Graph Gradient, Divergence, Laplacian, Hessian

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I. Differential Functionals

We consider both differential functionals, $f : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \to \mathbb{R}^M$. Let's say that these functionals are dependent on $\{x_i\}_{i=1 \ldots N}$.

Gradient: Gradient, $\nabla : \mathbb{R} \to \mathbb{R}^N$ is defined as:

$$(\nabla f)_i = \frac{\partial f}{\partial x_i}$$

and gradient on vector valued function, $\nabla : \mathbb{R}^N \to \mathbb{R}^{N \times M}$ is defined as

$$(\nabla f)_{ij} = \frac{\partial f_i}{\partial x_j}$$

Divergence: Divergence, $\text{div} : \mathbb{R}^N \to \mathbb{R}$ is defined as:

$$\text{div} F = \sum_{i=1}^{N} \frac{\partial F_i}{\partial x_i}$$

and more generalized divergence, $\text{div} : \mathbb{R}^{N \times M} \to \mathbb{R}^N$ is defined as:

$$(\text{div} F)_i = \sum_{j=1}^{N} \frac{\partial F_{ij}}{\partial x_j}$$

Laplacian: Laplacian, $\Delta : \mathbb{R} \to \mathbb{R}$ is defined as:

$$\Delta f = \sum_{i=1}^{N} \frac{\partial^2 f}{\partial x_i^2}$$

and Laplacian on vector valued function, $\Delta : \mathbb{R}^N \to \mathbb{R}^N$ is defined as:

$$(\Delta f)_i = \sum_{j=1}^{N} \frac{\partial^2 f_i}{\partial x_j^2}$$

Hessian: Hessian, $H : \mathbb{R} \to \mathbb{R}^{N \times N}$ is defined as:

$$(Hf)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

II. Graph Signals

Graph Gradient: Graph gradient, $\nabla : \mathbb{R}^N \to \mathbb{R}^{N \times N}$, is defined as:

$$(\nabla f)_{ij} = \sqrt{w_{ij}}(f_i - f_j)$$

- In fact, this kind of gradient operator is called Jacobian operator.

Graph Divergence: Graph divergence, $\text{div} : \mathbb{R}^{N \times N} \to \mathbb{R}^N$, is defined as:

$$(\text{div} F)_i = \frac{1}{a_i} \sum_{j \in \mathcal{V}} \sqrt{w_{ij}} F_{ij}$$

Graph Laplacian: Graph Laplacian, $\Delta : \mathbb{R}^N \to \mathbb{R}^N$, is defined as:

$$(\Delta f)_i = \frac{1}{a_i} \sum_{j \in \mathcal{V}} w_{ij}(f_i - f_j)$$

- In fact, graph Laplacian is the graph divergence of graph gradient.
- When $a_i = 1$ is chosen, the graph Laplacian is called the combinatorial graph Laplacian, which is not normalized.

Graph Hessian: Graph Hessian, $H : \mathbb{R}^N \to \mathbb{R}^{N\times N\times N}$, is defined as:

$$(Hf)_{ijk} = \frac{w_{ij}(f_i - f_j) + w_{ik}(f_i - f_k)}{2}$$

Verification of Some Properties:

- Trace of Hessian is equal to Laplacian:

$$\begin{aligned}\mathrm{tr}(Hf)_i &= \sum_{j,k\in\mathcal{V}\,:\,j=k} \frac{w_{ij}(f_i - f_j) + w_{ik}(f_i - f_k)}{2} \\ &= \sum_{j\in\mathcal{V}} w_{ij}(f_i - f_j) = (\Delta f)_i\end{aligned}$$

III. Digital Images

A digital image can be seen as a 2D-grid graph where each node represents the pixel in the image. Each edge weight is equal to $1/4$ and the adjacency matrix is created from the adjacency of the pixels. Then, the metrics become as follows:

Gradient: Gradient on images, $\nabla : \mathbb{R}^N \to \mathbb{R}^{N\times N}$, is defined as:

$$(\nabla f)_{ij} = f_i - f_j$$

Divergence: Divergence on images, $\mathrm{div} : \mathbb{R}^{N\times N} \to \mathbb{R}^N$, is defined as:

$$(\mathrm{div}F)_i = \frac{1}{4}\sum_{j\in\mathcal{N}_i} F_{ij}$$

where $\mathcal{N}_i$ is neighborhood of vertex i. In this case, $\mathcal{N}_i = \{j \mid ||(i//N, i\%N) - (j//N, j\%N)|| = 1\}$. Note that $|\mathcal{N}_i| = 4$.

Laplacian: Laplacian on images, $\Delta : \mathbb{R}^N \to \mathbb{R}^N$, is defined as:

$$(\Delta f)_i = \frac{1}{4}\sum_{j\in\mathcal{N}_i}(f_i - f_j)$$

- It is equivalent to take the convolution of an image with the following kernel:

$$\begin{bmatrix} 0 & -1/4 & 0 \\ -1/4 & 1 & -1/4 \\ 0 & -1/4 & 0 \end{bmatrix}$$

Graph Hessian: Graph Hessian, $H : \mathbb{R}^N \to \mathbb{R}^{N\times N\times N}$, is defined as:

$$(Hf)_{ijk} = \frac{w_{ij}(f_i - f_j) + w_{ik}(f_i - f_k)}{2}$$

IV. Riemannian Manifolds

Gradient:
Divergence:
Laplacian:
Hessian:

V. Discrete Manifolds by Triangular Meshing

Gradient:
Divergence:
Laplacian:
Hessian: