Notes on Mechanism Design

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- This study notes are mainly based on the lecture note written by Valimaki in 2018.

1 Single Agent

- One principal v.s. one agent.
- $a \in A$: allocation, $\theta \in \Theta$: agent's private info. $\theta \sim F(\theta)$. $u^P(a,\theta)$, $u^A(a,\theta)$.
- We often assume quasi-linear payoff functions:
  - $a := (x,t)$, $u^P(a,\theta) := v^P(x,\theta) + t$, $u^A(a,\theta) := v^A(x,\theta) - t$.
- A mechanism is a pair $M := (\Sigma, \phi)$, where Σ is a message space and $\phi : \Sigma \to \Delta(A)$.
- Agent's strategy: $\sigma : \Theta \to \Delta(\Sigma)$. Principal commits to a mechanism M.
- Consider a social choice function $\psi : \Theta \to A$. We want to know whether ψ is implementable (, i.e., achievable in equilibrium,) or not.
- As for implementability, we can discuss it focusing only on direct mechanisms, assuming $\Sigma := \Theta$, w.l.o.g. (Revelation principle)

1.1 Revenue Equivalence

- In §1.1 and §1.2, we assume that the parameter space is a closed interval $\Theta := [\underline{\theta}, \overline{\theta}] \subseteq \mathbb{R}$.

1.1.1 Milgrom and Segal (2002), Envelope Theorem

- $\Theta := [\underline{\theta}, \overline{\theta}]$. $f(\cdot,\theta) : X \to \mathbb{R}$. $\{f(\cdot,\theta)\}_{\theta\in\Theta}$.
- $V(\theta) := \max_{x\in X} f(x,\theta)$. $X^*(\theta) := \operatorname{argmax}_{x\in X} f(x,\theta)$

Def. 1.1 (Selection). *A function $x^* : \Theta \to X$ is a selection from X^* if $x^*(\theta) \in X^*(\theta)$ for all $\theta \in \Theta$.*

Thm. 1.1 (Milgrom and Segal (2002)). *Assume the following:*

- *For any $x \in X$, $f(x,\cdot) : \Theta \to \mathbb{R}$ is absolutely continuous on Θ.*
- *For any $x \in X$, $f(x,\cdot) : \Theta \to \mathbb{R}$ is differentiable on Θ.*

Then, the following holds:

- *V is absolutely continuous.*
- *For any selection x^* from X^*, $V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} f_\theta(x^*(s),s)ds$.*

Proof. Note that the absolute continuity of $f(x,\theta)$ implies that $f_\theta(x,\theta) \in L^1(\Theta)$ for any $x \in X$.

(i) V is absolutely continuous. It is sufficient to show that V is Lipschitz continuous. Fix any θ', θ. Since any integrable function is bounded, for any x there exists $L > 0$ s.t. $|f_\theta(x,\theta)| \le L$ for almost all $\theta \in \Theta$.

$$\begin{aligned}|V(\theta') - V(\theta)| &= \left|\max_{x'} f(x',\theta') - \max_x f(x,\theta)\right| \\ &\le \max_x \left|f(x,\theta') - f(x,\theta)\right| = \max_x \left|\int_{\theta'}^{\theta} f_\theta(x,s)ds\right| \\ &\le L \cdot |\theta' - \theta|\end{aligned}$$

(ii) Fix any selection x^* from X^*. By the result of (i),

$$V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} V'(s)ds$$

Fix any selection x^* and θ', θ such that $\theta' > \theta$. By the definition of V and x^*,

$$\begin{aligned}V(\theta) = f(x^*(\theta),\theta) \ge f(x^*(\theta'),\theta) \\ V(\theta') = f(x^*(\theta'),\theta') \ge f(x^*(\theta),\theta')\end{aligned}$$

Hence,

$$\begin{aligned}V(\theta') - V(\theta) \le f(x^*(\theta'),\theta') - f(x^*(\theta'),\theta). \\ \frac{V(\theta') - V(\theta)}{\theta' - \theta} \le \frac{f(x^*(\theta'),\theta') - f(x^*(\theta'),\theta)}{\theta' - \theta}.\end{aligned}$$

Similarly,

$$\begin{aligned}V(\theta) - V(\theta') \le f(x^*(\theta'),\theta) - f(x^*(\theta'),\theta'). \\ \frac{V(\theta') - V(\theta)}{\theta - \theta'} \ge \frac{f(x^*(\theta'),\theta) - f(x^*(\theta'),\theta')}{\theta - \theta'}.\end{aligned}$$

Note that by assumption $f(x,\cdot)$ is differentiable at all $\theta \in \Theta$. Therefore, if V is differentiable at θ, we have $V'(\theta) = f_\theta(x^*(\theta),\theta)$. □

1.1.2 RET

- Focus on the agent's utility: $u := u^A$.
- $A := \phi(\Theta)$. $V(\theta) := \max_{a\in A} u(a,\theta)$. $A^*(\theta) := \operatorname{argmax}_{a\in A} u(a,\theta)$.
- Assume that $u(a,\cdot)$ is absolutely continuous and differentiable on Θ for all $a \in A$.
- By incentive compatibility, $\phi(\theta) \in A^*(\theta)$ for all $\theta \in \Theta$: ϕ is a selection from A^*.

Thm. 1.2 (Revenue Equivalence Theorem)**.**

$$V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_\theta(\phi(s),s)ds$$

In particular, under quasi-linear utility,

$$V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v_\theta(x(s),s)ds$$

$$t(\theta) = v(x(\theta),\theta) - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} v_\theta(x(s),s)ds$$

Proof. Milgrom and Segal.

As for quasi-linear cases, since $u((x,t),\theta) = v(x,\theta) - t$, we have $u_\theta = v_\theta$. Note that $V(\theta) = v(x(\theta),\theta) - t(\theta)$. □

- RET states that under any IC mechanism, except for the constant $V(\underline{\theta})$, the transfer from the agent to the principal is uniquely determined once the allocation rule x is fixed.

1.2 Characterization of IC

1.2.1 Monotone Comparative Statics

This subsection is based on the lecture slides by John K.-H. Quah:
`http://www.johnquah.com/lecture-slides.html`

- Consider parameterized optimization problems.
- We often want to know how optimizers and optimal values change according to the changes in parameters.
- comparative statics = Sensitivity analysis
- Implicit function theorem: Not only the direction of changes but also the rate of change. Many assumptions are required.
- Monotone comparative statics: Only the direction of changes. Fewer assumptions.
- $\Theta \subseteq \mathbb{R}$. Two functions $g : \Theta \to \mathbb{R}$ and $f : \Theta \to \mathbb{R}$.

Def. 1.2 (Single Crossing). *g dominates f by single crossing property (SCP), $g \succsim_{SC} f$, if for all $x'' > x'$,*

- $f(x'') - f(x') \geq 0 \implies g(x'') - g(x') \geq 0$
- $f(x'') - f(x') > 0 \implies g(x'') - g(x') > 0$

$\{f(\cdot, \theta)\}_{\theta \in \Theta}$ is an SCP family if

$$\forall \theta'' > \theta';\ f(\cdot, \theta'') \succsim_{SC} f(\cdot, \theta')$$

Def. 1.3 (Increasing Differences). *g dominates f by increasing differences, $g \succsim_{IN} f$, if for all $x'' > x'$,*

$$g(x'') - g(x') \geq f(x'') - f(x').$$

$\{f(\cdot, \theta)\}_{\theta \in \Theta}$ satisfies increasing differences if

$$\forall \theta'' > \theta';\ f(\cdot, \theta'') \succsim_{IN} f(\cdot, \theta')$$

Def. 1.4 (Strictly Increasing Differences). *g dominates f by strictly increasing differences, $g \succsim_{SID} f$, if for all $x'' > x'$,*

$$g(x'') - g(x') > f(x'') - f(x').$$

$\{f(\cdot, \theta)\}_{\theta \in \Theta}$ satisfies strictly increasing differences (SID) if

$$\forall \theta'' > \theta';\ f(\cdot, \theta'') \succsim_{SID} f(\cdot, \theta')$$

Rem. 1.1. *$g \succsim_{IN} f$ implies $g \succsim_{SC} f$.*

Rem. 1.2. *$\{f(\cdot, \theta)\}_{\theta \in \Theta}$ satisfies SID iff $\{f(x, \cdot)\}_{x \in X}$ satisfies SID.*

Thm. 1.3 (Milgrom and Shannon (1994)). *$X \subseteq \mathbb{R}$. $f, g : X \to \mathbb{R}$.*

$$[\forall Y \subseteq X;\ \operatorname*{argmax}_{x \in Y} g(x) \geq \operatorname*{argmax}_{x \in Y} f(x)] \iff g \succsim_{SC} f$$

Note that, for $Y, Z \subseteq \mathbb{R}$,

$$Y \geq Z \overset{\Delta}{\iff} [y \in Y,\ z \in Z \implies y \vee x \in Y,\ y \wedge z \in Z.]$$

Proof. .

$\Rightarrow$) We show contrapositive. Suppose that $g \not\succsim_{SC} f$. There exist x'', x' such that $x'' > x'$ and at least one of the following holds:

$$f(x'') \geq f(x'), g(x'') < g(x') \tag{1}$$

or

$$f(x'') > f(x'), g(x'') \leq g(x') \tag{2}$$

Let $Y := \{x', x''\}$, $G_Y := \text{argmax}_{x \in Y} g(x)$ and $F_Y := \text{argmax}_{x \in Y} f(x)$. In case of (1), $x' \vee x'' \notin G_Y$. In case of (2), $x' \wedge x'' \notin F_Y$.

$\Leftarrow$) Fix any $Y \subseteq X$ and $x'', x' \in Y$ such that $x' \in G_Y$ and $x'' \in F_Y$. We need to show that $x' \vee x'' \in G_Y$ and $x' \wedge x'' \in F_Y$. First, since $x'' \in F_Y$, we have $f(x'') \geq f(x')$. By assumption, $g(x'') \geq g(x')$. Since $x' \in G_Y$, we have $x'' \in G_Y$ and $x' \vee x'' \in G_Y$.

Next, we show $f(x'') = f(x')$. Note that this implies that $x' \wedge x'' \in F_Y$. Suppose toward contradiction that $f(x'') > f(x')$. Then, since $g \succsim_{SC} f$, we have $g(x'') > g(x')$. This contradicts $x' \in G_Y$. $\square$

1.2.2 Characterization of IC

Ass. 1. *Quasi-linear utility. $v(x, \theta)$ is absolutely continuous and differentiable on Θ for all x.*

Lem. 1.1. *Let $V(\theta) := v(x(\theta), \theta) - t(\theta)$. If a mechanism (x, t) is IC, then*

$$V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v_\theta(x(s), s) ds \tag{LIC}$$

Proof. RET. $\square$

Lem. 1.2. *If a mechanism (x, t) is IC and $\{v(\cdot, \theta)\}_{\theta \in \Theta}$ satisfies SID, then*

$$x(\theta) \text{ is non-decreasing in } \theta. \tag{M}$$

Proof. Fix θ'', θ' such that $\theta'' > \theta'$. Since $\{v(\cdot, \theta)\}_{\theta \in \Theta}$ satisfies SID, $v(\cdot, \theta'') \succsim_{SID} v(\cdot, \theta')$. Suppose toward contradiction that $x(\theta'') < x(\theta')$. Since $v(\cdot, \theta'') \succsim_{SID} v(\cdot, \theta')$,

$$v(x(\theta'), \theta'') - v(x(\theta''), \theta'') > v(x(\theta'), \theta') - v(x(\theta''), \theta') \geq 0$$

This violates IC. A contradiction. $\square$

- The lemmas above shows that, assuming $\{v(\cdot, \theta)\}_{\theta \in \Theta}$ satisfies SID, IC of (x, t) implies (LIC) and (M).
- We can show that the converse also holds.

Lem. 1.3. *Assume that $\{v(\cdot, \theta)\}_{\theta \in \Theta}$ satisfies (S)ID. If the conditions (LIC) and (M) hold, then (x, t) is IC.*

Proof. Fix any θ, θ'. We need to show that $v(x(\theta), \theta) - t(\theta) \geq v(x(\theta'), \theta) - t(\theta')$. Note that, by (LIC), we have

$$t(\theta) = v(x(\theta), \theta) - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} v_\theta(x(s), s) ds$$

Then,

$$\begin{aligned}
&[v(x(\theta), \theta) - t(\theta)] - [v(x(\theta'), \theta) - t(\theta')] \\
&= [v(x(\theta), \theta) - t(\theta)] - [v(x(\theta'), \theta) + v(x(\theta'), \theta') - v(x(\theta'), \theta') - t(\theta')] \\
&= \int_{\underline{\theta}}^{\theta} v_\theta(x(s), s) ds - \int_{\underline{\theta}}^{\theta'} v_\theta(x(s), s) ds - [v(x(\theta'), \theta) - v(x(\theta'), \theta')] \\
&= \int_{\theta'}^{\theta} v_\theta(x(s), s) ds - \int_{\theta'}^{\theta} v_\theta(x(s), \theta') ds = \int_{\theta'}^{\theta} \underbrace{[v_\theta(x(s), s) - v_\theta(x(\theta'), s)]}_{\geq 0 \because ID} ds \geq 0
\end{aligned}$$

$\square$

Thm. 1.4 (Characterization of IC). *Assume that $\{v(\cdot, \theta)\}_\theta$ satisfies SID. Then,*

$$(x, t) \text{ is IC} \iff x \text{ is non-decreasing, and } t \text{ is calculated by (LIC)}$$

1.3 General Case: Rochet's Theorem and Cyclical Monotonicity

- Consider quasi-linear utility cases.
- Characterize IC mechanisms.

Def. 1.5 (weak monotonicity)**.** *An allocation rule $x : \Theta \to A$ is weakly monotone if*

$$\forall \theta, \theta';\ [v(x(\theta), \theta') - v(x(\theta), \theta)] + [v(x(\theta'), \theta) - v(x(\theta'), \theta')] \leq 0$$

Prop. 1.1. *If (x, t) is IC, then x is weakly monotone.*

Def. 1.6 (cyclical monotonicity)**.**

$$S := \{(\theta^1, \cdots, \theta^{k+1}) \mid \forall i \in [k+1]; \theta^i \in \Theta,\ \theta^1 = \theta^{k+1},\ k \in \mathbb{Z}^+\}$$

An allocation rule x ie cyclically monotone if , for any $(\theta^1, \cdots, \theta^{k+1}) \in S$,

$$\sum_{i=1}^{k} [v(x^i, \theta^{i+1}) - v(x^i, \theta^i)] \leq 0 \text{ , where } x^i := x(\theta^i) \tag{CM}$$

Thm. 1.5 (Rochet (1987))**.**

$$\exists t;\ (x, t) : IC \iff x \text{ is cyclically monotone.}$$

Proof. .

$\Rightarrow$) Easy.

$\Leftarrow$) Fix $\theta_0 \in \Theta$.

$$S(\theta) := \{(\theta^1, \cdots, \theta^{k+1}) \mid \forall i \in [k+1]; \theta^i \in \Theta,\ \theta^1 = \theta_0,\ \theta^{k+1} = \theta,\ k \in \mathbb{Z}^+\}$$

$$V(\theta) := \sup_{(\theta^1, \cdots, \theta^{k+1}) \in S(\theta)} \sum_{i=1}^{k} [v(x^i, \theta^{i+1}) - v(x^i, \theta^i)]$$

(i) [$V(\theta_0) = 0$.] By CM, $V(\theta_0) \leq 0$. Considering the case where $k := 1$, we see that $(\theta_0, \theta_0) \in S(\theta_0)$ satisfies $[v(x^1, \theta^2) - v(x^1, \theta^1)] = 0$. Therefore, $V(\theta_0) = 0$.

(ii) [$V(\theta) < \infty$ for all $\theta \in \Theta$.] Fix any $(\theta^1, \cdots, \theta^{k+1}) \in S(\theta)$.

$$\begin{aligned}
0 = V(\theta_0) &\geq \sum_{i=1}^{k} [v(x^i, \theta^{i+1}) - v(x^i, \theta^i)] + [v(x^{i+1}, \theta_0) - v(x^{i+1}, \theta^{k+1})] \\
&= \sum_{i=1}^{k} [v(x^i, \theta^{i+1}) - v(x^i, \theta^i)] + [v(x(\theta), \theta_0) - v(x(\theta), \theta)]
\end{aligned}$$

$$\therefore \sum_{i=1}^{k} [v(x^i, \theta^{i+1}) - v(x^i, \theta^i)] \leq v(x(\theta), \theta) - v(x(\theta), \theta_0)$$

$$\therefore V(\theta) \leq v(x(\theta), \theta) - v(x(\theta), \theta_0)$$

(iii) [Construct the transfer rule] Fix any θ, θ'. By the same argument as in (ii), we can show that

$$V(\theta) \geq V(\theta') + v(x(\theta'), \theta) - v(x(\theta'), \theta')$$

Define $t(\theta) := v(x(\theta), \theta) - V(\theta)$. With this t, a mechanism (x, t) satisfies IC:

$$v(x(\theta), \theta) - t(\theta) - (v(x(\theta'), \theta) - t(\theta')) = V(\theta) - V(\theta') - v(x(\theta'), \theta) + v(x(\theta'), \theta') \geq 0$$

$\square$

1.4 Optimizing over Incentive Compatible Mechanisms

Ass. 2 (Assumptions for IC characterization)**.** *In §1.4, we assume that (1) utility function is quasi linear, (2) $v(x,\theta)$ is absolutely continuous and differentiable on Θ for all x, and (3) $\{v(\cdot,\theta)\}_\theta$ has SID.*

Ass. 3 (Private Values)**.** $v^P(x,\theta) \equiv v^P(x)$

Ass. 4 (Absolutely continuous distribution)**.** *The distribution function F is absolutely continuous, i.e., there exists $f : \Theta \to \mathbb{R}_+$ s.t. $F(x) := \int_{\underline{\theta}}^{x} f(s)ds$*

- Optimal Mechanism = Revenue Maximizing Mechanism
- By Thm. 1.4, (x,t) is IC iff x is nondecreasing and t is calculated by Envelope theorem.

$$\begin{aligned}
\text{[Expected Revenue]} &= \mathbb{E}_\theta\left[t(\theta) + v^P(x(\theta))\right] \\
&= \mathbb{E}_\theta\left[v(x(\theta),\theta) - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} v_\theta(x(s),s)ds + v^P(x(\theta))\right] \\
&= \mathbb{E}_\theta\left[S(x(\theta),\theta) - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} v_\theta(x(s),s)ds\right]
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_\theta\left[\int_{\underline{\theta}}^{\theta} v_\theta(x(s),s)ds\right] &= \int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\theta}}^{\theta} v_\theta(x(s),s)ds dF(\theta) \\
&= \int_{\underline{\theta}}^{\overline{\theta}}\int_{s}^{\overline{\theta}} v_\theta(x(s),s)dF(\theta)ds \\
&= \int_{\underline{\theta}}^{\overline{\theta}} (1-F(s))v_\theta(x(s),s)ds \\
&= \int_{\underline{\theta}}^{\overline{\theta}} (1-F(\theta))v_\theta(x(\theta),\theta)d\theta \\
&= \int_{\underline{\theta}}^{\overline{\theta}} v_\theta(x(\theta),\theta)\frac{1-F(\theta)}{f(\theta)}dF(\theta) \\
&= \mathbb{E}_\theta\left[v_\theta(x(\theta),\theta)\frac{1-F(\theta)}{f(\theta)}\right]
\end{aligned}$$

$$\therefore \quad \text{[Expected Revenue]} = \mathbb{E}_\theta\left[S(x(\theta),\theta) - V(\underline{\theta}) - v_\theta(x(\theta),\theta)\frac{1-F(\theta)}{f(\theta)}\right]$$

- The principal solves the following revenue maximization problem:

$$\max_{x(\cdot),V(\underline{\theta})} \mathbb{E}_\theta\left[S(x(\theta),\theta) - V(\underline{\theta}) - v_\theta(x(\theta),\theta)\frac{1-F(\theta)}{f(\theta)}\right] \text{ s.t. } x(\cdot) : \text{increasing.}$$

- It is optimal to set $V(\underline{\theta}) := 0$, assuming that the outside option value is zero. Then, the problem can be reduced to

$$\max_{x(\cdot)} \mathbb{E}_\theta\left[\underbrace{S(x(\theta),\theta) - v_\theta(x(\theta),\theta)\frac{1-F(\theta)}{f(\theta)}}_{(\star)}\right] \text{ s.t. } x(\cdot) : \text{nondecreasing.}$$

- If $v^P(\theta) \equiv 0$, $(\star) = v(x(\theta),\theta) - v_\theta(x(\theta),\theta)\frac{1-F(\theta)}{f(\theta)}$ =: [the virtual valuation of the bidder.]

Finding a Solution

- Just ignoring the monotonicity of x and solve the relaxed problem. Fix $\theta \in \Theta$, and solve maximization problem for each θ.
- [Is the argument below valid in case x is not C^1 on Θ?]

Ass. 5. *Assume that v is linear in θ.*

Ass. 6. *Assume the interior solution. (?)*

$$\max_x S(x,\theta) - v_\theta(x,\theta)\frac{1-F(\theta)}{f(\theta)}$$

$$S_x(x,\theta) - v_{\theta x}(x,\theta)\frac{1-F(\theta)}{f(\theta)} = 0 \tag{FOC}$$

$$\underbrace{\left(\underbrace{S_{x\theta}(x,\theta)}_{=v_{x\theta}(x,\theta)\ (\because PV)} - v_{\theta x}(x,\theta)\frac{d}{d\theta}\frac{1-F(\theta)}{f(\theta)} - \underbrace{v_{\theta x\theta}(x,\theta)}_{=0}\frac{1-F(\theta)}{f(\theta)}\right)}_{=v_{x\theta}(x,\theta)\left(1-\frac{d}{d\theta}\frac{1-F(\theta)}{f(\theta)}\right)} d\theta$$

$$+\left(\underbrace{S_{xx}(x,\theta) - v_{\theta xx}(x,\theta)\frac{1-F(\theta)}{f(\theta)}}_{\leq 0 \quad (\because SOC)}\right) dx = 0$$

$$\iff \left(v_{x\theta}(x,\theta) - v_{x\theta}(x,\theta)\frac{d}{d\theta}\frac{1-F(\theta)}{f(\theta)}\right) d\theta + (\cdots)dx = 0$$

$$\therefore \quad \operatorname{sgn}\left(\frac{dx}{d\theta}\right) = \operatorname{sgn}\left(v_{x\theta}(x,\theta) - v_{x\theta}(x,\theta)\frac{d}{d\theta}\frac{1-F(\theta)}{f(\theta)}\right)$$

- Note that since v has SID, $v_{x\theta} \geq 0$.
- The following condition (MHR: monotone hazard rate condition) is sufficient in order for x to be nondecreasing:

$$\frac{d}{d\theta}\frac{1-F(\theta)}{f(\theta)} \leq 0 \iff \frac{d}{d\theta}\frac{f(\theta)}{1-F(\theta)} \geq 0$$

2 Many Agents

2.1 Setup

Def. 2.1 (Solution Concepts)**.** (M, ϕ): *mechanism*

1. $\sigma = (\sigma_1, \ldots, \sigma_N)$ *is a dominant strategy equilibrium (DSE) if*

$$\forall i\ \forall \theta_i\ \forall m_i\ \forall m_{-i} \forall \theta_{-i}; \quad u_i(\phi(\sigma_i(\theta_i), m_{-i}; \theta_i, \theta_{-i})) \geq u_i(\phi(m_i, m_{-i}; \theta_i, \theta_{-i}))$$

2. σ *is ex-post equilibrium if*

$$\forall i\ \forall \theta_i\ \forall m_i \forall \theta_{-i}; \quad u_i(\phi(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}); \theta_i, \theta_{-i})) \geq u_i(\phi(m_i, \sigma_{-i}(\theta_{-i}); \theta_i, \theta_{-i}))$$

   In words, as long as the other players are truthful, their type is not important for a players' incentives for truthful reporting.

3. σ *is Bayes-Nash equilibrium (BNE) in pure strategies if*

$$\forall i\ \forall \theta_i\ \forall m_i; \quad \mathbb{E}_{\theta_{-i}|\theta_i}[u_i(\phi(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}); \theta_i, \theta_{-i}))] \geq \mathbb{E}_{\theta_{-i}|\theta_i}[u_i(\phi(m_i, \sigma_{-i}(\theta_{-i}); \theta_i, \theta_{-i}))]$$

Rem. 2.1. *A DSE is ex-post eqm. An ex-post eqm is BNE.*

Rem. 2.2. *The revelation principle holds for all three equilibrium concepts.*

2.2 Notions of Incentive Compatibility

- DSIC mechanisms, ex-post IC mechanisms, BNIC mechanisms.

2.3 Individual Rationality

Ass. 7. *The value of an outside option is constant on type and normalized to zero.*

Def. 2.2. *Consider an IC mechanism* (x, t)*.*

1. (x, t) *is ex-post individually rational (IR) if*

$$\forall i \forall \theta; \quad v_i(x(\theta), \theta) - t_i(\theta) \geq 0$$

2. (x, t) *is interim IR if*

$$\forall i \forall \theta; \quad \mathbb{E}_{\theta_{-i}|\theta_i}[v_i(x(\theta), \theta) - t_i(\theta)] \geq 0$$

3. (x, t) *is ex-ante IR if*

$$\forall i \forall \theta; \quad \mathbb{E}_{\theta}[v_i(x(\theta), \theta) - t_i(\theta)] \geq 0$$

Rem. 2.3. *An ex-post IR mechanism is interim IR and ex-ante IR.*

Rem. 2.4. *For ex-post IR, it is sometimes useful to impose the following restriction: consider the case where other players are not truthful:*

$$\forall i \forall m_{-i} \forall \theta; \quad v_i(x(\theta_i, m_{-i}), \theta) - t_i(\theta_i, m_{-i}) \geq 0$$

2.4 Bayesian and Dominant Strategy Mechanisms

??

Def. 2.3 (Standard Independent Private Value Model). *In the standard IPV model, we assume*

1. *For all i,* $\Theta_i := [\underline{\theta}_i, \overline{\theta}_i]$
2. *quasi-linear utility*
3. *IPV*
4. *The assumptions for the envelope theorem hold. (??)* [1]

Ass. 8. *For all i,* $\Theta_i := [\underline{\theta}_i, \overline{\theta}_i]$*,* $v_i(x_i, \theta_i)$ *satisfies SID, and is sufficiently smooth w.r.t.* θ_i*. IPV.*

- Define V_i as follows:

$$V_i(\theta_i) := \max_{m_i}\{\mathbb{E}_{\theta_{-i}}[v_i(x(m_i, \theta_{-i}), \theta_i) - t_i(m_i, \theta_{-i})]\} = \max_{m_i}\{\widetilde{v}_i(x(m_i), \theta_i) - \widetilde{t}_i(m_i)\}$$

- $\widetilde{v}_i(x(m_i), \theta_i) := \mathbb{E}_{\theta_{-i}}[v_i(x(m_i, \theta_{-i}), \theta_i)]$, $\widetilde{t}_i(m_i) := \mathbb{E}_{\theta_{-i}}[t_i(m_i, \theta_{-i})]$
- Then, we can apply the same argument as in one agent case.

Rem. 2.5. *Assume that* $v_i(x, \theta_i)$ *satisfies SID. Then,*

- (x, t)*: DSIC implies* $x(m_i, \theta_{-i})$ *is non-decreasing in* m_i *for all* $\theta_{-i} \in \Theta_i$*.*
- (x, t)*: BIC implies* $\mathbb{E}_{\theta_{-i}}[x(m_i, \theta_{-i})]$ *is non-decreasing in* m_i*.*

2.5 Efficient Mechanisms

Ass. 9. *Quasi-linear utilities, private values.*

Def. 2.4 (Private Values). *The model has private values if*

$$\forall i \in \{0, 1, \dots, N\}\ \forall \theta_{-i}\theta'_{-i} \in \Theta_{-i}; \quad v_i(x, \theta_i, \theta_{-i}) = v_i(x, \theta_i, \theta'_{-i}) \equiv v_i(x, \theta_i)$$

- Consider the case where $v_0 \equiv 0$, $t_0 = \sum_{i=1}^N t_i$.

$$x^*(\theta) \in \operatorname*{argmax}_{x \in X} \sum_{i=1}^N v_i(x, \theta) = \operatorname*{argmax}_{x \in X} \sum_{i=1}^N v_i(x, \theta_i)$$

Def. 2.5 (VCG mechanism). *Fix any function* $\tau_i : M_{-i} \to \mathbb{R}$ *for all i. Define the transfer rule:*

$$t_i^*(m) := -\sum_{j \neq i} v_j(x^*(m), m_j) + \tau_i(m_{-i})$$

The mechanism (x^*, t^*) *is called VCG mechanism.*

Rem. 2.6. *VDG mechanism is a DSIC mechanism. (∵ By definition of* x^**.)*

Rem. 2.7 (Green & Laffont(1979)). *If there are no restrictions on the domain of preferences for the players, then VDG mechanisms are the only mechanisms that make truthful revelation a dominant strategy for the efficient allocation rule* x^**.*

Def. 2.6 (Pivot mechanism (the externality mechanism)). *Pivot mechanism is a VCG mechanism that set*

$$\tau_i(m_i) := \max_{x \in X} \sum_{j \neq i} v_j(x^*(m), m_j)$$

[1] Let $V_i(\theta_i) := \max_{m_i}$

- $W_{\mathcal{S}}(\theta) := \max_{x \in X} \sum_{i \in \mathcal{S}} v_i(x, \theta_i)$
- In a pivot mechanism, player i pays her externality and gets her marginal contribution to the social welfare:
$$t_i(\theta) = W_{[N]\setminus\{i\}}(\theta) - \sum_{j \neq i} v_j(x^*(\theta), \theta_j)$$
$$v_i(x^*(\theta), \theta_i) - t_i(\theta) = W_{[N]}(\theta) - W_{[N]\setminus\{i\}}(\theta)$$
- Correlation in the agents' types makes no difference in the above argument: It works for all private values settings with quasi-linear payoffs.

2.5.1 Budget Balance

Def. 2.7 (Budget Balance)**.** *(x, t) : mechanism.*

1. *(x, t) : mechanism is ex-post budget balanced if $\forall \theta$; $\sum_{i=1}^{N} t_i(\theta) = 0$.*
2. *(x, t) : mechanism is ex-ante budget balanced if $\mathbb{E}_\theta[\sum_{i=1}^{N} t_i(\theta)] = 0$.*

Ass. 10. *independent type, i.e., $\forall i$; $\theta_i \perp\!\!\!\perp \theta_{-i}$.*

Def. 2.8 (AGV mechanism (expected externality mechanism))**.** *A mechanism (x^*, t), with the following transfer rule t, is called AGV mechanism:*

$$t_i(m) := -\underbrace{\mathbb{E}_{\theta_{-i}}\left[\sum_{j \neq i} v_j(x^*(m_i, \theta_{-i}), \theta_j)\right]}_{=:\mathcal{E}_i(m_i)} + \tau_i(m_{-i})$$

Rem. 2.8. *In AGV mechanism, truthful report is a BNIC. (NB: No need to use independence here.)*

Rem. 2.9. *AGV mechanism with the following $(\tau_i)_i$ is ex-post BB.*

$$\tau_i(m_{-i}) = \frac{\sum_{j \neq i} \mathcal{E}_j(m_j)}{N-1}$$

- ex-post BB does not hold for correlated private values. ($\mathcal{E}_i(m_i)$ should not depend on θ_i.)
- It is possible to design variants with similarities to AGV mechanism that work also with correlated values settings.

$$\widehat{t}_i(m) := -\sum_{j \neq i} v_j(x^*(m), m_j) + \mathbb{E}_{\theta_i}\left[\sum_{j=1}^{N} v_j(x^*(\theta_i, m_{-i}), m_j) \mid \theta_{-i} = m_{-i}\right]$$

- $(x^*, \widehat{t})$ is a VCG mechanism, and it is ex-ante BB and ex-ante IC.
- What can be said about incentives to participate interim?

2.5.2 BB in BNE

- What more can be done with BNE as the solution concept?
- As long as we assume IPV, all BNE implementing the efficient decision rule have the same expected payoffs.
- The expected payment etc. can be calculated using VCG.

2.5.3 IC, BB and Participation