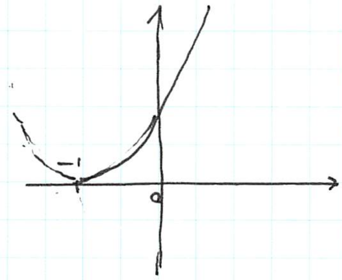
$x \to 0$ のとき. $f(x) \to 0$, $g(x) \to 0$.

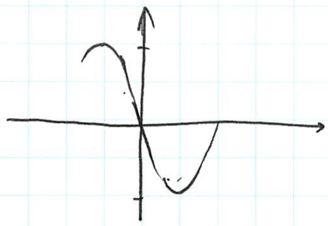
・ $f(x) = O(g(x)) \overset{\triangle}{\iff} \underbrace{\exists c, C \ \forall x}_{\exists \varepsilon > 0} : 0 < c < C,\ |x| < \varepsilon \Rightarrow c < \dfrac{f(x)}{g(x)} < C$

・ e.g. ・ $x^3 + 2x^2 + x = O(x)$.

∵ $\dfrac{x^3+2x^2+x}{x} = x^2+2x+1 = (x+1)^2$



・ $\sin x = O(x)$ ∵



・ $f(x) = o(g(x)) \overset{\triangle}{\iff} \dfrac{f(x)}{g(x)} \xrightarrow{x \to 0} 0$.

e.g. $x^2 = o(x)$. ∵ $\dfrac{x^2}{x} = x \to 0$.

・ $f(x)$ と $g(x)$ が同じオーダー $\iff f(x) = O(g(x))$, $g(x) = O(g(x))$.

e.g. ・ $f(x) = 2x^2 + x$, $g(x) = x^2 + 2x$ のとき.

$$\begin{cases} \dfrac{f(x)}{g(x)} = \dfrac{2x^2+x}{x^2+2x} = \cancel{\dfrac{2+\frac{1}{x}}{1+}} \ \dfrac{2x+1}{x+2} \ \cancel{\frac{x}{2}} \longrightarrow \dfrac{1}{2}. \\ \dfrac{g(x)}{f(x)} \longrightarrow 2. \end{cases}$$

∴ f と g は同じオーダー.

・ $f(x) = x^3 + 2x$, $g(x) = x^2$

・ $\dfrac{g(x)}{f(x)} = \dfrac{x^2}{x^3+2x} = \dfrac{x}{x^2+2} \longrightarrow 0$. ∴ $g(x) = O(f(x))$

・ $\dfrac{f(x)}{g(x)} = \dfrac{x^3+2x}{x^2} = \dfrac{x^2+2}{x} \longrightarrow \infty$ ∴ $f(x) \neq g(f(x))$

$$f'(p) = \lim_{h\to 0} \frac{f(p+h) - f(p)}{h}$$

$$\Longleftrightarrow \lim_{h\to 0} \left(\frac{f(p+h) - f(p) - f'(p)\,h}{h} \right) = 0.$$

$$\therefore \quad f(p+h) - f(p) - f'(p)\cdot h = o(h).$$

$$ah^3 \overset{?}{=} O(h^2) \qquad \frac{ah^3}{h^2} = ah$$

$$F(x_{k+1}) = F(y_k) + f(y_k)\cdot\frac{h}{2} + \frac{f'(y_k)}{2!}\cdot\left(\frac{h}{2}\right)^2 + \frac{f''(y_k)}{3!}\left(\frac{h}{2}\right)^3 + \cdots$$

$$I_k := \int_{x_k}^{x_{k+1}} f(x)\,dx = F(x_{k+1}) - F(x_k)$$

$$F(x_{k+1}) = F(y_k) + f(y_k)\frac{h}{2} + \frac{1}{2!}f'(y_k)\left(\frac{h}{2}\right)^2 \cdots$$

$$= \big(F(x_{k+1}) - F(y_k)\big) - \big(F(x_k) - F(y_k)\big)$$

$$= f(y_k)\cdot\frac{h}{2} + \frac{f'(y_k)}{2!}\left(\frac{h}{2}\right)^2 + \frac{f''(y_k)}{3!}\left(\frac{h}{2}\right)^3 + \frac{f'''(y_k)}{4!}\left(\frac{h}{2}\right)^4 + \cdots$$

$$-\left(-f(y_k)\cdot\frac{h}{2} + \frac{f'(y_k)}{2!}\left(\frac{h}{2}\right)^2 - \frac{f''(y_k)}{3!}\left(\frac{h}{2}\right)^3 + \cdots\right.$$

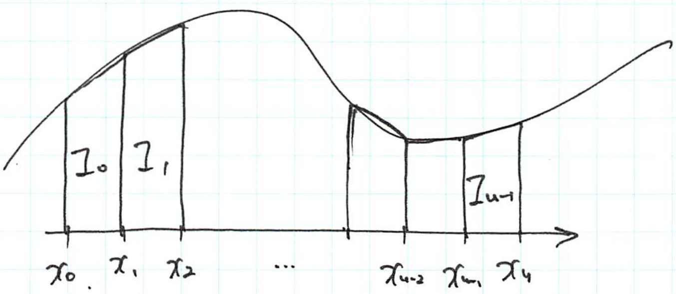
$$= f(y_k)\cdot h + \underbrace{\frac{f''(y_k)}{3!}\cdot\left(\frac{h}{2}\right)^3\cdot 2 + \frac{f''''(y_k)}{5!}\left(\frac{h}{2}\right)^5\cdot 2 + \cdots}_{(\star)}$$

$$\hat{I}_k := h\cdot f(y_k).$$

$$I_k - \hat{I}_k = (\star) = O(h^3).$$

Trapezoidal rule

$$\hat{I}_k := h \cdot \frac{f(\cancel{y_{k+1}}\,x_{k+1}) + f(\cancel{y_k}\,x_k)}{2}$$



$$I := \int_{x_0}^{x_n} f(x)\,dx.$$

$$\hat{I} := \sum_{k=0}^{n-1} \hat{I}_k, \quad \cancel{\hat{I}_k},$$

誤差 $d_k := \hat{I}_k - I_k = h \cdot \frac{f(x_{k+1}) - f(x_k)}{2} - I_k.$

$$= \underbrace{\left(h \cdot \frac{f(x_{k+1}) - f(x_k)}{2} - h \cdot f(y_k)\right)}_{d_k^1} - \underbrace{\left(I_k - h f(y_k)\right)}_{d_k^2}$$

・d_k^1 について．（$d_k^2 = O(h^3)$ は midpoint rule で示した）.

$$d_k^1 = \frac{f(x_{k+1}) + f(x_k)}{2} \cdot h - f(y_k) h.$$

$$= \frac{h}{2}\left(\left(f(x_{k+1}) - f(y_k)\right) + \left(f(x_k) - f(y_k)\right)\right)$$

$$= \frac{h}{2}\Big[\ \cancel{f(y_k)} + \frac{h}{2} f'(y_k) + \left(\frac{h}{2}\right)^2 f''(y_k) + \left(\frac{h}{2}\right)^3 f'''(y_k) + \cdots$$

$$+ \cancel{f(y_k)} - \frac{h}{2} f'(y_k) + \left(\frac{h}{2}\right)^2 f''(y_k) - \left(\frac{h}{2}\right)^3 f'''(y_k) + \cdots \Big]$$

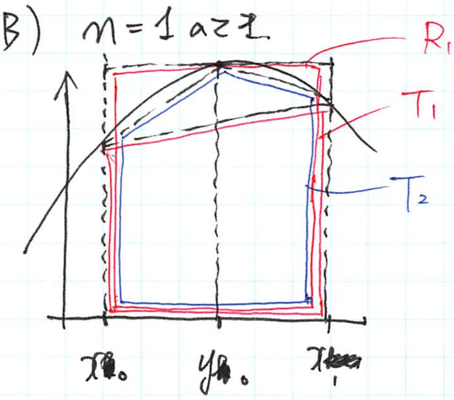
$$= \frac{h}{2}\left[\ 2 \cdot \left(\frac{h}{2}\right)^2 f''(y_k) + 2 \cdot \left(\frac{h}{2}\right)^4 f''''(y_k) + \cdots \right]$$

$$= O(h^3).$$

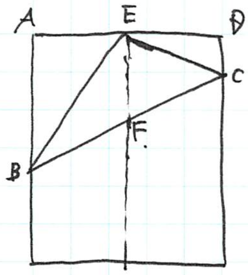
$$\therefore\ d_k = O(h^3).$$

$$T_{2n} \overset{①}{=} \frac{T_n + R_n}{2} \overset{②}{=} \frac{T_n}{2} + [\text{新しい分点での関数値の和}] \times \frac{b-a}{2n}$$

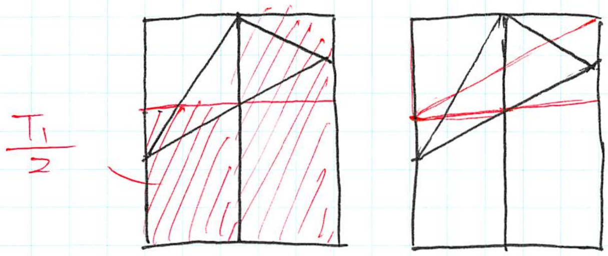
∵ (IB) $n=1$ のとき



$T_2 \overset{①}{=} \frac{T_1 + R_1}{2}$ を示したい.



□ABCD × $\frac{1}{2}$ = △BCE が示せればよい. OK.



Euler-MacLaurin 導出は森 pp.183-186.

$f(x)$ が区間 $[a, b]$ において $(2m)$ 回微分可能のとき.

台形則 の $T(f)_h$

$$\underbrace{\int_a^b f(x)\,dx}_{I} = \underbrace{h\left\{\frac{1}{2}f(a) + \sum_{k=1}^{n-1} f(a+kh) + \frac{1}{2}f(b)\right\}}$$
$$\underbrace{- \sum_{r=1}^{m} \underbrace{\frac{h^{2r} B_{2r}}{(2r)!}}_{C_{2r}} \left\{ f^{(2r-1)}(b) - f^{(2r-1)}(a) \right\}}_{(☆)} + R_m$$

・B_{2r} : ベルヌイ数.　　・$R_m := \dfrac{h^{2m+1}}{(2m)!} \displaystyle\int_0^1 B_{2m}(t) \left\{ \sum_{k=0}^{n-1} f^{(2m)}(a+kh+ht) \right\} dt.$

1°. 関数項の有限和を, 積分と端点の微係数から求める. (解析の計算)

e.g. $1^2 + 2^2 + 3^2 + \cdots + n^2$　　$f(x) := x^2$, $a=0$, $h=1$ とおけば ($b = n$ となる)

$$\underbrace{\int_0^n x^2\,dx}_{\frac{1}{3}n^3} = 1 \cdot \left\{ \frac{1}{2}\cdot 0 + 1^2 + 2^2 + \cdots + (n-1)^2 + \frac{1}{2}n^2 \right\}$$
$$- \frac{\overset{\frac{1}{6}}{B_2}}{2} \underbrace{\left(f'(n) - f'(0)\right)}_{2n}$$

$$\frac{1}{3}n^3 = 1^2 + \cdots + (n-1)^2 + \frac{1}{2}n^2 - \frac{1}{6}n$$
$$\therefore\ 1^2 + \cdots + (n-1)^2 = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n = \frac{1}{6}(2n^3 - 3n^2 + n)$$
$$= \frac{1}{6}n(2n^2 - 3n + 1) = \frac{1}{6}n(2n-1)(n-1)$$

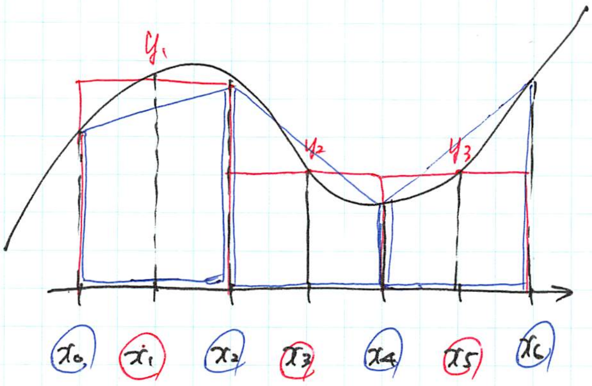
2°. 台形則とその誤差の関係.

特に(?), $f(x)$ が高階の微係数まで含めて $(b-a)$ を周期とする周期関数のとき

端点において $f^{(2r-1)}(b) = f^{(2r-1)}(a)$ となる (i.e. (☆) 部分がゼロに)

→ 著しく精度の高い解が得られる.

Simpson則



分割幅h.

→ 幅($2h$)と見なし中点則と台形則を適用.

→ $S(f)_h = \dfrac{2R(f)_{2h} + T(f)_{2h}}{3}$

e.g. 3.2 $I = \int_0^1 e^x dx$, $h = 0.25$.

$$I_4 = \frac{1.649 + \frac{(1.000 + 2.718)\times 1}{2}}{3} = \frac{1}{3}\left(1.000 + 4\times 1.284 + 2\times 1.649 + 4\times 2.117 + 2.718\right)$$

$$S(f)_h = \frac{1}{3}\left(2R(f)_{2h} + T(f)_{2h}\right).$$

$$= \frac{1}{3}\left[2\times 2h\left(f_1 + f_3 + \cdots + f_{2n-1}\right) + 2h\left(\frac{1}{2}f_0 + f_2 + \cdots + f_{2n-2} + \frac{1}{2}f_{2n}\right)\right]$$

$$= \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \cdots + 4f_{2n-1} + \cancel{4}f_{2n}\right]$$

Newton-Cotes

・小区間 $[x_{2k}, x_{2k+1}, x_{2(k+1)}]$ をLagrange補間で2次近似

　　$[x_0, x_1, x_2]$ について

・3点 $(x_0, f_0), (x_1, f_1), (x_2, f_2)$ を通る多項式

$$P(x) = f_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

（・一般に $(x_0, f_0), \dots, (x_n, f_n)$ を通る n次多項式

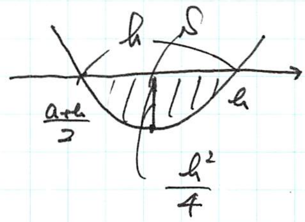
$$P(x) = \sum_{i=0}^{n} f_i \cdot \frac{\prod_{j=0, j\neq i}^{n} (x-x_j)}{\prod_{j=0, j\neq i}^{n} (x_i-x_j)}$$
）

$$\therefore \int_a^b P(x)\,dx = w_0 f_0 + w_1 f_1 + w_2 f_2.$$

$\cancel{x_0 - x_1} = x_1 - x_0 = x_2 - x_1 = h$ とおくと. $x_0 = a,\ x_1 = \frac{a+b}{2},\ x_2 = b$

$$\cdot\ w_0 = \int_a^b \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\,dx.$$

$$= \frac{1}{(-h)(-2h)} \int_a^b \left(x - \frac{a+b}{2}\right)(x-b)\,dx.$$



$$= \frac{1}{\cancel{2}\cancel{h^2}} \cdot h \cdot \frac{\cancel{h^2}}{\cancel{4}} \cdot \frac{\cancel{2}}{3} = \frac{h}{3}.$$

$$\therefore \int_a^b P(x)\,dx = \frac{h}{3} f_0 + \frac{4}{3} h f_1 + \frac{h}{3} \cdot f_2 = \frac{h}{3}(f_0 + \cancel{f} 4f_1 + f_2)$$

・閉じた公式　$I := \int_a^b f(x)\,dx.$　　$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$

$f(x) \simeq P_n(x)$: n次 Lagrange 多項式

誤差 $\Delta I_n = \int_a^b f(x)\,dx - \int_a^b P_n(x)\,dx = \int_a^b \big(\underbrace{f(x) - P_n(x)}_{= E_n(x)}\big)\,dx.$

p.23 の議論より、$\forall x$ $\exists \xi \in (a,b)$: $E_n(x) \big(= f(x) - P_n(x)\big) = \dfrac{\omega(x)}{n!} f^{(n+1)}(\xi)$.　$\xi(x)$

(ただし、$\omega(x) := \prod_{j=0}^{n} (x - x_j)$)

$$\therefore\ \Delta I_n = \int_a^b E_n(x)\,dx = \int_a^b \frac{\omega(x)}{\cancel{n!}\ (n+1)!} \cdot f^{(n+1)}(\xi(x))\,dx \quad \cdots (*)$$

右だと $\dfrac{1}{(n+1)!}$ になってる…?

・Lagrange補間の誤差の導出 (p.23)　$(x_0, f_0), \dots, (x_n, f_n)$, $P_n(x)$: n次

・$f(x_i) = P_n(x_i)$　$i = 0, \dots, n$.

・$F(z) := f(z) - P_n(z) - \big(f(x) - P_n(x)\big) \dfrac{\omega(z)}{\omega(x)}$ とおく。($\omega(x_i) = 0$ に注意)　($n+1$次)

・$F(z)$ は ~~can~~ $x, x_0, \dots, x_n$ の $n+2$ 点で 0 に。

(・$F(x) = \big(f(x) - P_n(x)\big) - \big(f(x) - P_n(x)\big) \cdot \underbrace{\dfrac{\omega(x)}{\omega(x)}}_{=1} = 0.$

・$F(x_i) = \underbrace{f(x_i) - P_n(x_i)}_{=0} - \big(f(x) - P_n(x)\big) \underbrace{\dfrac{\omega(x_i)}{\omega(x)}}_{=0} = 0.$)

・$\exists \xi \in (a,b)$: $F^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \underbrace{P_n^{(n+1)}(\xi)}_{=0} - \underbrace{\big(f(x) - P_n(x)\big)}_{E_n(x)} \dfrac{(n+1)!}{\omega(x)} = 0$

$$\therefore\ E_n(x) = \frac{\omega(x)}{(n+1)!} \cdot f^{(n+1)}(\xi)$$

$$|\Delta I_n| = \left| \int_a^b E_n(x)\,dx \right| = \left| \int_a^b \frac{\omega(x)}{(n+1)!} \cdot f^{(n+1)}(\xi(x))\,dx \right|$$

$M := \max_{a \le x \le b} \left| f^{(n+1)}(\xi(x))\,dx \right|$ とおけば

$$\le \frac{1}{(n+1)!} \cdot M \cdot \left| \int_a^b \omega(x)\,dx \right| \quad \cdots ?$$

・ Gauss p.50 導出.

$$I(f) := \sum_{k=1}^{n} w_k \cdot f(x_k). \qquad f(x) = x^j \quad (j=0,\dots,2n-1) \text{ のとき.}$$

$I(f) = \int_{-1}^{1} f(x)\,dx$ が成立するように. w_k, x_k を定める.

・ $n=3$ のとき. 変数6つ $(w_1, w_2, w_3, x_1, x_2, x_3)$, 式6本

$$\sum_{k=1}^{3} w_k \cdot x_k^j = \int_{-1}^{1} x^j\,dx$$

が $j=0,1,\dots,5$ について成立すればよい.

・ $\cancel{w_1 x_1 = \int_{-1}^{1} x\,dx}$

j: even のとき 0.

j: odd のとき $2\int_0^1 x^j\,dx = 2\left[\frac{1}{j+1}x^{j+1}\right]_0^1 = \frac{2}{j+1}$

$$\left\{\begin{array}{l}
\cdot\ w_1 x_1 + w_2 x_2 + w_3 x_3 = \frac{2}{3} \\
\cdot\ w_1 x_1^2 + w_2 x_2^2 + w_3 x_3^2 = 0 \\
\cdot\ w_1 x_1^3 + w_2 x_2^3 + w_3 x_3^3 = \frac{1}{2} \\
\quad \vdots \\
\cdot\ w_1 x_1^6 + w_2 x_2^6 + w_3 x_3^6 = 0.
\end{array}\right.$$