

Practice Session for week 2

In below table, CPU temperature measurements are shown. Calculate true error and relative true error.

Samples	Temperature (C)			
	Truth	Observed	True Error	Relative True Error
1	40,58	40,32	$40,58 - 40,32 = 0,26$	$0,26 / 40,58 = \% 0,6407$
2	40,65	40,77	$40,65 - 40,77 = -0,12$	$-0,12 / 40,65 = -\% 0,2952$
3	41,34	41,36	$41,34 - 41,36 = -0,02$	$-0,02 / 41,34 = -\% 0,0483$
4	41,91	42,20	$41,91 - 42,20 = -0,29$	$-0,29 / 41,91 = -\% 0,6919$

Derivative of a function f at a point x is defined by the equation;

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad f'(x) \approx \frac{f(x+h) - f(x)}{h} \quad \text{for small } h \text{ values.}$$

- If $f(x) = x^3 + 3x^2$ and $h = 0,6$
Find approximate value of $f'(3)$
Find true value of $f'(3)$
Find true error, relative true error.

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \quad f'(x) \approx \frac{f(3+0,6) - f(3)}{0,6} \approx \frac{f(3,6) - f(3)}{0,6}$$

$$f'(x) \approx \frac{(46,656 + 38,88) - (54)}{0,6} \approx \frac{31,536}{0,6} \approx 52,56 \text{ is approximate value.}$$

$$f'(x) = 3x^2 + 6x \text{ is derivative function of function } f(x)$$

$$f'(3) = 27 + 18 = 45 \text{ is true value.}$$

$$\text{True error} = \text{true value} - \text{approximate value} = 45 - 52,56 = -7,56$$

$$\text{Relative true error} = \text{true error} / \text{true value} = -7,56 / 45 = -0,168 (\% -16,8)$$

- Compute approximate error and relative approximate error for $f'(3)$ using $h = 0,6$ and $h = 0,4$

Approximate error = present approximation – previous approximation

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \quad f'(x) \approx \frac{f(3+0,4) - f(3)}{0,4} \approx \frac{f(3,4) - f(3)}{0,4}$$

$$f'(x) \approx \frac{(39,304 + 34,68) - (54)}{0,4} \approx \frac{73,984 - 54}{0,4} \approx \frac{19,984}{0,4} \approx 49,96 \text{ is present approximation.}$$

$$\text{Approximate error} = 49,96 - 52,56 = -2,56$$

Relative approximation error = approximation error / present approximation

$$= \frac{-2,56}{49,96} = -0,05204 \quad (\% -5,2041)$$

Error measurement on floating point format.

Floating decimal point scientific form : $sign \times mantissa \times 10^{exponent}$

$$\sigma \times m \times 10^e \quad -2,5678 \times 10^2 \quad \sigma = -1 \quad m = 2,5678 \quad e = 2$$

$[(1)_2 < m < (10)_2]$ 1 is not stored as it is always given to be 1.

Let we have a floating point format ten bit word;

sn	se	e	e	e	e	m	m	m	m
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- * 1 bit for sign of number
- * 1 bit for sign of exponent
- * 4 bit for exponent
- * 4 bit for mantissa

Display $(123,9631)_{10}$ floating point number for the given format and calculate the true error while representation if it occurs.

$$(123)_{10} = (1111011)_2 \quad (0 \times 2^7) + (1 \times 2^6) + (1 \times 2^5) + (1 \times 2^4) + (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (1 \times 2^0)$$

$$(0,9631)_{10} = (?)_2$$

	Result	Integer part	Fractional part
$0,9631 \times 2$	1,9262	1	0,9262
$0,9262 \times 2$	1,8524	1	0,8524
$0,8524 \times 2$	1,7048	1	0,7048
$0,7048 \times 2$	1,4096	1	0,4096

0,4096*2	0,8192	0	0,8192
0,8192*2	1,6384	1	0,6384
0,6384*2	1,2768	1	0,2768
0,2768*2	0,5536	0	0,5536
0,5536*2	1,1072	1	0,1072

$$1111011,111101101 = 1,111011111101101 * 2^6$$

sn	se	e	e	e	e	m	m	m	m
0	0	0	1	1	0	1	1	1	0

$$1,1110 * 2^6 = (1111000)_2 = 120$$

$$\text{True error} = 123,9631 - 120 = 3,9631$$

8 – bit mantissa example for the same question

sn	se	e	e	e	e	m	m	m	m	m	m	m	m
0	0	0	1	1	0	1	1	1	0	1	1	1	1

$$1,11101111 * 2^6 = (1111011,11)_2 = 123,75$$

$$\text{True error} = 123,9631 - 123,75 = 0,2131$$

Practice Session for week 3

Use Naive Gauss Elimination to solve the system of linear equations.

$$x_1 + 2x_2 + x_3 = -8$$

$$2x_1 - 2x_2 - 3x_3 = 0$$

$$-x_1 + x_2 + 2x_3 = 3$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \\ 3 \end{bmatrix}$$

Add -2 times row1 to row2

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \\ 3 \end{bmatrix}$$

Add -1 times row1 to row3

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ 0 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \\ -5 \end{bmatrix}$$

Add (1/2) times row2 to row3

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \\ 3 \end{bmatrix}$$

Now elimination stops.

$$0.5x_3 = 3$$

$$x_3 = 6$$

$$-6x_2 - 5 * 6 = 16$$

$$x_2 = -\frac{23}{3}$$

$$x_1 + 2x_2 + x_3 = -8$$

$$x_1 = \frac{4}{3}$$

Solve same linear system using LU decomposition.

$$Ax=B$$

Apply gauss elimination **only matrix A**.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \quad \text{same operation as gauss elimination } U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ 0 & 0 & 0.5 \end{bmatrix}$$

Now find [L] matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad l_{21} = \frac{a_{21}}{a_{11}}, \quad l_{31} = \frac{a_{31}}{a_{11}}, \quad l_{32} = \frac{a_{32}}{a_{22}}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \\ 3 \end{bmatrix}$$

$$Z = \begin{bmatrix} -8 \\ 16 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \\ 3 \end{bmatrix}$$

$$0.5x_3 = 3$$

$$x_3 = 6$$

$$-6x_2 - 5 * 6 = 16$$

$$x_2 = -\frac{23}{3}$$

$$x_1 + 2x_2 + x_3 = -8$$

$$x_1 = \frac{4}{3}$$

Details of LU decomposition.

$$A=LU$$

$$LUX=B$$

$$UX=L^{-1}B$$

$$x=U^{-1}L^{-1}B$$

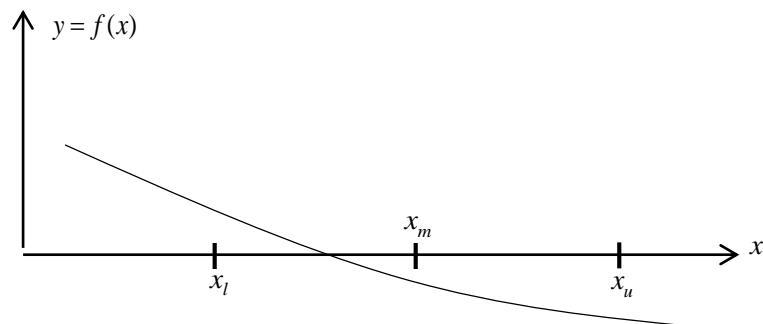
$$x=L^{-1}Z$$

Practice Session for week 4

Example of Bisection Method

A function $f(x)$ is defined as $f(x) = x^2 - 4x - 5$. For $y = f(x) = 0$ estimate a root of this function using Bisection method. Use $[0, 48]$ as initial estimation range points and use absolute relative approximate error notation for error of estimated root at each iteration.

Recall



$$x_m = \frac{x_l + x_u}{2}$$

$$|E| = \left| \frac{x_m^{new} - x_m^{old}}{x_m^{new}} \right| * 100$$

$f(x_l) * f(x_m) < 0$ the root lies between x_l and x_m

$f(x_l) * f(x_m) > 0$ the root lies between x_m and x_u

$f(x_l) * f(x_m) = 0$ x_m is root

$$\begin{aligned}
x_l &= 0 & f(x_l) &= f(0) = 0 - 0 - 5 = -5 \\
x_u &= 48 & f(x_u) &= f(48) = 2304 - 192 - 5 = 2107 & x_m &= (0 + 48) / 2 = 24 \\
x_m &= 24 & f(x_m) &= f(24) = 576 - 96 - 5 = 475 \\
f(x_l) * f(x_m) &< 0 & \text{the root lies between } x_l \text{ and } x_m & \text{ so new estimation range } [0, 24]
\end{aligned}$$

$$\begin{aligned}
x_l &= 0 & f(x_l) &= f(0) = 0 - 0 - 5 = -5 \\
x_u &= 24 & f(x_u) &= f(24) = 576 - 96 - 5 = 475 & x_m &= (0 + 24) / 2 = 12 \\
x_m &= 12 & f(x_m) &= f(12) = 144 - 48 - 5 = 91 \\
f(x_l) * f(x_m) &< 0 & \text{the root lies between } x_l \text{ and } x_m & \text{ so new estimation range } [0, 12]
\end{aligned}$$

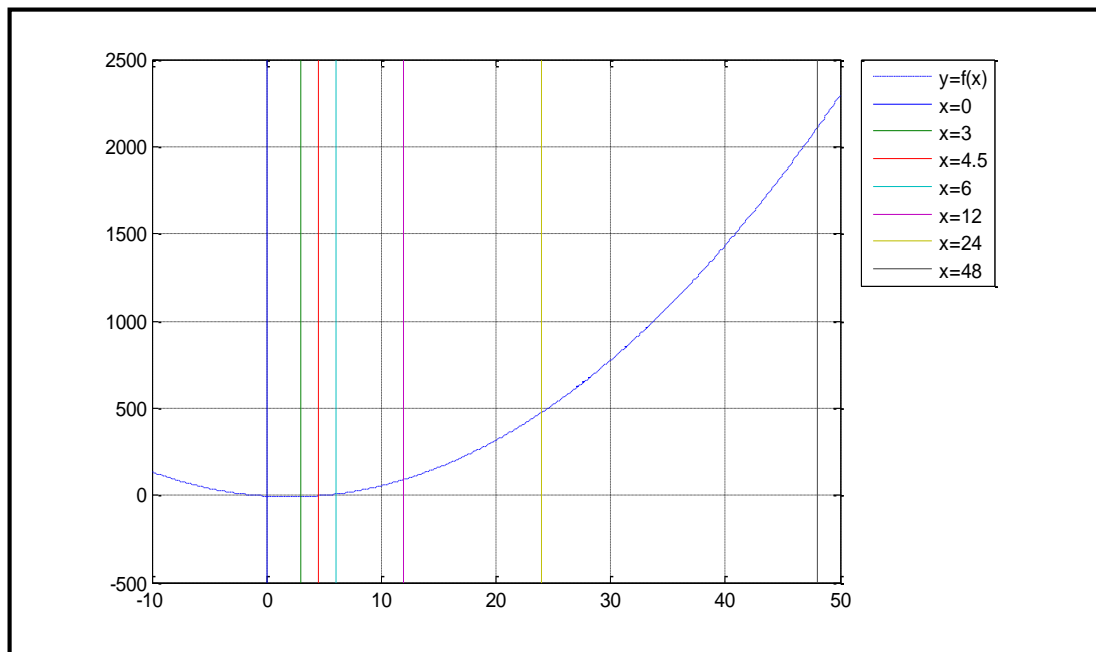
$$\begin{aligned}
x_l &= 0 & f(x_l) &= f(0) = 0 - 0 - 5 = -5 \\
x_u &= 12 & f(x_u) &= f(12) = 144 - 48 - 5 = 91 & x_m &= (0 + 12) / 2 = 6 \\
x_m &= 6 & f(x_m) &= f(6) = 36 - 24 - 5 = 7 \\
f(x_l) * f(x_m) &< 0 & \text{the root lies between } x_l \text{ and } x_m & \text{ so new estimation range } [0, 6]
\end{aligned}$$

$$\begin{aligned}
x_l &= 0 & f(x_l) &= f(0) = 0 - 0 - 5 = -5 \\
x_u &= 6 & f(x_u) &= f(6) = 36 - 24 - 5 = 7 & x_m &= (0 + 6) / 2 = 3 \\
x_m &= 3 & f(x_m) &= f(3) = 9 - 12 - 5 = -8 \\
f(x_l) * f(x_m) &> 0 & \text{the root lies between } x_m \text{ and } x_u & \text{ so new estimation range } [3, 6] \\
x_l &= 3 & f(x_l) &= f(3) = 9 - 12 - 5 = -8 \\
x_u &= 6 & f(x_u) &= f(6) = 36 - 24 - 5 = 7 & x_m &= (3 + 6) / 2 = 4,5 \\
x_m &= 4,5 & f(x_m) &= f(4,5) = 20,25 - 18 - 5 = -2,75 \\
f(x_l) * f(x_m) &> 0 & \text{the root lies between } x_m \text{ and } x_u & \text{ so new estimation range } [4,5, 6]
\end{aligned}$$

Table - Estimation of root for initial range given at each iteration. Iteration will be continued until root is estimated with an acceptable error.

Iteration	x_l	x_u	x_m	$f(x_l)$	$f(x_u)$	$f(x_m)$	E
1	0	48	24	-5	2107	475	-
2	0	24	12	-5	475	91	%100
3	0	12	6	-5	91	7	%100
4	0	6	3	-5	7	-8	%100
5	3	6	4,5	-8	7	-2,75	%33,33

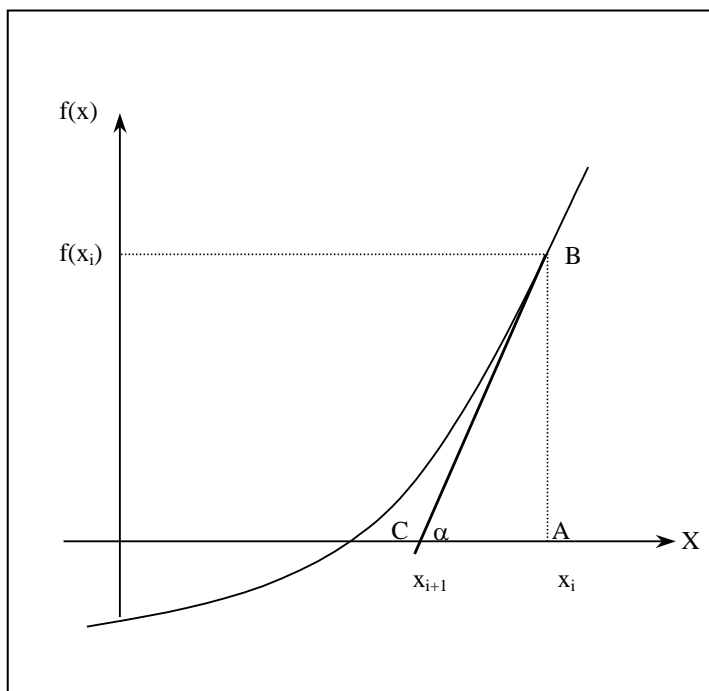
Figure - Bisection lines for given non-linear equation.



Example of Newton Raphson Method

A function $f(x)$ is defined as $f(x) = x^3 - 10x^2 + 100$. For $y = f(x) = 0$ estimate a root of this function using Newton-Raphson method. Initial guess value of root x_0 is 15 and use absolute relative approximate error notation for error of estimated root.

Recall



$$\tan(\alpha) = \frac{AB}{AC}$$

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Gist of prediction of x_{i+1}

$$|E| = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| * 100$$

Function : $f(x) = x^3 - 10x^2 + 100$

Derivative of function : $f'(x) = 3x^2 - 20x$

$$x_1 = 15 - \frac{f(15)}{f'(15)} = 15 - \frac{3375 - 2250 + 100}{675 - 300} = 15 - \frac{1225}{375} = 15 - 3,26 = 11,74$$

$$x_2 = 11,74 - \frac{f(11,74)}{f'(11,74)} = 11,74 - \frac{1618,09 - 1378,27 + 100}{413,48 - 234,8} = 11,74 - \frac{339,82}{178,68} = 11,74 - 1,89 = 9,85$$

$$x_3 = 9,85 - \frac{f(9,85)}{f'(9,85)} = 9,85 - \frac{955,67 - 970,22 + 100}{291,06 - 197} = 9,85 - \frac{85,45}{94,06} = 9,85 - 1,89 = 8,94$$

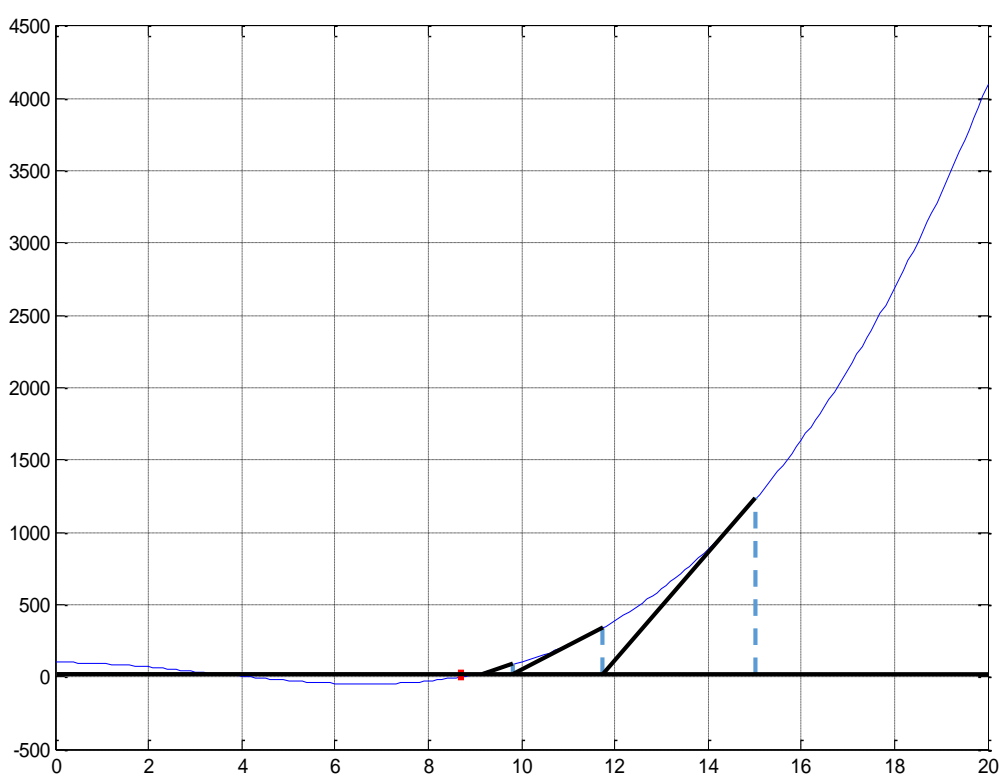
$$x_4 = 8,94 - \frac{f(8,94)}{f'(8,94)} = 8,94 - \frac{714,51 - 799,23 + 100}{239,77 - 178,8} = 8,94 - \frac{15,28}{60,97} = 8,94 - 0,25 = 8,69$$

$$x_5 = 8,69 - \frac{f(8,69)}{f'(8,69)} = 8,69 - \frac{656,23 - 755,16 + 100}{226,54 - 173,8} = 8,69 - \frac{1,07}{52,74} = 8,69 - 0,02 = 8,67$$

Table - Estimation of root for the function described in Q2 at each iteration. Iteration will be continued until root is estimated with an acceptable error.

Iteration	x	$f(x)$	$f'(x)$	E
0	$x_0 = 15$	1225	375	-
1	$x_1 = 11,74$	339,82	178,68	%27,76
2	$x_2 = 9,85$	85,45	94,06	%19,18
3	$x_3 = 8,94$	15,28	60,97	%10,17
4	$x_4 = 8,69$	1,07	52,74	%2,87
5	$x_5 = 8,67$	-	-	%0,23

Figure – Graphical analysis of Newton-Raphson method for given non-linear equation.



Practice Session for week 5

Direct Method Example

General formula for an n th order polynomial can be written as

$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ for (n+1) data points. Data points :

$(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$

Direct Method with linear interpolation

For given data table below model the linear interpolating polynomial and estimate the function value at given point.

x	5	12	6	9
f(x)	28	63	?	?

$$x_1 = 5 \quad f(5) = 28 \quad f(5) = a_0 + a_1 5 \quad 28 = a_0 + a_1 5 \quad \boxed{a_0 = 3}$$

$$x_2 = 12 \quad f(12) = 63 \quad f(12) = a_0 + a_1 12 \quad 63 = a_0 + a_1 12 \quad \boxed{a_1 = 5}$$

$$f(x) = 3 + 5x \quad f(6) = 3 + 5 \cdot 6 = 33$$

$$f(9) = 3 + 5 \cdot 9 = 48$$

Direct Method with quadratic interpolation

For given data table below model the quadratic interpolating polynomial and estimate the function value at given point.

x	5	12	20	17
f(x)	28	63	80	?

$$x_1 = 5 \quad f(5) = 28 \quad f(5) = a_0 + a_1 5 + a_2 25 \quad 28 = a_0 + a_1 5 + a_2 25$$

$$x_2 = 12 \quad f(12) = 63 \quad f(12) = a_0 + a_1 12 + a_2 144 \quad 63 = a_0 + a_1 12 + a_2 144$$

$$x_2 = 20 \quad f(20) = 80 \quad f(20) = a_0 + a_1 20 + a_2 400 \quad 80 = a_0 + a_1 20 + a_2 400$$

$$a_0 = -8.5$$

$$a_1 = 8.25$$

$$a_2 = -0.19$$

$$\boxed{f(x) = -8.5 + 8.25x - 0.19x^2}$$

$$\boxed{f(17) = -8.5 + 8.25 \cdot 17 - 0.19 \cdot (17 \cdot 17) = -8.25 + 140.25 - 54.91 = 77.09}$$

Lagrangian Method Example

General form of Lagrangian interpolating polynomial;

$$f_n(x) = \sum_{i=0}^n L_i(x) f(x_i) \quad \text{for } (n+1) \text{ data points.} \quad L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$$

Data points : $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$

Lagrangian Method with quadratic interpolation

For given data table below model the quadratic interpolating polynomial and estimate the function value at given point.

x	5	9	18	13
f(x)	21	34	49	?

$$f_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$f_2(x) = L_0(x) \cdot 21 + L_1(x) \cdot 34 + L_2(x) \cdot 49$$

$$L_0(x) = \left(\frac{x - x_1}{x_0 - x_1} \right) \left(\frac{x - x_2}{x_0 - x_2} \right) = \left(\frac{x - 9}{5 - 9} \right) \left(\frac{x - 18}{5 - 18} \right) = \left(\frac{9 - x}{4} \right) \left(\frac{18 - x}{13} \right)$$

$$L_1(x) = \left(\frac{x - x_0}{x_1 - x_0} \right) \left(\frac{x - x_2}{x_1 - x_2} \right) = \left(\frac{x - 5}{9 - 5} \right) \left(\frac{x - 18}{9 - 18} \right) = \left(\frac{x - 5}{4} \right) \left(\frac{18 - x}{9} \right)$$

$$L_2(x) = \left(\frac{x - x_0}{x_2 - x_0} \right) \left(\frac{x - x_1}{x_2 - x_1} \right) = \left(\frac{x - 5}{18 - 5} \right) \left(\frac{x - 9}{18 - 9} \right) = \left(\frac{x - 5}{13} \right) \left(\frac{x - 9}{9} \right)$$

$$f_2(x) = \left(\frac{9 - x}{4} \right) \left(\frac{18 - x}{13} \right) \cdot 21 + \left(\frac{x - 5}{4} \right) \left(\frac{18 - x}{9} \right) \cdot 34 + \left(\frac{x - 5}{13} \right) \left(\frac{x - 9}{9} \right) \cdot 49$$

$$f_2(13) = (-1) \left(\frac{5}{13} \right) \cdot 21 + (2) \left(\frac{5}{9} \right) \cdot 34 + \left(\frac{8}{13} \right) \left(\frac{4}{9} \right) \cdot 49 = -8,07 + 37,77 + 13,40 = 43,1$$

Newton's Method Example

General form of Newton's interpolating polynomials;

Given $(n+1)$ data points, $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ as

$$f_n(x) = b_0 + b_1(x - x_0) + \dots + b_n(x - x_0)(x - x_1)\dots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

$$b_{n-1} = f[x_{n-1}, x_{n-2}, \dots, x_0]$$

$$b_n = f[x_n, x_{n-1}, \dots, x_0]$$

and

$$f[x_n, x_{n-1}, \dots, x_0] = \frac{f[x_n, x_{n-1}, \dots, x_1] - f[x_{n-1}, x_{n-2}, \dots, x_0]}{x_n - x_0}$$

For given data table below model the quadratic interpolating polynomial and estimate the function value at given point.

x	5	9	18	13
f(x)	21	34	49	?

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f[x_0] = f(x_0) = f(5) = 21$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(9) - f(5)}{9 - 5} = \frac{34 - 21}{4} = \frac{13}{4} = 3,25$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$= \frac{\frac{f(18) - f(9)}{18 - 9} - \frac{f(9) - f(5)}{9 - 5}}{18 - 5} = \frac{\frac{49 - 34}{18 - 9} - \frac{34 - 21}{9 - 5}}{18 - 5} = \frac{\frac{15}{9} - \frac{13}{4}}{13} = \frac{60 - 117}{13 \cdot 36} = -\frac{57}{468} = -0,12 \quad \text{So,}$$

$$f_2(x) = 21 + 3,25(x - 5) - 0,12(x - 5)(x - 9)$$

$$f_2(13) = 21 + 3,25(13 - 5) - 0,12(13 - 5)(13 - 9)$$

$$f_2(13) = 21 + 26 - 3,84$$

$$f_2(13) = 43,16$$