Notes on Mechanism Design

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August 5, 2018

- This study notes are mainly based on the lecture note written by Valimaki in 2018.

1 Single Agent

- One principal v.s. one agent.
- $a \in A$: allocation, $\theta \in \Theta$: agent's private info. $\theta \sim F(\theta)$. $u^P(a,\theta)$, $u^A(a,\theta)$.
- We often assume quasi-linear payoff functions:
  - $a := (x,t)$, $u^P(a,\theta) := v^P(x,\theta) + t$, $u^A(a,\theta) := v^A(x,\theta) - t$.
- A mechanism is a pair $M := (\Sigma, \phi)$, where Σ is a message space and $\phi : \Sigma \to \Delta(A)$.
- Agent's strategy: $\sigma : \Theta \to \Delta(\Sigma)$. Principal commits to a mechanism M.
- Consider a social choice function $\psi : \Theta \to A$. We want to know whether ψ is implementable (, i.e., achievable in equilibrium,) or not.
- As for implementability, we can discuss it focusing only on direct mechanisms, assuming $\Sigma := \Theta$, w.l.o.g. (Revelation principle)

1.1 Revenue Equivalence

- In §1.1 and §1.2, we assume that the parameter space is a closed interval $\Theta := [\underline{\theta}, \overline{\theta}] \subseteq \mathbb{R}$.

1.1.1 Milgrom and Segal (2002), Envelope Theorem

- $\Theta := [\underline{\theta}, \overline{\theta}]$. $f(\cdot,\theta) : X \to \mathbb{R}$. $\{f(\cdot,\theta)\}_{\theta\in\Theta}$.
- $V(\theta) := \max_{x\in X} f(x,\theta)$. $X^*(\theta) := \operatorname{argmax}_{x\in X} f(x,\theta)$

Def. 1.1 (Selection). *A function $x^* : \Theta \to X$ is a selection from X^* if $x^*(\theta) \in X^*(\theta)$ for all $\theta \in \Theta$.*

Thm. 1.1 (Milgrom and Segal (2002)). *Assume the following:*

- *For any $x \in X$, $f(x,\cdot) : \Theta \to \mathbb{R}$ is absolutely continuous on Θ.*
- *For any $x \in X$, $f(x,\cdot) : \Theta \to \mathbb{R}$ is differentiable on Θ.*

Then, the following holds:

- *V is absolutely continuous.*
- *For any selection x^* from X^*, $V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} f_\theta(x^*(s), s)ds$.*

Proof. Note that the absolute continuity of $f(x,\theta)$ implies that $f_\theta(x,\theta) \in L^1(\Theta)$ for any $x \in X$.

(i) V is absolutely continuous. It is sufficient to show that V is Lipschitz continuous. Fix any θ', θ. Since any integrable function is bounded, for any x there exists $L > 0$ s.t. $|f_\theta(x,\theta)| \leq L$ for almost all $\theta \in \Theta$.

$$
\begin{aligned}
|V(\theta') - V(\theta)| &= \left| \max_{x'} f(x', \theta') - \max_{x} f(x, \theta) \right| \\
&\leq \max_{x} |f(x,\theta') - f(x,\theta)| = \max_{x} \left| \int_{\theta'}^{\theta} f_\theta(x,s)ds \right| \\
&\leq L \cdot |\theta' - \theta|
\end{aligned}
$$

(ii) Fix any selection x^* from X^*. By the result of (i),

$$
V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} V'(s)ds
$$

Fix any selection x^* and θ', θ such that $\theta' > \theta$. By the definition of V and x^*,

$$
\begin{aligned}
V(\theta) = f(x^*(\theta), \theta) \geq f(x^*(\theta'), \theta) \\
V(\theta') = f(x^*(\theta'), \theta') \geq f(x^*(\theta), \theta')
\end{aligned}
$$

Hence,

$$
\begin{aligned}
V(\theta') - V(\theta) \leq f(x^*(\theta'), \theta') - f(x^*(\theta'), \theta). \\
\frac{V(\theta') - V(\theta)}{\theta' - \theta} \leq \frac{f(x^*(\theta'), \theta') - f(x^*(\theta'), \theta)}{\theta' - \theta}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
V(\theta) - V(\theta') \leq f(x^*(\theta'), \theta) - f(x^*(\theta'), \theta'). \\
\frac{V(\theta') - V(\theta)}{\theta - \theta'} \geq \frac{f(x^*(\theta'), \theta) - f(x^*(\theta'), \theta')}{\theta - \theta'}.
\end{aligned}
$$

Note that by assumption $f(x, \cdot)$ is differentiable at all $\theta \in \Theta$. Therefore, if V is differentiable at θ, we have $V'(\theta) = f_\theta(x^*(\theta), \theta)$. □

1.1.2 RET

- Focus on the agent's utility: $u := u^A$.
- $A := \phi(\Theta)$. $V(\theta) := \max_{a \in A} u(a, \theta)$. $A^*(\theta) := \operatorname{argmax}_{a \in A} u(a, \theta)$.
- Assume that $u(a, \cdot)$ is absolutely continuous and differentiable on Θ for all $a \in A$.
- By incentive compatibility, $\phi(\theta) \in A^*(\theta)$ for all $\theta \in \Theta$: ϕ is a selection from A^*.

Thm. 1.2 (Revenue Equivalence Theorem)**.**

$$
V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_\theta(\phi(s), s)ds
$$

In particular, under quasi-linear utility,

$$
V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v_\theta(x(s), s)ds
$$

$$
t(\theta) = v(x(\theta), \theta) - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} v_\theta(x(s), s)ds
$$

Proof. Milgrom and Segal. As for quasi-linear cases, the results follow from

$$
V(\theta) = v(x(\theta), \theta) - t(\theta)
$$

□

- RET states that under any IC mechanism, except for the constant $V(\underline{\theta})$, the transfer from the agent to the principal is uniquely determined once the allocation rule x is fixed.

1.2 Characterization of IC

1.2.1 Monotone Comparative Statics

This subsection is based on the lecture slides by John K.-H. Quah:
`http://www.johnquah.com/lecture-slides.html`

- Consider parameterized optimization problems.
- We often want to know how optimizers and optimal values change according to the changes in parameters.
- comparative statics = Sensitivity analysis
- Implicit function theorem: Not only the direction of changes but also the rate of change. Many assumptions are required.
- Monotone comparative statics: Only the direction of changes. Fewer assumptions.
- $\Theta \subseteq \mathbb{R}$. Two functions $g : \Theta \to \mathbb{R}$ and $f : \Theta \to \mathbb{R}$.

Def. 1.2 (Single Crossing). *g dominates f by single crossing property (SCP), $g \succsim_{SC} f$, if for all $x'' > x'$,*

- $f(x'') - f(x') \geq 0 \implies g(x'') - g(x') \geq 0$
- $f(x'') - f(x') > 0 \implies g(x'') - g(x') > 0$

$\{f(\cdot, \theta)\}_{\theta \in \Theta}$ *is an SCP family if*

$$\forall \theta'' > \theta';\ f(\cdot, \theta'') \succsim_{SC} f(\cdot, \theta')$$

Def. 1.3 (Increasing Differences). *g dominates f by increasing differences, $g \succsim_{IN} f$, if for all $x'' > x'$,*

$$g(x'') - g(x') \geq f(x'') - f(x').$$

$\{f(\cdot, \theta)\}_{\theta \in \Theta}$ *satisfies increasing differences if*

$$\forall \theta'' > \theta';\ f(\cdot, \theta'') \succsim_{IN} f(\cdot, \theta')$$

Def. 1.4 (Strictly Increasing Differences). *g dominates f by strictly increasing differences, $g \succsim_{SID} f$, if for all $x'' > x'$,*

$$g(x'') - g(x') > f(x'') - f(x').$$

$\{f(\cdot, \theta)\}_{\theta \in \Theta}$ *satisfies strictly increasing differences (SID) if*

$$\forall \theta'' > \theta';\ f(\cdot, \theta'') \succsim_{SID} f(\cdot, \theta')$$

- Note that $g \succsim_{IN} f$ implies $g \succsim_{SC} f$.

Thm. 1.3 (Milgrom and Shannon (1994)). *$X \subseteq \mathbb{R}$. $f, g : X \to \mathbb{R}$.*

$$[\forall Y \subseteq X;\ \operatorname*{argmax}_{x \in Y} g(x) \geq \operatorname*{argmax}_{x \in Y} f(x)] \iff g \succsim_{SC} f$$

Note that, for $Y, Z \subseteq \mathbb{R}$,

$$Y \geq Z \overset{\Delta}{\iff} [y \in Y,\ z \in Z \implies y \vee x \in Y,\ y \wedge z \in Z.]$$

Proof. .

$\Rightarrow$) We show contrapositive. Suppose that $g \not\gtrsim_{SC} f$. There exist x'', x' such that $x'' > x'$ and at least one of the following holds:

$$f(x'') \geq f(x'), g(x'') < g(x') \tag{1}$$

or

$$f(x'') > f(x'), g(x'') \leq g(x') \tag{2}$$

Let $Y := \{x', x''\}$, $G_Y := \mathrm{argmax}_{x \in Y}\, g(x)$ and $F_Y := \mathrm{argmax}_{x \in Y}\, f(x)$. In case of (1), $x' \vee x'' \notin G_Y$. In case of (2), $x' \wedge x'' \notin F_Y$.

$\Leftarrow$) Fix any $Y \subseteq X$ and $x'', x' \in Y$ such that $x' \in G_Y$ and $x'' \in F_Y$. We need to show that $x' \vee x'' \in G_Y$ and $x' \wedge x'' \in F_Y$. First, since $x'' \in F_Y$, we have $f(x'') \geq f(x')$. By assumption, $g(x'') \geq g(x')$. Since $x' \in G_Y$, we have $x'' \in G_Y$ and $x' \vee x'' \in G_Y$.

Next, we show $f(x'') = f(x')$. Note that this implies that $x' \wedge x'' \in F_Y$. Suppose toward contradiction that $f(x'') > f(x')$. Then, since $g \gtrsim_{SC} f$, we have $g(x'') > g(x')$. This contradicts $x' \in G_Y$. $\square$

1.2.2 Characterization of IC

- Consider quasi-linear utility cases. Assume that $v(x, \theta)$ is absolutely continuous and differentiable on Θ for all x.

Lem. 1.1. *Let $V(\theta) := v(x(\theta), \theta) - t(\theta)$. If a mechanism (x, t) is IC, then*

$$V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v_\theta(x(s), s) ds \tag{LIC}$$

Proof. RET. $\square$

Lem. 1.2. *If a mechanism (x, t) is IC and $\{v(\cdot, \theta)\}_{\theta \in \Theta}$ satisfies SID, then*

$$x(\theta) \text{ is non-decreasing in } \theta. \tag{M}$$

Proof. Fix θ'', θ' such that $\theta'' > \theta'$. Since $\{v(\cdot, \theta)\}_{\theta \in \Theta}$ satisfies SID, $v(\cdot, \theta'') \gtrsim_{SID} v(\cdot, \theta')$. Suppose toward contradiction that $x(\theta'') < x(\theta')$. Since $v(\cdot, \theta'') \gtrsim_{SID} v(\cdot, \theta')$,

$$v(x(\theta'), \theta'') - v(x(\theta''), \theta'') > v(x(\theta'), \theta') - v(x(\theta''), \theta') \geq 0$$

This violates IC. A contradiction. $\square$

- The lemmas above shows that, assuming $\{v(\cdot, \theta)\}_{\theta \in \Theta}$ satisfies SID, IC of (x, t) implies (LIC) and (M).
- We can show that the converse also holds.

Lem. 1.3. *Assume that $\{v(\cdot, \theta)\}_{\theta \in \Theta}$ satisfies SID. If the conditions (LIC) and (M) hold, then (x, t) is IC.*

Proof. Fix any θ, θ'. We need to show that $v(x(\theta), \theta) - t(\theta) \geq v(x(\theta'), \theta) - t(\theta')$. Note that, by (LIC), we have

$$t(\theta) = v(x(\theta), \theta) - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} v_\theta(x(s), s) ds$$

Then,

$$\begin{aligned}
&[v(x(\theta), \theta) - t(\theta)] - [v(x(\theta'), \theta) - t(\theta')] \\
&= [v(x(\theta), \theta) - t(\theta)] - [v(x(\theta'), \theta) + v(x(\theta'), \theta') - v(x(\theta'), \theta') - t(\theta')] \\
&= \int_{\underline{\theta}}^{\theta} v_\theta(x(s), s) ds - \int_{\underline{\theta}}^{\theta'} v_\theta(x(s), s) ds - [v(x(\theta'), \theta) - v(x(\theta'), \theta')] \\
&= \int_{\theta'}^{\theta} v_\theta(x(s), s) ds - \int_{\theta'}^{\theta} v_\theta(x(s), \theta') ds = \int_{\theta'}^{\theta} [v_\theta(x(s), s) - v_\theta(x(\theta'), s)] ds \geq 0
\end{aligned}$$

$\square$

Thm. 1.4 (Characterization of IC). *Assume that $\{v(\cdot, \theta)\}_\theta$ satisfies SID. Then,*

$$(x, t) \text{ is IC} \iff x \text{ is non-decreasing, and } t \text{ is calculated by (LIC)}$$

1.3 General Case: Rochet's Theorem and Cyclical Monotonicity

- Consider quasi-linear utility cases.
- Characterize IC mechanisms.

Def. 1.5 (weak monotonicity)**.** *An allocation rule $x : \Theta \to A$ is weakly monotone if*

$$\forall \theta, \theta';\ [v(x(\theta), \theta') - v(x(\theta), \theta)] + [v(x(\theta'), \theta) - v(x(\theta'), \theta')] \leq 0$$

Prop. 1.1. *If (x, t) is IC, then x is weakly monotone.*

Def. 1.6 (cyclical monotonicity)**.**

$$S := \{(\theta^1, \cdots, \theta^{k+1}) \mid \forall i \in [k+1]; \theta^i \in \Theta,\ \theta^1 = \theta^{k+1},\ k \in \mathbb{Z}^+\}$$

An allocation rule x ie cyclically monotone if , for any $(\theta^1, \cdots, \theta^{k+1}) \in S$,

$$\sum_{i=1}^{k} [v(x^i, \theta^{i+1}) - v(x^i, \theta^i)] \leq 0 \text{ , where } x^i := x(\theta^i) \tag{CM}$$

Thm. 1.5 (Rochet (1987))**.**

$$\exists t;\ (x, t) : IC \iff x \text{ is cyclically monotone.}$$

Proof. .

$\Rightarrow$) Easy.

$\Leftarrow$) Fix $\theta_0 \in \Theta$.

$$S(\theta) := \{(\theta^1, \cdots, \theta^{k+1}) \mid \forall i \in [k+1]; \theta^i \in \Theta,\ \theta^1 = \theta_0,\ \theta^{k+1} = \theta,\ k \in \mathbb{Z}^+\}$$

$$V(\theta) := \sup_{(\theta^1, \cdots, \theta^{k+1}) \in S(\theta)} \sum_{i=1}^{k} [v(x^i, \theta^{i+1}) - v(x^i, \theta^i)]$$

(i) [$V(\theta_0) = 0$.] By CM, $V(\theta_0) \leq 0$. Considering the case where $k := 1$, we see that $(\theta_0, \theta_0) \in S(\theta_0)$ satisfies $[v(x^1, \theta^2) - v(x^1, \theta^1)] = 0$. Therefore, $V(\theta_0) = 0$.

(ii) [$V(\theta) < \infty$ for all $\theta \in \Theta$.] Fix any $(\theta^1, \cdots, \theta^{k+1}) \in S(\theta)$.

$$\begin{aligned}
0 = V(\theta_0) &\geq \sum_{i=1}^{k} [v(x^i, \theta^{i+1}) - v(x^i, \theta^i)] + [v(x^{i+1}, \theta_0) - v(x^{i+1}, \theta^{k+1})] \\
&= \sum_{i=1}^{k} [v(x^i, \theta^{i+1}) - v(x^i, \theta^i)] + [v(x(\theta), \theta_0) - v(x(\theta), \theta)]
\end{aligned}$$

$$\therefore \sum_{i=1}^{k} [v(x^i, \theta^{i+1}) - v(x^i, \theta^i)] \leq v(x(\theta), \theta) - v(x(\theta), \theta_0)$$

$$\therefore V(\theta) \leq v(x(\theta), \theta) - v(x(\theta), \theta_0)$$

(iii) [Construct the transfer rule] Fix any θ, θ'. By the same argument as in (ii), we can show that

$$V(\theta) \geq V(\theta') + v(x(\theta'), \theta) - v(x(\theta'), \theta')$$

Define $t(\theta) := v(x(\theta), \theta) - V(\theta)$. With this t, a mechanism (x, t) satisfies IC:

$$v(x(\theta), \theta) - t(\theta) - (v(x(\theta'), \theta) - t(\theta')) = V(\theta) - V(\theta') - v(x(\theta'), \theta) + v(x(\theta'), \theta') \geq 0$$

$\square$