TEL 252E
Practice Questions

08/04/2008

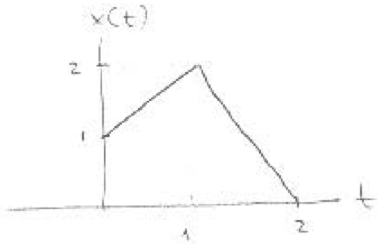
(1)

$$x(t) \longrightarrow \underset{\text{LTI}}{\boxed{h(t)}} \longrightarrow y(t)$$

$$x(t) = \begin{cases} t+1 & 0 \le t \le 1 \\ 4-2t & 1 < t \le 2 \\ 0 & \text{else} \end{cases} \qquad h(t) = \delta(t+2) + 2\delta(t+1)$$

Find $y(t)$

<u>Answer</u> Recall that $\delta(t-t_0) * x(t) = x(t-t_0)$

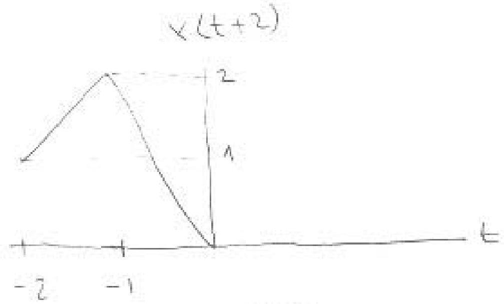


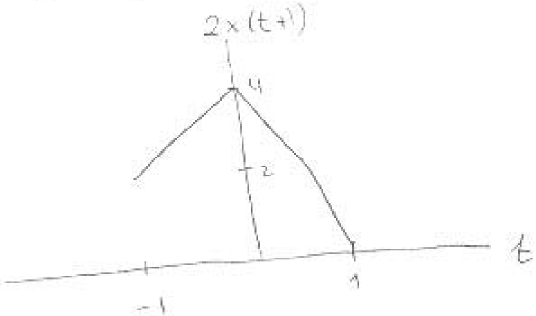
$$x(t) * h(t) = y(t)$$

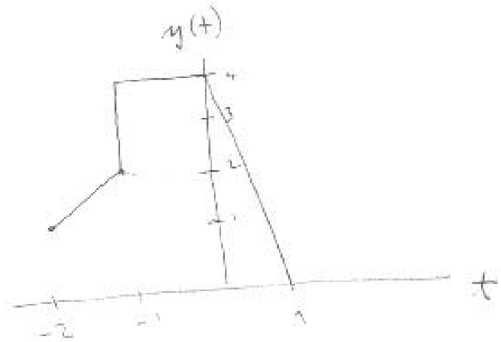
$$= x(t) * \{\delta(t+2) + 2\delta(t+1)\}$$

$$= x(t+2) + 2x(t+1)$$

Graphic solution







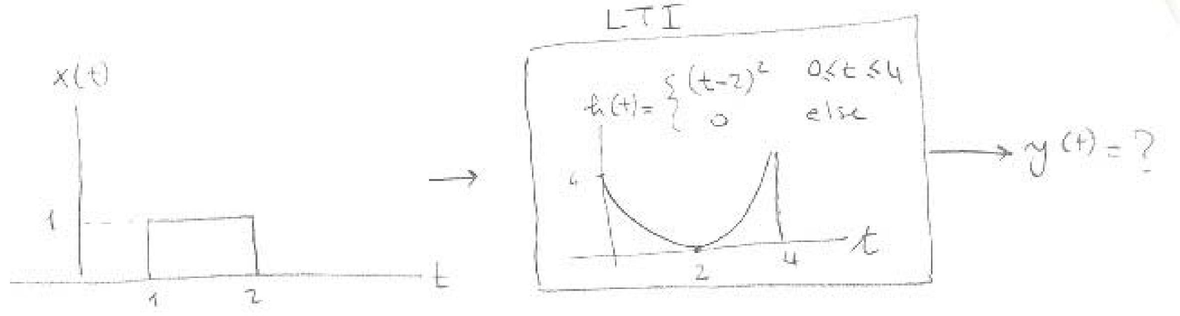
Mathematical Solution

$$x(t+2) = \begin{cases} t+3 & 0 \le t+2 \le 1 \\ 4-2(t+2) & 1 < t+2 \le 2 \\ 0 & \text{else} \end{cases}$$

$$2x(t+1) = \begin{cases} 2t+4 & 0 \le t+1 \le 1 \\ 8-4(t+1) & 1 < t+1 \le 2 \\ 0 & \text{else} \end{cases}$$

$$y(t) = \begin{cases} t+3 & -2 \le t \le -1 \\ 4 & -1 < t \le 0 \\ 8-4(t+1) & 0 < t \le 1 \\ 0 & \text{else} \end{cases}$$

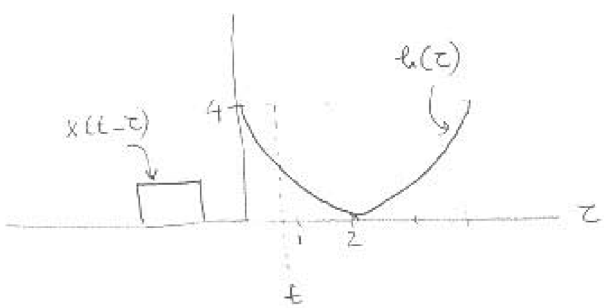
(2)



$y(t) = x(t) * h(t)$

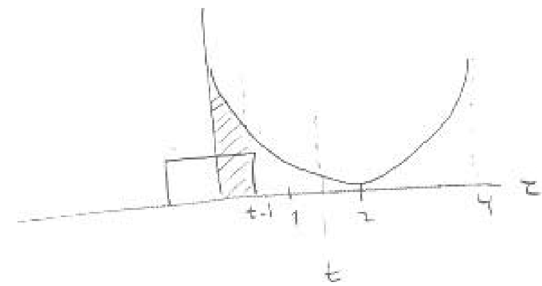
- Graphical convolution

$t<1$



$y(t)=0$

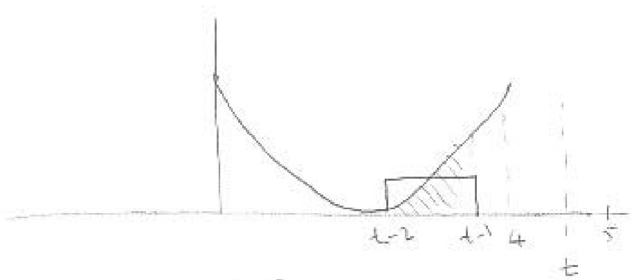
$1<t<2$



$$y(t) = \int_0^{t-1} h(\tau)\,d\tau$$

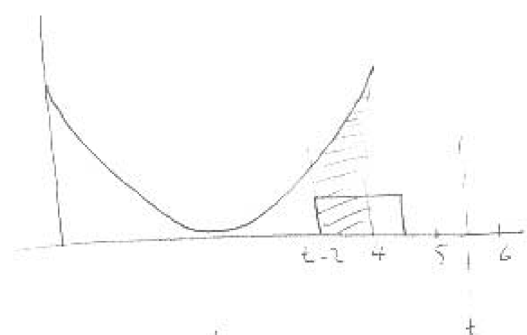
$$= \int_0^{t-1} (\tau-2)^2\,d\tau$$

$2<t<5$



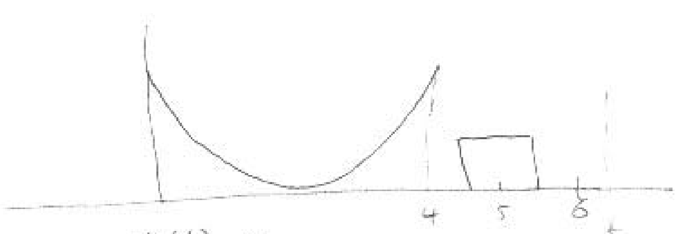
$$y(t) = \int_{t-1}^{t-2} h(\tau)\,d\tau$$

$5<t<6$



$$y(t) = \int_{t-2}^{4} h(\tau)\,d\tau$$

$6<t$



$y(t)=0$

③ $h[n] = \alpha^n u[n] \quad |\alpha| < 1 \Rightarrow H(e^{j\omega}) = ?$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n$$

$$= \frac{1}{1-\alpha e^{-j\omega}}$$

$\boxed{\text{if } |\alpha| \geq 1 \text{ then } \sum |h[n]| = \infty \Rightarrow \text{D.T.F.T does NOT exists}}$

④

$$\alpha^n u[n] \longrightarrow \underset{}{\overset{LTI}{\boxed{\beta^n u[n]}}} \longrightarrow y[n] = ?$$

- By convolution

$$y[n] = \sum_{k=-\infty}^{\infty} \alpha^k u[k] \beta^{n-k} u[n-k] = \sum_{k=0}^{n} \beta^n (\alpha/\beta)^k$$

$$= \beta^n \frac{1-(\alpha/\beta)^{n+1}}{1-\alpha/\beta} u[n]$$

$$= \beta^n \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} \frac{\beta}{\beta^{n+1}} u[n]$$

$$= \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} u[n]$$

- By DT-FT

$$Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$$

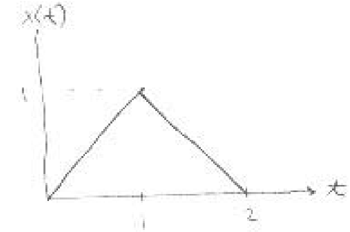
$$= \frac{1}{1-\alpha e^{-j\omega}} \cdot \frac{1}{1-\beta e^{-j\omega}} = \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}$$

$$A = \frac{\alpha}{\alpha - \beta} \qquad B = \frac{\beta}{\beta - \alpha}$$

$$y[n] = \frac{-\alpha}{\beta - \alpha} \alpha^n u[n] + \frac{\beta}{\beta - \alpha} \beta^n u[n] = \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} u[n]$$

4) Let $x(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \end{cases}$ be periodic with $T=2$, and Fourier series coef a_k.

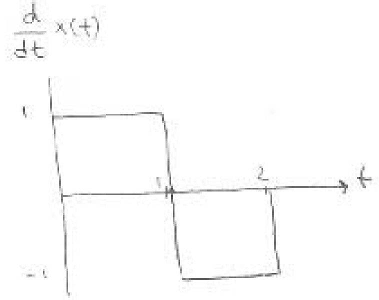
a) Find a_0



$$a_0 = \frac{1}{2}\int_0^2 x(t)\,dt$$

$$= \frac{1}{2}$$

b)



$$b_k = \frac{1}{2}\int_0^1 e^{-j\pi k t}\,dt - \frac{1}{2}\int_1^2 e^{-j\pi k t}\,dt$$

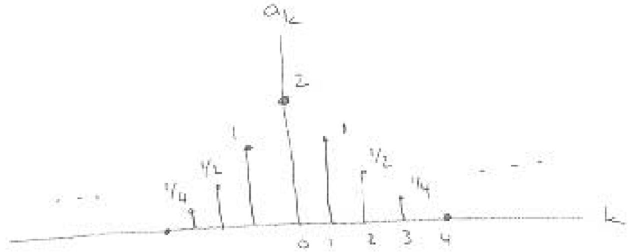
$$= \frac{1}{j\pi k}\left[1 - e^{-j\pi k}\right] \quad \forall k \neq 0$$

$$b_0 = 0$$

c) Find a_k using derivative property of F.S. coefficients

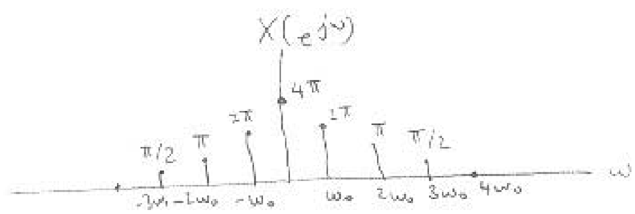
$$\frac{d}{dt}x(t) \xleftrightarrow{FS} b_k = jk\pi a_k \Rightarrow a_k = -\frac{1}{k^2\pi^2}\left(1 - e^{-j\pi k}\right)$$

5) If the F.S. coefficients of a periodic D.T. signal (with period $N=8$), find $x[n]$



$$x[n] = \sum_{k=\langle N\rangle} a_k e^{j\frac{2\pi}{8}kn} = 2 + 2\cos\frac{\pi n}{4} + \cos\frac{\pi n}{2} + \frac{1}{2}\cos\frac{3\pi n}{4}$$

F. T.

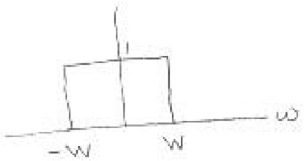


$$\omega_0 = \frac{2\pi}{8} = \pi/4$$

Find the Fourier Tr. of

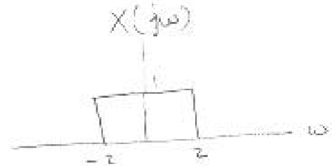
$$\left(\frac{\sin 2t}{\pi t}\right)^2 \xleftrightarrow{F.T} ?$$

From Example 4.5 in your textbook



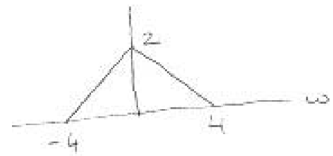
$$\xleftrightarrow{F.T.} \frac{\sin Wt}{\pi t}$$

Let $x(t) = \dfrac{\sin 2t}{\pi t} \xleftrightarrow{F.T.}$



Recall $x(t)\,y(t) \xleftrightarrow{F.T.} X(j\omega) * Y(j\omega)$

then $(x(t))^2 \longleftrightarrow X(j\omega) * X(j\omega)$



(7) Given a LTI system with a impulse response

$$H(j\omega) = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega}$$

a) Determine a differential equation relating input $x(t)$ with output $y(t)$

Recall that

$$\frac{d}{dt}x(t) \longleftrightarrow j\omega X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega} \Rightarrow -\omega^2 Y(j\omega) + 5j\omega Y(j\omega) + 6Y(j\omega)$$

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t)$$

On the other side of the equation

$$j\omega X(j\omega) + 4X(j\omega) \longleftrightarrow \frac{d}{dt}x(t) + 4x(t)$$

Then

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = \frac{d}{dt}x(t) + 4x(t)$$

b) Find $h(t)$

$$H(j\omega) = \frac{j\omega+4}{(j\omega+2)(j\omega+3)} = \frac{A}{j\omega+2} + \frac{B}{j\omega+3}$$

$$A = \left.\frac{j\omega+4}{j\omega+3}\right|_{j\omega=-2} = 2 \qquad B = \left.\frac{j\omega+4}{j\omega+2}\right|_{j\omega=-3} = -1$$

$$h(t) = 2e^{-2t}u(t) - e^{-3t}u(t)$$

c) Find $y(t)$ when $x(t) = e^{-4t}u(t) + te^{-4t}u(t)$

$$X(j\omega) = \frac{1}{j\omega+4} + \frac{-1}{(j\omega+4)^2}$$

$$Y(j\omega) = H(j\omega)\,X(j\omega) = \frac{1}{(4-j\omega)(2-j\omega)} \longleftrightarrow \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-4t}u(t)$$

(8) A causal and stable LTI system

$$\left(\frac{4}{5}\right)^n u[n] \longrightarrow \boxed{h[n]} \longrightarrow n\left(\frac{4}{5}\right)^n u[n]$$

a) Find $H(e^{j\omega})$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \qquad \left(\frac{4}{5}\right)^n u[n] \longleftrightarrow X(e^{j\omega}) = \frac{1}{1-\frac{4}{5}e^{-j\omega}}$$

$$n\left(\frac{4}{5}\right)^n u[n] \longleftrightarrow Y(e^{j\omega}) = j\frac{d}{d\omega}\left\{\frac{1}{1-\frac{4}{5}e^{-j\omega}}\right\}$$

$$= \frac{(4/5)\,e^{-j\omega}}{\left(1-\frac{4}{5}e^{-j\omega}\right)^2}$$

$$H(e^{j\omega}) = \frac{(4/5)\,e^{-j\omega}}{\left(1-\frac{4}{5}e^{-j\omega}\right)}$$

b) Determine the input output relation

$$Y(e^{j\omega})\left(1-\frac{4}{5}e^{-j\omega}\right) = X(e^{j\omega})\,\frac{4}{5}\,e^{-j\omega}$$

$$y[n] - \frac{4}{5}\,y[n-1] = \frac{4}{5}\,x[n-1]$$

9) Find the output of the following LTI system

$$(n+1)\left(\frac{1}{4}\right)^n u[n] \longrightarrow \underset{\text{LTI}}{\boxed{h[n] = \left(\frac{1}{2}\right)^n u[n]}} \longrightarrow y[n] = ?$$

$$H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2}$$

from Table 5.2 of the textbook

$$Y(e^{j\omega}) = H(e^{j\omega})\,X(e^{j\omega})$$

$$= \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)^2}$$

$$Y(e^{j\omega}) = \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B_1}{1-\frac{1}{4}e^{-j\omega}} + \frac{B_2}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2}$$

$$A = \left.\frac{1}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2}\right|_{e^{-j\omega}=2} = 4 \qquad B_2 = \left.\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right|_{e^{-j\omega}=4} = -1$$

$$\left(\frac{-1}{4}\right) B_1 = \frac{d}{dk}\left.\frac{1}{1-\frac{1}{2}k}\right|_{k=4} = \left.\frac{1/2}{\left(1-\frac{1}{2}k\right)^2}\right|_{k=4} = 1/2$$

be careful

$$B_1 = -2$$

$$y[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n] - (n+1)\left(\frac{1}{4}\right)^n u[n]$$

(9) Find D.T. Fourier transform of

$$h[n] = \left(\frac{1}{2}\right)^n \cos\frac{\pi n}{2}\, u[n]$$

$$h[n] = \left(\frac{1}{2}\right)^n \cdot \frac{1}{2} \cdot \left\{ e^{j\pi n/2} + e^{-j\pi n/2} \right\} u[n]$$

$$= \left\{ \frac{1}{2}\left(\frac{e^{j\pi/2}}{2}\right)^n + \frac{1}{2}\left(\frac{e^{-j\pi/2}}{2}\right)^n \right\} u[n]$$

$$= \frac{1/2}{1-\frac{e^{j\pi/2}}{2}e^{-j\omega}} + \frac{1/2}{1-\frac{e^{-j\pi/2}}{2}e^{-j\omega}}$$

(10) $y[n] + \frac{1}{2} y[n-1] = x[n]$

a) Find $H(e^{jw})$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{1}{1+\frac{1}{2}e^{-jw}}$$

b) If $x[n] = \left(\frac{1}{2}\right)^n u[n]$, find $y[n]$

$$X(e^{jw}) = \frac{1}{1-\frac{1}{2}e^{-jw}} \Rightarrow Y(e^{jw}) = \frac{A}{1+\frac{1}{2}e^{-jw}} + \frac{B}{1-\frac{1}{2}e^{-jw}}$$

$$A = \left.\frac{1}{1-\frac{1}{2}e^{-jw}}\right|_{e^{-jw}=-2} = 1/2 \quad . \quad B = \left.\frac{1}{1+\frac{1}{2}e^{-jw}}\right|_{e^{-jw}=2} = 1/2$$

then $y[n] = \frac{1}{2}\left(-\frac{1}{2}\right)^n u[n] + \frac{1}{2}\left(\frac{1}{2}\right)^n u[n]$

c) If $x[n] = \left(\frac{-1}{2}\right)^n u[n]$ find $y[n]$

$$X(e^{jw}) = \frac{1}{1+\frac{1}{2}e^{-jw}} \Rightarrow Y(e^{jw}) = \frac{1}{\left(1+\frac{1}{2}e^{-jw}\right)^2}$$

then $y[n] = (n+1)\left(\frac{-1}{2}\right)^n u[n]$

d) If $x[n] = \delta[n] + \frac{1}{2}\delta[n-1]$, find $y[n]$

$$X(e^{jw}) = 1 + \frac{e^{-jw}}{2} \Rightarrow Y(e^{jw}) = -1 + \frac{2}{1+\frac{1}{2}e^{-jw}}$$

then
$y[n] = -\delta[n] + 2\left(\frac{-1}{2}\right)^n u[n]$

(11)

$$(e^{-t}+e^{-3t})\,u(t) \rightarrow \overset{\text{LTI}}{\boxed{h(t)}} \rightarrow (2e^{-t}-2e^{-4t})\,u(t)$$

a) Find $h(t)$

$$\frac{2}{j\omega+1}-\frac{2}{j\omega+4}$$

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}$$

$$\frac{1}{j\omega+1}+\frac{1}{j\omega+3}$$

$$=\frac{\dfrac{8+\cancel{2j\omega}-\cancel{2j\omega}-2}{(\cancel{j\omega+1})(j\omega+4)}}{\dfrac{j\omega+3+j\omega+1}{(\cancel{j\omega+1})(j\omega+3)}}=\frac{6}{(j\omega+4)}\cdot\frac{(j\omega+3)}{4+2j\omega}$$

$$=\frac{3(j\omega+3)}{(j\omega+4)(2+j\omega)}=\frac{A}{j\omega+4}+\frac{B}{j\omega+2}$$

$$h(t)=\frac{3}{2}\left(e^{-4t}+e^{-2t}\right)u(t)$$

b) Find the corresponding differential equation

$$3X(j\omega)(j\omega+3)=Y(j\omega)(-\omega^2+6j\omega+8)$$

$$3\frac{d}{dt}x(t)+9x(t)=\frac{d^2}{dt^2}y(t)+6\frac{d}{dt}y(t)+8y(t)$$