Formal Logic: Handout 5

Propositional Logic – Semantics

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Key Concepts: *Meaning, Truth-Value, Truth, Falsehood, Semantics, Pragmatics, Model of Reality, Truth-Table.*

1 Introduction

The raison d'être of language is to encode meaning. This is not true only of natural languages but also of formal languages. Therefore, we cannot be content with the structural description of our formal language, L_1, which provides us with a set of basic expressions and a set of syntactic rules according to which larger expressions are generated. We also need to specify how the expressions of this language encode meaning. We will not delve into deep philosophical questions about that notion. We simply assume that the meaning of a proposition is a truth-value. Therefore, knowing whether a proposition is true or false amounts to knowing its meaning. It is due to this truth-based approach to propositional meaning that logical constants are required to be truth-functional (cf. Handout 3): in order for a compound proposition to receive a truth-value, the meaning of a logical constant has to be a function that yields a truth value as output when fed with truth-values. The negation operator is a function that takes a single truth-value as input and outputs the reverse of that value. As binary operators all the other logical constants take binary tuples of truth-values as input and generate a single truth-value as output. The conjunction operator generates truth only if both of its operands are true (otherwise, it generates falsehood). The disjunction operator yields falsehood only if both of its operands are false (otherwise, it generates truth). The implication operator outputs falsehood only if its antecedent is true and its consequent is false (otherwise, it generates truth). The equivalence operator yields truth if its operands have the same truth-value (otherwise, it generates falsehood).

The study of linguistic meaning (without taking pragmatic factors into account) is called semantics. Recall (from Handout 4) that that branch of science is concerned with the mapping between the syntax of a language and the adopted model of reality:

Figure 1. Semantics as a bridge between language and reality.

Last week we learned how to define the syntax of our formal language. Today we will see how to form a model of reality and how to construct a bridge between our language and that model in order to assign meanings to units of the language.

2 SEMANTICS OF L_1

2.1 … Specified in English

In model-theoretic approaches, the specification of the semantics of a formal language starts with a specification of a model.[1] A model for L_1 is a function F such that for all propositional variables ρ of L_1,

$$F(\rho) \in \{1, 0\}.$$

Relative to a particular model F of the propositional variables of L_1, we can inductively define a semantic value $[\varphi]^F$ for every formula φ of L_1:

1. $[\rho]^F = F(\rho)$, for all propositional variables ρ.
2. $[\neg\varphi]^F = 1$ iff $[\varphi]^F = 0$ (and $[\neg\varphi]^F = 0$ otherwise).
3. $[\varphi \vee \psi]^F = 1$ iff $[\varphi]^F = 1$ or $[\psi]^F = 1$.
4. $[\varphi \wedge \psi]^F = 1$ iff $[\varphi]^F = 1$ and $[\psi]^F = 1$.
5. $[\varphi \rightarrow \psi]^F = 1$ iff $[\varphi]^F = 0$ or $[\psi]^F = 1$.
6. $[\varphi \leftrightarrow \psi]^F = 1$ iff both $[\varphi]^F = 1$ and $[\psi]^F = 1$ or both $[\varphi]^F = 0$ and $[\psi]^F = 0$.

2.2 … Specified as truth-tables

How the truth value of a composite sentence is formed from the truth values of its constituents (i.e. their structural semantics) is prescribed in a truth table. The truth tables for our logical constants are as follows:

φ	$\neg\varphi$
0	1
1	0

Table 1. Truth-table for negation.

φ	ψ	$\varphi \wedge \psi$
0	0	0
0	1	0
1	0	0
1	1	1

Table 2. Truth-table for conjunction.

φ	ψ	$\varphi \vee \psi$
0	0	0
0	1	1
1	0	1
1	1	1

[1] 'Valuation' and 'interpretation' are two other terms meaning the same as 'model'

Table 3. Truth-table for disjunction.

φ	ψ	$\varphi \to \psi$
0	0	1
0	1	1
1	0	0
1	1	1

Table 4. Truth-table for implication.

Equivalence:

φ	ψ	$\varphi \leftrightarrow \psi$
0	0	1
0	1	0
1	0	0
1	1	1

Table 5. Truth-table for logical equivalence.

Each row in a truth table corresponds to a particular model in which we assign a particular truth-value to each logical variable.

2.3 Calculating the truth value of a formula with a (composite) a truth-table

Having a truth-table for each logical operator, we can calculate the truth-value of any given formula of L_1 in an inductive way, starting from (one of) the innermost operator(s) and working outward. Below is the truth-table to be constructed when calculating the value of $(p \to (q \wedge r))$:

p	q	r	$q \wedge r$	$p \to (q \wedge r)$
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	1	1
1	0	0	0	0
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

Table 6. Composite truth-table for $(p \to (q \wedge r))$.

For each row, first we calculate the truth value of $q \wedge r$ according to the truth-table for conjunction and looking at the truth values in the 2nd and 3rd columns and, then, we calculate the truth value of $p \to (q \wedge r)$ according to the truth-table for implication and looking at the truth values in the 1st and 4th columns.[2]

3 LOGICAL PROPERTIES OF FORMULAS

Below are some important logical properties of formulas of L_1:

1. φ is a **logical truth** (tautology), written $\models \varphi$, iff for all models F, $[\varphi]^F = 1$;

   e.g., $(p \vee \neg p)$
2. φ is a **logical falsehood** (contradiction) iff for all models F, $[\varphi]^F = 0$;

   e.g., $(p \wedge \neg p)$
3. φ is **contingent** iff there is a model F such that $[\varphi]^F = 1$, and a model F' such that

   $[\varphi]^{F'} = 0$; e.g., p, $(p \vee q)$
4. φ and ψ are **logically equivalent** iff for all models F, $[\varphi]^F = [\psi]^F$. If φ and ψ are logically equivalent, then $\models (\varphi \leftrightarrow \psi)$
5. ψ is a **logical consequence** of φ, written $\varphi \models \psi$, iff for all models F, *if* $[\varphi]^F = 1$, then $[\psi]^F = 1$. So given our formal definition of a model F for the language L_0, the foregoing yields a mathematically precise characterization of truth functionally valid arguments with φ as premise and ψ as conclusion. The original, intuitive idea that it is 'not possible' for the conclusion to be false and the premises true can now be expressed in terms of the nonexistence of formal counter-models.
6. $\varphi \models \psi$ iff $\models (\varphi \to \psi)$. Accordingly φ and ψ are logically equivalent iff $\varphi \models \psi$ and $\psi \models \varphi$.

4 TRANSLATION FROM A NATURAL LANGUAGE TO L_1

English is the language we use to talk about L_1.That is, English is our meta-language. As for L_1, it is the object-language (i.e. the language under examination). A useful exercise to gain insight into the question of what kind of meaning a formal language encodes is to translate natural language (NL) sentences into that formal language. When translating a NL sentence into a formal language, we can observe the expressive power of the formal language. In order to make this observation, the structure of the NL sentence should be preserved as much as possible. As an example, consider the translation of "if I don't know the answer, then I don't answer the question if I don't feel lucky" into L_1:

Translation: $(\neg p \to (\neg q \to \neg r))$

Key: p: I know the answer.
q: I feel lucky.
r: I answer the question.

Notice that the translation comes with a key whereby we state the propositional content of each propositional letter used.

[2] Notice that the right-most parantheses are omitted when writing the formulas. This is a convention that we will commonly use henceforth.

EXERCISES

A. Construct the composite truth-table for each of the following formulas.

1. $(\neg p \rightarrow q)$
2. $(\neg(\neg p \vee q) \rightarrow \neg\neg r)$.
3. $((\neg(\neg p \vee q) \rightarrow \neg\neg r) \leftrightarrow (p \wedge r))$.
4. $((p \wedge r) \leftrightarrow (r \wedge p))$
5. $((p \wedge r) \leftrightarrow (\neg r \vee \neg p))$

B. Translate the following sentences into L_1, preserving as much of the structure as possible and giving the key in each case.

1. John is not only stupid but nasty too.
2. Nobody laughed or applauded.
3. If it rains while the sun shines, a rainbow will appear.
4. Ahmet is going to the beach or the movies on foot or by bike.
5. If father and mother both go, then I won't, but if only father goes, then I will go too.
6. If you do not help me if I need you, I will not help you if you need me.

REFERENCE

- Introduction to Logic: Logic, Language, and Meaning (Volume 1), Gamut, L.T.F. Chicago: The University of Chicago Press, 1991.