1	2	3	4	Total

Name: Solutions
Number:

KOM505E - Probability Theory and Stochastic Processes
Final Exam

Jan. 11, 2016

Rules:
- Closed book & notes.
- Cell phones are not allowed.
- This exam will count for 40% of your final grade.
- Duration: 120 min.

1. Answer the following questions. Parts 1.a and 1.b are NOT related.

   (a) A biased random walk process is defined as $X[n] = \sum_{i=0}^{n} U[i]$ where $U[i]$ is a Bernoulli random process with

   $$p_U[k] = \begin{cases} 1/4 & \text{for } k = -1 \\ 3/4 & \text{for } k = 1 \end{cases}$$

   i. Find the expected value $E(X[n])$ and the variance $Var(X[n])$ of $X[n]$.

   ii. Is $X[n]$ stationary? Explain briefly.

   iii. Does $X[n]$ have stationary and independent increments? Explain briefly.

   (b) A Bernoulli random process $Y[n]$ that takes values 0 and 1, each with probability of $p = 1/2$ is transformed using: $Z[n] = (-1)^n Y[n]$. Is the random process independent and identically distributed (IID)?

a- i) $E(X[n]) = E\left(\sum_{i=0}^{n} U[i]\right) = \sum_{i=0}^{n} E(U[i]) = \sum_{i=0}^{n} \underbrace{(-1)\cdot\frac{1}{4} + (+1)\frac{3}{4}}_{1/2} = \frac{(n+1)}{2}$

$Var(X[n]) = Var\left(\sum_{i=0}^{n} U[i]\right) = \sum_{i=0}^{n} Var(U[i])$ as $U[i]$ are independent

$Var(U[i]) = E[U^2[i]] - E^2[U[i]] = 1 - \frac{1}{2^2} = \frac{3}{4}$

$E[U^2[i]] = (-1)^2\frac{1}{4} + (1)^2\frac{3}{4} = 1$

$Var(X(n)) = \sum_{i=0}^{n} \frac{3}{4} = \frac{3(n+1)}{4}$

ii) As 1^{st} & 2^{nd} moments are time varying (depending on n)
X[n] is NOT stationary

iii) $X[n] = X[n-1] + U[n]$
$X[n] - X[n-1] = U[n]$ ← increments are Bernoulli RP ↓ iid

iid processes are independent and stationary

b) $Z[n] = (-1)^n Y[n]$

* $Z[n]$ samples are independent as $Y[n]$'s are indep.

* However $Z[n]$ are not identically distributed

$$Z[1] = -Y[1] \Rightarrow P_{Z_1}(k) = \begin{cases} 1/2 & k=0 \\ 1/2 & k=-1 \end{cases}$$

$$Z[2] = Y[2] \Rightarrow P_{Z_2}(k) = \begin{cases} 1/2 & k=0 \\ 1/2 & k=1 \end{cases}$$

$P_{Z_1} \neq P_{Z_2} \longrightarrow$ not stationary
not identical

2. Suppose $X[n]$ is an IID Gaussian Random Process with mean μ and variance $\sigma_x^2 = 1$. $X[n]$ is input to a differencer to generate the output random process $Y[n] = X[n] - X[n-1]$.

   (a) Find the joint pdf of the samples $[Y[1], Y[2]]$

   (b) Are the samples independent? Is $Y[n]$ an IID random process?

   (c) Calculate the autocorrelation sequence $r_Y[k]$ of $Y[n]$. Plot $r_Y[k]$ versus k. (Recall $r_X[k] = E[X[n]X[n+k]]$)

   (d) Is $Y[n]$ wide-sense stationary?

a) $\left.\begin{array}{l} Y[1] = X[1] - X[0] \\ Y[2] = X[2] - X[1] \end{array}\right\}$ $\underbrace{\begin{bmatrix} Y[1] \\ Y[2] \end{bmatrix}}_{\tilde{Y}} = \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} X[0] \\ X[1] \\ X[2] \end{bmatrix}}_{\tilde{X}}$

$$\tilde{Y} \sim N(\underbrace{G\mu_{\tilde{X}}}_{\mu_{\tilde{Y}}}, \underbrace{G C_{\tilde{X}} G^T}_{C_{\tilde{Y}}}) \qquad \mu_{\tilde{X}} = \begin{bmatrix} \mu_x \\ \mu_x \\ \mu_x \end{bmatrix} = \begin{bmatrix} \mu \\ \mu \\ \mu \end{bmatrix}$$

$$C_{\tilde{X}} = \sigma_x^2 I = I$$

$$\mu_{\tilde{Y}} = G\mu_{\tilde{X}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \mu \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C_{\tilde{Y}} = G C_{\tilde{X}} G^T = G G^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

b) As $C_{\tilde{Y}}$ is not diagonal samples are correlated

Hence they are NOT independent.

c) $$r_Y[k] = E\{Y[n]Y[n+k]\}$$

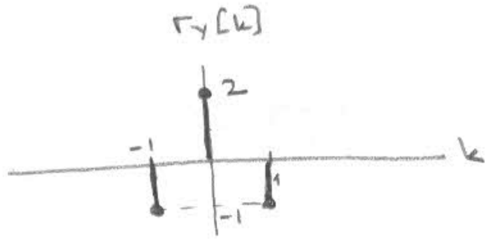
$$= E\{(X[n] - X[n-1])(X[n+k] - X[n+k-1])\}$$

$$= r_X[k] - r_X[k-1] - r_X[k+1] + r_X[k]$$

$$= 2r_X[k] - r_X[k-1] - r_X[k+1]$$

Since $X[n]$ is iid $\quad r_X[k] = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{else} \end{cases}$

$$r_X[k] = \delta[k] \Rightarrow r_Y[k] = 2\delta[k] - \delta[k-1] - \delta[k+1]$$



d) $E\{Y[n]\} = E\{X[n] - X[n-1]\} = E\{X[n]\} - E\{X[n-1]\}$

$$= \mu - \mu = 0$$

$r_Y[k] = 2\delta[k] - \delta[k-1] - \delta[k+1]$ from part (c)

Hence $Y[n]$ is WSS.

3. A_1, A_2 and A_3 are mutually exclusive and exhaustive set of events associated with a random experiment E_1. Similarly, events B_1, B_2 and B_3 are mutually exclusive and exhaustive set of events associated with a random experiment E_2. The joint probabilities of occurrence of these events and some marginal probabilities (at the last row) are listed in the table below.

	B_1	B_2	B_3
A_1	3/36	K_1	5/36
A_2	5/36	4/36	5/36
A_3	K_2	6/36	K_3
$P(B_i)$	12/36	14/36	K_4

(a) Find the values of K_1, K_2, K_3, and K_4.

(b) Find $P(B_3|A_1)$ and $P(A_1|B_3)$.

(c) Are events A_1 and B_1 independent? Show your reason.

a) $P(B_1) = \frac{12}{36} = \sum_{i=1}^{3} P(A_i, B_1) = \frac{3}{36} + \frac{5}{36} + K_2 \Rightarrow K_2 = \frac{4}{36}$

$P(B_2) = \frac{14}{36} = \sum_{i=1}^{3} P(A_i, B_2) = K_1 + \frac{4}{36} + \frac{6}{36} \Rightarrow K_1 = \frac{4}{36}$

$\sum_{i=1}^{3} P(B_i) = 1 = \frac{12}{36} + \frac{14}{36} + K_4 \Rightarrow K_4 = \frac{10}{36}$

$P(B_3) = \frac{10}{36} = \sum_{i=1}^{3} P(A_i, B_3) = \frac{5}{36} + \frac{5}{36} + K_3 \Rightarrow K_3 = 0$

b) $P(B_3|A_1) = \frac{P(B_3, A_1)}{P(A_1)} = \frac{5/36}{3/36 + 4/36 + 5/36} = \frac{5}{12}$

Similarly

$P(A_1|B_3) = \frac{5/36}{10/36} = 1/2$

c) $\left.\begin{array}{l} P(A_1) = 12/36 \\ P(B_1) = 12/36 \end{array}\right\}$ $P(A_1, B_1) = \frac{3}{36} \neq P(A_1)\,P(B_1) = \left(\frac{12}{36}\right)^2 = \frac{1}{9}$

A_1 & B_1 are NOT independent.

4. Let X be a discrete random variable with Poisson probability mass function (pmf): $F(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}$.

(a) Find the probability of $X > 2$ when $\lambda = 2$. Hint: $e^x = \sum_{k=0}^{\infty} x^k/k!$

(b) Find the Tchebyshev bound for the probability of $X > 5$ when $\lambda = 2$.
Hint: Tchebyshev inequality: $P(|X - \mu_x| \geq k) \leq \sigma_x^2/k^2$

a) $P(X>2) = \sum_{k=3}^{\infty} P(X=3) = \sum_{k=3}^{\infty} \frac{2^k e^{-2}}{k!} = e^{-2} \sum_{k=3}^{\infty} \frac{2^k}{k!}$

$= e^{-2}\left(e^2 - \sum_{k=0}^{2} \frac{2^k}{k!}\right)$

$= 1 - e^{-2}\underbrace{\left(1 + \frac{2}{1!} + \frac{4}{2!}\right)}_{5}$

$= 1 - 5e^{-2}$

b) $P(|X-2| \geq 4) \leqslant \frac{2}{9}$

Since $X \geqslant 0$ $\quad |X-2| \geqslant 4$ means $X \geqslant 6$

$P(X \geqslant 6) = P(X > 5) \leqslant \frac{2}{16} = \frac{1}{8}$