

Notes on Mechanism Design

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- This study notes are mainly based on the lecture note written by Valimaki in 2018.

1 Single Agent

- One principal v.s. one agent.
- $a \in A$: allocation, $\theta \in \Theta$: agent's private info. $\theta \sim F(\theta)$.
- $u^P(a, \theta), u^A(a, \theta)$
- We often assume quasi-linear payoff functions:
- $a := (x, t), u^P(a, \theta) := v^P(x, \theta) + t, u^A(a, \theta) := v^A(x, \theta) - t$.
- As for implementability, we can discuss it focusing only on direct mechanisms, (Θ, ϕ) , w.l.o.g. (Revelation principle)

1.1 Revenue Equivalence

1.1.1 Milgrom and Segal (2002), Envelope Theorem

- $\Theta := [\underline{\theta}, \bar{\theta}]$. $f(\cdot, \theta) : X \rightarrow \mathbb{R}$. $\{f(\cdot, \theta)\}_{\theta \in \Theta}$.
- $V(\theta) := \max_{x \in X} f(x, \theta)$. $X^*(\theta) := \operatorname{argmax}_{x \in X} f(x, \theta)$

Def. 1.1 (Selection). A function $x^* : \Theta \rightarrow X$ is a selection from X^* if $x^*(\theta) \in X^*(\theta)$ for all $\theta \in \Theta$.

Thm. 1.1 (Milgrom and Segal (2002)). Assume the following:

- For any $x \in X$, $f(x, \cdot) : \Theta \rightarrow \mathbb{R}$ is absolutely continuous on Θ .
- For any $x \in X$, $f(x, \cdot) : \Theta \rightarrow \mathbb{R}$ is differentiable on Θ .

Then, the following holds:

- V is absolutely continuous.
- For any selection x^* from X^* , $V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} f_{\theta}(x^*(s), s) ds$.

Proof. Note that the absolute continuity of $f(x, \theta)$ implies that $f_{\theta}(x, \theta) \in L^1(\Theta)$ for any $x \in X$.

(i) V is absolutely continuous. It is sufficient to show that V is Lipschitz continuous. Fix any θ', θ . Since any integrable function is bounded, for any x there exists $L > 0$ s.t. $|f_{\theta}(x, \theta)| \leq L$ for almost all $\theta \in \Theta$.

$$\begin{aligned} |V(\theta') - V(\theta)| &= \left| \max_{x'} f(x', \theta') - \max_x f(x, \theta) \right| \\ &\leq \max_x |f(x, \theta') - f(x, \theta)| = \max_x \left| \int_{\theta'}^{\theta} f_{\theta}(x, s) ds \right| \\ &\leq L \cdot |\theta' - \theta| \end{aligned}$$

(ii) Fix any selection x^* from X^* . By the result of (i),

$$V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} V'(s) ds$$

Fix any selection x^* and θ', θ such that $\theta' > \theta$. By the definition of V and x^* ,

$$\begin{aligned} V(\theta) &= f(x^*(\theta), \theta) \geq f(x^*(\theta'), \theta) \\ V(\theta') &= f(x^*(\theta'), \theta') \geq f(x^*(\theta), \theta') \end{aligned}$$

Hence,

$$\begin{aligned} V(\theta') - V(\theta) &\leq f(x^*(\theta'), \theta') - f(x^*(\theta'), \theta) \\ \frac{V(\theta') - V(\theta)}{\theta' - \theta} &\leq \frac{f(x^*(\theta'), \theta') - f(x^*(\theta'), \theta)}{\theta' - \theta}. \end{aligned}$$

Similarly,

$$\begin{aligned} V(\theta) - V(\theta') &\leq f(x^*(\theta'), \theta) - f(x^*(\theta'), \theta') \\ \frac{V(\theta) - V(\theta')}{\theta - \theta'} &\geq \frac{f(x^*(\theta'), \theta) - f(x^*(\theta'), \theta')}{\theta - \theta'}. \end{aligned}$$

Note that by assumption $f(x, \cdot)$ is differentiable at all $\theta \in \Theta$. Therefore, if V is differentiable at θ , we have $V'(\theta) = f_{\theta}(x^*(\theta), \theta)$. \square

1.1.2 RET

- Focus on the agent's utility: $u := u^A$.
- $A := \phi(\Theta)$. $V(\theta) := \max_{a \in A} u(a, \theta)$. $A^*(\theta) := \arg\max_{a \in A} u(a, \theta)$.
- Assume that $u(a, \cdot)$ is absolutely continuous and differentiable on Θ for all $a \in A$.
- By incentive compatibility, $\phi(\theta) \in A^*(\theta)$ for all $\theta \in \Theta$: ϕ is a selection from A^* .

Thm. 1.2 (Revenue Equivalence Theorem).

$$V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_{\theta}(\phi(s), s) ds$$

In particular, under quasi-linear utility,

$$V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v_{\theta}(x(s), s) ds$$

$$t(\theta) = v(x(\theta), \theta) - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} v_{\theta}(x(s), s) ds$$

Proof. Milgrom and Segal. As for quasi-linear cases, the results follow from

$$V(\theta) = v(x(\theta), \theta) - t(\theta)$$

\square

- RET states that under any IC mechanism, except for the constant $V(\underline{\theta})$, the transfer from the agent to the principal is uniquely determined once the allocation rule x is fixed.

1.2 Characterization of IC

1.2.1 Monotone Comparative Statics

- Consider parameterized optimization problem.
- We often want to know how optimizers and optimal values change according to the changes in parameters.
- Implicit function theorem: many assumptions are required.
- Monotone comparative analysis = Sensitivity analysis
- $\Theta \subseteq \mathbb{R}$. Two functions $g : \Theta \rightarrow \mathbb{R}$ and $f : \Theta \rightarrow \mathbb{R}$.

Def. 1.2 (Single Crossing). g dominates f by single crossing property (SCP), $g \succsim_{SC} f$, if for all $x'' > x'$,

- $f(x'') - f(x') \geq 0 \implies g(x'') - g(x') \geq 0$
- $f(x'') - f(x') > 0 \implies g(x'') - g(x') > 0$

$\{f(\cdot, \theta)\}_{\theta \in \Theta}$ is an SCP family if

$$\forall \theta'' > \theta'; f(\cdot, \theta'') \succsim_{SC} f(\cdot, \theta')$$

Def. 1.3 (Increasing Differences). g dominates f by increasing differences, $g \succsim_{IN} f$, if for all $x'' > x'$,

$$g(x'') - g(x') \geq f(x'') - f(x').$$

$\{f(\cdot, \theta)\}_{\theta \in \Theta}$ satisfies increasing differences if

$$\forall \theta'' > \theta'; f(\cdot, \theta'') \succsim_{IN} f(\cdot, \theta')$$

- Note that $g \succsim_{IN} f$ implies $g \succsim_{SC} f$.

Thm. 1.3 (Milgrom and Shannon). $X \subseteq \mathbb{R}$. $f, g : X \rightarrow \mathbb{R}$.

$$[\forall Y \subseteq X; \operatorname{argmax}_{x \in Y} g(x) \geq \operatorname{argmax}_{x \in Y} f(x)] \iff g \succsim_{SC} f$$

Note that

$$\operatorname{argmax}_{x \in Y} g(x) \geq \operatorname{argmax}_{x \in Y} f(x) \stackrel{\Delta}{\iff} [z \in \operatorname{argmax}_{x \in Y} g(x), w \in \operatorname{argmax}_{x \in Y} f(x) \implies z \geq w]$$

Proof. .

\Rightarrow) Fix any x'', x' such that $x'' > x'$. □

1.2.2 Characterization of IC