1	2	3	4	Total

Name: Answers
Number:

KOM505E - Probability Theory and Stochastic Processes
Midterm #1

Nov. 3, 2016

Rules:

- **Write your solutions to be considered for grading in the boxes given after each part of the problems.** Your solutions outside the boxes below will NOT BE GRADED.
- Closed book & notes.
- This exam has 4 problems. Duration: 120 min.
- This exam will count for 20% of your final grade.

Useful Formulae:

For $|r| < 1$: $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$, $\sum_{k=0}^{\infty} r^{2k} = \frac{1}{1-r^2}$, $\sum_{k=0}^{\infty} r^{2k+1} = \frac{r}{1-r^2}$

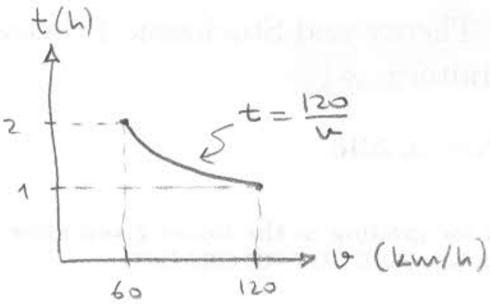
$\frac{d}{dx}\ln x = \frac{1}{x}$

	Values	PDF
Uniform	$a<x<b$	$\frac{1}{b-a}$
Exponential	$x \geq 0$	$\lambda \exp(-\lambda x)$
Gaussian	$-\infty<x<\infty$	$\frac{\exp[-(1/(2\sigma^2))(x-\mu)^2]}{\sqrt{2\pi\sigma^2}}$
Laplacian	$-\infty<x<\infty$	$\frac{1}{\sqrt{2\sigma^2}}\exp(-\sqrt{2/\sigma^2}\|x\|)$
Gamma	$x \geq 0$	$\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\exp(-\lambda x)$
Rayleigh	$x \geq 0$	$\frac{x}{\sigma^2}\exp[-x^2/(2\sigma^2)]$

	Values	PMF
Uniform	$k=-M,...,M$	$\frac{1}{2M+1}$
Bernoulli	$k=0,1$	$p^k(1-p)^{1-k}$
Binomial	$k=0,1,...,M$	$\binom{M}{k} p^k(1-p)^{M-k}$
Geometric	$k=1,2,...$	$(1-p)^{k-1}p$
Poisson	$k=0,1,...$	$\exp(-\lambda)\frac{\lambda^k}{k!}$

1. (25 pts) Buses are traveling between 2 cities that are 120 km apart. Assume that each bus travels with a constant speed during their trip. The speeds of the buses are uniformly distributed between 60 and 120 km/h.

   (a) Find the pdf of the durations. Note that speed times duration gives the distance.



$$f_V(v) = \begin{cases} 1/60 & \text{if } 60 \leq v \leq 120 \\ 0 & \text{elsewhere} \end{cases}$$

$$t = g(v) = \frac{120}{v} \qquad \left|\frac{d}{dv} g(v)\right| = \left|-\frac{120}{v^2}\right| = \frac{120}{v^2} = \frac{120}{\left(\frac{120}{t}\right)^2} = \frac{t^2}{120}$$

$g(v)$ is a one-to-one function. For each t value, there is a single v s.t. $t = \frac{120}{v}$.

Hence

$$f_T(t) = \frac{f_V(v)}{\left|\frac{d}{dv} g(v)\right|} = \frac{f_V(120/t)}{t^2/120} = \frac{120}{t^2} f_V(120/t)$$

and

$$f_V(120/t) = \begin{cases} 1/60 & \text{if } 1 \leq t \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

Then

$$f_T(t) = \begin{cases} 2/t^2 & \text{if } 1 \leq t \leq 2 \\ 0 & \text{, elsewhere} \end{cases}$$

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(b) Find the mean duration for this trip.

$$\mu_T = \int_{-\infty}^{\infty} t\, f_T(t)\, dt = \int_1^2 t \cdot \frac{2}{t^2}\, dt = \int_1^2 2/t\, dt$$

$$= 2 \ln t \Big|_1^2$$

$$= 2\ln 2 - 2\cancel{\ln 1}^{\,0}$$

$$\mu_T = 2 \ln 2$$

(c) Find the variance of duration for this trip.

$$\sigma_T^2 = E(t^2) - \mu_T^2$$

$$E(t^2) = \int_{-\infty}^{\infty} t^2 f_T(t)\, dt = \int_1^2 t^2 \cdot \frac{2}{t^2}\, dt = 2t\Big|_1^2 = 4 - 2 = 2$$

Hence

$$\sigma_T^2 = 2 - 4(\ln 2)^2$$

2. (25 pt) The cdf of a continuous random variable (X) is given as follows:

$$F(x) = K - \exp\{-\frac{x^2}{6}\}, \quad \text{for } x \geq 0, \text{where K is a real valued constant.}$$

a) Find the value of K.

For any cdf

$$\lim_{x \to \infty} F(x) = 1$$

$$\lim_{x \to \infty} \{K - \underbrace{e^{-x^2/6}}_{=0 \text{ as } x \to \infty}\} = K$$

Hence $K = 1$

b) Find the pdf of X: $f_X(x)$.

$$f_X(x) = \frac{d}{dx} F(x) = \frac{d}{dx} \{1 - e^{-x^2/6}\}$$

$$= (-e^{-x^2/6}) \cdot \underbrace{\frac{d}{dx}(-x^2/6)}_{-\frac{2x}{6}}$$

$$f_X(x) = \begin{cases} \frac{x}{3} e^{-x^2/6} & \text{for } x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

c) Find the probability of $3 \leq X \leq 5$.

$$P(3 \leq X \leq 5) = F(5) - F(3)$$

$$= (1 - e^{-25/6}) - (1 - e^{-9/6})$$

$$= e^{-9/6} - e^{-25/6}$$

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3. (25 pt) In last years probability course, 30% of the students were female. 70% of the male students and 75% of female students passed this course. If we select a random student and observe that this student failed the course, what is the probability of this student being a male?

$p = 0.7$ (male) $\quad (1-p) = 0.3$ (female)

$A = \{$student being male$\} \Rightarrow A^c = \{$student female$\}$

ne ...iables

$B = \{$student failed$\}$

$$P(A|B) = P(\text{male} \mid \text{failed})$$

$$= \frac{P(B|A)\,P(A)}{P(B)}$$

Use Bayes

$$= \frac{P(B|A)\,P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

$$= \frac{(0.3)(0.7)}{(0.3)(0.7) + (0.25)(0.3)}$$

(insert correct probabilities)

$$= \frac{0.21}{0.21 + 0.075} = \frac{0.21}{0.285}$$

...orrect calculations)

$$\cong 0.73$$

2100 | 285
1995 | 0.73
0105
855

0.83
1.70
2.53

4. (25 pts)

(a) You dial the number of a call center, and from your past experience you know that roughly the call center line is busy 7 times out of your 10 calls. You have a meeting starting just now, but you need to make this call before your meeting. Each time you call. if the line is busy, you call again after 2 minutes. What is the probability that you will be late to your meeting by more than 7 minutes? (Assume once you get to speak to the call center, you are done immediately).

Geom. prob. law w/ $p = 0.3$; $1-p = 0.7$ (busy = fail)

t=0 2min 4 min 6min

1 Fail 2 F 3 F 4 F

+2 $P[X > 4] = P$ (late more than 7 minutes)

$=$ 1st success at 5th or later

+2 $P[X>4] = 1 - P[X \le 4] = 1 - \{P[X=1] + P[X=2] + P[X=3] + P[X=4]\}$

+1 $= 1 - \{p + (1-p)p + (1-p)^2 p + (1-p)^3 p\} = 1 - \{0.3[1 + (0.7) + (0.7)^2 + (0.7)^3]\}$

+1 $= 1 - \{0.3[1.7 + 0.49 + 0.34]\} = 1 - 0.3(2.5) = 1 - (0.75) = 0.25$

(b) Continuing from part 4-(a) (answer independently): Suppose, you tried to call the call center m times with no success. Knowing that, what is the probability of first successful call at the (m+l)th trial? How does your result relate to the probability of first successful call at the (l)th trial?

+1 $P(1^{st}$ success at $(m+l)^{th}$ trial | no success at first m trials$) = ?$

+2 $= \dfrac{P(1^{st} \text{ success at } (m+l)^{th} \text{ trial})}{P(\text{no success at } m \text{ trials})} = \dfrac{(1-p)^{m+l-1}\, p}{(1-p)^m} = (1-p)^{l-1} p$

+2 $\Rightarrow$ also equal to $P(1^{st}$ success at l^{th} trial$)$ +2

(c) Part 4-(c) is independent of parts (a) and (b). Consider n+m independent Bernouilli trials, where the success probability is given by p for each trial. What is the probability of having one success in the first n trials given that there are k successes in all n+m trials?

Binomial law;

+1 $P(1$ success in first n trials | k successes in n+m trials$) = ?$

+2 $= \dfrac{P(1 \text{ success in first } n \text{ trials},\ k-1 \text{ successes in } m \text{ trials})}{P(k \text{ successes in } n+m \text{ trials})}$

+2 $= \dfrac{\binom{n}{1} p^1 (1-p)^{n-1} \binom{m}{k-1} (1-p)^{m-k+1} p^{(k-1)}}{\binom{n+m}{k} (1-p)^{n+m-k} p^k} = \dfrac{n \binom{m}{k-1}}{\binom{n+m}{k}}$

+2

(d) Assume that an infinite sequence of independent Bernouilli trials are performed, where each trial success probability is given by $p < 1$. What is the probability that all trials result in success ?

+2 $\underbrace{HHH \cdots H}_{n \text{ trials}} \cdots$ $P(\text{all heads in } n \text{ trials}) = p^n$

+1 $\lim_{n\to\infty} P(\text{all heads}) = \lim_{n\to\infty} p^n = 0$, $p<1$

+1 $\Rightarrow P(\text{all trials result in success for an infinite seq. of trials}) = 0$