1. (20 pts)

Consider two independent continuous random variables x and y. The (mean, variance) of these random variables are (μ_x, σ_x^2) for x, and (μ_y, σ_y^2) for y.

A new random variable z is formed as $z = \alpha_1 x - \alpha_2 y$, where α_1 and α_2 are constant real numbers. Compute the mean (μ_z) and variance (σ_z^2) of z in terms of α_1, α_2, μ_x, μ_y, σ_x^2, and σ_y^2.

$$\mu_Z = E\{Z\}$$

$$= E\{\alpha_1 X - \alpha_2 Y\} = \alpha_1 E\{X\} - \alpha_2\{Y\}$$

$$= \alpha_1 \mu_X - \alpha_2 \mu_Y$$

$$\sigma_Z^2 = E\{Z^2\} - \mu_Z^2$$

$$= E\{(\alpha_1 X - \alpha_2 Y)^2\} - (\alpha_1 \mu_X - \alpha_2 \mu_Y)^2$$

$$= E\{\alpha_1^2 X^2 - 2\alpha_1\alpha_2 XY + \alpha_2^2 Y^2\} - (\alpha_1\mu_X - \alpha_2\mu_Y)^2$$

$$= \alpha_1^2 \underbrace{E\{X^2\}}_{\sigma_X^2 + \mu_X^2} - 2\alpha_1\alpha_2 \underbrace{E\{XY\}}_{\mu_X\mu_Y} + \alpha_2^2 \underbrace{E\{Y^2\}}_{\sigma_Y^2 + \mu_Y^2} - (\alpha_1\mu_X - \alpha_2\mu_Y)^2$$

$$= \alpha_1^2\sigma_X^2 + \underbrace{\alpha_1^2\mu_X^2 - 2\alpha_1\alpha_2\mu_X\mu_Y + \alpha_2^2\sigma_Y^2 + \alpha_2^2\mu_Y} - (\alpha_1\mu_X - \alpha_2\mu_Y)^2$$

$$= \alpha_1^2\sigma_X^2 + \alpha_2^2\sigma_Y^2 + (\alpha_1\mu_X - \alpha_2\mu_Y)^2 - (\alpha_1\mu_X - \alpha_2\mu_Y)^2$$

$$= \alpha_1^2\sigma_X^2 + \alpha_2\sigma_Y^2$$

Hence

$$\boxed{\begin{aligned} \mu_Z &= \alpha_1\mu_X - \alpha_2\mu_Y \\ \sigma_Z^2 &= \alpha_1^2\sigma_X^2 + \alpha_2\sigma_Y^2 \end{aligned}}$$

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2. (20 pts)

A Gaussian mixture pdf (probability density function) is defined as follows:

$$f_x(x) = \alpha_1 \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{x^2}{2\sigma_1^2}\right) + \alpha_2 \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{x^2}{2\sigma_2^2}\right)$$

for $\sigma_1 \neq \sigma_2$. What is the relation between α_1 and α_2 such that $f_x(x)$ is a valid pdf.

A valid pdf: $f_x(x) \geq 0 \quad \forall x$ and $\int_{-\infty}^{\infty} f_x(x)\,dx = 1$.

(i) $$f_x(x) = \alpha_1 \underbrace{\frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-x^2/2\sigma_1^2}}_{\geq 0 \quad \forall x} + \alpha_2 \underbrace{\frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-x^2/2\sigma_2^2}}_{\geq 0 \quad \forall x}$$

Hence

$f_x(x) \geq 0 \quad \forall x$ if $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$

(ii) $$\int_{-\infty}^{\infty} f_x(x)\,dx = 1$$

$$= \alpha_1 \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-x^2/2\sigma_1}\,dx}_{=1} + \alpha_2 \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-x^2/2\sigma_2^2}\,dx}_{=1} = 1$$

Hence $\alpha_1 + \alpha_2 = 1$

$$\boxed{\begin{array}{l} f_x(x) \text{ is a valid pdf if} \\ \text{i) } \alpha_1 \geq 0 \text{ and } \alpha_2 \geq 0 \\ \text{ii) } \alpha_1 + \alpha_2 = 1 \end{array}}$$

3. (20 pts)

An experiment consists of two subexperiments. First a number (x) is chosen at random from the interval $(0, 1)$. Then, a second number (y) is chosen at random from the same interval.

(i) Determine and draw the sample space S^2 for the overall experiment.

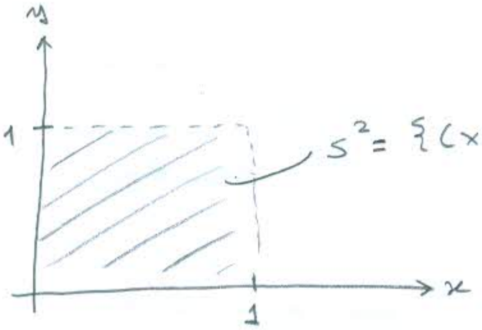
(ii) Consider the event

$$A = \{(x, y) : 1/4 \leq x \leq 1/2, 1/2 \leq Y \leq 3/4\}$$

and find the probability of event A.

(iii) Determine the conditional probability of event A given event B, where $B = \{(x, y) : y \leq x + 1/4\}$.

(i)

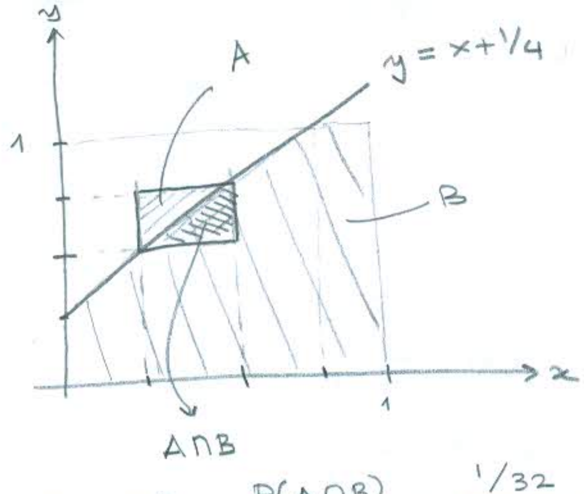


$S^2 = \{(x,y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$

(ii) $P(A) = P(\{x : 1/4 \leq x \leq 1/2\}) \cdot P(\{y : 1/2 \leq y \leq 3/4\})$

$$= \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$$

(iii)



$P(A \cap B) = \frac{1}{2} \cdot P(A)$

$= 1/32$

$P(B) = 1 - \dfrac{3/4 \cdot 3/4}{2}$

$= 1 - \frac{9}{32} = \frac{23}{32}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/32}{23/32} = \frac{1}{23}$$

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4. (20 pts)

Consider a random variable U with a uniform distribution in (0,1) range. Determine the function $g(.)$ such that $X = g(U)$ has a Rayleigh pdf with $\sigma = 1$.

$$f_U(u) = 1 \qquad u \in [0,1]$$

$$f_X(x) = \frac{x}{\sigma^2} \exp\left\{\frac{-x^2}{2\sigma^2}\right\} \qquad x \in [0,\infty)$$

$$= x e^{-x^2/2} \qquad \text{if } \sigma = 1$$

$$F_X(x) = \int_0^x x e^{-x^2/2}\, dx = -e^{-x^2/2}\Big|_0^x = 1 - e^{-x^2/2} \qquad x \geq 0$$

$$F_U(u) = u \qquad u \in [0,1]$$

$$F_X(x) = F_U(u)$$

$$F_X(g(u)) = F_U(u)$$

$$g(u) = F_X^{-1}\{F_U(u)\}$$

$$= F_X^{-1}\{u\}$$

$$x \underset{F_X^{-1}}{\overset{F_X}{\rightleftarrows}} 1 - e^{-x^2/2}$$

$$F_X^{-1}(x) = \sqrt{-2\ln(1-x)}$$

$$\boxed{g(u) = \sqrt{-2\ln(1-u)}}$$

5. (20 pts) Suppose you have a very old car battery, which occasionally fails to start in cold weather and makes you try starting it several times. You will use your probabilistic modeling knowledge from this class to make calculations on the probability of your car starting in a cold winter morning after trying it for either "even" or "odd" number of times.

After years of experience with your car, you guess that the probability that your car will not start at any trial is 0.2. Assume that your each trial of starting the car is independent from each other.

Do the following:

(5 pts) (i) Denote the car "start" and "not start" with binary digits "1" and "0" respectively, and model the possible outcomes in your sample space using the number of trials you go through until you start your car and drive. Draw the sample space S with explicitly stating the possible outcomes w_i in the sample space S.

(4 pts) (ii) Define a random variable (r.v.) X, which maps the outcomes in S (i.e. number of trials until you start the car) to the set of positive integers. Draw the space S^x with the possible values of the r.v. $X(w_i) : S \to S^x$, and show the mapping X in your drawing.

(5 pts) (iii) What is $P_X(X = k)$? (Hint: You should also use the information in the question given as your guess of probability.)

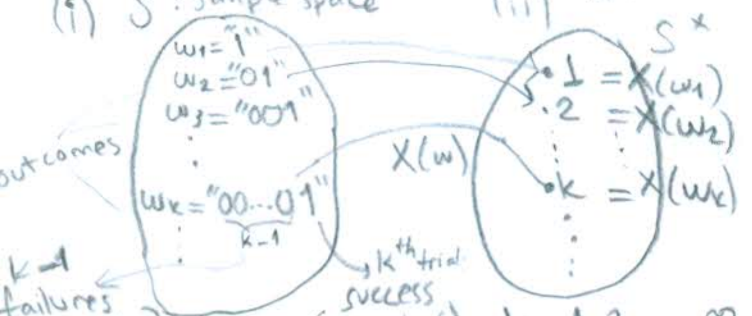
(6 pts) (iv) Now, assume that during your car start trials, you have a friend with you who suggests a game as follows. She says if the car starts at the trial with an even number, she wins, otherwise (i.e. the car starts at the trial with an odd number) you win. To model this situation, define a new random variable

$$Y = \begin{cases} 1 & \text{if } X = k \text{ is even.} \\ -1 & \text{if } X = k \text{ is odd.} \end{cases}$$

Calculate first the probability that you win: i.e. $P_Y(Y = -1)$? Then what is the probability that your friend wins: $P_Y(Y = 1)$? Comment based on the resulting probability values: who has more advantage and why?

5 pts

(i) S : Sample Space (ii) 4 pts



5 pts

(iii) Model by a geometric r.v. since we are mapping the outcome that the car starts at the k^{th} trial to number k.
(i.e. success at k^{th} trial)

$$P[X=k] = (1-p)^{k-1} p \quad , \quad k = 1, 2, \ldots$$

$$\begin{pmatrix} 1-p = 0.2 \\ p = 0.8 \end{pmatrix} \qquad = (0.2)^{k-1} \underbrace{(0.8)}_{p}$$

not inserted −2 pts

(iv) $P[Y=k] = \begin{cases} P[X \text{ is even}] , & k = 1 \\ P[X \text{ is odd}] , & k = -1 \end{cases}$

2 pts

$$P[Y=-1] = P[X \text{ is odd}] = \sum_{k=1, k \text{ odd}}^{\infty} (1-p)^{k-1} p = \sum_{k=1}^{\infty} (1-p)^{2k-2} p = p \sum_{k=0}^{\infty} \underbrace{(1-p)^{2k}}_{\alpha} = p \frac{1}{1-(1-p)^2}$$

2 pts

$$= \frac{p}{p(2-p)} = \frac{1}{2-p} = \frac{1}{1.2} = \frac{5}{6}$$

$(=0.83)$

1 pt $P[Y=1] = 1 - P[Y=-1] = 1 - \frac{5}{6} = \frac{1}{6}$

(This could be verified from : $P[Y=1] = p \sum_{k=0}^{\infty} (1-p)^{2k+1} = p \frac{(1-p)}{1-(1-p)^2} = \frac{1-p}{2-p} = \frac{0.2}{1.2} = \frac{1}{6}$)
Not needed.

1 pt You have more advantage, as starting from the 1st (odd) trial, prob of the car starting is much larger than its not starting, ∴ Picking odd trials has an obvious advantage.
The calculated probabilities for $P_Y[1]$ & $P_Y[-1]$ are thus expected (the large difference).