Graph Calculus

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I. Graph Node Signal and Graph Edge Signal

A finite weighted graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is defined where $\mathcal{V} = \{1, ..., N\}$ is the set of nodes with $|\mathcal{V}| = N$ and $\mathcal{E}$ is the set of edges.

This type of graphs can be represented by weight matrix $\mathbf{W} \in \mathbb{R}^{N \times N}$. Then, edge space can be defined as $\mathcal{E} = \{(i,j) : w_{ij} \neq 0, \forall i, j \in \mathcal{V}\}$ with $|\mathcal{E}| = N_e$. $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$.

Space of Real Vertex Signals:

Generic real vertex function can be given as $f : \mathcal{V} \to \mathbb{R}$. Then, Hilbert space of such functions can be defined as $\mathcal{H}(\mathcal{V}) = \{f : \mathcal{V} \to \mathbb{R}\}$. Since $|\mathcal{V}| = N$, $\mathcal{H}(\mathcal{V}) \approx \mathbb{R}^N$.

Inner product of $\mathcal{H}(\mathcal{V})$:

$$< f, g >_{\mathcal{H}(\mathcal{V})} = \sum_{i \in \mathcal{V}} f_i g_i, \quad \forall f, g \in \mathcal{H}(\mathcal{V})$$

ℓ_p-norm of $\mathcal{H}(\mathcal{V})$:

$$||f||_p = (\sum_{i \in \mathcal{V}} |f_i|^p)^{1/p} \quad 1 \leq p < \infty$$

$$||f||_\infty = \max_{i \in \mathcal{V}} |f_i|$$

ℓ_2-norm is induced by the inner product.

Space of Real Edge Signals:

Generic real edge function can be given as $F : \mathcal{E} \to \mathbb{R}$. Then, Hilbert space of such functions can be defined as $\mathcal{H}(\mathcal{E}) = \{F : \mathcal{E} \to \mathbb{R}\}$. Since $|\mathcal{E}| = N_e$, $\mathcal{H}(\mathcal{E}) \approx \mathbb{R}^{N_e}$.

Inner product of $\mathcal{H}(\mathcal{E})$:

$$< F, G >_{\mathcal{H}(\mathcal{E})} = \sum_{(i,j) \in \mathcal{E}} F_{ij} G_{ij}, \quad \forall F, G \in \mathcal{H}(\mathcal{E})$$

ℓ_p-norm of $\mathcal{H}(\mathcal{E})$:

$$||F||_p = (\sum_{(i,j) \in \mathcal{E}} |F_{ij}|^p)^{1/p} \quad 1 \leq p < \infty$$

$$||F||_\infty = \max_{(i,j) \in \mathcal{E}} |F_{ij}|$$

ℓ_2-norm is induced by the inner product.

II. Differential Graph Operators

Graph Gradient: Graph gradient operator $\nabla : \mathcal{H}(\mathcal{V}) \to \mathcal{H}(\mathcal{E})$ can be defined as:

$$(\nabla f)_{ij} = \sqrt{w_{ij}}(f_i - f_j)$$

For undirected graph, $(\nabla f)_{ij} = -(\nabla f)_{ji}$.

Local variation of a vertex function f in a vertex $i \in \mathcal{V}$ can be defined as:

$$||(\nabla f)_{i.}||_p = (\sum_{j \in \mathcal{V}} (w_{ij})^{p/2} |f_i - f_j|^p)^{1/p}$$

Graph Divergence: Graph divergence operator $\mathrm{div} : \mathcal{H}(\mathcal{E}) \to \mathcal{H}(\mathcal{V})$ can be defined as:

$$(\mathrm{div} F)_i = \sum_{j \in \mathcal{V}} \sqrt{w_{ij}}(F_{ji} - F_{ij})$$

Adjointness, $\nabla^* = -\text{div}$:

$$\begin{aligned}
<\nabla f, G>_{\mathcal{H}(\mathcal{E})} &= \sum_{(i,j)\in\mathcal{E}} (\nabla f)_{ij} G_{ij} \\
&= \sum_{(i,j)\in\mathcal{E}} \sqrt{w_{ij}}(f_i - f_j) G_{ij} \\
&= \sum_{(i,j)\in\mathcal{E}} \sqrt{w_{ij}} f_i G_{ij} - \sum_{(i,j)\in\mathcal{E}} \sqrt{w_{ij}} f_j G_{ij} \\
&= \sum_{(i,j)\in\mathcal{E}} \sqrt{w_{ij}} f_i G_{ij} - \sum_{(i,j)\in\mathcal{E}} \sqrt{w_{ij}} f_i G_{ji} \\
&= \sum_{(i,j)\in\mathcal{E}} \sqrt{w_{ij}} f_i (G_{ij} - G_{ji}) \\
&= \sum_{i\in\mathcal{V}} f_i \sum_{j\in\mathcal{V}} \sqrt{w_{ij}} (G_{ij} - G_{ji}) \\
&= \sum_{i\in\mathcal{V}} f_i (-\text{div} G)_i =< f, -\text{div} G >_{\mathcal{H}(\mathcal{V})}
\end{aligned}$$

Graph Laplacian: Graph laplacian operator $L : \mathcal{H}(\mathcal{V}) \to \mathcal{H}(\mathcal{V})$ can be defined as:

$$\begin{aligned}
(Lf)_i &= (\text{div}(\nabla f))_i \\
&= \sum_{i\in\mathcal{V}} w_{ij}(f_i - f_j)
\end{aligned}$$