Formal Logic: Handout 6

Propositional Logic – Semantic Validity

Yılmaz Kılıçaslan
April 05, 2022

<u>Key Concepts:</u> *Semantic Validity, Tacit versus Explicit Knowledge, Mechanical / Algoritmic Decision, Decidability.*

1 INTRODUCTION

Recall that logic is the science of reasoning and that a deductive train of reasoning, which is usually called an argument, is, most simply, a sequence of sentences, with the last sentence being the conclusion deduced from the rest, which are usually referred to as premises. Here is an example (which we previously saw as an exercice in week 2):

(1) *If Socrates is a philosopher, then he is human and he is wise.*
If Socrates is human, then he is not wise.

Socrates is not a philosopher.

where it is argued that that *Socrates is not a philosopher* follows from the premises that *if Socrates is a philosopher, then he is human and he is wise* and that *if Socrates is human, then he is not wise*.

When we have an argument the first question we should ask is that of whether the argument is valid or not. Remember also that an argument is said to be valid if its conclusion is true wherever its premises are true. In the second week of the course, when we made an introduction to logic, we evaluated the validity of several arguments expressed in English. We were somehow able to determine whether these arguments were valid or not. In addition, you were given the argument above as an exercise and were expected to detect that it is valid.

Our decisions as to the validity of an argument have been rather intuition-based so far. This had to be so because all the arguments we examined were expressed in a natural language, the syntax and semantics of which we **know** only **tacitly** (i.e., subconsciously). However, we have **explicit knowledge** of the syntax and semantics of L_1, which is our formal language of propositional logic, because it is us that have defined them. What is more important is that resting on the formal specification of L_1, we can **algorithmically** (i.e., **mechanically**) decide on the validity of arguments expressed in this language.

The subject matter of this lecture is **semantic validity**. We will try to decide on the on the validity of arguments semantically and in a way as algorithmic as possible.

2 SEMANTIC VALIDITY USING TRUTH-TABLES

The semantic way of deciding whether an argument is valid or not is to do this by considering the semantic values (truth values in our paradigm) of the premises and the conslusion of the argument: to repeat once again, an argument is valid if the truth of the conclusion necessitates the truth of each of the premises; otherwise, it is invalid. One common way of semantically deciding on validity is by using truth-tables. Let us apply this approach to argument (1). The first thing to do is translate it to L_1. Argument (1) can be taken to be a translation of argument (1) into L_1:

(2) $p \rightarrow (q \wedge r)$
$q \rightarrow \neg r$

$\neg p$

Now the task is to decide on the validy of that formally specified argument. Resting on the traditional semantic interpretations of the operators, we can construct the following truth-table by means of which we can make this decision:

				PREMISE (1)		PREMISE (2)	CONCLUSION
p	q	r	$q \wedge r$	**$p \rightarrow (q \wedge r)$**	**$\neg r$**	**$q \rightarrow \neg r$**	**$\neg p$**
0	0	0	0	1	1	1	1
0	0	1	0	1	0	1	1
0	1	0	0	1	1	1	1
0	1	1	1	1	0	0	1
1	0	0	0	0	1	1	0
1	0	1	0	0	0	1	0
1	1	0	0	0	1	1	0
1	1	1	1	1	0	0	0

Table 1. Truth-table to check the validity of an argument.

Let us make a few retrospective remarks on model-theoretic issues. Each row of this table corresponds to a model of the world with respect to which the propositions are interpreted. That is, in a model assumed by propositional logic, the reality is a set of facts, true propositions. A truth-table contains all possible models. The number of possible models is 2^N where N is the number of propositional variables used in the propositions. Hence, we have eight (8) (= 2^3) rows in this example, since we have three distinct propositional variables (namely; p, q, and r).

It is also noteworthy that if a truth-table is costructed for validity checking, it is expected to include a separate column for each premise and for the conclusion, as is the case in the table above. Given these, we can easily make our decision. In table 1, we observe that wherever the premises are true (i.e., in the models represented by the first three rows), the conclusion is true, too. These points are highlighted in the following table.

p	q	r	q ∧ r	p → (q ∧ r)	¬r	q → ¬r	¬p
				PREMISE (1)		PREMISE (2)	CONCLUSION
0	0	0	0	1	1	1	1
0	0	1	0	1	0	1	1
0	1	0	0	1	1	1	1
0	1	1	1	1	0	0	1
1	0	0	0	0	1	1	0
1	0	1	0	0	0	1	0
1	1	0	0	0	1	1	0
1	1	1	1	1	0	0	0

Table 2. Truth-table for a valid argument.

Therefore, we can safely conclude that argument (2) is valid.

Another imoprtant point to make is that when argument (1), which is expressed in English, is translated into a language of propositional logic, its validity is preserved in its new form. That is, when it comes to arguments like argument (1) L1 in particular, and any formal language of propositional logic in general, is exspressive enough to decide on the validity question.

3 Decidabilty in Propositional Logic

Officially, $\varphi_1, \ldots, \varphi_n / \psi$ is said to be (semantically) valid just in case for all models *M* such that $[\varphi_1]^M = \ldots = [\varphi_n]^M = 1$, $[\psi]^M = 1$. In that case, ψ is said to be a semantic consequence of $\varphi_1, \ldots, \varphi_n$. A shorter notation for this relation is this: $\varphi_1, \ldots, \varphi_n \models \psi$.

To solve a problem algorithmically is to solve it in a finite number of steps, where is step is taken to be discrete (i.e., clearly sperated from the others). The validity of every argument schema in propositional logic can be algoritimically decided by means of truth tables. Therefore, prpositional logic is said to be **decidable** as to semantic validity.

3 Problenms with Propositional Logic

As the set of propositional symbols in a language of propositional logic is infinite in size, an infinite number of statements and valid arguments can be constructed in this logical system. However, propositional logic still fails to capture an infinitely large portion of valid arguments. The following is one of them:

(3) *All philosophers are mortal*.
Socrates is a philosopher.

Socrates is mortal.

This argument is valid simply because the truth of its premises involves the truth of its conclusion. But, its validity is inevitably lost if the medium of expression is changed from English into a formal language of propositional logic. Take this argument:

(4) p
q
---
r

This is one of the forms the argument expressed in (3) may have in propositional logic. As this truth table shows, this representation does not force the conclusion to be true wherever the premises are true:

Premises		Conclusion
p	*q*	*r*
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Table 3. Truth-table for an invalid argument.

which straightforwardly means that the represented argument is invalid.

It should be clear that the truth-table (i.e., semantic) method will allow us to detect any invalid argument. As we can decide whether any given argument is valid or not using the semantic method, propositional logic is said to be decidable

3 Conclusion

Below are the main points made in this lecture:

- $\varphi_1, \ldots, \varphi_n / \psi$ is said to be (semantically) valid just in case for all models M such that $[\varphi_1]^M = \ldots = [\varphi_n]^M = 1$, $[\psi]^M = 1$.

- In that case, ψ is said to be a semantic consequence of $\varphi_1, \ldots, \varphi_n$: $\varphi_1, \ldots, \varphi_n \models \psi$ (shorter notation)

- The validity of every argument schema in propositional logic can be decided by means of truth tables.

  - We only have to consider the models in which the premises are all true.

    . The conclusion has to be true in all such models for the argument to be valid.

    . A model where the premises are all true whereas the conclusion is false is called a counterexample.

- As we can decide whether any given argument is valid or not using the truth-table (i.e., semantic) method, propositional logic is said to be decidable.

REFERENCE

- Introduction to Logic: Logic, Language, and Meaning (Volume 1), Gamut, L.T.F. Chicago: The University of Chicago Press, 1991.