Lecture 5

- **Read:** Chapter 3.5-3.8.
- Multiple Discrete RVs
  - Conditioning
    - Conditioning a Joint PMF by an Event
    - Conditional PMF
  - Independent Random Variables
  - More Than Two Discrete Random Variables

Conditioning

- **Definition:(Conditioning Joint PMF by an Event)** The conditional joint PMF of (X, Y) given some event B, a set on the x-y plane, is defined as

$$\begin{aligned} p_{X,Y|B}(x,y) &= P[X=x, Y=y|B] \\ &= \frac{P[\{X=x\} \cap \{Y=y\} \cap B]}{P[B]} \\ &= \begin{cases} \frac{p_{X,Y}(x,y)}{P[B]} & , (x,y) \in B \\ 0 & , \text{otherwise} \end{cases} \end{aligned}$$

- **Definition:(Conditioning an RV Based on Another)** The conditional PMF of X given Y is

$$\begin{aligned} p_{X|Y}(x|y) &= P[X=x|Y=y] \\ &= \frac{P[\{X=x\} \cap \{Y=y\}]}{P[Y=y]} = \frac{p_{X,Y}(x,y)}{p_Y(y)} \end{aligned}$$

Conditional Expectation

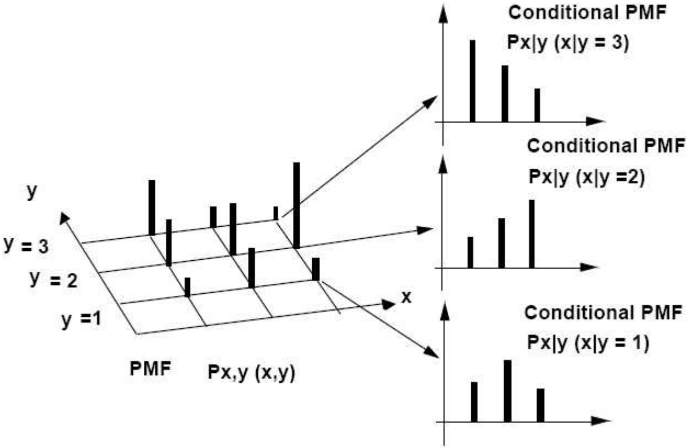
- **Definition:(Conditional Expectation of a Function)** Conditional expectation of $g(X, Y)$ given B

$$E[g(X, Y)|B] = \sum_x \sum_y g(x, y) \cdot p_{X,Y|B}(x, y)$$

- **Definition:(Conditional Expectation)** Conditional expectation of X given $Y = y$

$$E[X|Y = y] = \sum_x x p_{X|Y}(x|y)$$

Visualizing Conditioning



Summary

Joint PMF: $p_{X,Y}(x,y) = P[X = x, Y = y]$

if independent

marginal PMF

$$p_X(x) = \sum_{y \in S_Y} p_{X,Y}(x,y)$$

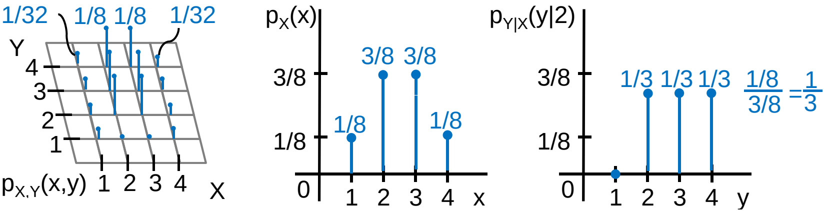
$$p_Y(y) = \sum_{x \in S_X} p_{X,Y}(x,y)$$

conditional PMF

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

$$p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$$

Summary: Visualizing Conditioning



- If independent, then given the value of X, the distribution of Y would not change.

Conditional Expected Values

- **Idea:** Given $Y = y$, the PMF of X is $p_{X|Y}(x|y)$, so the average of X is now given by

$$E[X|Y = y] = \sum_{x \in S_X} x p_{X|Y}(x|y)$$

We can generalize this definition and define the conditional expectation of a function of the random variable X...

Conditional Expected Value of a Function

- **Definition:(Conditional Expected Value of a Function)**
  For any $y \in S_Y$, the conditional expected value of $g(X)$ given $Y = y$ is

$$E[g(X)|Y = y] = \sum_{x \in S_X} g(x) p_{X|Y}(x|y)$$

Special Cases:

1. E[X|Y= y]
2. Conditional Variance

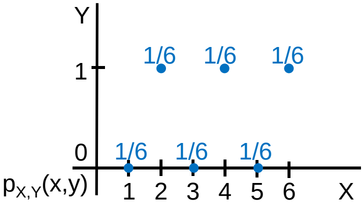
$$\begin{aligned} Var[X|Y = y] &= E[(X - E[X|Y = y])^2 | Y = y] \\ &= \sum_{x \in S_X} (x - E[X|Y = y])^2 p_{X|Y}(x|y) \end{aligned}$$

Conditional Expectation: Example (I)

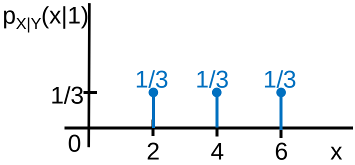
- Let $X =$ outcome of roll of die

$$Y = \begin{cases} 1 & \text{, if } X \text{ was even} \\ 0 & \text{, if } X \text{ was odd} \end{cases}$$

- What are $p_{X,Y}(x,y)$, $p_{X|Y}(x|1)$?



$$p_{X|Y}(x|1) = \frac{p_{X,Y}(x,1)}{p_Y(1)} = \frac{p_{X,Y}(x,1)}{1/2}$$



Conditional Expectation: Example (II)

- Let $X =$ outcome of roll of die

$$Y = \begin{cases} 1 & \text{, if } X \text{ was even} \\ 0 & \text{, if } X \text{ was odd} \end{cases}$$

- What are $E[X|Y = 1]$ and $Var[X]$?

. . . .

$$E[X|Y = 1] = \sum_{x \in S_X} x p_{X|Y}(x|1) = 2 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} + 6 \cdot \frac{1}{3} = 4$$

$$\begin{aligned} Var[X] &= E[(X - E[X])^2] = E[(X - 3.5)^2] = E[X^2] - (E[X])^2 \\ &= \frac{1}{6}[(1 - 3.5)^2 + (2 - 3.5)^2 + (3 - 3.5)^2 + (4 - 3.5)^2 \\ &\qquad + (5 - 3.5)^2 + (6 - 3.5)^2] \end{aligned}$$

Conditional Expectation: Example (III)

- Let $X =$ outcome of roll of die

$$Y = \begin{cases} 1 & \text{, if } X \text{ was even} \\ 0 & \text{, if } X \text{ was odd} \end{cases}$$

- What is $Var[X|Y = 1]$?

. . . .

$$\begin{aligned} Var[X|Y = y] &= E[(X - E[X|Y = y])^2 | Y = y] \\ &= \sum_{x \in S_X} (x - E[X]|Y = y)^2 p_{X|Y}(x|y) \\ Var[X|Y = 1] &= E[X^2|Y = 1] - (E[X|Y = 1])^2 \\ &= 4 \cdot \frac{1}{3} + 16 \cdot \frac{1}{3} + 36 \cdot \frac{1}{3} - 16 \end{aligned}$$

Conditional Expectation

- **Definition:(Conditional Expectation)** Let $g(y) = E[X|Y = y]$. Then, $g(Y)$ is called the conditional expectation of X given Y and is written as $E[X|Y]$.
- **Warning:** $E[X|Y] = g(Y)$ is a function of an RV, Y, so it is also an RV, i.e., necessarily just a number.

  Since $E[X|Y]$ is an RV, we should be able to take its average!
- **Theorem:**(Iterated Expectation) $E[E[X|Y]] = E[X]$
  - The significance of the iterated expectation is that there are problems in which calculating $E[X]$ directly is much more difficult than first calculating $E[X|Y]$ and then using the iterated expectation to find $E[X]$.

Iterated Expectation: Proof of Theorem

- **Theorem:** E[E[X|Y]] = E[X]
- **Proof:**

$$
\begin{aligned}
E[E[X|Y]] &= E[g(Y)] \\
&= \sum_y g(y) p_Y(y) \\
&= \sum_y E[X|Y = y] p_Y(y) \\
&= \sum_y \left(\sum_x x p_{X|Y}(x|y) \right) p_Y(y) \\
\sum_y \sum_x x p_{X|Y}(x|y) p_Y(y) &= \sum_y \sum_x x p_{X,Y}(x, y) \\
&= \sum_x x \sum_y p_{X,Y}(x, y) \\
&= \sum_x x p_X(x) = E[X]
\end{aligned}
$$

Iterated Expectation: Example

- Suppose
  $Y = 0 \Rightarrow$ Male, P[Y$=$ 0]$=$ 1/2
  $Y = 1 \Rightarrow$ Female

- Let $X =$ age of person
  $E[X|Y = 0] =$ average age of males
  $Var[X|Y = 1] =$ variance of ages of females

. . . .

$$E[X|Y] = g(Y) = \begin{cases} E[X|Y = 0] & \text{, with probability } 1/2 \\ E[X|Y = 1] & \text{, with probability } 1/2 \end{cases}$$

$$\begin{aligned} g(Y) &= E[X|Y = y] \\ E[E[X|Y]] &= E[X|Y = 0]P[Y = 0] + E[X|Y = 1]P[Y = 1] \\ &= E[X] \end{aligned}$$

Independent Random Variables

- **Definition:** X and Y are independent if $\{\mathrm{X}=\mathrm{x}\}$ and $\{\mathrm{Y}=\mathrm{y}\}$ are independent events for all x, y.

$$P[\{X=x\}\cap\{Y=y\}] = P[X=x]\cdot P[Y=y]$$
$$\Rightarrow p_{X,Y}(x,y) = p_X(x)\cdot p_Y(y)$$

Equivalently, since for all x, y such that $p_Y(y)>0$

$$p_{X,Y}(x,y) = p_{X|Y}(x|y)\cdot p_Y(y) \underbrace{=}_{\text{independent}} p_X(x)p_Y(y)$$

we have independence when $p_{X|Y}(x|y)=p_X(x)$.

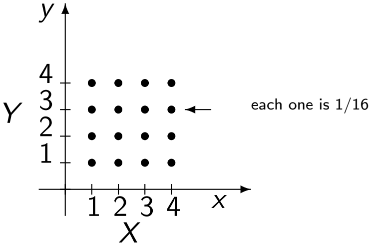
  - **Interpretation:** knowledge of the experimental value of Y does not affect PMF of X.
  - Given two functions g and h, if X and Y are independent, so are $g(X)$ and $h(Y)$.
  - **Remark:** Independence iff can factorize the joint PMFs into functions of x and y alone.

Summary: Independent RVs

Independent RVs: $p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y)$

$$
\begin{aligned}
\text{Alternatively, } p_{X|Y}(x|y) &= p_X(x) \\
p_{Y|X}(y|x) &= p_Y(y)
\end{aligned}
$$

Independent RVs: Example

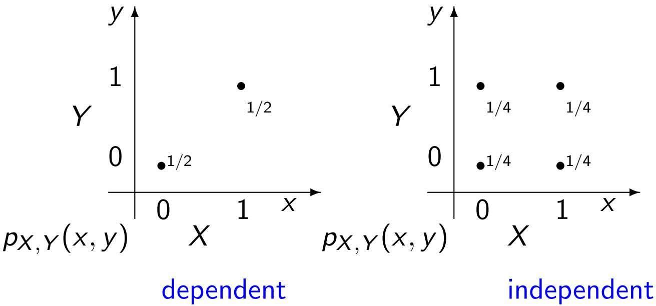


$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} = \frac{1/16}{1/4} = p_X(x)$$

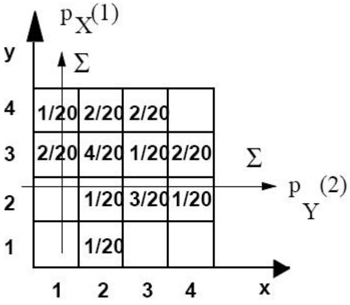
independent RVs

Support (Range) of Independent RVs

- **Note:** Independent RVs will have a range (support) $S_{X,Y} = S_X \times S_Y$, i.e., rectangular.



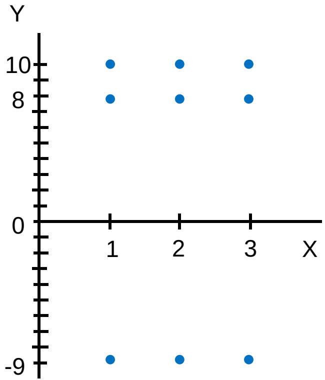
Support of RVs



- Consider the joint PMF of the two RVs in the table above.
- Are they independent?
- Does a choice of $X = x$, e.g., 1 or 2, impact the possible values Y can take?

Do Only Independent RVs Have Rectangular Supports?

- **Example:**



$$X = \begin{cases} 1 & \text{, with probability } 1/6 \\ 2 & \text{, with probability } 2/6 \\ 3 & \text{, with probability } 3/6 \end{cases}$$

$$Y = \begin{cases} -9 & \text{, with probability } 1/3 \\ 8 & \text{, with probability } 1/3 \\ 10 & \text{, with probability } 1/3 \end{cases}$$

- **Warning:** Ranges that are "rectangular" in this sense may or may not correspond to independent random variables!

Joint Distribution from Marginals

- **Question:** Given the marginal distributions of two RVs, can you reconstruct the joint distribution?
- **Answer:** Yes, if they are independent!

Facts About Independent RVs

If X and Y are independent

- $E[XY] = E[X]E[Y]$, so independent $\implies$ uncorrelated, i.e., $Cov[X, Y] = 0$.
- $E[X|Y = y] = E[X]$ for all $y \in S_Y$
- $E[Y|X = x] = E[Y]$ for all $x \in S_X$
- $E[f(X)g(Y)] = E[f(X)]E[g(Y)]$
- $Var[X + Y] = Var[X] + Var[Y]$

Independence of Several Random Variables

- X, Y, and Z are independent if
$$p_{X,Y,Z}(x,y,z) = p_X(x)p_Y(y)p_Z(z)$$
for all x, y, and z.
- If so, $f(X)$, $g(Y)$, and $h(Z)$ would also be independent.
- Typically, $f(X,Y)$ and $g(Y,Z)$ would not.
- Also, $Var[X+Y+Z] = Var[X] + Var[Y] + Var[Z]$.
- **Example:** Suppose $X_i \sim$ Bernoulli(p), and let $Y = \sum_{i=1} X_i$. Find its variance.

More Than Two Discrete Random Variables: Joint PMF

- **Definition:(Joint PMF of N Random Variables)** The joint PMF of the discrete random variables $X_1, ..., X_n$ is

$$p_{X_1,...,X_n}(x_1, ..., x_n) = P[X_1 = x_1, ..., X_n = x_n]$$

More Than Two Discrete Random Variables: Probability of an Event A

- **Theorem:** The probability of an event A expressed in terms of the discrete random variables $X_1, \ldots, X_n$ is

$$P[A] = \sum_{(x_1,\ldots,x_n)\in A} p_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$$

More Than Two Discrete Random Variables: Marginal PMFs

- **Theorem:** For a joint PMF of four random variables, $p_{W,X,Y,Z}(w,x,y,z)$, some marginal PMFs are

$$p_{X,Y,Z}(x,y,z) = \sum_{w \in S_W} p_{W,X,Y,Z}(w,x,y,z)$$

$$p_{W,Z}(w,z) = \sum_{x \in S_X} \sum_{y \in S_Y} p_{W,X,Y,Z}(w,x,y,z)$$

$$p_{Y,Z}(w,z) = \sum_{w \in S_W} \sum_{x \in S_X} p_{W,X,Y,Z}(w,x,y,z)$$

$$p_X(x) = \sum_{w \in S_W} \sum_{y \in S_Y} \sum_{z \in S_Z} p_{W,X,Y,Z}(w,x,y,z)$$

N Independent Random Variables

- **Definition:(N Independent Random Variables)** The discrete random variables $X_1, ..., X_n$ are independent if and only if

$$p_{X_1,...,X_n}(x_1, ..., x_n) = p_{X_1}(x_1) \cdots p_{X_n}(x_n)$$

for all $x_1, ..., x_n$.

Expectations

- **Theorem:** The expected value of $g(X_1, ..., X_n)$ satisfies

$$E[g(X_1, ..., X_n)] = \sum_{x_1 \in S_{X_1}} ... \sum_{x_n \in S_{X_n}} g(x_1, ..., x_n) p_{X_1,...,X_n}(x_1, ..., x_n)$$

- **Theorem:** When $X_1, ..., X_n$ are independent discrete random variables,

$$E[g(X_1)g(X_2)\cdots g(X_n)] = E[g(X_1)]E[g(X_2)]\cdots E[g(X_n)]$$