BLG 335E – Analysis of Algorithms I
Fall 2013, Recitation 2
23.10.2013

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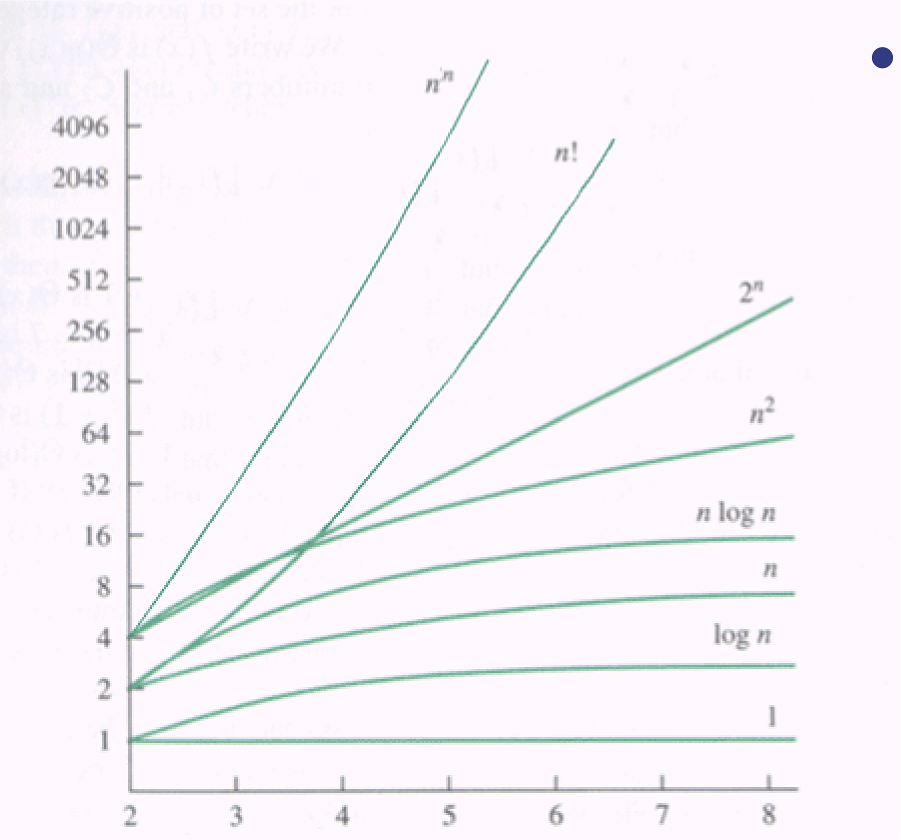
Warm-up Problem

- Order the following functions by asymptotic growth rate:
  - $n^2 + 5n + 7$
  - $\log_2 n^3$
  - 95^{17}
  - $2^{\log_2 n}$
  - n^3
  - $nlog_2n + 9n$
  - $4 \log_2 n$
  - $\log_2 n + 3n$

Warm-up Problem



- Solution:
  - 95^{17}
  - $\log_2 n^3$
  - $4 \log_2 n$
  - $2^{\log_2 n}$
  - $\log_2 n + 3n$
  - $nlog_2n + 9n$
  - $n^2 + 5n + 7$
  - n^3

Problem 1

- Give tight asymptotic bounds for $T(n)$ in each of the following recurrences.
  a. $T(n) = T(n-1) + n$
  b. $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n$
  c. $T(n) = T\left(\frac{9n}{10}\right) + n$
  d. $T(n) = 16T\left(\frac{n}{4}\right) + n^2$
  e. $T(n) = 7T\left(\frac{n}{2}\right) + n^2$

Problem 1

- Give tight asymptotic bounds for $T(n)$ in each of the following recurrences.

a. $T(n) = T(n-1) + n$

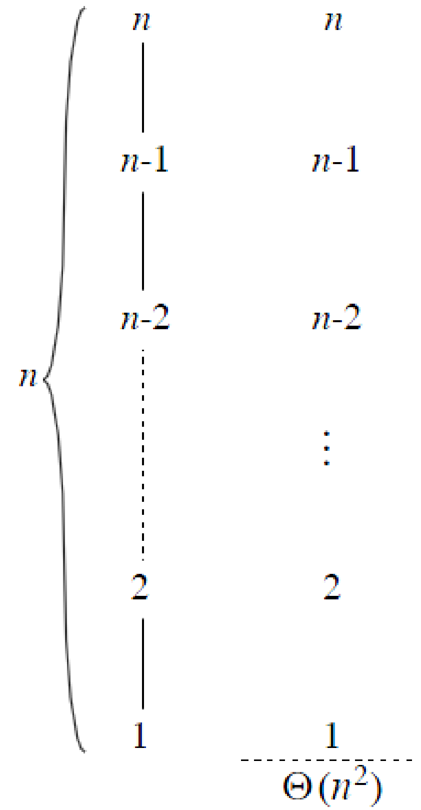
Lower bound (Ω):

$T(n) \geq cn^2 \; for \; some \; c > 0$

$T(n) \geq c(n-1)^2 + n$

$= cn^2 - 2cn + c + n \geq cn^2$

$true \; if \; 0 < c < \frac{1}{2} \, and \, n \geq 0$



Problem 1

- Give tight asymptotic bounds for $T(n)$ in each of the following recurrences.

$a.\quad T(n) = T(n-1) + n$

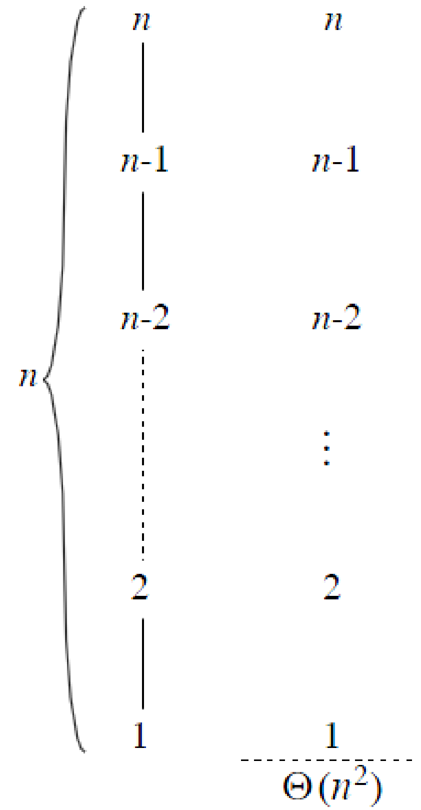
Upper bound (*O*):

$T(n) \leq cn^2 \; for\ some\ c > 0$

$T(n) \leq c(n-1)^2 + n$

$= cn^2 - 2cn + c + n \leq cn^2$

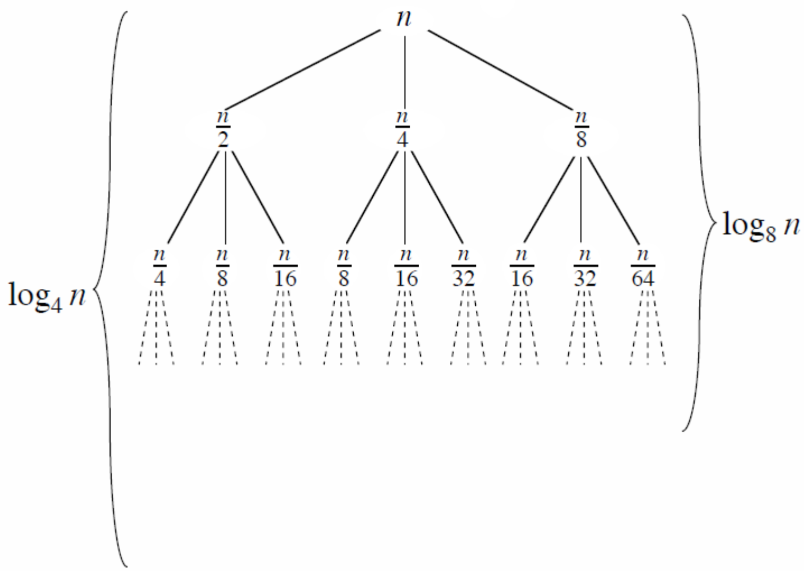
$true\ if\ c = 1\ and\ n \geq 1$



Problem 1

- Give tight asymptotic bounds for $T(n)$ in each of the following recurrences.

  b. $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n$



$$n$$

$$n\left(\frac{4+2+1}{8}\right) = \frac{7}{8}n$$

$$n\left(\frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{2}{32} + \frac{1}{64}\right)$$
$$= n\frac{16+16+12+4+1}{64}$$
$$= n\frac{49}{64} = \frac{7}{8}^2 n$$

$$\vdots$$

$$\sum_{i=1}^{\log n} \left(\frac{7}{8}\right)^i n = \Theta(n)$$

Problem 1

- Give tight asymptotic bounds for $T(n)$ in each of the following recurrences.

  b. $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n$

  Upper bound (O):

  $$T(n) \le \frac{cn}{2} + \frac{cn}{4} + \frac{cn}{8} + n = \frac{7cn}{8} + n \le cn$$

  $true\ if\ c \ge 8$

Problem 1

- Give tight asymptotic bounds for $T(n)$ in each of the following recurrences.

$$b.\ \ T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n$$

Lower bound (Ω):

$$T(n) \geq \frac{cn}{2} + \frac{cn}{4} + \frac{cn}{8} + n = \frac{7cn}{8} + n \geq cn$$

$$true\ if\ 0 < c \leq 8$$

Master Method

$$T(n) = aT(n/b) + f(n)$$

1 $f(n) = O\left(n^{\log_b a-\varepsilon}\right) \Rightarrow T(n) = \Theta\left(n^{\log_b a}\right)$

2 $f(n) = \Theta\left(n^{\log_b a}\right) \Rightarrow T(n) = \Theta\left(n^{\log_b a} \log_2 n\right)$

3 $f(n) = \Omega\left(n^{\log_b a+\varepsilon}\right) \quad and \quad af(n/b) \le cf(n),$

$for \quad \exists c \quad c < 1 \quad and \quad n > n_0$

$\Rightarrow T(n) = \Theta\left(f(n)\right)$

Problem 1

- Give tight asymptotic bounds for $T(n)$ in each of the following recurrences.

c. $T(n) = T\left(\frac{9n}{10}\right) + n$

$a = 1, b = \frac{10}{9}, f(n) = n = \Omega\left(\mathrm{n}^{\log_{\frac{10}{9}} 1 + 1}\right)$

$possibly\ case\ 3, let's\ check\ c$

$1\frac{9n}{10} \leq cn\ holds\ for\ c = \frac{9}{10} \leq 1$

$certainly\ case\ 3:$

$T(n) = \Theta(\mathrm{n})$

Problem 1

- Give tight asymptotic bounds for $T(n)$ in each of the following recurrences.

d. $T(n) = 16T\left(\frac{n}{4}\right) + n^2$ $a = 16, b = 4, f(n) = n^2$

$$n^2 = \Theta\left(n^{\log_4 16}\right), case\ 2:$$

$$T(n) = \Theta(n^2 \log_2 n)$$

e. $T(n) = 7T\left(\frac{n}{2}\right) + n^2$ $a = 7, b = 2, f(n) = n^2$

$$n^2 = O\left(n^{\log_2 7 - \varepsilon}\right), case\ 1:$$

$$T(n) = \Theta(n^{\log_2 7})$$

Problem 2

- **Hat-check problem**
- Each of n customers gives a hat to a hat-check person at a restaurant.
- The hat-check person gives the hats back to the customers in a random order.
- What is the expected number of customers that get back their own hat?

Problem 2

- **Counting Solution:**
  - Count number of fixed points in all n! possible permutations
  - Divide by n! To find average number
  - Answer is 1
- **With Indicator Random Variables:**
  - $X_i = I\{customer\ i\ gets\ his\ own\ hat\}$
    $$for\ i = 1,2,\ldots,n$$
  - $X = X_1 + X_2 + \cdots + X_n$
    $number\ of\ customers\ that\ get\ their\ own\ hat$

Problem 2

- $E[X] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$
- $E[X_i] = \frac{1}{n}$
- $E[X] = \sum_{i=1}^{n} \frac{1}{n} = 1$

Problem 3

- Searching for a value x in an unsorted array A consisting of n elements.
- Pick a random index i into A.
- If $A[i] = x$, then we terminate;
- Otherwise, we continue the search by picking a new random index into A.
- We continue picking random indices into A until we find x or until we have checked every element of A.

Problem 3

- What is the expected number of indices into A must be picked before x is found? (There is exactly one x in A)
- What if there are $k \geq 1$ x values in A?
- If there is no x in A, what is the expected number of indices into A that must be picked before search terminates?