まえがき

微分積分学 ～ 18世紀 ニュートンら

19世紀 反省期 (デデキント・ワイエルシュトラス・コーシー)

高校の微積分と大学の〃

高校 ～ 手法の導入

大学 ～ 基礎づけと応用がプラス

本書の特色

(1) 高校のカリキュラムに接続

(2) 応用力がつく

(3) 基礎的 (変るテクニックにやらない)

学習法

① 書いて学習

② 覚える (基本的な概念)

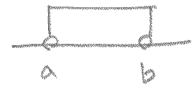
③ まねる

1. 関数，極限と連続，1変数の微分法

1-1. 関数

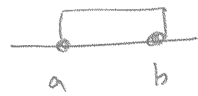
◇ <u>区間</u>

① $a < x < b$



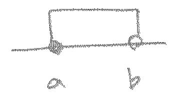
(a, b) 開区間という

② $a \leqq x \leqq b$



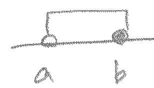
$[a, b]$ 閉区間という

③ $a \leqq x < b$



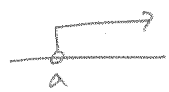
$[a, b)$

④ $a < x \leqq b$



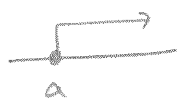
$(a, b]$

⑤ $a < x$



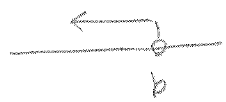
(a, ∞)

⑥ $a \leqq x$



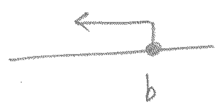
$[a, \infty)$

⑦ $x < b$



$(-\infty, b)$

⑧ $x \leqq b$



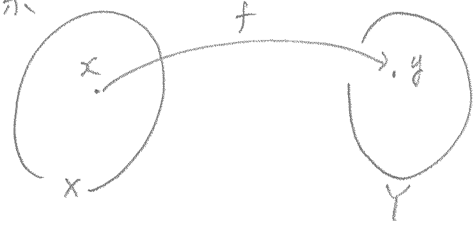
$(-\infty, b]$

⑨ 実数全体 $\mathbb{R} = (-\infty, \infty)$
(real number)

◇ 関数

$x \in X$ ← 実数の集合

要素



Xの要素xに実数yを1つずつ対応させる規則fがあるとき

yをxの関数であるといい

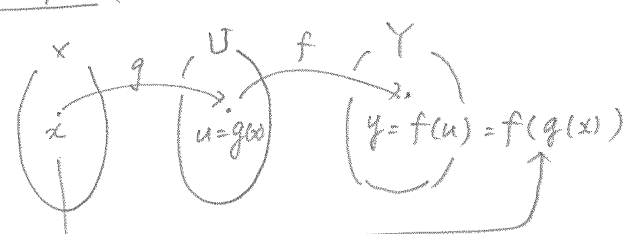
$y = f(x)$ と表す。

y ↑ 従属変数　　x ↑ 独立変数

X：定義域

$Y = \{y ; y = f(x), x \in X\}$　値域（3p. 注1-1）

◇ 合成関数



合成関数 $y = f(g(x))$

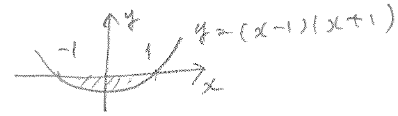
xを1つきめると $f(g(x))$ が1つきまる。

例 1.1.

$y=\sqrt{1-x^2}$ の定義域と値域を求め，区間で示せ.

[解] $\underset{0}{\overset{\sqrt{1-x^2}}{\vee\!\!\shortparallel}}$ でないと y が虚数となってしまうから，

$\Leftrightarrow$ $1-x^2 \geqq 0$ $\quad x^2-1 \leqq 0.$ $\quad (x-1)(x+1) \leqq 0$



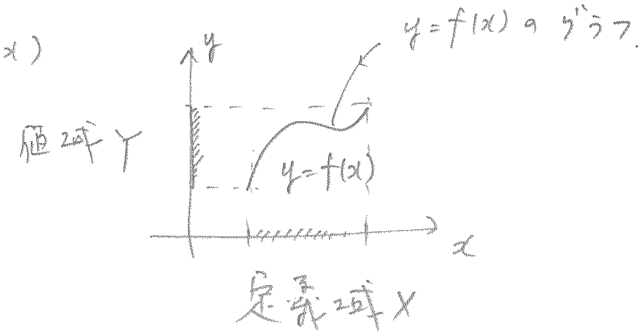
$-1 \leqq x \leqq 1$

したがって定義域は $[-1, 1]$

$x \in [-1, 1]$ のとき $0 \leqq 1-x^2 \leqq 1$ $\quad$ ∴ (よって) $\quad 0 \leqq \sqrt{1-x^2} \leqq 1$

したがって値域は $[0, 1]$ である.

注 1.1 $\quad y=f(x)$



$Y = \{ y \,;\, y=f(x),\ x \in X \}$

$y=f(x)$ のグラフ $= \{ (x, y) \,;\, y=f(x),\ x \in X \}$

例1.2

$y=f(u)=u^2$ と $u=g(x)=2x+1$ の合成関数 $y=f(g(x))$ は?

$y=g(u)=2u+1$ と $u=f(x)=x^2$ 〃 $y=g(f(x))$ は?

(解)

$$y=\underset{\substack{\uparrow\\ 2x+1}}{u}{}^2=(2x+1)^2$$

$$y=2\underset{\substack{\uparrow\\ x^2}}{u}+1=2x^2+1$$

注1.2　$f(g(x))$ と $g(f(x))$ は一般に一致しない!

問1.1　$f(u)=\sqrt{4-u^2}$, $u=\dfrac{x}{x-1}$ の合成関数の定義域は?

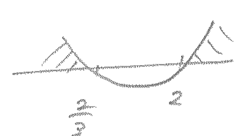
$$y=f(u)=\sqrt{4-\underset{\substack{\uparrow\\ \frac{x}{x-1}}}{u}{}^2}=\sqrt{4-\left(\frac{x}{x-1}\right)^2}=\sqrt{\frac{4(x-1)^2-x^2}{(x-1)^2}}$$

← ゼロで割れないから

$x\neq 1$ ……(*)

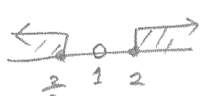
$$=\sqrt{\frac{(2x-2-x)(2x-2+x)}{(x-1)^2}}=\sqrt{\frac{(x-2)(3x-2)}{(x-1)^2}}$$

$\sqrt{\ }$ 中身はゼロ以上　$3(x-2)(x-\frac{2}{3})\geq 0$



$x\leq\frac{2}{3}$, $2\leq x$ ……(**)

(*) (**) より　$x\leq\frac{2}{3}$, $2\leq x$



問 1.2.

$f(x) = (x-1)(x-2)$ のとき

$$f(a-3) = (a-3-1)(a-3-2) = \underline{(a-4)(a-5)}\ //$$

$$f(\underbrace{\boxed{f(a)}}_{(a-1)(a-2)}) = f((a-1)(a-2)) = \{\underbrace{(a-1)(a-2)}_{a^2-3a+2}-1\}\{\underbrace{(a-1)(a-2)}_{a^2-3a+2}-2\}$$

$$= \underline{(a^3-3a-3)(a^2-3a)}\ //$$

omit.

$\boxed{e^{\square} \text{ の後で}}$

問 1.3. $f(x) = e^{x} + e^{-x}$ のとき.

$f(x+y) \cdot f(x-y) = f(2x) + f(2y)$ が成立することを示せ.

$$\text{左辺} = f(x+y) \cdot f(x-y)$$

$$= \left(e^{x+y} + e^{-x-y}\right)\left(e^{x-y} + e^{-x+y}\right)$$

$$= \underbrace{e^{x+y} e^{x-y}}_{e^{2x}} + \underbrace{e^{x+y} e^{-x+y}}_{e^{2y}} + \underbrace{e^{-x-y} e^{x-y}}_{e^{-2y}} + \underbrace{e^{-x-y} e^{-x+y}}_{e^{-2x}}$$

$$= \underbrace{e^{2x} + e^{-2x}}_{f(2x)} + \underbrace{e^{2y} + e^{-2y}}_{f(2y)} = \text{右辺}$$

$\boxed{1.2}$ 逆関数

◇ 逆関数

(例)　$x \geqq 0$ を定義域として　$y = x^2$ ① を考える

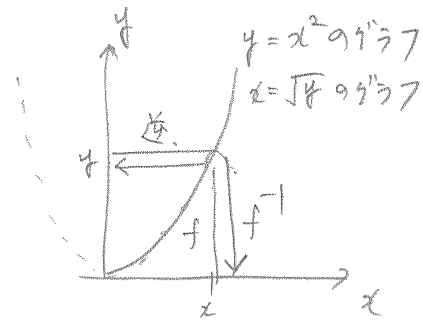
① を x について解くと、　$x^2 = y \Rightarrow x = \pm\sqrt{y} = \sqrt{y}$ … ②

($x \geqq 0$ より)

(つまり y が1つ決まると x が1つ決まる)

② は、x が y の関数であることを示している。

$$\sqrt{y} = x \underset{\text{逆関数}}{\leftarrow} y = x^2$$



② を ① の逆関数という

一般に　$y = f(x)$ を　x について解き、ただ1つの解　$x = g(y)$ が得られたとする。このとき、

$$x = g(y) = f^{-1}(y) \quad \cdots \quad ③$$

と書き

f の逆関数という

x を独立変数として、y を従属変数として f^{-1} を表せば

$$y = g(x) = f^{-1}(x)$$

↑ これを　$y = f(x)$ の逆関数ということが多い

◇ 逆関数の性質

(1) $y=f(x)$ のグラフと $x=f^{-1}(y)$ のグラフは一致

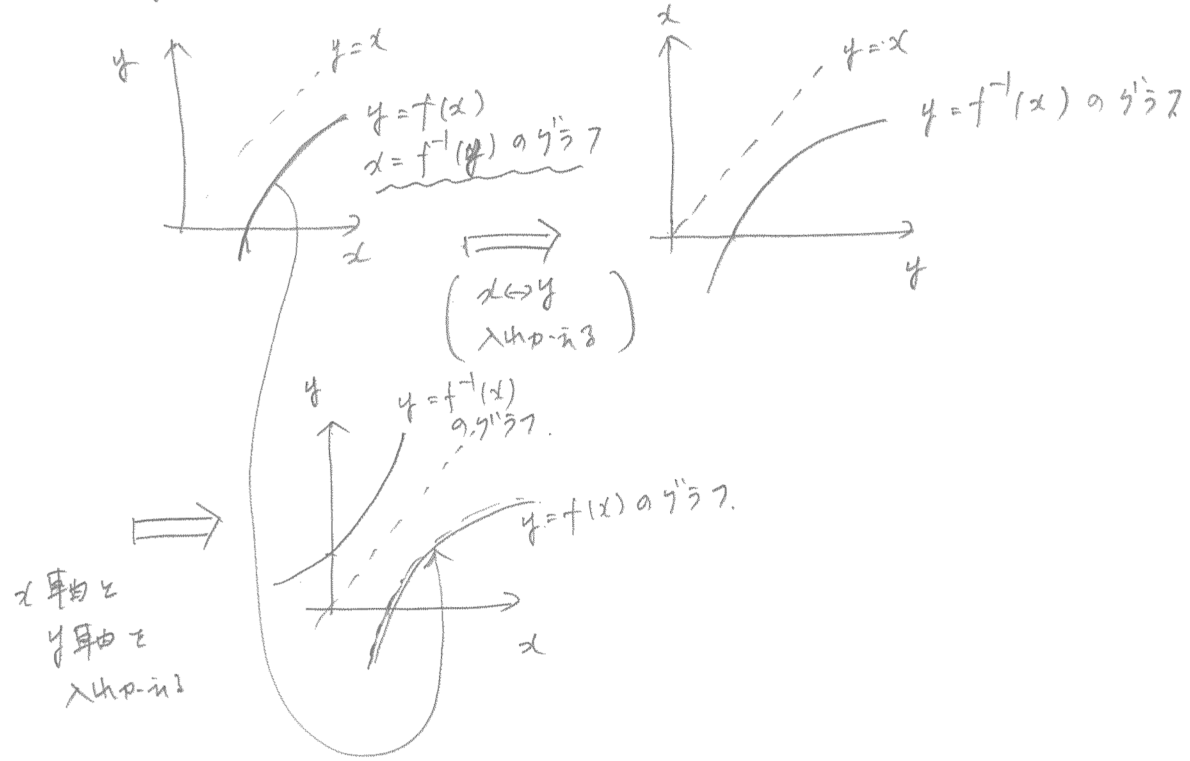
∵ なぜなら

$$\{(x,y)\,;\ \underbrace{y=f(x)}_{\Updownarrow \atop x=f^{-1}(y)},\ x\in X,\ y\in Y\}$$

$$=\{(x,y)\,;\ x=f^{-1}(y),\ y\in Y,\ x\in X\}$$

(2) $y=f(x)$ の逆関数 $y=f^{-1}(x)$ のグラフは $y=x$ 対称

∵



(3) ← いろんな逆関数あるよ.

略

(4) ← p.22

例 1.3

$y = 1 + \sqrt{x+1}$ の逆関数は？（定義域と値域も求めよ.）

[解] $\sqrt{x+1}$ → 0以上　　$x \geqq -1$ …

$\sqrt{x+1} \geqq 0$ より　　$y = 1 + \sqrt{\ } \geqq 1.$

したがって.　$y = f(x) = 1 + \sqrt{x+1}$ の定義域は $x \geqq -1$

値域は $y \geqq 1.$

逆関数 f^{-1} は.　x について解くと.

$y - 1 = \sqrt{x+1} \rightarrow (y-1)^2 = x+1$　　$x = (y-1)^2 - 1$

$= y^2 - 2y + 1 - 1$

$= y^2 - 2y = f^{-1}(y)$

定義域 $y \geqq 1$

値域は $x \geqq -1$

独立変数を x にすると.　$y = f^{-1}(x) = x^2 - 2x$

定義域は $x \geqq 1$　　$[1, \infty)$

値域は $y \geqq -1.$　　$[-1, \infty)$

グラフの描き方

$y = 1 + \sqrt{x+1}$ のグラフは.

$y = \sqrt{x}$ のグラフを x方向に -1, y方向に 1 平行移動したもの

$y = x^2 - 2x = (x-1)^2 - 1$

平方完成

$y = x^2$ のグラフを x方向に 1, y方向に -1

平行移動

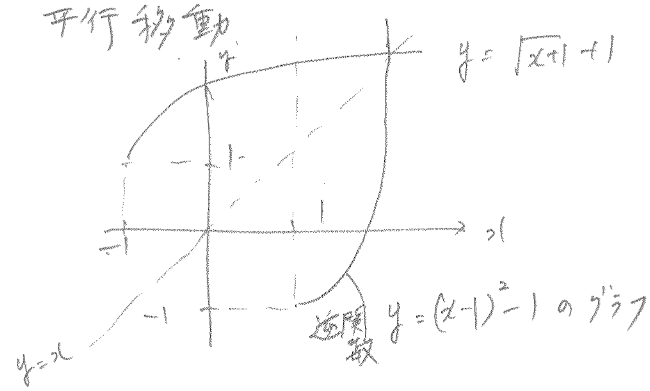


図 1.8

メモ　$y = f(x)$ のグラフを

x方向に α, y 〃 に β

平行移動したグラフの

方程式は

$y - \beta = f(x - \alpha)$

or

$y = f(x - \alpha) + \beta$

例　平方完成

$x^2 + ax + b$

$\left(x + \frac{a}{2}\right)^2 = x^2 + ax + \left(\frac{a}{2}\right)^2$

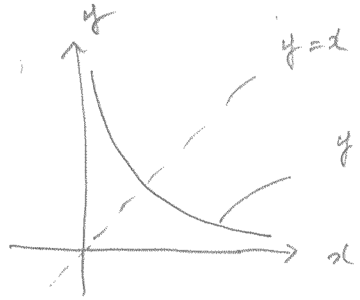
より

$\left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2$ におきかえると

$\rightarrow \left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 + b$

例1.4　$y=f(x)$ と $y=f^{-1}(x)$ が同じ関数となる f はどんな関数か?

[解]



$y=x$ に対して対称なグラフは $y=f(x)$ と $y=f^{-1}(x)$ が同じグラフ.

$y=x$ に対して対称なグラフは

$$g(x,y)=g(y,x)$$ ← (x と y の入れかえに対して形の変わらない式 $g(x,y)$)

を用いて $g(x,y)=0$ と定められる。

(例1)　$\underbrace{x+y-2}=0$

$g(x,y)=x+y-2$ のとき

実際,

$y=-x+2=f(x)$

$x=-y+2=f^{-1}(y) \longrightarrow y=f^{-1}(x)=-x+2$　一致!

(例2)　$\underbrace{xy-x-y}=0.$

$g(x,y)=xy-x-y$

$(x-1)y=x \longrightarrow y=\dfrac{x}{x-1}=f(x)$

$(y-1)x=y \longrightarrow x=\dfrac{y}{y-1}=f^{-1}(y) \Longrightarrow y=f^{-1}(x)=\dfrac{x}{x-1}$　一致!

問1-4　　f^{-1}と定義域と値域は?

(1)　$y = 4x - 2$　　$x \in \mathbb{R}$

$= f(x)$　　$y \in \mathbb{R}$

$4x = y + 2$　より　$x = \frac{y+2}{4} = f^{-1}(y)$　　$y \in \mathbb{R}$　$x \in \mathbb{R}$

$y = f^{-1}(x) = \frac{x+2}{4}$　　$x \in \mathbb{R}$　定義域

$y \in \mathbb{R}$　値域

(2)　$y = \frac{1}{\sqrt{x}}$　　$x > 0$

$y > 0$.

$= f(x)$

$\sqrt{x} = \frac{1}{y}$　より　$x = \frac{1}{y^2} = f^{-1}(y)$　　$y > 0$　$x > 0$

$y = f^{-1}(x) = \frac{1}{x^2}$　　$x > 0$　定義域

$y > 0$　値域

(3)　$y = x^2 + 2x - 4$　　$(x \geqq -1)$

$= (x+1)^2 - 1 - 4$

$= (x+1)^2 - 5$　　$(y \geqq -5)$

$= f(x)$

$x^2 + 2x - y - 4 = 0$.　　$x = -1 \pm \sqrt{1^2 - (-y-4)}$　　$y \geqq -5$

$= -1 + \sqrt{y+5}$　　$x \geqq -1$

$= f^{-1}(y)$

$y = f^{-1}(x) = -1 + \sqrt{x+5}$　　$x \geqq -5$　定義域

$y \geqq -1$　値域

1.3 有理関数

◇有理整関数（n次関数）

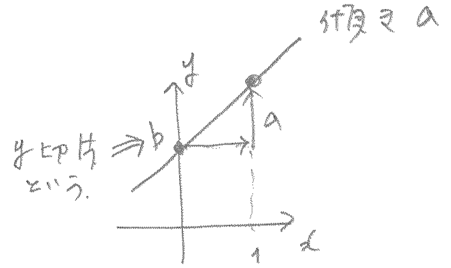
$$y = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

a_0 ↑ ゼロでない

（例）1次関数

$$y = ax + b$$

$$(a \neq 0)$$



（例）2次関数

$$y = ax^2 + bx + c$$

$$(a \neq 0)$$

$$= a\left(x^2 + \frac{b}{a}x\right) + c$$

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$$

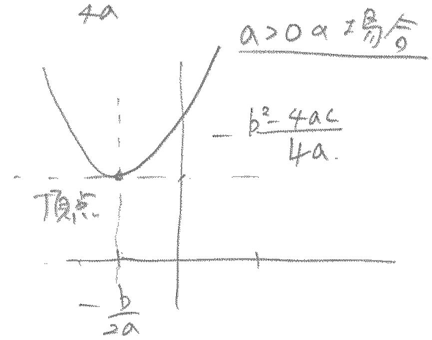
$y = ax^2$ のグラフを $\left\{\begin{array}{ll} x\text{方向に} & -\dfrac{b}{2a} \\ y \quad〃 & -\dfrac{b^2-4ac}{4a} \end{array}\right\}$ 平行移動

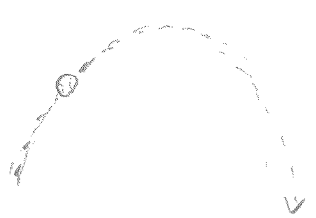
頂点 $\left(-\dfrac{b}{2a}, -\dfrac{b^2-4ac}{4a}\right)$

軸 $x = -\dfrac{b}{2a}$

の

放物線





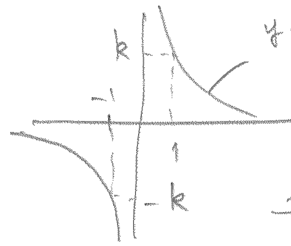
◇ 有理関数

$$y=\frac{g(x)}{h(x)}=\frac{a_0x^n+a_1x^{n-1}+\cdots+a_n}{b_0x^m+b_1x^{m-1}+\cdots+b_m} \qquad a_0\neq 0,\ b_0\neq 0$$

(例) $h(x)=1$ のとき $y=g(x)$ となり 有理整関数

(例) $y=\dfrac{ax+b}{cx+d}$ $(c\neq 0,\ ad-bc\neq 0)$

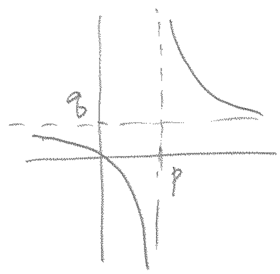
特に $y=\dfrac{k}{x}$ $(k\neq 0)$ は.



$y=\frac{k}{x}$ のグラフ

直角双曲線

$y=\dfrac{k}{x-p}+q$ は.

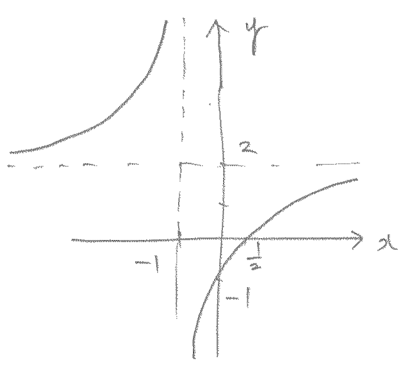


例1.5 $y=\dfrac{2x-1}{x+1}$ のグラフは?

$$\begin{array}{r} 2 \\ x+1\,\overline{)\,2x-1} \\ 2x+2 \\ \hline -3 \end{array}$$

$$y=2+\frac{-3}{x+1}$$

$y=\dfrac{-3}{x}$ のグラフと $\begin{cases} x方向\ -1 \\ y\ 〃\ 2 \end{cases}$ だけ平行移動したもの

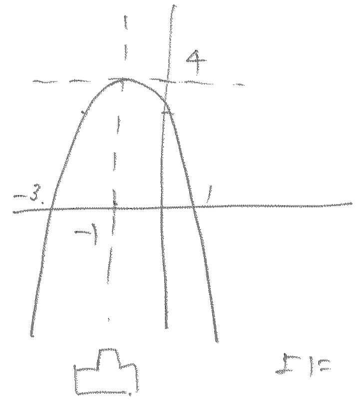


例1.6　$y=-x^2-2x+3$ のグラフ

$$y=-(\underbrace{x^2+2x}_{(x+1)^2-1^2})+3$$

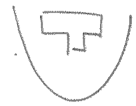
$$=-(x+1)^2+1+3$$

$$=-(x+1)^2+4$$



凸　上に凸(トツ)という

比較



下に凸という

問1.5　$x^2-3x-4 \le y \le -\frac{x}{2}+5$ をみたす (x,y) の範囲は?

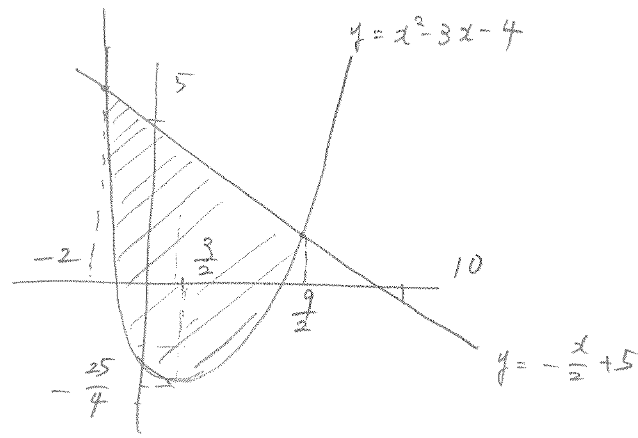
$$y=x^2-3x-4$$

$$=\left(x-\frac{3}{2}\right)^2-\frac{9}{4}-4$$

$$=\left(x-\frac{3}{2}\right)^2+\boxed{\frac{-16-9}{4}}$$

$$\frac{-25}{4}$$

$$y=-\frac{x}{2}+5$$



交点は　$x^2-3x-4=-\frac{x}{2}+5$

$$2x^2-6x-8+x-10$$

$$=2x^2-5x-18=0 \qquad 3\cdot 6$$

$$(2x-9)(x+2)=0 \qquad 2\cdot 9$$

$$x=-2,\ \frac{9}{2}$$

問1.6.

$$\begin{cases} f(-x)=f(x) & \cdots\ f(x)\text{は偶関数} \\ f(-x)=-f(x) & \cdots\ f(x)\text{は奇関数} \end{cases}$$ という.

(1) $f(x)=x^2+2$

$f(-x)=(-x)^2+2=x^2+2=f(x)$ … 偶

(2) $f(x)=x^3-3x$

$f(-x)=(-x)^3-3(-x)=-x^3+3x=-(x^3-3x)=-f(x)$ … 奇

(3) $f(x)=x^2+x+1$

$f(-x)=x^2-x+1$ ‥ どちらでもない.

(4) $f(x)=\dfrac{x-1}{x+1}$

$f(-x)=\dfrac{-x-1}{-x+1}=\dfrac{x+1}{x-1}$ どちらでもない

(5) $f(x)=x+\dfrac{1}{x}$

$f(-x)=-x-\dfrac{1}{x}=-\left(x+\dfrac{1}{x}\right)=-f(x)$ ‥ 奇

(6) $f(x)=\dfrac{1}{x+1}-\dfrac{1}{x-1}$

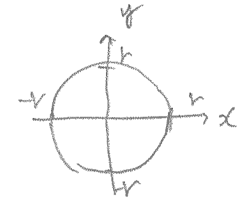
$f(-x)=\dfrac{1}{-x+1}-\dfrac{1}{-x-1}=\dfrac{1}{x+1}-\dfrac{1}{x-1}=f(x)$ 偶

1.4 2次曲線（円, 楕円, 双曲線）, 無理関数のグラフ

◇ **2次曲線** $\underbrace{f(x, y)}_{x,\,y\text{の2次多項式}} = 0$ で表される曲線

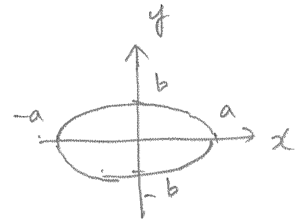
◇ **円**

$$\underbrace{x^2 + y^2}_{(\text{原点からの距離})^2} = r^2 \Rightarrow$$



◇ **だ円**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow$$



$x = 0$ のとき $y = \pm b$

$y = 0$ のとき $x = \pm a$

◇ **双曲線** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ または $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

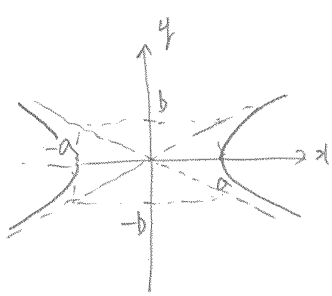
$y = 0$ のとき $x = \pm a$ ｜ $x = 0$ のとき $y = \pm b$

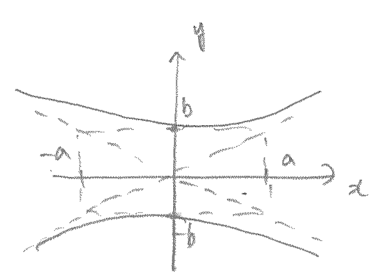
$|x|, |y| >> 1$ となると

$\frac{b^2}{a^2}x^2 - y^2 = \frac{1}{x^2} \sim 0$ ｜ $\frac{b^2}{a^2}x^2 - y^2 = \frac{-1}{x^2} \sim 0$

$$\underbrace{y^2 - \frac{b^2}{a^2}x^2}_{\left(y - \frac{b}{a}x\right)\left(y + \frac{b}{a}x\right)} = 0 \text{ に近づく.}$$

$y = \pm \frac{b}{a}x$ に近づく ← 漸近線（ぜんきん）という





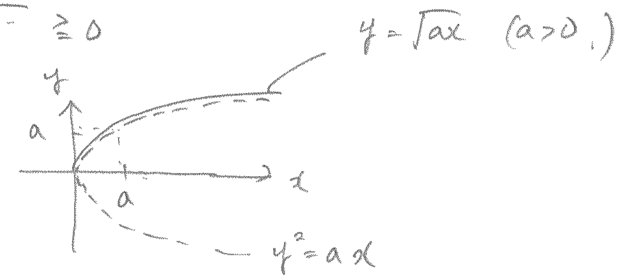
◇ 無理関数　$y=\sqrt{ax+b}$ のグラフ

$y=\sqrt{a\left(x+\frac{b}{a}\right)}$ より　$y=\sqrt{ax}$ のグラフを x 方向に $-\frac{b}{a}$ 平行移動

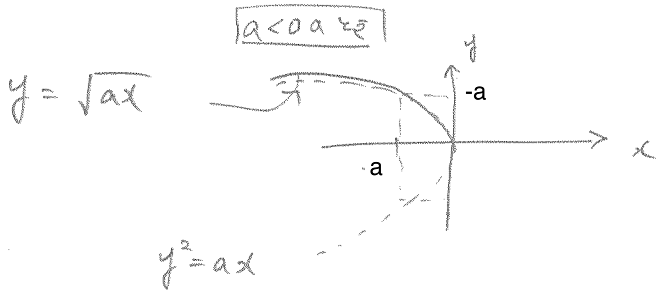
$y=\sqrt{ax}$ の定義域 $\begin{cases} a>0 \text{ のとき } x \geqq 0 \\ a<0 \text{ のとき } x \leqq 0 \end{cases}$

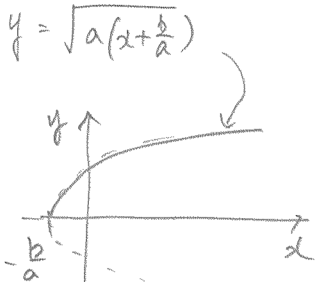
値域　$y=\sqrt{\cdots} \geqq 0$

$y^2=(\sqrt{ax})^2=ax$　より

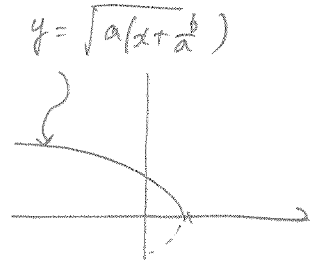


$\boxed{a<0 \text{ のとき}}$





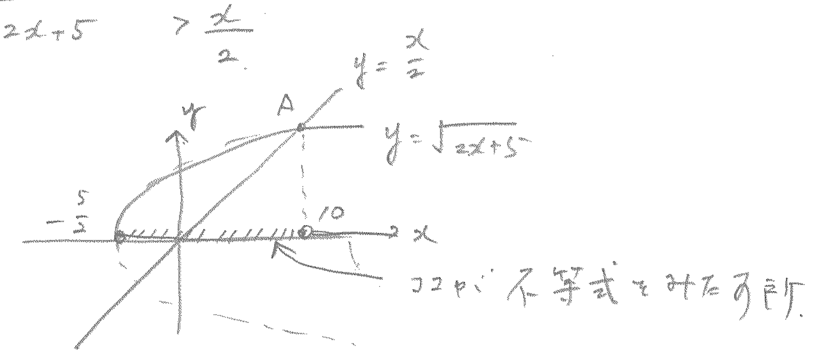
$a>0$



$a<0$

例 1.7

$$\sqrt{2x+5} > \frac{x}{2}$$



交点Aを求める

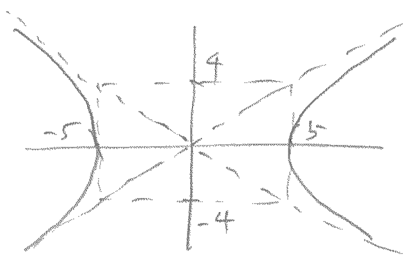
$$\sqrt{2x+5} = \frac{x}{2} \qquad 2x+5 = \frac{x^2}{4} \qquad \underbrace{x^2-8x-20}_{(x-10)(x+2)} = 0$$

$x = 10, \cancel{-2}$　　　図より　$-\frac{5}{2} \leqq x < 10$

例 1.8

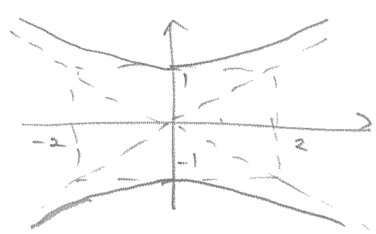
(1) $\frac{x^2}{25} - \frac{y^2}{16} = 1$

$y=0$ のとき $x=\pm 5$



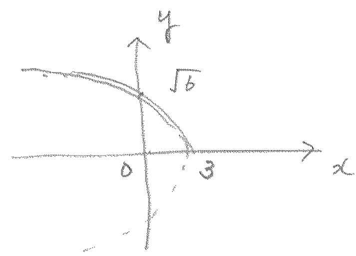
(2) $\frac{x^2}{4} - y^2 = -1$

$x=0$ のとき $y=\pm 1$



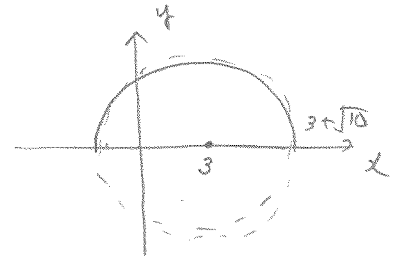
問1-7.

(1) $y=\sqrt{6-2x}=\sqrt{-2(x-3)}$



(2) $y=\sqrt{-x^2+6x+1}$

$=\sqrt{-(x-3)^2+10}$



$y \geqq 0.$ $\quad y^2=-(x-3)^2+10$ $\quad (x-3)^2+y^2=10$

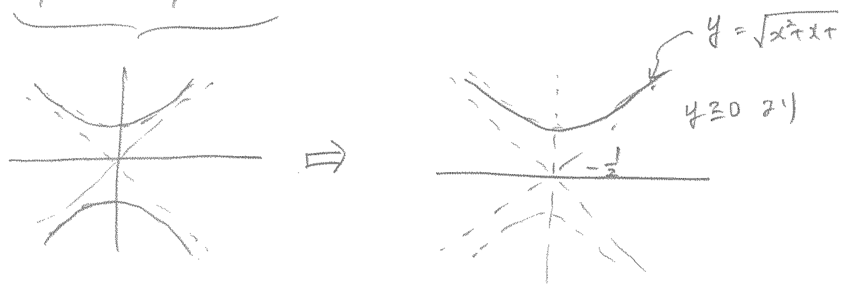
(3) $y=\sqrt{x^2+x+1}$

$=\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}$

$y \geqq 0$ にて $\quad y^2=(x+\frac{1}{2})^2+\frac{3}{4}$ $\quad (x+\frac{1}{2})^2-y^2=-\frac{3}{4}$

$$\frac{(x+\frac{1}{2})^2}{\frac{3}{4}}-\frac{y^2}{\frac{3}{4}}=-1.$$

双曲線 $\dfrac{x^2}{\frac{3}{4}}-\dfrac{y^2}{\frac{3}{4}}=-1$ を x 方向に $-\frac{1}{2}$ 平行移動



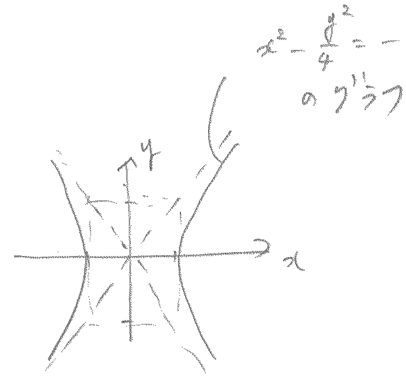
(4) $y = 2\sqrt{(x-1)^2-1}$

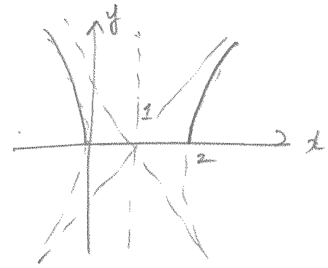
$y \geqq 0$ $\quad \dfrac{y^2}{4} = \cancel{4}\{(x-1)^2-1\}$

$(x-1)^2 - \dfrac{y^2}{4} = 1.$

は $x^2 - \dfrac{y^2}{4} = -1$ のグラフを

x方向に1平行移動.



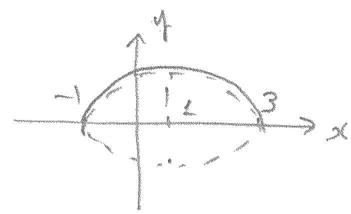


(5) $2y = \sqrt{3+2x-x^2}$

$y \geqq 0$ として $\quad 4y^2 = 3+2x-x^2$

$\quad\quad = 4-(x-1)^2$

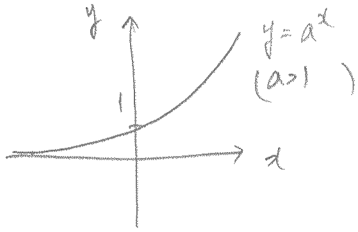
$(x-1)^2 + 4y^2 = 4 \Rightarrow \dfrac{(x-1)^2}{4} + y^2 = 1$

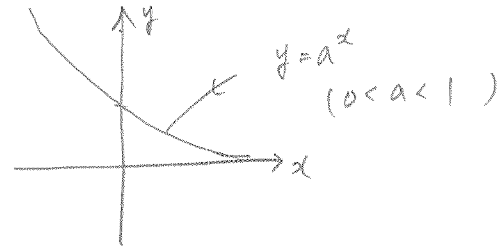


$\boxed{1.5}$ 指数関数，対数関数

◇ 指数関数　　$a>0,\ a\neq 1$ のとき

$$y = a^x \quad \cdots\cdots$$ a を底とする指数関数という





◇ 指数法則

$$a^m a^n = \underbrace{\underbrace{a \cdots a}_{m\text{回}}\ \underbrace{a \cdots a}_{n\text{回}}}_{m+n\text{回}} = a^{m+n}$$

$$\frac{a^m}{a^n} = \frac{\cancel{a}\cdots\cancel{a}\ a\cdots a}{\underbrace{\cancel{a}\cdots\cancel{a}}_{n\text{回}}} = a^{m-n}$$

$$a^0 = \frac{a^m}{a^m} = 1.$$

$$(a^m)^n = \underbrace{a^m \cdots a^m}_{a\text{が}\ m\times n\text{回}} = a^{mn}$$

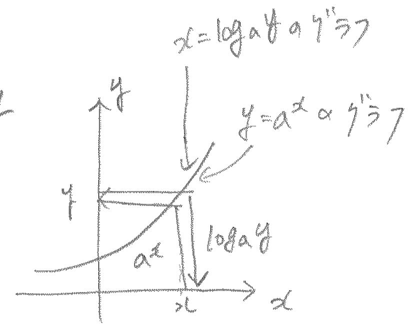
$$(ab)^m = ab\,ab \cdots ab = a^m b^m$$

m, n が実数でも成立する！

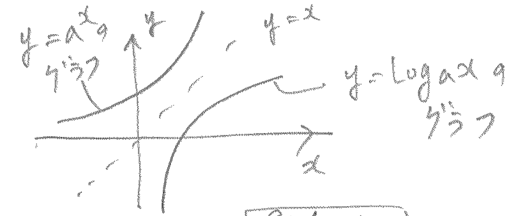
$$a^{m/n} = \sqrt[n]{a^m}\ ,\quad \text{特に}\quad a^{\frac{1}{n}} = \sqrt[n]{a}$$

◇ 対数関数

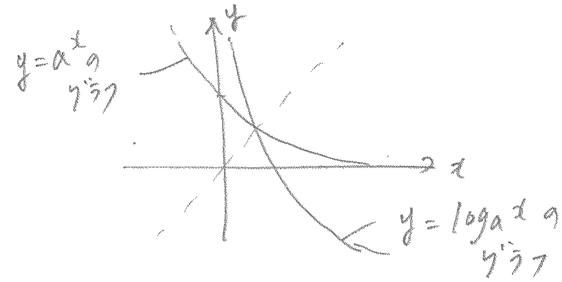
$y=a^x$ の逆関数を $x=\log_a y$ と書く。



$y=\log_a x$ の $\begin{cases} \text{定義域は } x>0 \\ \text{値域は } (-\infty,\infty) \end{cases}$



$\boxed{a>1 \text{ のとき}}$



$\boxed{0<a<1 \text{ のとき}}$

◇ 対数の性質：指数法則から出てくる

○ $M=a^{\log_a M}$，$N=a^{\log_a N}$ より $MN=a^{\log_a M+\log_a N}$

また $MN=a^{\log_a(MN)}$ よって $\underline{\log_a MN=\log_a M+\log_a N}$

○ $1=a^{\log_a 1}$，$1=a^0$ よって $\underline{\log_a 1=0}$

○ $a=a^{\log_a a}$，$a=a^1$ よって $\underline{\log_a a=1}$

○ $M^r=a^{\log_a M^r}$ 一方 $M^r=(a^{\log_a M})^r=a^{r\log_a M}$

よって $\underline{\log_a M^r=r\log_a M}$

∘ $\dfrac{M}{N} = a^{\log_a \frac{M}{N}}$ 　一方 $\dfrac{M}{N} = \dfrac{a^{\log_a M}}{a^{\log_a N}} = a^{\log_a M - \log_a N}$

よって $\log_a \dfrac{M}{N} = \log_a M - \log_a N$

∘ $b = a^{\log_a b} = \left(c^{\log_c a}\right)^{\log_a b} = c^{\log_a b \cdot \log_c a}$

一方 $b = c^{\log_c b}$

よって $\log_a b \log_c a = \log_c b$

$\Rightarrow$ $\log_a b = \dfrac{\log_c b}{\log_c a}$ 　底の変換

◇ 自然対数

$\lim_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n = e = 0.278\ldots$ 　ネイピア数

$\log_e x = \log x$ $\leftarrow$ 自然対数

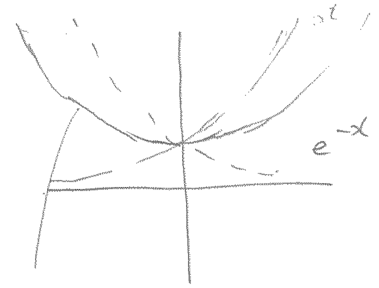
◇ $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ 増加関数

$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ 減少 〃 　単調関数

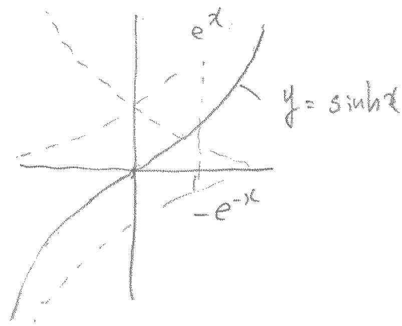
◇ 双曲線関数

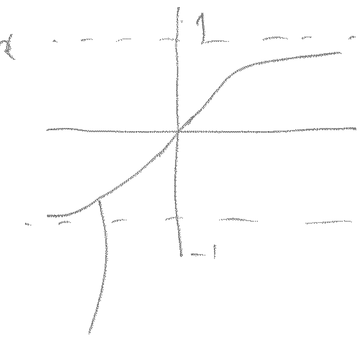
$$\cosh x = \frac{e^x+e^{-x}}{2}, \quad \sinh x = \frac{e^x-e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x-e^{-x}}{e^x+e^{-x}}$$

(ハイパボリックコサイン)　(〃サイン)　(〃タンジェント)



$y = \cosh x$





$y = \tanh x$

例1.9. $a^x = e^{x\log a}$ を示せ.

$a = e^{\log a}$ より $a^x = (e^{\log a})^x = e^{x\log a}$ //

例1.10 $\log_{10}$ を用いて $\sqrt{2}$ と $\sqrt[3]{3}$ の大小

$$\log_{10}\sqrt{2} - \log_{10}\sqrt[3]{3} = \frac{1}{2}\log_{10}2 - \frac{1}{3}\log_{10}3$$

$$= \frac{1}{6}\left(\log_{10}2^3 - \log_{10}3^2\right) = \frac{1}{6}\underbrace{\boxed{\log_{10}\frac{8}{9}}}_{\text{マイナス}} < 0$$

よって $\log_{10}\sqrt{2} < \log_{10}\sqrt[3]{3}$

$$\underline{\sqrt{2} < \sqrt[3]{3}}.$$

問1-8.

$8^n > 10^{100}$ となる最小の n は？.

$$\log_{10} \underset{\substack{\shortparallel \\ 2^{3n}}}{\boxed{8^n}} > \log_{10} 10^{100}$$

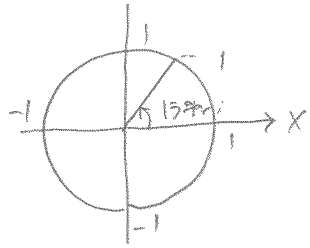
$$3n \underbrace{\log_{10} 2}_{0.3010} > 100$$

$$n > \frac{100}{3 \times 0.3010} = 110.7..$$

$\underline{n = 111}$ 回.

1.6 三角関数

◇ 弧度法

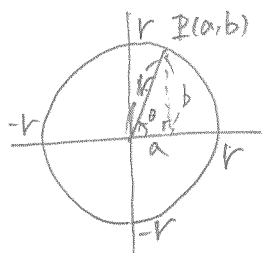


1周 360° の円周の長さは 2π

⇒ $360^\circ = 2\pi$ [ラジアン]

$180^\circ = \pi$ ラジアン

◇ 三角関数

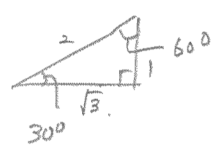


$\cos\theta = a/r$

$\sin\theta = b/r$

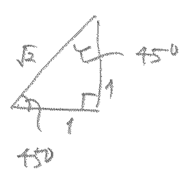
$\tan\theta = \dfrac{b}{a}$

・ $\theta = \dfrac{\pi}{6}$ のとき



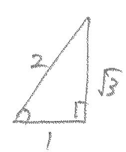
$\sin\dfrac{\pi}{6} = \dfrac{1}{2}$ 　 $\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$, $\tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}$

・ $\theta = \dfrac{\pi}{4}$ のとき



$\sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$, $\cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$, $\tan\dfrac{\pi}{4} = 1$

・ $\theta = \dfrac{\pi}{3}$ のとき

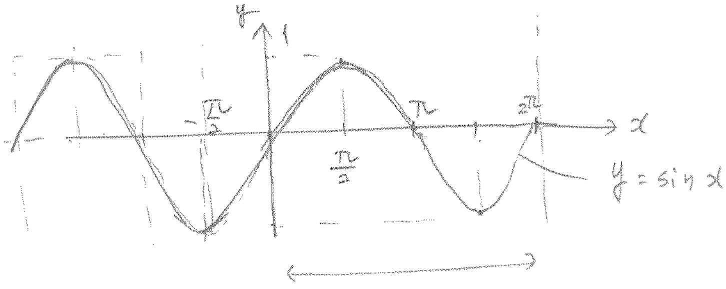


$\sin\dfrac{\pi}{3} = \dfrac{1}{2}$, $\cos\dfrac{\pi}{3} = \dfrac{1}{2}$, $\tan\dfrac{\pi}{3} = \sqrt{3}$

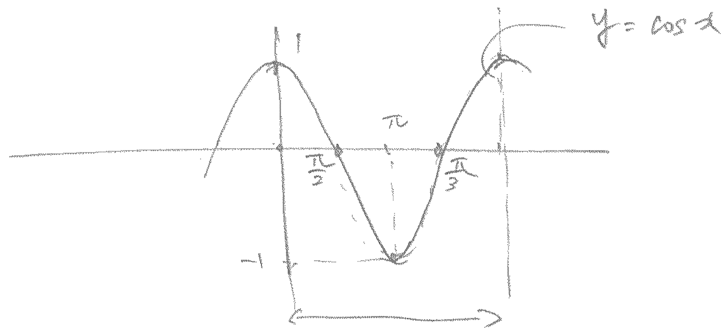
◇ 三角関数の逆数

$\mathrm{cosec}\,\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, $\cot\theta = \dfrac{1}{\tan\theta}$

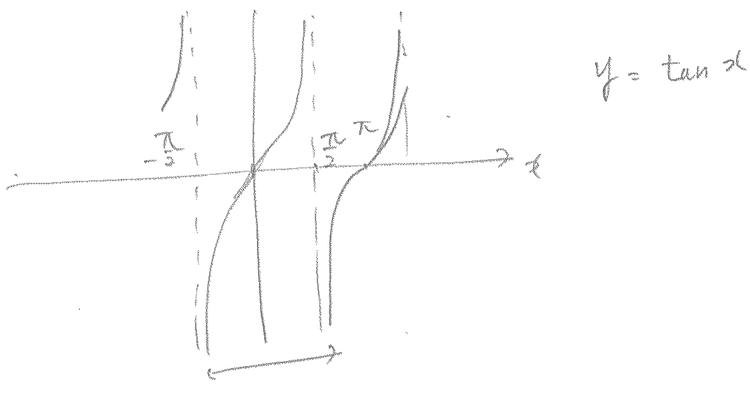
◇ 三角関数のグラフ



周期 2π



周期 2π



周期 π

◇ 周期 p： $f(x+p)=f(x)$ が成立する $p\ (\neq 0)$ のこと

任意の x で

◇ 基本公式

1. 相互関係 $\tan x = \dfrac{\sin x}{\cos x}$, $\sin^2 x + \cos^2 x = 1$

※ $\sin^2 x = (\sin x)^2$ の略記

両辺を $\cos^2 x$ でわると

$$\tan^2 x + 1 = \frac{1}{\cos^2 x} = \sec^2 x$$

2. 偶奇性 グラフより

$$\begin{cases} \sin(-x) = -\sin x & \text{奇} \\ \cos(-x) = \cos x & \text{偶} \\ \tan(-x) = -\tan x & \text{奇} \end{cases}$$

3. 周期性

$$\sin(x + 2\pi n) = \sin x$$

$$\cos(x + 2\pi n) = \cos x$$

$$\tan(x + \pi n) = \tan x$$

4. 加法定理

(1)

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

(2)

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

5. 倍角の公式

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 y = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

6. 半角の公式

$$\cos x = 2\cos^2 \frac{x}{2} - 1 \Rightarrow \underline{\cos^2 \frac{x}{2} = \frac{1+\cos x}{2}} \;//$$

$$\cos x = 1 - 2\sin^2 \frac{x}{2} \Rightarrow \underline{\sin^2 \frac{x}{2} = \frac{1-\cos x}{2}} \;//$$

7 和・差を積に

(6) $$\sin(x+y) = \sin x\cos y + \cos x \sin y$$

$$\qquad - \qquad\qquad\qquad\quad -$$

+)

$$\sin(\underbrace{\boxed{x+y}}_{\alpha}) + \sin(\underbrace{\boxed{x-y}}_{\beta}) = 2\sin x\cos y$$

$$x = \frac{\alpha+\beta}{2}$$

$$y = \frac{\alpha-\beta}{2}$$

$$\Rightarrow \quad \sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

(7) $$\sin(x+y) - \sin(\alpha-\beta) = 2\cos x\sin y$$

$$\Rightarrow \quad \sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

(8) $$\cos(x+y) = \cos x\cos y - \sin x \sin y$$

$$\qquad - \qquad\qquad\qquad\quad +$$

$$\cos(x+y) + \cos(x-y) = 2\cos x\cos y$$

$$\Rightarrow \quad \cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

(9) $$\cos(x+y) - \cos(x-y) = -2\sin x\sin y$$

$$\Rightarrow \quad \cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

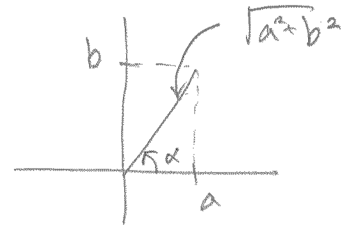
8 積和(差)公式

(10) $$\sin A\cos B = \frac{1}{2}\left\{\sin(A+B) + \sin(A-B)\right\}$$

(11) $$\cos A\cos B = \frac{1}{2}\left\{\cos(A+B) + \cos(A-B)\right\}$$

(12) $$\sin A\sin B = \frac{-1}{2}\left\{\cos(A+B) - \cos(A-B)\right\}$$

9. 三角関数の合成



$$a\sin x + b\cos x$$

$$= \sqrt{a^2+b^2}\,\{\underbrace{\sin x\cos\alpha + \cos x\sin\alpha}_{\sin(x+\alpha)}\}$$

$$= \underline{\sqrt{a^2+b^2}\,\sin(x+\alpha)}$$ //

$$a = \sqrt{a^2+b^2}\cos\alpha$$

$$b = \sqrt{a^2+b^2}\sin\alpha$$

問1.9 済

問1.10 $\tan\frac{x}{2} = t$ とおくとき, $\sin x$, $\cos x$ を t で表せ

$$\sin\left(\frac{x}{2}\cdot 2\right) = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\overbrace{\boxed{\tan\frac{x}{2}}}^{t}\cos^2\frac{x}{2} = \underline{\frac{2t}{t^2+1}}$$ 答)

$$\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2} = 1 \qquad \underbrace{\tan^2\frac{x}{2}}_{t^2} + 1 = \frac{1}{\cos^2\frac{x}{2}} \Rightarrow \cos^2\frac{x}{2} = \frac{1}{t^2+1}\right)$$

$$\cos\left(2\cdot\frac{x}{2}\right) = 2\boxed{\cos^2\frac{x}{2}} - 1 = 2\,\frac{1}{t^2+1} - 1 = \frac{2-t^2-1}{t^2+1} = \underline{\frac{1-t^2}{1+t^2}}$$ 答)

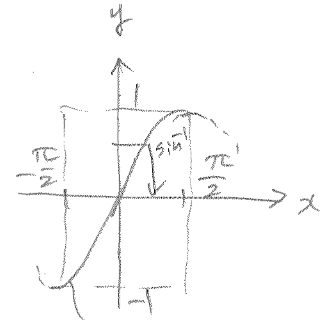
1.7 逆三角関数

◇ $y = \sin x \quad \left(-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2},\ -1 \leqq y \leqq 1\right)$

の逆関数は、

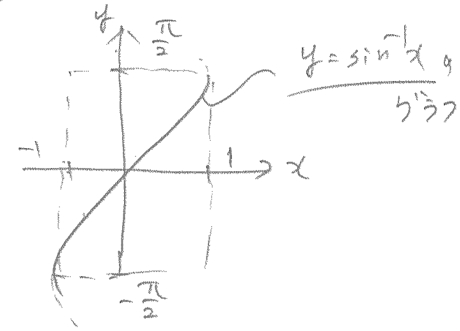
$x = \sin^{-1} y \quad \left(-1 \leqq y \leqq 1,\ -\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}\right)$

アークサイン、インバースサイン と読む



$\begin{cases} y = \sin x \\ x = \sin^{-1} y \end{cases}$ のグラフ

$y = \sin^{-1} x \quad \left(-1 \leqq x \leqq 1,\ -\frac{\pi}{2} \leqq y \leqq \frac{\pi}{2}\right)$ のグラフ



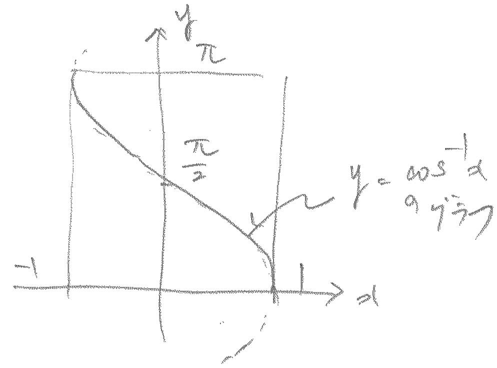
$y = \cos x \quad (0 \leqq x \leqq \pi,\ -1 \leqq y \leqq 1)$

の逆関数は、

$x = \cos^{-1} y \quad (-1 \leqq y \leqq 1,\ 0 \leqq x \leqq \pi)$

アークコサイン、インバースコサイン とよむ

$y = \cos^{-1} x$ のグラフは

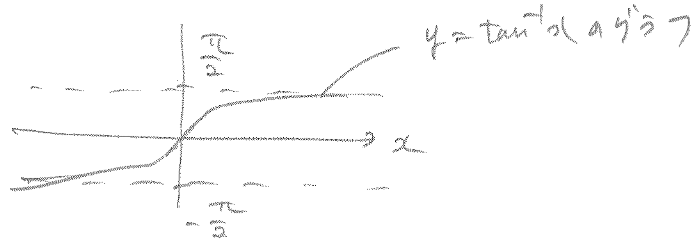


$y = \tan x$ $\left(-\frac{\pi}{2} < x < \frac{\pi}{2},\ -\infty < y < \infty\right)$ の逆関数は、

$x = \underset{\sim\sim}{\tan^{-1}} y$ $\left(-\infty < y < \infty,\ -\frac{\pi}{2} < x < \frac{\pi}{2}\right)$

アークタンジェント，インバースタンジェントとよむ

$y = \tan^{-1} x$ のグラフは、

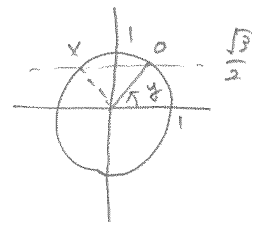


例 1.11.

(1) $\sin^{-1}\frac{\sqrt{3}}{2} = y$?　　$-\frac{\pi}{2} \leqq y \leqq \frac{\pi}{2}$ であり、

$\sin\left(\sin^{-1}\frac{\sqrt{3}}{2}\right) = \sin y$. ⇒　$\sin y = \frac{\sqrt{3}}{2}$

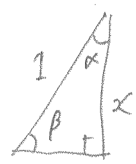
$\underline{y = \frac{\pi}{3}}$ (答)



(2) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

別解.

$0 \leqq x \leqq 1$ のときは.



$\sin^{-1}x = \beta$

$\cos^{-1}x = \alpha$

よって　$\alpha + \beta + 90^\circ = 180^\circ$　　$\alpha + \beta = 90^\circ = \frac{\pi}{2}$

$-1 \leqq x \leqq 0$ のときは　$-x = x'$ とおくと　$0 \leqq x' \leqq 1$ で

$\sin^{-1}x = \sin^{-1}(-x') = -\sin^{-1}x'$
$\cos^{-1}x = \cos^{-1}(-x') = \pi - \cos^{-1}x'$
} グラフより明らか.

$\frac{\cos^{-1}(-x') + \cos^{-1}x'}{2} = \frac{\pi}{2}$

$\pi - \sin^{-1}x' - \cos^{-1}x' = \frac{\pi}{2}$

$\sin^{-1}x' + \cos^{-1}x' = \frac{\pi}{2}$　上記で示してある　図

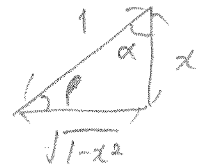
$\sin^{-1}x = y$　　$-\frac{\pi}{2} \leqq y \leqq \frac{\pi}{2}$

$z = \frac{\pi}{2} - y$　　$0 \leqq z \leqq \pi$

$\cos z = \underset{\substack{\parallel \\ \sin y \\ \parallel \\ x}}{\cos\left(\frac{\pi}{2} - y\right)}$　⇒　$z = \cos^{-1}x$　つまり $\frac{\pi}{2} - y = \cos^{-1}x$

$\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

(3) $0 \leqq x \leqq 1$ のとき $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$



$$\left.\begin{array}{l} \alpha = \cos^{-1}x \\ \alpha = \sin^{-1}\sqrt{1-x^2} \end{array}\right\} \Rightarrow \underline{\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}}$$

問1.1

(1) $\underbrace{\cos^{-1}\left(-\frac{1}{2}\right)}_{120^\circ} + \underbrace{\tan^{-1}\left(-\sqrt{3}\right)}_{-60^\circ} + \underbrace{\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)}_{-45^\circ} = 120 - 105 = 15^\circ = \underline{\frac{\pi}{12}}$

(2) $2\underbrace{\sin^{-1}1}_{\frac{\pi}{2}} - \underbrace{\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)}_{135^\circ} + \underbrace{\tan^{-1}(-1)}_{-45^\circ} + \underbrace{\tan^{-1}0}_{0^\circ}$

$= 180^\circ - 180^\circ = \underline{0}$

(3) $\underbrace{\sin^{-1}(-1)}_{-90^\circ} + \underbrace{\cos^{-1}\frac{\sqrt{3}}{2}}_{30^\circ} - \underbrace{\tan^{-1}1}_{45^\circ} + \underbrace{\sin^{-1}0}_{0^\circ} = -90^\circ + 30^\circ - 45^\circ$

$= \underline{-105^\circ} = -\frac{\cancel{105}\ \cancel{21}\ 7}{\cancel{180}\ \cancel{36}\ 12}\pi = \underline{-\frac{7}{12}\pi}$

(4) $\underbrace{\sin\left(\underbrace{\cos^{-1}\frac{\sqrt{3}}{2}}_{30^\circ}\right)}_{\frac{1}{2}} + \underbrace{\cos\ \underbrace{\tan^{-1}-\frac{1}{\sqrt{3}}}_{-30^\circ}}_{\frac{\sqrt{3}}{2}} + \underbrace{\overset{\tan}{\cancel{\ }}\left(\underbrace{\sin^{-1}\frac{-1}{\sqrt{2}}}_{-45^\circ = -\frac{\pi}{4}}\right)}_{-1}$

$= \underline{\frac{1+\sqrt{3}}{2} - 1}$

$= \underline{\frac{\sqrt{3}-1}{2}}$

1.8 関数の極限

◇ Aに収束する場合

xがaに限りなく近づくとき，$f(x)$が一定の値Aに近づくとき

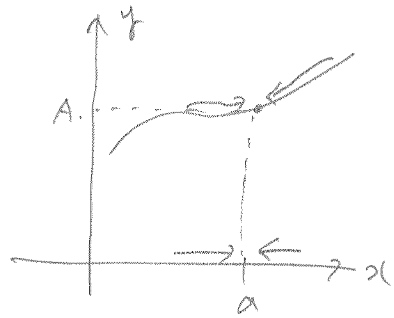
$$f(x) \longrightarrow A \quad (x \to a)$$

or

$$\lim_{x \to a} f(x) = A$$ と表す。

A：極限値という

「$x \to a$ のとき $f(x)$ は A に収束する」という



定理 1.1

$$\lim_{x \to a} f(x) = A, \quad \lim_{x \to a} g(x) = B$$

⇒ (1) $\displaystyle\lim_{x \to a} \{f(x) \pm g(x)\} = A \pm B$

$f(x) \to A$, $g(x) \to B$

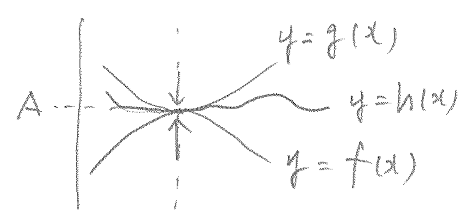
(2) $\displaystyle\lim_{x \to a} \boxed{f(x)}\, g(x) = AB$

$f(x) \to A$, $g(x) \to B$

(3) $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{A}{B}$ （$B \neq 0$ のとき）

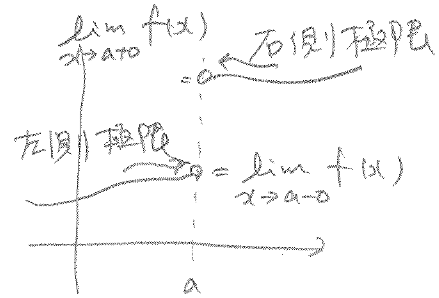
(4) xがaの近くで $f(x) \leqq h(x) \leqq g(x)$ で

$\displaystyle\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = A$ のとき



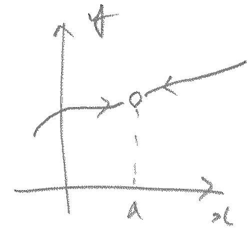
$\lim h(x) = A$

◇ 右側極限値，左側極限値



$$A = \lim_{x\to a} f(x) \iff \begin{cases} \lim_{x\to a+0} f(x) = A \\ \text{かつ} \\ \lim_{x\to a-0} f(x) = A \end{cases}$$

$a = 2.$



例題 1.12

(1) $\displaystyle \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = \lim_{x\to 2} (x+2) = \underline{4}.$

注 $x \neq 2$ で限りなく近づける

(2) $\displaystyle \lim_{x\to 1-0} \frac{x^2-1}{|x-1|} = \lim_{x\to 1-0} \frac{\cancel{(x-1)}(x+1)}{-\cancel{(x-1)}} = -(1+1) = \underline{-2}.$

$x<1$ なので $|x-1| = -(x-1)$

(3) $\displaystyle \lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$ ($\frac{0}{0}$ 不定形)

⇓

くふう 分子有理化

$$\frac{(\sqrt{1+x} - \sqrt{1-x})(\sqrt{1+x} + \sqrt{1-x})}{x\,(\sqrt{1+x} + \sqrt{1-x})}$$

$$= \frac{1+x-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})} = \frac{2\cancel{x}}{\cancel{x}(\sqrt{1+x}+\sqrt{1-x})} \to \frac{2}{\sqrt{1}+\sqrt{1}} = 1.$$

問 1.12.

(1) $\lim_{x \to 1} \dfrac{x^3-2x^2+2x-1}{x^2-3x+2}$

$x^3-2x^2+2x-1 = \underbrace{x^3-1}_{(x-1)(x^2+x+1)} - 2x(x-1) = (x-1)\{x^2+x+1-2x\}$

$= (x-1)(x^2-x+1)$

$x^2-3x+2 = (x-1)(x-2)$

$\lim_{x \to 1} \dfrac{x^2-x+1}{x-2} = \dfrac{1-1+1}{-1} = \underline{-1}.$

(2) $\lim_{x \to 4} \dfrac{\sqrt{x}-2}{x-4} = \lim_{x \to 4} \dfrac{\cancel{(\sqrt{x}-2)(\sqrt{x}+2)}}{\cancel{(x-4)}(\sqrt{x}+2)} = \dfrac{1}{2+2} = \underline{\dfrac{1}{4}}.$

(3) $\lim_{x \to 0} \dfrac{1-\cos x}{x}$

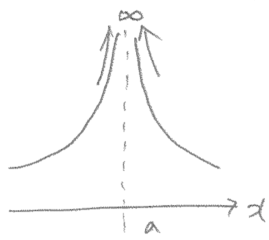
メモ $\boxed{\sin^2\theta = \dfrac{1-\cos 2\theta}{2}}$

$\dfrac{1-\cos x}{x} = \dfrac{2}{x}\sin^2\dfrac{x}{2} = \underbrace{\boxed{\dfrac{\sin\frac{x}{2}}{\frac{x}{2}}}}_{\downarrow \, 1} \cdot \underbrace{\sin\dfrac{x}{2}}_{\downarrow \, 0} \to \underline{0}.$

メモ $\boxed{\lim_{x \to 0} \dfrac{\sin x}{x} = 1}$

(4) $\lim_{x \to -0} \dfrac{x}{|x|} = \lim_{x \to -0} \dfrac{\cancel{x}}{-\cancel{x}} = \underline{-1}.$

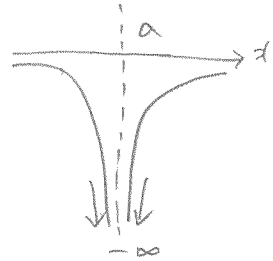
◇ 無限大に発散する場合



$\lim_{x \to a} f(x) = \infty$

$f(x) \to \infty \ (x \to a)$

「正の無限大に発散する」という



$\lim_{x \to a} f(x) = -\infty$

$f(x) \to -\infty \ (x \to a)$

「負の無限大に 〃 」という

◇ 極限値なし

(例1) $y = \sin \frac{1}{x}$

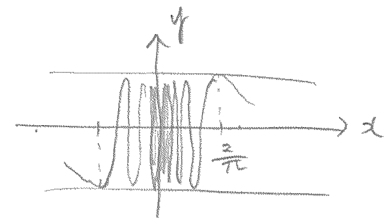
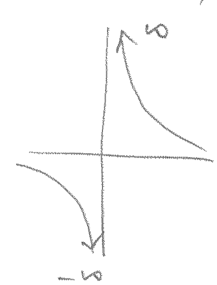


図 1.37

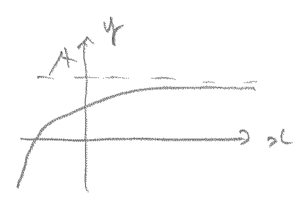
(例2) $y = \frac{1}{x}$



$\lim_{x \to +0} f(x) = \infty$

$\lim_{x \to -0} f(x) = -\infty$

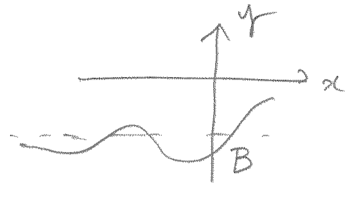
◇ $x \to \infty$, $x \to -\infty$ の場合



$x \to \infty$ のとき $f(x) \to A$ のとき

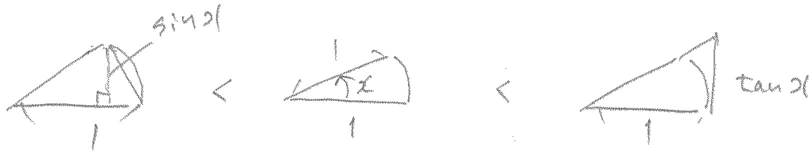
$\lim_{x \to \infty} f(x) = A$ と書く.

$\lim_{x \to -\infty} f(x) = B$ も同様



定理1.2

(1) $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}=1.$



$$\Rightarrow \quad \frac{\sin x}{2} \underset{①}{<} \frac{x}{2} \underset{②}{<} \frac{\tan x}{2}$$

$$\left.\begin{array}{ll} ① \Rightarrow & \dfrac{\sin x}{x} < 1 \\[2ex] ② \Rightarrow & \cos x < \dfrac{\sin x}{x} \end{array}\right\} \quad \cos x < \frac{\sin x}{x} < 1$$

$$\cos x \to 1 \quad (x \to +0).$$

よって $\displaystyle\lim_{x\to +0}\frac{\sin x}{x}=1.$

$$\lim_{x\to -0}\frac{\sin x}{x} = \lim_{z\to +0}\frac{\sin(-z)}{-z} = \lim_{z\to +0}\frac{\sin z}{z} = 1$$

$$\underline{\lim_{x\to 0}\frac{\sin x}{x} = 1}.$$

(2) $\displaystyle\lim_{x \to \pm\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x \to 0}(1+x)^{1/x} = e$

∵ 追記1-1

$$e = \lim_{n \to \infty}\left(1+\frac{1}{n}\right)^n = \lim_{x \to \infty}\left(1+\frac{1}{x}\right)^x = \lim_{z \to +0}(1+z)^{1/z} \quad \cdots\cdots ①$$

$$e = \lim_{n \to -\infty}\left(1-\frac{1}{n}\right)^{-n} = \lim_{n \to -\infty}\left(\frac{n-1}{n}\right)^{-n} = \lim_{n \to -\infty}\left(\frac{n}{n-1}\right)^n$$

$$= \lim_{n \to -\infty}\left(1+\frac{1}{n-1}\right)^n = \lim_{n \to -\infty}\left\{\left(1+\frac{1}{n-1}\right)^{n-1}\cdot\underbrace{\left(1+\frac{1}{n-1}\right)}_{\downarrow\ 1}\right\}$$

$$= \lim_{n \to -\infty}\left(1+\frac{1}{n}\right)^n = \lim_{x \to -\infty}\left(1+\frac{1}{x}\right)^x = \lim_{z \to -0}(1+z)^{1/z} \quad \cdots\cdots ②$$

①, ② より

$$e = \lim_{z \to 0}(1+z)^{1/z} = \lim_{x \to \infty}\left(1+\frac{1}{x}\right)^x = \lim_{x \to -\infty}\left(1+\frac{1}{x}\right)^x$$

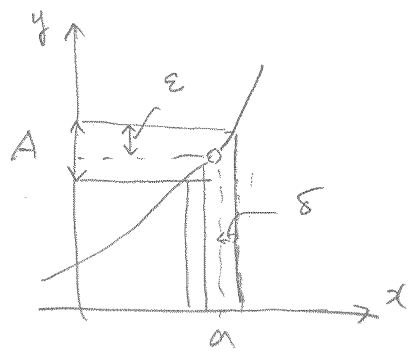
//

追記1.2　　$\varepsilon-\delta$ 論法

$$\lim_{x\to a} f(x) = A \iff$$ 「x が a に限りなく近づくとき $f(x)$ が A に限りなく近づく」

をより精密に述べると、

「任意に与えられた $\varepsilon\,(>0)$ に対して、
適当な δ をとると、$0<|x-a|<\delta$ なすべての x に対して、
$|f(x)-A|<\varepsilon$ となる」



例. 1-13

(1) $\displaystyle\lim_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)$

$\displaystyle=\lim_{z\to\infty}\left(\sqrt{z^2-z+1}-z\right)$

$\displaystyle=\lim_{z\to\infty}\frac{\left(\sqrt{z^2-z+1}\right)^2-z^2}{\left(\sqrt{z^2-z+1}+z\right)}=\lim_{z\to\infty}\frac{-z+1}{\sqrt{z^2-z+1}+z}$

$\displaystyle=\lim_{z\to\infty}\frac{-1+\frac{1}{z}}{\sqrt{1-\frac{1}{z}+\frac{1}{z^2}}+1}=\frac{-1}{1+1}=\underline{-\frac{1}{2}}.$

(2) $\displaystyle\lim_{x\to\infty}e^{-x}\sin x$

$-1\leqq\sin x\leqq 1$ より

$$\underset{\substack{\downarrow\\0}}{-e^{-x}}\leqq e^{-x}\sin x\leqq\underset{\substack{\downarrow\\0}}{e^{-x}}$$

より

$\displaystyle\lim_{x\to\infty}e^{-x}\sin x=0$

(3). $\displaystyle\lim_{x\to0}\sin\frac{1}{x}=$ 存在しない 補

TexT 18P
1-1. 42p.

$$\begin{array}{r} 2.33 \\ 18\,\overline{)\,42} \\ 36 \\ \hline 60 \\ 54 \\ \hline 60 \end{array}$$

問 1.13.

(1) $\displaystyle \lim_{x\to\infty} \frac{3x^2-6x-1}{-x^2-4x+2} = \lim_{x\to\infty} \frac{3-\frac{6}{x}-\frac{1}{x^2}}{-1-\frac{4}{x}+\frac{2}{x^2}} = -3$

(2) $\displaystyle \lim_{x\to 0} \frac{1}{x^3}$ 存在しない

(3) $\displaystyle \lim_{x\to 0} \frac{\sin ax}{x} = \lim_{x\to 0} a \underbrace{\frac{\sin ax}{ax}}_{\downarrow\, 1} = a$

(4) $\displaystyle \lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{x\to 0} \log \underbrace{\boxed{(1+x)^{\frac{1}{x}}}}_{\downarrow\, e} = \underline{1}$

(5) $\displaystyle \lim_{x\to 0} \frac{\boxed{e^x-1}^{\,\to k}}{x} =$

$e^x - 1 = k$ とおくと $k \to 0$

$k+1 = e^x \qquad x = \log(k+1)$

$\displaystyle = \lim_{k\to 0} \frac{k}{\log(1+k)} = \lim_{k\to 0} \frac{1}{\underbrace{\boxed{\frac{1}{k}\log(1+k)}}_{\downarrow\, 1}} = \underline{1}$

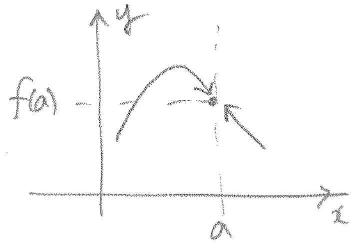
(6) $\displaystyle \lim_{x\to 0} (1+ax)^{1/x} = \lim_{x\to 0} (1+ax)^{\frac{a}{ax}}$

$\displaystyle = \lim_{x\to 0} \left\{\underbrace{\boxed{(1+ax)^{\frac{1}{ax}}}}_{\downarrow\, e}\right\}^a = \underline{e^a}$

(7) $\displaystyle \lim_{x\to 0} \frac{\tan ax}{\tan bx} = \lim_{x\to 0} \underbrace{\boxed{\frac{\cos bx}{\cos ax}}}_{\downarrow\, 1} \cdot \underbrace{\boxed{\frac{\sin(ax)/ax}{\sin bx/bx}}}_{\downarrow\, 1} \cdot \frac{ax}{bx} = \underline{\frac{a}{b}}$

1.9　関数の連続性

◇ 1点で連続な関数

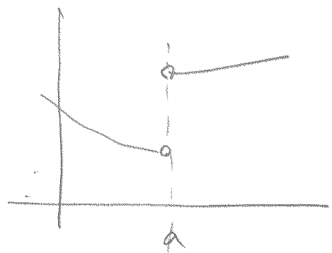


$\lim_{x \to a} f(x) = f(a)$ のとき

「$f(x)$ は $x=a$ で連続である」という.

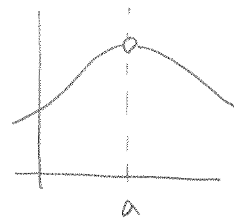
$x=a$ で不連続な場合

(i)

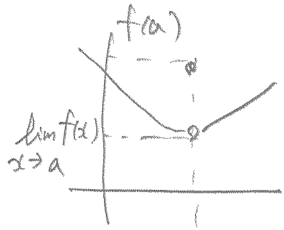


$\lim_{x \to a+0} f(x) \neq \lim_{x \to a-0} f(x)$

のとき

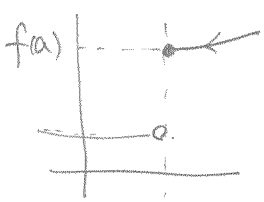


$f(a)$ が定義されていないとき



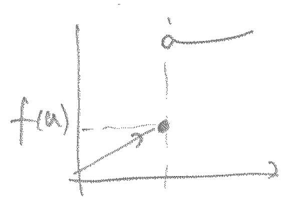
$\lim_{x \to a} f(x) \neq f(a)$

◇ 右側連続, 左側連続



$\lim_{x \to a+0} f(x) = f(a)$

のとき



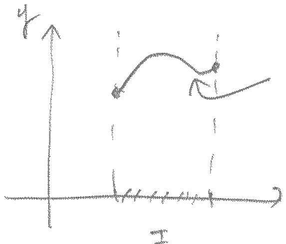
$\lim_{x \to a-0} f(x) = f(a)$

のとき

片側連続

◇ 区間 I で連続

すべての $a \in I$ で $f(x)$ が連続なとき、「区間 I で連続」という

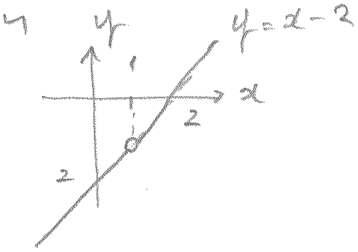


きれ目のない曲線

例1.14　不連続にする x は?

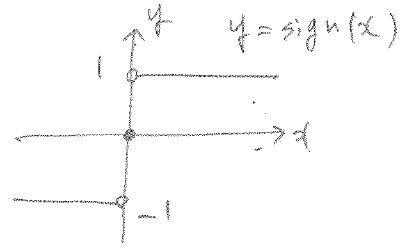
(1) $\dfrac{x^2-3x+2}{x-1}$　　$x=1$ では定義されない

$x \neq 1$ では $\dfrac{\cancel{(x-1)}(x-2)}{\cancel{(x-1)}} = x-2$



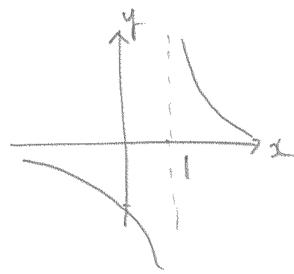
$x=1$ で不連続

(2) sign(x)
符号



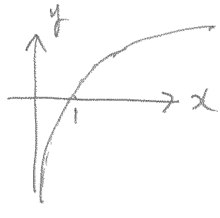
$x=0$ で不連続

(3) $\dfrac{3}{x-1}$



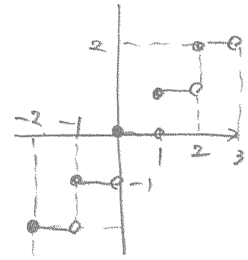
$x=1$ で不連続

(4) $\log x$



$x=0$ で不連続

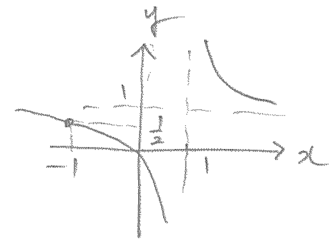
(5) $[x]$
↑
xの"整数部分"
$[1.1]=1$.
$[-1.2]=-2$.



$x=n$(整数)
で不連続

問 1.14.

(1) $f(x) = \begin{cases} \dfrac{x(x+1)}{x^2-1} & x \neq -1 \\ \dfrac{1}{2} & x = -1 \end{cases}$



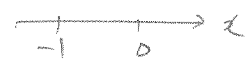
$x \neq -1$ のとき

$$f(x) = \frac{x\cancel{(x+1)}}{(x-1)\cancel{(x+1)}} = 1 + \frac{1}{x-1}$$

$x = 1$ で不連続

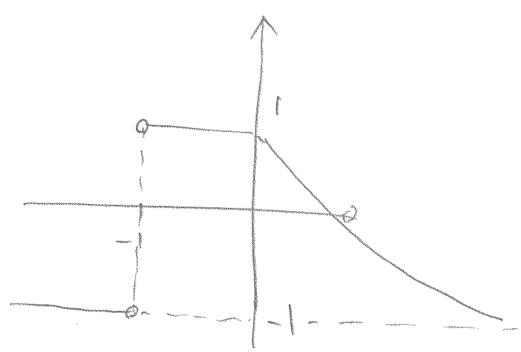
$x \in (-\infty, 1)$
$x \in (1, \infty)$ で連続

(2) $f(x) = \dfrac{1-|x|}{|1+x|} = \begin{cases} (x \geqq 0) & \dfrac{1-x}{1+x} \\ (-1 < x < 0) & \dfrac{1+x}{1+x} = 1 \\ (x = -1) & f(x) \text{ 存在しない} \end{cases}$



$(x < -1)$

$$\frac{1+x}{-(1+x)} = -1$$



$x = -1$ で不連続

$(-\infty, -1)$
$(-1, \infty)$ で連続

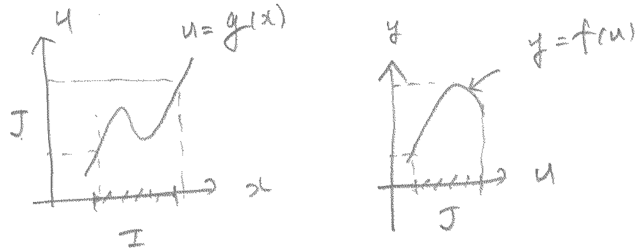
定理 1.3.

$f(x), g(x)$ が 区間 I で 連続

$\Rightarrow$ $kf(x)$, $f(x) \pm g(x)$, $f(x)g(x)$, $\dfrac{f(x)}{g(x)}$ (ただし $g(x) \neq 0$)

も 区間 I で 連続

定理 1.4.



$\Rightarrow$ $y = f(g(x))$ は. $x \in I$ において 連続

合成関数

$\because$ $\forall a \in I$ に対して. $x \to a$ なら $u \to g(a) = u_0$

$u \to u_0$ なら $y \to f(u_0) = f(g(a))$

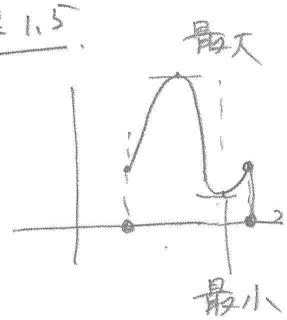
つまり

$x \to a$ なら $y \to f(g(a))$

よって $y = f(g(x))$ は. I で 連続.

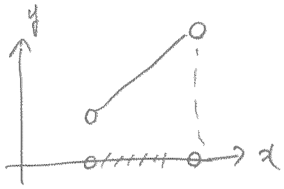
◇ 閉区間で連続な関数

定理1.5

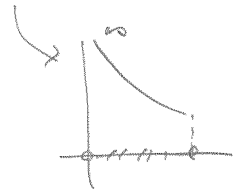


最大値・最小値をとる.

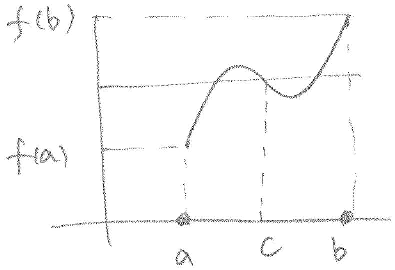
(注)



開区間のとき 最大・最小
を持つとは限らない.



定理1.6 (中間値の定理 I)

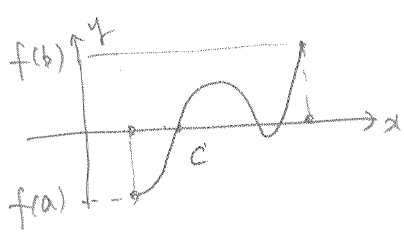


$f(a) \neq f(b)$ のとき

$f(a)$ と $f(b)$ の間の $\forall k$ に対して

$f(c)=k$ となる c が 少なくとも1つ存在

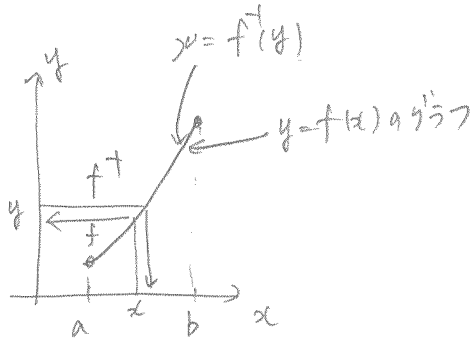
定理1.6' (〃 II)



$f(a)$ と $f(b)$ が異符号

$f(c)=0$ となる 〃

定理 1.7



fの逆f^{-1}が存在する。 $y=f(x)$が連続 $\Rightarrow$ $x=f^{-1}(y)$も連続

例 1.15

$f(x)=\dfrac{x^2-4}{x-2}$ の連続性は?

$x\neq 2$のとき $f(x)=\dfrac{(x-2)(x+2)}{x-2}=x+2$ $\Rightarrow$ $x\neq 2$のとき 連続.

$x=2$のとき. $\lim_{x\to 2} f(x)=2+2=4$ だが $f(2)$が定義されないので不連続.

注 $f(x)=\begin{cases}\dfrac{x^2-4}{x-2} & (x\neq 2)\\ 4 & (x=2)\end{cases}$ とすれば連続にできる.

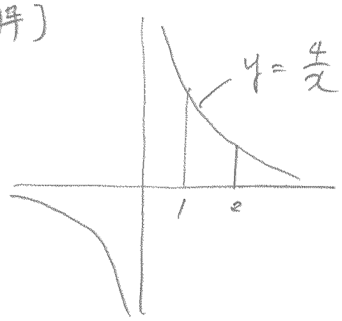
$\Rightarrow$ $x=2$は除去可能な不連続点という

例1.16.

$f(x)=\dfrac{4}{x}$ は $(0,\infty)$ で最大値・最小値を持つか？

$[1,2]$ 〃 ?

［解］



$x\in(0,\infty)$ で

$0<y<\infty$ 最大(小)持たない

$x\in[1,2]$ では $\max y=4$

$\min y=2$

例1.17

「$f(x)=3^x-6x+2=0$ は $(2,3)$ で少なくとも1つ解を持つ」示せ.

$$\begin{cases} f(2)=3^2-12+2=-1<0 \\ f(3)=27-18+2=11>0 \end{cases}$$

$\Rightarrow$ $f(c)=0$ なる c が $(2,3)$ に少なくとも1つ存在する

問1.15

「$f(x)=x^3-3x^2-2x+5=0$ は 2より小さい正の解を持つ」を示せ

$$\begin{cases} f(0)=5>0 \\ f(2)=8-12-4+5=-3<0 \end{cases}$$

$\Rightarrow$

$f(c)=0$ なる c が $(0,2)$ に少なくとも1つ存在.

$\boxed{1.10}$ 微分係数

◇ $f(x)$ が $x=a$ の近くで定義されているとき

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \quad \cdots\cdots (*)$$

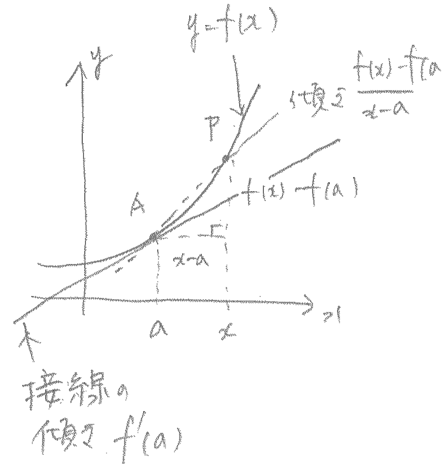
$x=a+h$

「$x=a$ における $f(x)$ の微分係数」という

$$= \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

が存在するとき

「$f(x)$ は $x=a$ で微分可能である」という。



$(*)$ のとき，

$$\lim_{x \to a} \left\{ \underbrace{\frac{f(x)-f(a)}{x-a} - f'(a)}_{\varepsilon \text{ とおくと.}} \right\} = 0$$

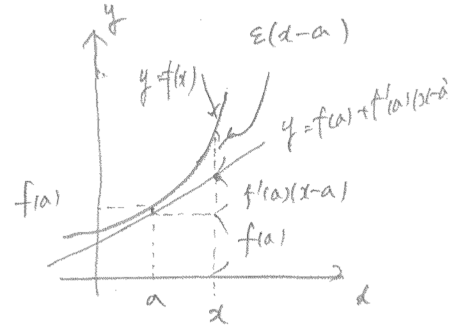
$$\frac{f(x)-f(a)}{x-a} - f'(a) = \varepsilon, \quad \varepsilon \to 0 \quad (x \to a)$$

よって $f(x)-f(a) - f'(a)(x-a) = \varepsilon(x-a)$

$$f(x) = f(a) + f'(a)(x-a) + \varepsilon(x-a)$$

$[$ここで ε は $\varepsilon \to 0 \ (x \to a)$ $]$

と書ける。



◇ $\underbrace{\text{右微分係数，左微分係数}}^{x=a\text{ における}}$

$$f'_{+}(a) = \lim_{h \to +0} \frac{f(a+h)-f(a)}{h}$$

$$f'_{-}(a) = \lim_{h \to -0} \frac{f(a+h)-f(a)}{h}$$

$f(x)$ が $x=a$ で微分可能である $\Longleftrightarrow$ $f'_{+}(a) = f'_{-}(a)$

定理1.8.

$f(x)$ が $x=a$ で微分可能 $\Rightarrow$ $f(x)$ は $x=a$ で連続

∵ $\lim_{x\to a}\{f(x)-f(a)\} = \lim_{x\to a} f'(a)(x-a)+\varepsilon(x-a) = 0$ ☒

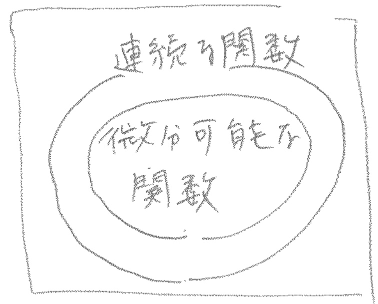


図1.50
1変数関数 $y=f(x)$ の集合

◇ 接線

$x=a$ における $y=f(x)$ の接線の方程式は

$$y = f(a)+f'(a)(x-a)$$

例1.18.

(1) $f(x)=\frac{1}{x}$ のとき $f'(a)$ $(a\neq 0)$ を求める.

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{1}{h}\left\{\frac{1}{a+h}-\frac{1}{a}\right\}$$

$$= \lim_{h\to 0}\frac{1}{h}\,\frac{-h}{a(a+h)} = \frac{-1}{a^2}$$

(2) $f(x)=|x|$ は $x=0$ で連続か? 微分可能か?

$f(+0) = \lim_{x\to +0}|x| = \lim_{x\to +0}x = 0$. $f(-0) = \lim_{x\to -0}|x| = \lim_{x\to -0}-x = 0$ 連続

$$\left.\begin{aligned} f'_+(0) &= \lim_{h\to +0}\frac{|0+h|-|0|}{h} = \lim_{h\to +0}\frac{h}{h} = 1 \\ f'_-(0) &= \lim_{h\to -0}\frac{|h|-|0|}{h} = \lim_{h\to -0}\frac{-h}{h} = -1 \end{aligned}\right\} \text{微分不可能}$$

例1.19.

$f(x)=\sqrt{x}$ 上の $(4,2)$ における接線の方程式を求めよ.

$$f'(4)=\lim_{h\to 0}\frac{\sqrt{4+h}-\sqrt{4}}{h}=\lim_{h\to 0}\frac{(\sqrt{4+h}-\sqrt{4})(\sqrt{4+h}+\sqrt{4})}{h\,(\sqrt{4+h}+\sqrt{4})}$$

$$=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{4+h}+\sqrt{4})}=\frac{1}{\sqrt{4}+\sqrt{4}}=\frac{1}{4}.$$

よって

$$y=\frac{1}{4}(x-4)+2$$

$$=\underline{\frac{1}{4}x+1}$$

問1.16　$y=\frac{1}{x}$ 上の $(2,\frac{1}{2})$ における接線は?

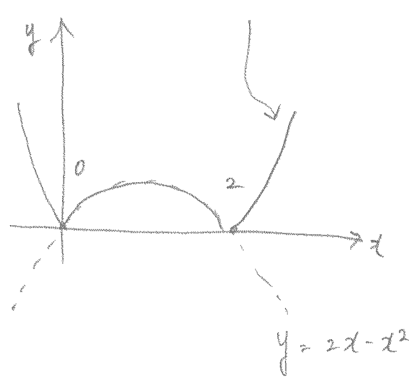
$$f'(2)=\lim_{h\to 0}\frac{\frac{1}{2+h}-\frac{1}{2}}{h}=\lim_{h\to 0}\frac{1}{h}\,\frac{-h}{2(2+h)}=-\frac{1}{4}$$

$$\underline{y}=-\frac{1}{4}(x-2)+\frac{1}{2}=\underline{-\frac{1}{4}x+1}$$

問1.17.

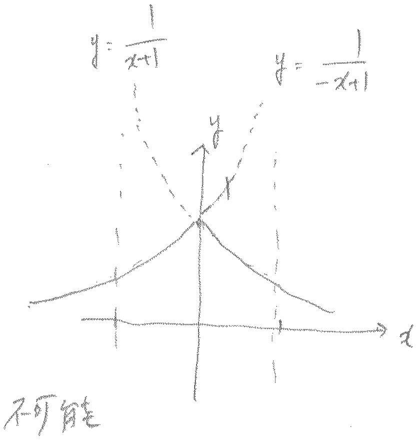
(1)　$f(x)=|2x-x^2|$ のとき, $f(x)$ は $x=2$ で微分可能か?

$y=|2x-x^2|$ のグラフ



$x=2$ で微分不可能

(2)　$f(x)=\frac{1}{|x|+1}$ の $x=0$ における連続性



$x=0$ で連続, 微分可能

1.11 導関数

◇ 区間 I で $y=f(x)$ が微分可能なとき、「区間 I で微分可能」という
$\underbrace{\text{区間 } I \text{ で}}_{\text{の各点}}$

◇ 導関数
区間 I の各点 x に対して、その点の微分係数 $f'(x)$ を対応させる関数を
$y=f(x)$ の導関数 といって

$$f'(x),\ y',\ f',\ \frac{dy}{dx},\ \frac{d}{dx}f(x)$$ などと表す.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$ だったから

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim\frac{\Delta y}{\Delta x}$$

Δy：「y の増分」という
Δx：「x の増分」〃

となる.

$f(x)$ から $f'(x)$ を求めることを
「$f(x)$ を微分する」という.

定理 1.9

$f(x)$, $g(x)$ が 微分可能

⇒ (1) $(kf(x))' = kf'(x)$

∵ $$\lim_{\Delta x \to 0} \frac{kf(x+\Delta x) - kf(x)}{\Delta x} = k \underbrace{\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}_{f'(x)} \quad \boxtimes$$

(2) $(f(x) + g(x))' = f'(x) + g'(x)$

∵ $$\lim_{\Delta x \to 0} \frac{\{f(x+\Delta x) + g(x+\Delta x)\} - \{f(x) + g(x)\}}{\Delta x} = f'(x) + g'(x) \quad \boxtimes$$

(3) $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

∵ $$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\{f(x+\Delta x) - f(x)\}g(x+\Delta x) + f(x)\{g(x+\Delta x) - g(x)\}}{\Delta x}$$

$$= f'(x)g(x) + f(x)g'(x) \quad \boxtimes$$

(4) $$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

∵ $$\lim_{\Delta x \to 0} \frac{1}{\Delta x}\left\{\frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}\right\} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{f(x+\Delta x)g(x) \overbrace{\ }^{-f(x)g(x) + f(x)g(x)} - f(x)g(x+\Delta x)}{g(x)g(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{g(x)g(x+\Delta x)} \left(\frac{1}{\Delta x}(f(x+\Delta x) - f(x))g(x) - f(x)\frac{(g(x+\Delta x) - g(x))}{\Delta x} \right)$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \quad \boxtimes$$

◇ 基本的な関数の導関数

(1) $(c)' = 0$

∵ $\displaystyle\lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = 0$

(2) $(x^{\alpha})' = \alpha x^{\alpha - 1}$

omit ∵ $x^{\alpha} = e^{\alpha \log x}$ より $(x^{\alpha})' = e^{\alpha \log x} \dfrac{\alpha}{x} = \dfrac{x^{\alpha}}{x}\alpha = \alpha x^{\alpha-1}$

合成関数の微分 [定理1.10]　　log の微分. 公式II (11)

(3) $(\sin x)' = \cos x.$

∵ $\displaystyle\lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} \underset{\boxed{13p.\ 7}}{=} \lim_{\Delta x \to 0} \frac{1}{\Delta x} \, 2\cos\frac{2x + \Delta x}{2} \sin\frac{\Delta x}{2}$

$\displaystyle = \lim_{\Delta x \to 0} \frac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)} \cdot \cos\left(x + \frac{\Delta x}{2}\right) = \cos x.$

(4) $(\cos x)' = -\sin x$

∵ $\displaystyle\lim_{\Delta x \to 0} \frac{\cos(x + \Delta x) - \cos x}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} (-2) \sin\frac{2x + \Delta x}{2} \sin\frac{\Delta x}{2}$

$\displaystyle = \lim_{\Delta x \to 0} \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \left(-\sin\left(x + \frac{\Delta x}{2}\right)\right) = -\sin x$

(5) $(\tan x)' = \dfrac{1}{\cos^2 x}$

∵ $\displaystyle\left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x}.$

(6) $(\cot x)' = -\dfrac{1}{\sin^2 x}$

∵ $\displaystyle\left(\frac{\cos x}{\sin x}\right)' = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = -\frac{1}{\sin^2 x}$

(7) $(\sec x)' = \sec x \cdot \tan x$

∵ $\displaystyle\left(\frac{1}{\cos x}\right)' = \frac{+\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \tan x$

(8) $(\operatorname{cosec} x)' = -\operatorname{cosec} x \cdot \cot x$

例. 1.20.

$f(x) = \sqrt{a^2 - x^2} \quad (a \neq 0)$ の $f'(x)$ は?

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{a^2-(x+h)^2} - \sqrt{a^2-x^2}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{a^2 - (x+h)^2 - (a^2 - x^2)}{\sqrt{\quad} + \sqrt{\quad}}$$

$$= \lim_{h \to 0} \frac{-2x\cancel{h} - h^{\cancel{2}}}{\cancel{h}} \frac{1}{\sqrt{a^2-(x+h)^2} + \sqrt{a^2-x^2}} = \frac{-\cancel{2}x}{\cancel{2}\sqrt{a^2-x^2}} = \frac{-x}{\sqrt{a^2-x^2}}$$

例 1.21.

(1) $\left(x - \frac{1}{\sqrt{x}}\right)^2 = x^2 - 2\sqrt{x} + \frac{1}{x}$ より

$$\left(x^2 - 2\sqrt{x} + \frac{1}{x}\right)' = \underbrace{(x^2)'}_{= 2x} - 2\underbrace{(x^{\frac{1}{2}})'}_{= \frac{1}{2}x^{-\frac{1}{2}}} + \underbrace{(x^{-1})'}_{-x^{-2}} = 2x + x^{-\frac{1}{2}} - x^{-2}$$

(2) $$\left((x^3 + 4x)\sqrt{1-x^2}\right)' = \underbrace{(x^3+4x)'}_{3x^2+4}\sqrt{1-x^2} + (x^3+4x)\boxed{\frac{-x}{\sqrt{1-x^2}}}$$

前問より ($a=1$ にて)

$$= (3x^2+4)\sqrt{1-x^2} + \frac{-(x^3+4x)x}{\sqrt{1-x^2}}$$

(3) $$\left(\frac{\sin x}{1+\tan x}\right)' = \frac{(\sin x)'(1+\tan x) - \sin x\,(1+\tan x)'}{(1+\tan x)^2}$$

$$= \frac{\cos x\,(1+\tan x) - \sin x \,\frac{1}{\cos^2 x}}{(1+\tan x)^2} = \frac{\cos x + \sin x - \sin x \sec^2 x}{(1+\tan x)^2}$$

問1.18

$f(x) = \sqrt{3x-4}$ の $f'(x)$ は?

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{1}{h}\left(\sqrt{3x+3h-4} - \sqrt{3x-4}\right)$$

$$= \lim_{h \to 0} \frac{\cancel{3x}+3\cancel{h}-\cancel{4}-(\cancel{3x}-\cancel{4})}{\cancel{h}\ \sqrt{3x+3h-4}+\sqrt{3x-4}} = \frac{3}{\sqrt{3x-4}+\sqrt{3x-4}} = \underline{\frac{3}{2\sqrt{3x-4}}}$$

問1.19.

(1)
$$\left(\frac{2x+5}{\sqrt{3x-4}}\right)' = \frac{\overset{2}{\boxed{(2x+5)'}}\sqrt{3x-4} - (2x+5)\left(\sqrt{3x-4}\right)'}{3x-4}$$

$$= \left(2\sqrt{} - (2x+5)\frac{3}{2\sqrt{3x-4}}\right)\frac{1}{3x-4}$$

$$= \underline{\frac{2}{\sqrt{3x-4}} - \frac{3(2x+5)}{(3x-4)^{\frac{3}{2}}}}$$

(2)
$$\left(\frac{3x}{\sqrt{a^2-x^2}}\right)' = \frac{\overset{3}{\boxed{(3x)'}}\sqrt{a^2-x^2} - 3x\overset{-\frac{x}{\sqrt{a^2-x^2}}}{\boxed{\left(\sqrt{a^2-x^2}\right)'}}}{a^2-x^2}$$

(問1.20)

$$= \frac{1}{a^2-x^2}\left\{3\sqrt{a^2-x^2} + \frac{3x^2}{\sqrt{a^2-x^2}}\right\} = \underline{\frac{3}{\sqrt{a^2-x^2}} + \frac{3x^2}{(a^2-x^2)^{\frac{3}{2}}}}$$

(3)
$$(x^2\cos x)' = \underbrace{(x^2)'}_{2x}\cos x + x^2\underbrace{(\cos x)'}_{-\sin x}$$

$$= \underline{2x\cos x - x^2\sin x}$$

(4)
$$\left((x^2-1)\sin x\right)' = (x^2-1)'\sin x + (x^2-1)(\sin x)'$$

$$= \underline{2x\sin x + (x^2-1)\cos x}$$

(5)
$$\left((4x-3)\sqrt{1-x}\right)' = \underbrace{(4x-3)'}_{4}\sqrt{1-x} + (4x-3)\underbrace{\left(\sqrt{1-x}\right)'}$$

合成関数の微分 (Th 1.10)

$$\left(\begin{aligned} \frac{d}{dx}(1-x)^{\frac{1}{2}} &= \frac{du^{\frac{1}{2}}}{du}\,\frac{d(1-x)}{dx} \\ &= \frac{1}{2}u^{-\frac{1}{2}}(-1) \\ &= -\frac{1}{2}(1-x)^{-\frac{1}{2}} \end{aligned}\right)$$

$$= \underline{4\sqrt{1-x} + \frac{(4x-3)\,(1-x)^{-\frac{1}{2}}}{2}}$$

[定理 1.10]
$$x \longrightarrow u = g(x) \longrightarrow y = f(u) = f(g(x))$$
合成関数

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}.$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{h} = \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{\underbrace{\boxed{g(x+h)-g(x)}}_{\Delta g\text{ とおく}}}\;\frac{g(x+h)-g(x)}{h}.$$

$$\left(\begin{aligned} &\Delta g = g(x+h) - g(x) \text{ とおくと } h\to 0 \text{ で } \Delta g \to 0 \\ &g(x+h) = g(x) + \Delta g \end{aligned}\right)$$

$$= \lim_{\substack{h\to 0\\ \Delta g\to 0}} \underbrace{\left[\frac{f(g(x)+\Delta g) - f(g(x))}{\Delta g}\right]}_{\downarrow\; f'(\boxed{g(x)}) \;\; g(x) = u} \cdot \underbrace{\left[\frac{g(x+h)-g(x)}{h}\right]}_{\downarrow\; g'(x)}$$

$$= \frac{dy}{du}\cdot\frac{du}{dx} \quad ☒$$

細かいこと

$\Delta g \neq 0$ のとき

$\Delta g = 0$ のときの式

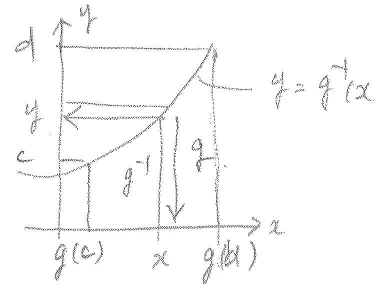
$f'(g(x))$

$$\frac{f'(g(x)+\Delta g) - f'(g(x))}{\Delta g}$$

として
式でおきかえた方が.

定理1.11

$y=g^{-1}(x)$ の微分が求めたいとき



$x=g(y)$ の微分可能のとき

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}} \quad \left(\text{ただし } \frac{dx}{dy}\neq 0\right)$$

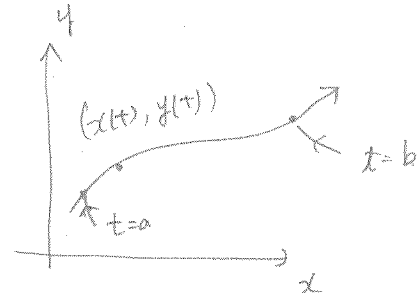
$x=g(y)=g(g^{-1}(x))$ の両辺をxで微分

$$\underbrace{(x)'}_{=1}=\left(g(g^{-1}(x))\right)'=\underbrace{\boxed{g'(\underbrace{\boxed{g^{-1}(x)}}_{=y})}}_{\frac{dx}{dy}}\ \underbrace{\boxed{\left(g^{-1}(x)\right)'}}_{=\frac{dy}{dx}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$$

定理1.12

$x=f(t)$, $y=g(t)$ は微分可能



$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{g'(t)}{f'(t)}$$

$x=f(t)$ を t について解くと $t=f^{-1}(x)$

$$y=g(f^{-1}(x)) \quad \text{より} \quad \frac{dy}{dx}=\underbrace{\boxed{g'(\underbrace{\boxed{f^{-1}(x)}}_{=t})}}_{\frac{dy}{dt}}\ \underbrace{\boxed{\left(f^{-1}(x)\right)'}}_{=\frac{dt}{dx}=\frac{1}{\frac{dx}{dt}}}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \blacksquare$$

$\frac{dt}{dx}=\frac{1}{\frac{dx}{dt}}$ ← (Th. 1.11)

◇ 導関数の公式 II

(9) $(e^x)' = e^x$

∵ $$\lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = e^x \boxed{\lim_{h\to 0}\frac{e^h-1}{h}} = e^x \quad ☒$$

$\boxed{\cdots} = 1$ 問1.13(5)より

(10) $(a^x)' = a^x \log a$

∵ $a^x = e^{x\log a}$ より $$\left(e^{\overset{u}{\boxed{x\log a}}}\right)' = \frac{de^u}{du}\,\frac{d\,x\log a}{dx} = e^u\cdot\log a \quad ☒$$

(11) $(\log x)' = \frac{1}{x}$, $(\log(-x))' = \frac{1}{x}$

$$\lim_{h\to 0}\frac{1}{h}\left(\log(x+h)-\log x\right) = \lim_{h\to 0}\frac{1}{h}\left(\log\frac{x+h}{x}\right) = \lim_{h\to 0}\log\left(1+\frac{h}{x}\right)^{\frac{1}{h}}$$

$$= \lim_{h\to 0}\log\left\{\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right\}^{\frac{1}{x}} = \frac{1}{x}\underbrace{\lim_{h\to 0}\log\underbrace{\boxed{\left(1+\frac{h}{x}\right)^{\frac{x}{h}}}}_{\to e}}_{\to 1} = \frac{1}{x} \quad ∴$$

$$\left(\log(\underset{u}{\boxed{-x}})\right)' = \frac{d\log u}{du}\,\frac{d(-x)}{dx} = \frac{-1}{u} = \frac{-1}{-x} = \frac{1}{x}. \quad ☒$$

(12) $(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}$

∵ $y = \sin^{-1}x$, $x = \sin(y)$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}} \quad ☒$$

$\left(-\frac{\pi}{2} \le y \le \frac{\pi}{2}\right.$ より $\cos y \ge 0$ よって $\underbrace{\cos y = \sqrt{1-\sin^2 y}}_{プラス} = \left.\sqrt{1-x^2}\right)$

(13)

$$(\cos^{-1}x)' = \frac{-1}{\sqrt{1-x^2}}$$

∵ $y = \cos^{-1}x$ $\qquad x = \cos y$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{-\sin y} = \frac{-1}{\sqrt{1-x^2}} \qquad \boxtimes$$

($0 \leq y \leq \pi$. より $\sin y \geq 0$ よって $\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-x^2}$)

(14) $(\tan^{-1}x)' = \dfrac{1}{1+x^2}$

$y = \tan^{-1}x$, $x = \tan y$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{1}{\cos^2 y}} = \cos^2 y = \frac{1}{1+x^2} \qquad \square$$

$$\left(x^2 = \frac{1-\cos^2 y}{\cos^2 y} \rightarrow (x^2+1)\cos^2 y = 1 \rightarrow \cos^2 y = \frac{1}{x^2+1} \right)$$

(15) $(\cot^{-1}x)' = \dfrac{-1}{1+x^2}$

∵ $y = \cot^{-1}x$, $x = \cot y$.

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{-1}{\sin^2 y}} = -\sin^2 y = -\frac{1}{1+x^2} \qquad \boxtimes$$

$$\left(x^2 = \frac{1-\sin^2 y}{\sin^2 y} \Rightarrow (1+x^2)\sin^2 y = 1 \right)$$

(16) $(\sec^{-1}x)' = \dfrac{1}{|x|\sqrt{x^2-1}}$

∵ $y = \sec^{-1}x \Rightarrow x = \sec y = \dfrac{1}{\cos y}$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{+(\cos y)^{-2}\cdot \sin y} = \frac{1}{\frac{x^2}{|x|}\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}} \quad \boxtimes$$

$$\begin{cases} (\cos y)^{-2} = x^2 \\ \sin y = \sqrt{1-\cos^2 y} = \sqrt{1-\frac{1}{x^2}} = \frac{1}{|x|}\sqrt{x^2-1} \end{cases}$$

(17) $(\operatorname{cosec}^{-1}x)' = \dfrac{-1}{|x|\sqrt{x^2-1}}$

∵ $y = \operatorname{cosec}^{-1}x \Rightarrow x = \operatorname{cosec} y = \dfrac{1}{\sin y}$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{-(\sin y)^{-2}\cos y} = \frac{-1}{\frac{x^2}{|x|}\sqrt{x^2-1}} = \frac{-1}{|x|\sqrt{x^2-1}} \quad \boxtimes$$

$$\begin{cases} x^2 = \frac{1}{\sin^2 y} \\ \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{1}{x^2}} = \sqrt{\frac{x^2-1}{x^2}} = \frac{\sqrt{x^2-1}}{|x|} \end{cases}$$

例 1.22. y' は?

(1) $y = \sqrt[3]{(x^2+1)^2} = (x^2+1)^{\frac{2}{3}}$

$$\frac{dy}{dx} = \Big((\underbrace{\boxed{x^2+1}}_{u})^{\frac{2}{3}} \Big)' = \frac{du^{\frac{2}{3}}}{du} \frac{d(x^2+1)}{dx} = \frac{2}{3} u^{-\frac{1}{3}} \cdot 2x = \underline{\frac{4x}{3} (x^2+1)^{-\frac{1}{3}}}$$

(2) $y = \cos^{-1}\left(\underbrace{\boxed{\frac{x^2-1}{x^2+1}}}_{u} \right) \quad (x > 0)$

$$\frac{dy}{dx} = \frac{d \cos^{-1} u}{du} \frac{du}{dx}$$

$$= \frac{-1}{\sqrt{1-u^2}} \cdot \frac{4x}{(x^2+1)^2}$$

$$\left(u = \frac{x^2+1-2}{x^2+1} = 1 - \frac{2}{x^2+1} \qquad \frac{du}{dx} = -2 \frac{(-1)}{(x^2+1)^2} 2x = \frac{4x}{(x^2+1)^2} \right)$$

$$\left(\sqrt{1-u^2} = \sqrt{1 - \frac{x^4-2x^2+1}{(x^2+1)^2}} = \frac{\sqrt{\cancel{x^4}+2x^2+\cancel{1}-(\cancel{x^4}-2x^2+\cancel{1})}}{x^2+1} = \frac{\sqrt{4x^2}}{x^2+1} = \frac{2x}{x^2+1} \right)$$

$$= -\frac{\cancel{x^2+1}}{\cancel{2x}} \frac{\overset{2}{\cancel{4x}}}{(x^2+1)^{\cancel{2}}} = \underline{\frac{-2}{x^2+1}}$$ //

例 1.23. $f(x)$ 微分可能で $f(x) \neq 0$ のとき

$$\Big(\log|\underbrace{\boxed{f(x)}}_{u}|\Big)' = \frac{f'(x)}{f(x)} \text{ を示せ}$$

[解]

$$\frac{dy}{dx} = \underbrace{\frac{d\log|u|}{du}}_{\frac{1}{u}} \cdot \underbrace{\frac{du}{dx}}_{f'(x)} = \underline{\frac{f'(x)}{f(x)}} \quad \boxtimes$$

問 1.20.

(1) $(\sin^3 x)' = \frac{d}{du}u^3 \frac{d \sin x}{dx} = 3u^2 \cos x = \underline{3\sin^2 x \cos x}$

(2) $\left(\tan\left(\underbrace{\boxed{2x+\frac{\pi}{6}}}_{u}\right)\right)' = \frac{d \tan u}{du} \frac{d\left(2x+\frac{\pi}{6}\right)}{dx} = \frac{1}{\cos^2 u} \cdot 2 = \underline{\frac{2}{\cos^2\left(2x+\frac{\pi}{6}\right)}}$

(3) $\left(\sqrt{1+x+x^2}\right)' = \frac{d}{du}u^{\frac{1}{2}} \frac{d}{dx}(1+x+x^2) = \frac{1}{2}u^{-\frac{1}{2}} (1+2x)$

$= \underline{\frac{1+2x}{2\sqrt{1+x+x^2}}}$

(4) $(e^{ax}\sin bx)' = \underbrace{\boxed{(e^{ax})'}}_{e^{ax}\cdot a}\sin bx + e^{ax}\underbrace{\boxed{(\sin bx)'}}_{\cos bx \cdot b}$

$= \underline{a e^{ax}\sin bx + b e^{ax}\cos bx}$

(5) $\left(\tan^{-1}\left(\frac{1}{\sqrt{2}}\tan\frac{x}{2}\right)\right)'$

$= \frac{d}{du}\tan^{-1}u \quad \frac{d}{dx}\frac{1}{\sqrt{2}}\tan\frac{x}{2}$

$= \frac{1}{1+u^2} \quad \frac{1}{\sqrt{2}} \quad \frac{1}{\cos^2\frac{x}{2}} \cdot \frac{1}{2} = \frac{1}{1+\frac{\tan^2\frac{x}{2}}{2}} \quad \frac{}{\cos^2\frac{x}{2}} \quad \frac{1}{2\sqrt{2}}$

$= \frac{1}{2\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} \quad \frac{1}{\sqrt{2}} = \underline{\frac{1}{1+\cos^2\frac{x}{2}} \cdot \frac{1}{\sqrt{2}}}$

(6) $\left(\sin^{-1}\frac{x}{2}\right)'$

$$= \frac{d}{du}\sin^{-1}u \ \frac{d}{dx}\frac{x}{2} = \frac{1}{\sqrt{1-u^2}}\cdot\frac{1}{2} = \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\frac{1}{2} = \underline{\frac{1}{\sqrt{4-x^2}}}$$

(7) $\left(\log(\underbrace{\boxed{x^2-3x+2}}_{u})\right)' \quad (x>2)$

$$= \frac{d}{du}\log u \ \frac{d}{dx}(x^2-3x+2) = \frac{1}{u}\cdot 2x-3 = \underline{\frac{2x-3}{x^2-3x+2}}$$

問 1.21

$$\sinh x = \frac{e^x-e^{-x}}{2} \Rightarrow (\sinh x)' = \frac{e^x+e^{-x}}{2} = \cosh x$$

$$\cosh x = \frac{e^x+e^{-x}}{2} \Rightarrow (\cosh x)' = \frac{e^x-e^{-x}}{2} = \sinh x$$

$$\tanh x = \frac{\sinh x}{\cosh x} \Rightarrow (\tanh x)' = \frac{\cosh x\cosh x - \sinh x\sinh x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

例 1.24.

$y = \sqrt{x^2+1}\ \sqrt[3]{x^3+1}$　$x \neq -1$ のとき y' は?

[解.] そのまま微分すると

$$y' = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} 2x \sqrt[3]{x^3+1} + \sqrt{x^2+1}\ \frac{1}{3}(x^3+1)^{-\frac{2}{3}} 3x^2$$

$$= \frac{x}{\sqrt{x^2+1}} \sqrt[3]{x^3+1} + \sqrt{x^2+1}\ \frac{x^2}{\sqrt[3]{(x^3+1)^2}}$$

$$= \frac{x}{x^2+1} \sqrt{x^2+1}\ \sqrt[3]{x^3+1} + \frac{x^2}{x^3+1} \sqrt{x^2+1}\ \sqrt[3]{x^3+1}$$

$$= \left(\frac{x}{x^2+1} + \frac{x^2}{x^3+1}\right) \sqrt{x^2+1}\ \sqrt[3]{x^3+1}$$

[解]

$$\log|y| = \log\left(\sqrt{x^2+1}\ \sqrt[3]{x^3+1}\right)$$

$$= \frac{1}{2}\log(x^2+1) + \frac{1}{3}\log|x^3+1|$$

両辺を x で微分すると

$$\frac{y'}{y} = \frac{1}{2}\frac{2x}{x^2+1} + \frac{1}{3}\frac{3x^2}{x^3+1}$$

$$y' = \left(\frac{x}{x^2+1} + \frac{x^2}{x^3+1}\right) y$$

$$= \left(\frac{x}{x^2+1} + \frac{x^2}{x^3+1}\right) \sqrt{x^2+1}\ \sqrt[3]{x^3+1}$$

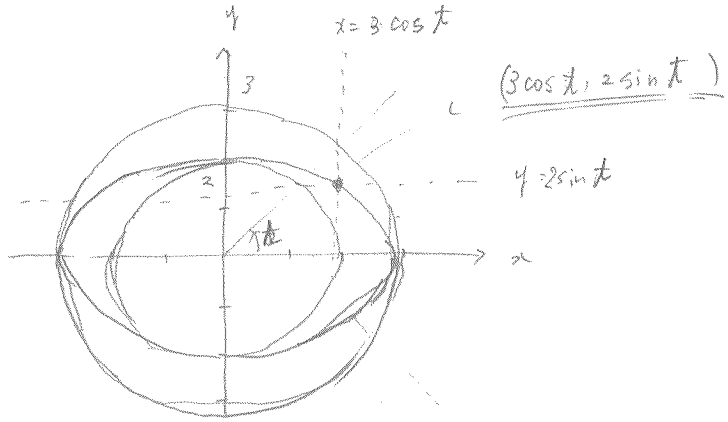
例 1.25

$$\begin{cases} x = 3\cos t \\ y = 2\sin t \end{cases}$$ はどんな曲線？　$t=\pi/3$ における接線の方程式は？

[解]

$\frac{x}{3} = \cos t$ ， $\frac{y}{2} = \sin t$

$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = (\cos t)^2 + (\sin t)^2 = 1$ …… だ円の方程式



$t=\pi/3$ における接線は？

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t}{-3\sin t}$$

$$\left.\frac{dy}{dx}\right|_{t=\pi/3} = -\left.\frac{2\cos t}{3\sin t}\right|_{t=\pi/3} = -\frac{2}{3}\frac{1}{2}\frac{1}{\frac{\sqrt{3}}{2}} = -\frac{2}{3\sqrt{3}}$$

よって $\left.(3\cos t, 2\sin t)\right|_{t=\pi/3} = \left(3\frac{1}{2}, 2\cdot\frac{\sqrt{3}}{2}\right) = \left(\frac{3}{2}, \sqrt{3}\right)$ を通る傾き $-\frac{2}{3\sqrt{3}}$

の直線の方程式

$$y - \sqrt{3} = -\frac{2}{3\sqrt{3}}\left(x - \frac{3}{2}\right)$$

が 接線の方程式である

問 1.22

(1) $(\tan x)^{\sin x} = \left(e^{\log \tan x}\right)^{\sin x} = e^{\sin x \log \tan x}$ より ($0<x<\frac{\pi}{2}$ より $\tan x > 0$) に注意.

$$(与式)' = \underbrace{e^{\sin x \log \tan x}} \left(\sin x \log \tan x\right)'$$

$$= \quad 〃 \quad \left(\cos \log \tan x + \sin x \frac{1}{\tan x} \cdot \frac{1}{\cos^2 x}\right)$$

$$= \underline{(\tan x)^{\sin x}\left(\cos x \log \tan x + \frac{1}{\cos x}\right)} \ //$$

(2) $x^{\tan^{-1} x} = \left(e^{\log x}\right)^{\tan^{-1} x} = e^{\tan^{-1} x \log x}$ より

$$(与式)' = \underbrace{e^{\tan^{-1} x \log x}} \left(\tan^{-1} x + \log x\right)'$$

$$= \quad 〃 \quad \left(\frac{1}{1+x^2} + \frac{1}{x}\right)$$

$$= \underline{x^{\tan^{-1} x}\left(\frac{1}{1+x^2} + \frac{1}{x}\right)} \ //$$

(3) $e^{\overset{u}{\boxed{x^x}}} \quad (x>0)$

$$(与式)' = \frac{de^u}{du}\frac{du}{dx} = e^u \boxed{\frac{d x^x}{dx}} = \underline{e^{x^x} x^x (\log x + 1)}$$

$$\left(\boxed{} = \left(e^{x \log x}\right)' = e^{x \log x}(x \log x)' = x^x(\log x + 1) \right)$$

(4) $y = x^2 \sqrt{\frac{1-x^2}{1+x^2}} \quad (-1<x<1,\ x \neq 0)$

$$\log y = 2x + \frac{1}{2}\log(1-x^2) - \frac{1}{2}\log(1+x^2)$$

$$\frac{y'}{y} = 2 + \frac{1}{2}\frac{-2x}{1-x^2} - \frac{1}{2}\frac{2x}{1+x^2} = 2 - \frac{x}{1-x^2} - \frac{x}{1+x^2}$$

$$y' = \underline{x^2\sqrt{\frac{1-x^2}{1+x^2}} \cdot \left(2 - \frac{x}{1-x^2} - \frac{x}{1+x^2}\right)} = \underline{x^2\sqrt{〃}\left(2 - x\frac{2}{(1-x^2)(1+x^2)}\right)}$$

or.

問 1.23

(1) $\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1-\cos\theta) \end{cases}$ $(a>0)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1-\cos\theta)} = \underline{\frac{\sin\theta}{1-\cos\theta}}$$

or

$$= \frac{2\sin\theta/_2 \cos\theta/_2}{2\sin^2\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \underline{\cot\frac{\theta}{2}}$$

(2) $\begin{cases} x = 2t^2 \\ y = 4t \end{cases}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4}{4t} = \underline{\frac{1}{t}}$$

2 平均値の定理と導関数の応用

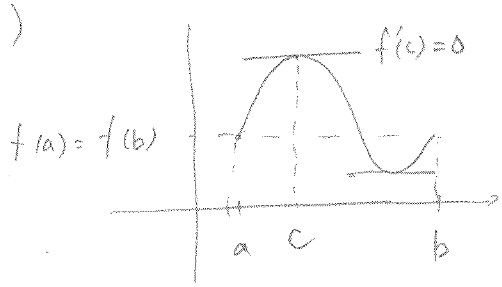
2.1 平均値の定理

定理2.1 (ロルの定理)

$f(x)$ が $[a, b]$ で連続, (a, b) で微分可能で $f(a)=f(b)$

$\Rightarrow$ $f'(c)=0$ となる c が $a<c<b$ に少なくとも1つ存在する.

(説明)



(証明) (a, b) の最大値又は最小値をとる場合を考える.

(一定値のときは 常に $f(a)=f(b)=f(c)$, $f'(c)=0$)

一般性を失うことなく $f(x)$ は $x=c$ で最大値を持つとする.

(最小値のみ持つときは, $-f(x)$ を考えればよい)

$$f'_+(c) = \lim_{h \to +0} \frac{f(x+h)-f(x)}{h} \leqq 0 \qquad f(x+h)-f(x)<0 \text{ より}$$

$$f'_-(c) = \lim_{h \to -0} \frac{f(x+h)-f(x)}{h} \geqq 0$$

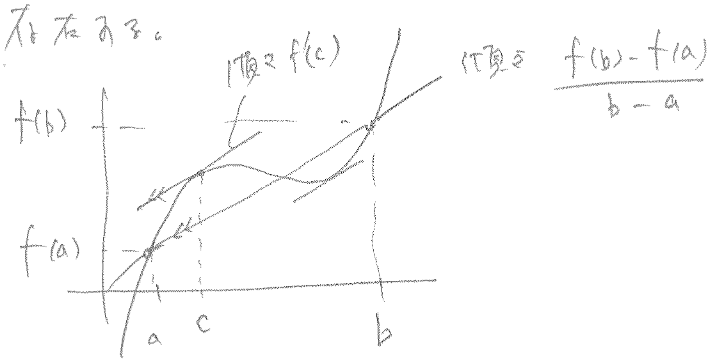
$f'(c) = f'_+(c) = f'_-(c)$ より $f'(c)=0$ をみたす ∎

定理 2.2 (平均値の定理)

$f(x)$ が $[a,b]$ 連続, (a,b) で微分可能

$\Rightarrow$ $\dfrac{f(b)-f(a)}{b-a} = f'(c)$ をみたす c が $a<c<b$ に少なくとも1つ存在する。

(証明)



$$\frac{f(b)-f(a)}{b-a} = k \quad \text{とおく.}$$

$$f(b)-f(a)-k(b-a) = 0 \quad \cdots \quad (*)$$

左辺で $a \to x$ とした式を

$$F(x) = f(b)-f(x)-k(b-x) \quad \text{とおく}$$

$$F(b) = f(b)-f(b)-k(b-b) = 0$$

$$F(a) = f(a)-f(a)-k(b-a) = 0 \quad \leftarrow (*) \text{より}$$

$$F'(x) = -f'(x)+k$$

ロルの定理より $a<c<b$ のある c において

$$F'(c) = 0$$

$$\Rightarrow \quad +f'(c) = k = \frac{f(b)-f(a)}{b-a}$$

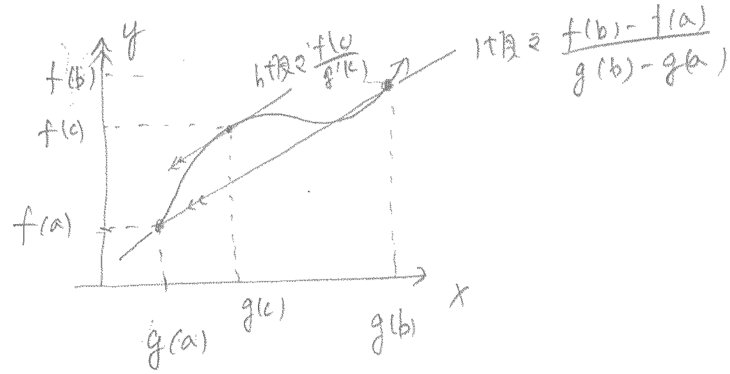
図

定理 2.3 (コーシーの平均値の定理)

$f(x), g(x)$ が $[a, b]$ で連続, (a, b) で微分可能

$$\Rightarrow \quad \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \quad \text{ただし } g'(c) \neq 0$$

∵



$$\frac{f(b)-f(a)}{g(b)-g(a)} = k$$

$$f(b)-f(a)-k\{g(b)-g(a)\} = 0$$

左辺で $a \to x$ とした式を $F(x)$ とおく

$$F(x) = f(b)-f(x)-k\{g(b)-g(x)\}$$

$$F(b) = 0, \quad F(a) = 0$$

$$F'(x) = -f'(x) + k\,g'(x)$$

ロルの定理より

$F'(c) = -f'(c) + k\,g'(c) = 0$ となる c が $a < c < b$ に存在する.

よって $g'(c) \neq 0$ のとき

$$\frac{f'(c)}{g'(c)} = k = \frac{f(b)-f(a)}{g(b)-g(a)} \quad \boxtimes$$

(注) $g(x) = x$ とおけば

$$g'(x) = 1 \quad \Rightarrow \quad f'(c) = \frac{f(b)-f(a)}{b-a}$$

平均値の定理となる

例2.1
$f(x)$ が $[a,b]$ で連続，(a,b) で $f'(x)=0$ ならば

$f(x)$ は $[a,b]$ で定数.

(∵) $$\frac{f(x)-f(a)}{x-a}=f'(c)=0 \qquad a<c<x\leqq b$$

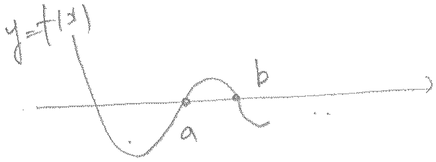
よって $f(x)=f(a)$ for $a<x\leqq b$

よって一定である.

例2.2
$$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n=0 \quad (a_0\neq 0)$$

の隣りあわせの2実数解 a, b の間には

$f'(x)=0$ の実数解が少なくとも1つ存在

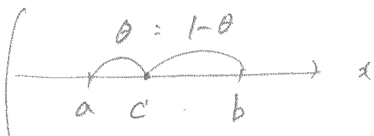


$f(x)$ は $[a,b]$ で連続 (a,b) で微分可能である.

よって $f(c)=0$ となる c が $a<c<b$ に少なくとも1つ存在する

追記2.1 $a<c<b$ のような c $\Leftrightarrow$ $0<\theta<1$, $c=a+\theta(b-a)$

とも表すことが出来る



(c は a, b を $\theta:1-\theta$ に内分する点.)

$$\frac{f(b)-f(a)}{b-a}=f'(a+\theta(b-a))$$ となる θ が $0<\theta<1$ に少なくとも1つ存在

$\xrightarrow[b=a+h]{}$ $f(a+h)=f(a)+hf'(a+\theta h)$ をみたす $\underbrace{\theta}_{(0<\theta<1)}$ が少なくとも1つ存在

問2.1

$f(x)=e^x$ のとき $[0,1]$ で入れて c を求める.

$$\frac{f(1)-f(0)}{1}=f'(c)$$

より $e-1=e^c$ $\quad \underline{c=\log(e-1)}$.

問2.1

(1) $f(x)=x^2-2x$ の区間 $[-1,2]$ での平均値の定理の c と θ は?

$$\frac{f(2)-f(-1)}{3}=f'(c)$$

$f(2)=4-4=0$, $f(-1)=1+2=3$

$f'(x)=2x-2$,

$f'(c)=2c-2$

$$2c-2=\frac{-3}{3}=-1$$

$2c=1$, $\underline{c=\frac{1}{2}}$ //

$$c=a+\theta(b-a)=-1+\theta\cdot 3=\frac{1}{2}$$

$3\theta=\frac{3}{2}$, $\underline{\theta=\frac{1}{2}}$ //

(2) $f(x)=\sqrt{x}$ 区間 $[0,4]$

$f(0)=0$, $f(4)=2$

$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}$

$$\frac{1}{2}c^{-\frac{1}{2}}=\frac{2-0}{4-0}=\frac{1}{2}$$

$c^{-\frac{1}{2}}=1$. $\underline{c=1}$

$c=a+\theta(b-a)=$ $\quad 4\theta=1$ $\quad \underline{\theta=\frac{1}{4}}$

2.3 テイラーの定理

◇ 高次導関数

$$y=f(x)\xrightarrow{\text{微分}}f'(x)\xrightarrow{\text{微分}}f''(x)\to\cdots\to f^{(n)}(x)$$

$f''(x)$, $\dfrac{d^2}{dx^2}f(x)$, y'', $\dfrac{d^2y}{dx^2}$ ⇑ 第2次導関数

$f^{(n)}(x)$, $\dfrac{d^n f(x)}{dx^n}$, $y^{(n)}$, $\dfrac{d^n y}{dx^n}$ ⇑ 第n次導関数

◇ (1) $(e^x)^{(n)}=e^x$

∵ $y=e^x\to y'=(e^x)'=e^x\to y''=(e^x)'=e^x\to\cdots\ y^{(n)}=e^x$

(2) $\{\log(1+x)\}^{(n)}=(-1)^{n-1}\dfrac{(n-1)!}{(1+x)^n}$

∵ $y=\log(1+x)$

$y'=\dfrac{1}{1+x}=(1+x)^{-1}$

$y''=(-1)(1+x)^{-2}$

$y'''=(-1)(-2)(1+x)^{-3}$

$\vdots$

$y^{(n)}=(-)^{n-1}(n-1)!\,(1+x)^{-n}$

(3) $(\sin x)^{(n)}=\sin(x+\frac{\pi}{2})$

∵ $y=\sin x$

$y'=\cos x=\sin(x+\frac{\pi}{2})$

$y''=(\sin(x+\frac{\pi}{2}))'=\cos(x+\frac{\pi}{2})=\sin(x+\frac{\pi}{2}\times 2)$

$\vdots$

$y^{(n)}=\sin(x+\frac{\pi}{2}n)$

(4) $(\cos x)^{(n)}=\cos(x+\frac{\pi}{2}\cdot n)$

∵ $y=\cos x$

$y'=-\sin x=\cos(x+\frac{\pi}{2})$

$y''=-\sin(x+\frac{\pi}{2})=\cos(x+\frac{\pi}{2}\cdot 2)$

$\vdots$

$y^{(n)}=\cos(x+\frac{\pi}{\ }\ )$

(5) $\{(1+x)^\alpha\}^{(n)} = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$

$\because$ $y = (1+x)^\alpha$

$y' = \alpha(1+x)^{\alpha-1}$

$y'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$

$\vdots$

$y^{(n)} = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$

定理2-4.（ライプニッツの公式）

$u(x)$, $v(x)$ が n回微分可能

$\Rightarrow$ uvも n回微分可能

$(u\cdot v)^{(n)} = u^{(n)}v^{(0)} + {}_nC_1 u^{(n-1)}v^{(1)} + {}_nC_2 u^{(n-2)}v^{(2)} + \cdots + u^{(0)}v^{(n)}$

$\because$ $(uv)' = u'v + uv'$

$(uv)'' = (u'v + uv')' = u''v + u'v' + u'v' + uv''$

$= u''v + 2u'v' + uv''$

$(uv)^k = u^{(k)}v^{(0)} + {}_nC_1 u^{(k-1)}v^{(1)} + \cdots + u^{(0)}v^{(k)}$ と仮定し

$n=k+1$ の式を示す

$(uv)^{k+1} = \left(u^{(k)}v^{(0)} + {}_nC_1 u^{(k-1)}v^{(1)} + \cdots + u^{(0)}v^{(k)}\right)'$

$= u^{(k+1)}v^{(0)} + \underbrace{u^{(k)}v^{(1)} + {}_nC_1 u^{(k)}v^{(1)}}_{{}_nC_0 + {}_nC_1 = {}_{n+1}C_1} + \underbrace{u^{k-1}v^2}_{{}_nC_1} + \underbrace{\cdots}_{{}_nC_2} + u^{(1)}v^{(k)} + u^{(0)}v^{(k+1)})$

メモ ${}_nC_i + {}_nC_{i+1} = \dfrac{n!}{(n-i)!\,i!} + \dfrac{n!}{(n-i-1)!(i+1)!}$

$= \dfrac{n!}{(n-i-1)!\,i!}\left(\dfrac{1}{n-i} + \dfrac{1}{i+1}\right) = \dfrac{(n+1)!}{(n-i)!(i+1)!} = {}_{n+1}C_{i+1}$

$\dfrac{i+1+n-i}{(n-i)(i+1)}$

$= u^{(k+1)}v^{(0)} + {}_{n+1}C_1 u^{(k)}v^{(1)} + {}_{n+1}C_2 u^{(k-1)}v^{(2)} + \cdots + u^{(0)}v^{(k+1)})$

これは $n=k+1$ の式である ■

定理2.5 （テイラーの定理）

$f(x), f^{(1)}(x), \ldots, f^{(n-1)}(x)$ が $[a, b]$ で連続

$f^{(n)}$ が (a, b) で存在

$$\Rightarrow f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \frac{f'''(a)}{3!}(b-a)^3 + \cdots$$

$$+ \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + \underbrace{\boxed{\frac{f^{(n)}(c)}{n!}(b-a)^n}}_{= R_n}$$

$a < c < b$ をみたす c が少なくとも1つ存在。

R_n：ラグランジュの剰余項

∵ k を

$$f(b) = f(a) + f'(a)(b-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + k(b-a)^n$$

とおく。

$$F(x) = f(b) - \left\{ f(x) + f'(x)(b-x) + \cdots + \frac{f^{(n-1)}(x)}{(n-1)!}(b-x)^{n-1} + k(b-x)^n \right\}$$

$F(b) = f(b) - f(b) = 0$

$$F(a) = f(b) - \left\{ f(a) + f'(a)(b-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + k(b-a)^n \right\} = 0$$

$$F'(x) = -\left\{ \cancel{f'(x)} + \cancel{f''(x)(b-x)} + \cancel{f'(x)(-1)} + \cdots + \frac{f^{(n)}(x)}{(n-1)!}(b-x)^{n-1} + \cancel{\frac{f^{(n-1)}(x)}{(n-1)!}(b-x)^{n-2}(-1)} + kn(b-x)^{n-1}(-1) \right\}$$

$F'(c) = 0$ となる c が $a < c < b$ に存在。

$$\Rightarrow \frac{f^{(n)}(c)}{(n-1)!}(b-c)^{n-1} = nk(b-c)^{n-1} \longrightarrow k = \frac{f^{(n)}(c)}{n!}$$

よって

$$f(b) = f(a) + f'(a)(b-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + \frac{f^{(n)}(c)}{n!}(b-a)^n$$

□

（追記）

$n = 1$ とすると

$f(b) = f(a) + f'(c)(b-a)$ 平均値の定理となる

追記 2.3. 一般に

$m=1,2,\cdots n$ のどれか

$$f(b)=f(a)+f'(a)(b-a)+\cdots+\frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1}+k(b-a)^{m}$$ と

として ロルの定理用いると

$$F(x)=f(b)-\left\{f(x)+f'(x)(b-x)+\cdots+\frac{f^{(n-1)}(x)}{(n-1)!}(b-x)^{n-1}+k(b-x)^{m}\right\}$$

$$F(b)=0,\quad F(a)=0$$

$$F'(x)=-\left\{\frac{f^{(n)}(x)}{(n-1)!}(b-x)^{n-1}+km(b-x)^{m-1}(-1)\right\}$$

$$F'(c)=0 \Rightarrow m(b-c)^{m-1}k=\frac{f^{(n)}(c)}{(n-1)!}(b-c)^{n-1}$$

$$\therefore k=\frac{f^{(n)}(c)}{m(n-1)!}(b-c)^{n-m}$$

$m=1$ のとき

$$f(b)=f(a)+f'(a)(b-a)+\cdots+\frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1}+\boxed{\frac{f^{(n)}(c)}{(n-1)!}(b-c)^{n-1}(b-a)}$$

コーシーの剰余項

⋮

$m=n$ のとき

$$f(b)=f(a)+\cdots+\boxed{\frac{f^{(n)}(c)}{n!}(b-a)^{n}}$$

ラグランジュの剰余項

例 2.3

$y = x^3 \sin x$ の第n次導関数は?

(解) $(x^3)' = 3x^2$, $(x^3)'' = 6x$, $(x^3)''' = 6$. $(x^3)^{(4)} = 0$

ライプニッツの公式より

$$y^{(n)} = \underbrace{(\sin x)^{(n)}}_{\sin(x+\frac{\pi}{2}n)} x^3 + {}_nC_1 \underbrace{(\sin x)^{(n-1)}}_{\sin(x+\frac{n-1}{2}\pi)} \overbrace{(x^3)'}^{3x^2} + {}_nC_2 \underbrace{(\sin x)^{(n-2)}}_{\sin(x+\frac{n-2}{2}\pi)} \overbrace{(x^3)''}^{6x} + {}_nC_3 \underbrace{(\sin x)^{(n-3)}}_{\sin(x+\frac{n-3}{2}\pi)} \overbrace{(x^3)'''}^{6}$$

$$= x^3 \sin\left(x+\frac{n}{2}\pi\right) + n \cdot 3x^2 \sin\left(x+\frac{n-1}{2}\pi\right)$$

$$+ \frac{n(n-1)}{\cancel{2}} \overset{3}{\cancel{6}} x \sin\left(x+\frac{n-2}{2}\pi\right)$$

$$+ \frac{n(n-1)(n-2)}{\cancel{3!}} \cancel{6} \sin\left(x+\frac{n-3}{2}\pi\right)$$

例 2.4

$f(x)$ が aを含む区間で2回微分可能で $f''(x)$ が連続のとき

$$\lim_{h\to 0} \frac{f(a+h)+f(a-h)-2f(a)}{h^2} = f''(a)$$

(∵) $f(a+h)-f(a) = f'(c_1)\,h$ $\quad c_1 = a+\theta_1 h \quad 0<\theta_1<1$

$f(a-h)-f(a) = f'(c_2)(-h)$ $\quad c_2 = a-\theta_2 h \quad 0<\theta_2<1$

$$f(a+h)+f(a-h)-2f(a) = f'(c_1)h - f'(c_2)h$$

$$\lim_{h\to 0} \frac{f(a+h)+f(a-h)-2f(a)}{h^2} = \lim_{h\to 0} \frac{f'(c_1)-f'(c_2)}{h}$$

$f(a+h) = f(a) + f'(a)h + \frac{f''(c_1)}{2}h^2$ $\quad c_1 = a+\theta_1 h \quad 0<\theta_1<1$

$f(a-h) = f(a) + f'(a)(-h) + \frac{f''(c_2)}{2}h^2$ $\quad c_2 = a+\theta_2(-h) \quad 0<\theta_2<1$

$$\text{左辺} = \lim_{h\to 0} \frac{\frac{f''(c_1)}{2}h^2 + \frac{f''(c_2)}{2}h^2}{h^2} = \lim_{\substack{h\to 0 \\ (c_1\to a \\ c_2\to a)}} \frac{f''(c_1)+f''(c_2)}{2} = \frac{f''(a)+f''(a)}{2} = \underline{f''(a)}$$

($f''(x)$ が連続なので)

問 2.3

$y = (x^2 + x + 1)e^x$ の $y^{(n)}$ は?

$v = x^2 + x + 1$, $v^{(1)} = 2x + 1$, $v^{(2)} = 2$, $v^{(3)} = 0$ ..

$u = e^x$, $u^{(n)} = e^x$

$$y^n = e^x(x^2 + x + 1) + n\,e^x(2x+1) + \frac{n(n-1)}{2}\,e^x \cdot 2$$

$$= \underline{e^x\{x^2 + x + 1 + n(2x+1) + n(n-1)\}}$$

問 2.4

$y = e^x \cos x$ のとき $y^{(n)} = 2^{n/2}\, e^x \cos\left(x + \frac{n\pi}{4}\right)$ を示せ.

$y^{(0)} = 2^0\, e^x \cos x = y$ より $n=0$ は OK.

$n = k$ で成立とすると

$$y^{(k+1)} = \left(2^{k/2}\, e^x \cos\left(x + \frac{k}{4}\pi\right)\right)'$$

$$= 2^{k/2}\left\{(e^x)'\cos\left(x + \frac{k}{4}\pi\right) + e^x\left(\cos\left(x + \frac{k}{4}\pi\right)\right)'\right\}$$

$$= 2^{k/2}\left\{e^x \cos\left(x + \frac{k}{4}\pi\right) - e^x \sin\left(x + \frac{k}{4}\pi\right)\right\}$$

$$= 2^{k/2}\, e^x \sqrt{2}\,\{\underbrace{\cos\left(x + \frac{k}{4}\pi\right)\frac{1}{\sqrt{2}} - \sin\left(x + \frac{k}{4}\pi\right)\frac{1}{\sqrt{2}}}_{\cos\left(x + \frac{k\pi}{4} + \frac{\pi}{4}\right)}\}$$

$$= 2^{\frac{k+1}{2}}\, e^x \cos\left(x + \frac{k+1}{4}\pi\right)$$

$n = k+1$ でも成立 よって 示せた。

2.3 マクローリンの定理，関数の多項式近似

定理2.6 $f(x)$ が $x=0$ を含む区間で，n回微分可能

$$\Rightarrow f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \underbrace{\boxed{\frac{f^{(n)}(c)}{n!}x^n}}_{=R_n}$$

ただし c は 0 と x の間の実数. $c=\theta x \ (0<\theta<1)$

∵ テイラーの定理で $a=0,\ b=x$ とおいたもの.

◇ 関数の多項式近似　nが大きいとき $|R_n| \ll 1$ となるとき

$$f(x) \fallingdotseq f(0) + f'(0)x + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}$$

◇ (1) $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \boxed{\frac{e^{\theta x}}{n!}x^n} = R_n$

∵ $e^x = e^0 + e^0 x + \frac{e^0}{2!}x^2 + \cdots + \frac{e^0}{(n-1)!}x^{n-1} + R_n.$ ■

(2) $\sin x \fallingdotseq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-)^{m-1}\frac{x^{2m-1}}{(2m-1)!} + 0 + \overbrace{\boxed{\frac{\sin(\theta x + \frac{2m+1}{2})}{(2m+1)!}x^{2m+1}}}^{R_{2m+1}}$

∵ $(\sin x)^{(n)}\Big|_{x=0} = \sin(x + \frac{n}{2}\pi)\Big|_{x=0} = \sin(\frac{n}{2}\pi) = 0, 1, 0, -1$ for $0+4m$, $1+4m$, $2+4m$, $3+4m$

(3) $\cos x \fallingdotseq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-)^{m-2}\frac{x^{2m-2}}{(2m-2)!} + 0 + \underbrace{\boxed{\frac{\cos(\theta x + \frac{2m}{2}\pi)}{(2m)!}x^{2m}}}_{R_{2m}}$

∵ $(\cos x)^{(n)} = \cos(x + \frac{n}{2}\pi)$　2-1

(4) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-)^{n-2}\frac{x^{n-1}}{n-1} + \boxed{\frac{x^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(\theta x)}$　コーシー剰余項

$$= \frac{x^n}{\cancel{(n-1)!}}(1-\theta)^{n-1}(-)^{n-1}\frac{\cancel{(n-1)!}}{(1+\theta x)^n}$$

$$= \boxed{(\theta-1)^{n-1}\left(\frac{x}{1+\theta x}\right)^n} = R_n \quad \left(\begin{array}{c}-1<x\leqq 1 \\ \alpha \text{ には} \\ R_n \to 0\end{array}\right)$$

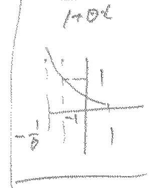
∵ $(\log 1+x)^{(n)} = (-)^{n-1}\frac{(n-1)!}{(1+x)^n}$

(5) $(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots \frac{\alpha(\alpha-1)\cdots(\alpha-n+2)}{(n-1)!}x^{n-1}$

$$+ \boxed{\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}(1+\theta x)^{\alpha-n}x^n} = R_n$$

∵ $\{(1+x)^\alpha\}^{(n)} = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$

$\frac{1-\theta}{1+\theta x}$



例2.5

$\tan x$ のマクローリン 3次まで（R_4 は不要）

$y = \tan x \longrightarrow y|_{x=0} = 0$

$y' = \frac{1}{\cos^2 x} \longrightarrow y'|_{x=0} = 1$

$y'' = +2\cos^{-3}x \sin x \longrightarrow y''|_{x=0} = 0$

$y''' = +2(-3)\cos^{-4}x \sin x \sin x + 2\cos^{-3}x \cos x \longrightarrow y'''|_{x=0} = 2.$

$$\tan x = 0 + 1\cdot x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + R_4$$

$$= \underline{x + \frac{x^3}{3} + R_4}$$

例2.6.

(1) e^x の近似式 $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^9}{9!}$ で $x=1$ とおいて

e の近似値を求めよ。

$$R_{10} = \frac{e^{\theta x}}{10!}x^{10} \to |R_{10}| = \frac{|x|^{10}}{10!}e^{\theta x} \leq \frac{|x|^{10}e^1}{10!} = \frac{e}{10!} < \frac{3}{10!} = 0.00000082$$

$$e^1 \fallingdotseq \underbrace{1 + 1 + \frac{1}{2!}}_{2.5} + \underbrace{\frac{1}{3!}}_{0.166666} + \underbrace{\frac{1}{4!}}_{0.041666} + \underbrace{\frac{1}{5!}}_{0.008333} + \underbrace{\frac{1}{6!}}_{0.001388} + \underbrace{\frac{1}{7!}}_{0.0001984} + \underbrace{\frac{1}{9!}}_{0.00000275}$$

$= 2.7182818\cdots$

(2) $\log\frac{1+x}{1-x}$ $|x| < 1$ の x の $2m-1$ 次近似式は?

与式 $= \log(1+x) - \log(1-x)$ なので

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots + \frac{x^{2m-1}}{(2m-1)}$$

$$-)\ \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^{2m-1}}{(2m-1)}$$

$$\log(1+x) - \log(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2m-1}}{2m-1}\right)$$

//

問2.5 $(1+x)^{1/3}$ の x の 2次の近似式は?

$$(1+x)^{1/3} \fallingdotseq 1 + \frac{1}{3}x + \frac{1}{3}\left(-\frac{2}{3}\right)\frac{x^2}{2}$$

$= 1 \quad x \quad x^2$

2.4 マクローリン級数

◇ $x=0$ の近くで $f(x)$ が何回でも微分可能ならば

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + R_n$$

において、$R_n \to 0 \quad (n \to \infty)$ ならば

$$f(x) = \underbrace{f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots}$$

この x の整級数を

マクローリン級数という

(1) $e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \cdots \qquad (-\infty < x < \infty)$

∵ $|R_n| = \left|\frac{x^n}{n!}e^{\theta x}\right| = \frac{|x|^n e^{|x|}}{n!} \longrightarrow 0 \qquad (-\infty < x < \infty)$

(2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (-\infty < x < \infty)$

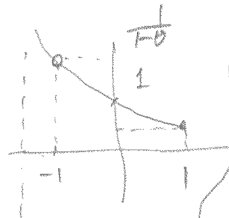
∵ $|R_{2m+1}| = \left|\frac{|x|^{2m+1}}{(2m+1)!}\sin(\quad)\right| \leq \frac{|x|^{2m+1}}{(2m+1)!} \to 0$

(3) $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad (-\infty < x < \infty)$

∵ $|R_{2m}| = \left|\frac{x^{2m}}{(2m)!}\cos(\quad)\right| \leq \frac{|x|^{2m}}{2m!} \to 0$

(4) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \qquad (-1 < x \leq 1)$

∵ $|R_n| = \left|(\theta-1)^{n-1}\left(\frac{x}{1+\theta x}\right)^n\right| \leq \frac{|x|^n}{1+\theta x}\left|\frac{1-\theta}{1+\theta x}\right|^{n-1} < \frac{1}{1-\theta}\left|\frac{1-\theta}{1+\theta x}\right|^{n-1} \longrightarrow 0$



$$\frac{1}{1+\theta} \leq \frac{1}{1+\theta x} < \frac{1}{1-\theta}$$

$$0 < \frac{1-\theta}{1+\theta} \leq \frac{1-\theta}{1+\theta x} < 1$$

より

$$\left|\frac{1-\theta}{1+\theta x}\right| < 1$$

$|x| \leq 1$

(5) $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots \qquad -1 < x \leqq 1$

∵ $|R_n| = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}(1+\theta x)^{\alpha-n}x^n$ ×

コーシーの剰余項を用いると

$$R_n = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}x^n(1-\theta)^{n-1}(1+\theta x)^{\alpha-n}$$

$$|R_n| = \left|\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}x^n\right| \left|\frac{1-\theta}{1+\theta x}\right|^{n-1} \underbrace{(1+\theta x)^{\alpha-1}}_{\text{有界}}$$

$0 < \frac{1-\theta}{1+\theta x} < 1$ より

$$\leqq \underbrace{\left|\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}x^n\right|}_{C_n} \left|\frac{1-\theta}{1+\theta x}\right|^{n-1}(1+\theta x)^{\alpha-1} \longrightarrow 0.$$

$$\left|\frac{C_{n+1}}{C_n}\right| = \left|\frac{\alpha-n}{n}\cancel{(n-1)!}\, x\right| \longrightarrow |x| \leqq 1$$

[(4) と (5) は

注　$|x|<1$ と $x=1$ に分けて示す方がすっきりする

$|x|<1 \to$ コーシー　　$x=1 \to$ ラグランジュ.]

注 2.2

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

において $x = i\theta$ $(i^2 = -1)$ とおくと

$$\underline{e^{i\theta}} = \underset{\sim}{1} + i\theta + \underset{\sim}{\frac{-x^2}{2!}} + \frac{-ix^3}{3!} + \underset{\sim}{\frac{x^4}{4!}} + \frac{ix^5}{5!} + \cdots$$

$$= \underline{\cos\theta + i\sin\theta}$$ オイラーの関係式

補助定理

$$\lim_{n\to\infty} \underbrace{\frac{x^n}{n!}}_{=C_n} = 0 \qquad (-\infty < x < \infty)$$

$\because$ $$\frac{C_{n+1}}{C_n} = \frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n} = \frac{x}{n+1} \xrightarrow[(n\to\infty)]{} 0. \qquad (-\infty < x < \infty)$$

例 2.7 (済)

例 2.8 $f(x) = \sqrt{1+x}$ をマクローリン級数で表せ.

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{x^2}{2!} + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^3}{3!}$$

$$+ \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{x^4}{4!} + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{x^2}{4\cdot 2} + \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 + \cdots$$

問2.6. 済

問2.7.

$$f(x) = \frac{2}{1-2x} - \frac{1}{1-x}$$

$$= 2\left(1 + 2x + (2x)^2 + (2x)^3 + \cdots\right) \quad \cdots \quad \left(|x| < \tfrac{1}{2}\right)$$

$$- \left(1 + x + x^2 + x^3 + \cdots\right) \quad \cdots \quad \left(|x| < 1.\right)$$

$$= \underline{(2-1) + (2^2-1)x + (2^3-1)x^2 + (2^4-1)x^3 + \cdots} \qquad \left(|x| < \tfrac{1}{2}\right)$$

$\boxed{2.5}$ 関数の増減，極値

定理2.7 $f(x)$ が $[a,b]$ で連続 (a,b) で微分可能.

(1) (a,b) で常に $f'(x)>0 \Rightarrow$ $a \leq x_1 < x_2 \leq b$

$$\frac{f(x_1)-f(x_2)}{x_1-x_2} = f'(c) > 0$$

$\uparrow$
$x_1 < c < x_2$

$f(x_1)-f(x_2) < 0$.

すなわち $[a,b]$ で増加関数.

(2) (a,b) で常に $f'(x)<0 \Rightarrow$ $f(x_1)-f(x_2) = f'(c)(x_1-x_2) < 0$.

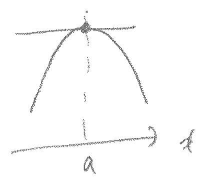
$f(x_1)-f(x_2) < 0$

$[a,b]$ で減少関数

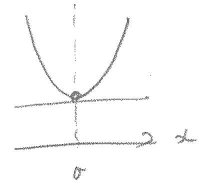
◇ 極値.

$x=a$ 近くで連続な関数 $f(x)$ が a の近くの $x\ (\neq a)$ に対して

。 $f(x) < f(a)$ であるとき $x=a$ で極大



・ $f(x) > f(a)$ であるとき $x=a$ で極小



「極値」という

定理2.8 $f(x)$ が微分可能で $f(a)$ が極値 $\Rightarrow f'(a)=0$

(∵) $x=a$ で極大とすると $f(x) < f(a)$

$$f_-(a) = \lim_{h\to -0}\frac{f(a+h)-f(a)}{h} \geq 0$$

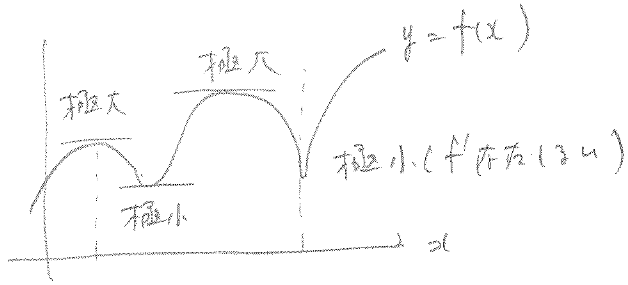
$$f_+(a) = \lim_{h\to +0}\frac{f(a+h)-f(a)}{h} \leq 0$$

$f'(a) = f'_-(a) = f'_+(a)$ より $f'(a)=0$.

$x=a$ で極小の場合も同様 ☒

◇ 極値の求め方

$f'(x)=0$ or $f'(x)$ が存在しない x の値を求める



増減表を作ると

x		a	
$f'(x)$	$+$		$-$
$f(x)$	↗		↘

極大

x		a	
$f'(x)$	$-$		$+$
$f(x)$	↘		↗

極小

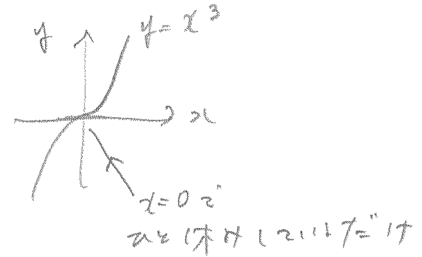
注 2.3

$f'(a)=0$ だからといって極値である訳ではない。

ex $f(x)=x^3$　$f'(x)=3x^2$ より $x=0$ で $f'(x)=0$ だが

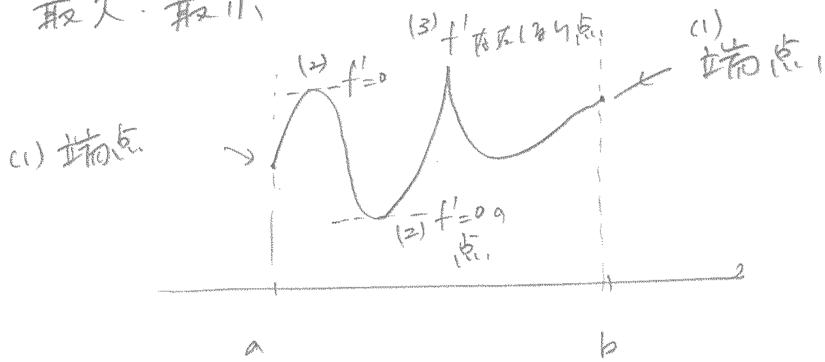
x		0	
$f'(x)$	$+$		$+$
$f(x)$	↗		↗

極値ではない



追記 2.5　変域の端点は極値の対象にならないことが多い

◇ 最大・最小



(1) (2) (3) の候補の $f(x)$ の値と比較して求める

例2.9.

$f(x)=(x-5)\sqrt[3]{x^2}$ の増減を調べ，極値を求めよ.

$$f'(x)=\sqrt[3]{x^2}+(x-5)\frac{2}{3}x^{-\frac{1}{3}}$$

$$=\frac{3x^{\frac{2}{3}}x^{\frac{1}{3}}+(x-5)2}{3\sqrt[3]{x}}=\frac{3x+2(x-5)}{3\sqrt[3]{x}}=\frac{5x-10}{〃}$$

$$=\frac{5(x-2)}{3\sqrt[3]{x}}$$

x		0		2	
$f'(x)$	+		−	0	+
$f(x)$	↗	0	↘		↗

$f(2)=-3\sqrt[3]{4}$

極大 $x=0$ のとき $f(0)=0$

極小 $x=2$ のとき $f(2)=-3\sqrt[3]{4}$

例 2.10

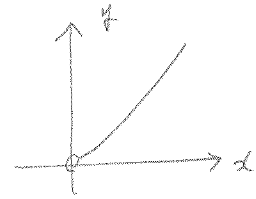
$\log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ $\quad (x>0)$ を示せ.

解 $\quad y = f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \log(1+x)$

$$f'(x) = 1 - x + x^2 - \frac{1}{1+x} = \frac{1+x^3-1}{1+x} = \frac{x^3}{1+x} > 0$$

よって

x	0	
$f'(x)$	0	$+$
$f(x)$	0	$\nearrow$



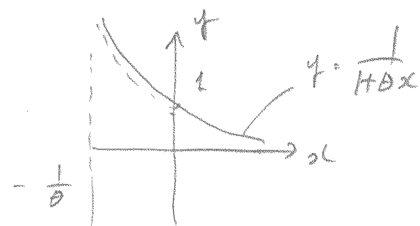
よって $\quad f(x) > 0 \quad (x>0)$

$$\Leftrightarrow \quad x - \frac{x^2}{2} + \frac{x^3}{3} > \log(1+x)$$

(別解) $\quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + R_4$

$$R_4 = (\theta-1)^{n-1}\left(\frac{x}{1+\theta x}\right)^n \quad (n=4)$$

$$= (\theta-1)^3\left(\frac{x}{1+\theta x}\right)^4$$



$$\left.\begin{array}{l} 0 < \frac{1}{1+\theta x} < 1 \\ 0 < x < 1 \\ (\theta-1)^3 < 0 \end{array}\right\} \text{より} \quad R_4 < 0.$$

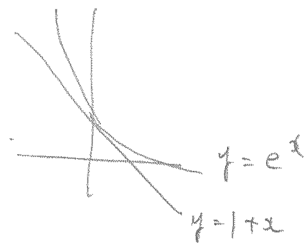
よって $\quad -R_4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \log(1+x) > 0 \quad \boxtimes$

問2.8

(1) $e^x > 1+x+\frac{x^2}{2}$ $(x>0)$ を示せ.

解) $f(x)=e^x-\left(1+x+\frac{x^2}{2}\right)$ $(x>0)$

$f'(x)=e^x-(1+x)$



x		0	
$f'(x)$	+	0	+
$f(x)$		0	↗

よって $x>0$ のとき $f(x)>0$

つまり $e^x > 1+x+\frac{x^2}{2}$

(2) $x \underset{①}{>} \sin x \underset{②}{>} x-\frac{x^3}{6}$ $\left(0<x<\frac{\pi}{2}\right)$

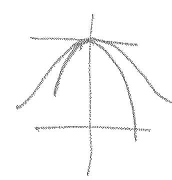
①より $f(x)=\sin x - x$

$f'(x)=\cos x - 1$

x	0	$\frac{\pi}{2}$
$f'(x)$	0	−
$f(x)$	0	↘

$f(x)<0$ つまり

$\sin x < x$ … ①'



②より $g(x)=\sin x-\left(x-\frac{x^3}{6}\right)$

$g'(x)=\cos x-\left(1-\frac{x^2}{2}\right)$

x	0	$\frac{\pi}{2}$
$g'(x)$	0	+
$g(x)$	0	↗

$g(x)>0$ つまり $\sin x-\left(x-\frac{x^3}{6}\right)>0$ … ②'

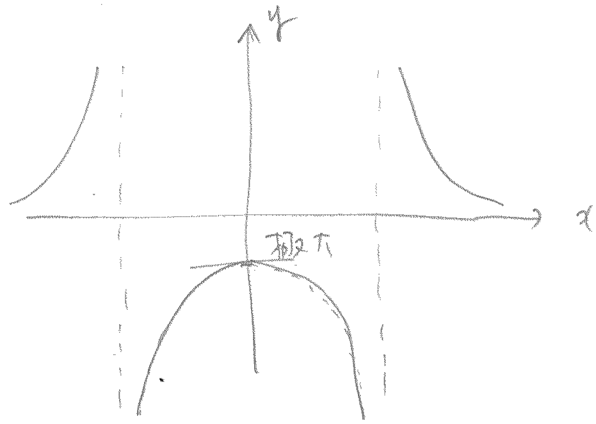
①'②'より 与式示せた。

問 2.9 グラフ

(1) $f(x) = \dfrac{1}{x^2-1}$

$f'(x) = \dfrac{-1}{(x^2-1)^2} 2x$

x		-1		0		1	
$f'(x)$	$+$	$\times$	$+$	0	$-$	$\times$	$-$
$f(x)$	↗		↗	-1	↘		↘



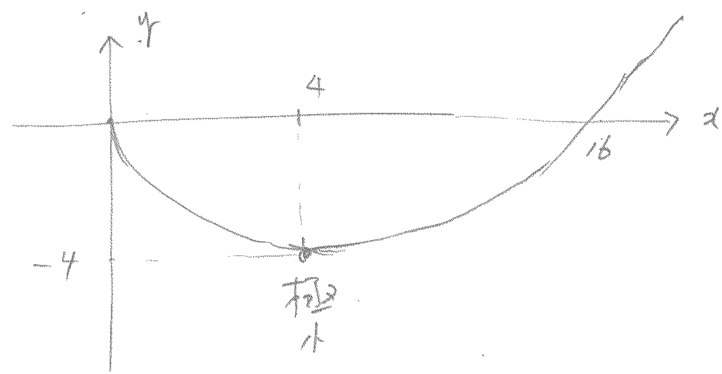
(2) $f(x) = x - 4\sqrt{x}$ $(x \geqq 0)$

$f'(x) = 1 - 4 \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}}$

$= 1 - \dfrac{2}{\sqrt{x}} = \dfrac{\sqrt{x}-2}{\sqrt{x}}$

x	0		4	
$f'(x)$	$\times$	$-$	0	$+$
$f(x)$	0	↘	-4	↗
			⇑ 極小	

$f(x) = \sqrt{x}(\sqrt{x} - 4)$



(3) $f(x) = x\sqrt{2x - x^2} \qquad (0 \leqq x \leqq 2)$

$$f'(x) = \sqrt{2x - x^2} + x\frac{1}{2}(2x - x^2)^{-\frac{1}{2}}(2 - 2x)$$

$$= \sqrt{2x - x^2} + \frac{x(1-x)}{\sqrt{2x - x^2}}$$

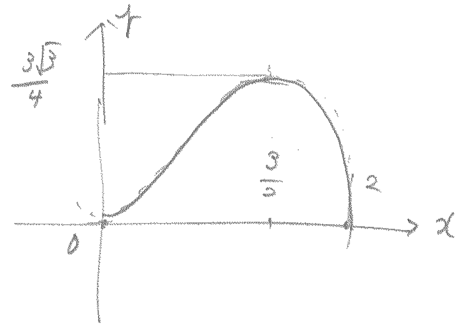
$$= \frac{2x - x^2 + x - x^2}{\sqrt{2x - x^2}} = \frac{3x - 2x^2}{\sqrt{2x - x^2}}$$

$$= \frac{-2x\left(x - \frac{3}{2}\right)}{\sqrt{2x - x^2}}$$

x	0		$\frac{3}{2}$		2
$f'(x)$	0	$+$	0	$-$	$\times$
$f(x)$	0	$\nearrow$	$\boxed{\ }$	$\searrow$	0

$$\frac{3}{2}\sqrt{3 - \frac{9}{4}}$$

$$= \frac{3}{2}\sqrt{\frac{3}{4}} = \frac{3\sqrt{3}}{4}$$



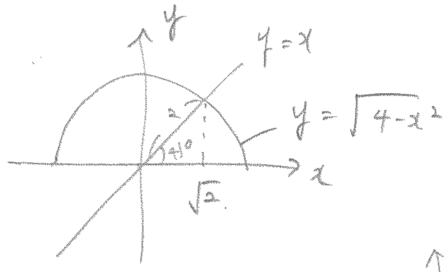
問 2.10

$f(x) = x + \sqrt{4 - x^2} \qquad 0 \leqq x^2 \leqq 4$ より $\underline{-2 \leqq x \leqq 2}$

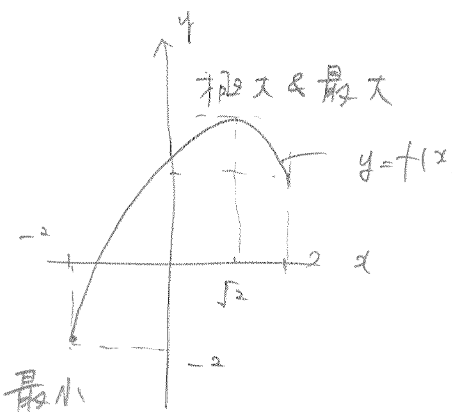
$$f'(x) = 1 + \frac{1}{2}(4 - x^2)^{-\frac{1}{2}}(-2x)$$

$$= 1 + \frac{-x}{\sqrt{4 - x^2}}$$

$$= \frac{\sqrt{4 - x^2} - x}{\sqrt{4 - x^2}}$$ → 分子

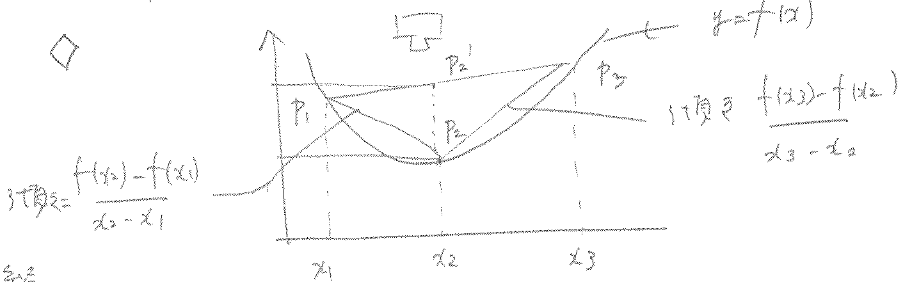


x	-2		$\sqrt{2}$		2
$f'(x)$	$\times$	$+$	0	$-$	$\times$
$f(x)$	-2	$\nearrow$	$\sqrt{2}+\sqrt{2}$ $= 2\sqrt{2}$	$\searrow$	2



$\boxed{2.6}$ 曲線の凹凸と変曲点

◇



$f(x)$ …連続

区間 I に含まれる $\forall x_1, x_2, x_3$ $(x_1 < x_2 < x_3)$ に対して

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leqq \frac{f(x_3)-f(x_2)}{x_3-x_2} \quad ①$$

が成立 $\Longleftrightarrow$「区間 I で下に凸」

〃 $\geqq$ 〃 $\Longleftrightarrow$「〃 I に凸」

①の図形的性.

$$\frac{f(x_2)}{x_2-x_1} + \frac{f(x_2)}{x_3-x_2} \leqq \frac{f(x_1)}{x_2-x_1} + \frac{f(x_3)}{x_3-x_2}$$

$$(x_3 - \cancel{x_2} + \cancel{x_2} - x_1)\, f(x_2) \leqq (x_3-x_2) f(x_1) + (x_2-x_1) f(x_3)$$

$$\underbrace{f(x_2)}_{\Downarrow\ x_2P_2} \leqq \frac{(x_3-x_2)f(x_1)+(x_2-x_1)f(x_3)}{x_3-x_1}$$

P_1, P_3 を $x_2-x_1 : x_3-x_2$ に内分する点 P_2' の y 座標

〃

x_2P_2'

$\Rightarrow$ グラフの点が P_1P_3 の下にある.

さらに　$x_2P_2 \leqq x_2P_2'$ より

$$P_1P_2 \text{の傾き} \leqq \underbrace{P_1P_2' \text{の傾き} = P_2'P_3 \text{の傾き}}_{P_1P_3 \text{の傾き}} \leqq P_2P_3 \text{の傾き}$$

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leqq \frac{f(x_3)-f(x_1)}{x_3-x_1} \leqq \frac{f(x_3)-f(x_2)}{x_3-x_2} \quad \cdots\cdots (*)$$

定理2.9

(1) $f(x)$ が (a,b) で 下に凸 $\Leftrightarrow f''(x) \geqq 0,\ x \in (a,b)$

(2) 上に凸 $\Leftrightarrow f''(x) \leqq 0,\ x \in (a,b)$

(証) (1)

$p_2 \to p_1$ として

$$f'(x_1) \leqq \frac{f(x_3)-f(x_1)}{x_3-x_1}$$

$p_2 \to p_3$ として

$$\frac{f(x_3)-f(x_1)}{x_3-x_1} \leqq f'(x_3)$$

$\Big\}\ f'(x_1) \leqq f'(x_3)$

$x_1 < x_3$ のとき $f'(x_1) \leqq f'(x_3)$ なので $f'(x)$ は単調増加である.

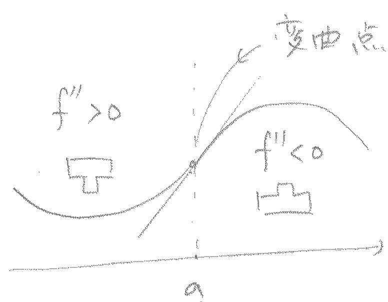
$f'(x)$ が Iで 微分可能であるとき ($f''(x)$ が存在するとき)

また $f'(x)$ が単調増加 $\underset{\text{Iで}}{\Leftrightarrow}$ $f''(x) \geqq 0$

ここらへんは 平均値の定理の系で 用いている

(2) 同様

◇ 変曲点



$f''(a)=0$

◇ グラフの描き方

(1) ~(7) 例題で練習

例 2.1

・増減，凹凸，グラフ

(1) $y = e^{-x^2}$

$y' = e^{-x^2}(-2x)$

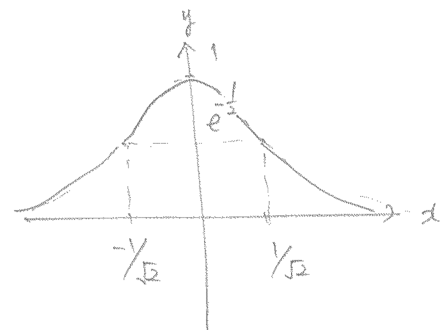
$y'' = e^{-x^2}(-2x)^2 + e^{-x^2}(-2)$

$= e^{-x^2}(4x^2 - 2)$

$= 4e^{-x^2}(x^2 - \frac{1}{2})$

x	$-\infty$		0		∞
y'		$+$	0	$-$	
y	0	$\nearrow$	1	$\searrow$	0

x		$\frac{-1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$	
y''	$+$	0	$-$	0	$+$
	$\cup$	$e^{-\frac{1}{2}}$	$\cap$	$e^{-\frac{1}{2}}$	$\cup$



(2) $y = \log(1+x^2)$

$y' = \frac{2x}{1+x^2}$

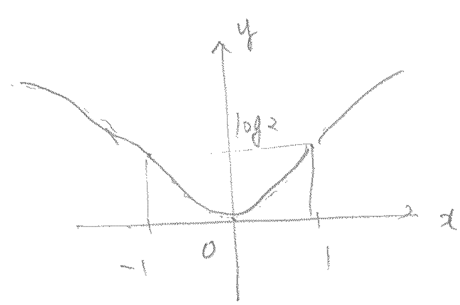
$y'' = \frac{2(1+x^2) - 2x(2x)}{(1+x^2)^2}$

$= \frac{2 - 2x^2}{(1+x^2)^2}$

$= \frac{2(1-x^2)}{(1+x^2)^2}$

x		0	
y'	$-$	0	$+$
y	$\searrow$	0	$\nearrow$

x		-1		1	
y''	$-$	0	$+$	0	$-$
y	$\cap$	$\log 2$	$\cup$	$\log 2$	$\cap$



問 2-11

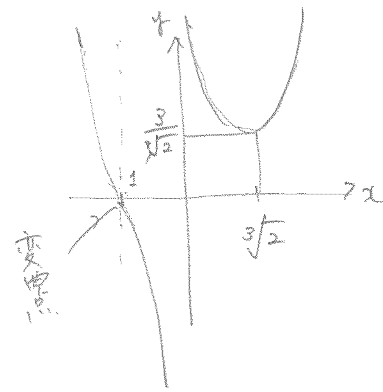
(1) $y = x^2 + \frac{1}{x} = \frac{x^3+1}{x}$

$y' = 2x - x^{-2} = \frac{2x^3 - 1}{x^2}$

$y'' = 2 + 2x^{-3} = \frac{2x^3 + 2}{x^3}$

x		0		$\frac{1}{\sqrt[3]{2}}$	
y'		$\times$	$-$	0	$+$
y	$\searrow$		$\searrow$	$\frac{3}{\sqrt[3]{2}}$	$\nearrow$

x		-1		0	
y''	$-$	0	$+$	$\times$	$+$
y	$\cap$	0	$\cup$		$\cup$



(2) $y = \frac{x}{x^2+1}$

$y' = \frac{x^2+1 - x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$

$y'' = \frac{-2x}{(x^2+1)^2} + (-x^2+1)(-2)(x^2+1)^{-3}2x$

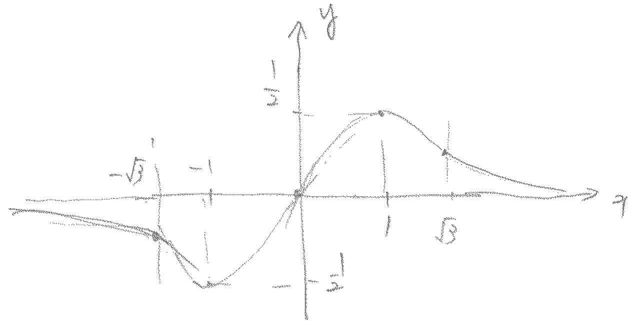
$= \frac{-2x(x^2+1) - 4x(-x^2+1)}{(x^2+1)^3}$

$= \frac{-2x\{x^2+1-2x^2+2\}}{〃}$

$= \frac{-2x(-x^2+3)}{〃}$

x	$-\infty$		-1		1		∞
y'		$-$	0	$+$	0	$-$	
y	0	$\searrow$	$-\frac{1}{2}$	$\nearrow$	$\frac{1}{2}$	$\searrow$	0

x		$-\sqrt{3}$		0		$\sqrt{3}$	
y''	$-$	0	$+$	0	$-$	0	$+$
y	$\cap$	$-\frac{\sqrt{3}}{4}$	$\cup$	0	$\cap$	$\frac{\sqrt{3}}{4}$	$\cup$



(3) $y = \dfrac{8}{x^2+4}$

$$y' = 8 \cdot \frac{-2x}{(x^2+4)^2} = \frac{-16x}{(x^2+4)^2}$$

$$y'' = \frac{-16}{(x^2+4)^2} - 16x\,\frac{-2\cdot 2x}{(x^2+4)^3}$$

$$= \frac{16}{(x^2+4)^3}\left\{-\cancel{16}(x^2+4) + 4x^2\right\}$$

$$= \frac{+16}{(x^2+4)^3}(3x^2-4)$$

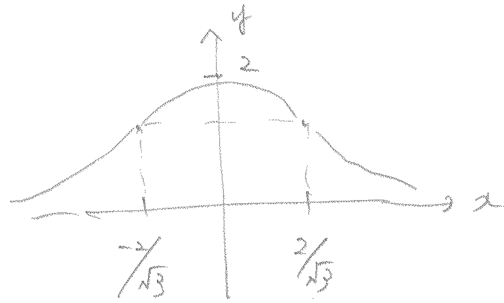
x	∞		0		∞
y'		$+$	0	$-$	
y		$\nearrow$	2	$\searrow$	

x		$\frac{-2}{\sqrt{3}}$		$\frac{2}{\sqrt{3}}$	
y''	$+$	0	$-$	0	$+$
y	$\cup$	$\frac{3}{2}$	$\cap$	$\frac{3}{2}$	$\cup$

$$\frac{8}{\frac{4}{3}+4}$$

$$= \frac{2}{\frac{1}{3}+1} \quad \frac{4}{3}$$

$$= \frac{\cancel{2}\cdot 3}{\cancel{4}_2}$$



(4) $y = x^3 + 6x^2 + 9x + 2$

$$y' = 3x^2 + 12x + 9$$
$$= 3(x^2+4x+3)$$
$$= 3(x+3)(x+1)$$

$$y'' = 6x + 12$$
$$= 6(x+2)$$

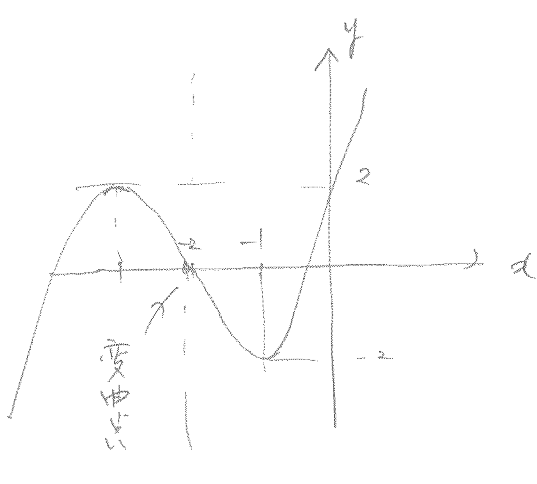
x		-3		-1	
y'	$+$	0	$-$	0	$+$
y	$\nearrow$	$\boxed{}$	$\searrow$	$\circ$	$\nearrow$

$$-\cancel{27} + \underbrace{6\cdot 9}_{54} - \cancel{27} + 2 = \boxed{2}$$

$$-1+6-9+2 = -2$$

x		-2	
y''	$-$	0	$+$
y	$\cap$	$\triangle$	$\cup$

$$-8+24-18+2 = 0$$



2.7 不定形とロピタルの定理

◇ 不定形

例) $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$ $\left(\frac{0}{0}\text{不定形}\right)$, $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x}$ $\left(\frac{0}{0}\text{不定形}\right)$

定理 2.10

$f(x)$, $g(x)$ は $x=a$ の近くで微分可能, $f(a)=g(a)=0$, $g'(x)\neq 0$ とする

もし $\displaystyle\lim_{x\to a}\frac{f'(x)}{g'(x)}=A$ が存在するなら $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=A$

有限

(∵) コーシーの平均値の定理より

$x\neq a$ に対して

$$\frac{f(x)-\overset{0}{\boxed{f(a)}}}{g(x)-\underset{0}{\boxed{g(a)}}}=\frac{f'(c)}{g'(c)}$$ (c は a と x の間の実数) が成立

$x\to a$ のとき $c\to a$ なので

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{c\to a}\frac{f'(c)}{g'(c)}=A.\quad\boxtimes$$

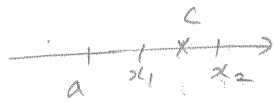
定理 2.1

$f(x)$, $g(x)$ は $x=a$ の近くで $x=a$ を除いて微分可能

$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$ とする。$\lim_{x\to a} \dfrac{f'(x)}{g'(x)} = A$ $(-\infty \leqq A \leqq \infty)$

が存在するなら $\lim_{x\to a} \dfrac{f(x)}{g(x)} = A$.

まず $x\to a+0$ の極限を考える。



$a < x_1 < x_2$ に対して

$$\frac{f(x_1)-f(x_2)}{g(x_1)-g(x_2)} = \frac{f'(c)}{g'(c)} \qquad (x_1 < c < x_2)$$

$$\frac{\dfrac{f(x_1)}{g(x_1)} - \dfrac{f(x_2)}{g(x_1)}}{1-\dfrac{g(x_2)}{g(x_1)}} = \frac{f'(c)}{g'(c)}$$

$$\frac{f(x_1)}{g(x_1)} = \frac{f'(c)}{g'(c)}\left\{1-\frac{g(x_2)}{g(x_1)}\right\} + \frac{f(x_2)}{g(x_1)}$$

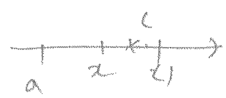
ここで $x_1 \to a+0$ とすると $g(x_1)\to\infty$ より

$$\lim_{x_1\to a+0} \frac{f(x_1)}{g(x_1)} = \lim_{\underset{?}{c\to}} \frac{f'(c)}{g'(c)},$$

ε δ 論法使わないとツメがあまい　$x\to a+0$ で考える

まず $\lim_{x\to a+0} \dfrac{f'(x)}{g'(x)} = A \neq \pm\infty$ のとき を考える

$a < x < x_1$ に対して $\left|\dfrac{f'(x)}{g'(x)} - A\right| < \varepsilon$ と出来る。



$$\frac{f(x)-f(x_1)}{g(x)-g(x_1)} = \frac{f'(c)}{g'(c)} \qquad \frac{\dfrac{f(x)}{g(x)} - \dfrac{f(x_1)}{g(x)}}{1-\dfrac{g(x_1)}{g(x)}} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}\left\{1-\frac{g(x_1)}{g(x)}\right\} + \frac{f(x_1)}{g(x)}$$

よって

$$\left|\frac{f(x)}{g(x)}-A\right| \leqq \left|\frac{f(x)}{g(x)}-\frac{f'(c)}{g'(c)}\right| + \left|\frac{f'(c)}{g'(c)}-A\right| \leqq \varepsilon+\varepsilon=2\varepsilon.$$

これは $\displaystyle\lim_{x\to a+0}\frac{f(x)}{g(x)}=A$ を示す.

$\displaystyle\lim_{x\to a-0}\frac{f(x)}{g(x)}=A$ も同様

$A=\infty$ の場合は.

$\forall L,\quad a<{}^{\forall}x<{}^{\exists}x_1 \quad \dfrac{f'(x)}{g'(x)}>L$ と出来る.

$$\frac{f(x)}{g(x)}=\frac{f'(c)}{g'(c)}\left\{1-\frac{g(x_1)}{g(x)}\right\}+\frac{f(x_1)}{g(x)}$$

$a<x<\delta \quad 1-\dfrac{g(x_1)}{g(x)}>\dfrac{1}{2}, \quad \dfrac{f(x_1)}{g(x)}>-\dfrac{1}{2}$ と出来る

$$\frac{f(x)}{g(x)}>L\frac{1}{2}-\frac{1}{2}=\frac{L-1}{2}$$

これは $\displaystyle\lim_{x\to a+0}\frac{f(x)}{g(x)}=\infty$ を示している.

$A=-\infty$ の場合も同様

追記 2.6

(1) $0\cdot\infty$ の不定形のとき、 $f\cdot g = \dfrac{f}{\frac{1}{g}}$ ($\frac{0}{0}$ 不定形) $= \dfrac{g}{\frac{1}{f}}$ ($\frac{\infty}{\infty}$ 不定形)

(2) $\infty^0, 1^\infty, 0^0$ など指数の入った不形形のときは対数を考えるとよい

ex $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

$$\log\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right) = \lim_{x\to\infty}\log\left(1+\frac{1}{x}\right)^x = \lim \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$= \lim \frac{\frac{1}{1+\frac{1}{x}}\cdot\cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1$$

よって $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$

(3) $\infty-\infty$ 不定形のとき

$$f-g = \frac{\frac{1}{g}-\frac{1}{f}}{\frac{1}{fg}} \quad \left(\frac{0}{0}\text{ 不定形}\right)$$

注意 2.4

$$\infty\cdot\infty = \infty,\quad \infty+\infty = \infty,\quad \frac{1}{+0} = \infty,\quad \frac{1}{-0} = -\infty,\quad \frac{1}{\infty} = 0$$

例 2-12

(1) $\displaystyle\lim_{x\to 0}\frac{e^x - e^{-x}}{\sin x}$ $\left(\frac{0}{0}\text{不定形}\right)$

$\displaystyle = \lim_{x\to 0}\frac{e^x + e^{-x}}{\cos x} = \frac{1+1}{1} = \underline{2}$ //

(2) $\displaystyle\lim_{x\to +0}\left(\frac{1}{x}\right)^{\sin x}$ の log をとると

$\displaystyle\lim_{x\to +0}\log\left(\frac{1}{x}\right)^{\sin x} = \lim_{x\to +0} -\sin x\log x$ $(0\cdot\infty\ \text{不定形})$

$\displaystyle = \lim_{x\to +0}\frac{\log x}{-\frac{1}{\sin x}} = \lim\frac{\frac{1}{x}}{\frac{\cos x}{\sin^2 x}} = \lim_{x\to +0}\underbrace{\frac{\sin x}{x}}_{\to 1}\underbrace{\frac{\sin x}{\cos x}}_{\to 0} = \underline{0}.$

よって $\displaystyle\underline{\lim_{x\to +0}\left(\frac{1}{x}\right)^{\sin x} = e^0 = 1}$

(3) $\displaystyle\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\log x}\right)$ $(\infty - \infty\ \text{不定形})$

$\displaystyle = \lim\frac{\log x - \frac{x-1}{x}}{\frac{1}{x}(x-1)\log x} = \lim_{x\to 1}\frac{\log x - \frac{x-1}{x}}{\frac{x-1}{x}\log x}$ $\left(\frac{0}{0}\text{不定形}\right)$

$\displaystyle = \lim\frac{\frac{1}{x} - \frac{x-(x-1)}{x^2}}{\frac{\left(\log x + (x-1)\frac{1}{x}\right)x - (x-1)\log x}{x^2}} = \lim\frac{\frac{1}{x} - \frac{1}{x^2}}{\frac{\cancel{x\log x} + x - 1 - \cancel{x\log x} + \log x}{x^2}}$

$\displaystyle = \lim\frac{x-1}{x-1+\log x} = \lim_{x\to 1}\frac{1}{1+\frac{1}{x}} = \underline{\frac{1}{2}}$ //

問2.12.

(1) $\displaystyle \lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{\sin x}{2x} = \underline{\frac{1}{2}}$ //

(2) $\displaystyle \lim_{x\to 0}\frac{x-\log(1+x)}{x^2} = \lim_{x\to 0}\frac{1-\frac{1}{1+x}}{2x} = \lim_{x\to 0}\frac{\frac{x}{1+x}}{2x} = \lim_{x\to 0}\frac{\cancel{x}}{2\cancel{x}(1+x)} = \underline{\frac{1}{2}}$ //

(3) $\displaystyle \lim_{x\to \infty} x\left(e^{1/x}-1\right)$ （$\infty\cdot 0$ 不定形）

$\displaystyle = \lim_{x\to\infty}\frac{\left(e^{1/x}-1\right)'}{\left(\frac{1}{x}\right)'} = \lim_{x\to\infty}\frac{e^{1/x}\cdot\cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}} = e^0 = \underline{1}$ //

(4) $\displaystyle \lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x$

$\displaystyle \lim_{x\to\infty} x\log\left(\frac{x+1}{x-1}\right) = \lim_{x\to\infty}\frac{\log\left(\frac{x+1}{x-1}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{x-1}{x+1}\cdot\frac{x-1-(x+1)}{(x-1)^2}}{\frac{-1}{x^2}}$

$\displaystyle = \lim_{x\to\infty}\frac{1}{x+1}\cdot\frac{-2}{x-1}\cdot\frac{x^2}{-1} = \underline{2}$ //

(5) $\displaystyle \lim_{x\to +0} x^x$

$\displaystyle \left(\lim_{x\to +0} x\log x = \lim\frac{\log x}{\frac{1}{x}} = \lim_{x\to +0}\frac{\frac{1}{x}}{\frac{-1}{x^2}} = 0\right)$

$\displaystyle = \lim_{x\to +0} e^{x\log x} = e^0 = 1$

(6) $\displaystyle \lim_{x\to\infty}\frac{x^3}{e^x} = \lim\frac{3x^2}{e^x} = \lim\frac{6x}{e^x} = \lim_{x\to\infty}\frac{6}{e^x} = 0$

3. 不定積分と定積分

3.1 不定積分

◇ $f(x)$ に対して $F'(x)=f(x)$ となる $F(x)$ のことを **原始関数** という
or **不定積分**

$$F(x)=\int f(x)\,dx$$ と書く。

$F(x)$ を1つの不定積分とすると他の不定積分はすべて $F(x)+\underbrace{C}_{\text{積分定数という}}$ の形に書ける

$F(x)$ を求めることを $f(x)$ を積分するという

$$F(x)\ \underset{\text{積分}}{\overset{\text{微分}}{\rightleftarrows}}\ f(x)$$

微分公式を読みかえることで積分公式が導かれる.

◇ 公式

(1) $$\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1}$$

∵ $x^{p}\ \underset{\text{積分}}{\overset{\text{微分}}{\rightleftarrows}}\ p x^{p-1}$

$p\neq 0$ のとき $\dfrac{x^{p}}{p} \rightleftarrows x^{p-1}$

$\alpha=p-1$ として $\alpha\neq -1$ のとき $\dfrac{x^{\alpha+1}}{\alpha+1}\ \underset{\text{積分}}{\overset{\text{微分}}{\rightleftarrows}}\ x^{\alpha}$ よって $\displaystyle\int x^{\alpha}\,dx=\frac{x^{\alpha+1}}{\alpha+1}$

(2) $$\int \frac{1}{x}\,dx = \log|x|$$

∵ $\log|x|\ \underset{\text{積分}}{\overset{\text{微分}}{\rightleftarrows}}\ \dfrac{1}{x}$

(3) $$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x$$

$\tan^{-1}x\ \underset{\text{積分}}{\overset{\text{微分}}{\rightleftarrows}}\ \dfrac{1}{1+x^2}$

おぼえる必要なし

(4) $$\int \frac{1}{x^2-1}\,dx = \int \frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx = \frac{1}{2}\left(\log|x-1|-\log|x+1|\right) = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right|$$

(15) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x$

(:) $\sin^{-1}x \rightleftarrows \dfrac{1}{\sqrt{1-x^2}}$

(16) $\displaystyle\int \frac{1}{\sqrt{x^2+a}}\,dx = \log\left|x+\sqrt{x^2+a}\right|$ ただし $(a\neq 0)$

(:) $\log\left|x+\sqrt{x^2+a}\right| \underset{積分}{\overset{微分}{\rightleftarrows}} \dfrac{1+\frac{1}{2}(x^2+a)^{-\frac{1}{2}}\cdot 2x}{x+\sqrt{x^2+a}} = \dfrac{1}{\sqrt{x^2+a}}\cdot\dfrac{\cancel{\sqrt{x^2+a}+x}}{\cancel{x+\sqrt{x^2+a}}} = \dfrac{1}{\sqrt{x^2+a}}$

(7) $\displaystyle\int \sin x\,dx = -\cos x$

(:) $\cos x \underset{積分}{\overset{微分}{\rightleftarrows}} -\sin x \iff -\cos x \underset{積分}{\rightleftarrows} \sin x$

(8) $\displaystyle\int \cos x\,dx = \sin x$

導出 (9) $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-(\cos x)'}{\cos x}\,dx = -\log|\cos x|$

〃 (10) $\displaystyle\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{(\sin x)'}{\sin x}\,dx = \log|\sin x|$

〃 (11) $\displaystyle\boxed{\int \sqrt{1-x^2}\,dx} = \int x'\sqrt{1-x^2}\,dx = x\sqrt{1-x^2} - \int x\,\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\cdot -2x\,dx$

$\displaystyle = x\sqrt{1-x^2} + \int \frac{x^2}{\sqrt{1-x^2}}\,dx = x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx$

$\displaystyle = x\sqrt{1-x^2} - \boxed{\int \sqrt{1-x^2}\,dx} + \underbrace{\int \frac{1}{\sqrt{1-x^2}}\,dx}_{\sin^{-1}x}$

$\iff$ $\displaystyle 2\int \sqrt{1-x^2}\,dx = \frac{x\sqrt{1-x^2}+\sin^{-1}x}{2}$

よって $\displaystyle\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left\{x\sqrt{1-x^2}+\sin^{-1}x\right\}$

(12.) $$\int \sqrt{x^2+a}\,dx = x\sqrt{x^2+a} - \boxed{\int \frac{x \cdot 2x}{2\sqrt{x^2+a}}\,dx}$$

$$\int \frac{x^2+a-a}{\sqrt{x^2+a}}\,dx = \int \sqrt{x^2+a} - \frac{a}{\sqrt{x^2+a}}\,dx$$

日 $$2\int \sqrt{x^2+a}\,dx = x\sqrt{x^2+a} + a\underbrace{\boxed{\int \frac{1}{\sqrt{x^2+a}}\,dx}}_{\text{〃}\ \log\left|x+\sqrt{x^2+a}\right|}$$

よって $$\int \sqrt{x^2+a}\,dx = \frac{1}{2}\left\{x\sqrt{x^2+a} + a\log\left|x+\sqrt{x^2+a}\right|\right\}$$

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(13) $$\int \sec^2 x\,dx = \int \frac{1}{\cos^2 x}\,dx = \tan x$$

(∵) $$\tan x \underset{\text{積}}{\overset{\text{微分}}{\rightleftarrows}} \frac{1}{\cos^2 x}$$

(14) $$\int \operatorname{cosec}^2 x\,dx = \int \frac{1}{\sin^2 x}\,dx = -\cot x$$

(∵) $$\cot x \underset{\text{積}}{\overset{\text{微分}}{\rightleftarrows}} \frac{-\sin x\sin x - \cos x\cos x}{\sin^2 x} = -\frac{1}{\sin^2 x}$$

(15) $$\int \sec x\,dx = \int \frac{1}{\cos x}\,dx = \int \frac{\cos x}{1-\sin^2 x}\,dx = \int \frac{(\sin x)'}{1-\sin^2 x}\,dx$$

$u = \sin x$

$$= \frac{1}{2}\int \frac{1}{1-u} + \frac{1}{1+u}\,du = \frac{1}{2}\left\{-\log|u-1| + \log|u+1|\right\}$$

$$= \frac{1}{2}\log\left|\frac{u+1}{u-1}\right| = \frac{1}{2}\log\left|\frac{\sin x+1}{\sin x-1}\right|$$

$$= \frac{1}{2}\log\left|\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{1-2\sin\frac{x}{2}\cos\frac{x}{2}}\right| = \frac{1}{2}\log\left|\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\sin\frac{x}{2}-\cos\frac{x}{2}}\right|^2$$

$$= \log\left|\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right| = \log\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| = \log\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$$ 〃

(16) $\int \operatorname{cosec} x \, dx = \log \left| \tan \frac{x}{2} \right|$

$\because$ $\int \frac{1}{\sin x} dx = \int \frac{\sin x}{1-\cos^2 x} dx = -\int \frac{(\cos x)'}{1-\cos^2 x} dx$ $\quad u = \cos x$

$= -\int \frac{1}{1-u^2} dx = -\frac{1}{2} \log \left| \frac{\cos x + 1}{\cos x - 1} \right|$

$= \frac{1}{2} \log \left| \frac{1-\cos x}{1+\cos x} \right| = \frac{1}{2} \log \left| \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} \right| = \log \left| \frac{\sin x/2}{\cos x/2} \right| = \log \left| \tan \frac{x}{2} \right|$

(17) $\int e^x dx = e^x$

$\because$ $e^x \underset{積}{\overset{微}{\rightleftarrows}} e^x$

定理3.1

(1) $\int k f(x) \, dx = k \int f(x) \, dx$ $\quad \because \left(k \int f(x) dx\right)' = k f(x)$

k: 定数

(2) $\int \{f(x) \pm g(x)\} dx = \int f(x) dx \pm \int g(x) dx$

$\because$ $\left(\int f(x) dx \pm \int g(x) dx\right)' = f(x) + g(x)$

例3.1, 問3.1は 置換・部分積分法($\boxed{3.2}$)をやってから.

問3.2 済

$\boxed{3.2}$ 置換積分法・部分積分法・有理関数の積分法

◇ $$\int f(x)\,dx \underset{\uparrow}{=} \int f(g(t))\,g'(t)\,dt$$

$x = g(t)$

$dx = g'(t)\,dt$

∵ $\int f(x)\,dx = F(x)$ とおいて $F(g(t)) = \int f(g(t))\,g'(t)\,dt$

両辺を t で微分

$$(\text{右辺})' = \underbrace{F'(g(t))}_{f(g(t))}\,g'(t) = f(g(t))$$ よって 示せた

(例) $$\int \frac{f'(x)}{f(x)}\,dx \underset{\uparrow}{=} \int \frac{du}{u} = \log|u| = \log|f(x)|$$

$u = f(x)$

$du = f'(x)\,dx$

◇ 部分積分法

定理3.3 $\displaystyle\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$

∵ $(右辺)' = \cancel{f'(x)g(x)} + f(x)g'(x) - \cancel{f'(x)g(x)} = f(x)g'(x)$ ☒

例.3.3.

(1) $\displaystyle\int x\log x\,dx = \int \left(\frac{x^2}{2}\right)'\log x\,dx = \frac{x^2}{2}\log x - \int \frac{x^2}{2}\frac{1}{x}\,dx$

$\displaystyle = \underline{\frac{x^2}{2}\log x - \frac{x^2}{4}}$

$\left(\frac{x^2}{2}\right)'$ と見なす

(2) $\displaystyle\int \underbrace{x}_{=\left(\frac{x^2}{2}\right)'}\sin^{-1}x\,dx = \frac{x^2}{2}\sin^{-1}x - \boxed{\int \frac{x^2}{2}\underbrace{(\sin^{-1}x)'}_{=\frac{1}{\sqrt{1-x^2}}}\,dx}$

$\displaystyle\boxed{} = \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\int \sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}}\,dx$

$\displaystyle = \frac{1}{2}\underbrace{\boxed{\int \sqrt{1-x^2}\,dx}}_{=\frac{1}{2}\left(x\sqrt{1-x^2}+\sin^{-1}x\right)} - \frac{1}{2}\underbrace{\boxed{\int \frac{dx}{\sqrt{1-x^2}}}}_{=\sin^{-1}x}$

公式(11)

与式 $\displaystyle = \frac{x^2}{2}\sin^{-1}x + \frac{1}{4}\left(x\sqrt{1-x^2}+\sin^{-1}x\right) - \frac{1}{2}\sin^{-1}x$

$\displaystyle = \underline{\frac{x^2}{2}\sin^{-1}x - \frac{1}{4}\sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2}}$

例題 3.2

(1) $\displaystyle\int \frac{1}{x^2-2x+5}\,dx = \int \frac{1}{4\left\{\left(\frac{x-1}{2}\right)^2+1\right\}}\,dx$

$\left(x^2-2x+5 = (x-1)^2+4 = 4\left\{\left(\frac{x-1}{2}\right)^2+1\right\}\right)$

$u = \frac{x-1}{2}$ とおくと. $du = \frac{1}{2}\,dx$ より

$\displaystyle = \int \frac{1}{4(u^2+1)}\,2\,du = \frac{1}{2}\int \frac{1}{u^2+1}\,du = \frac{1}{2}\tan^{-1} u = \underline{\frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)}$ //

(2) $\displaystyle\int x(x^2-3)^5\,dx$

$\left(u = x^2-3 \text{ とおくと } du = 2x\,dx,\ x\,dx = \frac{du}{2}\right)$

$\displaystyle = \int u^5 \frac{du}{2} = \frac{u^6}{12} = \underline{\frac{(x^2-3)^6}{12}}$ //

◇ 有理関数の積分法

多項式 → $\dfrac{f(x)}{g(x)}$ の積分 ← 多項式

⇒ (I) 割り算実行して

$$f(x) = \underbrace{Q(x)}_{\text{これはすぐに積分出来る}} + \frac{R(x)}{g(x)}$$

← $R(x)$の次数 < $g(x)$の 〃

⇒ (II) $\dfrac{R(x)}{g(x)}$ を部分分数に分解する。(復習)

$$\frac{R(x)}{g(x)} = \underbrace{\bigcirc + \bigcirc + \cdots + \bigcirc}_{\text{分解されて}}$$

⇒ 分解された項の積分は

どちらか or

$$\begin{cases} \displaystyle\int \frac{A}{(x-a)^n}\,dx & \text{← 公式(I)(2)で求まる（すぐ）} \\[2ex] \displaystyle\underbrace{\int \frac{Bx+C}{(x^2+px+q)^m}\,dx}_{(*)} & \text{ただし } (p^2-4q<0) \end{cases}$$

$$\left(x+\frac{p}{2}\right)^2 - \frac{p^2-4q}{4} = \Big(\underbrace{\boxed{x+\frac{p}{2}}}_{t\text{ とおく}}\Big)^2 + \underbrace{q-\frac{p^2}{4}}_{a^2\text{ とおく}}$$

正.

$$(*) = \int \frac{Bt + C - \frac{pB}{2}}{(t^2+a^2)^m}\,dt = B\int \frac{t}{(t^2+a^2)^m}\,dt - \frac{pB}{2}\underbrace{\boxed{\int \frac{dt}{(t^2+a^2)^m}}}_{I_m}$$

$$B\int \frac{t}{(t^2+a^2)^m}\,dt \to \begin{cases} \frac{1}{2}\log|t^2+a^2| & (m=-1) \\[1ex] \frac{1}{2}\frac{1}{(-m+1)}(t^2+a^2)^{-m+1} & (m\neq -1) \end{cases}$$

$$I_m = \int \frac{dt}{(t^2+a^2)^m} = \int \frac{t'}{(t^2+a^2)^m}\,dt = \frac{t}{(t^2+a^2)^m} - \int \frac{t\,(-m)}{(t^2+a^2)^{m+1}}\cdot 2t\,dt$$

$$= \frac{t}{(t^2+a^2)^m} + 2m\int \frac{t^2+a^2-a^2}{(t^2+a^2)^{m+1}}\,dt$$

$$= \frac{t}{(t^2+a^2)^m} + 2m\left(I_m - a^2 I_{m+1}\right)$$

(I_m つづき)

$m \to m-1$ にして

$$I_m = \frac{1}{2(m-1)a^2}\left\{\frac{t}{(t^2+a^2)^{m-1}} + (2m-3)I_{m-1}\right\} \quad \cdots \text{漸化式} \quad (m-1>0)$$

$$I_m \to I_{m-1} \to \cdots \to I_1 = \int \frac{dt}{t^2+a^2} = \frac{1}{a}\tan^{-1}\frac{t}{a} \quad [\text{公式}(3)]$$

I_m は $I_1 = \dfrac{1}{a}\tan^{-1}\dfrac{t}{a}$ から求まる.

漸化式を用いて

例 3-1

(1) $\displaystyle\int \frac{5}{4x^2+3}\,dx = \frac{5}{3}\int \frac{1}{\frac{4}{3}x^2+1}\,dx$

$\left(\dfrac{2x}{\sqrt{3}} = t \text{ とおくと} \qquad \dfrac{2\,dx}{\sqrt{3}} = dt \;,\; dx = \dfrac{\sqrt{3}}{2}\,dt \right)$

$\displaystyle = \frac{5}{3}\int \frac{1}{t^2+1}\,\frac{\sqrt{3}}{2}\,dt = \frac{5}{2\sqrt{3}}\tan^{-1} t = \underline{\frac{5}{2\sqrt{3}}\tan^{-1}\left(\frac{2x}{\sqrt{3}}\right)}$

(2) $\displaystyle\int \frac{3}{\sqrt{5x^2+4}}\,dx = \frac{3}{2}\int \frac{1}{\sqrt{\frac{5}{4}x^2+1}}\,dx$

$\left(\dfrac{\sqrt{5}}{2}x = t, \qquad \dfrac{\sqrt{5}}{2}\,dx = dt, \quad dx = \dfrac{2}{\sqrt{5}}\,dt \right)$

$\displaystyle = \frac{3}{2}\int \frac{1}{\sqrt{t^2+1}}\,\frac{2}{\sqrt{5}}\,dt = \frac{3}{\sqrt{5}}\int \frac{1}{\sqrt{t^2+1}}\,dt = \frac{3}{\sqrt{5}}\log\left|t+\sqrt{t^2+1}\right|$

$\displaystyle = \frac{3}{\sqrt{5}}\log\left|\frac{\sqrt{5}}{2}x+\sqrt{\frac{5}{4}x^2+1}\right|$

$\displaystyle = \frac{3}{\sqrt{5}}\log\left(\frac{\sqrt{5}}{2}\left|x+\sqrt{x^2+\frac{4}{5}}\right|\right)$

$\displaystyle = \frac{3}{\sqrt{5}}\log\left|x+\sqrt{x^2+\frac{4}{5}}\right| + \cancel{\frac{3}{\sqrt{5}}\log\frac{\sqrt{5}}{2}}$

定数はなくてもいい

(3) $\displaystyle\int \frac{1}{1-x^2}\,dx = \int \frac{1}{2}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)dx$

$\displaystyle = \frac{1}{2}\left\{\log|1+x| - \log|1-x|\right\} = \underline{\frac{1}{2}\log\left|\frac{1+x}{1-x}\right|}$

$\displaystyle = -\frac{1}{2}\log\left(\left|\frac{1+x}{1-x}\right|^{-1}\right) = -\frac{1}{2}\log\left|\frac{1-x}{1+x}\right|$

(4) $\displaystyle\int \frac{1}{\sqrt{16-9x^2}}\,dx$

$\displaystyle = \frac{1}{4}\int \frac{1}{\sqrt{1-\frac{9}{16}x^2}}\,dx = \frac{1}{4}\int \frac{1}{\sqrt{1-t^2}}\,\frac{4}{3}\,dt = \frac{1}{3}\int \frac{dt}{\sqrt{1-t^2}}$

$\left(t = \frac{3}{4}x \qquad dt = \frac{3}{4}dx \right)$

$\displaystyle = \frac{1}{3}\sin^{-1} t = \underline{\frac{1}{3}\sin^{-1}\left(\frac{3}{4}x\right)}$

(5) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{x^2-1+1}{\sqrt{1-x^2}}\,dx$

$\displaystyle = \int \frac{1}{\sqrt{1-x^2}} - \sqrt{1-x^2}\,dx$

$\displaystyle = \sin^{-1}x - \boxed{\int x'\sqrt{1-x^2}\,dx}$

$\displaystyle \boxed{} = x\sqrt{1-x^2} - \int x\,\frac{1}{2}\,\frac{2x}{\sqrt{1-x^2}}\,dx$

$\displaystyle = \sin^{-1}x - x\sqrt{1-x^2} - \int \frac{x^2}{\sqrt{1-x^2}}\,dx$

$\displaystyle \int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\left\{ \sin^{-1}x - x\sqrt{1-x^2} \right\}$

問 3-1

(1) $\displaystyle\int \frac{1}{2x}\,dx = \underline{\frac{1}{2}\log|x|}$.

(2) $\displaystyle\int \sin \underbrace{\boxed{\frac{x}{2}}}_{t}\,dx = \int \sin t \; 2\,dt = 2(-\cos t) = \underline{-2\cos\frac{x}{2}}$.

$\left(t = \frac{x}{2} \qquad dt = \frac{dx}{2} \right)$

(3) $\displaystyle\int \cos \underbrace{\boxed{2x}}_{t}\,dx = \int \cos t \,\frac{dt}{2} = \frac{\sin t}{2} = \underline{\frac{\sin 2x}{2}}$.

$(t = 2x \qquad dt = 2\,dx)$

(4) $\displaystyle\int e^{\overbrace{\boxed{-x}}^{t}}\,dx = \int e^{t}(-)\,dt = -e^{t} = \underline{-e^{-x}}$.

$(t = -x. \qquad dt = -dx)$

(5) $\displaystyle\int \cot x + \tan x \; dx$

$\displaystyle = \int \frac{\cos x}{\underbrace{\boxed{\sin x}}_{t}}\,dx + \int \frac{\sin x}{\underbrace{\boxed{\cos x}}_{u}}\,dx$

$t = \sin x. \quad dt = \cos x\,dx, \qquad u = \cos x \quad du = -\sin x\,dx$

$\displaystyle = \int \frac{dt}{t} + \int \frac{-du}{u} = \log|t| - \log|u|$

$\displaystyle = \underline{\log\left|\frac{\sin x}{\cos x}\right|}$.

(6) $\displaystyle\int \frac{4}{\sqrt{3x^2-6}}dx = \frac{4}{\sqrt{6}}\int \frac{1}{\sqrt{\frac{x^2}{2}-1}}dx = \frac{4}{\sqrt{6}}\int \frac{1}{\sqrt{t^2-1}}\sqrt{2}\,dt$

($t = \dfrac{x}{\sqrt{2}}$ とおく。 $dt = \dfrac{dx}{\sqrt{2}}$)

$\displaystyle = \frac{4}{\sqrt{3}}\int \frac{1}{\sqrt{t^2-1}}dt = \frac{4}{\sqrt{3}}\log\left|t+\sqrt{t^2-1}\right|$

$\displaystyle = \frac{4}{\sqrt{3}}\log\left|\frac{x}{\sqrt{2}}+\sqrt{\frac{x^2}{2}-1}\right|$

$\displaystyle = \frac{4}{\sqrt{3}}\log\left|x+\sqrt{x^2-2}\right|$ + 定数

(7) $\displaystyle\int \frac{x^2}{\sqrt{x^2+4}}dx = \frac{1}{2}\int \frac{x^2}{\sqrt{\left(\frac{x}{2}\right)^2+1}}dx = \frac{1}{2}\int \frac{4t^2}{\sqrt{t^2+1}}2\,dt$

($t = \left(\dfrac{x}{2}\right)$ とおく。 $dt = \dfrac{1}{2}dx$)

$\displaystyle = 4\boxed{\int \frac{t^2}{\sqrt{t^2+1}}dt}$

($\displaystyle\boxed{} = \int \frac{t^2+1-1}{\sqrt{t^2+1}}dt = \underbrace{\int \sqrt{t^2+1}\,dt}_{\frac{1}{2}\left\{t\sqrt{t^2+1}+\log\left|t+\sqrt{t^2+1}\right|\right\}\ (公式12)} - \underbrace{\int \frac{1}{\sqrt{t^2+1}}dt}_{\log\left|t+\sqrt{t^2+1}\right|\ (公式(6))}$)

与式 $= 2\left\{t\sqrt{t^2+1}+\log\left|t+\sqrt{t^2+1}\right|\right\} - 4\log\left|t+\sqrt{t^2+1}\right|$

$\displaystyle = 2\frac{x}{2}\sqrt{\frac{x^2}{4}+1} - 2\log\left|\frac{x}{2}+\sqrt{\frac{x^2}{4}+1}\right|$

$\displaystyle = \frac{1}{2}x\sqrt{x^2+4} - 2\log\left|x+\sqrt{x^2+4}\right|$

(8) $\displaystyle\int \frac{1-x^2}{1+x^2}dx = \int \frac{2-(x^2+1)}{1+x^2}dx = \int \frac{2}{1+x^2}dx - \int dx$

$= 2\tan^{-1}x - x$

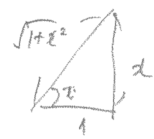
問 3.3

(1) $\displaystyle\int \frac{1}{(1+x^2)^{3/2}}\,dx = \int \frac{1}{(1+\tan^2 t)^{3/2}}\,\frac{1}{\cos^2 t}\,dt$

$\left(x = \tan t,\quad dx = \dfrac{1}{\cos^2 t}\,dt\right)$

$\boxed{1+\tan^2 x = \dfrac{1}{\cos^2 t}}$

$\displaystyle = \int \frac{1}{(\cos^{-2} t)^{\frac{3}{2}}}\,\frac{1}{\cos^2 t}\,dt$



$\displaystyle = \int \frac{1}{\cos^{-3} t \cdot \cos^2 t}\,dt = \int \cos t\,dt = \sin t = \underline{\frac{x}{\sqrt{1+x^2}}}$

(2) $\displaystyle\int \frac{3x}{\sqrt{1-x^4}}\,dx = \int \frac{3/2}{\sqrt{1-t^2}}\,dt = \frac{3}{2}\int \frac{1}{\sqrt{1-t^2}}\,dt = \frac{3}{2}\sin^{-1} t$

$\left(x^2 = t,\quad 2x\,dx = dt\right)$

$\displaystyle = \underline{\frac{3}{2}\sin^{-1} x^2}$

(3) $\displaystyle\int \frac{x^2}{x^3-1}\,dx = \int \frac{1}{t^2-1}\,\frac{dt}{3} = \frac{1}{3}\int \frac{1}{t^2-1}\,dt = \frac{1}{3}\int \left(\frac{1}{t-1} - \frac{1}{t+1}\right)dt$

$\left(x^3 = t,\quad 3x^2\,dx = dt\right)$

$\displaystyle = \frac{1}{3}\left\{\log|t-1| - \log|t+1|\right\} = \frac{1}{3}\log\left|\frac{t-1}{t+1}\right| = \underline{\frac{1}{3}\log\left|\frac{x^3-1}{x^3+1}\right|}$

問 3.4

(1) $\int x^2 \underbrace{\cos x}_{(\sin x)'} dx$

$= x^2 \sin x - \int 2x \underbrace{\sin x}_{(-\cos x)'} dx = x^2 \sin x - \left\{2x(-\cos x) - \int 2(-\cos x)\, dx\right\}$

$= \underline{x^2 \sin x + 2x\cos x - 2\sin x}$

(2) $\int \underbrace{1}_{x'} \cdot \log x\, dx = x\log x - \int x \frac{1}{x} dx = \underline{x\log x - x}$

(3) $\int x \underbrace{e^x}_{(e^x)'} dx = xe^x - \int x' e^x dx = \underline{xe^x - e^x}$

(4) $\int x^3\sqrt{1-x^2}\, dx = \int x \cdot x^2\sqrt{1-x^2}\, dx = \int (1-t)\sqrt{t}\, \frac{dt}{-2}$

($t = 1-x^2$, $x^2 = 1-t$ とおくと, $dt = -2x\, dx$)

$= \int \left(t^{\frac{1}{2}} - t^{\frac{3}{2}}\right)\frac{1}{-2}\, dt = \left(\frac{2}{3}t^{\frac{3}{2}} - \frac{2}{5}t^{\frac{5}{2}}\right)\frac{1}{-2} = \frac{1}{5}t^{\frac{5}{2}} - \frac{1}{3}t^{\frac{3}{2}}$

$= \underline{\frac{1}{5}(1-x^2)^{\frac{5}{2}} - \frac{1}{3}(1-x^2)^{\frac{3}{2}}}$.

$= \frac{3(1-x^2) - 5}{15}(1-x^2)^{\frac{3}{2}} = \underline{\frac{-3x^2-2}{15}(1-x^2)^{\frac{3}{2}}}$.

◇ 部分分数に分解.

$$\frac{P(x)}{(x-a)^n} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}$$

$$\frac{P(x)}{(x^2+px+q)^m} = \frac{B_1x+C_1}{x^2+px+q} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \cdots + \frac{B_mx+C_m}{(x^2+px+q)^m}$$

例題 3.4.

$$\frac{x^4-3x^2+3x-7}{(x+2)(x-1)^2}$$

分母 $=$

$$\begin{array}{r} x^2-2x+1 \\ x+2 \\ \hline x^3-2x^2+x \\ +2x^2-4x+2 \\ \hline x^3-3x+2 \end{array}$$

$\underline{x^3-3x+2}$ 3 次.

$$\begin{array}{r} x \\ x^3-3x+2 \;\big)\; x^4-3x^2+3x-7 \\ x^4-3x^2+2x \\ \hline x-7 \end{array}$$

$$x + \underbrace{\frac{x-7}{(x+2)(x-1)^2}}$$

$$\sim\sim = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\underset{①}{\underline{x-7}} = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$$

$$= A(x^2-2x+1) + B(x^2+x-2) + C(x+2)$$

$$= \underbrace{(A+B)x^2 + (-2A+B+C)x + (A-2B+2C)}_{②}$$

すべての x で ① = ② が成立するには

$$\begin{cases} A+B=0 \longrightarrow B=-A \\ -2A+B+C=1 \xrightarrow{\ \ \ } -3A+C=1 \\ A-2B+2C=-7 \longrightarrow 3A+2C=-7 \end{cases} \longrightarrow 3C=-6 \quad C=-2$$

$$A = \frac{-7+4}{3} = -1, \quad B = 1$$

よって

$$与式 = x + \frac{-1}{x+2} + \frac{1}{x-1} + \frac{-2}{(x-1)^2}$$

(3) 3.5

$$\int x + \frac{-1}{x+2} + \frac{1}{x-1} + \frac{-2}{(x-1)^2}\, dx$$

$$= \frac{x^2}{2} - \log|x+2| + \log|x-1| - 2\cdot (x-1)^{-1}$$

(3) 3.6

$$\int \frac{1}{x^3+1}\, dx$$

$$\frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} = \frac{1}{x^3+1}$$

$$\text{左辺} = A(x^2-x+1) + \underbrace{(Bx+C)(x+1)}_{Bx^2+(B+C)x+C}$$

$$= (A+B)x^2 + (-A+B+C)x + A+C = 1\ (\text{右辺})$$

$$\begin{cases} A+B=0 \longrightarrow B=-A \\ -A+B+C=0 \longrightarrow -A-A+1-A=0 \\ A+C=1 \longrightarrow C=1-A \end{cases} \qquad B=-\frac{1}{3},\quad A=\frac{1}{3},\quad C=\frac{2}{3}.$$

$$\text{与式} = \int \frac{\frac{1}{3}}{x+1} + \frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}\, dx = \frac{1}{3}\log|x+1| - \frac{1}{3}\boxed{\int \frac{x-2}{x^2-x+1}\, dx} \quad \cdots (*)$$

$$\int \frac{x-2}{(x-\frac{1}{2})^2+\frac{3}{4}}\, dx = \frac{4}{3}\int \frac{x-2}{\frac{4}{3}(x-\frac{1}{2})^2+1}\, dx$$

$$\left(t = \frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right) \qquad x = \frac{\sqrt{3}}{2}t + \frac{1}{2} \qquad dx = \frac{\sqrt{3}}{2}\, dt \right)$$

$$\frac{\sqrt{3}}{2}\cdot\frac{4}{3}\int \frac{\frac{\sqrt{3}}{2}t + \frac{1}{2} - 2}{t^2+1}\, dt = \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{3}}\underbrace{\int \frac{2t}{t^2+1}\, dt}_{\log(t^2+1)} + \frac{\sqrt{3}}{2}\cdot\frac{4}{3}\left(\frac{-3}{2}\right)\underbrace{\int \frac{1}{t^2+1}\, dt}_{\tan^{-1} t}$$

$$(*) = \frac{1}{3}\log|x+1| - \frac{1}{6}\log(t^2+1) + \frac{1}{\sqrt{3}}\tan^{-1} t$$

$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log\left(\frac{4}{3}\left(x-\frac{1}{2}\right)^2 + \cdots\right)$$

問35

(1)

$$\begin{array}{r} x^2+x+4 \\ x^3-4x \enclose{longdiv}{x^5+x^4 \qquad -8} \\ \underline{x^5-4x^3 \qquad\qquad} \\ x^4+4x^3 \qquad\quad \\ \underline{x^4-4x^2 \qquad} \\ 4x^3+4x^2 \qquad \\ \underline{4x^3-16x} \\ 4x^2+16x-8 \end{array}$$

$$\int x^2+x+4+\frac{4x^2+16x-8}{x^3-4x}\,dx = \frac{x^3}{3}+\frac{x^2}{2}+4x+4\boxed{\int \frac{x^2+4x-2}{x(x^2-4)}\,dx} \cdots (*)$$

$$\frac{x^2+4x-2}{x(x-2)(x+2)} = \frac{a}{x-2}+\frac{b}{x}+\frac{c}{x+2}$$

$$\begin{aligned} x^2+4x-2 &= a(x^2+2x)+b(x^2-4)+c(x^2-2x) \\ &= (a+b+c)x^2+(2a-2c)x+(-4b) \end{aligned}$$

$a+b+c=1 \rightarrow a+c=\frac{1}{2}$ $\qquad 2a=\frac{5}{2} \qquad \underline{a=\frac{5}{4}}$

$2a-2c=4 \rightarrow a-c=2$

$-4b=-2 \rightarrow \underline{b=\frac{1}{2}}$ $\qquad \underline{c}=\frac{1}{2}-\frac{5}{4}=\underline{\frac{-3}{4}}$

$$\boxed{} = \frac{5}{4}\log|x-2|+\frac{1}{2}\log|x|-\frac{3}{4}\log|x+2|$$

$$\begin{aligned} (*) &= \frac{x^3}{3}+\frac{x^2}{2}+4x+4\left(\frac{5}{4}\log|x-2|+\frac{1}{2}\log|x|-\frac{3}{4}\log|x+2|\right) \\ &= \frac{x^3}{3}+\frac{x^2}{2}+4x+5\log|x-2|+2\log|x|-3\log|x+2| \end{aligned}$$

(2) $$\frac{x}{(1+x)^2(1+x^2)} = \frac{a}{1+x} + \frac{b}{(1+x)^2} + \frac{cx+d}{1+x^2}$$

$$= \frac{a(1+x)(1+x^2) + b(1+x^2) + (cx+d)(x^2+2x+1)}{(1+x)^2(1+x^2)}$$

$$= \underline{(a+c)x^3 + (a+b+2c+d)x^2 + (a+c+2d)x + (a+b+d)}$$

$a+c=0 \longrightarrow \boxed{a=0}$

$a+b+2c+d=0 \longrightarrow a+b+2c = -\frac{1}{2}$

$a+c+2d=1 \longrightarrow 2d=1,\ \boxed{d=\frac{1}{2}} \longrightarrow 2c=0,\ \boxed{c=0}$

$a+b+d=0 \longrightarrow a+b=-\frac{1}{2} \longrightarrow \boxed{b=-\frac{1}{2}}$

$$\int 与式\, dx = \int \frac{0}{1+x} + \frac{-\frac{1}{2}}{(1+x)^2} + \frac{\frac{1}{2}}{1+x^2}\, dx$$

$$= -\frac{1}{2}(-1)(1+x)^{-1} + \frac{1}{2}\tan^{-1}x$$

$$= \underline{\frac{1}{2(1+x)} + \frac{\tan^{-1}x}{2}}$$

(3) $\dfrac{1}{x(1+x^2)} = \dfrac{a}{x} + \dfrac{bx+c}{1+x^2}$

$= \dfrac{a(1+x^2)+(bx+c)x}{x(1+x^2)}$

$1 = (a+b)x^2 + (c)x + a$

$\begin{cases} a+b=0 \longrightarrow b=-1. \\ \boxed{c=0} \\ \boxed{a=1} \end{cases}$

$\displaystyle\int \frac{1}{x} + \frac{-x}{1+x^2}\,dx = \log|x| - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$

$\log(1+x^2)$

$= \underline{\log|x| - \frac{1}{2}\log(1+x^2)} = \underline{\log\left|\dfrac{x}{\sqrt{1+x^2}}\right|}$

問3.6 済

3.3 三角関数，無理関数，指数関数，対数関数の積分

$f(X)$ … Xの有理関数
$f(X,Y)$ … X,Yの 〃
とする

三角関数

(1) $\int f(\sin x)\underbrace{\cos x}_{(\sin x)'}\,dx \Rightarrow t=\sin x$

(2) $\int f(\cos x)\underbrace{\sin x}_{(-\cos x)'}\,dx \Rightarrow t=\cos x$

(3) $\int f(\sin^2 x, \cos^2 x)\,dx \Rightarrow t=\tan x$

$\because\ t^2=\dfrac{1-\cos^2 x}{\cos^2 x}$ $\quad (t^2+1)\cos^2 x=1$

$\cos^2 x=\dfrac{1}{t^2+1}$

$\sin^2 x=\dfrac{t^2}{t^2+1}$

$dt=\dfrac{1}{\cos^2 x}dx \Rightarrow dx=\cos^2 x\,dt=\dfrac{1}{t^2+1}dt$

tの有理関数の積分に帰着するから

(4) $\int f(\sin x,\cos x)\,dx \Rightarrow t=\tan\dfrac{x}{2}$

$\because\ \sin x=2\sin\dfrac{x}{2}\cos\dfrac{x}{2}=2\underbrace{\tan\dfrac{x}{2}}_{t}\underbrace{\cos^2\dfrac{x}{2}}_{\frac{1}{t^2+1}}=\dfrac{2t}{t^2+1}$

$\cos x=2\cos^2\dfrac{x}{2}-1=\dfrac{2}{t^2+1}-1=\dfrac{2-(t^2+1)}{t^2+1}=\dfrac{1-t^2}{1+t^2}$

$dt=\dfrac{1}{\cos^2\frac{x}{2}}\cdot\dfrac{1}{2}dx$

$dx=2\underbrace{\cos^2\left(\dfrac{x}{2}\right)}_{\frac{1}{t^2+1}}dt=\dfrac{2}{1+t^2}dt$

tの有理関数の積分に帰着するから

無理関数

$$\int f(x, \sqrt[n]{ax+b})\, dx \Rightarrow t = \sqrt[n]{ax+b}$$

$\because$

$$t^n = ax+b$$

$$x = \frac{t^n - b}{a}$$

$$dx = \frac{n}{a} t^{n-1}\, dt$$

tの有理関数の積分に帰着

$$\int f\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx \Rightarrow t = \sqrt[n]{\frac{ax+b}{cx+d}}$$

$\because$

$$t^n = \frac{ax+b}{cx+d}$$

$$t^n(cx+d) = ax+b$$

$$(ct^n - a)x = b - dt^n$$

$$x = -\frac{b-dt^n}{a-ct^n}$$

$$\frac{dx}{dt} = \frac{dnt^{n-1}(a-ct^n) + (b-dt^n)(-cnt^{n-1})}{(a-ct^n)^2}$$

$$= \frac{(\cancel{-dnc} + \cancel{dnc})t^{2n-1} + (adn - bcn)t^{n-1}}{〃}$$

$$= \frac{(ad-bc)\,n\,t^{n-1}}{(a-ct^n)^2}$$

$$dx = \frac{ad-bc}{(a-ct^n)^2} n t^{n-1}\, dt$$

tの有理関数の積分に帰着

$$\int f(x,\sqrt{ax^2+bx+c})\,dx$$

(i) $a>0$ のとき $\sqrt{ax^2+bx+c}=t-\sqrt{a}\,x$ とおく

(例) $\int \sqrt{x^2+A}\,dx$

$t-x=\sqrt{x^2+A}$, $t^2-2tx+x^2=x^2+A$

$2tx=t^2-A$

$x=\dfrac{t^2-A}{2t}$

$$\frac{dx}{dt}=\frac{2t\cdot 2t-(t^2-A)\cdot 2}{4t^2}=\frac{4t^2-2t^2+2A}{4t^2}=\frac{2t^2+2A}{4t^2}$$

$$=\int\left(t-\frac{t^2-A}{2t}\right)\left(\frac{2t^2+2A}{4t^2}\right)dt$$

$$=\int\left(\underbrace{t-\frac{t}{2}}_{\frac{t}{2}}+\frac{A}{2t}\right)\left(\frac{1}{2}+\frac{A}{2t^2}\right)dt$$

$$=\int\left(\frac{t}{4}+\underbrace{\frac{A}{4t}+\frac{A}{4t}}_{\frac{A}{2t}}+\frac{A^2}{4t^3}\right)dt=\frac{t^2}{8}+\frac{A}{2}\log|t|-\frac{A^2}{8}t^{-2}$$

$$=\frac{1}{8}\left(t^2-\frac{A^2}{t^2}\right)+\frac{A}{2}\log|t|$$

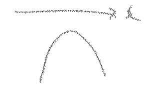
$t=x+\sqrt{x^2+A}$ $\longrightarrow$ $\dfrac{A}{t}=A\,\dfrac{\sqrt{x^2+A}-x}{A}$ $\rightarrow$ $\left(\dfrac{A}{t}\right)^2=x^2+A+x^2-2x\sqrt{x^2+A}$

$\rightarrow t^2=x^2+x^2+A+2x\sqrt{x^2+A}$

$=2x^2+A+2x\sqrt{x^2+A}$

$\sim = 4x\sqrt{x^2+A}$

$$=\frac{1}{2}x\sqrt{x^2+A}+\frac{A}{2}\log\left|x+\sqrt{x^2+A}\right|$$

($D<0$ のとき  なので 実で積分出来ない)

(ii) $a<0$, $D>0$ のとき

$$ax^2+bx+c = a(x-\alpha)(x-\beta) \qquad (\alpha<\beta)$$

$$\sqrt{\frac{x-\alpha}{\beta-x}} = t$$

x	α	β
t	0	∞

(例) $\displaystyle\int \sqrt{1-x^2}\,dx \qquad t=\sqrt{\frac{x-(-1)}{1-x}}=\sqrt{\frac{x+1}{1-x}}$

$$t^2=\frac{x+1}{1-x}=-1+\frac{2}{1-x} \qquad t^2+1=\frac{2}{1-x} \qquad 1-x=\frac{2}{t^2+1}$$

$$x=1-\frac{2}{t^2+1}=\frac{t^2-1}{t^2+1}$$

$$\sqrt{1-x^2}=\sqrt{\frac{(t^2+1)^2-(t^2-1)^2}{(t^2+1)^2}}=\sqrt{\frac{4t^2}{(t^2+1)^2}}=\frac{2t}{t^2+1}$$

$$\frac{dx}{dt}=\frac{2t(t^2+1)-(t^2-1)\cdot 2t}{(t^2+1)^2}=\frac{4t}{(t^2+1)^2}$$

$$=\int \frac{2t}{t^2+1}\frac{4t}{(t^2+1)^2}\,dt=\int\frac{8t^2}{(t^2+1)^3}\,dt = 8\underbrace{\boxed{\int\frac{1}{(t^2+1)^2}\,dt}}_{I_2}-8\underbrace{\boxed{\int\frac{1}{(t^2+1)^3}\,dt}}_{I_3} \cdots (*)$$

$$\left(\frac{8t^2}{(t^2+1)^3}=8\left(\frac{1}{(t^2+1)^2}-\frac{1}{(t^2+1)^3}\right)\right)$$

P.66 (8) より $\displaystyle I_n=\frac{1}{2(n-1)}\left\{\frac{t}{(t^2+1)^{n-1}}+(2n-3)I_{n-1}\right\}$

$$I_3=\frac{1}{2\cdot 2}\left\{\frac{t}{(t^2+1)^2}+3I_2\right\}, \qquad I_2=\frac{1}{2}\left\{\frac{t}{(t^2+1)}+I_1\right\}$$

$$\frac{x\sqrt{1-x^2}}{2}+\frac{\sin^{-1}x}{2} \quad (答)$$

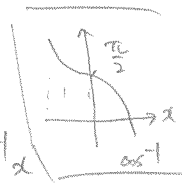
$$(*)=8I_2-2\frac{t}{(t^2+1)^2}-6I_2=2I_2-\frac{2t}{(t^2+1)^2}=\frac{t}{t^2+1}+I_1-\frac{2t}{(t^2+1)^2}$$

$$=I_1+t\frac{t^2-1}{(t^2+1)^2}=I_1+\sqrt{\frac{x+1}{1-x}}\,\frac{\frac{2x}{1-x}}{\frac{2^2}{(1-x)^2}}=I_1+\sqrt{\frac{x+1}{1-x}}\,\frac{x(1-x)}{2}=\frac{x}{2}\sqrt{(x+1)(1-x)}+I_1$$

$$\left(t^2-1=\frac{2}{1-x}-2=\frac{2-2+2x}{1-x}=\frac{2x}{1-x}\right)$$

$$I_1=\int\frac{dt}{t^2+1}=\tan^{-1}t=\tan^{-1}\sqrt{\frac{x+1}{1-x}}=\theta=\frac{1}{2}\sin^{-1}x$$

$$2\theta=\cos^{-1}(-x)=\frac{\pi}{2}-\cos^{-1}x$$



$$=\sin^{-1}x$$

$\sqrt{1-x^2}$

公式 $\cos^{-1}+\sin^{-1}=\frac{\pi}{2}$

(8) $\int f(x, \sqrt{a^2-x^2})\, dx = \int f(a\sin\theta, a\cos\theta)\, a\cos\theta\, d\theta$ → 三角(4) 利用

$$\left(\begin{array}{l} x = a\sin\theta, \quad \sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta \\ \left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right) \\ \frac{dx}{d\theta} = a\cos\theta \end{array} \right)$$

(9) $\int f(x, \sqrt{x^2+a^2})\, dx = \int f\left(a\tan\theta, \frac{a}{\cos\theta}\right) \frac{a}{\cos^2\theta}\, d\theta$ → 三角(4) 利用

$$\left(\begin{array}{l} x = a\tan\theta, \quad \sqrt{x^2+a^2} = \sqrt{a^2(\tan^2\theta+1)} = \frac{a}{\cos\theta} \\ \frac{dx}{d\theta} = a\frac{1}{\cos^2\theta} \Rightarrow dx = \frac{a}{\cos^2\theta}\, d\theta \end{array} \right)$$

(10) $\int f(x, \sqrt{x^2-a^2})\, dx = \int f\left(a\sec\theta, \frac{a\sin\theta}{\cos\theta}\right) \frac{\sin\theta}{\cos^2\theta}\, d\theta$ → 〃

$\boxed{\text{メモ} \quad \sec\theta = \frac{1}{\cos\theta}}$

$x = a\sec\theta$, $\sqrt{x^2-a^2} = \sqrt{\frac{a^2}{\cos^2\theta} - a^2} = \frac{a\sin\theta}{\cos\theta}$

$(0 \leq \theta \leq \pi,\ \theta \neq \frac{\pi}{2})$

$\frac{dx}{d\theta} = \frac{+1}{\cos^2\theta}\sin\theta \Rightarrow dx = \frac{\sin\theta}{\cos^2\theta}\, d\theta$

(11) $\displaystyle\int f(e^x)\, e^x\, dx = \int f(t)\, dt$

$\left(t = e^x. \quad \dfrac{dt}{dx} = e^x \Rightarrow dt = e^x dx \right)$

(12) $\displaystyle\int f(\log x)\, \frac{1}{x}\, dx = \int f(t)\, dt$

$\left(t = \log x, \quad \dfrac{dt}{dx} = \dfrac{1}{x} \quad dt = \dfrac{1}{x} dx \right)$

例 3.7

(1) $\displaystyle\int \frac{1}{2+\cos x}\,dx$

$t = \tan\frac{x}{2}$ $\qquad \cos\left(2\cdot\frac{x}{2}\right) = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = 2\cos^2\frac{x}{2} - 1 = \frac{2}{t^2+1} - 1 = \frac{1-t^2}{t^2+1}$

$t^2 = \dfrac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} \rightarrow \cos^2\frac{x}{2}\, t^2 = 1 - \cos^2\frac{x}{2} \rightarrow (t^2+1)\cos^2\frac{x}{2} = 1 \qquad \cos^2\frac{x}{2} = \frac{1}{t^2+1}$

$\dfrac{dt}{dx} = \dfrac{1}{\cos^2\frac{x}{2}}\,\dfrac{1}{2} = \dfrac{1}{2\,\frac{1}{t^2+1}} \longmapsto \dfrac{2}{1+t^2}\,dt = dx$

$$= \int \frac{1}{2 + \frac{1-t^2}{1+t^2}}\,\frac{2}{1+t^2}\,dt = \int \frac{2}{2+2t^2+1-t^2}\,dt = \int \frac{2}{t^2+3}\,dt$$

$$= \frac{2}{3}\int \frac{1}{\left(\frac{t}{\sqrt{3}}\right)^2+1}\,dt = \frac{2}{\sqrt{3}}\int \frac{1}{\left(\frac{t}{\sqrt{3}}\right)^2+1}\,d\left(\frac{t}{\sqrt{3}}\right) = \underline{\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + C} \;//$$

(2) $\displaystyle\int \frac{\sin x}{1+\sin x}\,dx$

$\left(t = \tan\frac{x}{2}, \qquad , \quad dx = \frac{2}{1+t^2}\,dt \right)$

$\sin\left(2\,\frac{x}{2}\right) = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\tan\frac{x}{2}\cos^2\frac{x}{2} = \frac{2t}{t^2+1}$

$$= \int \frac{\frac{2t}{t^2+1}}{1+\frac{2t}{t^2+1}}\,\frac{2}{1+t^2}\,dt = \int \frac{2t}{t^2+2t+1}\,\frac{2}{1+t^2}\,dt$$

$$= 4\int \frac{t}{(t+1)^2(t^2+1)}\,dt = 2\int \frac{1}{t^2+1} - \frac{1}{(t+1)^2}\,dt$$

$\left(\dfrac{t}{(t+1)^2(t^2+1)} = \dfrac{1}{2}\left(\dfrac{1}{t^2+1} - \dfrac{1}{(t+1)^2}\right) = \dfrac{1}{2}\left(\dfrac{2t}{\;〃\;}\right) \right)$

$$= 2\tan^{-1} t - \frac{2}{-1}(t+1)^{-1} = \underline{2\tan^{-1} t + 2\,\frac{1}{t+1}}$$

$$(3)\ \int \frac{\overset{(\sin x)'}{\overset{\shortparallel}{\boxed{\cos x}}}}{1+\sin^2 x}\,dx = \int \frac{dt}{1+t^2} = \underline{\tan^{-1} t}_{\,//}$$

$$\left(t = \sin x \qquad \frac{dt}{dx} = \cos x \rightarrow dt = \cos x\,dx \right)$$

$$(4)\ \int \frac{1}{\cos^2 x + 4\sin^2 x}\,dx$$

三角(3) より $t = \tan x$ とおく

$$t^2 = \frac{\sin^2 x}{\cos^2 x} \rightarrow t^2\cos^2 x = 1-\cos^2 x \rightarrow (t^2+1)\cos^2 x = 1 \rightarrow \cos^2 x = \frac{1}{t^2+1}$$

$$\sin^2 x = \frac{t^2}{t^2+1}$$

$$\frac{dt}{dx} = \frac{1}{\cos^2 x} = t^2+1 \rightarrow \frac{dt}{t^2+1} = dx$$

$$= \int \frac{1}{\frac{1}{t^2+1} + \frac{4t^2}{t^2+1}} \frac{1}{t^2+1}\,dt = \int \frac{1}{4t^2+1}\,dt = \frac{1}{2}\int \frac{1}{(2t)^2+1}\,d(2t))$$

$$= \underline{\frac{1}{2}\tan^{-1}(2t)}_{\,/}$$

問3.7

(1) $\displaystyle\int \tan^3 x \, dx$

$$= \int \frac{\sin^3 x}{\cos^3 x} dx = \int \frac{1-\cos^2 x}{\cos^3 x} \sin x \, dx = \int \frac{1-t^2}{t^3} (-dt)$$

$$\left(t = \cos x \qquad \frac{dt}{dx} = -\sin x \rightarrow -dt = \sin x \, dx \right)$$

$$= \int \frac{t^2-1}{t^3} dt = \int \frac{1}{t} - t^{-3} dt = \log|t| - \frac{1}{-2} t^{-2} = \underline{\log|t| + \frac{1}{2} t^{-2}}$$

(2) $\displaystyle\int \frac{1+\sin x}{\sin x(1+\cos x)} dx$

ココ

$$\left(t = \tan\frac{x}{2} \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2} \qquad \frac{2\,dt}{t^2+1} = dx \right)$$

$$= \int \frac{1+\frac{2t}{1+t^2}}{\frac{2t}{\cancel{1+t^2}}\left(1+\frac{2}{1+t^2}\right)} \frac{1}{\cancel{t^2+1}} dt = \int \frac{t^2+1+2t}{2t(t^2+3)} dt$$

$$\left(\frac{t^2+2t+1}{t(t^2+3)} = \frac{a}{t} + \frac{bt+c}{t^2+3} = \frac{at^2+3a+bt^2+ct}{t(t^2+3)} = \frac{(a+b)t^2+ct+3a}{〃} \right)$$

$$\begin{cases} a+b=1 \rightarrow b=\frac{2}{3} \\ c=2 \\ 3a=1 \rightarrow a=\frac{1}{3} \end{cases}$$

$$= \frac{1}{2}\int \frac{\frac{1}{3}}{t} + \frac{\frac{2}{3}t+\frac{1}{3}}{t^2+3} dt = \frac{1}{6}\underbrace{\int \frac{1}{t} dt}_{\log|t|} + \frac{1}{6}\underbrace{\int \frac{2t}{t^2+3} dt}_{\log(t^2+3)} + \frac{1}{\sqrt{3}}\underbrace{\int \frac{1}{\left(\frac{t}{\sqrt{3}}\right)^2+1} \frac{dt}{\sqrt{3}}}_{\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)}$$

$$= \frac{1}{6}\log|t| + \frac{1}{6}\log(t^2+3) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)$$

$$= \int \frac{1+\frac{2t}{1+t^2}}{\frac{2t}{\cancel{1+t^2}}\left(1+\frac{1-t^2}{1+t^2}\right)} \frac{2}{\cancel{t^2+1}} dt = \int \frac{t^2+2t+1}{t \cdot 2} dt = \int \frac{t}{2} + 1 + \frac{1}{2t} dt$$

$$= \frac{t^2}{4} + t + \frac{1}{2}\log|t|$$

(3) $\displaystyle\int \frac{1}{1+\sin x}\,dx$

$$= \int \frac{1-\sin x}{\underset{\underset{\cos^2 x}{\shortparallel}}{\boxed{1-\sin^2 x}}}\,dx = \int \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\,dx = \underbrace{\int \frac{1}{\cos^2 x}\,dx}_{①} + \underset{\underset{-1(\cos x)^{-1}}{\shortparallel}}{\boxed{\int \frac{(\cos x)'}{\cos^2 x}\,dx}} \cdots (*)$$

(①は $t = \tan x$. $\quad t^2 = \dfrac{1-\cos^2 x}{\cos^2 x} \quad (t^2+1)\cos^2 x = 1$

$\cos^2 x = \dfrac{1}{t^2+1}$

$\dfrac{dt}{dx} = \dfrac{1}{\cos^2 x} \qquad dt = \dfrac{1}{\cos^2 x}\,dx$)

$① = \displaystyle\int dt = t.$

$(*) = \underline{\tan x - \dfrac{1}{\cos x}}$ //

(4) $\displaystyle\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \underline{\frac{x}{2} + \frac{\sin 2x}{4}}$

(5) $\displaystyle\int \frac{1}{2\sin x + \cos x}\,dx = \int \frac{1}{\frac{4t}{1+t^2} + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$

($t = \tan\frac{x}{2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$, $\sin x = \dfrac{2t}{1+t^2}$, $dx = \dfrac{2}{1+t^2}\,dt$)

$$= \int \frac{1}{-t^2+4t+1}\,2\,dt = -2\int \frac{1}{t^2-4t-1}\,dt$$

($t^2-4t-1 = 0 \qquad t = 2 \pm \sqrt{4+1} = 2 \pm \sqrt{5} = \alpha, \beta \quad (\alpha < \beta)$

$\dfrac{1}{t^2-4t-1} = \dfrac{1}{\beta-\alpha}\left(\dfrac{1}{t-\beta} - \dfrac{1}{t-\alpha}\right) = \dfrac{1}{\beta-\alpha}\left(\dfrac{\beta-\alpha}{〃}\right)$)

$$= -\frac{2}{2\sqrt{5}}\int \frac{1}{t-\beta} - \frac{1}{t-\alpha}\,dt = -\frac{1}{\sqrt{5}}\left\{\log|t-\beta| - \log|t-\alpha|\right\}$$

$$= \underline{-\frac{1}{\sqrt{5}}\log\left|\frac{t-2-\sqrt{5}}{t-2+\sqrt{5}}\right|} \text{ //}$$

例 3.8.

(1) $\displaystyle\int \frac{1}{1+\sqrt[3]{1+x}}\,dx = \int \frac{1}{1+t}\,3t^2\,dt$

$\left(t = \sqrt[3]{1+x} \rightarrow t^3 = 1+x \rightarrow 3t^2 dt = dx \right)$

無理(5)より

$$\begin{array}{r} t-1 \\ t+1\,\overline{)\,t^2} \\ t^2+t \\ \hline -t \\ -t-1 \\ \hline 1. \end{array}$$

$\displaystyle = 3\int t-1+\frac{1}{t+1}\,dt = 3\left(\frac{t^2}{2} - t + \log|t+1|\right)$

(2) $\displaystyle\int \frac{1}{\sqrt{(x-1)(2-x)}}\,dx$

無理(7)ii より $\displaystyle t = \sqrt{\frac{x-1}{2-x}} \rightarrow t^2 = \frac{x-1}{2-x} \rightarrow (2-x)t^2 = x-1$

$(t^2+1)x = 2t^2+1$

$\displaystyle x = \frac{2t^2+1}{t^2+1} = 2 - \frac{1}{t^2+1}$

$\displaystyle \frac{dx}{dt} = (t^2+1)^{-2}\cdot 2t \qquad dx = \frac{2t}{(t^2+1)^2}\,dt$

$\displaystyle \sqrt{(x-1)(2-x)} = \sqrt{t^2(2-x)^2} = t(2-x) = t\cdot\frac{1}{t^2+1}$

与式 $\displaystyle = \int \cancel{\frac{t^2+1}{t}}\,\frac{2\cancel{t}}{(t^2+1)^{\cancel{2}}}\,dt = 2\int\frac{dt}{t^2+1} = 2\tan^{-1}t$

(3) $\displaystyle\int \frac{1}{(x-1)\sqrt{x^2-4x-2}}\,dx$

$t - x = \sqrt{x^2-4x-2} \quad \rightarrow \quad t^2 - 2tx + \cancel{x^2} = \cancel{x^2} - 4x - 2$

$$(2t-4)x = t^2+2$$

$$x = \frac{t^2+2}{2(t-2)}$$

$$x-1 = \frac{t^2+2-2t+4}{2t-4} = \frac{t^2-2t+6}{〃}.$$

$$\sqrt{x^2-4x-2} = t - x = t - \frac{t^2+2}{2t-4} = \frac{2t^2-4t-t^2-2}{2t-4} = \frac{t^2-4t-2}{2t-4}$$

$$\frac{dx}{dt} = \frac{2t(2t-4)-(t^2+2)\cdot 2}{(2t-4)^2} = \frac{4t^2-8t-2t^2-4}{〃} = \frac{2t^2-8t-4}{(2t-4)^2}$$

$$与式 = \int \frac{1}{\dfrac{t^2-2t+6}{\cancel{2t-4}}\ \cancel{\dfrac{t^2-4t-2}{2t-4}}}\ \cancel{\frac{2t^2-8t-4}{(2t-4)^2}} \rightarrow 2\ dt$$

$$= \int \frac{2}{t^2-2t+6}\,dt.$$

$$= 2\int \frac{1}{(t-1)^2+5}\,dt$$

$$= \frac{2}{5}\int \frac{1}{\left(\frac{t-1}{\sqrt5}\right)^2+1}\,dt = \frac{2}{\sqrt5}\int \frac{1}{\left(\frac{t-1}{\sqrt5}\right)^2+1}\,d\left(\frac{t-1}{\sqrt5}\right)$$

$$= \frac{2}{\sqrt5}\tan^{-1}\left(\frac{t-1}{\sqrt5}\right)$$

X

(3) $\displaystyle\int \frac{1}{(x-1)\sqrt{x^2-4x-2}}\,dx$ $\quad \boxed{\sqrt{x^2-4x-2} = t-x}$ が"正しい"

無理(7)(i) より

$t=\sqrt{x^2-4x-2}-x \rightarrow \quad t^2+2tx+\cancel{x^2}=\cancel{x^2}-4x-2$

$(2t+4)x=-t^2-2, \quad x=\dfrac{-t^2-2}{2t+4}$

$x-1 = \dfrac{-t^2-2-2t-4}{2t+4} = \dfrac{-t^2-2t-6}{2t+4}$

$\sqrt{x^2-4x-2} = t+x = t+\dfrac{-t^2-2}{2t+4} = \dfrac{2t^2+4t-t^2-2}{2t+4} = \dfrac{t^2+4t-2}{2t+4}$

$\dfrac{dx}{dt} = \dfrac{-2t(2t+4)-(-t^2-2)\cdot 2}{(2t+4)^2} = \dfrac{-4t^2-8t+2t^2+4}{〃} = \dfrac{-2t^2-8t+4}{〃}$

$= \dfrac{-2}{4}\,\dfrac{t^2+4t-2}{(t+2)^2}$

与式 $= \displaystyle\int \frac{1}{\frac{1}{4}\,\frac{-t^2-2t-6}{\cancel{2t+4}\,2}\;\frac{\cancel{t^2+4t-2}}{\cancel{2t+4}\,2}}\;\left(-\frac{1}{2}\right)\frac{\cancel{t^2+4t-2}}{\cancel{(t+2)^2}}\,dt$

$= \displaystyle\int \frac{+\cancel{4}\,2}{t^2+2t+6}\,dt = 2\int \frac{1}{(t+1)^2+5}\,dt = \frac{2}{\sqrt{5}}\int \frac{1}{\left(\frac{t+1}{\sqrt{5}}\right)^2+1}\,\frac{dt}{\sqrt{5}}$

$= \dfrac{2}{\sqrt{5}}\tan^{-1}\left(\dfrac{t+1}{\sqrt{5}}\right)$

問 3.8

(1) $\displaystyle\int \frac{1}{x}\sqrt{\frac{1-x}{x}}\,dx$

$t=\sqrt{\dfrac{1-x}{x}}$ $\quad t^2=\dfrac{1-x}{x}$ $\quad t^2x=1-x$, $\quad (t^2+1)x=1$, $\quad x=\dfrac{1}{t^2+1}$

$\dfrac{dx}{dt}=\dfrac{-1}{(t^2+1)^2}(2t)$ $\rightarrow$ $dx=\dfrac{-2t}{(t^2+1)^2}dt$

$$=\int (t^2+1)t\,\frac{-2t}{(t^2+1)^2}\,dt=\int \frac{-2t^2-2+2}{t^2+1}\,dt=\int -2+\frac{2}{t^2+1}\,dt$$

$$=\underline{-2t+2\tan^{-1}t}\ /\!/$$

(2) $$\int \frac{1}{\sqrt{x^2+2}}\,dx=\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(\frac{x}{\sqrt{2}}\right)^2+1}}\,dx=\int \frac{1}{\sqrt{\left(\frac{x}{\sqrt{2}}\right)^2+1}}\,d\left(\frac{x}{\sqrt{2}}\right)$$

$$=\underline{\log\left|\frac{x}{\sqrt{2}}+\sqrt{\left(\frac{x}{\sqrt{2}}\right)^2+1}\right|=\log\left|x+\sqrt{x^2+2}\right|+C}\ /\!/$$

(別)

$\sqrt{x^2+2}=t-x$ とおくと $\quad \cancel{x^2}+2=t^2-2tx+\cancel{x^2}$

$2tx=t^2-2$. $\quad x=\dfrac{t^2-2}{2t}$

$\sqrt{x^2+2}=t-\dfrac{t^2-2}{2t}=\dfrac{2t^2-t^2+2}{2t}=\dfrac{t^2+2}{2t}$

$\dfrac{dx}{dt}=\dfrac{2t\cdot 2t-(t^2-2)\cdot 2}{4t^2}=\dfrac{2t^2+4}{4t^2}=\dfrac{t^2+2}{2t^2}$

与式 $=\displaystyle\int \frac{1}{\cancel{\frac{t^2+2}{2t}}}\,\frac{\cancel{t^2+2}}{2t^2}\,dt=\int\frac{dt}{t}=\underline{\log|t|}=\underline{\log\left|x+\sqrt{x^2+2}\right|}$

(3) $\displaystyle\int \frac{1}{(x-1)\sqrt{2+x-x^2}}\,dx$

$$-x^2+x+2 = -(x^2-x-2) = -(x+1)(x-2) \qquad \begin{cases} \alpha = -1 \\ \beta = 2. \end{cases}$$

$$t = \sqrt{\frac{x-\alpha}{\beta-x}} = \sqrt{\frac{x+1}{2-x}}$$

$$t^2 = \frac{x+1}{2-x} \rightarrow \quad t^2(2-x) = x+1 \rightarrow \quad (1+t^2)x = 2t^2-1$$

$$x = \frac{2t^2-1}{1+t^2}$$

$$x-1 = \frac{2t^2-1-1-t^2}{1+t^2} = \frac{t^2-2}{t^2+1}$$

$$\sqrt{2+x-x^2} = \sqrt{(x+1)(2-x)} = \sqrt{t^2(2-x)^2} = t(2-x)$$

$$= t\left(\frac{2+\cancel{2t^2}-\cancel{2t^2}+1}{1+t^2}\right) = \underline{\frac{3t}{t^2+1}}$$

$$\frac{dx}{dt} = \frac{4t(1+t^2)-(2t^2-1)2t}{(1+t^2)^2} = \frac{2t(2+\cancel{2t^2}-\cancel{2t^2}+1)}{(1+t^2)^2} = \frac{6t}{(1+t^2)^2}$$

$$= \int \frac{1}{\frac{t^2-2}{\cancel{t^2+1}}\ \frac{\cancel{3t}}{\cancel{t^2+1}}}\ \frac{\cancel{6t}\ 2}{(\cancel{t^2+1})^2}\,dt = \int \frac{2}{t^2-2}\,dt$$

$$= \frac{2}{2\sqrt{2}}\int \frac{1}{t-\sqrt{2}} - \frac{1}{t+\sqrt{2}}\,dt = \frac{1}{\sqrt{2}}\log|t-\sqrt{2}| - \log|t+\sqrt{2}|$$

$$= \frac{1}{\sqrt{2}}\log\left|\frac{t-\sqrt{2}}{x+\sqrt{2}}\right|$$

//

(3) $\displaystyle\int \frac{1}{(x-1)\sqrt{2-x-x^2}}\,dx$ → しまった $\displaystyle\int^{1} \frac{1}{(x-1)\sqrt{2+x-x^2}}\,dx$ だった

無理(7)(ii)より $-x^2-x+2=-(x^2+x-2)=-(x-1)(x+2)$ $\begin{cases}\alpha=-2\\ \beta=1\end{cases}$

$t=\sqrt{\dfrac{x-\alpha}{\beta-x}}=\sqrt{\dfrac{x+2}{1-x}}$ とおくと

$t^2=\dfrac{x+2}{1-x} \rightarrow (1-x)t^2=x+2$ $\quad (1+t^2)x=t^2-2,\ x=\dfrac{t^2-2}{1+t^2}$

$x-1=\dfrac{t^2-2}{t^2+1}-1=\dfrac{t^2-2-(t^2+1)}{t^2+1}=\dfrac{-3}{t^2+1}$

$\sqrt{2-x-x^2}=\sqrt{(1-x)(x+2)}=\sqrt{(1-x)^2t^2}=(1-x)t=\dfrac{3}{t^2+1}\cdot t$

$\dfrac{dx}{dt}=\dfrac{2t(t^2+1)-(t^2-2)\cdot 2t}{(1+t^2)^2}=\dfrac{2t}{(1+t^2)^2}\cdot 3$

与式 $=\displaystyle\int \frac{1}{\dfrac{-3}{\cancel{t^2+1}}\ \dfrac{3\cancel{t}}{\cancel{t^2+1}}}\ \frac{6\cancel{t}}{\cancel{(1+t^2)^2}}\,dt$

$=\displaystyle\int \frac{\cancel{6}2}{-\cancel{9}3}\,dt=-\frac{2}{3}t$

(4) $\displaystyle\int \frac{1}{x^2(x^2-1)^{3/2}}\,dx$

$x = \sec\theta$ とおくと　$x^2 = \dfrac{1}{\cos^2\theta}$,　$x^2-1 = \dfrac{1-\cos^2\theta}{\cos^2\theta} = \dfrac{\sin^2\theta}{\cos^2\theta}$

$(x^2-1)^{\frac{3}{2}} = \dfrac{\sin^3\theta}{\cos^3\theta}$

$\dfrac{dx}{d\theta} = \dfrac{-1\cdot(-\sin\theta)}{\cos^2\theta} = \dfrac{\sin\theta}{\cos^2\theta}$

$$与式 = \int \frac{1}{\cancel{\frac{1}{\cos^2\theta}}\,\frac{\sin^3\theta}{\cos^3\theta}}\,\frac{\sin\theta}{\cancel{\cos^2\theta}}\,d\theta = \int \frac{\cos^3\theta}{\sin^2\theta}\,d\theta$$

$$= \int \frac{1-\sin^2\theta}{\sin^2\theta}\,\underbrace{\boxed{\cos\theta}}_{(\sin\theta)'}\,d\theta = \int \frac{1-t^2}{t^2}\,dt = \int \frac{1}{t^2}-1\,dt = \underline{\frac{-1}{t} - t}$$

$t = \sin\theta$

(5) $\displaystyle\int \frac{1}{x\sqrt{4-x^2}}\,dx = \int \frac{1}{\frac{1}{t}\sqrt{4-\frac{1}{t^2}}}\,\frac{-1}{t^2}\,dt$

$\left(x = \frac{1}{t} \right.$ とおく． $\left. \frac{dx}{dt} = \frac{-1}{t^2}. \right)$

$\displaystyle = \int \frac{-1}{\sqrt{4t^2-1}}\,dt = \frac{-1}{2}\int \frac{1}{\sqrt{(2t)^2-1}}\,d(2t)$

$\displaystyle = -\frac{1}{2}\log\left|2t+\sqrt{(2t)^2-1}\right| = -\frac{1}{2}\log\left|t+\sqrt{t^2-\frac{1}{4}}\right| + C$ //

例 3.9

(1). $\displaystyle\int \frac{1}{e^{2x}-2e^{x}}\,dx = \int \frac{1}{t^2-2t}\,\frac{dt}{t} = \int \frac{1}{t^2(t-2)}\,dt \quad \cdots (*)$

$\left(t=e^x \qquad \frac{dt}{dx}=e^x,\quad dx=\frac{dt}{e^x}=\frac{dt}{t} \right)$

$$\frac{1}{t^2(t-2)} = \frac{a}{t}+\frac{b}{t^2}+\frac{c}{t-2} = \frac{at(t-2)+b(t-2)+ct^2}{t^2(t-2)}$$

分子 $=(a+c)t^2+(-2a+b)t+(-2b)$

$$\begin{cases} a+c=0 \longrightarrow c=\frac{1}{4} \\ -2a+b=0 \longrightarrow a=\frac{b}{2}=-\frac{1}{4} \\ -2b=1 \to b=-\frac{1}{2} \end{cases}$$

$$(*)=\int \frac{-\frac{1}{4}}{t}+\frac{-\frac{1}{2}}{t^2}+\frac{\frac{1}{4}}{t-2}\,dt = -\frac{1}{4}\log|t|+\frac{1}{2}t^{-1}+\frac{1}{4}\log|t-2|$$

$$=\underline{\frac{1}{4}\log\left|\frac{t-2}{t}\right|+\frac{1}{2}\frac{1}{t}}_{\,//}$$

(2) $\displaystyle\int \frac{\sqrt{1+\log x}}{x}\,dx = \int \sqrt{t}\,dt = \frac{2}{3}t^{\frac{3}{2}}$

$\left(\frac{1}{x}=(1+\log x)' \right.$ より $t=1+\log x$ とおく $\left.\right)$

(3) $\displaystyle\int \frac{1}{\sqrt{e^{3x}+4}}\,dx$

$\left(\sqrt{e^{3x}+4}=t \longrightarrow e^{3x}+4=t^2 \longrightarrow e^{3x}\cdot 3\frac{dx}{dt}=2t \right.$

$\left. \frac{dx}{dt}=\frac{2}{3e^{3x}}t=\frac{2}{3(t^2-4)}t \right)$

与式 $\displaystyle=\int \frac{1}{\cancel{t}}\,\frac{2}{3(t^2-4)}\cancel{t}\,dt = \frac{2}{3}\frac{1}{4}\int \frac{1}{t-2}-\frac{1}{t+2}\,dt$

$$=\underline{\frac{1}{6}\log\left|\frac{t-2}{t+2}\right|}_{\,//}$$

問3.9.

(1) $\displaystyle\int \frac{(\log x)^n}{x}\,dx \quad (n \neq -1) = \int t^n\,dt = \underline{\frac{t^{n+1}}{n+1}}$ //

$\left(t = \log x. \quad \dfrac{dt}{dx} = \dfrac{1}{x} \rightarrow \dfrac{1}{x}\,dx = dt \right)$

(2) $\displaystyle\int \frac{e^{2x}}{\sqrt[4]{e^x+1}}\,dx$

$$\left(\begin{aligned} & t = \sqrt[4]{e^x+1} \longrightarrow t^4 = e^x + 1 \longrightarrow e^x \frac{dx}{dt} = 4t^3 \\ & dx = \frac{4t^3}{e^x}\,dt = \frac{4t^3}{t^4-1}\,dt \end{aligned} \right)$$

$$与式 = \int \frac{(t^4-1)^{\cancel{2}}}{\cancel{t}} \frac{4t^{\cancel{3}\,2}}{\cancel{(t^4-1)}} = \int (t^4-1)\cdot 4t^2\,dt$$

$$= \int 4t^6 - 4t^2\,dt = \underline{\frac{4}{7}t^7 - \frac{4}{3}t^3} = \frac{4}{7}(e^x+1)^{\frac{7}{4}} - \frac{4}{3}(e^x+1)^{\frac{3}{4}}$$

$\boxed{\text{check.}}$
微分 $\longrightarrow$ $\displaystyle\frac{4}{7}\cdot\frac{7}{4}(e^x+1)^{\frac{3}{4}}\cdot e^x - \frac{4}{3}\cdot\frac{3}{4}(e^x+1)^{-\frac{1}{4}} e^x$

$$= (e^x+1)^{1-\frac{1}{4}} e^x - e^x (e^x+1)^{-\frac{1}{4}}$$

$$= \frac{e^x}{\sqrt[4]{e^x+1}} (e^x + 1 - 1) = \frac{e^{2x}}{\sqrt[4]{e^x+1}} \qquad \boxed{\text{OK}}$$

$$(3)\ \int \frac{\overset{(e^x+e^{-x})'}{\overset{\shortparallel}{\boxed{e^x - e^{-x}}}}}{e^x + e^{-x}}\,dx$$

$$= \underline{\log\left(e^x + e^{-x}\right)}$$

$$(4)\ \int x e^{-x^2}\,dx = \int e^{-x^2}\frac{(-x^2)'}{-2}\,dx = \underline{\frac{e^{-x^2}}{-2}}\ /\!/$$

$$(5)\ \int \log(1+\sqrt{x})\,dx = \int \log(1+t)\overset{(t^2)'}{2t}\,dt$$

$$\left(t = \sqrt{x} \qquad \frac{dt}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2t}, \quad 2t\,dt = dx\right)$$

$$= t^2\log(1+t) - \underbrace{\int t^2\frac{1}{t+1}\,dt}_{①} \cdots (*)$$

$$\begin{array}{r} t-1 \\ t+1\,\overline{)\,t^2} \\ t^2+t \\ \hline -t \\ -t-1 \\ \hline 1 \end{array}$$

$$① = \int t-1+\frac{1}{t+1}\,dt$$

$$= \frac{t^2}{2} - t + \log|t+1|$$

$$(*) = \underline{t^2\log(1+t) - \left(\frac{t^2}{2} - t + \log|t+1|\right)}$$

$\boxed{3.4}$ 定積分

◇ $f(x)$ がある区間で不定積分を持ち，その1つを $F(x)$ とすると，

($F'(x) = f(x)$)

$$\underbrace{\int_a^b f(x)\,dx}_{a\text{から}b\text{までの定積分}} = F(b) - F(a) = [F(x)]_a^b$$

上端

下端

◇ 基本的な性質

(1) $\int_a^a f(x)\,dx = F(a) - F(a) = 0$

(1)' $\int_a^b f(x)\,dx = F(b) - F(a) = -(F(a) - F(b)) = -\int_b^a f(x)\,dx$

(2) $\int_a^b f(x)\,dx = F(b) \underbrace{\;}_{-F(c)+F(c)} - F(a) = \int_c^b f(x)\,dx + \int_a^c f(x)\,dx$

$= \int_a^c + \int_c^b$

線形性

(3) $\int_a^b \{f(x) \pm g(x)\}\,dx = [F(x) \pm G(x)]_a^b = [F(x)]_a^b \pm [G(x)]_a^b$

$= \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(4) $\int_a^b k f(x)\,dx = [kF(x)]_a^b = k\,[F(x)]_a^b = k\int_a^b f(x)\,dx$

定理3.5

$f(x)$ は $[a, b]$ で連続　$x = g(t)$ は $[\alpha, \beta]$ で微分可能

$g'(t)$ は連続　$a = g(\alpha)$, $b = g(\beta)$

$$\Rightarrow \int_a^b f(x)\,dx = \int_\alpha^\beta f(g(t))\,g'(t)\,dt$$

$$\left(\begin{array}{l} x = g(t), \quad dx = g'(t)\,dt \\ \begin{array}{c|c} x & a \longrightarrow b \\ \hline t & \alpha \longrightarrow \beta \end{array} \end{array} \right)$$

定理3.6　$f(x), g(x)$ が C^1 級

$$\int_a^b f(x) g'(x)\,dx = [f(x) g(x)]_a^b - \int_a^b f'(x) g(x)\,dx$$

$$\because \int_a^b f(x) g'(x)\,dx = \left[f(x) g(x) - \int f'(x) g(x)\,dx \right]_a^b$$

$$= [f(x) g(x)]_a^b - \int_a^b f'(x) g(x)\,dx$$

例 3.10.

$$\int_0^{\pi/2} \frac{\sin x}{1+\sin x}\,dx = \int_0^1 \frac{\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2}} \frac{2}{1+t^2}\,dt$$

$$t = \tan\frac{x}{2} \qquad \sin x = \frac{2t}{1+t^2}$$

x	$0 \to \pi/2$
t	$0 \to 1$

$$\frac{dt}{dx} = \frac{1}{\cos^2\frac{x}{2}}\frac{1}{2} = \frac{(1+t^2)}{2} \to \frac{2\,dt}{1+t^2} = dx$$

$$\underbrace{1+\tan^2\frac{x}{2}}_{1+t^2} = \frac{1}{\cos^2\frac{x}{2}}$$

$$= \int_0^1 \frac{4t}{(t^2+2t+1)(1+t^2)}\,dt = \int_0^1 2\left(\frac{1}{1+t^2} - \frac{1}{t^2+2t+1}\right)dt$$

$$= 2\left[\tan^{-1} t\right]_0^1 + 2\left[(t+1)^{-1}\right]_0^1$$

$$= 2\cdot\frac{\pi}{4} + 2\Big\{\underbrace{\frac{1}{2} - 1}_{-\frac{1}{2}}\Big\} = \underline{\frac{\pi}{2} - 1}$$

例 3.11

$$\int_1^e x\log x\,dx = \int_1^e \left(\frac{x^2}{2}\right)'\log x\,dx = \left[\frac{x^2}{2}\log x\right]_1^e - \int_1^e \frac{x^{\cancel{2}}}{2}\frac{1}{\cancel{x}}\,dx$$

$$= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e = \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right) = \underline{\frac{e^2}{4} + \frac{1}{4}}$$

問 3.10

(1) $\displaystyle\int_2^3 \frac{1}{1-x^2}\,dx = \int_2^3 \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right)dx = \left[\frac{1}{2}\left(\underbrace{-\log|1-x| + \log|x+1|}_{\log\left|\frac{x+1}{x-1}\right|}\right)\right]_2^3$

$\displaystyle = \frac{1}{2}\left\{\log\frac{4}{2} - \log\frac{3}{1}\right\} = \underline{\frac{1}{2}\log\frac{2}{3}.}$

(2) $\displaystyle\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{1/2} \frac{-1+x^2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\,dx$

$\displaystyle = \int_0^{1/2} -\sqrt{1-x^2} + \frac{1}{\sqrt{1-x^2}}\,dx = \Big[\underbrace{-\frac{1}{2}\left(x\sqrt{1-x^2} + \sin^{-1}x\right) + \sin^{-1}x}_{-\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x}\Big]_0^{1/2}$

$\displaystyle = -\frac{1}{4}\sqrt{\frac{3}{4}} + \frac{1}{2}\underbrace{\sin^{-1}\frac{1}{2}}_{\frac{\pi}{6}} = \underline{\frac{\pi}{12} - \frac{\sqrt{3}}{8}}$ //

(3) $\displaystyle\int_0^{\sqrt{3}/2} \frac{5}{4x^2+3}\,dx = \frac{5}{3}\int_0^{\sqrt{3}/2} \frac{1}{\frac{4}{3}x^2+1}\,dx = \frac{5}{3}\int_0^1 \frac{1}{t^2+1}\frac{\sqrt{3}}{2}\,dt$

$\left(\begin{array}{l} x = \frac{2}{\sqrt{3}}x \qquad dt = \frac{2}{\sqrt{3}}dx \\ \begin{array}{c|c} x & 0 \to \sqrt{3}/2 \\ t & 0 \to 1. \end{array}\end{array}\right)$

$\displaystyle = \frac{5\sqrt{3}}{6}\left[\tan^{-1}t\right]_0^1 = \frac{5}{6}\sqrt{3}\,\frac{\pi}{4} = \underline{\frac{5\sqrt{3}}{24}\pi}.$

(4) $\displaystyle\int_0^1 \frac{3}{\sqrt{5x^2+4}}\,dx = \frac{3}{2}\int_0^1 \frac{1}{\sqrt{\left(\frac{\sqrt{5}}{2}x\right)^2+1}}\,dx = \frac{3}{2}\int_0^{\frac{\sqrt{5}}{2}} \frac{1}{\sqrt{t^2+1}}\,\frac{2}{\sqrt{5}}\,dt$

$\left(\begin{array}{l} t = \frac{\sqrt{5}}{2}x, \qquad dt = \frac{\sqrt{5}}{2}dx \\ \begin{array}{c|c} x & 0 \to 1 \\ t & 0 \to \frac{\sqrt{5}}{2} \end{array}\end{array}\right)$

$\displaystyle = \frac{3}{\sqrt{5}}\int_0^{\frac{\sqrt{5}}{2}} \frac{1}{\sqrt{t^2+1}}\,dt$

$\displaystyle = \frac{3}{\sqrt{5}}\left[\log\left|t + \sqrt{t^2+1}\right|\right]_0^{\sqrt{5}/2}$

$\displaystyle = \frac{3}{\sqrt{5}}\Big(\log\Big|\underbrace{\frac{\sqrt{5}}{2} + \sqrt{\frac{5}{4}+1}}\Big| - \underbrace{\log 1}_{0}\Big) = \underline{\frac{3}{\sqrt{5}}\log\left(\frac{\sqrt{5}}{2} + \frac{3}{2}\right)}$ //

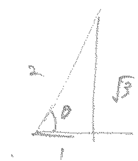
(5) $\displaystyle\int_0^{2/\sqrt{3}} \frac{1}{\sqrt{16-9x^2}}\,dx = \frac{1}{4}\int_0^{2/\sqrt{3}} \frac{1}{\sqrt{1-\left(\frac{3}{4}x\right)^2}}\,dx$

$\left(\begin{array}{l} t = \frac{3}{4}x \qquad dt = \frac{3}{4}dx \\ \begin{array}{c|c} x & 0 \longrightarrow 2/\sqrt{3} \\ \hline t & 0 \longrightarrow \frac{\sqrt{3}}{2} \end{array} \end{array} \right)$

$$= \frac{1}{4}\int_0^{\sqrt{3}/2} \frac{1}{\sqrt{1-t^2}}\,\frac{4}{3}\,dt$$

$$= \frac{1}{3}\Big[\sin^{-1} t\Big]_0^{\sqrt{3}/2}$$

$$= \frac{1}{3}\underbrace{\sin^{-1}\frac{\sqrt{3}}{2}}_{\theta} = \frac{1}{3}\cdot\frac{\pi}{3} = \underline{\frac{\pi}{9}}_{\,//}$$



(6) $\displaystyle\int_0^1 e^{-2x}\,dx = \left[\frac{e^{-2x}}{-2}\right]_0^1 = -\frac{1}{2}\left(e^{-2} - e^{0}\right) = \underline{\frac{1-e^{-2}}{2}}_{\,//}$

(7) $\displaystyle\int_0^1 \frac{x^2\overset{+4-4}{}}{\sqrt{x^2+4}}\,dx = \int_0^1 \sqrt{x^2+4} - \frac{4}{\sqrt{x^2+4}}\,dx$

$$= \left[\ \frac{1}{2}\left\{x\sqrt{x^2+4} + 4\log\left(x+\sqrt{x^2+4}\right)\right\} - 4\log\left|x+\sqrt{x^2+4}\right|\ \right]_0^1$$

$$= \frac{1}{2}\left\{\sqrt{5} + 4\log\left(1+\sqrt{5}\right)\right\} - 4\log\left(1+\sqrt{5}\right)$$

$$-\left(\frac{1}{2}\left\{+4\log 2\right\} - 4\log 2\right)$$

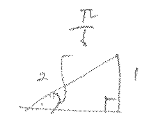
$$= \frac{\sqrt{5}}{2} - 2\log\left(1+\sqrt{5}\right) + 2\log 2$$

$$= \underline{\frac{\sqrt{5}}{2} - 2\log\frac{1+\sqrt{5}}{2}}_{\,//}$$

(8) $\displaystyle\int_0^{\cancel{\sqrt{3}/2}\ 1/\sqrt{2}} \frac{3x}{\sqrt{1-x^4}}\,dx = \frac{3}{2}\int_0^{\cancel{3/4}\ \frac{1}{2}} \frac{1}{\sqrt{1-t^2}}\,dt = \frac{3}{2}\left[\sin^{-1} t\right]_0^{\cancel{3/4}\ \frac{1}{2}} = \frac{3}{2}\cdot\frac{\pi}{6}$

$\displaystyle = \frac{\pi}{4}$ //

$$\left(\begin{array}{l} t = x^2 \qquad \dfrac{dt}{dx} = 2x. \qquad \dfrac{dt}{2} = x\,dx \\[2ex] \begin{array}{c|l} x & 0 \longrightarrow \cancel{\sqrt{3}/2}\ 1/\sqrt{2} \\ \hline t & 0 \longrightarrow \cancel{3/4}\ \frac{1}{2} \end{array} \end{array}\right) = \frac{3}{2}$$



(9) $\displaystyle\int_0^{\pi/2} x^2 \underset{(\sin x)'}{\boxed{\cos x}}\,dx = \underbrace{\left[x^2 \sin x\right]_0^{\pi/2}}_{\frac{\pi^2}{4}} - \underbrace{\int_0^{\pi/2} 2x \overset{(-\cos x)'}{\boxed{\sin x}}\,dx}_{①} \quad \cdots (*)$

$\displaystyle ① = \left[2x(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} 2(-\cos x)\,dx$

$\displaystyle = \pi\big(\underbrace{-\cos \tfrac{\pi}{2}}_{0}\big) - 2\left[\sin x\right]_0^{\frac{\pi}{2}}$

$= \cancel{\pi} - 2$

$\displaystyle (*) = \frac{\pi^2}{4} - \cancel{\pi} + 2$ // $\displaystyle = \frac{\pi^2}{4} + 2$ //

(10) $\displaystyle\int_0^{\pi/2} \frac{1}{2+\cos x}\,dx = \int_0^1 \frac{1}{2 + \frac{1-t^2}{1+t^2}} \frac{2}{1+t^2}\,dt$

$$\left(\begin{array}{l} t = \tan\dfrac{x}{2} \qquad \cos x = \dfrac{1-t^2}{1+t^2} \qquad \dfrac{dx}{dt} = \dfrac{2}{1+t^2}\,dt \\[2ex] \begin{array}{c|l} x & 0 \longrightarrow \pi/2 \\ \hline t & 0 \longrightarrow 1 \end{array} \end{array}\right)$$

$\displaystyle = \int_0^1 \frac{2}{2t^2+2+1-t^2}\,dt = \int_0^1 \frac{2}{t^2+3}\,dt = \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$

$\displaystyle = \frac{2}{\sqrt{3}} \tan^{-1}\frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}\,\frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}$ //

(11) $\displaystyle\int_{-1}^{1} \frac{1}{(1+x^2)^2}\,dx$

$$\left(\text{P.69. 問3.6} \quad I_n = \int \frac{dx}{(x^2+a^2)^n} = \frac{1}{2(n-1)a^2}\left\{ \frac{x}{(x^2+a^2)^{n-1}} + (2n-3) I_{n-1} \right\} \right)$$

$$= \frac{1}{2}\left\{ \underbrace{\left[\frac{x}{(x^2+1)} \right]_{-1}^{1}}_{\frac{1}{2} - \frac{-1}{2} = 1} + \underbrace{\int_{-1}^{1} \frac{1}{1+x^2}\,dx}_{\underbrace{\underbrace{\tan^{-1}1}_{\frac{\pi}{4}} - \underbrace{\tan^{-1}(-1)}_{-\frac{\pi}{4}}}_{\frac{\pi}{2}}} \right\} = \underline{\frac{1}{2}\left\{ 1 + \frac{\pi}{2} \right\}}\ //$$

(12) $$\int_0^1 x \underbrace{\boxed{e^x}}_{= (e^x)'} dx = \underbrace{\left[x e^x \right]_0^1}_{e} - \underbrace{\int_0^1 e^x\,dx}_{\left[e^x \right]_0^1} = e - \{ e - 1 \} = \underline{1}\ //$$

(13) $$\int_0^1 x^3 \sqrt{1-x^2}\,dx = \int_0^1 x^2 \sqrt{1-x^2}\, x\,dx = \int_1^0 (1-t)\sqrt{t}\left(-\frac{1}{2}\right) dt$$

$$\left(\begin{array}{l} t = 1 - x^2 \qquad \frac{dt}{dx} = -2x \\ \begin{array}{c|c} x & 0 \to 1 \\ \hline t & 1 \to 0 \end{array} \end{array} \right)$$

$$= \frac{1}{2}\int_0^1 t^{\frac{1}{2}} - t^{\frac{3}{2}}\,dt$$

$$= \frac{1}{2}\left[\frac{2}{3} t^{\frac{3}{2}} - \frac{2}{5} t^{\frac{5}{2}} \right]_0^1 = \frac{1}{2}\left(\frac{2}{3} - \frac{2}{5} \right)$$

$$= \underline{\frac{2}{15}}\ //$$

(14) $$\int_0^1 \frac{1 - x^2 \boxed{-1+1}}{1+x^2}\,dx = \int_0^1 -1 + \frac{2}{1+x^2}\,dx = \left[-x + 2\tan^{-1}x \right]_0^1$$

$$= -1 + 2\tan^{-1}1 = \underline{-1 + \frac{\pi}{2}}\ //$$

(15) $\int_0^1 \log(1+\sqrt{x})\,dx \quad = \int_0^1 \log(1+t)\,2t\,dt$

$\Big($ $t = \sqrt{x}$ $\quad dt = \frac{1}{2}x^{-\frac{1}{2}}dx \quad \underset{t}{\sqrt{x}}\,2dt = dx$

x	$0 \to 1$
t	$0 \to 1$

$= \underbrace{\left[\log(1+t)\,t^2\right]_0^1}_{\log 2} - \underbrace{\int_0^1 \underbrace{\frac{1}{1+t}\cdot t^2}_{t-1+\frac{1}{t+1}}\,dt}_{①} \quad \cdots (*)$

$$\begin{array}{r} t-1 \\ t+1 \overline{)\, t^2 \qquad} \\ t^2+t \\ \hline -t \\ -t-1 \\ \hline 1 \end{array}$$

$① = \left[\frac{t^2}{2} - t + \log|t+1|\right]_0^1 = \frac{1}{2} - 1 + \log 2 = -\frac{1}{2} + \log 2$

$(*) = \underline{\frac{1}{2}}$ //

(16) $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx = \int_0^1 \frac{dt}{1+t^2} = \left[\tan^{-1} t\right]_0^1 = \underline{\frac{\pi}{4}}$ //

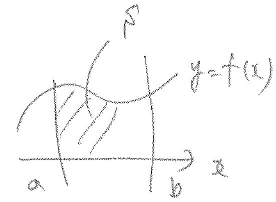
$\Big($ $t = \sin x \quad \frac{dt}{dx} = \cos x. \quad dt = \cos x\,dx$

x	$0 \to \pi/2$
t	$0 \to 1$

$\Big)$

$\boxed{3.5}$ 面積，不等式，微分と積分の関係

定理 3.7　　$f(x)$ は $[a,b]$ で連続，$f(x) \geqq 0$



$$S = \int_a^b f(x)\,dx$$

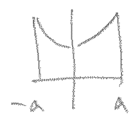
定理 3.8　　$f(x)$, $g(x)$，$[a,b]$ で連続

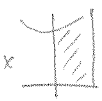
(1)　$g(x) \leqq f(x)$ ならば　$\displaystyle\int_a^b g(x)\,dx \leqq \int_a^b f(x)\,dx$

(2)　$\displaystyle\left|\int_a^b f(x)\,dx\right| \leqq \int_a^b |f(x)|\,dx$

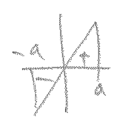
定理 3.9.　$f(x)$ が $[a,b]$ で連続 $\Rightarrow$ $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

◇ 公式

(3)　$f(x)$ が偶関数 $\Rightarrow$ 

$= 2 \times$ 

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)$$

(4)　$f(x)$ が奇〃　$\Rightarrow$ 

$= 0$

$$\int_{-a}^a f(x)\,dx = 0$$

(5) $$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \dfrac{n-1}{n}\,\dfrac{n-3}{n-2}\cdots\dfrac{4\cdot 2}{5\cdot 3} & n \geqq 2\ (\text{奇数}) \\ \dfrac{n-1}{n}\,\dfrac{n-3}{n-2}\cdots\dfrac{3}{4}\,\dfrac{1}{2}\times\dfrac{\pi}{2} & n \geqq 2\ (\text{偶数}) \end{cases}$$

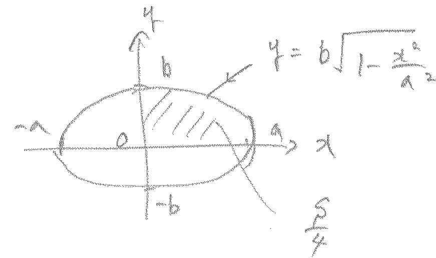
フーリエ級数で用いる

(6) $$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

(7) $$\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0$$

例 3.12

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ の囲む面積 S について.



(解) $\frac{y}{b} = \pm\sqrt{1-\frac{x^2}{a^2}}$

$$\frac{S}{4} = \int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx = \int_0^1 b\sqrt{1-t^2}\,a\,dt = ab\int_0^1\sqrt{1-t^2}\,dt$$

$$\left(t=\frac{x}{a} \quad \begin{array}{c|c} x & 0\to a \\ t & 0 \to 1 \end{array} \quad dt=\frac{dx}{a}\right) = \frac{ab}{2}\left[t\sqrt{1-t^2}+\sin^{-1}t\right]_0^1$$

$$= \frac{ab}{2}\underbrace{\boxed{\sin^{-1}1}}_{=\frac{\pi}{2}} = \frac{ab}{4}\pi$$

$$\underline{S = ab\pi}$$

例 3.13. $\log(1+\sqrt{2}) < \int_0^1 \frac{dx}{\sqrt{1+x^n}} < 1$ $(n>2)$ を示せ

$n>2$ のとき $0<x<1$ で $0<x^n<x^2<1$

$$\underset{①}{1<}\sqrt{1+x^n}\underset{②}{<}\sqrt{1+x^2}$$

① $\Rightarrow \frac{1}{\sqrt{1+x^n}} < 1$ ② $\Rightarrow \frac{1}{\sqrt{1+x^2}} < \frac{1}{\sqrt{1+x^n}}$

よって $\frac{1}{\sqrt{1+x^2}} < \frac{1}{\sqrt{1+x^n}} < 1$

$$\underbrace{\int_0^1 \frac{1}{\sqrt{1+x^2}}dx}_{} < \int_0^1 \frac{1}{\sqrt{1+x^n}}dx < \underbrace{\int_0^1 dx}_{=1}$$

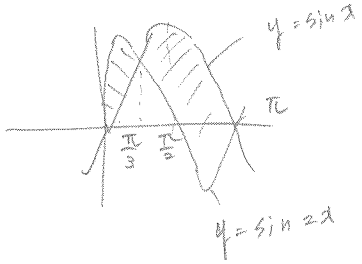
$\left[\log|x+\sqrt{1+x^2}|\right]_0^1$

$= \log(1+\sqrt{2})$

よって $\log(1+\sqrt{2}) < \int_0^1 \frac{dx}{\sqrt{1+x^n}} < 1$

問3.11

$y=\sin 2x$ と $y=\sin x$ が $0 \leqq x \leqq \pi$ において囲む部分の面積 S_1, S_2 の和



交点は $\sin 2x = \sin x$

$2\sin x \cos x = \sin x$

$\sin x\left(\cos x - \frac{1}{2}\right) = 0$

$\sin x = 0$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$

$$\int_0^{\frac{\pi}{3}} \sin 2x - \sin x \, dx + \int_{\frac{\pi}{3}}^{\pi} \sin x - \sin 2x \, dx \quad \cdots (*)$$

$F(x) = \dfrac{-\cos 2x}{2} + \cos x$ とおくと

$$(*) = F\left(\frac{\pi}{3}\right) - F(0) - F(\pi) + F\left(\frac{\pi}{3}\right)$$

$$= 2F\left(\frac{\pi}{3}\right) - F(0) - F(\pi)$$

$$= 2\left(\underbrace{-\frac{\cos\frac{2}{3}\pi}{2}}_{\substack{= -\frac{1}{2}\left(-\frac{1}{2}\right) \\ = \frac{1}{4}}} + \underbrace{\cos\frac{\pi}{3}}_{\frac{1}{2}}\right) - \underbrace{\left(\frac{-1}{2} + 1\right)}_{\frac{1}{2}} - \underbrace{\left(\frac{-1}{2} - 1\right)}_{-\frac{3}{2}}$$

$$= \underbrace{2 \times \frac{3}{4}}_{1} \underbrace{- \frac{1}{2} + \frac{3}{2}}_{1} = \frac{6-1}{2} = \frac{5}{2}$$

問3.12

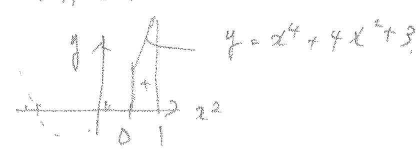
$0 < x < 1$ のとき

$$\sqrt{1-x^2} \underset{①}{<} \sqrt{1-x^4} \underset{②}{<} 2\sqrt{1+x^2} \quad (*)$$ を示し，

$$\frac{\pi}{2\sqrt{2}} < \int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx < \frac{\pi}{2}$$ を示せ。

(解) ①は $x^2 > x^4$ より $0 < 1-x^2 < 1-x^4 \Rightarrow \sqrt{1-x^2} \underset{①}{<} \sqrt{1-x^4}$

② $\Leftrightarrow$ $1-x^4 < 4(1+x^2)$ $\Leftrightarrow$ $\underbrace{x^4+4x^2+3}_{(x^2+3)(x^2+1)} > 0$ を示せばOK.



(*) 示せた。

このとき $\dfrac{1}{\sqrt{1-x^4}} < \dfrac{1}{\sqrt{1-x^2}}$ $\quad \dfrac{1}{2\sqrt{1+x^2}} < \dfrac{1}{\sqrt{1-x^4}}$

よって

$$\underbrace{\int_0^1 \frac{1\,dx}{2\sqrt{1+x^2}}}_{\frac{1}{2}\left[\log|x+\sqrt{1+x^2}|\right]_0^1} < \int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx < \underbrace{\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx}_{\left[\sin^{-1}x\right]_0^1}$$

$= \frac{1}{2}\log(1+\sqrt{2})$

$= \underbrace{\sin^{-1}1}_{\frac{\pi}{2}} - \underbrace{\sin^{-1}0}_{0} = \frac{\pi}{2}$

問題がまちがっているだろ!

$$\sqrt{1-x^2} < \sqrt{1-x^4} < \sqrt{2}\sqrt{1-x^2}$$

$$1-x^2 < (1-x^2)(1+x^2) < 2(1-x^2)$$

より

$$\sqrt{1-x^2} < \sqrt{1-x^4} < \sqrt{2}\sqrt{1-x^2}$$

$$\frac{1}{\sqrt{2}}\underbrace{\int_0^1 \frac{1\,dx}{\sqrt{1-x^2}}}_{\frac{\pi}{2}} < \int \frac{1}{\sqrt{1-x^4}}\,dx < \underbrace{\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx}_{\frac{\pi}{2}}$$ より $$\underline{\frac{\pi}{2\sqrt{2}} < \int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx < \frac{\pi}{2}}$$

問 3.13

$$F(x) = \int_0^x (x-t)\cos(3t)\,dt$$

$F''(x)$ は?

$$F'(x) = \underbrace{(x-x)\cos 3x}_{=0} + \underbrace{\int_0^x \cos 3t\,dt}_{\left[\frac{\sin 3t}{3}\right]_0^x} = \frac{\sin 3x}{3}$$

$$F''(x) = \frac{\cos 3x}{\cancel{3}}\cdot\cancel{3} = \underline{\cos 3x}$$

別) $F(x) = \int_0^x x\cos 3t - t\cos 3t\,dt$

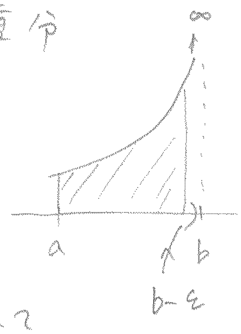
$$= x\int_0^x \cos 3t\,dt - \int_0^x t\cos 3t\,dt$$

$$F'(x) = \int_0^x \cos 3t\,dt + x\cancel{\cos 3x} - x\cancel{\cos 3x}$$

$$\underline{F''(x) = \cos 3x}$$

$\boxed{3.6}$ 広義積分

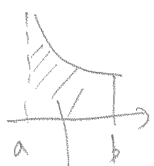
◇特異積分

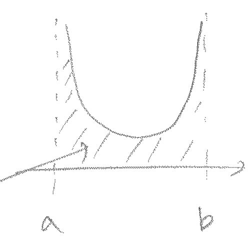


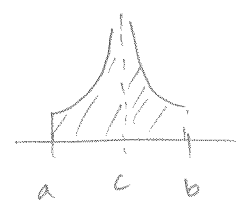
$\varepsilon > 0$ に対して

$$\int_a^{b-\varepsilon} f(x)\,dx \text{ は存在する}$$

$$\lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)\,dx \text{ が存在するとき } \underbrace{\int_a^b f(x)\,dx}_{\text{特異積分（広義積分）}} \text{ という.}$$







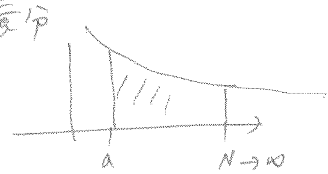
$$\int_a^b f(x)\,dx = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon' \to 0}} \int_{a+\varepsilon}^{b-\varepsilon'} f(x)\,dx$$

$$\int_a^b f(x)\,dx$$

$$= \int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon'}^{b} f(x)\,dx$$

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} f(x)\,dx$$

◇ 無限積分



無限積分 $\int_a^\infty f(x)\,dx = \lim_{N\to\infty}\int_a^N f(x)\,dx$

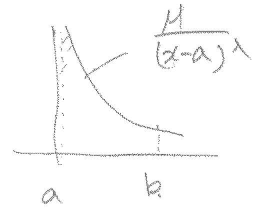
$\int_{-\infty}^a f(x)\,dx$, $\int_{-\infty}^\infty f(x)\,dx$ も同様

定理3.10

(1) $f(x)$ は $(a,b]$ で 連続

$M>0$, $\lambda<1$ に対して $|f(x)|(x-a)^\lambda \leqq M \quad (a<x<b)$

$\Rightarrow \int_a^b f(x)\,dx$ 存在



∵

$$0 < \underbrace{\int_{a+\varepsilon}^b |f(x)|\,dx}_{\downarrow\ 存在する} < \underbrace{\int_{a+\varepsilon}^b \frac{M}{(x-a)^\lambda}\,dx}_{\downarrow\ \varepsilon\to 0\ 存在する}$$

$\int_a^b |f(x)|$ が存在すれば $\int_a^b f(x)\,dx$ 存在.

(2) $f(x)$ が $[a,\infty)$ で 連続 $M>0$, $\lambda>1$ に対して $x^\lambda|f(x)| \leqq M$

$\Rightarrow \int_a^\infty f(x)\,dx$ は存在する.

例 3.14

(1) $$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \lim_{\varepsilon\to 0}\left[\sin^{-1}x\right]_0^{1-\varepsilon} = \underbrace{\sin^{-1}(1)}_{=\frac{\pi}{2}} - \underbrace{\sin^{-1}(0)}_{=0} = \frac{\pi}{2}$$

(2) $$\int_{-1}^0 \frac{1}{1-x^2}\,dx = \int_{-1+\varepsilon}^0 \frac{1}{1-x^2}\,dx$$

$$= \int_{-1+\varepsilon}^0 \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right)dx = \frac{1}{2}\Big[-\log|x-1| + \log|x+1|\Big]_{-1+\varepsilon}^0$$

$$= \frac{1}{2}\left[\log\left|\frac{x+1}{x-1}\right|\right]_{-1+\varepsilon}^0 = \frac{1}{2}\left(0 - \log\frac{\varepsilon}{2}\right) \xrightarrow[(\varepsilon\to 0)]{} +\infty$$ //

(3) $$\int_{-\infty}^{\infty} \frac{1}{x^2+4}\,dx = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-\infty}^{\infty} = \frac{1}{2}\Big(\underbrace{\tan^{-1}\infty}_{=\frac{\pi}{2}} - \underbrace{\tan^{-1}(-\infty)}_{=-\frac{\pi}{2}}\Big) = \frac{\pi}{2}$$ //

問 3.14

(1) $\displaystyle \int_0^1 x\log x\,dx = \underbrace{\left[\frac{x^2}{2}\log x\right]_0^1}_{0} - \underbrace{\int_0^1 \frac{x^2}{2}\frac{1}{x}\,dx}_{\left[\frac{x^2}{4}\right]_0^1}$

$\displaystyle = \frac{1}{4}$

(2) $\displaystyle \int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}}$

×

$\displaystyle t=\sqrt{\frac{2-x}{x-1}} \qquad t^2=\frac{2-x}{x-1} \qquad \sqrt{(x-1)(2-x)}=\sqrt{(x-1)^2t^2}=(x-1)t$

$\displaystyle 2t\frac{dt}{dx}=\left(-1+\frac{1}{x-1}\right)'=\frac{-1}{(x-1)^2} \Rightarrow -(x-1)^2\,2t\,dt=dx$

x	1	2
t	∞	0

○

$\displaystyle t=\sqrt{\frac{x-1}{2-x}} \qquad t^2=\frac{x-1}{2-x} \qquad x-1=t^2(2-x)$

$(t^2+1)x=2t^2+1$

$\displaystyle x=\frac{2t^2+1}{t^2+1}$

$\displaystyle \sqrt{(x-1)(2-x)}=\sqrt{t^2(2-x)^2}=t(2-x)$

x	$1\to 2$
t	$0\to\infty$

$\displaystyle t^2=\frac{x-2+1}{2-x}=-1+\frac{1}{2-x}=-1-\frac{1}{x-2}$

$\displaystyle 2t\frac{dt}{dx}=-1\cdot\frac{-1}{(x-2)^2} \qquad (x-2)^2\,2t\,dt=dx$

与式 $\displaystyle =\int_0^\infty \frac{(x-2)^2\,2\cancel{t}}{\cancel{t}(2-x)}\,dt=\int_0^\infty (2-x)\,dt=\int_0^\infty 2-\frac{2t^2+1}{t^2+1}\,dt$

$\displaystyle =\int_0^\infty \frac{\cancel{2t^2}+2-\cancel{2t^2}-1}{t^2+1}\,dt=\int_0^1\frac{1}{t^2+1}\,dt=\left[\tan^{-1}t\right]_0^1=\underline{\frac{\pi}{4}}$

(3) $\displaystyle\int_0^3 \frac{x}{(x^2-1)^{2/3}}\,dx$ 　　　 $x=1$ で不連続

$\left(t = x^2-1. \quad \frac{dt}{dx} = 2x \quad \frac{dt}{2} = x\,dx \right)$

x	$0 \to 3$
t	$-1 \to 8$

$$= \int_{-1}^{8} \frac{dt}{2t^{2/3}} = \frac{1}{2}\int_{-1}^{8} t^{-2/3}\,dt = \frac{1}{2}\,3\left[t^{\frac{1}{3}}\right]_{-1}^{8}$$

$$= \frac{3}{2}\left\{(2^3)^{\frac{1}{3}} - (-1)^{\frac{1}{3}}\right\} = \frac{3}{2}\left(2-(-1)\right) = \underline{\frac{9}{2}}$$

(4) $\displaystyle\int_1^\infty \frac{1}{x^\alpha}\,dx$ 　　 $(\alpha>0)$

$$\cancel{= \frac{1}{1-\alpha}\left[x^{1-\alpha}\right]_1^\infty = \left(\lim_{x\to\infty} x^{1-\alpha}\right) - 1 \quad \begin{cases} 0<\alpha<1 & \infty \\ \alpha=1 & 0 \\ \alpha>1 & -1. \end{cases}}$$

$\alpha \neq 1$ のとき $\displaystyle \left[\frac{x^{1-\alpha}}{1-\alpha}\right]_1^\infty = \begin{cases} 0<\alpha<1 \text{ のとき } \infty \\ 1<\alpha \text{ のとき } \dfrac{-1}{1-\alpha} = \dfrac{1}{\alpha-1} \end{cases}$

$\alpha=1$ のとき $\left[\log x\right]_1^\infty = \infty$

(5) $\displaystyle\int_1^\infty \frac{1}{x(1+x^2)}\,dx = \int_1^\infty \frac{x}{x^2(1+x^2)}\,dx$

$\left(t = x^2 \quad \frac{dt}{dx} = 2x \quad x\,dx = \frac{dt}{2} \quad \begin{array}{c|c} x & 1\to\infty \\ \hline t & 1\to\infty \end{array} \right)$

$$= \int_1^\infty \frac{1}{t(t+1)}\,\frac{dt}{2} = \frac{1}{2}\int_1^\infty \left(\frac{1}{t} - \frac{1}{t+1}\right)dt$$

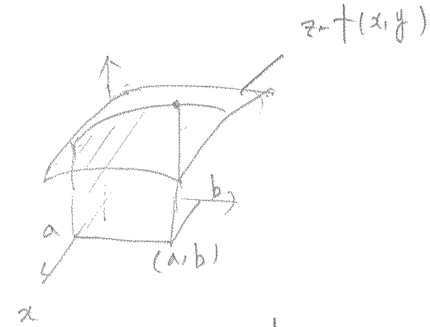
$$= \frac{1}{2}\left[\log\frac{|t|}{|t+1|}\right]_1^\infty = \frac{1}{2}\left(\log 1 - \log\frac{1}{2}\right) = \underline{\frac{1}{2}\log 2}$$

(P.100)

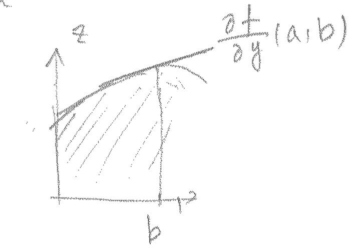
$\boxed{5.2}$ 偏導関数

◇ 偏微分係数

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$



$$f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k)-f(a,b)}{k}$$



◇ 偏導関数 $f_x(x,y)$, $f_y(x,y)$

$$\frac{\partial f}{\partial x}(x,y) , \quad \frac{\partial f}{\partial y}(x,y)$$

例 5.2

$f(x,y) = e^{2x}\sin y$ のとき $(x,y)=(1,\frac{\pi}{2})$ の偏微分係数を求めよ.

$$f_x(x,y) = \frac{\partial}{\partial x}\left(e^{2x}\sin y\right) = 2e^{2x}\sin y$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left(e^{2x}\sin y\right) = e^{2x}\cos y$$

$$f_x(1,\tfrac{\pi}{2}) = 2e^2\sin\tfrac{\pi}{2} = 2e^2$$

$$f_y(1,\tfrac{\pi}{2}) = e^2\cos\tfrac{\pi}{2} = 0$$

問 5.3

(1) $$z = \frac{4x-5y}{2x+3y}$$

$$z_x = \frac{\partial}{\partial x}\,\frac{4x-5y}{2x+3y} = \frac{4(2x+3y)-(4x-5y)2}{(2x+3y)^2} = \frac{22y}{(2x+3y)^2}$$

$$z_y = \frac{\partial}{\partial y}\,\frac{4x-5y}{2x+3y} = \frac{-5(2x+3y)-(4x-5y)3}{(2x+3y)^2} = \frac{-22x}{\text{〃}}$$

(2) $$z = \sin^{-1}\frac{x}{y} \qquad (x>0,\ y>0)$$

$$z_x = \frac{\partial}{\partial x}\sin^{-1}\frac{x}{y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\,\frac{1}{y} = \underline{\frac{1}{\sqrt{y^2-x^2}}}$$ 〃

$$z_y = \frac{\partial}{\partial y}\sin^{-1}\frac{x}{y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\,\frac{x(-1)}{y^2} = \underline{-\frac{x}{y\sqrt{y^2-x^2}}}$$ 〃

(3) $z = x^y \quad (x>0)$

$$z_x = \frac{\partial}{\partial x}x^y = \underline{y\,x^{y-1}}$$ 〃

$$z_y = \frac{\partial}{\partial y}x^y = \frac{\partial}{\partial y}e^{y\log x} = e^{y\log x}\log x = \underline{x^y \log x}$$ 〃

問 12, 5.4, 5.5 omit

p.140. 微分方程式の解法

◇ 微分方程式

独立変数 x と関数 y と $y', y'', \cdots y^{(n)}$ の間の方程式

↑ 方程式の階数 $= n$

y y' .. $y^{(n)}$ について 1次式のもの

線形微分方程式という $y^{(n)} + P_1(x)y^{(n-1)} + P_2(x)y^{(n-2)} + \cdots + P_n(x)y = Q(x)$

$Q(x)=0$ のものを 同次（斉次）方程式

◇ 微分方程式の解

微分方程式をみたす関数 = 解, 解を求めることを「方程式を解く」という

n 階方程式の解で n 個の任意定数を含むもの = 一般解

任意定数に特定の値を代入して得られる解 = 特殊解.

(例) $y = A\cos 2x + B\sin 2x$ ① (A, B は定数)

$y' = -2A\sin 2x + 2B\cos 2x$

$y'' = -4A\cos 2x - 4B\sin 2x = -4(A\cos 2x + B\sin 2x) = -4y$

∴ $y'' = -4y$ ② ← 2階微分方程式

①は一般解

②から ①を求めることを微分方程式を解くという

◇ 1階微分方程式

I　変数分離形

$$\frac{dy}{dx} = \frac{P(x)}{Q(y)} \text{ のもの} \qquad \underbrace{Q(y)\,dy}_{y\text{だけ}} = \underbrace{P(x)\,dx}_{x\text{だけ}}$$

$$\int Q(y)\,dy = \int P(x)\,dx + C$$

II　同次形

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

[解法]　$u = \frac{y}{x}$ とおいて　$\frac{du}{dx}$ の微分方程式に変形

例A.1

(1)　$x^3 \frac{dy}{dx} + y^2 = 0$

$$x^3 \frac{dy}{dx} = -y^2 \rightarrow -\int \frac{dy}{y^2} = \int \frac{1}{x^3}\,dx$$

$$\Rightarrow \quad y^{-1} = \frac{1}{-2}x^{-2} + C = \frac{-1 + 2x^2 C}{2x^2}$$

$$y = \frac{2x^2}{2x^2C - 1}$$

(2)　$2xy\frac{dy}{dx} = x^2 + y^2$　　x^2で割ると　$2\frac{y}{x}\frac{dy}{dx} = 1 + \left(\frac{y}{x}\right)^2$

$u = \frac{y}{x}$ とおくと

$$\frac{du}{dx} = \frac{y'x - y}{x^2} = \frac{\frac{1+u^2}{2u}x - ux}{x^2} = \frac{\frac{1+u^2}{2u} - u}{x} = \frac{\frac{1-u^2}{2u}}{x}$$

$$2u\,y' = 1 + u^2 \rightarrow y' = \frac{1+u^2}{2u} \quad \text{と } y = ux$$

$$\underbrace{\int \frac{2u}{1-u^2}\,du}_{-\int \frac{(u^2-1)'}{u^2-1}du \,=\, -\log|u^2-1|} = \underbrace{\int \frac{1}{x}\,dx}_{\log|x|} \quad \Rightarrow \quad \log|u^2-1| = -\log|x| + \log C'$$

$$|u^2 - 1| = \frac{C}{|x|} \quad (C>0)$$

$$u^2 - 1 = \frac{C}{x}$$

$$\frac{y^2}{x^2} - 1 = \frac{C}{x} \qquad y^2 - x^2 = Cx$$

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VI　定数係数の2階同次線形微分方程式

$$\frac{d^2y}{dx^2} + \underbrace{a\frac{dy}{dx} + b}_{\text{定数}}y = 0 \quad \cdots (*)$$

[解法]　$y = e^{tx}$ の形を仮定すると

$y' = e^{tx}t$, $y'' = e^{tx}t^2$ を

(*)に代入.　$e^{tx}(t^2 + at + b) = 0$

$t^2 + at + b = 0$ … 特性方程式という

⇒ 解を α, β とおくと

(i) $\underbrace{\alpha \neq \beta}_{\text{(実数)}}$ のとき　$y = \underbrace{c_1 e^{\alpha x} + c_2}_{\text{任意定数}} e^{\beta x}$

(ii) $\alpha = \beta$ のとき　$y = c_1 e^{\alpha x} + c_2 x e^{\alpha x}$

(iii) $\alpha = p + qi$, $\beta = p - qi$ のとき
$y = c_1 e^{px} \cos qx + c_2 e^{px} \sin qx$

(13) A-3.

(1) $y'' - 7y' + 12y = 0$

$t^2 - 7t + 12 = (t-4)(t-3) = 0$

$t = 3, 4$ $\quad$ $\underline{y = c_1 e^{3x} + c_2 e^{4x}}$

(2) $y'' + 2y' + 2y = 0$

$t^2 + 2t + 2 = 0$ $\quad$ $t = -1 \pm \sqrt{1^2 - 2} = -1 \pm i$

$\underline{y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x}$

(3) $y'' - 2y' + y = 0$

$t^2 - 2t + 1 = 0$ $\quad$ $t = 1$

$\underline{y = c_1 e^{x} + c_2 x e^{x}}$