# Notes on Mechanism Design

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- This study notes are mainly based on the lecture note written by Valimaki in 2018.

## 1 Single Agent

- One principal v.s. one agent.
- $a \in A$: allocation, $\theta \in \Theta$: agent's private info. $\theta \sim F(\theta)$. $u^P(a, \theta)$, $u^A(a, \theta)$.
- We often assume quasi-linear payoff functions:
  - $a := (x, t)$, $u^P(a, \theta) := v^P(x, \theta) + t$, $u^A(a, \theta) := v^A(x, \theta) - t$.
- A mechanism is a pair $M := (\Sigma, \phi)$, where $\Sigma$ is a message space and $\phi : \Sigma \to \Delta(A)$.
- Agent's strategy: $\sigma : \Theta \to \Delta(\Sigma)$. Principal commits to a mechanism $M$.
- Consider a social choice function $\psi : \Theta \to A$. We want to know whether $\psi$ is implementable (, i.e., achievable in equilibrium,) or not.
- As for implementability, we can discuss it focusing only on direct mechanisms, assuming $\Sigma := \Theta$, w.l.o.g. (Revelation principle)

### 1.1 Revenue Equivalence

- In §1.1 and §1.2, we assume that the parameter space is a closed interval $\Theta := [\underline{\theta}, \overline{\theta}] \subseteq \mathbb{R}$.

#### 1.1.1 Milgrom and Segal (2002), Envelope Theorem

- $\Theta := [\underline{\theta}, \overline{\theta}]$. $f(\cdot, \theta) : X \to \mathbb{R}$. $\{f(\cdot, \theta)\}_{\theta \in \Theta}$.
- $V(\theta) := \max_{x \in X} f(x, \theta)$. $X^*(\theta) := \mathrm{argmax}_{x \in X} f(x, \theta)$

**Def. 1.1** (Selection)**.** *A function $x^* : \Theta \to X$ is a selection from $X^*$ if $x^*(\theta) \in X^*(\theta)$ for all $\theta \in \Theta$.*

**Thm. 1.1** (Milgrom and Segal (2002))**.** *Assume the following:*

- *For any $x \in X$, $f(x, \cdot) : \Theta \to \mathbb{R}$ is absolutely continuous on $\Theta$.*
- *For any $x \in X$, $f(x, \cdot) : \Theta \to \mathbb{R}$ is differentiable on $\Theta$.*

*Then, the following holds:*

- *$V$ is absolutely continuous.*
- *For any selection $x^*$ from $X^*$, $V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} f_\theta(x^*(s), s) ds$.*

*Proof.* Note that the absolute continuity of $f(x, \theta)$ implies that $f_\theta(x, \theta) \in L^1(\Theta)$ for any $x \in X$.

**(i) $V$ is absolutely continuous.** It is sufficient to show that $V$ is Lipschitz continuous. Fix any $\theta', \theta$. Since any integrable function is bounded, for any $x$ there exists $L > 0$ s.t. $|f_\theta(x, \theta)| \leq L$ for almost all $\theta \in \Theta$.

$$
\begin{aligned}
|V(\theta') - V(\theta)| &= \left| \max_{x'} f(x', \theta') - \max_{x} f(x, \theta) \right| \\
&\leq \max_x |f(x, \theta') - f(x, \theta)| = \max_x \left| \int_{\theta'}^{\theta} f_\theta(x, s) ds \right| \\
&\leq L \cdot |\theta' - \theta|
\end{aligned}
$$

**(ii)** Fix any selection $x^*$ from $X^*$. By the result of (i),

$$V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} V'(s) ds$$

Fix any selection $x^*$ and $\theta', \theta$ such that $\theta' > \theta$. By the definition of $V$ and $x^*$,

$$
\begin{aligned}
V(\theta) &= f(x^*(\theta), \theta) \geq f(x^*(\theta'), \theta) \\
V(\theta') &= f(x^*(\theta'), \theta') \geq f(x^*(\theta), \theta')
\end{aligned}
$$

Hence,

$$
\begin{aligned}
V(\theta') - V(\theta) &\leq f(x^*(\theta'), \theta') - f(x^*(\theta'), \theta). \\
\frac{V(\theta') - V(\theta)}{\theta' - \theta} &\leq \frac{f(x^*(\theta'), \theta') - f(x^*(\theta'), \theta)}{\theta' - \theta}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
V(\theta) - V(\theta') &\leq f(x^*(\theta'), \theta) - f(x^*(\theta'), \theta'). \\
\frac{V(\theta') - V(\theta)}{\theta - \theta'} &\geq \frac{f(x^*(\theta'), \theta) - f(x^*(\theta'), \theta')}{\theta - \theta'}.
\end{aligned}
$$

Note that by assumption $f(x, \cdot)$ is differentiable at all $\theta \in \Theta$. Therefore, if $V$ is differentiable at $\theta$, we have $V'(\theta) = f_\theta(x^*(\theta), \theta)$. □

### 1.1.2 RET

- Focus on the agent's utility: $u := u^A$.
- $A := \phi(\Theta)$. $V(\theta) := \max_{a \in A} u(a, \theta)$. $A^*(\theta) := \operatorname{argmax}_{a \in A} u(a, \theta)$.
- Assume that $u(a, \cdot)$ is absolutely continuous and differentiable on $\Theta$ for all $a \in A$.
- By incentive compatibility, $\phi(\theta) \in A^*(\theta)$ for all $\theta \in \Theta$: $\phi$ is a selection from $A^*$.

**Thm. 1.2** (Revenue Equivalence Theorem)**.**

$$V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_\theta(\phi(s), s) ds$$

*In particular, under quasi-linear utility,*

$$V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v_\theta(x(s), s) ds$$

$$t(\theta) = v(x(\theta), \theta) - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} v_\theta(x(s), s) ds$$

*Proof.* Milgrom and Segal. As for quasi-linear cases, the results follow from

$$V(\theta) = v(x(\theta), \theta) - t(\theta)$$

□

- RET states that under any IC mechanism, except for the constant $V(\underline{\theta})$, the transfer from the agent to the principal is uniquely determined once the allocation rule $x$ is fixed.

## 1.2 Characterization of IC

### 1.2.1 Monotone Comparative Statics

This subsection is based on the lecture slides by John K.-H. Quah:
`http://www.johnquah.com/lecture-slides.html`

- Consider parameterized optimization problems.
- We often want to know how optimizers and optimal values change according to the changes in parameters.
- comparative statics = Sensitivity analysis
- Implicit function theorem: Not only the direction of changes but also the rate of change. Many assumptions are required.
- Monotone comparative statics: Only the direction of changes. Fewer assumptions.
- $\Theta \subseteq \mathbb{R}$. Two functions $g : \Theta \to \mathbb{R}$ and $f : \Theta \to \mathbb{R}$.

**Def. 1.2** (Single Crossing). *g dominates f by single crossing property (SCP), $g \succsim_{SC} f$, if for all $x'' > x'$,*

- $f(x'') - f(x') \geq 0 \implies g(x'') - g(x') \geq 0$
- $f(x'') - f(x') > 0 \implies g(x'') - g(x') > 0$

$\{f(\cdot, \theta)\}_{\theta \in \Theta}$ *is an SCP family if*

$$\forall \theta'' > \theta';\ f(\cdot, \theta'') \succsim_{SC} f(\cdot, \theta')$$

**Def. 1.3** (Increasing Differences). *g dominates f by increasing differences, $g \succsim_{IN} f$, if for all $x'' > x'$,*

$$g(x'') - g(x') \geq f(x'') - f(x').$$

$\{f(\cdot, \theta)\}_{\theta \in \Theta}$ *satisfies increasing differences if*

$$\forall \theta'' > \theta';\ f(\cdot, \theta'') \succsim_{IN} f(\cdot, \theta')$$

**Def. 1.4** (Strictly Increasing Differences). *g dominates f by strictly increasing differences, $g \succsim_{SID} f$, if for all $x'' > x'$,*

$$g(x'') - g(x') > f(x'') - f(x').$$

$\{f(\cdot, \theta)\}_{\theta \in \Theta}$ *satisfies strictly increasing differences (SID) if*

$$\forall \theta'' > \theta';\ f(\cdot, \theta'') \succsim_{SID} f(\cdot, \theta')$$

- Note that $g \succsim_{IN} f$ implies $g \succsim_{SC} f$.

**Thm. 1.3** (Milgrom and Shannon (1994)). $X \subseteq \mathbb{R}$. $f, g : X \to \mathbb{R}$.

$$[\forall Y \subseteq X;\ \operatorname*{argmax}_{x \in Y} g(x) \geq \operatorname*{argmax}_{x \in Y} f(x)] \iff g \succsim_{SC} f$$

*Note that, for $Y, Z \subseteq \mathbb{R}$,*

$$Y \geq Z \overset{\Delta}{\iff} [y \in Y,\ z \in Z \implies y \vee x \in Y,\ y \wedge z \in Z.]$$

*Proof.* .

$\Rightarrow$) We show contrapositive. Suppose that $g \not\gtrsim_{SC} f$. There exist $x'', x'$ such that $x'' > x'$ and at least one of the following holds:

$$f(x'') \geq f(x'), g(x'') < g(x') \tag{1}$$

or

$$f(x'') > f(x'), g(x'') \leq g(x') \tag{2}$$

Let $Y := \{x', x''\}$, $G_Y := \text{argmax}_{x \in Y} g(x)$ and $F_Y := \text{argmax}_{x \in Y} f(x)$. In case of (1), $x' \vee x'' \notin G_Y$. In case of (2), $x' \wedge x'' \notin F_Y$.

$\Leftarrow$) Fix any $Y \subseteq X$ and $x'', x' \in Y$ such that $x' \in G_Y$ and $x'' \in F_Y$. We need to show that $x' \vee x'' \in G_Y$ and $x' \wedge x'' \in F_Y$. First, since $x'' \in F_Y$, we have $f(x'') \geq f(x')$. By assumption, $g(x'') \geq g(x')$. Since $x' \in G_Y$, we have $x'' \in G_Y$ and $x' \vee x'' \in G_Y$.

Next, we show $f(x'') = f(x')$. Note that this implies that $x' \wedge x'' \in F_Y$. Suppose toward contradiction that $f(x'') > f(x')$. Then, since $g \gtrsim_{SC} f$, we have $g(x'') > g(x')$. This contradicts $x' \in G_Y$. □

### 1.2.2 Characterization of IC

- Consider quasi-linear utility cases. Assume that $v(x, \theta)$ is absolutely continuous and differentiable on $\Theta$ for all $x$.

**Lem. 1.1.** *Let $V(\theta) := v(x(\theta), \theta) - t(\theta)$. If a mechanism $(x, t)$ is IC, then*

$$V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v_\theta(x(s), s) ds \tag{LIC}$$

*Proof.* RET. □

**Lem. 1.2.** *If a mechanism $(x, t)$ is IC and $\{v(\cdot, \theta)\}_{\theta \in \Theta}$ satisfies SID, then*

$$x(\theta) \text{ is non-decreasing in } \theta. \tag{M}$$

*Proof.* Fix $\theta'', \theta'$ such that $\theta'' > \theta'$. Since $\{v(\cdot, \theta)\}_{\theta \in \Theta}$ satisfies SID, $v(\cdot, \theta'') \gtrsim_{SID} v(\cdot, \theta')$. Suppose toward contradiction that $x(\theta'') < x(\theta')$. Since $v(\cdot, \theta'') \gtrsim_{SID} v(\cdot, \theta')$,

$$v(x(\theta'), \theta'') - v(x(\theta''), \theta'') > v(x(\theta'), \theta') - v(x(\theta''), \theta') \geq 0$$

This violates IC. A contradiction. □

- The lemmas above shows that, assuming $\{v(\cdot, \theta)\}_{\theta \in \Theta}$ satisfies SID, IC of $(x, t)$ implies (LIC) and (M).
- We can show that the converse also holds.

**Lem. 1.3.** *Assume that $\{v(\cdot, \theta)\}_{\theta \in \Theta}$ satisfies SID. If the conditions (LIC) and (M) hold, then $(x, t)$ is IC.*

*Proof.* Fix any $\theta, \theta'$. We need to show that $v(x(\theta), \theta) - t(\theta) \geq v(x(\theta'), \theta) - t(\theta')$. Note that, by (LIC), we have

$$t(\theta) = v(x(\theta), \theta) - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} v_\theta(x(s), s) ds$$

Then,

$$\begin{aligned}
&[v(x(\theta), \theta) - t(\theta)] - [v(x(\theta'), \theta) - t(\theta')] \\
&= [v(x(\theta), \theta) - t(\theta)] - [v(x(\theta'), \theta) + v(x(\theta'), \theta') - v(x(\theta'), \theta') - t(\theta')] \\
&= \int_{\underline{\theta}}^{\theta} v_\theta(x(s), s) ds - \int_{\underline{\theta}}^{\theta'} v_\theta(x(s), s) ds - [v(x(\theta'), \theta) - v(x(\theta'), \theta')] \\
&= \int_{\theta'}^{\theta} v_\theta(x(s), s) ds - \int_{\theta'}^{\theta} v_\theta(x(s), \theta') ds = \int_{\theta'}^{\theta} [v_\theta(x(s), s) - v_\theta(x(\theta'), s)] ds \geq 0
\end{aligned}$$

□

**Thm. 1.4** (Characterization of IC)**.** *Assume that $\{v(\cdot, \theta)\}_\theta$ satisfies SID. Then,*

$$(x, t) \text{ is IC} \iff x \text{ is non-decreasing, and } t \text{ is calculated by (LIC)}$$

### 1.3 General Case: Rochet's Theorem and Cyclical Monotonicity

- Consider quasi-linear utility cases.
- Characterize IC mechanisms.

**Def. 1.5** (weak monotonicity)**.** *An allocation rule $x : \Theta \to A$ is weakly monotone if*

$$\forall \theta, \theta'; \ [v(x(\theta), \theta') - v(x(\theta), \theta)] + [v(x(\theta'), \theta) - v(x(\theta'), \theta')] \leq 0$$

**Prop. 1.1.** *If $(x, t)$ is IC, then $x$ is weakly monotone.*

**Def. 1.6** (cyclical monotonicity)**.**

$$S := \{(\theta^1, \cdots, \theta^{k+1}) \mid \forall i \in [k+1]; \theta^i \in \Theta, \ \theta^1 = \theta^{k+1}, \ k \in \mathbb{Z}^+\}$$

*An allocation rule $x$ ie cyclically monotone if , for any $(\theta^1, \cdots, \theta^{k+1}) \in S$,*

$$\sum_{i=1}^{k} [v(x^i, \theta^{i+1}) - v(x^i, \theta^i)] \leq 0 \text{ , where } x^i := x(\theta^i) \tag{CM}$$

**Thm. 1.5** (Rochet (1987))**.**

$$\exists t; \ (x, t) : IC \iff x \text{ is cyclically monotone.}$$

*Proof.* .

$\Rightarrow$) Easy.

$\Leftarrow$) Fix $\theta_0 \in \Theta$.

$$S(\theta) := \{(\theta^1, \cdots, \theta^{k+1}) \mid \forall i \in [k+1]; \theta^i \in \Theta, \ \theta^1 = \theta_0, \ \theta^{k+1} = \theta, \ k \in \mathbb{Z}^+\}$$

$$V(\theta) := \sup_{(\theta^1, \cdots, \theta^{k+1}) \in S(\theta)} \sum_{i=1}^{k} [v(x^i, \theta^{i+1}) - v(x^i, \theta^i)]$$

**(i) [$V(\theta_0) = 0$.]** By CM, $V(\theta_0) \leq 0$. Considering the case where $k := 1$, we see that $(\theta_0, \theta_0) \in S(\theta_0)$ satisfies $[v(x^1, \theta^2) - v(x^1, \theta^1)] = 0$. Therefore, $V(\theta_0) = 0$.

**(ii) [$V(\theta) < \infty$ for all $\theta \in \Theta$.]** Fix any $(\theta^1, \cdots, \theta^{k+1}) \in S(\theta)$.

$$\begin{aligned} 0 = V(\theta_0) &\geq \sum_{i=1}^{k} [v(x^i, \theta^{i+1}) - v(x^i, \theta^i)] + [v(x^{i+1}, \theta_0) - v(x^{i+1}, \theta^{k+1})] \\ &= \sum_{i=1}^{k} [v(x^i, \theta^{i+1}) - v(x^i, \theta^i)] + [v(x(\theta), \theta_0) - v(x(\theta), \theta)] \end{aligned}$$

$$\therefore \sum_{i=1}^{k} [v(x^i, \theta^{i+1}) - v(x^i, \theta^i)] \leq v(x(\theta), \theta) - v(x(\theta), \theta_0)$$

$$\therefore V(\theta) \leq v(x(\theta), \theta) - v(x(\theta), \theta_0)$$

**(iii) [Construct the transfer rule]** Fix any $\theta, \theta'$. By the same argument as in (ii), we can show that

$$V(\theta) \geq V(\theta') + v(x(\theta'), \theta) - v(x(\theta'), \theta')$$

Define $t(\theta) := v(x(\theta), \theta) - V(\theta)$. With this $t$, a mechanism $(x, t)$ satisfies IC:

$$v(x(\theta), \theta) - t(\theta) - (v(x(\theta'), \theta) - t(\theta')) = V(\theta) - V(\theta') - v(x(\theta'), \theta) + v(x(\theta'), \theta') \geq 0$$

$\square$

### 1.4 Optimizing over Incentive Compatible Mechanisms

**Ass. 1** (Assumptions for IC characterization). *In §1.4, we assume that (1) utility function is quasi linear, (2) $v(x,\theta)$ is absolutely continuous and differentiable on $\Theta$ for all $x$, and (3) $\{v(\cdot,\theta)\}_\theta$ has SID.*

**Ass. 2** (Private Values). $v^P(x,\theta) \equiv v^P(x)$

**Ass. 3** (Absolutely continuous distribution). *The distribution function $F$ is absolutely continuous, i.e., there exists $f : \Theta \to \mathbb{R}_+$ s.t. $F(x) := \int_{\underline{\theta}}^{x} f(s)ds$*

- Optimal Mechanism = Revenue Maximizing Mechanism
- By Thm. 1.4, $(x,t)$ is IC iff $x$ is nondecreasing and $t$ is calculated by Envelope theorem.

$$\begin{aligned}
\text{[Expected Revenue]} &= \mathbb{E}_\theta \left[ t(\theta) + v^P(x(\theta)) \right] \\
&= \mathbb{E}_\theta \left[ v(x(\theta),\theta) - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} v_\theta(x(s),s)ds + v^P(x(\theta)) \right] \\
&= \mathbb{E}_\theta \left[ S(x(\theta),\theta) - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} v_\theta(x(s),s)ds \right]
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_\theta \left[ \int_{\underline{\theta}}^{\theta} v_\theta(x(s),s)ds \right] &= \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\theta} v_\theta(x(s),s)dsdF(\theta) \\
&= \int_{\underline{\theta}}^{\overline{\theta}} \int_{s}^{\overline{\theta}} v_\theta(x(s),s)dF(\theta)ds \\
&= \int_{\underline{\theta}}^{\overline{\theta}} (1-F(s))v_\theta(x(s),s)ds \\
&= \int_{\underline{\theta}}^{\overline{\theta}} (1-F(\theta))v_\theta(x(\theta),\theta)d\theta \\
&= \int_{\underline{\theta}}^{\overline{\theta}} v_\theta(x(\theta),\theta)\frac{1-F(\theta)}{f(\theta)}dF(\theta) \\
&= \mathbb{E}_\theta \left[ v_\theta(x(\theta),\theta)\frac{1-F(\theta)}{f(\theta)} \right]
\end{aligned}$$

$$\therefore \quad \text{[Expected Revenue]} = \mathbb{E}_\theta \left[ S(x(\theta),\theta) - V(\underline{\theta}) - v_\theta(x(\theta),\theta)\frac{1-F(\theta)}{f(\theta)} \right]$$

- The principal solves the following revenue maximization problem:

$$\max_{x(\cdot),V(\underline{\theta})} \mathbb{E}_\theta \left[ S(x(\theta),\theta) - V(\underline{\theta}) - v_\theta(x(\theta),\theta)\frac{1-F(\theta)}{f(\theta)} \right] \text{ s.t. } x(\cdot) : \text{increasing.}$$

- It is optimal to set $V(\underline{\theta}) := 0$, assuming that the outside option value is zero. Then, the problem can be reduced to

$$\max_{x(\cdot)} \mathbb{E}_\theta \left[ \underbrace{S(x(\theta),\theta) - v_\theta(x(\theta),\theta)\frac{1-F(\theta)}{f(\theta)}}_{(\star)} \right] \text{ s.t. } x(\cdot) : \text{nondecreasing.}$$

- If $v^P(\theta) \equiv 0$, $(\star) = v(x(\theta),\theta) - v_\theta(x(\theta),\theta)\frac{1-F(\theta)}{f(\theta)}$ =: [the virtual valuation of the bidder.]

### Finding a Solution

- Just ignoring the monotonicity of $x$ and solve the relaxed problem. Fix $\theta \in \Theta$, and solve maximization problem for each $\theta$.
- [Is the argument below valid in case $x$ is not $C^1$ on $\Theta$?]

**Ass. 4.** *Assume that $v$ is linear in $\theta$.*

**Ass. 5.** *Assume the interior solution. (?)*

$$\max_x S(x,\theta) - v_\theta(x,\theta)\frac{1-F(\theta)}{f(\theta)}$$

$$S_x(x,\theta) - v_{\theta x}(x,\theta)\frac{1-F(\theta)}{f(\theta)} = 0 \tag{FOC}$$

$$\underbrace{\left(\underbrace{S_{x\theta}(x,\theta)}_{=v_{x\theta}(x,\theta)\ (\because\ PV)} - v_{\theta x}(x,\theta)\frac{d}{d\theta}\frac{1-F(\theta)}{f(\theta)} - \underbrace{v_{\theta x\theta}(x,\theta)}_{=0}\frac{1-F(\theta)}{f(\theta)}\right)}_{=v_{x\theta}(x,\theta)\left(1-\frac{d}{d\theta}\frac{1-F(\theta)}{f(\theta)}\right)} d\theta$$

$$+\left(\underbrace{S_{xx}(x,\theta) - v_{\theta xx}(x,\theta)\frac{1-F(\theta)}{f(\theta)}}_{\leq 0 \quad (\because\ SOC)}\right) dx = 0$$

$$\iff \left(v_{x\theta}(x,\theta) - v_{x\theta}(x,\theta)\frac{d}{d\theta}\frac{1-F(\theta)}{f(\theta)}\right) d\theta + (\cdots)dx = 0$$

$$\therefore \quad \operatorname{sgn}\left(\frac{dx}{d\theta}\right) = \operatorname{sgn}\left(v_{x\theta}(x,\theta) - v_{x\theta}(x,\theta)\frac{d}{d\theta}\frac{1-F(\theta)}{f(\theta)}\right)$$

- Note that since $v$ has SID, $v_{x\theta} \geq 0$.
- The following condition (MHR: monotone hazard rate condition) is sufficient in order for $x$ to be nondecreasing:

$$\frac{d}{d\theta}\frac{1-F(\theta)}{f(\theta)} \leq 0 \iff \frac{d}{d\theta}\frac{f(\theta)}{1-F(\theta)} \geq 0$$

## 2 Many Agents