2[n], \tag{10.44}$$

where

$$x_1[n] \overset{z}{\longleftrightarrow} \frac{1}{1 - \frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4}, \tag{10.45}$$

$$x_2[n] \overset{z}{\longleftrightarrow} \frac{2}{1 - \frac{1}{3}z^{-1}} \quad |z| > \frac{1}{3}. \tag{10.46}$$

From Example 10.1, we can identify by inspection that

$$x_1[n] = \left(\frac{1}{4}\right)^n u[n] \tag{10.47}$$

and

$$x_2[n] = 2\left(\frac{1}{3}\right)^n u[n], \tag{10.48}$$

and thus,

$$x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]. \tag{10.49}$$