First-Order Predicate Logic

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PRESENTATION PLAN

- Need for quantifiers
- The *universal* and *existential* quantifiers
- A new logical system containing the universal and existential quantifiers: First-Order Predicate Logic
- A language for First-Order Predicate Logic: L_3

AN IMPLICIT PREMISE IN ARGUMENT_5

- ARGUMENT_5 contains an implicit premise which is necessary for its validity. This is a generalization about the relation '*taller than*':

  "If x is taller than y and y is taller than z, then x is taller than y."

- It should be clear that this generalization holds for all individuals that might happen to be the values of the variables used above.
- Therefore, the generalization can be paraphrased as follows:

  "For every x, every y, and every z, if x is taller than y and y is taller than z, then x is taller than y."

- To speak more technically, in order to formally handle this implicit generalization we need to introduce the *universal quantifer* into our logical system as a new logical constant.

QUANTIFIERS

- In fact, we need to define a new logical system with two additional logical constants:
  - **The universal quantifier: ∀, and**
  - **The existential quantifer: ∃.**

  Exercise: Why do you think these symbols are used for the quantifiers? (10 points each)
- The quantifers always appear together with a variable.
- The combination of a quantifer and a variable (e.g., **∀x** or **∃y**) is also referred to as a (universal or existential) quantifier.
- **∀x** … reads: For every x (in the domain we have) …
- **∃x** … reads: There is at least one x (in the domain we have) such that …

A Language of First-Order Predicate Logic

- The logical system we arrive at by adding the quantifiers over individuals is referred to as *First-Order (Predicate) Logic.*
- With this new logical system, we are smarter: We are able to construct and understand a larger set of valid arguments.
- Let us define a formal language, called L_3, that is expressive enough to capture the strength of First-Order Predicate Logic.
- The language L_3 will contain L_2, in the sense that all the sentences of L_2 will also be the sentences of L_3 and these sentences of L_2 will be interpreted just as before.
- The set of new sentences of L_3 will contain those sentences having quantifiers and individual variables as their constituents.
- As the newly introduced individual variables behave syntactically just like names for individuals, we will introduce the new syntactic category of *individual terms* (or simply *terms*) to include both variables and names for individuals.

Exercise: What might be the natural-language counterpart of a variable? (15 points)

SYNTAX OF L_3

A. *Basic Expressions:*

1. **Terms:**

   a. **Names:** *r*, *n*, *j*, and *m.*

   b. **Variables:** v_1, v_2, v_3, …

   (To avoid having too many subscripts in our formulas we will sometimes use *x* to stand for v_1, y to stand for v_2, and *z* to stand for v_3.)

2. **One-place predicates:** *M and B.*

3. **Two-place predicates:** *T and L.*

B. *Formation rules:*

1. If δ is a one-place predicate and α is a term, then $\delta(\alpha)$ is a formula.
2. If γ is a two-place predicate and α and β are terms, then $\gamma(\alpha, \beta)$ is a formula.

   If ϕ and ψ are formulas, then so are:

3. $\neg\phi$
4. $(\phi \wedge \psi)$
5. $(\phi \vee \psi)$
6. $(\phi \rightarrow \psi)$
7. $(\phi \leftrightarrow \psi)$
8. If ϕ is a formula, and *u* is a variable, then $\forall u\phi$ is a formula.
9. If ϕ is a formula, and *u* is a variable, then $\exists u\phi$ is a formula.

Translation of Argument 5 into L_3

- We can capture the validity of argument (5) in L_3, as it contains the quantifier that allows for the explicit statement of the implicit transitivity generalization.

Argument_5

Muhammad Ali is taller than Richard Nixon.
Richard Nixon is taller than Noam Chomsky.
———
Muhammad Ali is taller than Noam Chomsky.

Translation into L_3

$\forall x \forall y \forall z(((T(x, y) \wedge T(y, z)) \rightarrow T(x, z))$
$T(m, r)$
$T(r, n)$
———
$T(m, n)$

SOME REMARKS ON THE SYNTAX OF L_3 (1)

- So far, we have used the terms 'formula' and 'sentence' interchangeably. In fact, it is a common practice to reserve the term 'sentence' for a formula containing no free occurrences of variables.
- The letter u is used as a meta-language variable ranging over the variables of L_3; this is the only case of a non-Greek letter being used as a meta-language variable.
- An occurrence of a variable u in a formula ϕ can be defined as *bound in* ϕ if it occurs in ϕ within a sub-formula of the form: $\forall u\psi$ or $\exists u\psi$; otherwise, that occurrence is *free* in ϕ.

  ψ is called the *scope* of the occurrence of the quantifier $\forall u$ or $\exists u$ in ϕ.

SOME REMARKS ON THE SYNTAX OF L_3 (2)

- We distinguish between different *occurrences* of a quantifier. We may have formulas like (1):

  (1) $\forall x M(x) \wedge \forall x B(x)$

- We may also have situations in which bound variables occur within the scope of quantifiers with the same variable:

  (2) $\forall x(M(x) \wedge \forall x B(x))$

- A quantifier can only bind free variables within its scope. In fact, the formula in (2) is equivalent to the one in (3):

  (3) $\forall x(M(x) \wedge \forall y B(y))$

Some Remarks on the Syntax of L_3 (2)

- The syntactic rules do nothing to avoid so-called *vacuous quantification*, quantification over a formula with respect to a variable that does not occur in it. We can form

  $\forall x K(j, m)$ from $K(j, m)$

  or

  $\exists y B(x)$ from $B(x)$.

- It would complicate the syntax greatly to prohibit the formation of formulas with vacuous quantifiers.
- It is left to the semantic component to handle vacuous quantifiers: they are designed to treat vacuously quantified formulas as if the vacuous quantifier simply were not there.

SEMANTICS OF L_3

A. Semantic values of basic expressions:

1. If u is an individual variable of L_3, then $[u]^{M,g} = g(u)$.

2. If α is a non-logical constant of L_3, then $[\alpha]^{M,g} = F(\alpha)$.

B. Truth conditions for formulas of L_3 relative to M and g:

1. If δ is a one-place predicate and α is a term, then $[\delta(\alpha)]^{M,g} = [\delta]^{M,g}([\alpha]^{M,g})$.

2. If γ is a two-place predicate and α and β are terms, then $[\gamma(\alpha, \beta)]^{M,g} = ([\gamma]^{M,g}([\beta]^{M,g}))([\alpha]^{M,g}))$.

3.-7. If ϕ is a formula, then $[\neg\phi]^{M,g} = 1$ iff $[\neg\phi]^{M,g} = 0$; otherwise $[\neg\phi]^{M,g} = 0$. Similarly for $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$.

8. If ϕ is a formula and u is a variable, then $[\forall u\phi]^{M,g} = 1$ iff for every value assignment g' such that g' is exactly like g except possibly for the individual assigned to u by g', $[\phi]^{M,g'} = 1$.

9. If ϕ is a formula and u is a variable, then $[\exists u\phi]^{M,g} = 1$ iff for some value assignment g' such that g' is exactly like g except possibly for the individual assigned to u by g', $[\phi]^{M,g'} = 1$.

C. We adopt the following truth definition for formulas of L_3 relative to M:

For any formula ϕ of L_3, $[\phi]^{M} = 1$ if $[\phi]^{M,g} = 1$ for all value assignments g.

For any formula ϕ of L_3, $[\phi]^{M} = 0$ if $[\phi]^{M,g} = 0$ for all value assignments g.

Some Remarks on the Semantics of L_3 (2)

- The major novelty on the interpretation of L_3 over L_2 and L_1 lies in the notion of a function assigning to each variable of L_3 some value from the domain of individuals:

  g: variables → *individuals.*

- The definition of truth relative to a model is given in two stages:
  - First, we give a recursive defintion of *true formula of* L_3 *with respect to a model M and value assignment g.*
  - Then, on the basis of this intermediate definition, we state the final definition of *true sentence of* L_3 *with respect to a model M.*

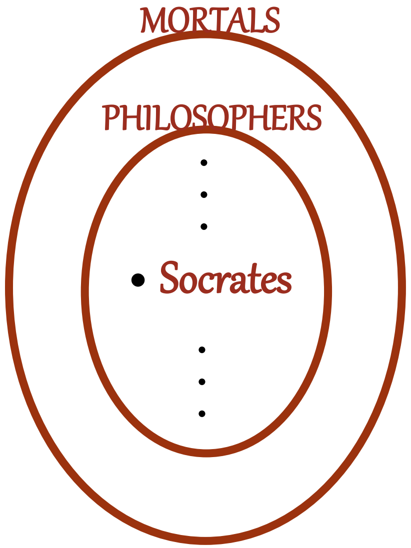
Example (1)

All philosophers are mortal.
Socrates is a philosopher.

Socrates is mortal.

$\forall x[\text{philosopher}(x) \longrightarrow \text{mortal}(x)]$
$\text{philosopher}(\text{socrates})$

$\text{mortal}(\text{socrates})$



EXAMPLE (2)

A dog is barking. → $\exists x[dog(x) \wedge bark(x)]$

Example (3)

A dog is chasing a cat.

$\exists x \exists y[[dog(x) \wedge cat(y)] \wedge chase(x, y)]$

$= \exists x\ [dog(x) \wedge \exists y[cat(y) \wedge chase(x, y)]]$

Example (4)

Every dog is chasing a cat.

$\forall x[dog(x) \longrightarrow \exists y\,[cat(y) \wedge chase(x, y)]]$

$\exists y\,[cat(y) \wedge \forall x[dog(x) \longrightarrow chase(x, y)]]$

Example (5)

A dog is not barking.

$\exists x[dog(x) \wedge \neg bark(x)]$

$\neg\exists x[dog(x) \wedge bark(x)]$

$= \forall x\neg[dog(x) \wedge bark(x)]$

EXAMPLE (6)

If a man owns a donkey he beats it.

$$\forall x \forall y\, (((man(x) \wedge donkey(y)) \wedge own(x, y)) \rightarrow beat(x, y))$$

EXERCISE 1

- **Translate the following sentences into a FOL language with equality, preserving as much of the structure as possible and giving the key in each case. Suppose that the set of basic expressions is large enough to include the non-logical constants that you need to use.**

1. Every dog chased a cat. (Two translations. Each 20 points)
2. Nobody laughed or applauded. (20 points)
3. Someone is friendly. (20 points)
4. Everyone is friendly. (20 points)
5. John doesn’t know a language. (Two translations. Each 20 points)
6. A student who is late is to be punished. (20 points)
7. A student, who is late, is to be punished. (30 points)
8. Every teacher admires himself. (20 points)
9. There is someone who is loved by everyone. (20 points)
10. If John has a donkey, he beats it. (25 points)
11. Every man who owns a donkey beats it. (20 points)
12. Someone who owns no car does own a motorbike. (40 points)
13. Everyone loves anyone who does not love anybody but himself or herself. (100 points)
14. Just two students passed. (100 points)

EXERCISE 2

- Determine the truth values of the following sentences (which are assumed to belong to a language of Firs-Order Predicate Logic) according to the universal domain of $U = \{1, 2, 3\}$. (20 points each)
  1. $\exists x \, \forall y \, [\, x^2 < y + 1 \,]$
  2. $\forall x \, \exists y \, [\, x^2 + y^2 < 12 \,]$
  3. $\forall x \, \forall y \, [\, x^2 + y^2 < 12 \,]$

References

- Introduction to Logic: Logic, Language, and Meaning (Volume 1), Gamut, L.T.F. Chicago: The University of Chicago Press, 1991.
- Dowty D.R., Wall R. E. and Peters S. (1981) Introduction to Montague Semantics. Dordrecht, The Netherlands: Kluwer Academic Publishers.