MAT 271E Probability and Statistics

Homework 1 Solutions

Assigned: February 11, 2012

Due: ~~February 15, 2012 (in class, before class starts)~~ **postponed to February 17, 2012 (5 PM)**

No late homework will be accepted!

Do not copy from solutions from your classmates. All work must be your own!

Read: "Probability and Stochastic Processes", Yates and Goodman, Ch. 1 and Ch. 2

1) An integrated circuit factory has three machines X, Y, and Z. Test one integrated circuit produced by each machine. Either a circuit is acceptable (*a*) or it fails (*f*). An observation is a sequence of three test results corresponding to the circuits from machines X, Y, and Z, respectively. For example, *aaf* is the observation that the circuits from X and Y pass the test and the circuit from Z fails the test.

a) What are the elements of the sample space of this experiment?

The elements of the sample space of this experiment are aaa, aaf, afa, aff, faa, faf, ffa, fff.

b) What are the elements of the sets:

Z_F = {circuit from Z fails}

The elements of the set Z_F are aaf, aff, faf, fff.

X_A = {circuit from X is acceptable} .

The elements of the set X_A are aaa, aaf, afa, aff.

c) Are Z_F and X_A mutually exclusive?

$\boldsymbol{Z_F \cap X_A = \{aaf, aff\} \neq \emptyset}$

∴ No, Z_F ve X_A are not mutually exclusive.

d) Are Z_F and X_A collectively exhaustive?

$\boldsymbol{Z_F \cup X_A = \{aaa, aaf, afa, aff, faf, fff\} \neq S}$

(S: Sample space of experiment)

∴ No, Z_F ve X_A are not collectively exhaustive.

e) What are the elements of the sets

C = {more than one circuit acceptable}

The elements of C are aaf, afa, faa, aaa.

D = {at least two circuits fail}

The elements of D are ffa, faf, aff, fff.

f) Are C and D mutually exclusive?

$C \cap D = \emptyset$

∴ Yes, C and D are mutually exclusive.

g) Are C and D collectively exhaustive?

$C \cup D = \{aaa, aaf, afa, aff, faa, faf, ffa, fff\} = S$

(S: Sample space of experiment)

∴ Yes, C and D are collectively exhaustive.

2) Computer programs are classified by the length of the source code and by the execution time. Programs with more than 150 lines in the source code are big (*B*). Programs with ≤ 150 lines are little (*L*). Fast programs (*F*) run in less than 0.1 seconds. Slow programs (*W*) require at least 0.1 seconds. Monitor a program executed by a computer. Observe the length of the source code and the run time. The probability model for this experiment contains the following information: P[LF] = 0.5, P[BF] = 0.2, and P[BW] = 0.2. What is the sample space of the experiment?

The sample space of the experiment is S = {BF, BW, LF, LW}.

Calculate the following probabilities:

a) P[W]

P[W] = P[BW] + P[LW]

P[LW] = 1 – (P[LF] + P[BF] + P[BW]) = 1 – (0.5 + 0.2 + 0.2) = 0.1

P[W] = 0.2 + 0.1 = 0.3

b) P[B]

P[B] = P[BF] + P[BW] = 0.2 + 0.2 = 0.4

c) P[W ∪ B]

P[W ∪ B] = P[W] + P[B] - P[W ∩ B] = 0.3 + 0.4 – 0.2 = 0.5

3) Cellular telephones perform *handoffs* as they move from cell to cell. During a call, a telephone either performs zero handoffs (H_0), one handoff (H_1), or more than one handoff (H_2). In addition, each call is either long (L), if it lasts more than 3 minutes, or brief (B). The following table describes the probabilities of the possible types of calls.

	H_0	H_1	H_2
L	0.1	0.1	0.2
B	0.4	0.1	0.1

What is the probability $P[H_0]$ that a phone makes no handoffs?

$P[H_0] = P[H_0L] + P[H_0B] = 0.1 + 0.4 = 0.5$

What is the probability a call is brief?

$P[B] = P[BH_0] + P[BH_1] + P[BH_2] = 0.4 + 0.1 + 0.1 = 0.6$

What is the probability a call is long or there are at least two handoffs?

$P[L \cup H_2] = \ = P[L] + P[H_2] - P[L \cap H_2] = (0.1 + 0.1 + 0.2) + (0.2 + 0.1) - 0.2 = 0.5$

4) You have a six-sided die that you roll once. Let R_i denote the event that the roll is i. Let G_j denote the event that the roll is greater than j. Let E denote the event that the roll of the die is even-numbered.

Let s_i denote the outcome that the roll is i. So, for $1 \le i \le 6$, $R_i = \{s_i\}$. Similarly, $G_j = \{s_{j+1}, \ldots, s_6\}$.

a) What is $P[R_3|G_1]$, the conditional probability that 3 is rolled given that the roll is greater than 1?

Since $G_1 = \{s_2, s_3, s_4, s_5, s_6\}$ and all outcomes have probability 1/6, $P[G_1] = 5/6$. The event $R_3G_1 = \{s_3\}$ and $P[R_3B_1] = 1/6$, so that

$$P[R_3|G_1] = \frac{P[R_3G_1]}{P[G_1]} = \frac{1}{5}$$

b) What is the conditional probability that 6 is rolled given that the roll is greater than 3?

$$P[R_6|G_3] = \frac{P[R_6G_3]}{P[G_3]} = \frac{P[\{s_6\}]}{P[\{s_4, s_5, s_6\}]} = \frac{1/6}{3/6} = \frac{1}{3}$$

c) What is P[G_3|E], the conditional probability that the roll is greater than 3 given that the roll is even?

The event E that the roll is even is E = { s_2, s_4, s_6} and has probability 3/6. The joint probability of G_3 and E is

$$P[G_3E] = P[\{s_4, s_6\}] = 1/3$$

The conditional probability of G_3 given E is

$$P[G_3|E] = \frac{P[G_3E]}{P[E]} = \frac{1/3}{1/2} = \frac{2}{3}$$

d) Given that the roll is greater than 3, what is the conditional probability that the roll is even?

The conditional probability that the roll is even given that it is greater than 3 (i.e., E given G_3) is

$$P[E|G_3] = \frac{P[EG_3]}{P[G_3]} = \frac{1/3}{1/2} = \frac{2}{3}$$

5) Assume that M, N, Q, and R are events with probabilities P[M ∪ N] = 5/8, P[M] = 3/8, P[Q ∩ R] = 1/3, and P[Q] = 1/2. M and N are disjoint, and Q and R are independent.

a) What is P[M ∩ N]?

Since M and N are disjoint, M ∩ N = ∅, P[M ∩ N] = 0.

b) What is P[N]?

To find P[N], we can write

P[M ∪ N] = P[M] + P[N] - P[M ∩ N]

5/8 = 3/8 + P[N] – 0

Thus, P[N] = 1/4.

c) What is P[M ∩ N^c]?

Since M and N are disjoint, M is a subset of N^c. Thus,

P[M ∩ N^c] = P[M] = 3/8

d) What is P[M ∪ N^c]?

Since M and N are disjoint, M is a subset of N^c. Thus,

P[M ∪ N^c] = P[N^c] = 1 - P[N] = 1 – 1/4 = 3/4

e) Are M and N independent?

$$P[MN] \stackrel{?}{=} P[M]P[N]$$

$$0 \stackrel{?}{=} \frac{3}{8} \cdot \frac{1}{4} \quad ✗$$

No, M and N are not independent.

f) What is P[R]?

Since Q and R are independent, P[QR] = P[Q] P[R]. So,

$$P[R] = \frac{P[QR]}{P[Q]} = \frac{1/3}{1/2} = \frac{2}{3}$$

g) What is $P[Q \cup R]$?

$P[Q \cup R] = P[Q] + P[R] - P[Q \cap R] = 1/2 + 2/3 - 1/3 = 5/6$

h) What is P[Q|R]?

Since Q and R are independent events, P[Q|R] = P[Q] = 1/2.

i) What is $P[Q \cap R^c]$?

$P[Q \cap R^c] = P[Q] - P[Q \cap R] = 1/2 - 1/3 = 1/6$

j) What is $P[Q \cup R^c]$?

$P[Q \cup R^c] = P[Q] + P[R^c] - P[Q \cap R^c]$
$= P[Q] + (1 - P[R]) - P[Q \cap R^c]$
$= 1/2 + (1 - 2/3) - 1/6$
$= 2/3$

k) What is $P[Q^c \cap R^c]$?

By De Morgan's Law, $Q^c \cap R^c = (Q \cup R)^c$. This implies

$P[Q^c \cap R^c] = P[(Q \cup R)^c] = 1 - P[Q \cup R] = 1/6$

l) Are Q and R^c independent?

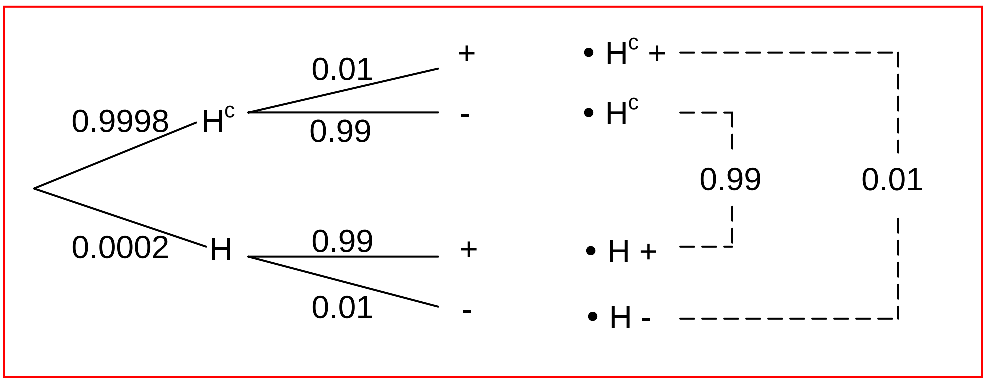
$$P[QR^c] \stackrel{?}{=} P[Q]P[R^c]$$

$$\frac{1}{6} \stackrel{?}{=} \frac{1}{2} \cdot \frac{1}{3} \quad \checkmark$$

Yes, Q and R^c are independent.

6) Suppose that for the general population, 1 in 5000 people carries the human immunodeficiency virus (HIV). A test for the presence of HIV yields either a positive (+) or negative (-) response. Suppose the test gives the correct answer 99% of the time.

The tree diagram for this experiment is shown in the figure below:



a) What is P[-|H], the conditional probability that a person tests negative given that the person does have the HIV virus?

The P[-|H] is the probability that a person who has HIV tests negative for the disease. This is referred to as a "false negative" result. The case where a person who does not have HIV but tests positive for the disease, is called a "false positive" result and has probability $P[+|H^c]$'dir. Since the test is correct 99% of the time,

$$P[-|H] = P[+|H^c] = 0.1$$

Alternatively, we could write

$$P[-|H] = \frac{P[H-]}{P[H]} = \frac{0.002 \cdot 0.1}{0.002} = 0.01$$

b) What is P[H|+], the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive?

The probability that a person who has tested positive for HIV actually has the disease is

$$P[H|+] = \frac{P[H+]}{P[+]} = \frac{P[H+]}{P[H\,+] + P[H^c+]}$$

We can use Bayes' theorem to evaluate the probabilities in the denominator.

$$P[H|+] = \frac{P[+|H]P[H]}{P[+|H]P[H] + P[+|H^c]P[H^c]}$$

$$= \frac{(0.99)(0.0002)}{(0.99)(0.0002) + (0.01)(0.9998)} = 0.0194$$

Thus, even though the test is correct 99% of the time, the probability that a random person who tests positive actually has HIV is less than 0.02. The reason this probability is so low is that the a priori probability that a person has HIV is very small.