Spiegler (2016, QJE)
Bayesian Networks and Boundedly Rational Expectations

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1 Motivation

- 限定合理性，特に nonrational expectation が形成されるメカニズムを明らかにする．
- nonrational expectations の下で何が起こるかを分析する．

2 Approach

- DAG(directed acyclic graph) を人々が心の内に抱いている Causality model の表現と考える．
- 人々は，objective probability distributions $p(x_1, \ldots, x_n)$ に DAG R を fit させることで subjective belief $p_R(x_1, \ldots, x_n)$ を形成し，その上で意思決定を行う　と考える．
- 定常状態 (i.e. $p_R(x) \equiv p(x)$ となっている状態) を分析する．そのために，均衡概念 (personal equilibrium) を定義．

3 Contribution

1. Bayesian network factorization formula (bayesian network の記述する条件付独立性に基づいて，確率分布を条件付確率分布の積に分解する公式) を，意思決定の均衡モデルに統合させた初の試み．
    - 既存のモデルに容易に限定合理性を導入できる．
    - 限定合理的な因果関係を R を用いて記述した上で，$p(x_{-1} \mid x_1)$ を $p_R(x_{-1} \mid x_1)$ で置換．
2. Causal/Statistical reasoning の誤りを記述する簡便な枠組みを与える．
    - reverse causation (因果関係の勘違い): DAG で矢印を逆に張ることに対応．
    - removal of a link (変数間の関係の見落とし): DAG で枝を消去することに対応．
    - この２つは，典型的な人々の勘違いを描写しているのでは？と考え，Illustrations と General Analysis の項で少し詳し目に分析．
3. General characterizations of choice behavior
    - rational なときと irrational なときで行動が変わるための条件は？
    - ある causality model R が，常に他の causality model R' より優れているといったことはあるのか？(答: ない．)
4. Bayesian networks as a unifying framework
    - nonrational expectation を分析した既存の議論をある程度整理して議論できそう？

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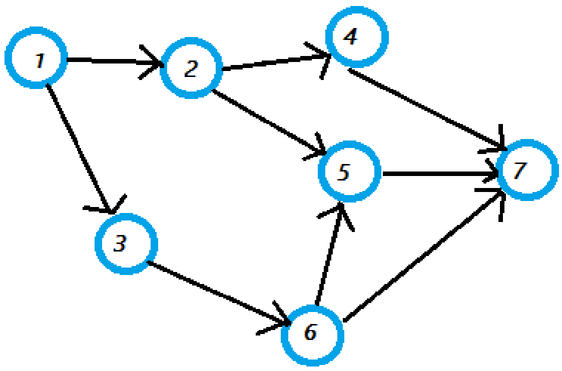
4 Model

- $X := X_1 \times \cdots \times X_N$: a finite set of states. しばしば $X_1 = A$ と表す.
- $p \in \Delta(X_1, \ldots, X_n)$: objective probability distribution
- X_i を値域に持つ確率変数 $\widetilde{X}_i$ $(i \in [n])$
- causality model: DAG (N, R), (しばしば，N を省略して R で DAG を表す．)
  - the set of nodes $N := \{1, \ldots, n\}$ ($i \in N$ は，確率変数 $\widetilde{X}_i$ に対応)
  - the set of edges $R := N \times N$.
  - $(i, j) \in N$ は，node i から node j の間に有向辺が存在することを表す．$i \to j$ と表すことも.
  - イメージ:「i が j の直接の原因」
  - $R(i) := \{j \in N \mid (j, i) \in R\}$: node i の親の集合.
  - $M \subseteq N$ のとき，$x_M := (x_i)_{i \in M}$.

e.g. 4.1 (DAG の例)**.** $N := \{1, 2, \ldots, 7\}$,
$R := \{(1,2), (1,3), (2,4), (2,5), (3,6), (4,7), (5,7), (6,5), (6,7)\}$
$R(5) = \{2, 6\}$, $x_{R(5)} = (x_2, x_6)$, $Descendants(6) = \{5, 7\}$, $NonDescendants(6) = \{1, 2, 3, 4\}$



- DM は，p を元に，R を通して信念 (主観的確率)p_R を形成した上で，最適化問題を解く.
- 一般には，$p = p_R$ とは限らず.

$$p_R(x) := \prod_{i=1}^{n} p(x_i \mid x_{R(i)})$$

$$\max_{p(x_1)} \sum_{x_{-1}} p_R(x_{-1} \mid x_1) u(x)$$

e.g. 4.2 (p_R の構成例)**.** *Fix $N := \{1, 2, 3\}$ and $p \in \Delta(X_1, X_2, X_3)$. Suppose that DM has his subjective DAG $R : 1 \to 2 \leftarrow 3$. Then, he constructs his subjective belief p_R as follows:*

$$p_R(x_1, x_2, x_3) := p(x_1) p(x_3) p(x_2 \mid x_1, x_3)$$

Lem. 4.1 (p_R is a probability distribution)**.** *For any $p \in \Delta(X)$, the function $p_R : X \to [0, 1]$ is also a probability distribution, i.e., $p_R \in \Delta(X)$.*

Proof. Assume w.l.o.g. that $(1,\dots,n)$ are topologically sorted. [1] Then,

$$
\begin{aligned}
\sum_x p_R(x) &= \sum_{x_1}\cdots\sum_{x_n}\prod_{i=1}^{n} p(x_i \mid x_{R(i)}) \\
&= \sum_{x_1}\cdots\sum_{x_{n-1}}\prod_{i\le n-1} p(x_i \mid x_{R(i)}) \underbrace{\sum_{x_n} p(x_n \mid x_{R(n)})}_{=1} \\
&= \sum_{x_1}\cdots\sum_{x_{n-1}}\prod_{i\le n-1} p(x_i \mid x_{R(i)}) \\
&= \cdots \\
&= 1
\end{aligned}
$$

□

Def. 4.1 (consistent)**.** *p is consistent with R(, or p factorizes over R)*
$\overset{\Delta}{\iff} p(x) = \prod_{i=1}^{n} p(x_i \mid x_{R(i)}) \iff p = p_R$

- objective probability distribution p is consistent with the true DAG R^*.

Ass. 4.1. *node 1 is ancestral in both R and R^*, i.e., $R(1) = R^*(1) = \emptyset$.*

Def. 4.2 (Conditional Independence)**.** *$V := \{V_1,\dots,V_n\}$: a set of random variables, $X, Y, Z \subseteq V$.*

$$X \perp Y \mid Z \overset{\Delta}{\iff} [p(Y=y, Z=z) > 0 \implies p(X=x \mid Y=y, Z=z) = p(X=x \mid Z=z)]$$

Lem. 4.2 (local independencies)**.** *p factorizes over R iff the following holds:*

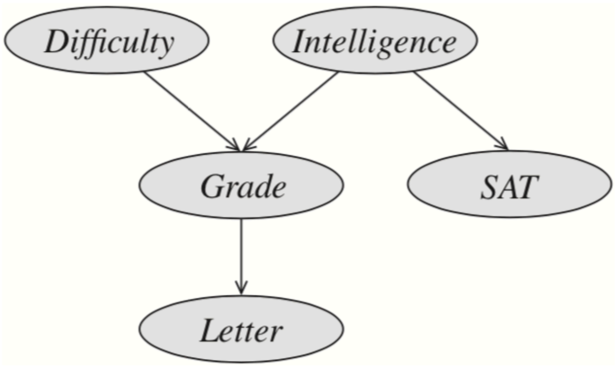
$$\widetilde{X}_{NonDescendants(i)} \perp \widetilde{X}_i \mid \widetilde{X}_{R(i)}$$

Cor. 4.1. *Let R be a DAG. Suppose that $R' \supseteq R$ and R' is also a DAG. If p is consistent with R, then p is also consistent with R'.*

Proof. Suppose that $R' \supseteq R$, and p is consistent with R. Assume w.l.o.g that (N, R) is topologically sorted. Since p is consistent with R, $p(x) = \prod_i p(x_i \mid x_{R(i)})$. Consider the term $p(x_i \mid x_{R(i)})$ for each i. Since R' is a DAG, $x_{R'(i)} = x_{R(i)}$, or $x_{R'(i)} = x_{R(i)} \sqcup x_{N'}$, where $N' \subseteq \text{NonDescendants}(i)$; otherwise, R' has a cycle. Then, by Lem.4.2, $p(x_i \mid x_{R'(i)}) = p(x_i \mid x_{R(i)})$. □

e.g. 4.3 (local independencies)**.** 下図のような *bayesian network structure R (i.e. DAG)* を考える．確率変数の従う分布を p とする．*Difficulty:* 受けた授業の難易度, *Intelligence:* 生徒の賢さ, *Grade:* 生徒の成績, *SAT: SAT* の成績, *Letter:* 推薦状の強さ．

$R(Letter) = \{Grade\}$, $NonDescendants(Letter) = \{Difficulty, Intelligence, SAT\}$. いま，*p is consistent with R* とする.[2] このとき，例えば，p は，$Difficulty \perp Letter \mid Grade$ という関係を満たすような分布になっている．つまり，推薦状の強さは，成績を所与としたとき，授業の難易度とは独立に決まる．これは,「成績が推薦状の強さの直接の原因である」ことを表している．



[1]DAG において，node の順番をうまく並び替えて，$i \to j \implies i < j$ とできることが知られている．(i.e. ある関数 $f: N \to N$ が存在し，$(i,j) \in R \implies f(i) < f(j)$ となる．) このとき明らかに，任意の i について，$j > i$ ならば，$j \notin R(i)$.

[2]p factorizes over (N, R) のとき，DAG と分布の組 $((N,R), p)$ を *bayesian network* と呼ぶ．

- historical database interpretation
  - 新しい DM が，自分より前の DMs 達が生成した膨大なデータを元に意思決定することを考える．
  - 膨大なデータは true distribution p に対応．
  - DM は，自分の causal model に基づいて，各 i について，$p_R(x_i \mid x_{R(i)})$ を学ぶ．
  - その上で，真の分布を $p_R(x)$ だと思って戦略 $(p(a))_a$ をとる．その結果が，$p_R \equiv p$ となっている．(定常状態)
  - 定常状態においては，$p = p_R$ が成立しており，おかしな causality model R に整合的な data(objective distrib.) が社会全体として実現してしまっている．
- 定常状態を考えるため，均衡概念を定義する必要．
  - $p_R(y \mid a)$ が $(p(a))_a$ にも依存するため，$p_R(y \mid a)$ を given として好き勝手に $(p(a))_a$ を動かすことはできない．
  - "trembling" を用いた定義: 均衡である以上，最適でない行動と比較した結果の行動であってほしいが，他の行動と比較するためには，全ての行動 a について，条件付期待値 $p_R(y \mid a)$ が定義されている必要があり，そのためには $p(a) > 0$ が必要．

Def. 4.3 (ε-perturbed personal equilibrium). *Fix R and $\varepsilon > 0$. A distribution $p \in \Delta(X)$ with full support on A is an ε-perturbed personal equilibrium*

$\overset{\Delta}{\iff}$

$$\forall a \in A;\ p(a) > \varepsilon \implies a \in \operatorname*{argmax}_{a'} \sum_y p_R(y \mid a')u(a', y)$$

Def. 4.4 (personal eqm.). *$p^* \in \Delta(X)$ is a personal eqm.*

$\overset{\Delta}{\iff}$

$$\exists(\varepsilon_k)_k\ \exists(p_k)_k;\ \varepsilon_k \to 0,\ p_k : \varepsilon_k\text{-perturbed personal equilibrium}, p_k \to p^*$$

Prop. 4.1 (Proposition 2). *For any DAG R, there exists a personal equilibrium.*

Proof. We show the following statement:

$$\forall (p(y \mid a))_{y,a} \exists (p(a))_a;\ p \text{ is PE, where } p(a, y) := p(y \mid a)p(a)$$

Fix $(p(y \mid a))_{y,a}$. Define $Q^\varepsilon \subseteq \Delta(A)$ as follows:

$$Q^\varepsilon := \{\pi \in \Delta(A) \subseteq \mathrm{R}^{|A|} \mid \forall a \in A; \pi(a) \geq \varepsilon\}$$

For each $\pi \in Q^\varepsilon$, define p^π, $p_R^\pi(a, y)$ as

$$p^\pi(a, y) := \pi(a)p(y \mid a),\ p_R^\pi(a, y) := \prod_{i=1}^n p^\pi(x_i \mid x_{R(i)})$$

Next, define a correspondence $\mathrm{BR} : Q^\varepsilon \rightrightarrows Q^\varepsilon$ as follows:

$$\mathrm{BR}(\pi) := \operatorname*{argmax}_{\rho \in Q^\varepsilon} \underbrace{\sum_a \rho(a) \sum_y p_R^\pi(y \mid a)u(a, y)}_{=:h(\rho,\pi)}.$$

Lem. 4.3 (Kakutani's theorem). *Suppose the following conditions:*

- *$F : X \rightrightarrows X$ is convex-valued, nonempty-valued and has a closed graph.*
- *X is convex, compact, nonempty.*

Then, there exists $x \in X$ such that $x \in F(x)$.

Lem. 4.4 (Berge's theorem). • *$f : X \times \Theta \to \mathbb{R}$: continuous.*

- *$\Gamma : \Theta \rightrightarrows X$: compact-valued, continuous.*
- $v(\theta) := \max_{x \in \Gamma(\theta)} f(x, \theta)$
- $x^*(\theta) := \operatorname{argmax}_{x \in \Gamma(\theta)} f(x, \theta)$
  Then, v is continuous, and x^* is u.s.c.

Lem. 4.5 (Sufficient condition for the closed graph). *$F : X \rightrightarrows X$ has a closed graph if F is closed-valued and F is u.s.c.*

Step 1: BR has a fixed point. For sufficiently small $\varepsilon > 0$, Q^ε is convex, compact, and nonempty. $h(\rho) \equiv h(\rho, \pi)$ is linear in ρ; hence, ρ is continuous and quasi-concave in ρ.

- Since h is continuous in ρ and Q^ε is compact, $\mathrm{BR}(\pi) \neq \emptyset$ for all $\pi \in Q^\varepsilon$.
- Since h is continuous, $\mathrm{BR}(\pi)$ is closed.
- Since h is quasi-concave, $\mathrm{BR}(\pi)$ is convex.

Then, we need to show that $\mathrm{BR}(\pi)$ has a closed graph. Since $\mathrm{BR}(\pi)$ is closed-valued, it is sufficient to show that $\mathrm{BR}(\pi)$ is u.s.c. Let $X \times \Theta := Q^\varepsilon \times Q^\varepsilon$ is the statement of Berge's theorem. Since $\Gamma(\theta) \equiv Q^\varepsilon$ (constant), Γ is continuous and compact. We can show that $h(\rho, \pi)$ is continuous not only in ρ but also in pi. ($\because$ $p^\pi(a, y)$ is continuous in π, and then $p^\pi_R(a, y)$ and $p^\pi(y \mid a)$ are also continuous in π.) As h is a function defined on a finite dimensional Euclidean space, h is continuous in (ρ, π). By Berge's theorem, $\mathrm{BR}(\pi)$ is u.s.c. in π; therefore, BR has a fixed point, i.e.,

$$\exists \pi \in Q^\varepsilon;\ \pi \in \mathrm{BR}(\pi).$$

Step 2: p^π is ε-PE. Note that

$$\pi \in \operatorname*{argmax}_{\rho \in Q^\varepsilon} \sum_a \rho(a) \sum_y p^\pi_R(y \mid a) u(a, y).$$

Consider the slightly modified version of the definition of ε-PE:

Def. 4.5 (ε-PE ($\star$)). *$p \in \Delta(X)$ s.t. $\forall a \in A; p(a) \geq \varepsilon$ is ε-PE ($\star$)*
$\overset{\Delta}{\iff}$

$$\forall a \in A;\ p(a) \geq \varepsilon \implies a \in \operatorname*{argmax}_{a'} \sum_y p_R(y \mid a') u(a', y) \tag{1}$$

Lem. 4.6 (The set of PEs remains the same). *Consider two sets of PEs: one is the set of PEs under the original definition of ε-PE, $\mathcal{E}$; the other is the set of PEs under the original definition of ε-PE ($\star$), $\mathcal{E}'$. Then, $\mathcal{E} = \mathcal{E}'$.*

$\mathcal{E}' \subseteq \mathcal{E}$ clearly holds. Fix $p^* \in \mathcal{E}$ and a corresponding sequence $(\varepsilon_k, p_k)_k$. Let $\varepsilon'_k := \min\{\varepsilon_k, p_k(a)\}$. Then, $p'_k \to p^*$ and p'_k is ε'_k-PE. This completes the proof of Lem.4.6.

Here, we show that p^π is a ε-PE ($\star$). Note that π satisfies the condition that $\pi(a) \geq \varepsilon$ for all $a \in A$. Suppose toward contradiction that

$$\exists a \in A; \pi(a) > \varepsilon,\ a \notin \operatorname*{argmax}_{a'} \underbrace{\sum_y p^\pi_R(y \mid a') u(a', y)}_{=:U(a')}$$

Pick some $a^* \in \operatorname{argmax}_{a'} U(a')$. (Since A is finite, we can pick such a^*.) Define $\widetilde{\pi} \in Q^\varepsilon$ as follows:

$$\widetilde{\pi}(a') = \begin{cases} \pi(a') + \frac{\pi(a)-\varepsilon}{2} & (a' = a^*) \\ \pi(a') - \frac{\pi(a)-\varepsilon}{2} & (a' = a) \\ \pi(a') & \text{o.w.} \end{cases}$$

Note that $\widetilde{\pi} \in Q^\varepsilon$ certainly holds. It suffices to check $\widetilde{\pi}(a) \geq \varepsilon$:

$$\widetilde{\pi}(a) = \frac{2\pi(a) - \pi(a) + \varepsilon}{2} = \frac{\pi(a) + \varepsilon}{2} \geq \varepsilon \quad (\because \pi \in Q^\varepsilon)$$

Observe that $\sum_a \widetilde{\pi}(a)U(a) > \sum_a \pi(a)U(a)$. This contradicts $\pi \in \mathrm{BR}(\pi)$. Therefore, p^π is a ε-PE ($\star$).

Step 3: At least one PE p^* exists. So far, we have shown that ε-PE exists (as long as ε is small enough.) Fix some sequence $(\varepsilon^k)_k \subseteq \mathbb{R}$ such that $\varepsilon^k \to 0$. Let p^k be a ε-PE for each k. Note that $(p^k)_k \subseteq \Delta(X) \subseteq \mathbb{R}^{|X|}$. Since $(p^k)_k$ is a sequence in a compact subset of a finite dimensional Euclidean space, $(p^k)_k$ has a convergent subsequence $(p^{k_m})_m$ such that $(p^{k_m})_m \to p^* \in \Delta(X)$. This p^* is PE. □

5 Illustrations

- Reverse causation: Dieter's dilemma
- Coarseness I: Demand for Education
- Coarseness II: Public Policy

5.1 Reverse causation: Dieter's dilemma

- Three variables: a, h, c:
  - DM's choice(diet or not), health outcome(good or bad), chemical level(high or low)
- DM は意思決定する時点では c, h の実現値については知らない.

5.1.1 Rational DM の場合

- True DAG: $R^* : a \to c \leftarrow h$
  - このとき，p は $p(a,h,c) = p(a)p(h)p(c \mid a,h)$ を満たす.
  - もし DM が rational(i.e. causality を正しく認識している) なら，彼が解く問題は,

$$\max_a \sum_h \sum_c p(h)p(c \mid a,h)u(a,h,c)$$

5.1.2 Irrational DM の場合

- DM の causality model が $R : a \to c \to h$ の場合を考える.
- p が personal eqm. なら，$p(a') > 0$ となる a' は以下の式を満たす.

$$a' \in \operatorname*{argmax}_a \sum_h \sum_c p(h \mid c)p(c \mid a)u(a,h,c)$$

5.1.3 Solving for the personal eqm.

- R の下での personal eqm. を求めてみる.
- もう少し構造を入れて考える.
  - $a, c, h \in \{0, 1\}$
  - $u(a, h, c) = u(a, h) := h - \kappa a$
  - $p(h = 1) = p(h = 0) = 1/2, h \perp a, c = (1 - h)(1 - a)$
- DM が rational な場合は，$p_{R^*}(h \mid a) = p(h)$ なので，常に $a^* := 0$ を選択することに注意.

Prop. 5.1 (personal eqm. in Dieter's dillemma)**.** *In this case, there is a unique personal eqm p:*

$$p(a = 0) = \begin{cases} 0 & (\kappa \leq 1/4) \\ 2 - \frac{1}{2\kappa} & (\kappa \in (1/4, 1/2)) \\ 1 & (\kappa \geq 1/2) \end{cases}$$

Proof. personal eqm. p を任意にとり，$\beta := p(a = 0) \in [0, 1]$ とする．まず，p についての specification より，

$$p(c = 0 \mid a = 1) = 1, p(c = 0 \mid a = 0) = \frac{1}{2}, p(h = 1 \mid c = 1) = 0, p(h = 1 \mid c = 0) = \frac{1}{2 - \beta}$$

がわかる.

$$\begin{aligned} p_R(h = 1 \mid a = 0) &= p(h = 1 \mid c = 0)p(c = 0 \mid a = 0) + p(h = 1 \mid c = 1)p(c = 1 \mid a = 0) \\ &= \frac{1}{2 - \beta}\frac{1}{2} \end{aligned}$$

$$\begin{aligned} p_R(h = 1 \mid a = 1) &= p(h = 1 \mid c = 0)p(c = 0 \mid a = 1) + p(h = 1 \mid c = 1)p(c = 1 \mid a = 1) \\ &= \frac{1}{2 - \beta} \end{aligned}$$

であり，また，$\sum_h p(h \mid a)u(a, h)$ の値は，a の取りうる各値についてそれぞれ以下のようになる.

$$\begin{aligned} \sum_h p_R(h \mid a' = 0)u(a' = 0, h) &= p_R(h = 1 \mid a' = 0) \cdot 1 \\ &= \frac{1}{2}\frac{1}{2 - \beta} \quad \text{(E0)} \\ \sum_h p_R(h \mid a' = 1)u(a' = 1, h) &= \frac{1}{2 - \beta}(1 - \kappa) + \left(1 - \frac{1}{2 - \beta}\right) \\ &= \frac{1}{2 - \beta} - \kappa \quad \text{(E1)} \end{aligned}$$

Case (i): $\beta \in (0, 1)$ のとき $\beta > \varepsilon, 1 - \beta > \varepsilon$ を満たすような十分小さい $\varepsilon > 0$ を一つとり固定する. personal eqm. の定義より，このとき，(E0) = (E1) が必要.

$$\therefore \quad \beta = 2 - \frac{1}{2\kappa}$$

これが personal eqm. になることは，$\varepsilon_k \to 0$ となるような点列を任意にとり，$p_k := (\beta, 1 - \beta)$ とすれば，十分大きい k について p_k は ε_k-perturbed personal eqm であり，$p_k \to p$ となることより ok.

Case (ii): $\beta = 0$ のとき $1-\beta > \varepsilon$ となるような ε を任意にとり固定する．このとき，(E0) $\leq$ (E1) が必要．(E0) $\leq$ (E1) $\Longleftrightarrow \kappa \leq 1/4$. これが personal eqm. になることは，$\kappa \leq 1/4$ のとき，$\varepsilon_k \to 0$ となるような点列を任意にとり，$p_k := (0,1)$ とすれば，十分大きい k について p_k は ε_k-perturbed personal eqm であり，$p_k \to p$ となることより ok.

Case (iii): $\beta = 1$ のとき Case (ii) のときと同様に示せる． $\square$

Interpretation:

- diet のコストが高すぎない限り，定常状態において，irrational DM は正の確率で diet をしてしまう．なぜか？
- 仮にいま DM が $a = 0$ を選んでいたとする．このとき，DM は c, h の間に negative correlation があることに気づく．
- 彼は今 $a \to c \to h$ だと思っているので，$a \uparrow \to c \downarrow \to h \uparrow$ とできると勘違いしてしまう．
- その結果，$p(a = 1) > 0$ となってしまう．
- $a = 1$ の頻度が下がると c, h 間の負の相関を強く認識．($p(h = 1 \mid c = 0) = \frac{1}{2-\beta}$)

5.2 Coarseness I: Demand for Education

- a, θ, s, w: parent's investment, child's innate ability, school performance, wage
- true DAG R^* :

$$\begin{tikzcd} a \arrow[r] & s \arrow[dr] & \theta \arrow[l] \arrow[d] \\ & & w \end{tikzcd}$$

$$\max_a \sum_\theta p(\theta) \sum_s p(s \mid a, \theta) \sum_w p(w \mid \theta, s) u(a, w)$$

- DM's subjective DAG R :

$$\begin{tikzcd} a \arrow[r] & s \arrow[dr] & \theta \\ & & w \end{tikzcd}$$

$$\max_a \sum_s p(s \mid a) \sum_w p(w \mid s) u(a, w)$$

- 「目に見えない変数 θ の影響を無視してしまう」ような間違い．
- $a \in [0,1], s, \theta, w \in \{1, 0\}$
- $u(a, w) := w - \kappa(a)$
- κ: twice-differentiable, increasing, weakly convex. (i.e. $\kappa' > 0, \kappa'' \leq 0$), $\kappa'(0) = 0, \kappa'(1) \geq 1$.
- $p(s = 1 \mid a, \theta) = a\theta$, $p(w = 1 \mid s, \theta) = \theta\beta_s$ ($\beta_1 > \beta_0$), $p(\theta = 1) = \delta > 0$.

5.2.1 rational DM's choice

$$\max_a \{\delta[a\beta_1 + (1-a)\beta_0] - \kappa(a)\}$$

- $\kappa'(a^*) = \delta(\beta_1 - \beta_0)$ を満たす a^* が optimal.

5.2.2 irrational DM's choice

Prop. 5.2. *In this case, the parent assigns probability one to some action a^{**} such that*

$$\kappa'(a^{**}) = \delta\left[\delta\beta_1 - \beta_0 \cdot \frac{\delta(1-a^{**})}{\delta(1-a^{**})+1-\delta}\right]$$

If κ' is either weakly convex or weakly concave, then a^{**} is unique.
Note that since $\kappa'(a^{**}) < \kappa'(a^*)$, we have $a^{**} > a^*$: the parent overinvests in personal eqm.

Interpretation:

- The parent overinvests because he overly estimates the positive correlation b/w a and w:
  - DM は s と w の間には pure causal effect しかないと考えているが，実際は θ が影響．
  - 投資が効くときは，θ が高いときであり，そのとき，w は高くなりやすくなっている．(しかしそのことに気づいていない．)

  $\to$ 投資の効果を過大評価．
- the perceived marginal benefit of investment $\kappa'(a^{**})$ が，eqm. investment a^{**} の関数に．
  - DM は常に $w \perp_R a \mid s$ だと思っているが，実際はそうではない．
  - perceived causal effect of s on w は，a の分布に依存する．
  - i.e. true DAG に consistent な p について，一般には $p(w \mid s, a) \neq p(w \mid s)$
  - 例えば，$s = 0$ を所与としたとき，$a = 1$ であったとすると，そこから $\theta = 0$ の確率が高いことが推測される．
  - $\mathbb{E}[w \mid s = 1] - \mathbb{E}[w \mid s = 0]$ increases in long-run investment. (a が大きいことを given にすると，$s = 0$ のとき，$\theta = 0$ の確率が高まるので，$\mathbb{E}[w \mid s = 0]$ は a が低いときと比べて小さくなる．)
  - 以上の議論は true distribution の下で考えている．personal eqm. では，true DAG, subjective DAG 両方の性質が満たされることに注意．

Proof of Prop.5.2.

$$\sum_s p(s \mid a) \sum_w p(w \mid s) u(a, w) = \sum_s p(s \mid a) p(w = 1 \mid s) - \kappa(a)$$
$$p(s = 1 \mid a) = \sum_\theta p(\theta) p(s = 1 \mid a, \theta) = \delta a$$
$$p(s = 1 \mid a) = \sum_\theta p(\theta) p(s = 1 \mid a, \theta) = \delta a$$
$$p(w = 1 \mid s = 1) = \delta\beta_1$$
$$p(w = 1 \mid s = 0) = \frac{p(w = 1, s = 0)}{p(s = 0)}$$

$$
\begin{aligned}
p(w=1,s=0) &= \sum_{\theta}\sum_{a} p(w=1,s=0,a,\theta) \\
&= \sum_{\theta}\int_a p(\theta)p(w=1\mid s=0,\theta)p(s=0\mid\theta,a)d\mu(a) \\
&= (1-\delta)\int_a \underbrace{p(w=1\mid s=0,\theta=0)}_{0}p(s=0\mid\theta=0,a)d\mu(a) \\
&+\delta\int_a \underbrace{p(w=1\mid s=0,\theta=1)}_{(\beta_0)}\underbrace{p(s=0\mid\theta=1,a)}_{(1-a)}d\mu(a) \\
&= \delta\beta_0\int_a(1-a)d\mu(a)
\end{aligned}
$$

$$
\begin{aligned}
p(s=0) &= \sum_{\theta}\sum_{a} p(a,s=0,\theta) \\
&= \sum_{\theta}\sum_{a} p(\theta)p(a)p(s=0\mid a,\theta) \\
&= (1-\delta)\int_a \underbrace{p(s=0\mid a,\theta=0)}_{1}d\mu(a)+\delta\int_a\underbrace{p(s=0\mid a,\theta=1)}_{(1-a)}d\mu(a) \\
&= (1-\delta)+\delta\int_a(1-a)d\mu(a)
\end{aligned}
$$

Then,

$$p(w=1\mid s=0)=\underbrace{\frac{\delta\int_a(1-a)d\mu(a)}{(1-\delta)+\delta\int_a(1-a)d\mu(a)}}_{=:\gamma}\beta_0$$

Note that $\gamma<\delta$. Hence,

$$\sum_s p(s\mid a)p(w=1\mid s)-\kappa(a)=\delta a\cdot\delta\beta_1+(1-\delta a)\gamma\beta_0-\kappa(a)$$

FOC is $\kappa'(a)=\delta(\delta\beta_1-\gamma\beta_0)$ $(\in(0,1))$. $\square$

5.3 Coarseness II: Public Policy

- a,y,e,z: policy, two macro variables, private sector's expectation of y.
- true DAG R^* :

$$
\begin{tikzcd}
a \arrow[r] \arrow[dr] & y \arrow[r] & z \\
& e \arrow[ur] &
\end{tikzcd}
$$

- DM's DAG R :

$$
\begin{tikzcd}
a \arrow[r] & y \arrow[r] & z \\
& e &
\end{tikzcd}
$$

6 General Analysis

6.1 Consequentialist Rationality

- personal equilibrium が最適化問題の解として記述できるための条件は？
- (そもそもなんでこんなことを議論したいの？)

6.1.1 Preliminaries

Def. 6.1 (skeleton). *Fix a DAG $\mathcal{G} := (N, R)$. The skeleton of $\mathcal{G}$, $\tilde{\mathcal{G}} := (N, \tilde{R})$, is an indirected version of $\mathcal{G}$: formally, $\tilde{R} := \{(i,j) \in N \times N \mid (i,j) \in R, \text{ or } (j,i) \in R\}$. $(i,j) \in \tilde{R}$ is sometimes denoted by $i\tilde{R}j$, or $i - j$.*

e.g. 6.1 (skeleton). $R : i \to j \to k$, $\tilde{R} : i - j - k$.

Def. 6.2 (clique, ancestral clique). *Fix a DAG (N, R). $M \subseteq N$ is a clique in R*

$\overset{\Delta}{\iff}$

$$\forall i, j \in M;\ i \neq j \implies i\tilde{R}j.$$

A clique M in R is an ancestral clique when $\forall i \in M;\ R(i) \subseteq M$.

e.g. 6.2 (clique).

- $M_1 := \{5, 6, 7\}$: *clique, but not ancestral clique.*
- $M_2 := \{2, 4, 5, 7\}$: *not clique.*
- $M_3 := \{1, 3\}$: *ancestral clique.*

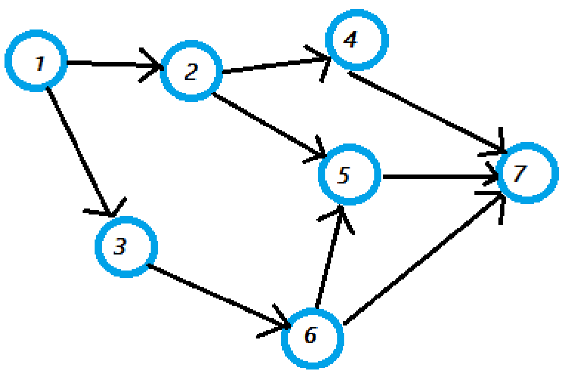


Figure 1: DAG

Def. 6.3 (equivalent). *Fix N. Two DAGs R and Q are equivalent, denoted as $R \sim Q$,*

$\overset{\Delta}{\iff}$

$$\forall p \in \Delta(X); p_R(x) = p_Q(x)$$

We sometimes denote the equivalence class of R as $[R]$.

e.g. 6.3 (equivalent). *$R : 1 \to 2$ and $Q : 2 \to 1$ are equivalent: For any $p \in \Delta(X)$,*

$$p(x_1, x_2) = p(x_2 \mid x_1)p(x_1) = p(x_1 \mid x_2)p(x_2).$$

Def. 6.4 (v-structure). *The v-structure of a DAG R, $v(R)$, is defined as follows:*

$$v(R) := \{(i, j, k) \mid i \to j, j \to k, i \nrightarrow j, j \nrightarrow i\}$$

e.g. 6.4 (v-structure). *Consider the DAG R in Figure 1. $(2, 5, 6)$ is a v-structure of R; $(5, 7, 6)$ is not a v-structure in R.*

Prop. 6.1 (Verma and Pearl, 1991). *$R \sim Q \iff [\tilde{R} = \tilde{Q}$ and $v(R) = v(Q)]$.*

e.g. 6.5. *$R : 1 \to 2 \to 3$ and $Q : 3 \to 2 \to 1$ are equivalent: $\tilde{R} = \tilde{Q} = 1 - 2 - 3$ and $v(R) = v(Q) = \emptyset$. However, $S : 1 \to 2 \leftarrow 3 \nsim R$ because $v(S) = \{(1, 2, 3)\} \neq \emptyset$.*

6.1.2 Consequentialist Rationality

- $\Delta_R(X) := \{p \in \Delta(X) \mid p \text{ is consistent with } R\}$ とする．

Def. 6.5 (Consequentialistically rational). *A DAG R is C-rational w.r.t. true DAG R^**
$\overset{\Delta}{\iff}$

$$\forall p, q \in \Delta_{R^*}(X); [\forall x; p(x_{-1} \mid x_1) = q(x_{-1} \mid x_1) \implies \forall x; p_R(x_{-1} \mid x_1) = q_R(x_{-1} \mid x_1)]$$

- R: C-rational であれば，true distrib. p の $p(x_1)$ を $p(x_{-1} \mid x_1)$ を変えないようにいじっても，$p_R(x_{-1} \mid x_1)$ は変化しない．
- つまり，$p(x_{-1} \mid x_1)$:given として $p(x_1)$ を最適化問題の解として選んでも $p(x_{-1} \mid x_1)$ には無影響．

e.g. 6.6 (C-rationality in dieter's dilemma). 例えば $p_R(h = 1 \mid a = 0) = \frac{1}{2-\beta}\frac{1}{2}$ といった結果からわかるように，*dieter's dilemma* においては，R は *C-rational* ではない: いまある p を所与とし，$p_R(h \mid a)$ の下で最適化問題を解いて $p^*(a)$ を求めると，$p'(a,h,c) := p(h,c \mid a)p^*(a) \neq p(a,h,c)$ であり，$p'_R(h \mid a) \neq p_R(h \mid a)$ となる．

- R^* itself is C-rational w.r.t. R^*.

∵) Fix $p, q \in \Delta_{R^*}(X)$ s.t. $p(x_{-1} \mid x_1) = q(x_{-1} \mid x_1)$ for all x. Fix x.
$p_{R^*}(x) = p(x_1)p(x_{-1} \mid x_1)$. $p_{R^*}(x_1) = p(x_1)\sum_{x_{-1}} p(x_{-1} \mid x_1) = p(x_1)$.
Then, $p_{R^*}(x_{-1} \mid x_1) = p(x_{-1} \mid x_1)$. Similarly, $q_{R^*}(x_{-1} \mid x_1) = q(x_{-1} \mid x_1)$. □

- From now on, assume that $R \neq R^*$.

Prop. 6.2 (characterization of C-rationality (Proposition 6)). *R is C-rational w.r.t. R^**
$\iff$

$$\forall i > 1;\ 1 \notin R(i) \implies x_i \perp_{R^*} x_1 \mid x_{R(i)}$$

e.g. 6.7 (Dieter's dilemma).
- *True DAG: $R^* : 1 \to 3 \leftarrow 2$*
- *Subjective DAG: $R : 1 \to 2 \to 3$* について考える．
- $i := 3$ として考える．このとき，$1 \notin R(3), x_3 \not\perp_{R^*} x_1 \mid x_2$.
- よって，*R is not C-rational w.r.t. R^*.*
- 次に，$R' : 1 \to 3 \quad 2$ について考えてみる．*(fully coarsed/cursed)*
- *R' is C-rational w.r.t. R^*: $x_2 \perp_{R^*} x_1$.*

- DAG のサイズが大きいと，独立性の条件を調べるのは大変
- d-separation という概念を用いた効率的な判定アルゴリズムが存在．

Proof of Prop.6.2. [細部よくわからず．]

$$
\begin{aligned}
p_R(x_{-1} \mid x_1) &= \frac{p_R(x_1, x_{-1})}{p_R(x_1)} = \frac{p(x_1)\prod_{i\geq 2} p(x_i \mid x_{R(i)})}{\sum_{x'_{-1}} p(x_1)\prod_{i\geq 2} p(x'_i \mid x_{R(i)\cap\{1\}},\ x'_{R(i)-\{1\}})} \\
&= \frac{\prod_{i\geq 2} p(x_i \mid x_{R(i)})}{\sum_{x'_{-1}} \prod_{i\geq 2} p\left(x'_i \mid x_{R(i)\cap\{1\}},\ x'_{R(i)-\{1\}}\right)}
\end{aligned}
\tag{2}
$$

今，示したいのは次のような命題であることに注意する．

$$
\begin{aligned}
&p(x_{-1} \mid x_1) \text{ を保ちながら } p(x_1) \text{ を変えたときに } p_R(x_{-1} \mid x_1) \text{ が変化しない} \\
\Longleftrightarrow &\forall i > 1;\ 1 \notin R(i) \implies x_i \perp_{R^*} x_1 \mid x_{R(i)}
\end{aligned}
$$

$\Leftarrow$)　(2) において，分母の部分は，$p(x_1)$ に依存していない．よって，任意の $i \geq 2$ について，

$$
p\left(x'_i \mid x_{R(i)\cap\{1\}},\ x'_{R(i)-\{1\}}\right) \tag{$\star$}
$$

の部分が変化するかを見ればよい．

　$1 \in R(i)$ であれば，

$$
(\star) = p\left(x'_i \mid x_1, x'_{R(i)}\right)
$$

であるので，$(\star)$ は $p(x_1)$ には依存しない．(???)

　$1 \notin R(i)$ の場合，仮定より，$x_i \perp_{R^*} x_1 \mid x_{R(i)}$ であるので，

$$
\begin{aligned}
(\star) = p\left(x'_i \mid x'_{R(i)}\right) &= \sum_{x''_1} p(x''_1) p(x'_i \mid x''_1, x'_{R(i)}) \\
&= \sum_{x''_1} p(x''_1) p(x'_i \mid x'_{R(i)}) \\
&= p(x'_i \mid x'_{R(i)})
\end{aligned}
$$

であるので，この場合も $(\star)$ は $p(x_1)$ に依存しない．以上より，$p(x_1)$ を変えても $p_R(x_{-1} \mid x_1)$ が変化しないことが示された．

$\Rightarrow$)　$i > 1, 1 \notin R(i)$ をなる i を任意にとる．$1 \notin R(i)$ より，

$$
(\star) = p\left(x'_i \mid x'_{R(i)}\right) = \sum_{x''_1} p(x''_1) p(x'_i \mid x''_1, x'_{R(i)})
$$

いま，仮に $x_i \not\perp_{R^*} x_1 \mid x_{R(i)}$ だとする．このとき，$p(x'_i \mid x''_1, x'_{R(i)})$ は x''_1 に依存して変化する．(???) このとき，$(\star)$ は $p(x''_1)$ に依存して変化する．よって，(?)$p_R(x_{-1} \mid x_1)$ も $p(x''_1)$ に依存して変化．　□

6.2 Behavioral Rationality

- DAG R がどういう性質を満たしているとき，DM は rational な場合に最適である行動を選ぶか？ – all payoff-relevant variables are causally linked and have no other causes.
- よくある因果関係の勘違い (ここでは特に，link を一本逆にすることを考える) が behavioral rationality を violate するのはどういうときか？

6.2.1 Preliminaries

Def. 6.6 (fully connected). *A directed graph (N, R) is fully connected if $i \to j$ or $j \to i$ holds for all $i, j \in N$.*

Lem. 6.1 (fully connected DAG). *A DAG (N, R) is fully connected $\iff$ R is consistent for all $p \in \Delta(X)$.*

Proof. Assume w.l.o.g that $\{1, 2, \ldots, n\}$ are topologically sorted.

$\Rightarrow$) Fix any x. Then,

$$p(x) = \prod_i p(x_i \mid x_1, \ldots, x_{i-1}) = p_R(x)$$

$\Leftarrow$) We show contraposition. Suppose that R is not fully connected. Then, since R does not have enough its degree of freedom, we can construct p that is not consistent with R. For example, consider $R : 1 \to 2 \to 3$. R is not fully connected because $1 \nrightarrow 3$. Then, we can construct p such that

$$p(x) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2, x_1) \neq p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2) = p_R(x)$$

$\square$

Def. 6.7 (d-separation). *Let R be a DAG, and $X, Y, X \subseteq N$.*
A directed path P is d-separated by Z
$\overset{\Delta}{\iff}$

- *P contains a chain $i \to m \to j$ or a fork $i \leftarrow m \to j$ such that $m \in Z$.*
- *P contains an inverted fork $i \to m \leftarrow j$ such that m and the descendants of m are not in Z.*

Z d-separates X and Y $\overset{\Delta}{\iff}$ Z d-separates every path from a node in X to a node in Y. This is denoted by $(X \perp Y \mid Z)_R$. [3]

Prop. 6.3 (Probabilistic Implications of d-Separation). *For any three disjoint subsets of nodes X, Y, Z in a DAG R, and for all probability distributions p,*

1. *If p is consistent with R, then $(X \perp Y \mid Z)_R \implies (X \perp Y \mid Z)_p$*
2. *$(X \not\perp Y \mid Z)_R \implies \exists p;\ (X \not\perp Y \mid Z)_p$.*

6.2.2 Behavioral Rationality

- no restriction on $p \in \Delta(X)$, i.e., assume that true DAG R^* is fully connected.
- Impose some restriction on the set of possible utility functions.

Ass. 6.1 (Restriction on u). *$\exists M \subsetneq N$; $1 \in M$, and u is purely a function of x_M.*

[3] For a probability distribution p, $(X \perp Y \mid Z)_p$ denotes that X and Y are independent conditional on Z.

Def. 6.8 (Behaviorally Rational). *A DAG R is B-rational if in every personal eqm. p,*

$$p(x_1) \implies x_1 \in \operatorname*{argmax}_{x'_1} \sum_{x_{-1}} p(x_{-1} \mid x_1) u(x'_1, x_{-1})$$

Prop. 6.4 (Spiegler(2017), Proposition 2). *Let R be a DAG and let $C \subseteq N$.*

$$[\forall p \in \Delta(X) \forall x; p_R(x_C) = p(x_C)] \iff [\exists Q \in [R]; C \textit{ is an ancestral clique in } Q].$$

[2018/07/16: $\Leftarrow$ is correct; $\Rightarrow$ is not sure.]

e.g. 6.8. *$R : 1 \to 2 \leftarrow 3$. By Prop.6.1, we can see that $[R] = \{R\}$. Since $\{x_2\}$ is not an ancestral clique in R, by Prop.6.4, $\exists p \exists x_2; p_R(x_2) \neq p(x_2)$.*

Proof of Prop.6.4. See Appendix. よくわからず. □

Prop. 6.5. *The DM is behaviorally rational* $\iff \exists Q \in [R]$; *M is an ancestral clique in Q.*

Proof. [Prop.6.4 を修正しない限り，$\Rightarrow$ は不成立．]
Note that, by assumption, node 1 is an ancestral node in both R and R^*.

$\Leftarrow$) Assume that there exists $Q \in [R]$ such that M is an ancestral clique in Q. By Prop.(6.4), $p_R(x_M) = p(x_M)$. Fix any personal eqm. p. We need to show that p satisfies the following:

$$\forall x_1;\ p(x_1) > 0 \implies x_1 \in \operatorname*{argmax}_{x_1'} \sum_{x_{-1}} p(x_{-1} \mid x_1')u(x).$$

Fix x_1 such that $p(x_1) > 0$. Since u depends only on x_M,

$$\begin{aligned}
\operatorname*{argmax}_{x_1'} \sum_{x_{-1}} p(x_{-1} \mid x_1')u(x) &= \operatorname*{argmax}_{x_1'} \sum_{x_{M-\{1\}}} p(x_{M-\{1\}} \mid x_1')u(x_M) \\
&= \operatorname*{argmax}_{x_1'} \sum_{x_{M-\{1\}}} p_R(x_{M-\{1\}} \mid x_1')u(x_M)\ (\because p_R(x_M) = p(x_M)) \\
&= \operatorname*{argmax}_{x_1'} \sum_{x_{-1}} p_R(x_{-1} \mid x_1')u(x)
\end{aligned}$$

Since p is personal eqm., $x_1 \in \operatorname{argmax}_{x_1'} \sum_{x_{-1}} p_R(x_{-1} \mid x_1')u(x)$. Therefore, R is B-rational.

$\Rightarrow$) Assume that R is B-rational. By Prop.(6.4), we have $p_R(x_1) = p(x_1)$. Then,

$$p_R(x_{M-\{1\}} \mid x_1) = \frac{p_R(x_M)}{p_R(x_1)} = \frac{p_R(x_M)}{p(x_1)}, \quad p(x_{M-\{1\}} \mid x_1) = \frac{p(x_M)}{p(x_1)}.$$

Hence, $p_R(x_{M-\{1\}} \mid x_1) = p(x_{M-\{1\}} \mid x_1)$ holds if and only if $p_R(x_M) = p(x_M)$ holds.
By Prop.(6.4)[要修正], it is sufficient to show that $p(x_M) \equiv p_R(x_M)$; it suffices to show that $p_R(x_{M-\{1\}} \mid x_1) = p(x_{M-\{1\}} \mid x_1)$. Suppose toward contradiction that $p_R(x_{M-\{1\}} \mid x_1) \neq p(x_{M-\{1\}} \mid x_1)$. Then, we can construct the utility function u under which DM does not choose the optimal action w.r.t. p. (??) □

Interpretation:

- (1) all payoff-relevant variables are causally linked, (2) they have no other causes のときに，DM は rational な場合の最適行動を選択できる．
- ((1),(2) のどちらかが満たされなければ，ある p と u の下で suboptimal な行動をしてしまう．[要修正])
- 「簡単な operation が behavioral rationality を損なうか否か」みたいな議論も面白いかも？

Prop. 6.6 (Proposition 9). *Suppose that R departs from R^*, which is fully connected, by omitting one link $i \to j$. Then,*

$$\textit{DM is B-rational.} \iff j = n,\ i \neq 1.$$

e.g. 6.9.

- $R: 1 \to 3 \leftarrow 2$. $1 \to 2$ *omitted from* R^*. *DM is not B-rational. – double-counting.*
- $R: 1 \to 2 \to 3$. $1 \to 3$ *omitted from* R^*. *DM is not B-rational. – failed to perceive any effect of* x_1
- $R: 2 \leftarrow 1 \to 3$. $2 \to 3$ *omitted from* R^*. *DM is B-rational. – not distinguish direct and indirect effect.*

6.3 Payoff ranking of DAGs

- 矢印を一本増やす $\approx$ より賢くなる
- より賢い DAG を持つ人は，常に良い利得を達成できるか？ – No

e.g. 6.10. • *R: fully connected DAG,* $1 \to 2 \to 3 \to 4$*; u is purely a function of* x_1 *and* x_4*.*

- R': $2 \to 3$ *removed from R*
- *By Prop.6.6,* R' *is not B-rational:* R' *is weakly dominated by R in terms of expected performance.*
- R'': $2 \to 4$ *removed from* R'*.*

$$
\begin{tikzcd}
R'' : 1 \arrow[r] \arrow[d] \arrow[dr] & 2 \arrow[dl] , \\
3 \arrow[r] & 4
\end{tikzcd}
\quad
\begin{tikzcd}
Q : 1 \arrow[r] \arrow[d] \arrow[dr] & 2 \\
3 \arrow[ur] & 4 \arrow[l]
\end{tikzcd}
$$

- $Q \sim R''$ *(the same skeleton and v-structure).* $\{1, 4\}$ *is an ancestral clique in Q.*
- R'' *is B-rational w.r.t.* R^**;* R' *is weakly dominated by* R''*.*

Ass. 6.2 (For simplicity?)**.** 1 *is an isolated node in all relevant true and subjective DAGs.*

Def. 6.9 (Ranking of DAGs)**.** *R is more rational than* R'
$\overset{\Delta}{\iff} \forall p, u, a, a'$;

$$
\sum_y p_R(y) u(a, y) > \sum_y p_R(y) u(a', y), \tag{3}
$$

$$
\sum_y p_{R'}(y) u(a', y) > \sum_y p_{R'}(y) u(a, y) \tag{4}
$$

$$
\implies \sum_y p(y) u(a, y) > \sum_y p(y) u(a', y) \tag{5}
$$

- 「2 つの DAG で意見が割れたときは，常に片方が正しい」
- R: fully connected, R': not fully connected のときは正しい．

Prop. 6.7 (Proposition 10)**.** *Suppose both R and* R' *are not fully connected. Then, neither DAG is more rational than the other.*

Proof. Assume that both R and R' are not fully connected. If $R \sim R'$, the claim holds. Assume $R \nsim R'$.

Suppose toward contradiction that R is more rational than R'. Fix any $p \in \Delta(X)$. Let $q := (p_R(y))_y$ and $r := (p_R(y))_y$. Note that q and r are $k := |Y|$-length probability vectors. Fix any u, a, a'. Let $z^y := u(a, y) - u(a', y)$, $z := (z^y)_y$, and $D := [q \quad -r \quad -p]$. Note that D is a $k \times 3$ matrix. Fix any $\varepsilon > 0$. Let $b := (\varepsilon, \varepsilon, \varepsilon)^\top$.

First, we show the following:

$$
\nexists z \in \mathbb{R}^k;\ D^\top z > b \tag{6}
$$

Suppose not. Then there exists $z \in \mathbb{R}^k$ such that

$$
D^\top z = \begin{bmatrix} q^\top z \\ -r^\top z \\ -p^\top z \end{bmatrix} = \begin{bmatrix} \sum_y p_R(y)(u(a, y) - u(a', y)) \\ -\sum_y p_{R'}(y)(u(a, y) - u(a', y)) \\ -\sum_y p(y)(u(a, y) - u(a', y)) \end{bmatrix} > b
$$

This implies

$$\sum_y p_R(y)u(a,y) > \sum_y p_R(y)u(a',y)$$
$$\sum_y p_{R'}(y)u(a',y) > \sum_y p_{R'}(y)u(a,y)$$
$$\sum_y p(y)u(a',y) > \sum_y p(y)u(a,y)$$

This contradicts the assumption that R is more rational than R'. Therefore, (6) must hold.

Next, we apply Gale's theorem:

Lem. 6.2 (Gale's Theorem). *Let $A \in \mathbb{R}^{M\times N}$ and $b \in \mathbb{R}^N$. The following two statements are equivalent:*

1. $\exists x \in \mathbb{R}^M;\ A^\top x \leq b$
2. $\forall y \in \mathbb{R}^N;\ y \geq 0, Ay = 0 \implies b^\top y \geq 0$

By (6) and Gale's theorem, we have

$$\exists w \in \mathbb{R}^3;\ w \geq 0, Dw = 0, b^\top w < 0$$

[Spiegler(2016) では $w > 0$ になっているがなぜ？]

Since $b^\top w < 0$, there exists $j \in \{1,2,3\}$ such that $w_j > 0$. Since $Dw = 0$, for all $i \in [k]$, $w_1 q^i = w_2 r^i + w_3 p^i$, or

$$w_1 p_R(y) = w_2 p_{R'}(y) + w_3 p(y)$$

By summing up w.r.t. i, we have $w_1 = w_2 + w_3$. Hence,

$$w_1 > 0, (w_2 > 0 \text{ or } w_3 > 0)$$

Since $w_1 > 0$, for all y,

$$p_R(y) = \frac{w_2}{w_1} p_{R'}(y) + \frac{w_3}{w_1} p(y)$$

Let $\alpha := w_2/w_1$ and $\beta := w_3/w_1$. Then, by summing up w.r.t. y, we have $\alpha + \beta = 1$. Therefore, we have the following:

$$\forall p\ \exists \alpha \in [0,1]; p_R = \alpha p + (1-\alpha) p_{R'} \tag{7}$$

In case $\alpha < 1$, the proof is done: If p is consistent with R, or $p_R = p$, by (7), we have $p_R = p_{R'}$, and then $p = p_{R'}$; Similarly, if p is consistent with R', then p is also consistent with R: we have the following relationship:

$$p = p_R \iff p = p_{R'}$$

In addition, for any $p \in \Delta(X)$, p_R is consistent with R. Replace p with p_R and apply the procedure to p_R; we have $p_R = \alpha p_R + (1-\alpha)p_{R'}$, and then $p_R = p_{R'}$.

[$\alpha < 1$ for all p, or, $w > 0$ が言えれば ok だが …? , fully-connected の条件を用いていない．]

□

7 Variations and Relations to Other Concepts

7.1 Variations

- 複数の DAG を確率的に持つ (Partial cursedness)
- 社会に，異なる DAG を持つ主体が混在している．(e.g. Dieters′ dilemma)

7.2 Relations to Other Concepts

- Jehiel (2005) Analogy-based expectations
- Esponda (2008) Naive Behavioral Equilibrium
- Eyster and Rabin (2005) Partial cursedness
- Osborne and Rubinstein (1998) S(K) equilibrium

8 Concluding Remarks

8.1 Alternative interpretations of DAG

- Data limitations (cf: Spiegler (2017) Data Monkeys)
- Limited ability to ask the right questions

9 Appendix

Proof of Prop.6.4. [There is an error in the proof in Spiegler(2017).]

If C is empty, the proposition clearly holds; from now on, we assume $C \neq \emptyset$.

First, note that for any DAG R, the following holds:

$$\begin{aligned} p_R(x_C) &= \sum_{x'_{N-C}} p_R(x_C, x'_{N-C}) \\ &= \sum_{x_{N-C}} \prod_{i\in C} p(x_i \mid x_{R(i)\cap C}, x'_{R(i)-C}) \prod_{i \notin C} p(x'_i \mid x_{R(i)\cap C}, x'_{R(i)-C}) \end{aligned} \tag{8}$$

$\Leftarrow$) Fix C such that C is an ancestral clique in some $Q \in [R]$. Note that $R(i) - C = \emptyset$ for all $i \in C$. Then,

$$\prod_{i\in C} p(x_i \mid x_{R(i)\cap C}, x'_{R(i)-C}) = \prod_{i\in C} p(x_i \mid x_{R(i)\cap C}) = p(x_C) \ (\because \text{topological sort})$$

Hence, by (8),

$$p_R(x_C) = p_Q(x_C) = p(x_C) \underbrace{\sum_{x_{N-C}} \prod_{i\notin C} p(x'_i \mid x_{R(i)\cap C}, x'_{R(i)-C})}_{1} = p(x_C).$$

e.g. 9.1. *For example, consider the following DAG:*

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[dr] & 4 \\ & & 3 \end{tikzcd}$$

Let $C := \{1, 2\}$. *Then,*

$$p_R(x_1, x_2) = \sum_{x'_3, x'_4} p_R(x_1, x_2, x'_3, x'_4) = p(x_1, x_2) \sum_{x'_3, x'_4} p(x'_4) p(x'_3 \mid x_2) = p(x_1, x_2)$$

$\Rightarrow$) [We need to make some fix in this direction.]

We show contrapositive: we show the following:

$$[\forall Q \in [R];\ C \text{ is not an ancestral clique in } Q] \implies [\exists p \exists x;\ p_R(x_C) = p(x_C)]$$

Assume that C is not an ancestral clique in any $Q \in [R]$. Fix any $Q \in [R]$. We divide the proof into two cases:

Case (i): In case C is not a clique in Q. In this case, C is not a clique in any $R' \in [R]$. There must be two distinct nodes $i_0, i_1 \in C$ such that $(i_0, i_1) \notin Q$ and $(i_1, i_0) \notin Q$. Consider $p \in \Delta(X)$ such that for every $i \in C \setminus \{i_0, i_1\}$, x_i is independently distributed, whereas x_{i_0} and x_{i_1} are mutually correlated. Then,

$$\prod_{i\in C} p(x_i \mid x_{R(i)\cap C}, x'_{R(i)-C}) = \prod_{i\in C} p(x_i) \ (\because \text{there is no edge b/w } i_0 \text{ and } i_1)$$

$$\prod_{i\notin C} p(x'_i \mid x_{R(i)\cap C}, x'_{R(i)-C}) = \prod_{i\notin C} p(x'_i)$$

$$p_R(x_C) = (8) = \prod_{i\in C} p(x_i) \sum_{i\notin C} \prod_{i\notin C} p(x'_i) = \prod_{i\in C} p(x_i)$$

However,

$$p(x_C) = p(x_{i_0}) p(x_{i_1} \mid x_{i_0}) \prod_{i \in C\setminus\{i_0, i_1\}} p(x_i)$$

Therefore, for some p, $p_R(x_C) \neq p(x_C)$.

Case (ii): C is a clique, but not an ancestral clique in Q. For a DAG R, denote the set of the all v-structures in R as $v(R)$, i.e.,

$$v(R) := \{(i, j, k) \mid i \to j, k \to j, i \nrightarrow k, k \nrightarrow i\}$$

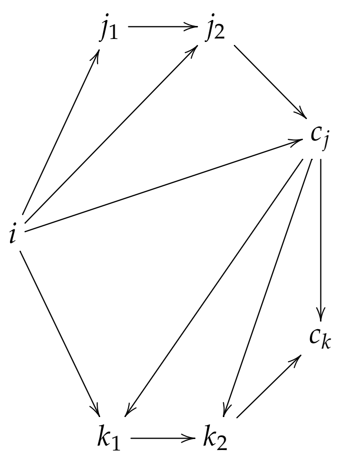
In the original proof, there is a lemma like the following, but the lemma is wrong:

Lem. 9.1. *Let R be a DAG and C be a clique in R. Assume the following two:*

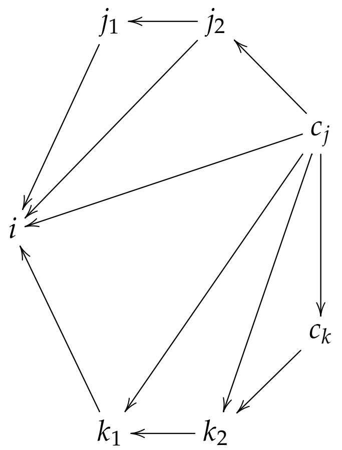
1. *$\forall j \in C$; j has no unmarried parents in R.*
2. *$\forall i \notin C$; if there is a directed path from i to some node $j \in C$ in R, then i has no unmarried parents in R.*

Transform R into another DAG R' by inverting every link along every such path; R and R' has the same v-structure.

e.g. 9.2 (Counter example for Lem.9.1)**.** *Let R be the graph below:*



Let $C := \{c_j, c_k\}$. Note that for all $k \in N \setminus C$ such that k has a path to some $c \in C$, k has no unmarried parents. R' is as follows:



Though $v(R) = \emptyset$, we have $v(R') = \{(j_1, i, k_1), (j_1, i, c_j), (j_2, i, k_1)\}$. Therefore, Lem.9.1 does not hold.

We can consider the modified version of the above lemma:

Lem. 9.2. *Let R be a DAG and C be a clique in R. Assume the following two:*

1. *$\forall j \in C$; j has no unmarried parents in R.*
2. *$\forall i, j \in N$; if there is a directed path from i to some node $c_i \in C$ and a path from j to some node $c_j \in C$ in R, then $i \to j$ or $j \to i$.*

Transform R into another DAG R' by inverting every link along the every path $i \rightsquigarrow c$ such that $i \notin C$ and $c \in C$; then, R and R' has the same v-structure.

For the moment, let us admit Lem.9.2. (I prove it later.)

[I tried to modified the condition in assumption 2 from $\forall i, j \in N$ to $\forall i, j \notin C$, but this does not hold: Below, $(i, j, k) \in v(R)$, but $(i, j, k) \notin v(R')$]

$$\begin{tikzcd} R: & i \arrow[dr] & \\ & & j \arrow[d] \\ & k \in C \arrow[ur] \arrow[r] & c_j \in C \end{tikzcd} \qquad \begin{tikzcd} R': & i & \\ & & j \arrow[ul] \\ & k \in C \arrow[ur] \arrow[r] & c_j \in C \arrow[u] \end{tikzcd}$$

The modified proof for Case (ii) By Lem.9.2, if the two assumptions in Lem.9.2 hold, there should exists $R' \in [R]$ such that C is an ancestral clique in R'; this contradicts the assumption we made at the beginning of the proof.

Hence, one of the following propositions holds:

$$\exists j \in C;\ j \text{ has an unmarried parents in } Q. \tag{P1}$$

$$\exists i, j \in N\ \exists c_i, c_j \in C;\ i \rightsquigarrow_Q c_i, j \rightsquigarrow_Q c_j, i \not\to_Q j, j \not\to_Q i \tag{P2}$$

In case of (P1), the original proof works. From now on, we assume (P1) does not hold and (P2) holds.

First of all, $i \notin C$ or $j \notin C$; otherwise there is an edge between them because C is a clique. Assume w.l.o.g. that $i \notin C$; Q contains the structure as below:

$$\begin{tikzcd} i \arrow[r, rightsquigarrow] & c_i \in C \arrow[d, dash] \\ j \arrow[r, rightsquigarrow] & c_j \in C \end{tikzcd}$$

Let $P_i \subseteq N$ and $P_j \subseteq N$ are the set of nodes contained in the directed paths from i to c_i and from j to c_j respectively.

Observations:

- $|P_i| \geq 2$. ($\because$ $i \notin C$.)
- $|P_j| \geq 1$. (j may be a member of C.)
- c_i and c_j may coincide.
- If $|P_j| = 1$, then $j \neq c_i$; otherwise, $i \to j$.

Consider $p \in \Delta(X)$ and a DAG R^* that satisfy

- p is consistent with R^*.
- $i \notin P_i \cup P_j \implies i$ is an isolated node in R^*.

Consider the subgraph of Q restricted on $P_i \cup P_j$. We name the subgraph Q'.

Case (ii-1): In case $Q'(j) = \emptyset$: Since C is a nonempty clique, $j \notin C$. Since $i \not\leftrightarrow j$, for all $p \in \Delta(X)$, we have $i \not\perp_{p'_Q} j$. Consider $p \in \Delta(X)$ such that $i \perp_p j$. Then, we can apply the same logic in the original proof in this case; we can show the existence of p such that $p(x_C) \neq p_Q(x_C)$ for some x_C.

Case (ii-2): In case $Q'(j) \neq \emptyset$: Fix $k \in Q'(j)$. Since Q' is a DAG, k is not a descendant of j in Q'. We also have $k \neq i$. Since all the nodes in Q' is either the descendant of node i or that of node j, node k is a descendant of node i. Assume w.l.o.g that there is no node along the path from i to k such that the node is a parent of j. (If $|Q'(j)| \geq 2$, then we can take the node k that is closest to i.)

$(i \perp j \mid k)_{Q'}$ **holds:** $\because$) First, take any path $i \rightsquigarrow j$, by the construction of k, k is on that path. Next, we need to check that neither of the following structure is contained in Q':

1. $i \to k \leftarrow j$
2. $$\begin{tikzcd} i \arrow[dr] & & \\ & m \arrow[r, rightsquigarrow] & k \\ j \arrow[ur] & & \end{tikzcd}$$

However, since Q' is a DAG and $k \to_{Q'} j$, neither of them holds.

cont. Therefore, there exists $p' \in \Delta(X_{P_i \cup P_j})$ such that $(x_i \not\perp x_j \mid x_k)_{p'}$. Consider the following probability distribution p:
$$p(x) := p'(x_{P_i \cup P_j}) \prod_{l \notin P_i \cup P_j} p(x_l)$$

p_R should satisfy $(x_i \perp x_j \mid x_k)_Q$. This implies

$$\exists p \exists x_C;\ p(x_C) = p_{Q'}(x_C)$$

□

もしかしたら，Spiegler が論文中でいっている主張は Lem.9.1 とは違うものかも．

Lem. 9.3. *Let R be a DAG and C be a non-ancestral clique in any $R' \in [R]$. Assume the following two:*

1. *$\forall j \in C$; j has no unmarried parents in R.*
2. *$\forall i \notin C$; if there is a directed path from i to some node $j \in C$ in R, then i has no unmarried parents in R.*

Transform R into another DAG R' by inverting every link along every such path; R and R' has the same v-structure.

しかし，この証明の仕方もよくわからず．

Proof of Lem.9.2. We show $v(R) = v(R')$.

Step 1: $v(R) \subseteq v(R')$ Fix any v-structure $(i, j, k) \in v(R)$, $i \to j \leftarrow k$. By assumption 1 in Lem.9.2, we can assume that $j \notin C$. We can also assume that $i \notin C$ or $k \notin C$; otherwise there is an edge between i and k because C is a clique. Assume w.l.o.g that $i \notin C$.

It is sufficient to show that (i, j, k) remains as a v-structure after the inversion. Suppose toward contradiction that (i, j, k) is not a v-structure any more after the inversion. It is necessary that at least one of the edges $i \to j$ and $k \to j$ should be inverted.

Case (1-1): In case $k \notin C$**:** Assume w.l.o.g that $i \to j$ is inverted. Then, there exists some node $c \in C$ such that $i \underset{R}{\rightsquigarrow} c$ [4]; this implies that $i \underset{R}{\rightsquigarrow} c$, and $k \underset{R}{\rightsquigarrow} c$. The graph below summarizes the relationships:

$$
\begin{tikzcd}
i \arrow[dr] & & \\
& j \arrow[r, rightsquigarrow] & c \in C \\
k \arrow[ur] & &
\end{tikzcd}
$$

However, by assumption 2 in Lem.9.2, there should be an edge between node i and node k; this contradicts the assumption that (i, j, k) is a v-structure in R.

Case (1-2) In case $k \in C$**:** In this case, $k \to j$ is not inverted; then, $i \to j$ should be inverted. Then, by the same logic as in Case (1-1), this leads to a contradiction.

Step 2: $v(R) \supseteq v(R')$ We show that the inversion does not create a new v-structure. Suppose toward contradiction that there exists a triple $(i, j, k) \in v(R) \setminus v(R')$. In this case, the structure as in the below graph should hold in R (c_i and c_k may be the same node.):

$$
\begin{tikzcd}
& i \arrow[r, rightsquigarrow] & c_i \in C \\
j \arrow[ur] \arrow[dr] & & \\
& k \arrow[r, rightsquigarrow] & c_k \in C
\end{tikzcd}
$$

However, by assumption 2 in Lem.9.2, there should be an edge between node i and node k. A contradiction. $\square$

[4] $i \underset{R}{\rightsquigarrow} j$ denotes that there is a directed path from node i to node j in a DAG R.