Homework #2
Solution Key

Q1) $n=0$

$$a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos(nx)\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)\,dx\right.$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \pi\int_{\pi/2}^{\pi}dx - \int_{\pi/2}^{\pi}x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\Big|_0^{\pi/2} + \pi x\Big|_{\pi/2}^{\pi} - \frac{x^2}{2}\Big|_{\pi/2}^{\pi}\right]\right.$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right] = \frac{2}{\pi}\left[\frac{\pi^2}{8}\right] \Rightarrow a_0 = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos(nx)\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos(nx)\,dx\right.$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos(nx)\,dx + \pi\int_{\pi/2}^{\pi}\cos nx\,dx - \int_{\pi/2}^{\pi}x\cos(nx)\,dx\right]$$

$$= \frac{2}{\pi}\left[x\frac{\sin(nx)}{n}\Big|_0^{\pi/2} + \frac{\cos(nx)}{n^2}\Big|_0^{\pi/2} + \frac{\pi}{n}\sin(nx)\Big|_{\pi/2}^{\pi}\right.$$

$$- \frac{x\sin(nx)}{n}\Big|_{\pi/2}^{\pi} - \frac{\cos(nx)}{n^2}\Big|_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[\cancel{\frac{\pi}{2n}\sin\left(\frac{\pi n}{2}\right)} + \frac{1}{n^2}\cos\left(\frac{\pi n}{2}\right) - \frac{1}{n^2} \cancel{-\frac{\pi}{n}\sin\left(\frac{\pi n}{2}\right)} \cancel{+\frac{\pi}{2n}\sin\left(\frac{\pi n}{2}\right)}\right.$$

$$-\frac{1}{n^2}\cos(\pi n) + \frac{1}{n^2}\cos\left(\frac{\pi n}{2}\right)$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\cos\left(\frac{\pi n}{2}\right) - \frac{1}{n^2} - \frac{1}{n^2}\cos(\pi n)\right] = \frac{2}{\pi n^2}\left[2\cos\left(\frac{\pi n}{2}\right) - 1 -\right.$$

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos(nx)$$

$$=\frac{\pi}{4}+\sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left[2\cos\left(\frac{n\pi}{2}\right)-1-(-1)^n\right]\cos nx$$

Q2) $x[n]$ is the periodic extension of $\{0,1,2,3\}$ with fudamental period $N_0=4$. thus, $\Omega_0=\frac{2\pi}{4}$ $\cdot N=4$ $\Omega_0=\frac{\pi}{2}$

$$C_k=\frac{1}{4}\sum_{n=0}^{3}x[n]\,e^{-jk\frac{\pi}{2}n} \qquad e^{-j\Omega_0}=e^{-j2\pi/4}=e^{-j\pi/2}=-j$$

$$C_k=\frac{1}{4}\left[x[0]+x[1]e^{-jk\frac{\pi}{2}}+x[2]e^{-jk\pi}+x[3]e^{-jk\frac{3\pi}{2}}\right]$$

$$C_k=\frac{1}{4}\left[e^{-jk\frac{\pi}{2}}+2e^{-jk\pi}+3e^{-jk\frac{3\pi}{2}}\right]$$

$$C_0=\frac{1}{4}[1+2+3]=\frac{3}{2}$$

$$C_1=\frac{1}{4}\left[e^{-j\frac{\pi}{2}}+2e^{-j\pi}+3e^{-j\frac{3\pi}{2}}\right]=\frac{1}{4}[-j-2-3j]=-\frac{1}{2}+\frac{j}{2}$$

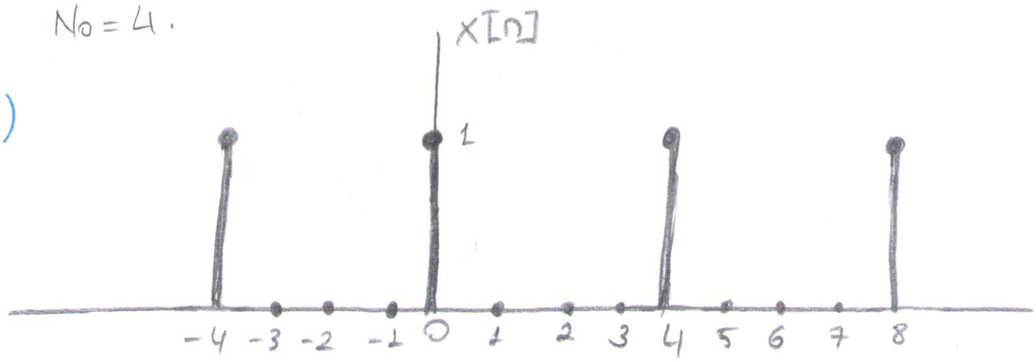
$$C_2=\frac{1}{4}\left[e^{-j\pi}+2e^{-j2\pi}+3e^{-j3\pi}\right]=\frac{1}{4}[-1+2-3]=-\frac{1}{2}$$

$$C_3=\frac{1}{4}\left[e^{-j\frac{3\pi}{2}}+2e^{-j3\pi}+3e^{-j\frac{3\pi}{2}}\right]=\frac{1}{4}[-j-2-3j]=-\frac{1}{2}-\frac{j}{2}$$

Note that $C_3=C_{4-1}=C_1$

Q3) $x[n]$ is the periodic extension of the sequence $\{1,0,0,0\}$ with period $N_0 = 4$.

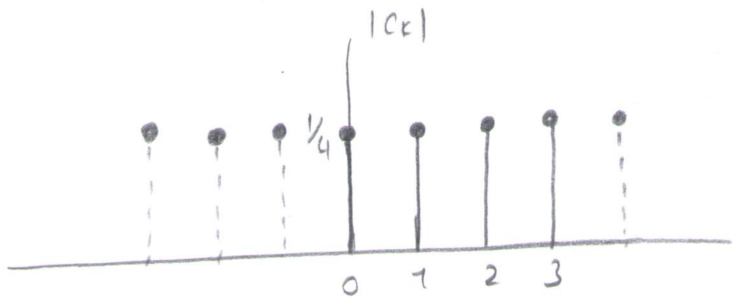
a)



b) $$x[n] = \sum_{k=0}^{3} c_k e^{jk(2\pi/4)n} = \sum_{k=0}^{3} c_k e^{jk(\pi/2)n}$$

$$c_k = \frac{1}{4}\sum_{n=0}^{3} x[n] e^{-jk(2\pi/4)n} = \frac{1}{4} x[0] = \frac{1}{4} \quad \text{for all } k$$

$x[1] = x[2] = x[3] = 0$. The fourier coefficients of $x[n]$ are sketched belo



Q4) $$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} -a^n u(n-11) e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{-11} -a^n e^{-j\Omega n} = \sum_{m=11}^{\infty} -a^{-m} e^{j\Omega m} = -\sum_{m=11}^{\infty} (a^{-1} e^{j\Omega})^m$$

$m = k+11$

$$= -\sum_{k=0}^{\infty} (a^{-1} e^{j\Omega})^{k+11}$$

$$= -(a^{-1}e^{j\Omega})^{11} \sum_{k=0}^{\infty} (a^{-1} e^{j\Omega})^k = \frac{-(a^{-1}e^{j\Omega})^{11}}{1 - a^{-1}e^{j\Omega}} ; \ |a e^{-1 j\Omega}| < 1$$

$|a| > 1$

$$|a| > 1 \Rightarrow X(e^{j\Omega}) = \frac{-(a^{-1}e^{j\Omega})^{11}}{1-(a^{-1}e^{j\Omega})}$$

Q5) from figure, we see that

$$x[n] = x_1[n+N_1]$$

Setting $N = 2N_1 + 1$ in equation $\boxed{e^{-j\Omega(N-1)/2}\dfrac{\sin(\Omega N/2)}{\sin(\Omega/2)}}$

$$X_1(\Omega) = e^{-j\Omega N_1}\frac{\sin\left[\Omega\left(N_1+\frac{1}{2}\right)\right]}{\sin(\Omega/2)}$$

from time-shifting property : $\boxed{x[n-n_0] \leftrightarrow e^{-j\Omega n_0}X(\Omega)}$

$$X(\Omega) = e^{j\Omega N_1}X_1(\Omega) = \frac{\sin\left[\Omega\left(N_1+\frac{1}{2}\right)\right]}{\sin(\Omega/2)}$$