Formal Logic: Handout 7

Propositional Logic – Syntactic Validity

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Key Concepts: *Syntactic Validity, Exponential Growth, Complexity, Natural Deduction, Derivation, Formal (Deduction / Derivation) Rules, Introduction Rule, Elimination Rule.*

1 INTRODUCTION

Recall that an argument is valid if and only if accepting its premises commits one to accept its conclusion. In the last lecture we saw how we can show whether an argument is valid or invalid using a semantic approach. In this approach, an argument is taken to be valid if and only if the conclusion is true in each model where the premises are all true. We can check whether this is the case using a truth-table: an argument is valid if and only if the conclusion is true in each row where all the premises are true.

However, the semantic approach is too costly simply because truth-tables grow too fast. We know that a truth-table will have 2^n rows to contain all possible models for formulas containing *n* propositional symbols. That is, a truth-table grows exponentially with the number of propositional symbols involved, which is one of the nightmare scenarios of complexity. In the examples we have seen so far we had at most three propositional symbols and, hence, the largest truth-table had *8* rows. But, suppose that we try to model a story. Even a short story is likely to contain more than *100* situations. If in the story there are *10* different situations, the truth table for it will have 2^{10} (*1024*) rows; if there are *20* different situations, we will have a truth table with 2^{20} (*1.048.576*) rows; and if there are *100* different situations, we will have a truth table with 2^{100} (*1.267.650.600.228.229.401.496.703.205.37*) rows. As can be easily noticed, trying to handle such problems with truth-tables would become practically hopeless very soon. Therefore, if we want to use validity checking in practical tasks (such as AI projects), we need to adopt an approach other than the truth-table-based one.[1]

What we require is, at least, an approach that will not rely on truth-tables. A syntactic approach to validity will meet this requirement as it will rest solely on certain formal criteria ignoring any issues concerning the truth-values of formulas.

The approach we have chosen to examine the notion of validity from a syntactic point of view is that of *natural deduction*. In what follows, we will describe this particular approach by specifying the formal criteria that it rests on.

2 NATURAL DEDUCTION: A SYNTACTIC APPROACH TO VALIDITY

2.1 Derivations

In the method of natural deduction, an argument is valid if and only if departing from the premises one can arrive at the conclusion following a sequence of steps defined by a finite set

[1] In other words, the procedure for semantic validity is effective but not efficient. That is, the problem of semantic validity is theoretically solvable (hence, decidable) but not practically usable.

of formal rules. To put it in other words, in this method the syntactic way of showing that an argument is valid is a journey with its source being the premises, its goal being the conclusion and its path being determined under the guidance of a set of formal rules. This journey is referred to as a derivation, the formal counterpart of an argument. Officially, a derivation is a finite numbered list of formulas like:

1. φ_1
   .
   .
   .

n. φ_n

Next to each formula φ_i a statement must be written saying how it was obtained:

a. 'Premise': if φ_i is a premise;

b. 'Assumption': if φ_i is an assumption;

c. The name of the rule, followed by the numbers of the formulas from which φ_i was obtained: if φ_i was obtained from the formulas occurring above it by means of one of the formal derivation rules.

The premises, if any, appear at the outset of the list. The last formula is the conclusion, the formula to be derived. $\varphi_1, \ldots, \varphi_{n-1} / \varphi_n$ is said to be (syntactically) valid (shorter notation: $\varphi_1, \ldots, \varphi_{n-1} \vdash \varphi_n$) just in case there is a derivation from the assumptions, $\varphi_1, \ldots, \varphi_{n-1}$, to the conclusion, φ_n. It should be noted that if the argument does not have any premises, then it is clear that the derived conclusion will be a tautology.

2.2 Steps in a Derivation

By the term 'formal' in the statements above, it is meant that the rules governing the derivation process rest solely on the forms of formulas. The truth-values of formulas have no place in the syntactic approach to the notion of validity. In the method of natural deduction, each step in the derivation is an answer to one of the following questions, which can be asked for each of the logical constants:

1. When can a formula with a constant $^\circ$ as its main operator be drawn as a conclusion?
2. What conclusions may be drawn from a formula with this constant as its main operator?

The answer to the first question gives an introduction rule, I°, for the logical constant under examination. The answer to the second question gives an elimination rule, E°, for the constant in question. The imtroduction and elimination rules formulated in this way will constitute a set of derivation rules. Let us now look at each of them one by one.

2.3 Derivation Rules

2.3.1 The Introduction Rule for the Conjunction Operator

This rule simply says that if we have two formulas appearing somewhere in the derivation, say φ and ψ, then we can add their conjunction, $\varphi \wedge \psi$, to the derivation:

1.	.	
	.	
	.	
$\mathbf{m_1.}$	φ	
	.	
	.	
	.	
$\mathbf{m_2.}$	ψ	
	.	
	.	
	.	
n.	$\varphi \wedge \psi$	$\mathrm{I}\wedge, m_1, m_2$

2.3.2 The Elimination Rules for the Conjunction Operator

The elimination rule for the conjunction operator states that if we can detach either the left or the right operand from a conjunction, say $\varphi \wedge \psi$, as a new formula to add to the derivation:

A.

1.	.	
	.	
	.	
m.	$\varphi \wedge \psi$	
	.	
	.	
	.	
n.	φ	$\mathrm{E}\wedge$, m

B.

1.	.	
	.	
	.	
m.	$\varphi \wedge \psi$	
	.	
	.	
	.	
n.	ψ	$\mathrm{E}\wedge$, m

2.3.3 The Introduction Rules for the Disjunction Operator

An already abaliable formula can be disjoined with any formula:

A.

1.	.	
	.	
	.	
m.	φ	
	.	
	.	
	.	
n.	$\varphi \vee \psi$	**I$\vee$, m**

B.

1.	.	
	.	
	.	
m.	ψ	
	.	
	.	
	.	
n.	$\varphi \vee \psi$	**I$\vee$, m**

2.3.4 The Elimination Rule for the Disjunction Operator

From the disjunction of two formulas, $\varphi \vee \psi$, we can obtain a new formula, say X, if can each of the former, φ or ψ, implies the latter:

1.	.	
	.	
	.	
m_1.	$\varphi \vee \psi$	
	.	
	.	
	.	
m_2.	$\varphi \to X$	
	.	
	.	
	.	
m_3.	$\psi \to X$	
	.	
	.	
	.	
n.	X	**E$\vee$, m_1, m_2 , m_3**

2.3.5 The Introduction Rule for the Implication Operator

If assuming a formula leads us to another, we can add an implication to the derivation with the former being the antecedent and the latter being the consequent:

1.	.	
	.	
	.	
m.	φ	***Assumption***
	.	
	.	
	.	
n-1.	ψ	
n.	$\varphi \to \psi$	**I→**

2.3.6 The Elimination Rule for the Implication Operator

The introduction rule for the implication operator is nothing more than the well-known *modus ponens* rule:

1.	.	
	.	
	.	
$\mathbf{m_1}$**.**	$\varphi \to \psi$	
	.	
	.	
	.	
$\mathbf{m_2}$**.**	φ	
	.	
	.	
	.	
n.	ψ	$\mathbf{E\to, m_1, m_2}$

2.3.7 The Introduction Rules for the Equivalence Operator

The introduction rules for the equivalence operator merely state that if two formulas imply each other, then the derivation can be augmented with an equivalence where these formulas are operands:

A.

1.	$\vdots$	
m1.	$\varphi \rightarrow \psi$	
	$\vdots$	
m2.	$\psi \rightarrow \varphi$	
	$\vdots$	
n.	$(\varphi \leftrightarrow \psi)$	**I$\leftrightarrow$, m1, m2**

B.

1.	$\vdots$	
m1.	$\varphi \rightarrow \psi$	
	$\vdots$	
m2.	$\psi \rightarrow \varphi$	
	$\vdots$	
n.	$(\psi \leftrightarrow \varphi)$	**I$\leftrightarrow$, m1, m2**

2.3.8 The Elimination Rule for the Equivalence Operator

1.	$\vdots$	
m.	$\varphi \leftrightarrow \psi$	
	$\vdots$	
n.	$(\Phi \rightarrow \psi) \wedge (\psi \rightarrow \Phi)$	**E$\leftrightarrow$, m**

2.3.9 Examples (Binary Operators)

In this subsubsection, we will see some examples in order to get insight into the questions of what we can do with the rules we have seen so far.

Here is a simple argument:

(1) $(p \wedge q)$ |-- $(q \wedge p)$

We, of course, immediately come to realize that $(q \wedge p)$ necessarily follows from $(p \wedge q)$ and, thus, the argument is valid. However, we are supposed to show this by deriving the former from the latter following a sequence of steps defined by the formal derivation rules. Below is such a derivation:

(2)

1.	$p \wedge q$	premise
2.	q	$E_{\wedge}$, 1
3.	p	$E_{\wedge}$, 1
4.	$q \wedge p$	$I_{\wedge}$, 2, 3

Our point of departure is the given premise, $p \wedge q$. We first detach the right conjunt and then the left conjunct from the premise in accordance with the elimination rules of the conjunction operator. The ordering of the detachment operations in this way is crucial because this will allow us to come to $q \wedge p$ in the next and last step in accordance with the introduction rule of the conjunction operator (that is, with q and p replacing their positions in the formula).

Here is another argument:

(3) $(p \wedge q) \rightarrow r$ |-- $(q \wedge p) \rightarrow r$

Syntactic derivation is based partly on observation and partly on intuition. We observe in this example that the conclusion is an implication. According to the introduction rule for the implication operator, if assuming φ takes us to ψ then we can encapsulate the part of the derivation from that φ to this ψ into $\varphi \rightarrow \psi$. So, it seems to be a good idea to assume the antecedent of the implication $(q \wedge p) \rightarrow r$ at some point of the derivation and try to arrive at r proceeding that point onward, as is done below:

(4)

1.	$(p \wedge q) \rightarrow r$	premise
2.	$q \wedge p$	assumption
3.	p	$E_{\wedge}$, 2
4.	q	$E_{\wedge}$, 2
5.	$p \wedge q$	$I_{\wedge}$, 3, 4
6.	r	$E_{\rightarrow}$, 1, 5
7.	$(q \wedge p) \rightarrow r$	$I_{\rightarrow}$

Notice that part of the derivation above is in essence the same as the derivation made for the first example, namely the part from step 2 to step 5, with the sole difference that the point of departure is not given to us as a premise but introduced by us as an assumption.

One thing we learn from the preceding example is that if we need a formula but this formula is neither given as a premise nor can be derived from formulas already made available by the preceding derivation, there is only one way left to get it: we have to assume it. Therefore, if an argument lacks any premises then we have to start the derivation process by making some assumption(s). Below is such an argument:

(5) |-- $((p \wedge q) \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r))$

and here is a derivation for it:

(6)

1.	$(p \wedge q) \rightarrow r)$	assumption
2.	p	assumption
3.	q	assumption
4.	$p \wedge q$	$I_{\wedge}$, 2, 3
5.	r	$E_{\rightarrow}$, 1, 4
6.	$q \rightarrow r$	$I_{\rightarrow}$
7.	$p \rightarrow (q \rightarrow r)$	$I_{\rightarrow}$
8.	$(p \wedge q) \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r)$	$I_{\rightarrow}$

Conclusions like that of this argument, those coming from nowhere, are called *tautologies*. These are propositions that are true in any possible model.

Assumptions need not, of course, be made at the outset of a derivation. They can be introduced at any point where they are required, as illustrated in (8) (which is a derivation for (7)).

(7) $(p \vee q) \vee r$ |-- $p \vee (q \vee r)$

(8)

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	assumption
3.	p	assumption
4.	$p \vee (q \vee r)$	$I_{\vee}$, 3
5.	$p \rightarrow (p \vee (q \vee r))$	$I_{\rightarrow}$
6.	q	assumption
7.	$q \vee r$	$I_{\vee}$, 6
8.	$p \vee (q \vee r)$	$I_{\vee}$, 7
9.	$q \rightarrow (p \vee (q \vee r))$	$I_{\rightarrow}$
10.	$p \vee (q \vee r)$	$E_{\vee}$, 2, 5, 9
11.	$(p \vee q) \rightarrow (p \vee (q \vee r))$	$I_{\rightarrow}$
12.	r	assumption
13.	$q \vee r$	$I_{\vee}$, 12
14.	$p \vee (q \vee r)$	$I_{\vee}$, 13
15.	$r \rightarrow (p \vee (q \vee r))$	$I_{\rightarrow}$
16.	$p \vee (q \vee r)$	$E_{\vee}$, 1, 11, 15

What is important is to get rid off any assumption made at some point before ending the derivation.

Up to this point, we have seen four arguments the validity of which can be demonstrated by using the introduction and/or elimination rules presented in the preceding subsubsections. However, there are also arguments that are valid but not provable by these rules. Below is argument of this sort:

(9) |-- (p → (q → p))

You can try to prove this argument with the rules at hand but any such attempt in this way is doomed to failure. We need is some extra derivation rules.

2.3.10 The Introduction Rule for the Bottom Operator

A.

1.	.	
	.	
	.	
m_1.	$\neg\varphi$	
	.	
	.	
	.	
m_2.	φ	
	.	
	.	
	.	
n.	$\bot$	**I$\bot$, m_1, m_2**

B.

1.	.	
	.	
	.	
m_1.	φ	
	.	
	.	
	.	
m_2.	$\neg\varphi$	
	.	
	.	
	.	
n.	$\bot$	**I$\bot$, m_1, m_2**

2.3.11 The Introduction Rule for the Negation Operator

1.	.	
	.	
	.	
m_1.	φ	***Assumption***
	.	
	.	
	.	
n-1.	$\bot$	
n.	$\neg\varphi$	**I$\neg$**

2.3.12 The Elimination Rule for the Negation Operator

1.	.	
	.	
	.	
m$_1$.	$\neg\varphi$	***Assumption***
	.	
	.	
	.	
n-1.	$\bot$	
n.	φ	**E¬**

2.3.13 Examples (Bottom and Negation Operators)

With these newly added derivation rules we can prove the argument in (9) as follows:

(10)

1.	p	assumption
2.	q	assumption
3.	¬p	assumption
4.	$\bot$	$I_\bot$, 1, 3
5.	p	E¬
6.	$q \rightarrow p$	$I_\rightarrow$
7.	$p \rightarrow (q \rightarrow p)$	$I_\rightarrow$

Consider now the following argument:

(11)

|-- $(\neg p \rightarrow (p \rightarrow q))$

This argument is not provable even with the recently augmented list of derivation rules. A rather interesting rule, named as Ex Falso Sequitur Quodlibet (EFSQ), need be added to this list to handle arguments like this.

2.3.14 Ex Falso Sequitur Quodlibet (EFSQ)

1.	.	
	.	
	.	
n-1.	$\bot$	
n.	φ	**EFSQ, n-1**

2.3.14 Examples (EFSQ)

We can now provide a derivation for (11):

(12)

1.	$\neg p$	assumption
2.	p	assumption
3.	$\perp$	$I_\perp$, 1, 2
4.	q	EFSQ, 3
5.	$p \rightarrow q$	$I_\rightarrow$
6.	$\neg p \rightarrow (p \rightarrow q)$	$I_\rightarrow$

3 Exercise (Translation and Syntactic Validity)

Let us end the discussion of syntactic validity with a last example that involves translation from natural language into propositional logic and most of the deduction rules we have seen. We are required in this example to translate the following argument into L_1 and prove it syntactically:

(13)

> *If the maid did it, then it was done with a revolver only if it was done in the parlor. But if the butler is innocent, then the maid did it unless it was done in the parlor. The maid did it only if it was done with a revolver, while the butler is guilty if it did happen in the parlor. So the butler is guilty.*

And, here is the solution to this exercise:

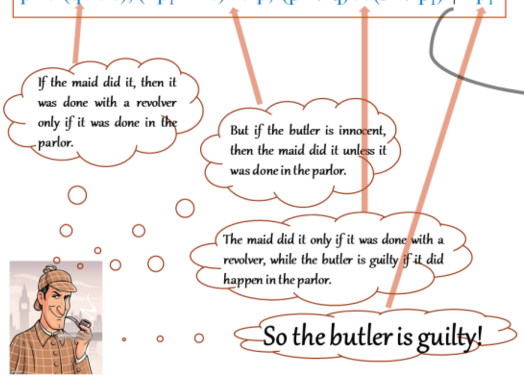
(14)

Key:

p: The maid committed the murder.
q: The murder was committed with a revolver.
r: The murder was committed in the parlor.
p_1: The butler committed the murder.

Translation:

$p \rightarrow (q \rightarrow r),\ (\neg p_1 \wedge \neg r) \rightarrow p,\ (p \rightarrow q) \wedge (r \rightarrow p_1) \vdash p_1$



1.	$p \rightarrow (q \rightarrow r)$	premise
2.	$(\neg p_1 \wedge \neg r) \rightarrow p$	premise
3.	$(p \rightarrow q) \wedge (r \rightarrow p_1)$	premise
4.	$\neg p_1$	assumption
5.	$\neg r$	assumption
6.	$\neg p_1 \wedge \neg r$	$I_\wedge$, 4, 5
7.	p	$E_\rightarrow$, 2, 6
8.	$q \rightarrow r$	$E_\rightarrow$, 1, 7
9.	$p \rightarrow q$	$E_\wedge$, 3
10.	q	$E_\rightarrow$, 7, 9
11.	r	$E_\rightarrow$, 8, 10
12.	$\perp$	$I_\perp$, 5, 11
13.	r	$E_\neg$
14.	$r \rightarrow p_1$	$E_\wedge$, 3
15.	p_1	$E_\rightarrow$, 13, 14
16.	$\perp$	$I_\perp$, 4, 15
17.	p_1	$I_\neg$

Reference

- Introduction to Logic: Logic, Language, and Meaning (Volume 1), Gamut, L.T.F. Chicago: The University of Chicago Press, 1991.