まえがき

微分積分学 ～ 18世紀 ニュートンら
19世紀 反省期 (デデキント－ワイエルシュトラス－コーシー)

高校の微積分と大学の〃
高校 ～ 手法の導入
大学 ～ 基礎づけと応用がプラス

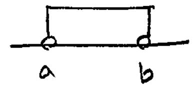
本書の特色
(1) 高校のカリキュラムに接続
(2) 応用力がつく
(3) 基礎的 (変なテクニックはやらない)

学習法
① 書いて学習
② 覚える (基本的な概念)
③ まねる

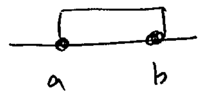
1. 関数，極限と連続，1変数の微分法

1.1. 関数

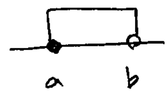
◇ 区間

① $a < x < b$ 

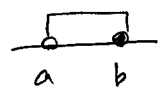
(a, b) 開区間という

② $a \leqq x \leqq b$ 

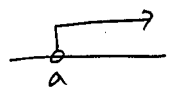
$[a, b]$ 閉区間という

③ $a \leqq x < b$ 

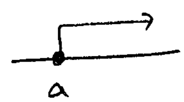
$[a, b)$

④ $a < x \leqq b$ 

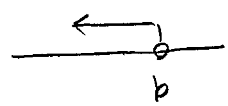
$(a, b]$

⑤ $a < x$ 

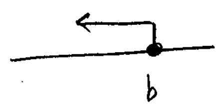
(a, ∞)

⑥ $a \leqq x$ 

$[a, \infty)$

⑦ $x < b$ 

$(-\infty, b)$

⑧ $x \leqq b$ 

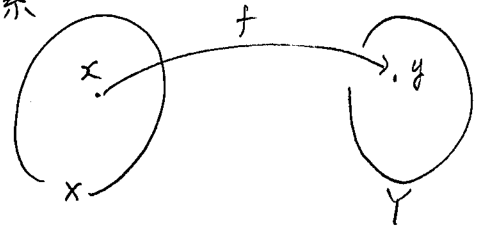
$(-\infty, b]$

⑨ 実数全体 $\mathbb{R} = (-\infty, \infty)$
(real number)

◇ 関数

$x \in X$ ← 実数の集合

要素



Xの要素xに実数yを1つずつ対応させる規則fがあるとき

yをxの関数であるといい

$y = f(x)$ と表す。

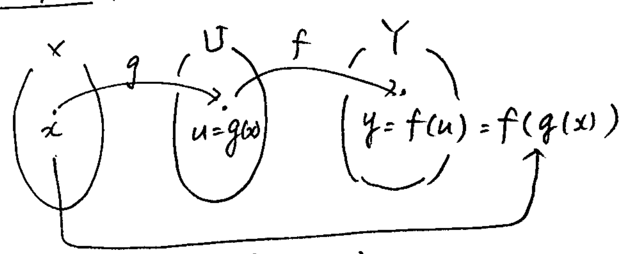
y：従属変数

x：独立変数

X：定義域

$Y = \{ y \,;\, y = f(x),\ x \in X \}$ 値域（3p. 注1.1）

◇ 合成関数



合成関数 $y = f(g(x))$

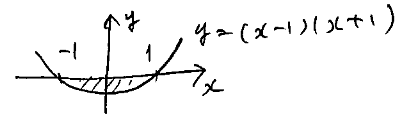
xを1つきめると $f(g(x))$ が1つきまる。

例 1.1.

$y=\sqrt{1-x^2}$ の定義域と値域を求め，区間で示せ.

[解] $\underset{0}{\overset{\sqrt{1-x^2}}{\lor\!\!|\!|}}$ でないと y が虚数となってしまうから，

⇒ $1-x^2 \geqq 0$ 　　 $x^2-1 \leqq 0$. 　　 $(x-1)(x+1) \leqq 0$



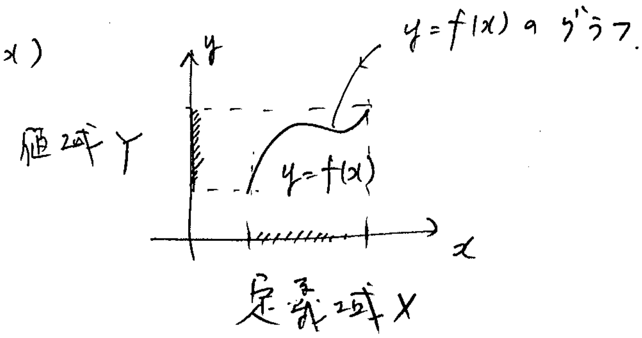
$-1 \leqq x \leqq 1$

したがって定義域は $[-1, 1]$

$x \in [-1, 1]$ のとき $0 \leqq 1-x^2 \leqq 1$ 　 ∴ 　 $0 \leqq \sqrt{1-x^2} \leqq 1$
(よって)

したがって値域は $[0, 1]$ である.

(p.3) 　 注 1.1 　 $y=f(x)$



$Y = \{ y \,;\, y=f(x),\ x \in X \}$

$y=f(x)$ のグラフ $= \{ (x, y) \,;\, y=f(x),\ x \in X \}$

例1.2

$y = f(u) = u^2$ と $u = g(x) = 2x+1$ の合成関数 $y = f(g(x))$ は?

$y = g(u) = 2u+1$ と $u = f(x) = x^2$ 〃 $y = g(f(x))$ は?

(解)

$$y = \underset{\substack{\uparrow \\ 2x+1}}{u}{}^2 = (2x+1)^2$$

$$y = 2\underset{\substack{\uparrow \\ x^2}}{u}+1 = 2x^2+1$$

注1.2　$f(g(x))$ と $g(f(x))$ は一般に一致しない!

問1.1　$f(u) = \sqrt{4-u^2}$, $u = \dfrac{x}{x-1}$ の合成関数の定義域は?

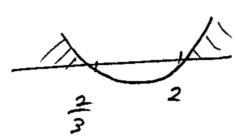
$$y = f(u) = \sqrt{4-\underset{\substack{\uparrow \\ \frac{x}{x-1}}}{u}{}^2} = \sqrt{4-\left(\frac{x}{x-1}\right)^2} = \sqrt{\frac{4(x-1)^2-x^2}{(x-1)^2}}$$

← ゼロで割れないから

$x \neq 1$ …(*)

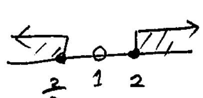
$$= \sqrt{\frac{(2x-2-x)(2x-2+x)}{(x-1)^2}} = \sqrt{\frac{(x-2)(3x-2)}{(x-1)^2}}$$

$\sqrt{\ \cdots\ }$ 内側はゼロ以上　$3(x-2)\left(x-\frac{2}{3}\right) \geq 0$



$x \leq \frac{2}{3}$, $2 \leq x$ …(**)

(*) (**) より　$x \leq \frac{2}{3}$, $2 \leq x$



第1回レポートで 済

問 1.2.

$f(x) = (x-1)(x+2)$ のとき

$$f(a-3) = (a-3-1)(a-3+2) = \underline{(a-4)(a-\cancel{?}1)}$$

$$f(\boxed{f(a)}) = f\big((a-1)(a+2)\big) = \{\underbrace{(a-1)(a+2)}_{\cancel{a^2-3a+2}\ \ a^2+a-2} - 1\}\{\underbrace{(a-1)(a-2)}_{\cancel{a^2-3a+2}\ \ a^2+a-2} + 2\}$$

$f(a) = (a-1)(a-2)$

$$= \cancel{(a^2-3a-3)(a^2-3a)}$$

$$= \underline{(a^2+a-3)(a^2+a)}$$

omit.

$e^{\square}$ の後で

問 1.3. $f(x) = e^{x} + e^{-x}$ のとき.

$f(x+y)\cdot f(x-y) = f(2x) + f(2y)$ が成立することを示せ.

$$\text{左辺} = f(x+y)\cdot f(x-y)$$

$$= \left(e^{x+y} + e^{-x-y}\right)\left(e^{x-y} + e^{-x+y}\right)$$

$$= \underbrace{e^{x+y}e^{x-y}}_{e^{2x}} + \underbrace{e^{x+y}e^{-x+y}}_{e^{2y}} + \underbrace{e^{-x-y}e^{x-y}}_{e^{-2y}} + \underbrace{e^{-x-y}e^{-x+y}}_{e^{-2x}}$$

$$= \underbrace{e^{2x} + e^{-2x}}_{f(2x)} + \underbrace{e^{2y} + e^{-2y}}_{f(2y)} = \text{右辺}$$

1.2 逆関数

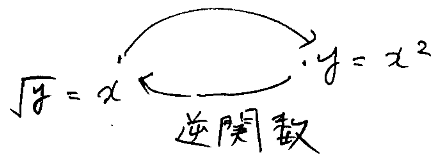
◇ 逆関数

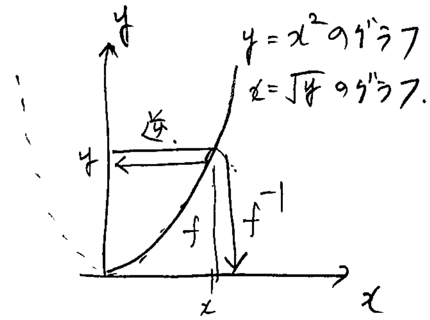
(例)　$x \geqq 0$ を定義域として $y = x^2$ ①を考える

①を x について解くと　$x^2 = y \Rightarrow x = \pm\sqrt{y} = \sqrt{y}$ …②

(つまり y が1つきまると x が1つきまる)　　($x \geqq 0$ より)

② は x が y の関数であることを示している。





② を ① の逆関数という

一般に　$y = f(x)$ を x について解き、ただ1つの解 $x = g(y)$ が得られたとする。このとき

$$x = g(y) = f^{-1}(y) \quad \cdots\cdots ③$$

と書き

f の逆関数という

x を独立変数として、y を従属変数として f^{-1} を表せば

$$y = g(x) = f^{-1}(x)$$

これを $y = f(x)$ の逆関数ということが多い

◇ 逆関数の性質

(1) $y=f(x)$ のグラフと $x=f^{-1}(y)$ のグラフは一致

略

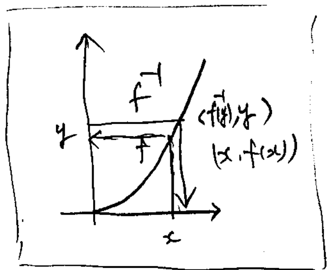
なぜなら

$$\{(x,y)\,;\ y=f(x),\ x\in X,\ y\in Y\}$$

$$\Updownarrow$$

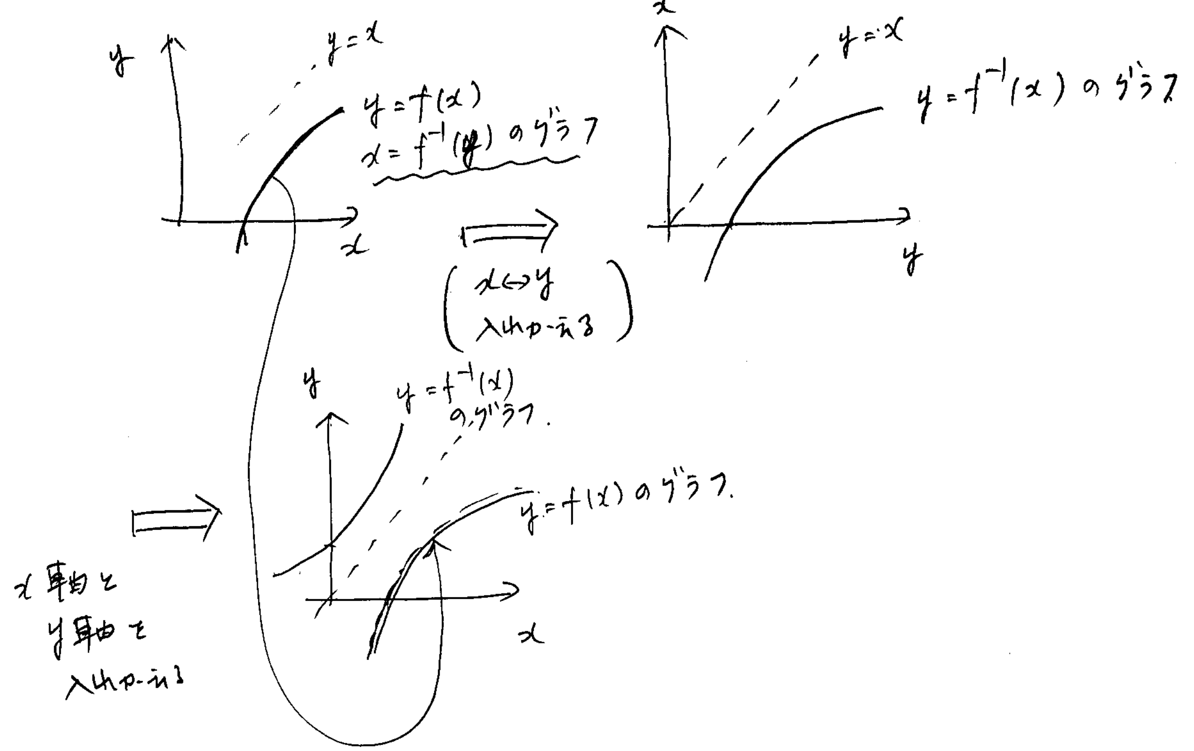
$$x=f^{-1}(y)$$

$$=\{(x,y)\,;\ x=f^{-1}(y),\ y\in Y,\ x\in X\}$$



から当然、

(2) $y=f(x)$ の逆関数 $y=f^{-1}(x)$ のグラフは $y=x$ 対称



(3) いろんな逆関数あるよ.

(4) 略

p.22

例 1.3

$y=1+\sqrt{x+1}$ の逆関数は？（定義域と値域も求めよ。）

[解] $\sqrt{x+1}$ → 0以上 $x \geq -1$ …

$\sqrt{x+1} \geq 0$ より $y=1+\sqrt{\ } \geq 1$.

したがって、$y=f(x)=1+\sqrt{x+1}$ の定義域は $x \geq -1$

値域は $y \geq 1$.

逆関数 f^{-1} は、x について解くと.

$$y-1=\sqrt{x+1} \to (y-1)^2=x+1 \qquad x=(y-1)^2-1$$
$$=y^2-2y+1-1$$
$$=y^2-2y=f^{-1}(y)$$

f^{-1} の 定義域 $y \geq 1$ 値域は $x \geq -1$

独立変数を x にすると、$y=f^{-1}(x)=x^2-2x$

定義域は $x \geq 1$ $[1,\infty)$

値域は $y \geq -1$. $[-1,\infty)$

グラフの描き方 $y=1+\sqrt{x+1}$ のグラフは、

$y=\sqrt{x}$ のグラフを x方向に -1, y方向に 1 平行移動したもの

$y=x^2-2x=(x-1)^2-1$

↑ 平方完成

$y=x^2$ のグラフを x方向に 1, y方向に -1

平行移動

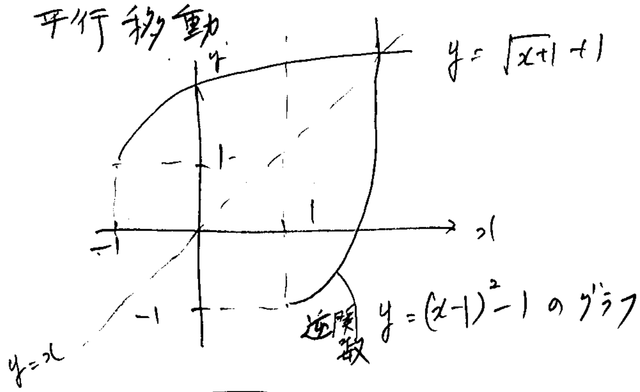


図 1.8

メモ $y=f(x)$ のグラフを x方向に α, y 〃 に β 平行移動したグラフの方程式は

$y-\beta=f(x-\alpha)$

or

$y=f(x-\alpha)+\beta$

メモ 平方完成

x^2+ax+b

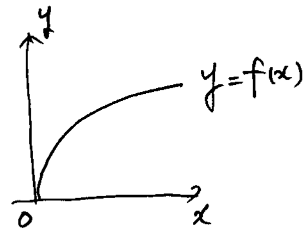
$\left(x+\frac{a}{2}\right)^2=x^2+ax+\left(\frac{a}{2}\right)^2$

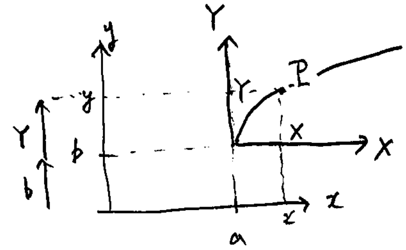
より

x^2+ax は $\left(x+\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2$ におきかえると

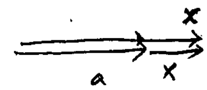
→ $\left(x+\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2+b$

$\boxed{XE}$





$Y = f(X) \Rightarrow y - b = f(x-a)$



$a + X = x$

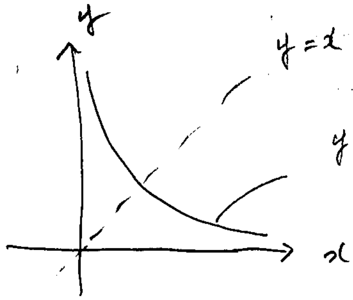
$\Rightarrow X = x - a$

$b + Y = y \Rightarrow Y = y - b$

omit

例1.4 $y=f(x)$ と $y=f^{-1}(x)$ が同じ関数となる f はどんな関数か?

[解]



$y=x$ に対して対称なグラフは $y=f(x)$ と $y=f^{-1}(x)$ が同じグラフ.

$y=x$ に対して対称なグラフは

$$g(x,y) = g(y,x)$$ ← (x と y の入れかえに対して形の変わらない式 $g(x,y)$)

を用いて $g(x,y)=0$ と定められる。

(例1) $\underbrace{x+y-2}=0$

$g(x,y)=x+y-2$ のとき

実際、

$y=-x+2=f(x)$

$x=-y+2=f^{-1}(y) \longrightarrow y=f^{-1}(x)=-x+2$

一致!

(例2) $\underbrace{xy-x-y}=0.$

$g(x,y)=xy-x-y$

$(x-1)y=x \rightarrow y=\dfrac{x}{x-1}=f(x)$

$(y-1)x=y \rightarrow x=\dfrac{y}{y-1}=f^{-1}(y) \Rightarrow y=f^{-1}(x)=\dfrac{x}{x-1}$

一致!

問 1-4　　f^{-1} と定義域と値域は？

(1)　$y = 4x - 2$　　$x \in \mathbb{R}$
$= f(x)$　　$y \in \mathbb{R}$

$4x = y + 2$ よって $x = \dfrac{y+2}{4} = f^{-1}(y)$　　$y \in \mathbb{R}$　$x \in \mathbb{R}$

$y = f^{-1}(x) = \dfrac{x+2}{4}$　　$\underline{x \in \mathbb{R}}$ 定義域　$\underline{y \in \mathbb{R}}$ 値域

(2)　$y = \dfrac{1}{\sqrt{x}}$　　$x > 0$
$= f(x)$　　$y > 0$

$\sqrt{x} = \dfrac{1}{y}$ より $x = \dfrac{1}{y^2} = f^{-1}(y)$　　$y > 0$　$x > 0$

$y = f^{-1}(x) = \dfrac{1}{x^2}$　　$\underline{x > 0}$ 定義域　$\underline{y > 0}$ 値域

(3)　$y = x^2 + 2x - 4$　$(x \geqq -1)$
$= (x+1)^2 - 1 - 4$
$= (x+1)^2 - 5$　　$(y \geqq -5)$
$= f(x)$

$x^2 + 2x - y - 4 = 0$.　　$x = -1 \pm \sqrt{1^2 - (-y-4)}$　　$y \geqq -5$
$= -1 + \sqrt{y+5}$　　$x \geqq -1$
$= f^{-1}(y)$

$y = f^{-1}(x) = -1 + \sqrt{x+5}$　　$\underline{x \geqq -5}$ 定義域　$\underline{y \geqq -1}$ 値域

1.3 有理関数

◇有理整関数（n次関数）

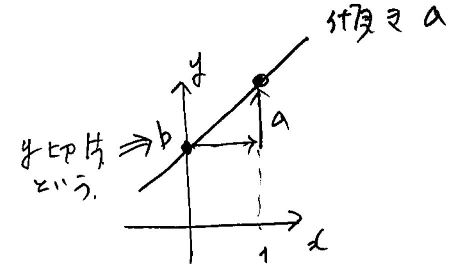
$$y = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

a_0 ↑ ゼロでない

（例） 1次関数

$$y = ax + b$$

$(a \neq 0)$



（例） 2次関数

$$y = ax^2 + bx + c$$

$(a \neq 0)$

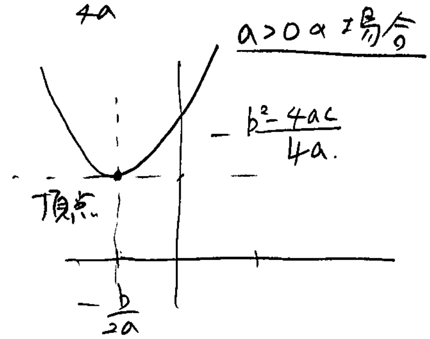
$$= a\left(x^2 + \frac{b}{a}x\right) + c$$

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$$

$y = ax^2$ のグラフを $\left\{ \begin{array}{ll} x\text{方向に} & -\frac{b}{2a} \\ y \quad \text{〃} & -\frac{b^2-4ac}{4a} \end{array} \right\}$ 平行移動



頂点 $\left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}\right)$

軸 $x = -\frac{b}{2a}$

の

放物線

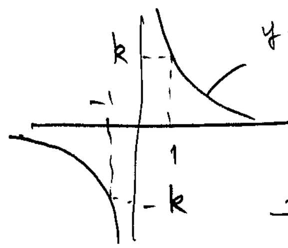
◇ 有理関数

$$y = \frac{g(x)}{h(x)} = \frac{a_0 x^n + a_1 x^{n-1} + \cdots + a_n}{b_0 x^m + b_1 x^{m-1} + \cdots + b_m} \qquad \begin{matrix} a_0 \neq 0 \\ b_0 \neq 0 \end{matrix}$$

(例) $h(x) = 1$ のとき $y = g(x)$ となり 有理整関数

(例) $y = \dfrac{ax+b}{cx+d}$ $\quad (c \neq 0 \ , \ ad - bc \neq 0)$

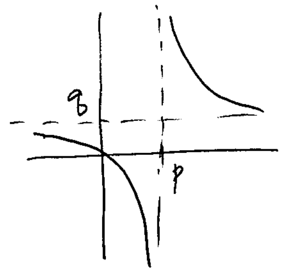
特に $y = \dfrac{k}{x}$ $(k \neq 0)$ は.



$y = \frac{k}{x}$ のグラフ

直角双曲線

$y = \dfrac{k}{x-p} + q$ は.

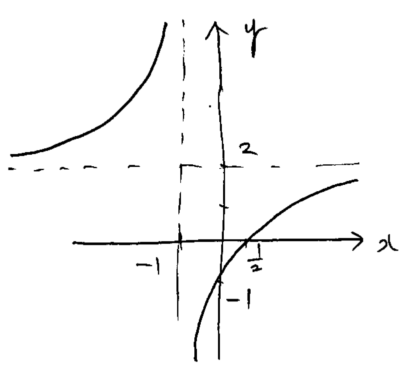


例1.5 $\quad y = \dfrac{2x-1}{x+1}$ のグラフは?

$$\begin{array}{r} 2 \\ x+1 \overline{\smash{\big)}\, 2x-1} \\ \underline{2x+2} \\ -3 \end{array}$$

$y = 2 + \dfrac{-3}{x+1}$

$y = \dfrac{-3}{x}$ のグラフを $\begin{cases} x\text{方向} & -1 \\ y \quad 〃 & 2 \end{cases}$ だけ平行移動したもの

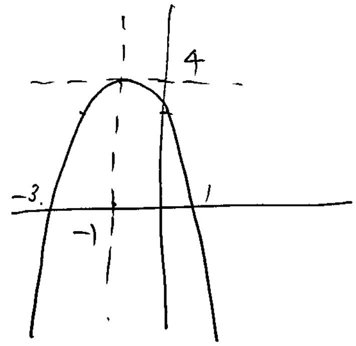


例1.6 $\quad y=-x^2-2x+3$ のグラフ

$$y=-(\underbrace{x^2+2x}_{(x+1)^2-1^2})+3$$

$$=-(x+1)^2+1+3$$

$$=-(x+1)^2+4$$



凸 $\quad$ 上に凸(トツ)という

比較 $\quad$ 下に凸という

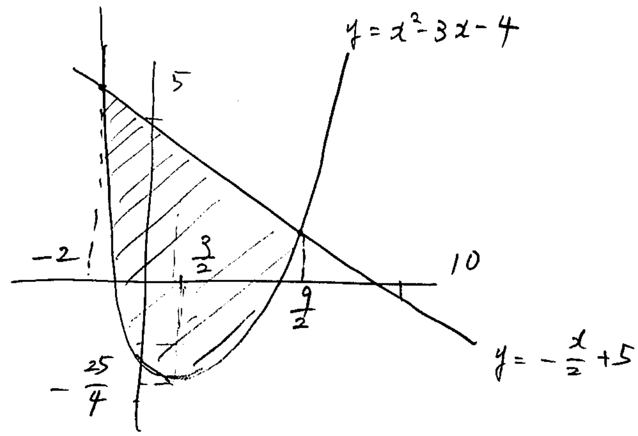
問1.5 $\quad x^2-3x-4 \le y \le -\frac{x}{2}+5$ をみたす (x,y) の範囲は?

$$y=x^2-3x-4$$

$$=\left(x-\frac{3}{2}\right)^2-\frac{9}{4}-4$$

$$=\left(x-\frac{3}{2}\right)^2+\underbrace{\boxed{\frac{-16-9}{4}}}_{=\frac{-25}{4}}$$

$$y=-\frac{x}{2}+5$$



交点は $\quad x^2-3x-4=-\frac{x}{2}+5$

$$2x^2-6x-8+x-10$$

$$=2x^2-5x-18=0 \qquad 3.6$$

$$(2x-9)(x+2)=0 \qquad 2.9$$

$$x=-2,\ \frac{9}{2}$$

問 1.6.

$$\begin{cases} f(-x) = f(x) & \cdots\ f(x) は偶関数 \\ f(-x) = -f(x) & \cdots\ f(x) は奇関数 \end{cases}$$ という。

(1) $f(x) = x^2 + 2$

$f(-x) = (-x)^2 + 2 = x^2 + 2 = f(x)$ … 偶

(2) $f(x) = x^3 - 3x$

$f(-x) = (-x)^3 - 3(-x) = -x^3 + 3x = -(x^3 - 3x) = -f(x)$ … 奇

(3) $f(x) = x^2 + x + 1$

$f(-x) = x^2 - x + 1$ … どちらでもない

(4) $f(x) = \dfrac{x-1}{x+1}$

$f(-x) = \dfrac{-x-1}{-x+1} = \dfrac{x+1}{x-1}$ どちらでもない

(5) $f(x) = x + \dfrac{1}{x}$

$f(-x) = -x - \dfrac{1}{x} = -\left(x + \dfrac{1}{x}\right) = -f(x)$ … 奇

(6) $f(x) = \dfrac{1}{x+1} - \dfrac{1}{x-1}$

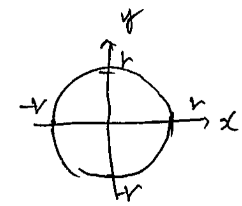
$f(-x) = \dfrac{1}{-x+1} - \dfrac{1}{-x-1} = \dfrac{1}{x+1} - \dfrac{1}{x-1} = f(x)$ 偶

$\boxed{1.4}$ 2次曲線(円, 楕円, 双曲線), 無理関数のグラフ

◇ **2次曲線**　$\underbrace{f(x,y)}_{x,\,y\text{の2次多項式}}=0$　で表される曲線

◇ **円**

$$\underbrace{x^2+y^2}_{(\text{原点からのキョリ})^2}=r^2 \quad \Rightarrow$$

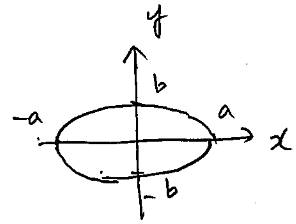


◇ **だ円**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \Rightarrow$$

$x=0$ のとき　$y=\pm b$

$y=0$ のとき　$x=\pm a$



◇ **双曲線**　$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$　または　$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=-1$

$y=0$ のとき　$x=\pm a$　|　$x=0$ のとき　$y=\pm b$

$|x|, |y| \gg 1$ となると

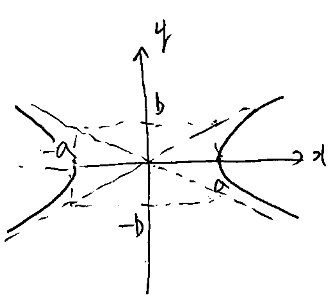
$$\frac{1}{a^2}-\frac{1}{b^2}\cdot\frac{y^2}{x^2}=\frac{1}{x^2}\sim 0 \quad \Big| \quad \frac{1\cancel{b}}{a^2}x^2-\frac{1y^2}{b^2x^2}=\frac{-1}{x^2}\sim 0$$

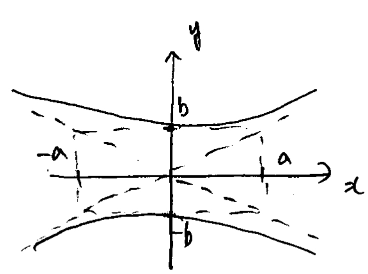
☺ $|x|\gg 1$ のとき　$\dfrac{y^2}{b^2x^2}-\dfrac{1}{a^2}=0$ に近づく。$\Leftrightarrow$ $y^2-\dfrac{b^2}{a^2}x^2=0$

因数分解

$$\left(y-\frac{b}{a}x\right)\left(y+\frac{b}{a}x\right)=0$$

$y=\pm\dfrac{b}{a}x$ に近づく ← 漸近線(ぜんきん)という





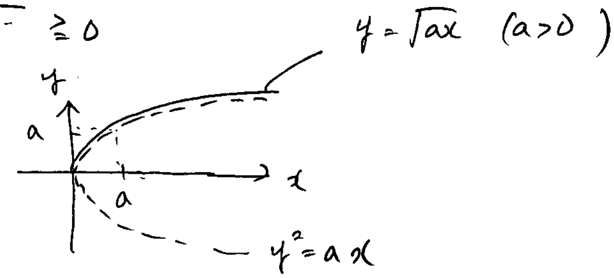
◁ 無理関数 $y=\sqrt{ax+b}$ のグラフ

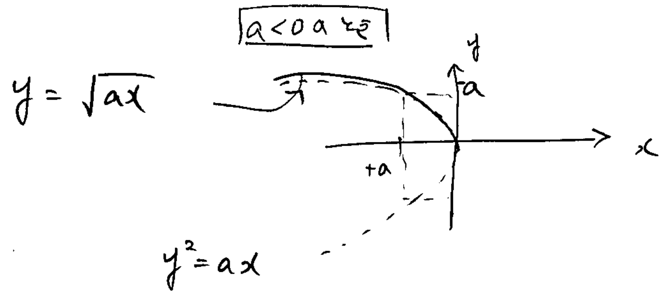
$y=\sqrt{a(x+\frac{b}{a})}$ より $y=\sqrt{ax}$ のグラフを x 方向に $-\frac{b}{a}$ 平行移動

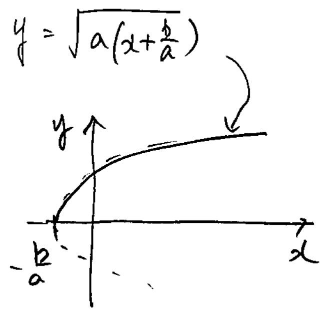
$y=\sqrt{ax}$ の定義域 $\begin{cases} a>0 \text{ のとき } x \geqq 0 \\ a<0 \text{ のとき } x \leqq 0 \end{cases}$

値域 $y=\sqrt{\cdots} \geqq 0$

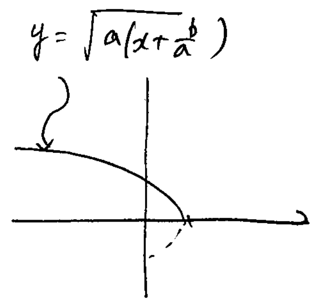
$y^2=(\sqrt{ax})^2=ax$ より







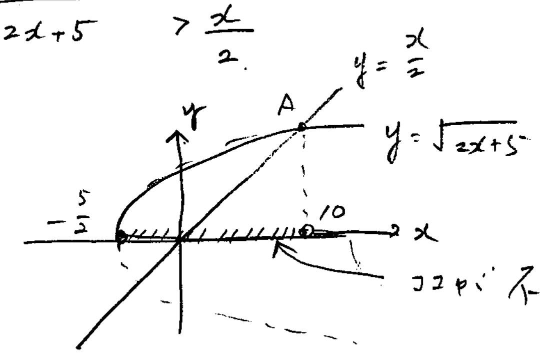
$a>0$



$a<0$

例 1.7

$$\sqrt{2x+5} > \frac{x}{2}$$



交点 A を求める

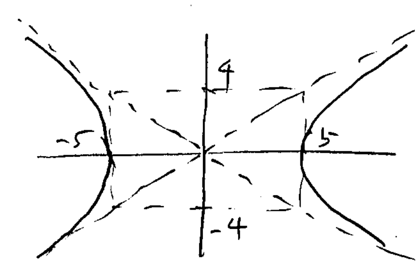
$$\sqrt{2x+5} = \frac{x}{2} \qquad 2x+5 = \frac{x^2}{4} \qquad \underbrace{x^2-8x-20}_{(x-10)(x+2)} = 0$$

$x = 10, \cancel{-2}$ $\times$　　図より $-\frac{5}{2} \leqq x < 10$

例 1.8

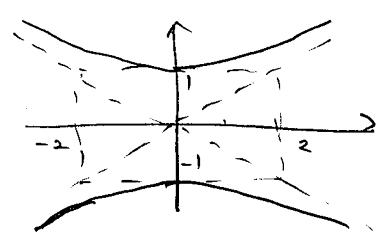
(1) $\frac{x^2}{25} - \frac{y^2}{16} = 1$

$y=0$ のとき $x=\pm 5$



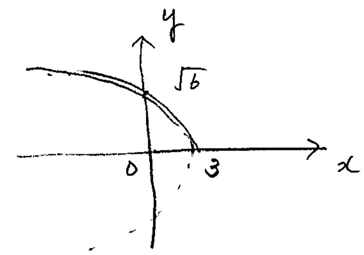
(2) $\frac{x^2}{4} - y^2 = -1$

$x=0$ のとき $y=\pm 1$



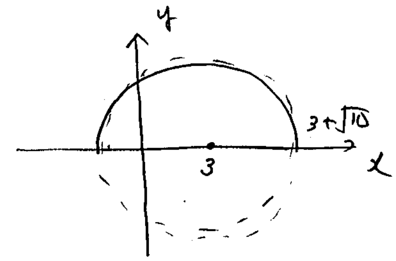
問1-7.

(1) $y=\sqrt{6-2x}=\sqrt{-2(x-3)}$



(2) $y=\sqrt{-x^2+6x+1}$

$=\sqrt{-(x-3)^2+10}$



$y \geqq 0$. $\quad y^2=-(x-3)^2+10 \quad (x-3)^2+y^2=10$

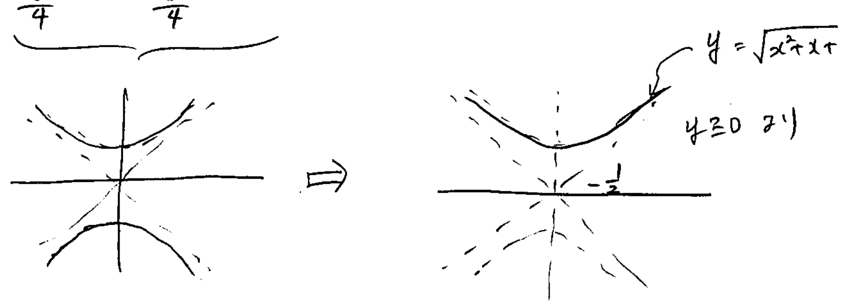
(3) $y=\sqrt{x^2+x+1}$

$=\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$

$y \geqq 0$ より $\quad y^2=\left(x+\frac{1}{2}\right)^2+\frac{3}{4} \quad \left(x+\frac{1}{2}\right)^2-y^2=-\frac{3}{4}$

$$\frac{\left(x+\frac{1}{2}\right)^2}{\frac{3}{4}}-\frac{y^2}{\frac{3}{4}}=-1.$$

双曲線 $\underbrace{\frac{x^2}{\frac{3}{4}}-\frac{y^2}{\frac{3}{4}}=-1}$ を x 方向に $-\frac{1}{2}$ 平行移動



(4) $y = 2\sqrt{(x-1)^2-1}$

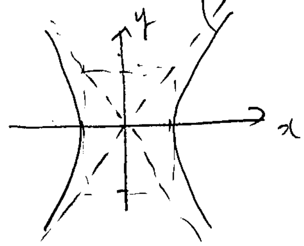
$y \geqq 0$ $\quad \frac{y^2}{4} = \cancel{4}\{(x-1)^2-1\}$

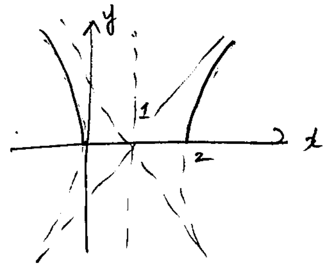
$$(x-1)^2 - \frac{y^2}{4} = 1.$$

は $x^2 - \frac{y^2}{4} = -1$ のグラフを

x方向に1平行移動.

$x^2 - \frac{y^2}{4} = -1$ のグラフ



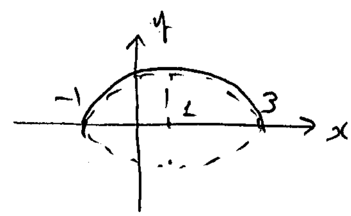


(5) $2y = \sqrt{3+2x-x^2}$

$y \geqq 0$ より $\quad 4y^2 = 3+2x-x^2$

$$= 4-(x-1)^2$$

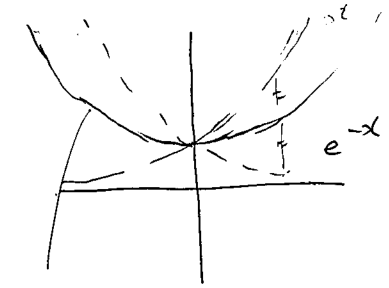
$(x-1)^2 + 4y^2 = 4 \Rightarrow \frac{(x-1)^2}{4} + y^2 = 1$



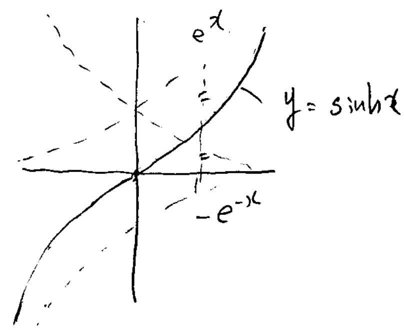
◇ 双曲線関数

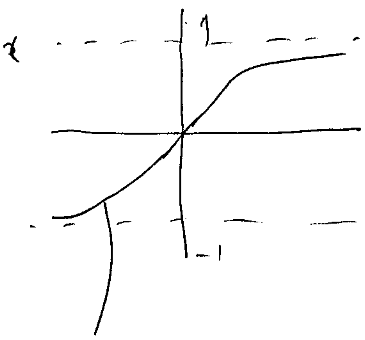
$$\cosh x = \frac{e^{x}+e^{-x}}{2}, \quad \sinh x = \frac{e^{x}-e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

(ハイパボリック コサイン)　　(〃 サイン)　　(〃 タンジェント)



$y = \cosh x$





$y = \tanh x$

例 1.9.

$a^{x} = e^{x \log a}$ を示せ.

$a = e^{\log a}$ より $a^{x} = (e^{\log a})^{x} = e^{x \log a}$ //

例 1.10

$\log_{10}$ を用いて $\sqrt{2}$ と $\sqrt[3]{3}$ の大小

$$\log_{10}\sqrt{2} - \log_{10}\sqrt[3]{3} = \frac{1}{2}\log_{10}2 - \frac{1}{3}\log_{10}3$$

$$= \frac{1}{6}\left(\log_{10}2^{3} - \log_{10}3^{2}\right) = \frac{1}{6}\underbrace{\boxed{\log_{10}\frac{8}{9}}}_{\text{マイナス}} < 0$$

よって $\log_{10}\sqrt{2} < \log_{10}\sqrt[3]{3}$

$$\underline{\sqrt{2} < \sqrt[3]{3}}.$$

問 1-8.

$8^n > 10^{100}$ となる最小の n は？

$$\log_{10} \underset{\underset{2^{3n}}{\shortparallel}}{\boxed{8^n}} > \log_{10} 10^{100}$$

$$3n \underbrace{\log_{10} 2}_{0.3010} > 100$$

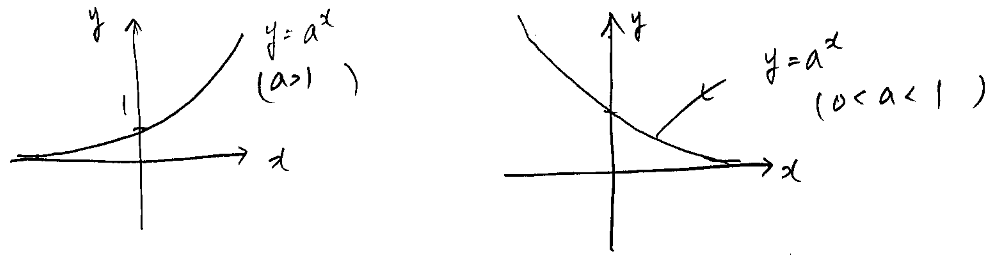
$$n > \frac{100}{3 \times 0.3010} = 110.7\cdots$$

$\underline{n = 111}$ 回

1.5 指数関数，対数関数

◇ 指数関数　　$a>0,\ a\neq 1$ のとき

$$y = a^x$$ …… a を底とする 指数関数 という



◇ 指数法則

$$a^m a^n = \underbrace{\underbrace{a\cdots a}_{m\text{個}}\ \underbrace{a\cdots a}_{n\text{個}}}_{m+n\text{個}} = a^{m+n}$$

$$\frac{a^m}{a^n} = \frac{\cancel{a}\cdots\cancel{a}\,a\cdots a}{\underbrace{\cancel{a}\cdots\cancel{a}}_{n\text{個}}} = a^{m-n}$$

$$a^0 = \frac{a^m}{a^m} = 1.$$

$$(a^m)^n = \underbrace{a^m \cdots a^m}_{a\text{が }m\times n\text{ 個}} = a^{mn}$$

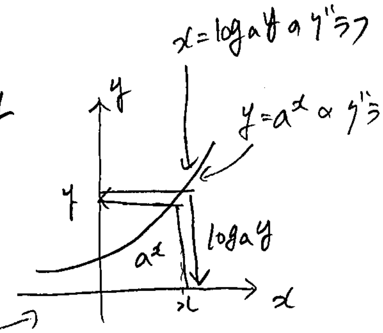
$$(ab)^m = abab\cdots ab = a^m b^m$$

m, n が実数でも成立する！

$$a^{m/n} = \sqrt[n]{a^m}$$，特に $a^{\frac{1}{n}} = \sqrt[n]{a}$

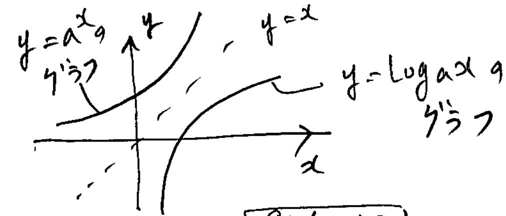
◇ 対数関数

$y = a^x$ の逆関数を $x = \log_a y$ と書く。

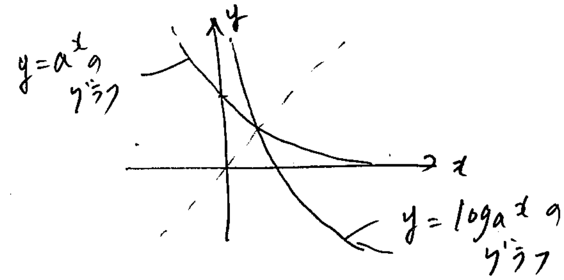


$y = \log_a x$ の $\begin{cases} \text{定義域は } x > 0 \\ \text{値域は } (-\infty, \infty) \end{cases}$

$$\boxed{y \xrightarrow{x=} \log_a y \to y = a^x \qquad y = a^{\log_a y}.}$$



$\boxed{a > 1 \text{ のとき}}$



$\boxed{0 < a < 1 \text{ のとき}}$

◇ 対数の性質：指数法則から出てくる

やってもらう

∘ $M = a^{\log_a M}$, $N = a^{\log_a N}$ より $MN = a^{\log_a M + \log_a N}$

また $MN = a^{\log_a (MN)}$ よって $\underline{\log_a MN = \log_a M + \log_a N}$

∘ $1 = a^{\log_a 1}$, $1 = a^0$ よって $\underline{\log_a 1 = 0}$

∘ $a = a^{\log_a a}$, $a = a^1$ よって $\underline{\log_a a = 1}$

∘ $M^r = a^{\log_a M^r}$ 一方 $M^r = (a^{\log_a M})^r = a^{r \log_a M}$

よって $\underline{\log_a M^r = r \log_a M}$

○ $\frac{M}{N} = a^{\log_a \frac{M}{N}}$ 　一方 $\frac{M}{N} = \frac{a^{\log_a M}}{a^{\log_a N}} = a^{\log_a M - \log_a N}$

よって $\log_a \frac{M}{N} = \log_a M - \log_a N$

○ $b = a^{\log_a b} = \left(c^{\log_c a}\right)^{\log_a b} = c^{\log_a b \cdot \log_c a}$

一方 $b = c^{\log_c b}$

よって $\log_a b \log_c a = \log_c b$

$\Rightarrow$ $\underline{\log_a b = \frac{\log_c b}{\log_c a}}$ 　底の変換

◇ 自然対数

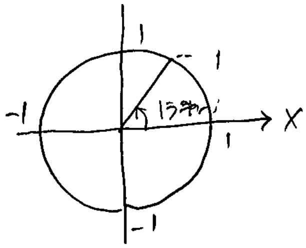
$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.718\cdots$ 　ネイピア数

$\log_e x = \log x$ 　← 自然対数

◇ $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ 増加関数
$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ 減少〃
} 単調関数

1.6 三角関数

◇ 弧度法

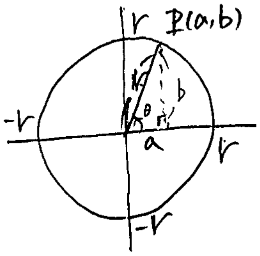


1周 360° の円周の長さは 2π

⇒ 360° = 2π [ラジアン]

180° = π ラジアン

◇ 三角関数

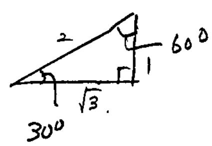


$\cos\theta = a/r$

$\sin\theta = b/r$

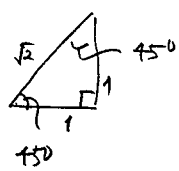
$\tan\theta = \dfrac{b}{a}$

・$\theta = \dfrac{\pi}{6}$ のとき



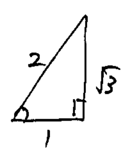
$\sin\dfrac{\pi}{6} = \dfrac{1}{2}$　$\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$, $\tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}$

・$\theta = \dfrac{\pi}{4}$ のとき



$\sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$, $\cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$, $\tan\dfrac{\pi}{4} = 1$

・$\theta = \dfrac{\pi}{3}$ のとき

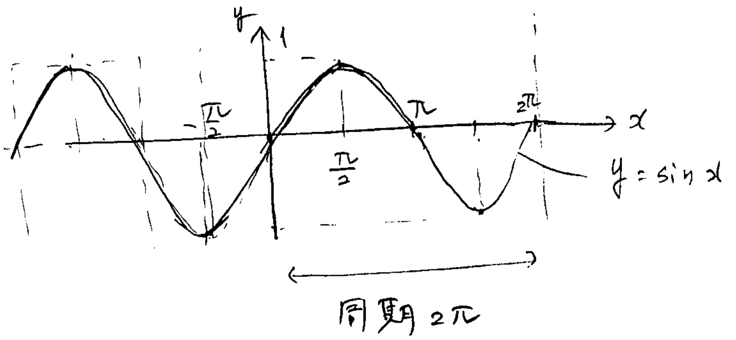


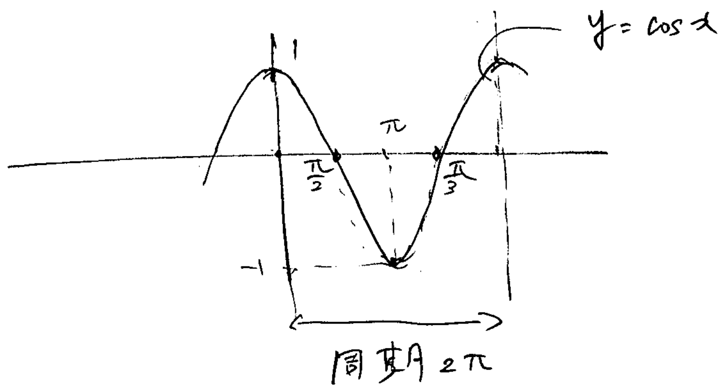
$\sin\dfrac{\pi}{3} = \dfrac{1}{2}$, $\cos\dfrac{\pi}{3} = \dfrac{1}{2}$, $\tan\dfrac{\pi}{3} = \sqrt{3}$

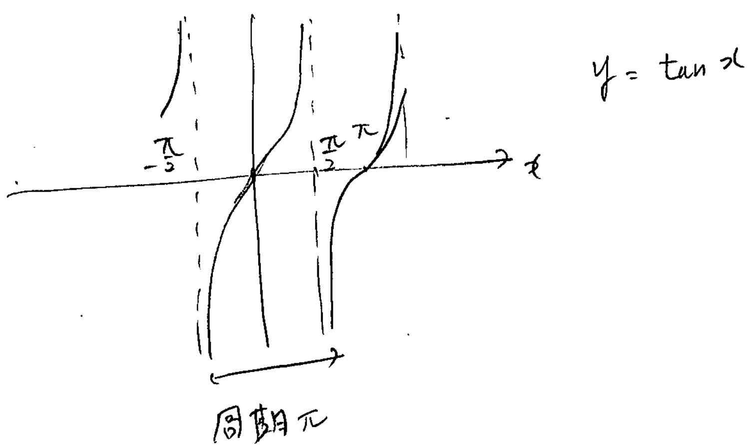
◇ 三角関数の逆数

$\mathrm{cosec}\,\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, $\cot\theta = \dfrac{1}{\tan\theta}$

◇ 三角関数のグラフ







◇ 周期 p：　$f(x+p)=f(x)$ が成立する $p(\neq 0)$ のこと

任意の x で

◇ 基本公式

1. 相互関係 $\tan x = \dfrac{\sin x}{\cos x}$, $\sin^2 x + \cos^2 x = 1$

※ $\sin^2 x = (\sin x)^2$ の略記

両辺を $\cos^2 x$ でわると

$$\tan^2 x + 1 = \frac{1}{\cos^2 x} = \sec^2 x$$

2. 偶奇性

グラフより

$$\begin{cases} \sin(-x) = -\sin x & 奇 \\ \cos(-x) = \cos x & 偶 \\ \tan(-x) = -\tan x & 奇 \end{cases}$$

3. 周期性

$$\sin(x + 2\pi n) = \sin x$$

$$\cos(x + 2\pi n) = \cos x$$

$$\tan(x + \pi n) = \tan x$$

4. 加法定理

(1)

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

(2)

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

5. 倍角の公式

$$\sin 2x = 2\sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 y = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

6. 半角の公式

$$\cos x = 2\cos^2 \frac{x}{2} - 1 \Rightarrow \underline{\cos^2 \frac{x}{2} = \frac{1+\cos x}{2}} \;//$$

$$\cos x = 1 - 2\sin^2 \frac{x}{2} \Rightarrow \underline{\sin^2 \frac{x}{2} = \frac{1-\cos x}{2}} \;//$$

7 和・差を積に

(6) $$\sin(x+y) = \sin x\cos y + \cos x \sin y$$

+)

$$\sin(\underbrace{\boxed{x+y}}_{\alpha}) + \sin(\underbrace{\boxed{x-y}}_{\beta}) = 2\sin x\cos y$$

$$x = \frac{\alpha+\beta}{2}$$
$$y = \frac{\alpha-\beta}{2}$$

$$\Rightarrow \quad \sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

(7) $$\sin(x+y) - \sin(\alpha-\beta) = 2\cos x\sin y$$

$$\Rightarrow \quad \sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

(8) $$\cos(x+y) = \cos x\cos y - \sin x\sin y$$

$$\cos(x+y) + \cos(x-y) = 2\cos x\cos y$$

$$\Rightarrow \quad \cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

(9) $$\cos(x+y) - \cos(x-y) = -2\sin x\sin y$$

$$\Rightarrow \quad \cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

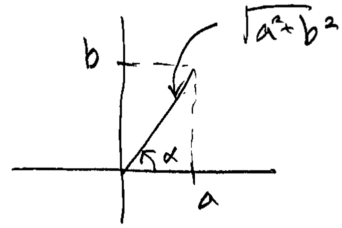
8 積和(差)公式

(10) $$\sin A\cos B = \frac{1}{2}\left\{\sin(A+B) + \sin(A-B)\right\}$$

(11) $$\cos A\cos B = \frac{1}{2}\left\{\cos(A+B) + \cos(A-B)\right\}$$

(12) $$\sin A\sin B = \frac{-1}{2}\left\{\cos(A+B) - \cos(A-B)\right\}$$

9. 三角関数の合成



$a = \sqrt{a^2+b^2}\cos\alpha$

$b = \sqrt{a^2+b^2}\sin\alpha$

$$a\sin x + b\cos x$$

$$= \sqrt{a^2+b^2}\{\underbrace{\sin x\cos\alpha + \cos x\sin\alpha}_{\sin(x+\alpha)}\}$$

$$= \underline{\sqrt{a^2+b^2}\sin(x+\alpha)}_{\,\text{〃}}$$

問1.9 済

問1.10 $\tan\frac{x}{2} = t$ とおくとき，$\sin x$，$\cos x$ を t で表せ

$$\sin\left(\frac{x}{2}\cdot 2\right) = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\boxed{\overset{t}{\tan\frac{x}{2}}}\cos^2\frac{x}{2} = \underline{\frac{2t}{t^2+1}} \text{ 答)}$$

$$\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2} = 1. \qquad \underbrace{\tan^2\frac{x}{2}}_{t^2} + 1 = \frac{1}{\cos^2\frac{x}{2}} \Rightarrow \cos^2\frac{x}{2} = \frac{1}{t^2+1}\right)$$

$$\cos\left(2\cdot\frac{x}{2}\right) = 2\boxed{\cos^2\frac{x}{2}} - 1 = 2\frac{1}{t^2+1} - 1 = \frac{2-t^2-1}{t^2+1} = \underline{\frac{1-t^2}{1+t^2}} \text{ 答)}$$

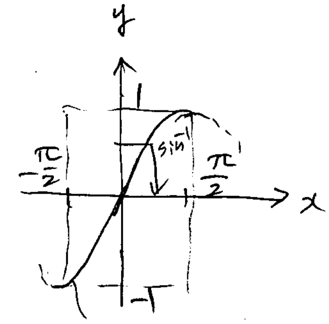
1.7 逆三角関数

◇ $y=\sin x$ $\left(-\frac{\pi}{2}\leqq x\leqq\frac{\pi}{2},\ -1\leqq y\leqq 1\right)$

の逆関数は、

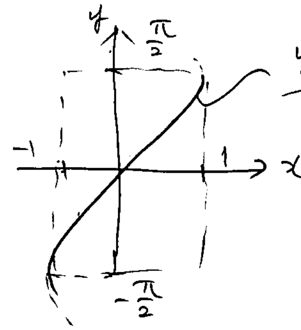
$x=\sin^{-1}y$ $\left(-1\leqq y\leqq 1,\ -\frac{\pi}{2}\leqq x\leqq\frac{\pi}{2}\right)$

アークサイン、インバースサイン と読む



$\begin{cases} y=\sin x \\ x=\sin^{-1}y \end{cases}$ のグラフ

$y=\sin^{-1}x$ $\left(-1\leqq x\leqq 1,\ -\frac{\pi}{2}\leqq y\leqq\frac{\pi}{2}\right)$ のグラフ



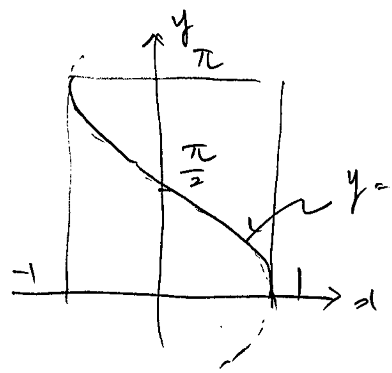
$y=\cos x$ $\left(0\leqq x\leqq\pi,\ -1\leqq y\leqq 1\right)$

の逆関数は、

$x=\cos^{-1}y$ $\left(-1\leqq y\leqq 1,\ 0\leqq x\leqq\pi\right)$

アークコサイン、インバースコサイン とよむ

$y=\cos^{-1}x$ のグラフは

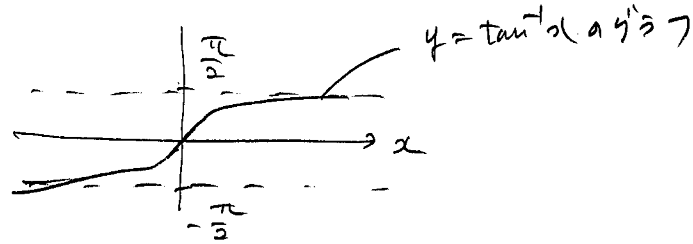


$y = \tan x \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2} \ , \ -\infty < y < \infty \right)$ の逆関数は.

$x = \tan^{-1} y \quad \left(-\infty < y < \infty \ , \ -\frac{\pi}{2} < x < \frac{\pi}{2} \right)$

アークタンジェント，インバースタンジェントとよみ

$y = \tan^{-1} x$ のグラフは.

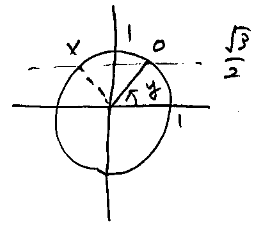


例 1.11.

(1) $\sin^{-1}\frac{\sqrt{3}}{2} = y$?　　　$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ であり、

$\sin\left(\sin^{-1}\frac{\sqrt{3}}{2}\right) = \sin y \Rightarrow \sin y = \frac{\sqrt{3}}{2}$

$\underline{y = \frac{\pi}{3}}$ 答)



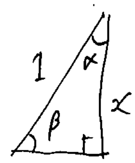
(2) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

別解.

$0 \leq x \leq 1$ のときは.

$\sin^{-1}x = \beta$

$\cos^{-1}x = \alpha$



よって　$\alpha + \beta + 90^\circ = 180^\circ$　　$\alpha + \beta = 90^\circ = \frac{\pi}{2}$

$-1 \leq x \leq 0$ のときは　$-x = x'$ とおくと　$0 \leq x' \leq 1$ で

$$\left.\begin{aligned} \sin^{-1}x &= \sin^{-1}(-x') = -\sin^{-1}x' \\ \cos^{-1}x &= \cos^{-1}(-x') = \pi - \cos^{-1}x' \end{aligned}\right\}$$ グラフより明らか.

$\frac{\cos^{-1}(-x') + \cos^{-1}x'}{2} = \frac{\pi}{2}$

$\pi - \sin^{-1}x' - \cos^{-1}x' = \frac{\pi}{2}$

$\sin^{-1}x' + \cos^{-1}x' = \frac{\pi}{2}$　上記で示してある　☒

$\sin^{-1}x = y$　　　$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

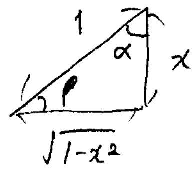
$z = \frac{\pi}{2} - y$　　　$0 \leq z \leq \pi$

$\cos z = \cos\left(\frac{\pi}{2} - y\right) \Rightarrow z = \cos^{-1}x$　つまり $\frac{\pi}{2} - y = \cos^{-1}x$

$\cos\left(\frac{\pi}{2} - y\right) = \sin y = x$

$\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

(3) $0 \leqq x \leqq 1$ のとき $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}$



$$\left.\begin{array}{l} \alpha = \cos^{-1} x \\ \alpha = \sin^{-1}\sqrt{1-x^2} \end{array}\right\} \Rightarrow \underline{\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}}$$

問 1-1

(1) $$\underbrace{\cos^{-1}\left(-\frac{1}{2}\right)}_{120^\circ} + \underbrace{\tan^{-1}\left(-\sqrt{3}\right)}_{-60^\circ} + \underbrace{\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)}_{-45^\circ} = 120 - 105 = 15^\circ = \underline{\frac{\pi}{12}}$$

(2) $$2\underbrace{\sin^{-1} 1}_{\frac{\pi}{2}} - \underbrace{\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)}_{135^\circ} + \underbrace{\tan^{-1}(-1)}_{-45^\circ} + \underbrace{\tan^{-1} 0}_{0^\circ}$$

$$= 180^\circ - 180^\circ = \underline{0}$$

(3) $$\underbrace{\sin^{-1}(-1)}_{-90^\circ} + \underbrace{\cos^{-1}\frac{\sqrt{3}}{2}}_{30^\circ} - \underbrace{\tan^{-1} 1}_{45^\circ} + \underbrace{\sin^{-1} 0}_{0^\circ} = -90^\circ + 30^\circ - 45^\circ$$

$$= \underline{-105^\circ} = -\frac{\cancel{105}}{\cancel{180}}\pi = \underline{-\frac{7}{12}\pi}$$

(4) $$\underbrace{\sin\left(\underbrace{\cos^{-1}\frac{\sqrt{3}}{2}}_{30^\circ}\right)}_{\frac{1}{2}} + \underbrace{\cos\ \underbrace{\tan^{-1}-\frac{1}{\sqrt{3}}}_{-30^\circ}}_{\frac{\sqrt{3}}{2}} + \underbrace{\tan\left(\underbrace{\sin^{-1}\frac{-1}{\sqrt{2}}}_{-45^\circ = -\frac{\pi}{4}}\right)}_{-1}$$

$$= \underline{\frac{1+\sqrt{3}}{2} - 1}$$

$$= \underline{\frac{\sqrt{3}-1}{2}}$$

$\boxed{1.8}$ 関数の極限

◇ Aに収束する場合

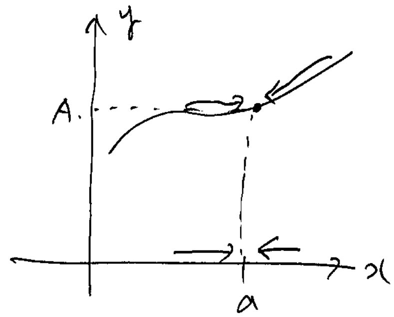
xがaに限りなく近づくとき，$f(x)$が一定の値Aに近づくとき

$$f(x) \longrightarrow A \quad (x \to a)$$

or

$$\lim_{x \to a} f(x) = \underset{\substack{\uparrow \\ \text{極限値という}}}{A}$$ と表す。

「$x \to a$ のとき $f(x)$ は A に収束する」という



定理 1.1

$$\lim_{x \to a} f(x) = A, \quad \lim_{x \to a} g(x) = B$$

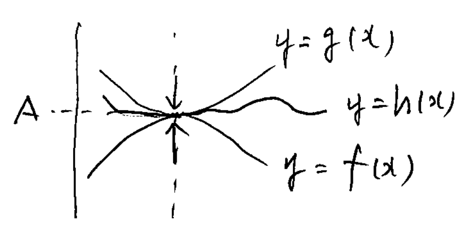
$\Rightarrow$ (1) $$\lim_{x \to a} \left(\underset{\substack{\downarrow \\ A}}{f(x)} \pm \underset{\substack{\downarrow \\ B}}{g(x)} \right) = A \pm B$$

(2) $$\lim_{x \to a} \underset{\substack{\downarrow \\ A}}{\boxed{f(x)}} \underset{\substack{\downarrow \\ B}}{g(x)} = AB$$

(3) $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{A}{B} \quad (B \neq 0 \text{ のとき})$$

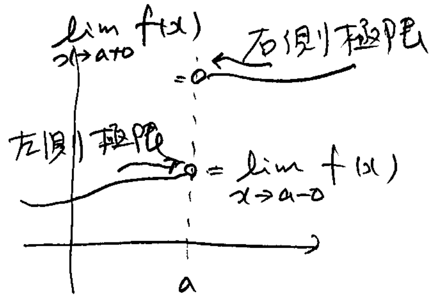
(4) xがaの近くで $f(x) \leqq h(x) \leqq g(x)$ で

$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = A$ のとき



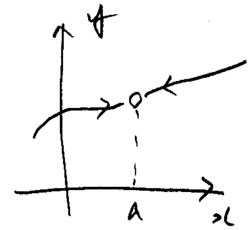
$$\lim h(x) = A$$

◇ 右側極限値，左側極限値．



$$A = \lim_{x\to a} f(x) \iff \begin{cases} \lim_{x\to a+0} f(x) = A \\ \text{かつ} \\ \lim_{x\to a-0} f(x) = A \end{cases}$$

のとき．



例題 1.12

(1) $\displaystyle \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = \lim_{x\to 2}(x+2) = \underline{4}.$

注) $x \neq 2$ で限りなく近づける

(2) $\displaystyle \lim_{x\to 1-0} \frac{x^2-1}{|x-1|} = \lim_{x\to 1-0} \frac{\cancel{(x-1)}(x+1)}{-\cancel{(x-1)}} = -(1+1) = \underline{-2}.$

$x<1$ なので $|x-1| = -(x-1)$

(3) $\displaystyle \lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$ $\left(\frac{0}{0}\text{ 不定形}\right)$

⇓

くふう 分子有理化

$$\frac{(\sqrt{1+x}-\sqrt{1-x})}{x}\frac{(\sqrt{1+x}+\sqrt{1-x})}{(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \frac{1+x-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})} = \frac{2\cancel{x}}{\cancel{x}(\sqrt{1+x}+\sqrt{1-x})} \to \frac{2}{\sqrt{1}+\sqrt{1}} = 1.$$

問 1.12.

(1) $\lim_{x \to 1} \frac{x^3-2x^2+2x-1}{x^2-3x+2}$

$x^3-2x^2+2x-1 = \underbrace{x^3-1}_{(x-1)(x^2+x+1)} - 2x(x-1) = (x-1)\{x^2+x+1-2x\}$

$= (x-1)(x^2-x+1)$

$x^2-3x+2 = (x-1)(x-2)$

$\lim_{x \to 1} \frac{x^2-x+1}{x-2} = \frac{1-1+1}{-1} = \underline{-1}.$

(2) $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x \to 4} \frac{\cancel{(\sqrt{x}-2)(\sqrt{x}+2)}}{\cancel{(x-4)}(\sqrt{x}+2)} = \frac{1}{2+2} = \underline{\frac{1}{4}}.$

(3) $\lim_{x \to 0} \frac{1-\cos x}{x}$

$\boxed{\text{メモ} \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}}$

$\frac{1-\cos x}{x} = \frac{2}{x}\sin^2\frac{x}{2} = \left|\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right| \cdot \sin\frac{x}{2} \to \underline{0}.$

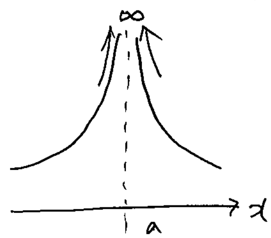
$\downarrow \qquad \downarrow$

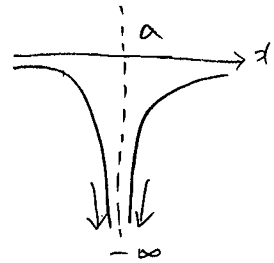
$1 \qquad 0$

$\boxed{\text{メモ} \quad \lim_{x \to 0} \frac{\sin x}{x} = 1}$

(4) $\lim_{x \to -0} \frac{x}{|x|} = \lim_{x \to -0} \frac{\cancel{x}}{-\cancel{x}} = \underline{-1}.$

◇ 無限大に発散する場合.





$$\lim_{x \to a} f(x) = \infty$$

$$f(x) \to \infty \quad (x \to a)$$

「正の無限大に発散する」という

$$\lim_{x \to a} f(x) = -\infty$$

$$f(x) \to -\infty \quad (x \to a)$$

「負の無限大に 〃 」という

◇ 極限値なし

(例1) $y = \sin \dfrac{1}{x}$

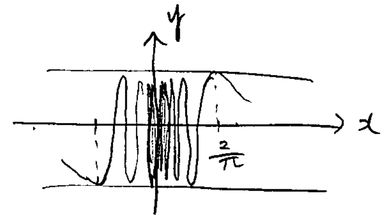
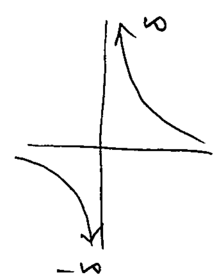


図 1.37

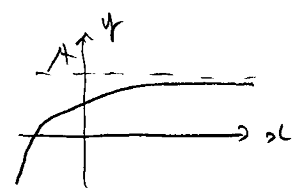
(例2) $y = \dfrac{1}{x}$



$$\lim_{x \to +0} f(x) = \infty$$

$$\lim_{x \to -0} f(x) = -\infty$$

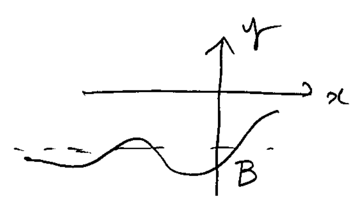
◇ $x \to \infty$, $x \to -\infty$ の場合



$x \to \infty$ のとき $f(x) \to A$ のとき

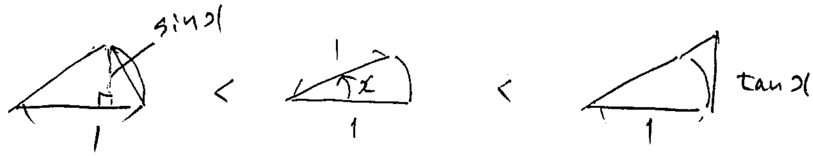
$\displaystyle\lim_{x \to \infty} f(x) = A$ と書く.

$\displaystyle\lim_{x \to -\infty} f(x) = B$ も同様



定理1.2.

(1) $\lim_{x\to 0} \dfrac{\sin x}{x} = 1.$



$$\Rightarrow \quad \frac{\sin x}{2} \underset{①}{<} \frac{x}{2} \underset{②}{<} \frac{\tan x}{2}$$

① $\Rightarrow$ $\dfrac{\sin x}{x} < 1$

② $\Rightarrow$ $\cos x < \dfrac{\sin x}{x}$

$$\left.\begin{array}{l} \\ \end{array}\right\} \quad \underset{\substack{\downarrow \\ 1. \\ x\to +0.}}{\cos x} < \frac{\sin x}{x} < 1$$

よって $\lim_{x\to +0} \dfrac{\sin x}{x} = 1.$

$$\lim_{x\to -0} \frac{\sin x}{x} = \lim_{z\to +0} \frac{\sin(-z)}{-z} = \lim_{z\to +0} \frac{\sin z}{z} = 1$$

$$\underline{\lim_{x\to 0} \frac{\sin x}{x} = 1}.$$

(2) $\displaystyle\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to 0}(1+x)^{1/x} = e$

∵ 追記1-1

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

$$= \lim_{z\to+0}(1+z)^{1/z} \quad \cdots\cdots ①$$

$$\lim_{n\to-\infty}\left(1-\frac{1}{n}\right)^{-n} = \lim_{n\to-\infty}\left(\frac{n-1}{n}\right)^{-n} = \lim_{n\to-\infty}\left(\frac{n}{n-1}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)^n = \lim_{n\to\infty}\left\{\left(1+\frac{1}{n-1}\right)^{n-1}\cdot\underbrace{\left(1+\frac{1}{n-1}\right)}_{\to 1}\right\}$$

$$= \lim_{n\to-\infty}\left(1+\frac{1}{n}\right)^n = \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x$$

$$= \lim_{z\to-0}(1+z)^{1/z} \quad \cdots\cdots ②$$

①, ② より

$$e = \lim_{z\to 0}(1+z)^{1/z}$$

$$= \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

$$= \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x$$

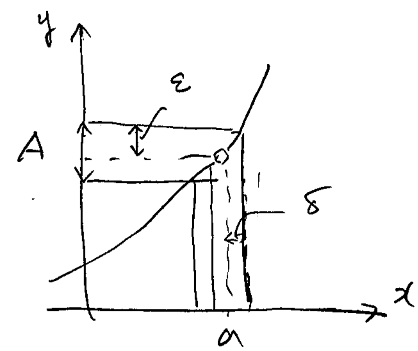
//

追記1.2　　$\varepsilon-\delta$ 論法

$\lim_{x \to a} f(x) = A \iff$「$x$が$a$に限りなく近づくとき
$f(x)$がAに限りなく近づく」

をより精密に述べると、

「任意に与えられた$\varepsilon(>0)$に対して、
適当なδをとると、$0<|x-a|<\delta$なすべてのxに対して、
$|f(x)-A|<\varepsilon$となる」



例. 1-13

(1) $\displaystyle\lim_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)$

$\displaystyle=\lim_{z\to\infty}\left(\sqrt{z^2-z+1}-z\right)$

$\displaystyle=\lim_{z\to\infty}\frac{\left(\sqrt{z^2-z+1}\right)^2-z^2}{\left(\sqrt{z^2-z+1}+z\right)}=\lim_{z\to\infty}\frac{-z+1}{\sqrt{z^2-z+1}+z}$

$\displaystyle=\lim_{z\to\infty}\frac{-1+\frac{1}{z}}{\sqrt{1-\frac{1}{z}+\frac{1}{z^2}}+1}=\frac{-1}{1+1}=\underline{-\frac{1}{2}}.$

(2) $\displaystyle\lim_{x\to\infty}e^{-x}\sin x$

$-1\leqq\sin x\leqq 1$ より

$$-e^{-x}\leqq e^{-x}\sin x\leqq e^{-x}$$

$-e^{-x}\to 0$, $e^{-x}\to 0$

より

$\displaystyle\lim_{x\to\infty}e^{-x}\sin x=0$

(3). $\displaystyle\lim_{x\to 0}\sin\frac{1}{x}$ = 存在しない 済

text 18P
1-1. 42p.

$$\begin{array}{r} 2.33 \\ 18\,\overline{)\,42} \\ 36 \\ \hline 60 \\ 54 \\ \hline 60 \end{array}$$

問 1.13.

(1) $$\lim_{x\to\infty} \frac{3x^2-6x-1}{-x^2-4x+2} = \lim_{x\to\infty} \frac{3-\frac{6}{x}-\frac{1}{x^2}}{-1-\frac{4}{x}+\frac{2}{x^2}} = -3$$

(2) $$\lim_{x\to 0} \frac{1}{x^3}$$ 存在しない

(3) $$\lim_{x\to 0} \frac{\sin ax}{x} = \lim_{x\to 0} a\underbrace{\frac{\sin ax}{ax}}_{\downarrow\, 1} = a$$

(4) $$\lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{x\to 0} \log \underbrace{\boxed{(1+x)^{\frac{1}{x}}}}_{\downarrow\, e} = \underline{1}.$$

(5) $$\lim_{x\to 0} \frac{\boxed{e^x-1}^{\,=k}}{x} =$$

$e^x-1=k$ とおくと $k\to 0$

$k+1=e^x \qquad x=\log(k+1)$

$$= \lim_{k\to 0} \frac{k}{\log(1+k)} = \lim_{k\to 0} \frac{1}{\underbrace{\boxed{\frac{1}{k}\log(1+k)}}_{\downarrow\, 1}} = \underline{1}.$$

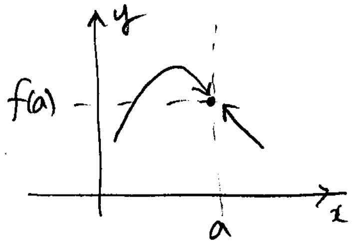
(6) $$\lim_{x\to 0} (1+ax)^{1/x} = \lim_{x\to 0} (1+ax)^{\frac{a}{ax}}$$

$$= \lim_{x\to 0} \Big\{\underbrace{\boxed{(1+ax)^{\frac{1}{ax}}}}_{\downarrow\, e}\Big\}^a = \underline{e^a}.$$

(7) $$\lim_{x\to 0} \frac{\tan ax}{\tan bx} = \lim_{x\to 0} \underbrace{\boxed{\frac{\cos bx}{\cos ax}}}_{\downarrow\, 1} \cdot \underbrace{\boxed{\frac{\sin(ax)/ax}{\sin bx/bx}}}_{\downarrow\, 1.} \cdot \frac{ax}{bx} = \underline{\frac{a}{b}}.$$

1.9 関数の連続性

◇ 1点で連続な関数

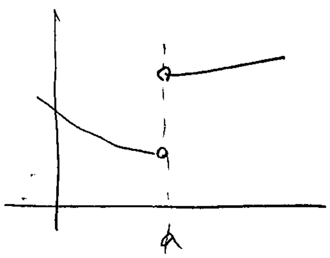


$\lim_{x \to a} f(x) = f(a)$ のとき

「f(x) は x=a で連続である」という。

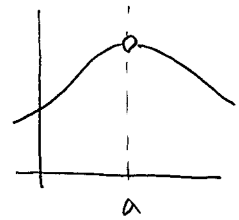
x=a で不連続な場合

(i)

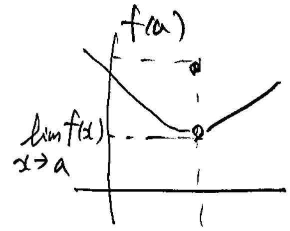


$\lim_{x \to a+0} f(x) \neq \lim_{x \to a-0} f(x)$

のとき

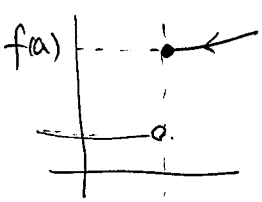


f(a) が定義されていないとき



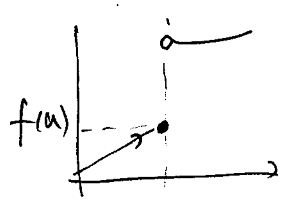
$\lim_{x \to a} f(x) \neq f(a)$

◇ 右側連続，左側連続



$\lim_{x \to a+0} f(x) = f(a)$

のとき



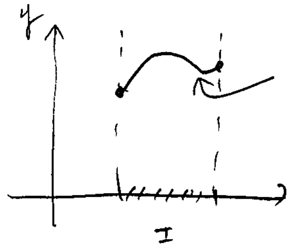
$\lim_{x \to a-0} f(x) = f(a)$

のとき

片側連続

◇ 区間Iで連続

すべての $a \in I$ で f(x) が連続なとき、「区間Iで連続」という

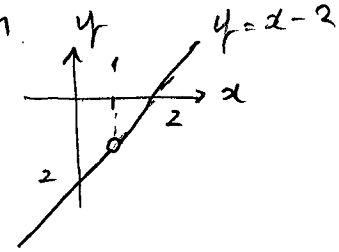


きれ目のない曲線

例1.14　　不連続にする x は？

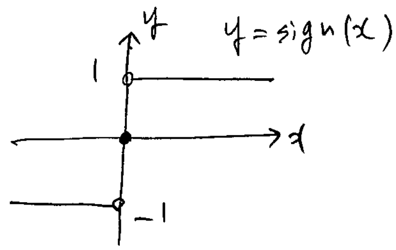
(1)　$\dfrac{x^2-3x+2}{x-1}$　　$x=1$ では定義されない

$x \neq 1$ では　$\dfrac{\cancel{(x-1)}(x-2)}{\cancel{(x-1)}} = x-2$



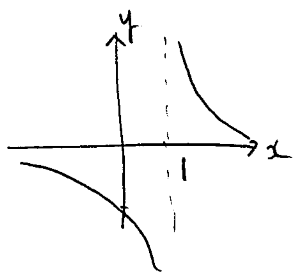
$x=1$ で不連続

(2)　$\underbrace{\mathrm{sign}}_{\text{符号}}(x)$



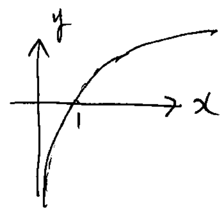
$x=0$ で不連続

(3)　$\dfrac{3}{x-1}$



$x=1$ で不連続

(4)　$\log x$

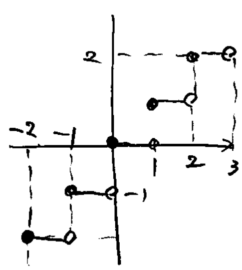


$x=0$ で不連続

(5)　$[x]$
x の"整数部分"

$[1.1]=1$

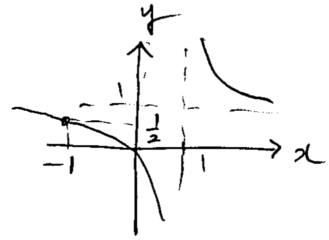
$[-1.2]=-2$



$x=n$（整数）で不連続

問1.14.

(1) $f(x) = \begin{cases} \dfrac{x(x+1)}{x^2-1} & x \neq -1 \\ \dfrac{1}{2} & x = -1 \end{cases}$



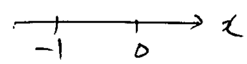
$x \neq -1$ のとき

$$f(x) = \frac{x\cancel{(x+1)}}{(x-1)\cancel{(x+1)}} = 1 + \frac{1}{x-1}$$

$x=1$ で不連続

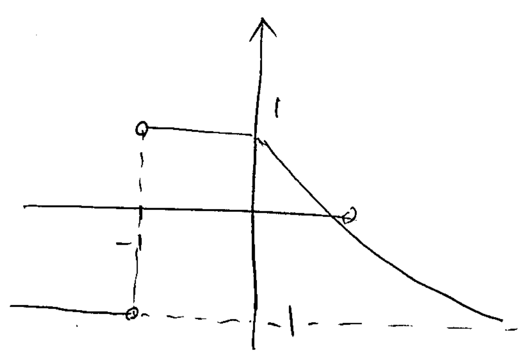
$x \in (-\infty, 1)$
$x \in (1, \infty)$ で連続

(2) $f(x) = \dfrac{1-|x|}{|1+x|} = \begin{cases} (x \geqq 0) & \dfrac{1-x}{1+x} \\ (-1 \leqq x < 0) & \dfrac{1+x}{1+x} = 1 \\ (x = -1) & f(x) \text{ 存在しない} \end{cases}$



$(x < -1)$

$$\frac{1+x}{-(1+x)} = -1$$



$x=-1$ で不連続

$(-\infty, -1)$
$(-1, \infty)$ で連続

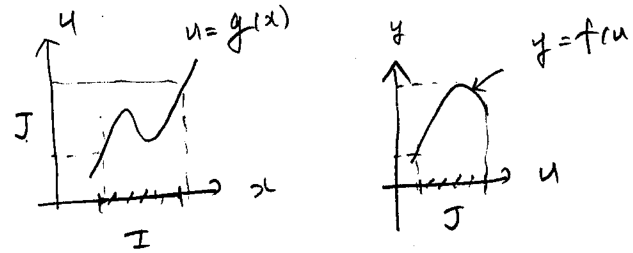
定理 1.3.

$f(x)$, $g(x)$ が 区間 I で 連続

$\Rightarrow$ $kf(x)$, $f(x) \pm g(x)$, $f(x)g(x)$, $\dfrac{f(x)}{g(x)}$ (ただし $g(x) \neq 0$)

も 区間 I で 連続

定理 1.4.



$\Rightarrow$ $y = f(g(x))$ は $x \in I$ において 連続

合成関数

∵ $\forall a \in I$ に対して

$x \to a$ なら $u \to g(a) = u_0$

$u \to u_0$ なら $y \to f(u_0) = f(g(a))$

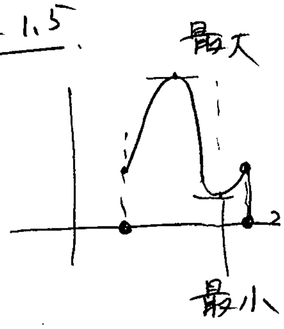
つまり

$x \to a$ なら $y \to f(g(a))$

よって $y = f(g(x))$ は I で 連続.

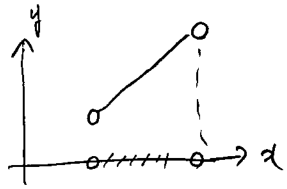
◇ 閉区間で連続な関数

定理 1.5

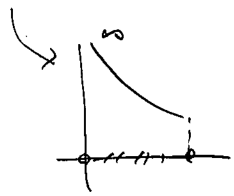


最大値・最小値をとる.

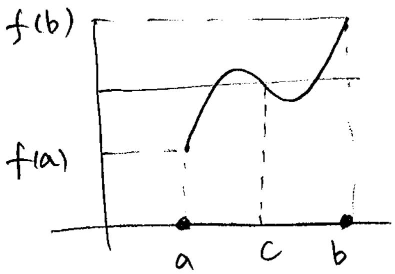
(注)



開区間のとき 最大・最小
を持つとは限らない.

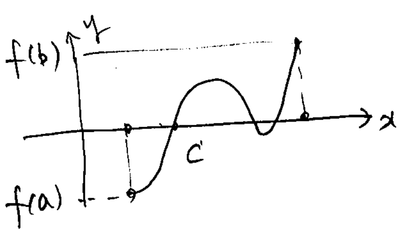


定理 1.6 (中間値の定理 I)



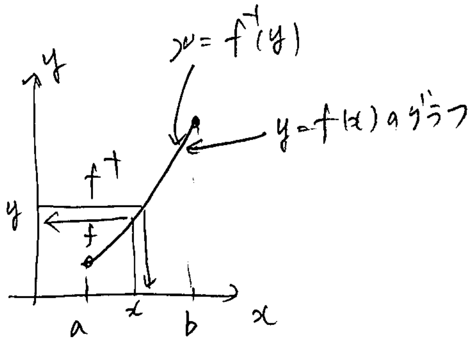
$f(a) \neq f(b)$ のとき
$f(a)$ と $f(b)$ の間の $\forall k$ に対して
$f(c)=k$ となる c が 少なくとも1つ存在

定理 1.6′ (〃 II)



$f(a)$ と $f(b)$ が異符号
$f(c)=0$ となる 〃.

定理 1.7



f の逆 f^{-1} が存在する。　$y=f(x)$ が連続 $\Rightarrow$ $x=f^{-1}(y)$ も連続

例 1.15

$f(x)=\dfrac{x^2-4}{x-2}$ の連続性は?

$x\neq 2$ のとき　$f(x)=\dfrac{(x-2)(x+2)}{x-2}=x+2$ $\Rightarrow$ $x\neq 2$ のとき 連続.

$x=2$ のとき.　$\displaystyle\lim_{x\to 2}f(x)=2+2=4$ だが $f(2)$ が定義されないので不連続.

注　$f(x)=\begin{cases}\dfrac{x^2-4}{x-2} & (x\neq 2)\\ 4 & (x=2)\end{cases}$ とすれば連続にできる.

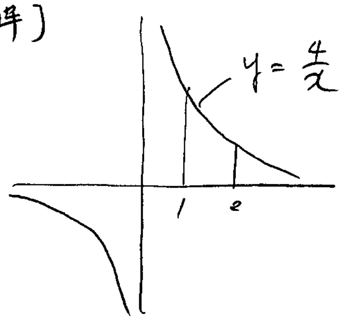
$\Rightarrow$ $x=2$ は除去可能な不連続点という

例 1.16.

$f(x) = \dfrac{4}{x}$ は $(0, \infty)$ で最大値・最小値を持つか？

$[1, 2]$ 〃 ？

[解]



$x \in (0, \infty)$ で

$0 < y < \infty$ 　最大(小)持たない

$x \in [1, 2]$ では　$\max y = 4$

$\min y = 2.$

例 1.17

「$f(x) = 3^x - 6x + 2 = 0$ は $(2, 3)$ で少なくとも1つ解を持つ」示せ.

$$\begin{cases} f(2) = 3^2 - 12 + 2 = -1 < 0 \\ f(3) = 27 - 18 + 2 = 11 > 0 \end{cases}$$

$\Rightarrow$ $f(c) = 0$ なる c が $(2, 3)$ に少なくとも1つ存在する

問 1.15

「$f(x) = x^3 - 3x^2 - 2x + 5 = 0$ は 2より小さい正の解を持つ」を示せ

$$\begin{cases} f(0) = 5 > 0 \\ f(2) = 8 - 12 - 4 + 5 = -3 < 0 \end{cases}$$

$\Rightarrow$

$f(c) = 0$ なる c が $(0, 2)$ に少なくとも1つ存在.

$\boxed{1.10}$ 微分係数

◇ $f(x)$ が $x=a$ の近くで定義されているとき

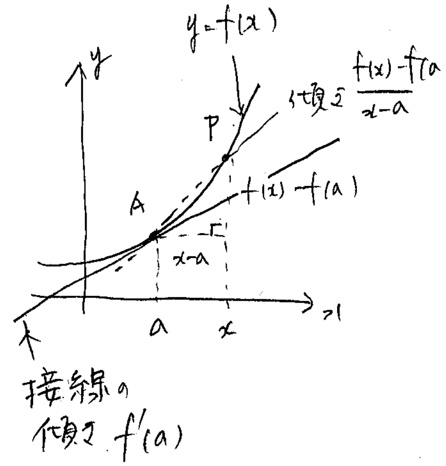
$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \quad \cdots\cdots (*)$$

$x=a+h$

$$= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$ が存在するとき

「$x=a$ における $f(x)$ の微分係数」という

「$f(x)$ は $x=a$ で微分可能である」という.



(*) のとき,

$$\lim_{x\to a}\Bigl\{\underbrace{\frac{f(x)-f(a)}{x-a}-f'(a)}_{\varepsilon\ とおくと.}\Bigr\}=0$$

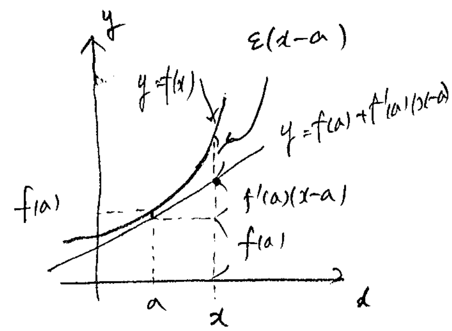
$$\frac{f(x)-f(a)}{x-a}-f'(a)=\varepsilon, \quad \varepsilon\to 0 \quad (x\to a)$$

よって $f(x)-f(a)-f'(a)(x-a)=\varepsilon(x-a)$

$$f(x)=f(a)+f'(a)(x-a)+\varepsilon(x-a)$$

[ここで ε は $\varepsilon\to 0\ (x\to a)$]

と書ける。



◇ $\underbrace{\text{右微分係数, 左微分係数}}$ $x=a$ における

$$f'_+(a)=\lim_{h\to +0}\frac{f(a+h)-f(a)}{h}$$

$$f'_-(a)=\lim_{h\to -0}\frac{f(a+h)-f(a)}{h}$$

$f(x)$ が $x=a$ で微分可能である $\Longleftrightarrow$ $f'_+(a)=f'_-(a)$

定理1.8.

$f(x)$ が $x=a$ で微分可能 $\Rightarrow$ $f(x)$ は $x=a$ で連続

(証) $\lim_{x\to a}\{f(x)-f(a)\} = \lim_{x\to a} f'(a)(x-a)+\varepsilon(x-a) = 0$ ☒

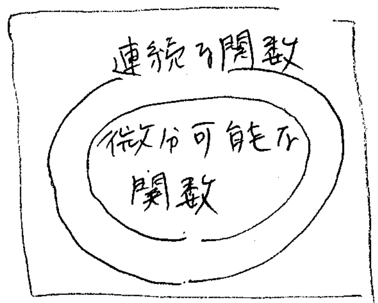


図1.50
1変数関数 $y=f(x)$ の集合

◇ 接線

$x=a$ における $y=f(x)$ の接線の方程式は.

$$y = f(a) + f'(a)(x-a)$$

例1.18.

(1) $f(x)=\frac{1}{x}$ のとき $f'(a)$ $(a\neq 0)$ を求める.

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{1}{h}\left\{\frac{1}{a+h}-\frac{1}{a}\right\}$$

$$= \lim_{h\to 0}\frac{1}{\cancel{h}}\frac{-\cancel{h}}{a(a+h)} = \frac{-1}{a^2}$$

(2) $f(x)=|x|$ は $x=0$ で連続か? 微分可能か?

$f(+0) = \lim_{x\to +0}|x| = \lim_{x\to +0} x = 0$. $f(-0) = \lim_{x\to -0}|x| = \lim_{x\to -0} -x = 0$ 連続

$$\left.\begin{aligned} f'_+(0) &= \lim_{h\to +0}\frac{|0+h|-|0|}{h} = \lim_{h\to +0}\frac{h}{h} = 1 \\ f'_-(0) &= \lim_{h\to -0}\frac{|h|-|0|}{h} = \lim_{h\to -0}\frac{-h}{h} = -1 \end{aligned}\right\} \text{微分不可能}$$

例 1.19.

$f(x)=\sqrt{x}$ 上の $(4,2)$ における接線の方程式を求めよ.

$$f'(4)=\lim_{h\to 0}\frac{\sqrt{4+h}-\sqrt{4}}{h}=\lim_{h\to 0}\frac{(\sqrt{4+h}-\sqrt{4})(\sqrt{4+h}+\sqrt{4})}{h\,(\sqrt{4+h}+\sqrt{4})}$$

$$=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{4+h}+\sqrt{4})}=\frac{1}{\sqrt{4}+\sqrt{4}}=\frac{1}{4}.$$

よって

$$y=\frac{1}{4}(x-4)+2$$

$$=\underline{\frac{1}{4}x+1}$$

問 1.16 $y=\frac{1}{x}$ 上の $(2,\frac{1}{2})$ における接線は?

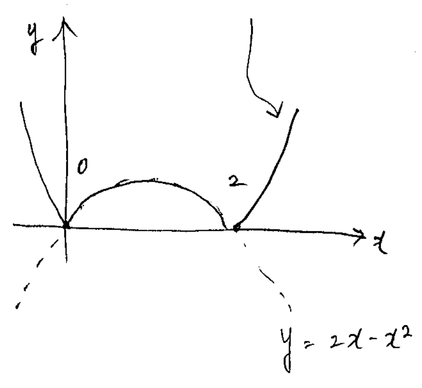
$$f'(2)=\lim_{h\to 0}\frac{\frac{1}{2+h}-\frac{1}{2}}{h}=\lim_{h\to 0}\frac{1}{h}\,\frac{-h}{2(2+h)}=-\frac{1}{4}$$

$$\underline{y}=-\frac{1}{4}(x-2)+\frac{1}{2}=\underline{-\frac{1}{4}x+1}.$$

問 1.17

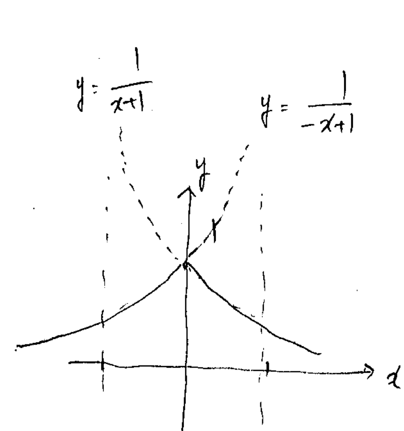
(1) $f(x)=|2x-x^2|$ のとき. $f(x)$ は $x=2$ で微分可能か?

$y=|2x-x^2|$ のグラフ



$x=2$ で微分不可能

(2) $f(x)=\frac{1}{|x|+1}$ の $x=0$ における連続性



$x=0$ で連続, 微分不可能

1.11 導関数

◇ 区間 I $\underbrace{}_{\text{の各点}}$ で $y=f(x)$ が微分可能なとき、「区間 I で微分可能」という

◇ 導関数

区間 I の各点 x に対して、その点の微分係数 $f'(x)$ を対応させる関数を

$y=f(x)$ の導関数 といって

$$f'(x),\ y',\ f',\ \frac{dy}{dx},\ \frac{d}{dx}f(x)$$ などと表す。

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$ だったから

「y の増分」という

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim\frac{\Delta y}{\Delta x}$$

「x の増分」〃

となる。

$f(x)$ から $f'(x)$ を求めることを

「$f(x)$ を微分する」という。

定理 1.9

$f(x), g(x)$ が 微分可能

$\Rightarrow$ (1) $\left(kf(x)\right)' = kf'(x)$

$\because$ $$\lim_{\Delta x \to 0} \frac{kf(x+\Delta x) - kf(x)}{\Delta x} = k \underbrace{\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}_{f'(x)} \quad \boxtimes$$

(2) $\left(f(x) + g(x)\right)' = f'(x) + g'(x)$

$\because$ $$\lim_{\Delta x \to 0} \frac{\{f(x+\Delta x) + g(x+\Delta x)\} - \{f(x) + g(x)\}}{\Delta x} = f'(x) + g'(x) \quad \boxtimes$$

(3) $\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$

$\because$ $$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\{f(x+\Delta x) - f(x)\}g(x+\Delta x) + f(x)\{g(x+\Delta x) - g(x)\}}{\Delta x}$$

$$= f'(x)g(x) + f(x)g'(x) \quad \boxtimes$$

(4) $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

$\because$ $$\lim_{\Delta x \to 0} \frac{1}{\Delta x}\left\{\frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}\right\} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{f(x+\Delta x)g(x) \overbrace{- f(x)g(x) + f(x)g(x)} - f(x)g(x+\Delta x)}{g(x)g(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{g(x)g(x+\Delta x)} \frac{1}{\Delta x}\left(f(x+\Delta x) - f(x)\right)g(x) - f(x)\left(\frac{g(x+\Delta x) - g(x)}{\Delta x}\right)$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \quad \boxtimes$$

◇ 基本的な関数の導関数

(1) $(c)' = 0$ ∵ $\displaystyle \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = 0$

(2) $(x^{\alpha})' = \alpha x^{\alpha - 1}$

omit ∵ $x^{\alpha} = e^{\alpha \log x}$ より $(x^{\alpha})' = e^{\alpha \log x} \dfrac{\alpha}{x} = \dfrac{x^{\alpha}}{x}\alpha = \alpha x^{\alpha-1}$

合成関数の微分 [定理 1.10]　　log の微分 公式II (11)

(3) $(\sin x)' = \cos x$.

∵ $\displaystyle \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \cdot 2\cos\frac{2x + \Delta x}{2} \sin\frac{\Delta x}{2}$　13p. 7

$\displaystyle = \lim_{\Delta x \to 0} \frac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)} \cdot \cos\left(x + \frac{\Delta x}{2}\right) = \cos x$.

(4) $(\cos x)' = -\sin x$

∵ $\displaystyle \lim_{\Delta x \to 0} \frac{\cos(x + \Delta x) - \cos x}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}(-2)\sin\frac{2x + \Delta x}{2} \sin\frac{\Delta x}{2}$

$\displaystyle = \lim_{\Delta x \to 0} \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \left(-\sin\left(x + \frac{\Delta x}{2}\right)\right) = -\sin x$

(5) $(\tan x)' = \dfrac{1}{\cos^2 x}$

∵ $\left(\dfrac{\sin x}{\cos x}\right)' = \dfrac{\cos x \cos x + \sin x \sin x}{\cos^2 x} = \dfrac{1}{\cos^2 x}$.

(6) $(\cot x)' = -\dfrac{1}{\sin^2 x}$

∵ $\left(\dfrac{\cos x}{\sin x}\right)' = \dfrac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = -\dfrac{1}{\sin^2 x}$

(7) $(\sec x)' = \sec x \cdot \tan x$

∵ $\left(\dfrac{1}{\cos x}\right)' = \dfrac{+\sin x}{\cos^2 x} = \dfrac{1}{\cos x} \cdot \tan x$

(8) $(\operatorname{cosec} x)' = -\operatorname{cosec} x \cdot \cot x$

例. 1.20.

$f(x) = \sqrt{a^2 - x^2}$ $(a \neq 0)$ の $f'(x)$ は?

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{a^2-(x+h)^2} - \sqrt{a^2-x^2}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\cancel{a^2} - (x+h)^2 - (\cancel{a^2} - x^2)}{\sqrt{} + \sqrt{}}$$

$$= \lim_{h\to 0} \frac{-2x\cancel{h} - h^{\cancel{2}}}{\cancel{h}} \frac{1}{\sqrt{a^2-(x+h)^2} + \sqrt{a^2-x^2}} = \frac{-\cancel{2}x}{\cancel{2}\sqrt{a^2-x^2}} = \frac{-x}{\sqrt{a^2-x^2}}$$

例 1.21.

(1) $\left(x - \frac{1}{\sqrt{x}}\right)^2 = x^2 - 2\sqrt{x} + \frac{1}{x}$ より

$$\left(x^2 - 2\sqrt{x} + \frac{1}{x}\right)' = \underbrace{(x^2)'}_{\substack{\parallel \\ 2x}} - 2\underbrace{(x^{\frac{1}{2}})'}_{\substack{\parallel \\ \frac{1}{2}x^{-\frac{1}{2}}}} + \underbrace{(x^{-1})'}_{-x^{-2}} = 2x + x^{-\frac{1}{2}} - x^{-2}$$

(2) $$\left((x^3 + 4x)\sqrt{1-x^2}\right)' = \underbrace{(x^3+4x)'}_{3x^2+4}\sqrt{1-x^2} + (x^3+4x)\underbrace{\left(\frac{-x}{\sqrt{1-x^2}}\right)}_{\text{前問より }(a=1\text{ に})}$$

$$= (3x^2+4)\sqrt{1-x^2} + \frac{-(x^3+4x)x}{\sqrt{1-x^2}}$$

(3) $$\left(\frac{\sin x}{1+\tan x}\right)' = \frac{(\sin x)'(1+\tan x) - \sin x\,(1+\tan x)'}{(1+\tan x)^2}$$

$$= \frac{\cos x\,(1+\tan x) - \sin x\,\frac{1}{\cos^2 x}}{(1+\tan x)^2} = \frac{\cos x + \sin x - \sin x \sec^2 x}{(1+\tan x)^2}$$

問 1.18

$f(x) = \sqrt{3x-4}$ の $f'(x)$ は?

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{h}\left(\sqrt{3x+3h-4}-\sqrt{3x-4}\right)$$

$$= \lim_{h\to 0}\frac{\cancel{3x}+3\cancel{h}-\cancel{4}-(\cancel{3x}-\cancel{4})}{\cancel{h}\cdot\left(\sqrt{3x+3h-4}+\sqrt{3x-4}\right)} = \frac{3}{\sqrt{3x-4}+\sqrt{3x-4}} = \underline{\frac{3}{2\sqrt{3x-4}}}.$$

問 1.19.

(1) $$\left(\frac{2x+5}{\sqrt{3x-4}}\right)' = \frac{\overset{2}{\boxed{(2x+5)'}}\sqrt{3x-4}-(2x+5)\left(\sqrt{3x-4}\right)'}{3x-4}$$

$$= \left(2\sqrt{}-(2x+5)\frac{3}{2\sqrt{3x-4}}\right)\frac{1}{3x-4}$$

$$= \underline{\frac{2}{\sqrt{3x-4}}-\frac{3(2x+5)}{(3x-4)^{\frac{3}{2}}}}$$

(2) $$\left(\frac{3x}{\sqrt{a^2-x^2}}\right)' = \frac{\overset{3}{\boxed{(3x)'}}\sqrt{a^2-x^2}-3x\overset{\frac{-x}{\sqrt{a^2-x^2}}}{\boxed{\left(\sqrt{a^2-x^2}\right)'}}}{a^2-x^2}$$ (問 1.20)

$$= \frac{1}{a^2-x^2}\left\{3\sqrt{a^2-x^2}+\frac{3x^2}{\sqrt{a^2-x^2}}\right\} = \underline{\frac{3}{\sqrt{a^2-x^2}}+\frac{3x^2}{(a^2-x^2)^{\frac{3}{2}}}}$$

(3) $$(x^2\cos x)' = \underbrace{(x^2)'}_{2x}\cos x + x^2\underbrace{(\cos x)'}_{-\sin x}$$

$$= \underline{2x\cos x - x^2\sin x}$$

(4) $\left((x^2-1)\sin x\right)' = (x^2-1)'\sin x + (x^2-1)(\sin x)'$

$= \underline{2x\sin x + (x^2-1)\cos x}$

(5) $\left((4x-3)\sqrt{1-x}\right)' = \underbrace{(4x-3)'}_{4}\sqrt{1-x} + (4x-3)\underbrace{\left(\sqrt{1-x}\right)'}$

合成関数の微分 (Th 1.10)

$$\left(\begin{aligned}\frac{d}{dx}(1-x)^{\frac{1}{2}} &= \frac{du^{\frac{1}{2}}}{du}\frac{d(1-x)}{dx}\\ &= \frac{1}{2}u^{-\frac{1}{2}}(-1)\\ &= -\frac{1}{2}(1-x)^{-\frac{1}{2}}\end{aligned}\right)$$

$= \underline{4\sqrt{1-x} + \frac{(4x-3)}{2}(1-x)^{-\frac{1}{2}}}$ //

[定理 1.10] $x \longrightarrow u=g(x) \longrightarrow y=f(u) = f(g(x))$

合成関数

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h} = \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{\underbrace{\boxed{g(x+h)-g(x)}}_{=\Delta g\ \text{とおく}}}\ \frac{g(x+h)-g(x)}{h}$$

$$\left(\begin{aligned}&\Delta g = g(x+h)-g(x) \text{ とおくと } h\to 0 \text{ で } \Delta g\to 0\\ &g(x+h) = g(x)+\Delta g\end{aligned}\right)$$

$$= \lim_{\substack{h\to 0\\ \Delta g\to 0}} \left|\frac{f(g(x)+\Delta g)-f(g(x))}{\Delta g}\right| \cdot \left|\frac{g(x+h)-g(x)}{h}\right|$$

$\downarrow$ $f'(\boxed{g(x)})$, $g(x) = u$ 　　 $\downarrow$ $g'(x)$

$$= \frac{dy}{du}\cdot\frac{du}{dx} \quad ☒$$

[細かいこと]

$\Delta g \neq 0$ のとき.

$\Delta g = 0$ のときの式

$f'(g(x))$

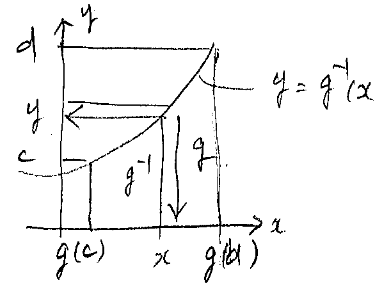
$\dfrac{f'(g(x)+\Delta g)-f'(g(x))}{\Delta g}$

とした式でおきかえた方.

定理 1.11

$y = g^{-1}(x)$ の微分がもとめたいとき

$x = g(y)$ の微分可能のとき



$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad \left(\text{ただし } \frac{dx}{dy} \neq 0\right)$$

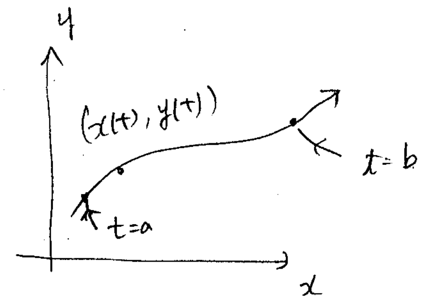
(∵) $x = g(y) = g(g^{-1}(x))$ の両辺を x で微分

$$\underbrace{(x)'}_{1} = (g(g^{-1}(x)))' = \underbrace{\boxed{g'(\underbrace{\boxed{g^{-1}(x)}}_{y})}}_{\frac{dx}{dy}} \underbrace{\boxed{(g^{-1}(x))'}}_{\frac{dy}{dx}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

定理 1.12

$x = f(t)$, $y = g(t)$ は微分可能



$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

(∵) $x = f(t)$ と t について解くと $t = f^{-1}(x)$

$y = g(f^{-1}(x))$ より

$$\frac{dy}{dx} = \underbrace{\boxed{g'(\underbrace{\boxed{f^{-1}(x)}}_{t})}}_{\frac{dy}{dt}} \underbrace{\boxed{(f^{-1}(x))'}}_{\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} \leftarrow (\text{Th. } 1.11)} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

☒

◇ 導関数の公式 II

(9) $(e^x)' = e^x$

∵ $\displaystyle\lim_{h\to0}\frac{e^{x+h}-e^x}{h} = e^x\boxed{\lim_{h\to0}\frac{e^h-1}{h}} = e^x$ ☒

$\underset{\text{1 問1.13(5)より}}{\parallel}$

(10) $(a^x)' = a^x\log a$

∵ $a^x = e^{x\log a}$ より $\big(e^{\overset{u}{\boxed{x\log a}}}\big)' = \dfrac{d\,e^u}{du}\,\dfrac{d\,x\log a}{dx} = e^u\cdot\log a$ ☒

(11) $(\log x)' = \dfrac{1}{x}$, $(\log(-x))' = \dfrac{1}{x}$

$\displaystyle\lim_{h\to0}\frac{1}{h}\left(\log(x+h)-\log x\right) = \lim_{h\to0}\frac{1}{h}\left(\log\frac{x+h}{x}\right) = \lim_{h\to0}\log\left(1+\frac{h}{x}\right)^{\frac{1}{h}}$

$\displaystyle = \lim_{h\to0}\log\left\{\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right\}^{\frac{1}{x}} = \frac{1}{x}\underbrace{\lim_{h\to0}\log\underset{\downarrow\, e}{\boxed{\left(1+\frac{h}{x}\right)^{\frac{x}{h}}}}}_{\downarrow\,1} = \frac{1}{x}$ ∴

$\big(\log(\underset{u}{\boxed{-x}})\big)' = \dfrac{d\log u}{du}\,\dfrac{d(-x)}{dx} = \dfrac{-1}{u} = \dfrac{-1}{-x} = \dfrac{1}{x}$. ☒

(12) $(\sin^{-1}x)' = \dfrac{1}{\sqrt{1-x^2}}$

∵ $y = \sin^{-1}x$, $x = \sin(y)$

$\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1-x^2}}$ ☒

$\left(-\frac{\pi}{2}\le y\le\frac{\pi}{2}\text{ より }\cos y\ge0\text{ よって }\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}\right)$

グラフ

(13)

$$(\cos^{-1}x)' = \frac{-1}{\sqrt{1-x^2}}$$

∵ $y = \cos^{-1}x \qquad x = \cos y$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{-\sin y} = \frac{-1}{\sqrt{1-x^2}} \quad \boxtimes$$

($0 \leqq y \leqq \pi$ より $\sin y \geqq 0$ よって $\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-x^2}$)

(14) $(\tan^{-1}x)' = \dfrac{1}{1+x^2}$

$y = \tan^{-1}x \ , \ x = \tan y$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{1}{\cos^2 y}} = \cos^2 y = \frac{1}{1+x^2} \quad \square$$

$$\left(x^2 = \frac{1-\cos^2 y}{\cos^2 y} \to (x^2+1)\cos^2 y = 1 \to \cos^2 y = \frac{1}{x^2+1} \right)$$

(15) $(\cot^{-1}x)' = \dfrac{-1}{1+x^2}$

∵ $y = \cot^{-1}x \ , \ x = \cot y$.

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{-1}{\sin^2 y}} = -\sin^2 y = -\frac{1}{1+x^2} \quad \boxtimes$$

$$\left(x^2 = \frac{1-\sin^2 y}{\sin^2 y} \Rightarrow (1+x^2)\sin^2 y = 1 \right)$$

(16) $(\sec^{-1}x)' = \dfrac{1}{|x|\sqrt{x^2-1}}$

$\because$ $y = \sec^{-1}x \Rightarrow x = \sec y = \dfrac{1}{\cos y}$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{+(\cos y)^{-2}\cdot \sin y} = \frac{1}{\frac{x^2}{|x|}\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}} \quad \boxtimes$$

$$\begin{cases} (\cos y)^{-2} = x^2 \\ \sin y = \sqrt{1-\cos^2 y} = \sqrt{1-\frac{1}{x^2}} = \frac{1}{|x|}\sqrt{x^2-1} \end{cases}$$

(17) $(\operatorname{cosec}^{-1}x)' = \dfrac{-1}{|x|\sqrt{x^2-1}}$

$\because$ $y = \operatorname{cosec}^{-1}x \Rightarrow x = \operatorname{cosec} y = \dfrac{1}{\sin y}$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{-(\sin y)^{-2}\cos y} = \frac{-1}{\frac{x^2}{|x|}\sqrt{x^2-1}} = \frac{-1}{|x|\sqrt{x^2-1}} \quad \boxtimes$$

$$\begin{cases} x^2 = \frac{1}{\sin^2 y} \\ \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{1}{x^2}} = \sqrt{\frac{x^2-1}{x^2}} = \frac{\sqrt{x^2-1}}{|x|} \end{cases}$$

例 1.22. y'は?

(1) $y = \sqrt[3]{(x^2+1)^2} = (x^2+1)^{\frac{2}{3}}$

$$\frac{dy'}{dx} = \Big(\big(\underbrace{\boxed{x^2+1}}_{u}\big)^{\frac{2}{3}}\Big)' = \frac{d u^{\frac{2}{3}}}{du}\,\frac{d(x^2+1)}{dx} = \frac{2}{3}u^{-\frac{1}{3}}\cdot 2x = \underline{\frac{4x}{3}(x^2+1)^{-\frac{1}{3}}}$$

(2) $y = \cos^{-1}\left(\underbrace{\boxed{\frac{x^2-1}{x^2+1}}}_{u}\right) \quad (x>0)$

$$\frac{dy}{dx} = \frac{d\cos^{-1}u}{du}\,\frac{du}{dx}$$

$$= \frac{-1}{\sqrt{1-u^2}}\cdot\frac{4x}{(x^2+1)^2}$$

$$\left(u = \frac{x^2+1-2}{x^2+1} = 1-\frac{2}{x^2+1}\right.$$

$$\left.\frac{du}{dx} = -2\frac{(-1)}{(x^2+1)^2}2x = \frac{4x}{(x^2+1)^2}\right)$$

$$\left(\sqrt{1-u^2} = \sqrt{1-\frac{x^4-2x^2+1}{(x^2+1)^2}}\right.$$

$$\left.= \frac{\sqrt{\cancel{x^4}+2x^2+\cancel{1}-(\cancel{x^4}-2x^2+\cancel{1})}}{x^2+1} = \frac{\sqrt{4x^2}}{x^2+1} = \frac{2x}{x^2+1}\right)$$

$$= -\frac{\cancel{x^2+1}}{\cancel{2x}}\cdot\frac{\overset{2}{\cancel{4x}}}{(x^2+1)^{\cancel{2}}} = \underline{\frac{-2}{x^2+1}}\,/\!/$$

例 1.23. $f(x)$ 微分可能で $f(x)\neq 0$ のとき

$$\big(\log|\underbrace{f(x)}_{u}|\big)' = \frac{f'(x)}{f(x)} \text{ を示せ}$$

[解]

$$\frac{dy}{dx} = \underbrace{\frac{d\log|u|}{du}}_{=\frac{1}{u}}\cdot\underbrace{\frac{du}{dx}}_{=f'(x)} = \underline{\frac{f'(x)}{f(x)}} \quad \boxtimes$$

問 1.20.

(1) $(\sin^3 x)' = \frac{d}{du}u^3 \frac{d\sin x}{dx} = 3u^2\cos x = \underline{3\sin^2 x\cos x}$.

(2) $\left(\tan\underbrace{\boxed{\left(2x+\frac{\pi}{6}\right)}}_{u}\right)' = \frac{d\tan u}{du}\frac{d\left(2x+\frac{\pi}{6}\right)}{dx} = \frac{1}{\cos^2 u}\cdot 2 = \underline{\frac{2}{\cos^2\left(2x+\frac{\pi}{6}\right)}}$.

(3) $\left(\sqrt{1+x+x^2}\right)' = \frac{d}{du}u^{\frac{1}{2}}\frac{d}{dx}(1+x+x^2) = \frac{1}{2}u^{-\frac{1}{2}}(1+2x)$

$= \underline{\frac{1+2x}{2\sqrt{1+x+x^2}}}$.

(4) $(e^{ax}\sin bx)' = \underbrace{\boxed{(e^{ax})'}}_{e^{ax}\cdot a}\sin bx + e^{ax}\underbrace{\boxed{(\sin bx)'}}_{\cos bx\cdot b}$

$= \underline{a e^{ax}\sin bx + b e^{ax}\cos bx}$

(5) $\left(\tan^{-1}\left(\frac{1}{\sqrt{2}}\tan\frac{x}{2}\right)\right)'$

$= \frac{d}{du}\tan^{-1}u\ \frac{d}{dx}\frac{1}{\sqrt{2}}\tan\frac{x}{2}$

$= \frac{1}{1+u^2}\ \frac{1}{\sqrt{2}}\ \frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2} = \frac{1}{1+\frac{\tan^2\frac{x}{2}}{2}}\ \frac{1}{\cos^2\frac{x}{2}}\ \frac{1}{2\sqrt{2}}$

$= \frac{1}{2\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}\ \frac{1}{\sqrt{2}} = \underline{\frac{1}{1+\cos^2\frac{x}{2}}\cdot\frac{1}{\sqrt{2}}}$.

(6) $\left(\sin^{-1}\frac{x}{2}\right)'$

$$= \frac{d}{du}\sin^{-1}u \ \frac{d}{dx}\frac{x}{2} = \frac{1}{\sqrt{1-u^2}}\cdot\frac{1}{2} = \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\frac{1}{2} = \underline{\frac{1}{\sqrt{4-x^2}}}$$

(7) $\left(\log(\underbrace{\boxed{x^2-3x+2}}_{u})\right)' \quad (x>2)$

$$= \frac{d}{du}\log u \ \frac{d}{dx}(x^2-3x+2) = \frac{1}{u}\cdot 2x-3 = \underline{\frac{2x-3}{x^2-3x+2}}$$

問 1.21

$$\sinh x = \frac{e^x-e^{-x}}{2} \Rightarrow (\sinh x)' = \frac{e^x+e^{-x}}{2} = \cosh x$$

$$\cosh x = \frac{e^x+e^{-x}}{2} \Rightarrow (\cosh x)' = \frac{e^x-e^{-x}}{2} = \sinh x$$

$$\tanh x = \frac{\sinh x}{\cosh x} \Rightarrow (\tanh x)' = \frac{\cosh x\cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

例1.24.

$$y = \sqrt{x^2+1}\ \sqrt[3]{x^3+1}$$ $x \neq -1$ のとき y' は?

[解.] そのまま微分すると

$$y' = \frac{1}{\cancel{2}}(x^2+1)^{-\frac{1}{2}}\cancel{2}x\ \sqrt[3]{x^3+1} + \sqrt{x^2+1}\ \frac{1}{\cancel{3}}(x^3+1)^{-\frac{2}{3}}\cancel{3}x^2$$

$$= \frac{x}{\sqrt{x^2+1}}\ \sqrt[3]{x^3+1} + \sqrt{x^2+1}\ \frac{x^2}{\sqrt[3]{(x^3+1)^2}}$$

$$= \frac{x}{x^2+1}\sqrt{x^2+1}\ \sqrt[3]{x^3+1} + \frac{x^2}{x^3+1}\sqrt{x^2+1}\ \sqrt[3]{x^3+1}$$

$$= \left(\frac{x}{x^2+1} + \frac{x^2}{x^3+1}\right)\sqrt{x^2+1}\ \sqrt[3]{x^3+1}$$

[解]

$$\log|y| = \log\left(\sqrt{x^2+1}\ \sqrt[3]{x^3+1}\right)$$

$$= \frac{1}{2}\log(x^2+1) + \frac{1}{3}\log|x^3+1|$$

両辺をxで微分すると

$$\frac{y'}{y} = \frac{1}{\cancel{2}}\frac{\cancel{2}x}{x^2+1} + \frac{1}{\cancel{3}}\frac{\cancel{3}x^2}{x^3+1}$$

$$y' = \left(\frac{x}{x^2+1} + \frac{x^2}{x^3+1}\right)y$$

$$= \left(\frac{x}{x^2+1} + \frac{x^2}{x^3+1}\right)\sqrt{x^2+1}\ \sqrt[3]{x^3+1}$$

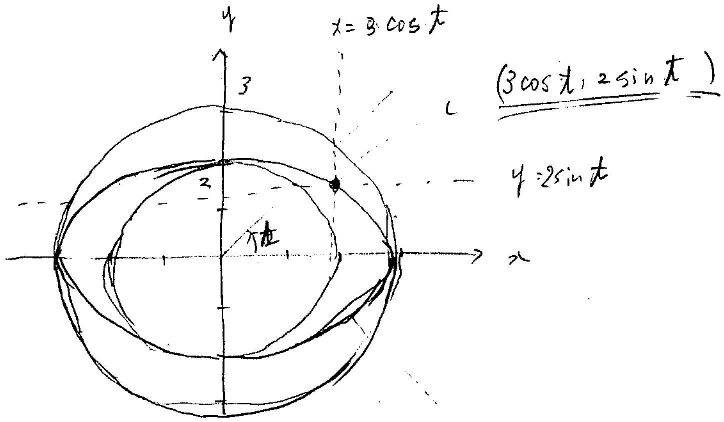
例 1.25

$$\begin{cases} x = 3\cos t \\ y = 2\sin t \end{cases}$$ はどんな曲線？　$t=\pi/3$ における接線の方程式は？

[解]

$\frac{x}{3} = \cos t$, $\frac{y}{2} = \sin t$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = (\cos t)^2 + (\sin t)^2 = 1 \quad \cdots\cdots \text{だ円の方程式}$$



$t=\pi/3$ における接線は？

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t}{-3\sin t}$$

$$\left.\frac{dy}{dx}\right|_{t=\pi/3} = \left.-\frac{2\cos t}{3\sin t}\right|_{t=\pi/3} = -\frac{2}{3}\,\frac{1}{2}\,\frac{1}{\frac{\sqrt{3}}{2}} = -\frac{2}{3\sqrt{3}}$$

よって $\left.(3\cos t, 2\sin t)\right|_{t=\pi/3} = \left(3\frac{1}{2}, 2\cdot\frac{\sqrt{3}}{2}\right) = \left(\frac{3}{2}, \sqrt{3}\right)$ を通る傾き $-\frac{2}{3\sqrt{3}}$

の直線の方程式

$$y - \sqrt{3} = -\frac{2}{3\sqrt{3}}\left(x - \frac{3}{2}\right)$$

が接線の方程式である

問 1.22

(1) $(\tan x)^{\sin x} = \left(e^{\log \tan x}\right)^{\sin x} = e^{\sin x \log \tan x}$ より （$0<x<\frac{\pi}{2}$ より $\tan x>0$）に注意.

$$(与式)' = \underbrace{e^{\sin x \log \tan x}} \left(\sin x \log \tan x\right)'$$

$$= \;〃\; \left(\cos \log \tan x + \sin x \frac{1}{\tan x}\cdot\frac{1}{\cos^2 x}\right)$$

$$= \underline{(\tan x)^{\sin x}\left(\cos x \log \tan x + \frac{1}{\cos x}\right)}\;//$$

(2) $x^{\tan^{-1}x} = \left(e^{\log x}\right)^{\tan^{-1}x} = e^{\tan^{-1}x \log x}$ より

$$(与式)' = \underbrace{e^{\tan^{-1}x \log x}} \left(\tan^{-1}x + \log x\right)'$$

$$= \;〃\; \left(\frac{1}{1+x^2} + \frac{1}{x}\right)$$

$$= \underline{x^{\tan^{-1}x}\left(\frac{1}{1+x^2} + \frac{1}{x}\right)}\;//$$

(3) $e^{\overset{u}{\boxed{x^x}}}$ $(x>0)$

$$(与式)' = \frac{de^u}{du}\frac{du}{dx} = e^u \boxed{\frac{dx^x}{dx}} = \underline{e^{x^x} x^x(\log x + 1)}$$

$$\left(\boxed{} = \left(e^{x\log x}\right)' = e^{x\log x}(x\log x)' = x^x(\log x + 1)\right)$$

(4) $y = x^2\sqrt{\frac{1-x^2}{1+x^2}}$ $(-1<x<1,\ x\neq 0)$

$$\log y = 2x + \frac{1}{2}\log(1-x^2) - \frac{1}{2}\log(1+x^2)$$

$$\frac{y'}{y} = 2 + \frac{1}{2}\frac{-2x}{1-x^2} - \frac{1}{2}\frac{2x}{1+x^2} = 2 - \frac{x}{1-x^2} - \frac{x}{1+x^2}$$

$$y' = \underline{x^2\sqrt{\frac{1-x^2}{1+x^2}}\cdot\left(2 - \frac{x}{1-x^2} - \frac{x}{1+x^2}\right)} = \underline{x^2\sqrt{〃}\left(2 - x\frac{2}{(1-x^2)(1+x^2)}\right)}$$

or.

問 1.23

(1) $\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1-\cos\theta) \end{cases}$ $(a>0)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1-\cos\theta)} = \underline{\frac{\sin\theta}{1-\cos\theta}}_{\,\,//}$$

or

$$= \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \underline{\cot\frac{\theta}{2}}_{\,\,//}$$

(2) $\begin{cases} x = 2t^2 \\ y = 4t \end{cases}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4}{4t} = \underline{\frac{1}{t}}$$

2　平均値の定理と導関数の応用

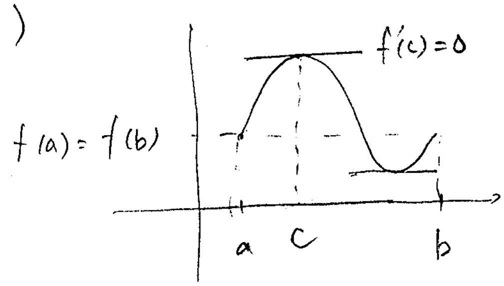
2.1　平均値の定理

定理 2.1（ロルの定理）

$f(x)$ が $[a, b]$ で連続，(a, b) で微分可能で $f(a)=f(b)$

$\Rightarrow$　$f'(c)=0$ となる c が $a<c<b$ に少なくとも1つ存在する。

（説明）



（証明）　(a, b) の最大値又は最小値をとる場合を考える。

（一定値のときは，常に $f(a)=f(b)=f(c)$，$f'(c)=0$）

一般性をうしなうことなく $f(x)$ は最大値を $x=c$ で持つとする。

（最小値のみ持つときは，$-f(x)$ を考えればよい）

$$f'_+(c) = \lim_{h\to+0}\frac{f(x+h)-f(x)}{h} \leqq 0 \qquad f(x+h)-f(x)<0 \text{ より}$$

$$f'_-(c) = \lim_{h\to-0}\frac{f(x+h)-f(x)}{h} \geqq 0$$

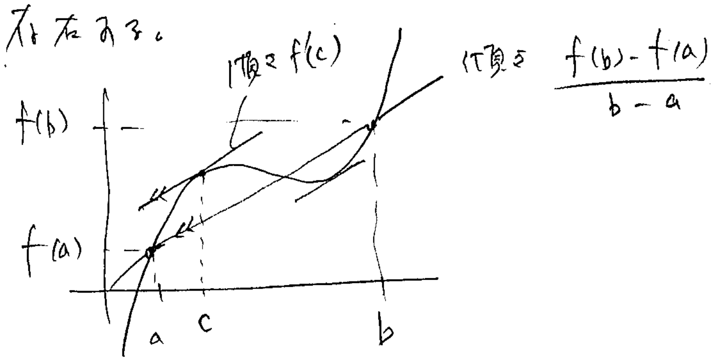
$f'(c) = f'_+(c) = f'_-(c)$ より　$f'(c)=0$ をみたす □

定理 2.2 (平均値の定理)

$f(x)$ が $[a,b]$ 連続, (a,b) で微分可能

$\Rightarrow$ $\dfrac{f(b)-f(a)}{b-a} = f'(c)$ をみたす c が $a<c<b$ に少なくとも1つ存在する。

(証明)



$$\frac{f(b)-f(a)}{b-a} = k \quad \text{とおく.}$$

$$f(b)-f(a)-k(b-a)=0 \quad \cdots \quad (*)$$

左辺で $a \to x$ にした式を

$$F(x) = f(b)-f(x)-k(b-x) \quad \text{とおく}$$

$$F(b) = f(b)-f(b)-k(b-b)=0$$

$$F(a) = f(b)-f(a)-k(b-a)=0 \quad \leftarrow (*) \text{ より}$$

$$F'(x) = -f'(x)+k$$

ロルの定理より $a<c<b$ のあるcにおいて

$$F'(c)=0$$

$$\Rightarrow \quad +f'(c) = k = \frac{f(b)-f(a)}{b-a}$$

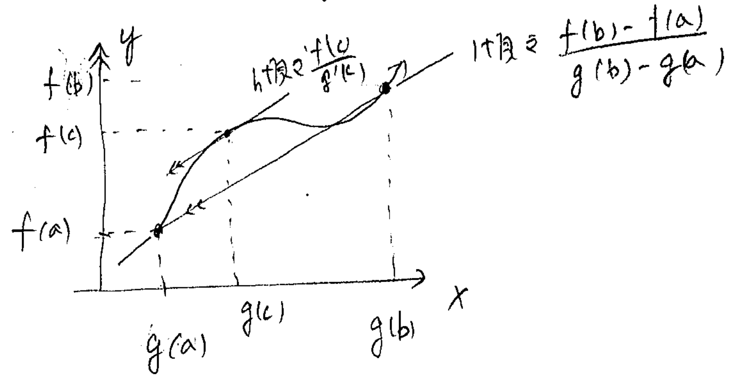
☒

定理 2.3 (コーシーの平均値の定理)

$f(x), g(x)$ が $[a,b]$ で連続, (a,b) で微分可能.

$$\Rightarrow \quad \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \qquad \text{ただし } g'(c) \neq 0$$

(証)



$$\frac{f(b)-f(a)}{g(b)-g(a)} = k$$

$$f(b)-f(a)-k\{g(b)-g(a)\} = 0$$

左辺で $a \to x$ とした式を $F(x)$ とおく

$$F(x) = f(b)-f(x)-k\{g(b)-g(x)\}$$

$$F(b) = 0, \quad F(a) = 0$$

$$F'(x) = -f'(x) + k\,g'(x)$$

ロルの定理より

$F'(c) = -f'(c) + k\,g'(c) = 0$ となる c が $a<c<b$ に存在する.

よって $g'(c) \neq 0$ のとき

$$\frac{f'(c)}{g'(c)} = k = \frac{f(b)-f(a)}{g(b)-g(a)} \quad \boxtimes$$

(注) $g(x) = x$ とおけば

$$g'(x) = 1. \quad \Rightarrow \quad f'(c) = \frac{f(b)-f(a)}{b-a}$$

平均値の定理となる

例2.1

$f(x)$ が $[a,b]$ で連続，(a,b) で $f'(x)=0$ ならば

$f(x)$ は $[a,b]$ で定数.

∵ $\dfrac{f(x)-f(a)}{x-a} = f'(c) = 0 \qquad a<c<x \leqq b$

よって $f(x)=f(a)$ for $a<x\leqq b$

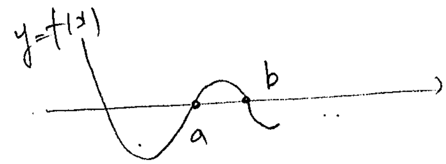
よって一定である.

例2.2

$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0 \quad (a_0 \neq 0)$

の隣りあわせの2実数解 a, b の間には

$f'(x)=0$ の実数解が少なくとも1つ存在

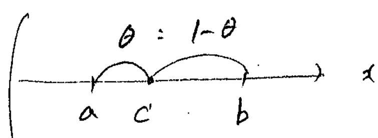


$f(x)$ は $[a,b]$ で連続 (a,b) で微分可能である.

よって $f'(c)=0$ となる c が $a<c<b$ に少なくとも1つ存在する

追記 2.1 $a<c<b$ のような c $\Leftrightarrow$ $0<\theta<1$，$c=a+\theta(b-a)$

とも表すことが出来る



c は a, b を $\theta : 1-\theta$ に内分する点.

$\dfrac{f(b)-f(a)}{b-a} = f'(a+\theta(b-a))$ となる θ が $0<\theta<1$ に少なくとも1つ存在

$\underset{b=a+h}{\Longrightarrow}$ $f(a+h) = f(a) + hf'(a+\theta h)$ をみたす $\underbrace{\theta}_{(0<\theta<1)}$ が少なくとも1つ存在

問2.1

$f(x)=e^x$ のとき $[0,1]$ で存在する c を求めよ.

$$\frac{f(1)-f(0)}{1} = f'(c)$$

より $e-1 = e^c$ $\underline{c = \log(e-1)}$.

問2.1

(1) $f(x)=x^2-2x$ の区間 $[-1,2]$ での平均値の定理の c と θ は?

$$\frac{f(2)-f(-1)}{3} = f'(c)$$

$f(2)=4-4=0$, $f(-1)=1+2=3$

$f'(x)=2x-2$,

$f'(c)=2c-2$

$2c-2 = \dfrac{-3}{3} = -1$

$2c=1$, $\underline{c=\frac{1}{2}}$ //

$c = a+\theta(b-a) = -1+\theta\cdot 3 = \frac{1}{2}$

$3\theta = \frac{3}{2}$, $\underline{\theta = \frac{1}{2}}$ //

(2) $f(x)=\sqrt{x}$ 区間 $[0,4]$

$f(0)=0$, $f(4)=2$

$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}$

$\frac{1}{2}c^{-\frac{1}{2}} = \dfrac{2-0}{4-0} = \frac{1}{2}$

$c^{-\frac{1}{2}}=1$. $\underline{c=1}$

$c = a+\theta(b-a) =$ $4\theta=1$ $\underline{\theta=\frac{1}{4}}$

2.2 テイラーの定理.

◇ 高次導関数

$$y = f(x) \xrightarrow{\text{微分}} f'(x) \xrightarrow{\text{微分}} f''(x) \rightarrow \cdots \rightarrow f^{(n)}(x)$$

$f''(x)$	$f^{(n)}(x)$
$\dfrac{d^2}{dx^2}f(x)$	$\dfrac{d^n f(x)}{dx^n}$
y''	$y^{(n)}$
$\dfrac{d^2 y}{dx^2}$	$\dfrac{d^n y}{dx^n}$
$\Uparrow$	$\Uparrow$
第2次導関数	第n次導関数

◇(1) $(e^x)^{(n)} = e^x$

∵ $y = e^x \rightarrow y' = (e^x)' = e^x \rightarrow y'' = (e^x)' = e^x \rightarrow \cdots \quad y^{(n)} = e^x$

(2) $\{\log(1+x)\}^{(n)} = (-1)^{n-1}\dfrac{(n-1)!}{(1+x)^n}$

∵ $y = \log(1+x)$

$y' = \dfrac{1}{1+x} = (1+x)^{-1}$

$y'' = (-1)(1+x)^{-2}$

$y''' = (-1)(-2)(1+x)^{-3}$

$\vdots$

$y^{(n)} = (-1)^{n-1}(n-1)!\,(1+x)^{-n}$

(3) $(\sin x)^{(n)} = \sin\left(x + \dfrac{\pi}{2}\right)$

∵ $y = \sin x$

$y' = \cos x = \sin\left(x + \dfrac{\pi}{2}\right)$

$y'' = \left(\sin\left(x + \dfrac{\pi}{2}\right)\right)' = \cos\left(x + \dfrac{\pi}{2}\right) = \sin\left(x + \dfrac{\pi}{2} \times 2\right)$

$\vdots$

$y^{(n)} = \sin\left(x + \dfrac{\pi}{2}n\right)$

(4) $(\cos x)^{(n)} = \cos\left(x + \dfrac{\pi}{2}\cdot n\right)$

∵ $y = \cos x$

$y' = -\sin x = \cos\left(x + \dfrac{\pi}{2}\right)$

$y'' = -\sin\left(x + \dfrac{\pi}{2}\right) = \cos\left(x + \dfrac{\pi}{2}\cdot 2\right)$

$\vdots$

$y^{(n)} = \cos\left(x + \dfrac{\pi}{2} \cdots\right)$

(5) $\{(1+x)^\alpha\}^{(n)} = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$

∵ $y = (1+x)^\alpha$

$y' = \alpha(1+x)^{\alpha-1}$

$y'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$

$\vdots$

$y^{(n)} = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$

定理 2-4.（ライプニッツの公式）

$u(x)$, $v(x)$ が n 回微分可能

⇒ uv も n 回微分可能

$(u\cdot v)^{(n)} = u^{(n)}v^{(0)} + {}_nC_1 u^{(n-1)}v^{(1)} + {}_nC_2 u^{(n-2)}v^{(2)} + \cdots + u^{(0)}v^{(n)}$

∵ $(uv)' = u'v + uv'$

$(uv)'' = (u'v + uv')' = u''v + u'v' + u'v' + uv''$

$= u''v + 2u'v' + uv''$

$(uv)^k = u^{(k)}v^{(0)} + {}_nC_1 u^{(k-1)}v^{(1)} + \cdots + u^{(0)}v^{(k)}$ と仮定し

$n=k+1$ の式を示す

$(uv)^{k+1} = \left(u^{(k)}v^{(0)} + {}_nC_1 u^{(k-1)}v^{(1)} + \cdots + u^{(0)}v^{(k)}\right)'$

$= u^{(k+1)}v^{(0)} + \underbrace{u^{(k)}v^{(1)} + {}_nC_1 u^{(k)}v^{(1)}}_{\substack{{}_nC_0 + {}_nC_1 \\ \| \\ {}_{n+1}C_1}} + \underbrace{u^{k-1}v^2}_{{}_nC_1} + \underbrace{}_{{}_nC_2} \cdots + u^{(1)}v^{(k)} + u^{(0)}v^{(k+1)})$

メモ
$${}_nC_i + {}_nC_{i+1} = \frac{n!}{(n-i)!\,i!} + \frac{n!}{(n-i-1)!\,(i+1)!}$$

$$= \frac{n!}{(n-i-1)!\,i!}\left(\frac{1}{n-i} + \frac{1}{i+1}\right) = \frac{(n+1)!}{(n-i)!\,(i+1)!} = {}_{n+1}C_{i+1}$$

$$\frac{\cancel{i}+1+n-\cancel{i}}{(n-i)(i+1)}$$

$= u^{(k+1)}v^{(0)} + {}_{n+1}C_1 u^{(k)}v^{(1)} + {}_{n+1}C_2 u^{(k-1)}v^{(2)} + \cdots + u^{(0)}v^{(k+1)})$

これは $n=k+1$ の式である ∎

定理2.5 （テイラーの定理）

$f(x), f^{(1)}(x), \ldots, f^{(n-1)}(x)$ が $[a,b]$ で連続

$f^{(n)}$ が (a,b) で存在

$$\Rightarrow f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a) + \frac{f'''(a)}{3!}(b-a)^3 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + \underbrace{\boxed{\frac{f^{(n)}(c)}{n!}(b-a)^n}}_{=R_n}$$

$a<c<b$ をみたす c が少なくとも1つ存在.

R_n ラグランジュの剰余項

∵ k を

$$f(b) = f(a) + f'(a)(b-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + k(b-a)^n$$

とおく.

$$F(x) = f(b) - \left\{ f(x) + f'(x)(b-x) + \cdots + \frac{f^{(n-1)}(x)}{(n-1)!}(b-x)^{n-1} + k(b-x)^n \right\}$$

$$F(b) = f(b) - f(b) = 0$$

$$F(a) = f(b) - \left\{ f(a) + f'(a)(b-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + k(b-a)^n \right\} = 0$$

$$F'(x) = -\left\{ \cancel{f'(x)} + f''(x)\cancel{(b-x)} + \cancel{f'(x)(-1)} + \cdots + \frac{f^{(n)}(x)}{(n-1)!}(b-x)^{n-1} + \frac{f^{(n-1)}(x)}{(n-1)!}\cancel{(b-x)^{n-2}}(-1) + kn(b-x)^{n-1}(-1) \right\}$$

$F'(c) = 0$ となる c が $a<c<b$ に存在.

$$\Rightarrow \frac{f^{(n)}(c)}{(n-1)!}(b-c)^{n-1} = nk(b-c)^{n-1} \longrightarrow k = \frac{f^{(n)}(c)}{n!}$$

よって

$$f(b) = f(a) + f'(a)(b-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + \frac{f^{(n)}(c)}{n!}(b-a)^n$$

図

（追記）

$n=1$ とすると

$f(b) = f(a) + f'(c)(b-a)$ 平均値の定理となる

追記 2.3. 一般に

$m=1,2,\cdots n$ のどれか.

$$f(b)=f(a)+f'(a)(b-a)+\cdots+\frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1}+k(b-a)^{m}$$

として ロルの定理用いると

$$F(x)=f(b)-\left\{f(x)+f'(x)(b-x)+\cdots+\frac{f^{(n-1)}(x)}{(n-1)!}(b-x)^{n-1}+k(b-x)^m\right\}$$

$$F(b)=0,\quad F(a)=0$$

$$F'(x)=-\left\{\frac{f^{(n)}(x)}{(n-1)!}(b-x)^{n-1}+km(b-x)^{m-1}(-1)\right\}$$

$$F'(c)=0 \Rightarrow m(b-c)^{m-1}k=\frac{f^{(n)}(c)}{(n-1)!}(b-c)^{n-1}$$

$$k=\frac{f^{(n)}(c)}{m(n-1)!}(b-c)^{n-m}$$

$m=1$ のとき.

$$f(b)=f(a)+f'(a)(b-a)+\cdots+\frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1}+\boxed{\frac{f^{(n)}(c)}{(n-1)!}(b-c)^{n-1}(b-a)}$$

コーシーの剰余項

$\vdots$

$m=n$ のとき

$$f(b)=f(a)+\cdots+\boxed{\frac{f^{(n)}(c)}{n!}(b-a)^{n}}$$

ラグランジュの剰余項

例 2.3.

$y=x^3\sin x$ の第n次導関数は?

(解) $(x^3)'=3x^2$, $(x^3)''=6x$, $(x^3)'''=6$, $(x^3)^{(4)}=0$

ライプニッツの公式より

$$y^{(n)} = \underbrace{(\sin x)^{(n)}}_{\sin(x+\frac{\pi}{2}n)} x^3 + {}_nC_1\underbrace{(\sin x)^{(n-1)}}_{\sin(x+\frac{n-1}{2}\pi)}\overbrace{(x^3)'}^{3x^2} + {}_nC_2\underbrace{(\sin x)^{(n-2)}}_{\sin(x+\frac{n-2}{2}\pi)}\overbrace{(x^3)''}^{6x} + {}_nC_3\underbrace{(\sin x)^{(n-3)}}_{\sin(x+\frac{n-3}{2}\pi)}\overbrace{(x^3)'''}^{6}$$

$$= x^3\sin\left(x+\frac{n}{2}\pi\right) + n\,3x^2\sin\left(x+\frac{n-1}{2}\pi\right)$$

$$+ \frac{n(n-1)}{\cancel{2}}\,\cancel{6}^{\,3}x\,\sin\left(x+\frac{n-2}{2}\pi\right)$$

$$+ \frac{n(n-1)(n-2)}{\cancel{3!}}\,\cancel{6}\,\sin\left(x+\frac{n-3}{2}\pi\right)$$

例 2.4

$f(x)$ が a を含む区間で2回微分可能で $f''(x)$ が連続のとき

$$\lim_{h\to 0}\frac{f(a+h)+f(a-h)-2f(a)}{h^2} = f''(a)$$

X

(∵) $f(a+h)-f(a) = f'(c_1)h$ 　　 $c_1 = a+\theta_1 h$ 　 $0<\theta_1<1$

$f(a-h)-f(a) = f'(c_2)(-h)$ 　　 $c_2 = a-\theta_2 h$ 　 $0<\theta_2<1$

$$f(a+h)+f(a-h)-2f(a) = f'(c_1)h - f'(c_2)h$$

$$\lim_{h\to 0}\frac{f(a+h)+f(a-h)-2f(a)}{h^2} = \lim_{h\to 0}\frac{f'(c_1)-f'(c_2)}{h}$$

$f(a+h) = f(a) + f'(a)h + \frac{f''(c_1)}{2}h^2$ 　　 $c_1 = a+\theta_1\times h$ 　 $0<\theta_1<1$

$f(a-h) = f(a) + f'(a)(-h) + \frac{f''(c_2)}{2}h^2$ 　　 $c_2 = a+\theta_2(-h)$ 　 $0<\theta_2<1$

$$\text{左辺} = \lim_{h\to 0}\frac{\frac{f''(c_1)}{2}h^2+\frac{f''(c_2)}{2}h^2}{h^2} = \lim_{\substack{h\to 0\\ \left(\substack{c_1\to a\\ c_2\to a}\right)}}\frac{f''(c_1)+f''(c_2)}{2} = \frac{f''(a)+f''(a)}{2} = \underline{f''(a)}$$

($f''(x)$ が連続なので)

問 2.3

$y=(x^2+x+1)e^x$ の $y^{(n)}$ は?

$v=x^2+x+1$, $v^{(1)}=2x+1$, $v^{(2)}=2$, $v^{(3)}=0$ ..

$u=e^x$, $u^{(n)}=e^x$

$$y^{n}= e^x(x^2+x+1)+n\,e^x(2x+1)+\frac{n(n-1)}{2}e^x\cdot 2$$

$$= e^x\{x^2+x+1+n(2x+1)+n(n-1)\}$$

問 2.4

$y=e^x\cos x$ のとき. $y^{(n)}=2^{n/2}e^x\cos\left(x+\frac{n\pi}{4}\right)$ を示せ.

$y^{(0)}=2^0.\ e^x\cos x = y$ より $n=0$ は OK.

$n=k$ で成立とすると

$$y^{(k+1)}=\left(2^{k/2}e^x\cos\left(x+\frac{k}{4}\pi\right)\right)'$$

$$=2^{k/2}\left\{(e^x)'\cos\left(x+\frac{k}{4}\pi\right)+e^x\left(\cos\left(x+\frac{k}{4}\pi\right)\right)'\right\}$$

$$=2^{k/2}\left\{e^x\cos\left(x+\frac{k}{4}\pi\right)-e^x\sin\left(x+\frac{k}{4}\pi\right)\right\}$$

$$=2^{k/2}e^x\sqrt{2}\left\{\underbrace{\cos\left(x+\frac{k}{4}\pi\right)\frac{1}{\sqrt{2}}-\sin\left(x+\frac{k}{4}\pi\right)\frac{1}{\sqrt{2}}}_{\cos\left(x+\frac{k\pi}{4}+\frac{\pi}{4}\right)}\right\}$$

$$=2^{\frac{k+1}{2}}e^x\cos\left(x+\frac{k+1}{4}\pi\right)$$

$n=k+1$ でも成立 よって示せた。

2.3. マクローリンの定理，関数の多項式近似

定理2-6 $f(x)$ が $x=0$ を含む区間で，n 回微分可能

$$\Rightarrow f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}+\underbrace{\boxed{\frac{f^{(n)}(c)}{n!}x^n}}_{= R_n}$$

c は 0 と x の間の定数．$c=\theta x\ (0<\theta<1)$

∵ テイラーの定理で $a=0$，$b=x$ とおいたもの．

◇ 関数の多項式近似　n が大きいとき $|R_n|\ll 1$ となるとき．

$$f(x)\fallingdotseq f(0)+f'(0)x+\cdots+\frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}$$

◇ (1) $e^x=1+x+\dfrac{x^2}{2!}+\cdots+\dfrac{x^{n-1}}{(n-1)!}+\boxed{\dfrac{e^{\theta x}}{n!}x^n}=R_n$

∵ $e^x=e^0+e^0x+\dfrac{e^0}{2!}x^2+\cdots+\dfrac{e^0}{(n-1)!}x^{n-1}+R_n$ ☒

(2) $\sin x\fallingdotseq x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\cdots+(-)^{m-1}\dfrac{x^{2m-1}}{(2m-1)!}+0+\overbrace{\boxed{\dfrac{\sin(\theta x+\frac{2m+1}{2}\pi)}{(2m+1)!}x^{2m+1}}}^{R_{2m+1}}$

∵ $(\sin x)^{(n)}\Big|_{x=0}=\sin(x+\frac{n}{2}\pi)\Big|_{x=0}=\sin\left(\frac{n}{2}\pi\right)=0,1,0,-1$ for $0+4m$, $1+4m$, $2+4m$, $3+4m$

(3) $\cos x\fallingdotseq 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\cdots+(-)^{m-2}\dfrac{x^{2m-2}}{(2m-2)!}+0+\underbrace{\boxed{\dfrac{\cos(\theta x+\frac{2m}{2}\pi)}{(2m)!}x^{2m}}}_{R_{2m}}$

∵ $(\cos x)^{(n)}=\cos(x+\frac{n}{2}\pi)$ より

(4) $\log(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\cdots+(-)^{n-2}\dfrac{x^{n-1}}{n-1}+\boxed{\dfrac{x^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(\theta x)}$ コーシー剰余項

∵ $(\log(1+x))^{(n)}=(-)^{n-1}\dfrac{(n-1)!}{(1+x)^n}$

$$=\frac{x^n}{\cancel{(n-1)!}}(1-\theta)^{n-1}(-)^{n-1}\frac{\cancel{(n-1)!}}{(1+\theta x)^n}$$

$$=\boxed{\frac{(\theta-1)^{n-1}}{}\left(\frac{x}{1+\theta x}\right)^n}=R_n$$

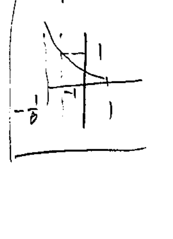
$-1<x\leqq 1$ のとき $R_n\to 0$?

(5) $(1+x)^\alpha=1+\dfrac{\alpha}{1!}x+\dfrac{\alpha(\alpha-1)}{2!}x^2+\cdots\dfrac{\alpha(\alpha-1)\cdots(\alpha-n+2)}{(n-1)!}x^{n-1}$

$$+\boxed{\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}(1+\theta x)^{\alpha-n}x^n}=R_n$$

∵ $\{(1+x)^\alpha\}^{(n)}=\alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$

$\dfrac{1-\theta}{1+\theta x}$



例 2.5
$\tan x$ の 2次マクローリン 3次まで（R_4 は不要）

$y = \tan x \longrightarrow y|_{x=0} = 0$

$y' = \dfrac{1}{\cos^2 x} \longrightarrow y'|_{x=0} = 1$

$y'' = +2\cos^{-3}x \sin x \longrightarrow y''|_{x=0} = 0$

$y''' = +2(-3)\cos^{-4}x \sin x \sin x + 2\cos^{-3}x \cos x \longrightarrow y'''|_{x=0} = 2.$

$\tan x = 0 + 1\cdot x + \dfrac{0}{2!}x^2 + \dfrac{2}{3!}x^3 + R_4$

$= x + \dfrac{x^3}{3} + R_4$

例 2.6.
(1) e^x の近似式 $e^x = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^9}{9!}$ で $x=1$ とおいて

e の近似値を求めよ.

$R_{10} = \dfrac{e^{\theta x}}{10!}x^{10} \rightarrow |R_{10}| = \dfrac{|x|^{10}}{10!}e^{\theta x} \leqq \dfrac{|x|^{10}}{10!}e^1 = \dfrac{e}{10!} < \dfrac{3}{10!} = 0.00000082$

$e^1 \fallingdotseq \underbrace{1+1+\frac{1}{2!}}_{2.5} + \underbrace{\frac{1}{3!}}_{0.166666} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{9!}$

0.041666　0.008333　0.001388　0.0001984　0.00000275

$= 2.7182818\cdots$

(2) $\log\dfrac{1+x}{1-x}$　$|x|<1$ の x の $2m-1$ 次近似式は？

与式 $= \log(1+x) - \log(1-x)$ なので

$\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} \cdots + \dfrac{x^{2m-1}}{(2m-1)}$

$-)\ \log(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} \cdots - \dfrac{x^{2m-1}}{(2m-1)}$

$\log(1+x) - \log(1-x) = 2\left(x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \cdots + \dfrac{x^{2m-1}}{2m-1}\right)$ //

問 2.5　$(1+x)^{1/3}$ の x の 2次の近似式は？

$(1+x)^{1/3} \fallingdotseq 1 + \dfrac{1}{3}x + \dfrac{1}{3}\left(-\dfrac{2}{3}\right)\dfrac{x^2}{2}$

$= 1 + x \quad x^2$

$\boxed{2.4}$ マクローリン級数

◇ $x=0$の近くで $f(x)$が何回でも微分可能なとき

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + R_n$$

において、 $R_n \to 0$ $(n \to \infty)$ ならば

$$f(x) = \underbrace{f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots}$$

この xの整級数を

マクローリン級数という

(1) $e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \cdots \quad (-\infty < x < \infty)$

∵ $|R_n| = \left|\frac{x^n e^{\theta x}}{n!}\right| = \frac{|x|^n e^{|x|}}{n!} \longrightarrow 0 \quad (-\infty < x < \infty)$

(2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad (-\infty < x < \infty)$

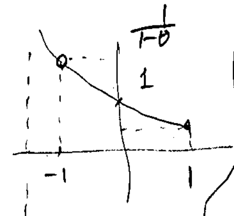
∵ $|R_{2m+1}| = \left|\frac{|x|^{2m+1}}{(2m+1)!}\sin(\quad)\right| \leqq \frac{|x|^{2m+1}}{(2m+1)!} \to 0$

(3) $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad (-\infty < x < \infty)$

∵ $|R_{2m}| = \left|\frac{x^{2m}}{(2m)!}\cos(\quad)\right| \leqq \frac{|x|^{2m}}{2m!} \to 0$

(4) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \quad (-1 < x \leqq 1)$

∵ $|R_n| = \left|(\theta-1)^{n-1}\left(\frac{x}{1+\theta x}\right)^n\right| \leqq \frac{|x|^n}{1+\theta x}\left|\frac{1-\theta}{1+\theta x}\right|^{n-1} < \frac{1}{1-\theta}\left|\frac{1-\theta}{1+\theta x}\right|^{n-1} \longrightarrow 0$



$\frac{1}{1+\theta} \leqq \frac{1}{1+\theta x} < \frac{1}{1-\theta}$

$0 < \frac{1-\theta}{1+\theta} \leqq \frac{1-\theta}{1+\theta x} < 1$ より

$\left|\frac{1-\theta}{1+\theta x}\right| < 1$

$|x| \leqq 1$

(5) $(1+x)^\alpha = 1+\alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots \qquad -1 < x \leqq 1$

(∵) $|R_n| = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}(1+\theta x)^{\alpha-n}x^n \quad \times$

コーシーの剰余項を用いると

$$R_n = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}x^n(1-\theta)^{n-1}(1+\theta x)^{\alpha-n}$$

$$|R_n| = \left|\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}x^n\right| \left|\frac{1-\theta}{1+\theta x}\right|^{n-1}(1+\theta x)^{\alpha-1}$$

有界

$0 < \frac{1-\theta}{1+\theta x} < 1$ より

$$\leqq \left|\underbrace{\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}}_{C_n}x^n\right| \left|\frac{1-\theta}{1+\theta x}\right|^{n-1}(1+\theta x)^{\alpha-1} \longrightarrow 0.$$

$$\left|\frac{C_{n+1}}{C_n}\right| = \left|\frac{\alpha-n}{n!}\cancel{(n-1)!}\,x\right| \longrightarrow |x| \leqq 1.$$

(4)と(5)は、

注 $|x|<1$ と $x=1$ に分けて示す方がすっきりする

$|x|<1$ → コーシー

$x=1$ → ラグランジュ.

注 2.2

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

において $x = i\theta$ $(i^2 = -1)$ とおくと

$$\underline{e^{i\theta}} = 1 + i\theta + \frac{-x^2}{2!} + \frac{-ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \cdots$$

$$= \underline{\cos\theta + i\sin\theta}$$ オイラーの関係式

補助定理

$$\lim_{n\to\infty} \underbrace{\frac{x^n}{n!}}_{=C_n} = 0 \qquad (-\infty < x < \infty)$$

$\because$ $$\frac{C_{n+1}}{C_n} = \frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n} = \frac{x}{n+1} \xrightarrow[(n\to\infty)]{} 0 \qquad (-\infty < x < \infty)$$

例 2.7 (済)

例 2.8 $f(x) = \sqrt{1+x}$ をマクローリン級数で表せ.

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{x^2}{2!} + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^3}{3!}$$

$$+ \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{x^4}{4!} + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{x^2}{4\cdot 2} + \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 + \cdots$$

問 2.6. 済

問 2.7.

$$f(x) = \frac{2}{1-2x} - \frac{1}{1-x}$$

$$= 2\left(1 + 2x + (2x)^2 + (2x)^3 + \cdots\right) \quad \cdots \quad \left(|x| < \tfrac{1}{2}\right)$$

$$- \left(1 + x + x^2 + x^3 + \cdots\right) \quad \cdots \quad \left(|x| < 1\right)$$

$$= \underline{(2-1) + (2^2-1)x + (2^3-1)x^2 + (2^4-1)x^3 + \cdots} \qquad \left(|x| < \tfrac{1}{2}\right)$$

$\boxed{2.5}$ 関数の増減，極値

定理2.7　$f(x)$ が $[a,b]$ で連続 (a,b) で微分可能.

(1) (a,b) で常に $f'(x)>0 \Rightarrow$

$a \leq x_1 < x_2 \leq b$

$$\frac{f(x_1)-f(x_2)}{x_1-x_2} = f'(c) > 0$$

$\uparrow$

$x_1 < c < x_2$

$f(x_1)-f(x_2) < 0$.

すなわち $[a,b]$ で増加関数.

(2) (a,b) で常に $f'(x)<0 \Rightarrow f(x_1)-f(x_2) = f'(c)\,(x_1-x_2) < 0$.

$f(x_1)-f(x_2) < 0$

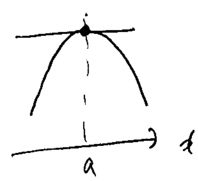
$[a,b]$ で減少関数

◇ 極値.

$x=a$ 近くで連続な関数 $f(x)$ が aの近くの $x\ (\neq a)$ に対して

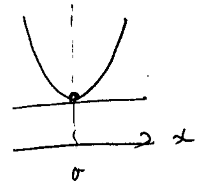
・ $f(x) < f(a)$ であるとき

$x=a$ で極大



・ $f(x) > f(a)$ であるとき

$x=a$ で極小



「極値」という

定理2.8　$f(x)$ が微分可能で $f(a)$ が極値 $\Rightarrow f'(a)=0$

∵ $x=a$で極大とすると　$f(x) < f(a)$

$$f_-(a) = \lim_{h\to -0}\frac{f(a+h)-f(a)}{h} \geq 0$$

$$f_+(a) = \lim_{h\to +0}\frac{f(a+h)-f(a)}{h} \leq 0$$

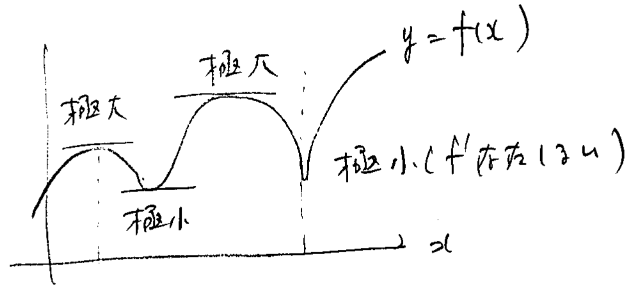
$f'(a) = f'_-(a) = f'_+(a)$ より

$f'(a)=0$.

$x=a$で極小の場合も同様 ☒

◇ 極値の求め方

$f'(x)=0$ or $f'(x)$ が存在しない x の値を求める



増減表を作ると

x		a	
$f'(x)$	$+$		$-$
$f(x)$	↗		↘

極大

x		a	
$f'(x)$	$-$		$+$
$f(x)$	↘		↗

極小

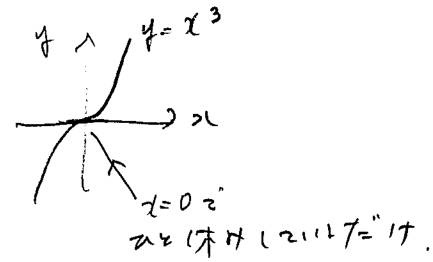
(注) 2.3.

$f'(a)=0$ だからといって極値である訳ではない。

ex $f(x)=x^3$ 　 $f'(x)=3x^2$ より $x=0$ で $f'(x)=0$ だが

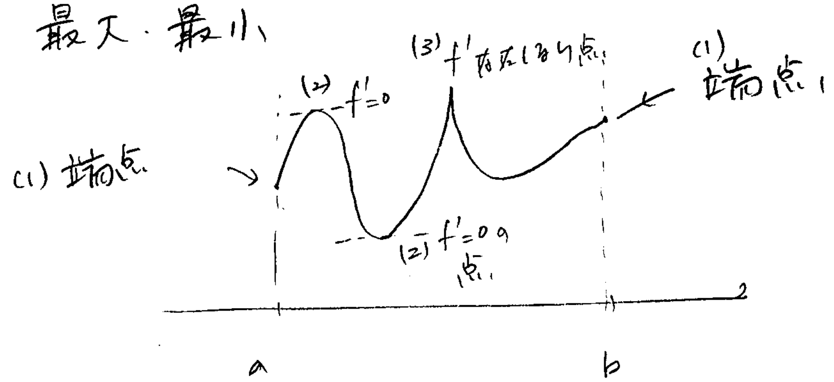
x		0	
$f'(x)$	$+$		$+$
$f(x)$	↗		↗

極値ではない



追記 2.5 　変域の端点は極値の対象にしないことが多い

◇ 最大・最小



(1)(2)(3) の候補の $f(x)$ の値を比較して求める

例 2.9.

$f(x)=(x-5)\sqrt[3]{x^2}$ の増減を調べ、極値を求めよ。

$$f'(x)=\sqrt[3]{x^2}+(x-5)\frac{2}{3}x^{-\frac{1}{3}}$$

$$=\frac{3x^{\frac{2}{3}}x^{\frac{1}{3}}+(x-5)2}{3\sqrt[3]{x}}=\frac{3x+2(x-5)}{3\sqrt[3]{x}}=\frac{5x-10}{〃}$$

$$=\frac{5(x-2)}{3\sqrt[3]{x}}$$

x		0		2	
$f'(x)$	+		−	0	+
$f(x)$	↗	0	↘	○	↗

$f(2)=-3\sqrt[3]{4}$

極大 $x=0$ のとき $f(0)=0$

極小 $x=2$ のとき $f(2)=-3\sqrt[3]{4}$

例 2.10

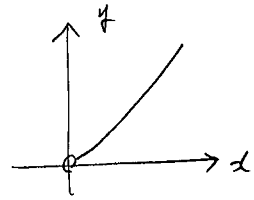
$\log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} \quad (x>0)$ を示せ.

解　$y = f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \log(1+x)$

$$f'(x) = 1 - x + x^2 - \frac{1}{1+x} = \frac{1+x^3-1}{1+x} = \frac{x^3}{1+x} > 0$$

よって

x	0	
$f'(x)$	0	$+$
$f(x)$	0	$\nearrow$



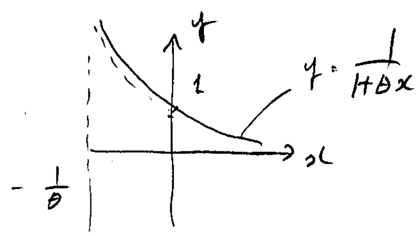
よって　$f(x) > 0 \quad (x>0)$

$$\Leftrightarrow \quad x - \frac{x^2}{2} + \frac{x^3}{3} > \log(1+x)$$

(別解)　$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + R_4$

$$R_4 = (\theta-1)^{n-1}\left(\frac{x}{1+\theta x}\right)^n \quad (n=4)$$

$$= (\theta-1)^3\left(\frac{x}{1+\theta x}\right)^4$$



$$\left.\begin{array}{l} 0 < \frac{1}{1+\theta x} < 1 \\ 0 < x < 1 \\ (\theta-1)^3 < 0 \end{array}\right\} \text{より} \quad R_4 < 0.$$

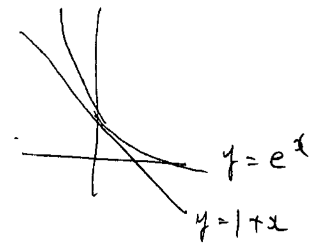
よって　$-R_4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \log(1+x) > 0$　☒

問 2.8

(1) $e^x > 1+x+\frac{x^2}{2}$ $(x>0)$ を示せ.

解) $f(x) = e^x - \left(1+x+\frac{x^2}{2}\right)$ $(x>0)$

$f'(x) = e^x - (1+x)$



x		0	
$f'(x)$	$+$	0	$+$
$f(x)$		0	↗

よって $x>0$ のとき $f(x)>0$

つまり $e^x > 1+x+\frac{x^2}{2}$

(2) $x \underset{①}{>} \sin x \underset{②}{>} x - \frac{x^3}{6}$ $\left(0<x<\frac{\pi}{2}\right)$

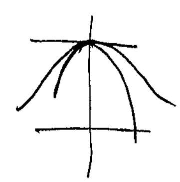
①より $f(x) = \sin x - x$

$f'(x) = \cos x - 1$

x	0		$\frac{\pi}{2}$
$f'(x)$	0	$-$	
$f(x)$	0	↘	

$f(x) < 0$ つまり

$\sin x < x$ … ①'



②より $g(x) = \sin x - \left(x - \frac{x^3}{6}\right)$

$g'(x) = \cos x - \left(1 - \frac{x^2}{2}\right)$

x	0		$\frac{\pi}{2}$
$g'(x)$	0	$+$	
$g(x)$	0	↗	

$g(x) > 0$ つまり $\sin x - \left(x - \frac{x^3}{6}\right) > 0$ … ②'

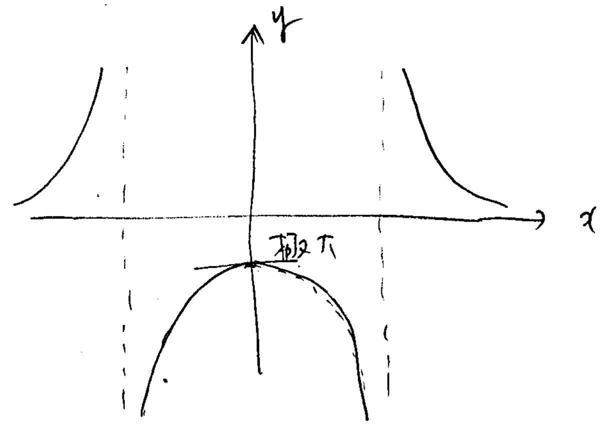
①'②'より 与式示せた。

問 29　グラフ

(1)　$f(x) = \dfrac{1}{x^2-1}$

$$f'(x) = \frac{-1}{(x^2-1)^2}\,2x$$

x		-1		0		1	
$f'(x)$	$+$	×	$+$	0	$-$	×	$-$
$f(x)$	↗		↗	-1	↘		↘



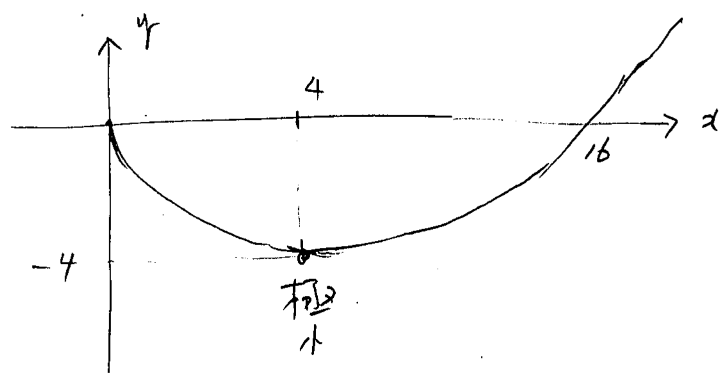
(2)　$f(x) = x - 4\sqrt{x}$　$(x \geqq 0)$

$$f'(x) = 1 - 4\cdot\frac{1}{2}\cdot x^{-\frac{1}{2}}$$

$$= 1 - \frac{2}{\sqrt{x}} = \frac{\sqrt{x}-2}{\sqrt{x}}$$

x	0		4	
$f'(x)$	×	$-$	0	$+$
$f(x)$	0	↘	-4	↗
			⇑ 極小	

$f(x) = \sqrt{x}(\sqrt{x}-4)$



(3) $f(x) = x\sqrt{2x - x^2} \qquad (0 \leqq x \leqq 2)$

$$f'(x) = \sqrt{2x - x^2} + x\frac{1}{2}(2x - x^2)^{-\frac{1}{2}}(2 - 2x)$$

$$= \sqrt{2x - x^2} + \frac{x(1-x)}{\sqrt{2x - x^2}}$$

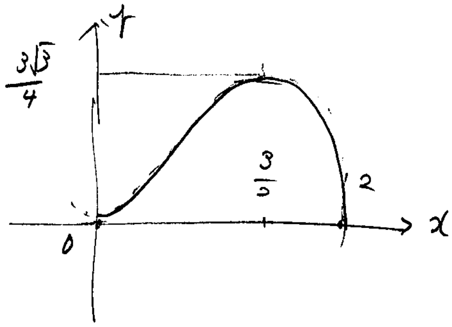
$$= \frac{2x - x^2 + x - x^2}{\sqrt{2x - x^2}} = \frac{3x - 2x^2}{\sqrt{2x - x^2}}$$

$$= \frac{-2x\left(x - \frac{3}{2}\right)}{\sqrt{2x - x^2}}$$

x	0		$\frac{3}{2}$		2
$f'(x)$	0	$+$	0	$-$	$\times$
$f(x)$	0	$\nearrow$	$\boxed{}$	$\searrow$	0

$$\frac{3}{2}\sqrt{3 - \frac{9}{4}}$$

$$= \frac{3}{2}\sqrt{\frac{3}{4}} = \frac{3\sqrt{3}}{4}$$



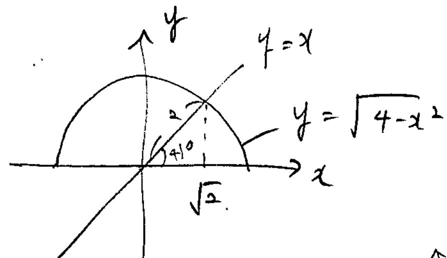
問 2.10.

$f(x) = x + \sqrt{4 - x^2} \qquad 0 \leqq x^2 \leqq 4$ より $\underline{-2 \leqq x \leqq 2}$

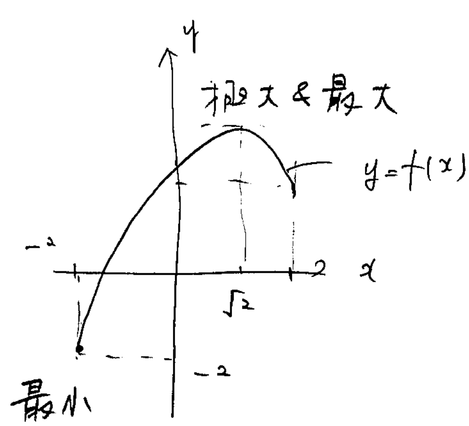
$$f'(x) = 1 + \frac{1}{2}(4 - x^2)^{-\frac{1}{2}}(-2x)$$

$$= 1 + \frac{-x}{\sqrt{4 - x^2}}$$

$$= \frac{\sqrt{4 - x^2} - x}{\sqrt{4 - x^2}} \longrightarrow \text{(分子)}$$

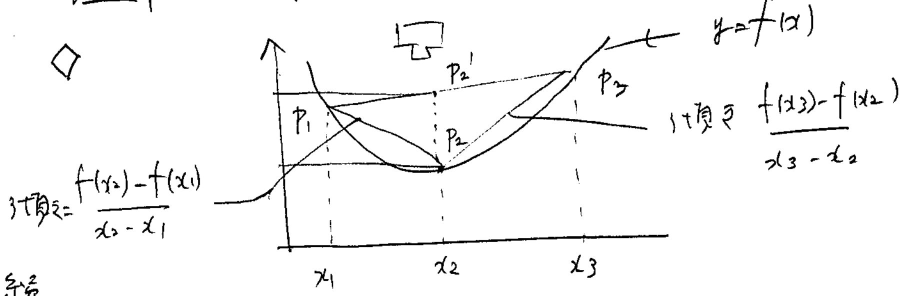


x	-2		$\sqrt{2}$		2
$f'(x)$	$\times$	$+$	0	$-$	$\times$
$f(x)$	-2	$\nearrow$	$\sqrt{2}+\sqrt{2} = 2\sqrt{2}$	$\searrow$	2



2.6 曲線の凹凸と変曲点

◇



$f(x)$ … 連続

区間 I に含まれる $\forall x_1, x_2, x_3$ $(x_1<x_2<x_3)$ に対して $\dfrac{f(x_2)-f(x_1)}{x_2-x_1} \leqq \dfrac{f(x_3)-f(x_2)}{x_3-x_2}$ ① が成立 $\Longleftrightarrow$ 「区間 I で下に凸」

〃 $\geqq$ 〃 $\Longleftrightarrow$ 「〃 I に凸」

① の図形的イミ.

$$\frac{f(x_2)}{x_2-x_1}+\frac{f(x_2)}{x_3-x_2} \leqq \frac{f(x_1)}{x_2-x_1}+\frac{f(x_3)}{x_3-x_2}$$

$$(x_3-\cancel{x_2}+\cancel{x_2}-x_1)f(x_2) \leqq (x_3-x_2)f(x_1)+(x_2-x_1)f(x_3)$$

$$f(x_2) \leqq \frac{(x_3-x_2)f(x_1)+(x_2-x_1)f(x_3)}{x_3-x_1}$$

$\Downarrow$

x_2P_2

P_1, P_3 を $x_2-x_1 : x_3-x_2$ に内分する点 P_2' の y 座標

〃

x_2P_2'

$\Rightarrow$ グラフの点が P_1P_3 の下にある.

さらに $x_2P_2 \leqq x_2P_2'$ より

$$P_1P_2\text{の傾き} \leqq \underbrace{P_1P_2'\text{の傾き} = P_2'P_3\text{の傾き}}_{P_1P_3\text{の傾き}} \leqq P_2P_3\text{の傾き}$$

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leqq \frac{f(x_3)-f(x_1)}{x_3-x_1} \leqq \frac{f(x_3)-f(x_2)}{x_3-x_2} \quad \cdots\cdots (*)$$

定理7.9

(1) $f(x)$が(a,b)で下に凸 $\Leftrightarrow$ $f''(x) \geqq 0$, $x \in (a,b)$

(2) 上に凸 $\Leftrightarrow$ $f''(x) \leqq 0$, $x \in (a,b)$

∵ (1)

$p_2 \to p_1$ より

$$f'(x_1) \leqq \frac{f(x_3)-f(x_1)}{x_3-x_1}$$

$p_2 \to p_3$ より

$$\frac{f(x_3)-f(x_1)}{x_3-x_1} \leqq f'(x_3)$$

$$\Bigg\} \quad f'(x_1) \leqq f'(x_3)$$

$x_1 < x_3$ のとき $f'(x_1) \leqq f'(x_3)$ なので $f'(x)$ は単調増加である.

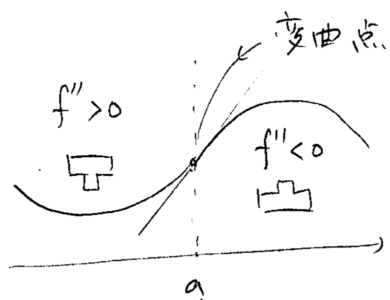
$f'(x)$ がIで微分可能であるとき ($f''(x)$ が存在するとき)

I上 $f'(x)$が単調増加 $\Leftrightarrow$ Iで $f''(x) \geqq 0$

ここらへんは平均値の定理の系で用いている

(2) 同様

◇ 変曲点



$f''(a)=0$

◇ グラフの描き方

(1)〜(7) 例題で練習

例 2.1)

・増減，凹凸，グラフ

(1) $y = e^{-x^2}$

$y' = e^{-x^2}(-2x)$

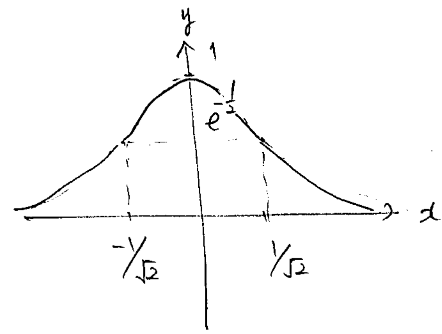
$y'' = e^{-x^2}(-2x)^2 + e^{-x^2}(-2)$

$= e^{-x^2}(4x^2 - 2)$

$= 4e^{-x^2}(x^2 - \frac{1}{2})$

x	$-\infty$		0		∞
y'		$+$	0	$-$	
y	0	$\nearrow$	1	$\searrow$	0

x		$\frac{-1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$	
y''	$+$	0	$-$	0	$+$
	$\cup$	$e^{-\frac{1}{2}}$	$\cap$	$e^{-\frac{1}{2}}$	$\cup$



(2) $y = \log(1+x^2)$

$y' = \frac{2x}{1+x^2}$

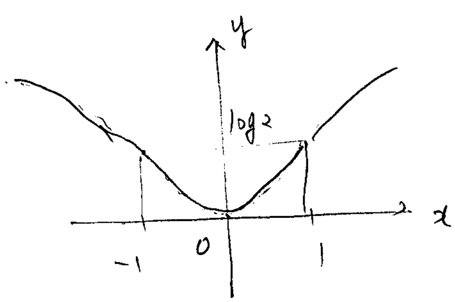
$y'' = \frac{2(1+x^2) - 2x(2x)}{(1+x^2)^2}$

$= \frac{2 - 2x^2}{(1+x^2)^2}$

$= \frac{2(1-x^2)}{(1+x^2)^2}$

x		0	
y'	$-$	0	$+$
y	$\searrow$	0	$\nearrow$

x		-1		1	
y''	$-$	0	$+$	0	$-$
y	$\cap$	$\log 2$	$\cup$	$\log 2$	$\cap$



問 2-11

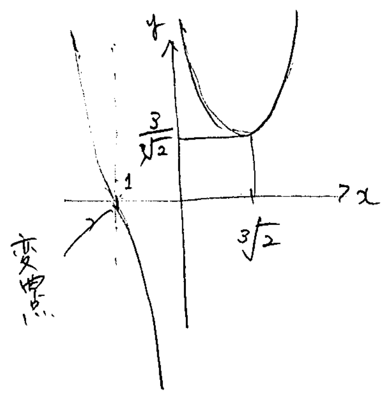
(1) $y = x^2 + \frac{1}{x} = \frac{x^3+1}{x}$

$y' = 2x - x^{-2} = \frac{2x^3-1}{x^2}$

$y'' = 2 + 2x^{-3} = \frac{2x^3+2}{x^3}$

x	0		$\frac{1}{\sqrt[3]{2}}$	
y'	×	$-$	0	$+$
y	↘	↘	$\frac{3}{\sqrt[3]{2}}$	↗

x		-1		0	
y''	$-$	0	$+$	×	$+$
y	∩	0	∪		∪



(2) $y = \frac{x}{x^2+1}$

$y' = \frac{x^2+1-x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$

$y'' = \frac{-2x}{(x^2+1)^2} + (-x^2+1)(-2)(x^2+1)^{-3}2x$

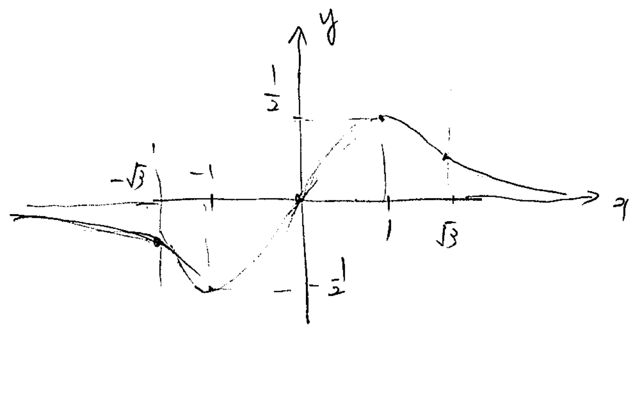
$= \frac{-2x(x^2+1) - 4x(-x^2+1)}{(x^2+1)^3}$

$= \frac{-2x\{x^2+1-2x^2+2\}}{〃}$

$= \frac{-2x(-x^2+3)}{〃}$

x	$-\infty$		-1		1		∞
y'		$-$	0	$+$	0	$-$	
y	0	↘	$-\frac{1}{2}$	↗	$\frac{1}{2}$	↘	0

x		$-\sqrt{3}$		0		$\sqrt{3}$	
y''	$-$	0	$+$	0	$-$	0	$+$
y	∩	$-\frac{\sqrt{3}}{4}$	∪	0	∩	$\frac{\sqrt{3}}{4}$	∪



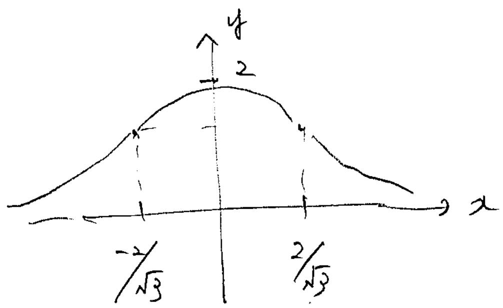
(3) $y = \dfrac{8}{x^2+4}$

$y' = 8 \cdot \dfrac{-2x}{(x^2+4)^2} = \dfrac{-16x}{(x^2+4)^2}$

$y'' = \dfrac{-16}{(x^2+4)^2} - 16x \dfrac{-2 \cdot 2x}{(x^2+4)^3}$

$= \dfrac{16}{(x^2+4)^3} \{ -(x^2+4) + 4x^2 \}$

$= \dfrac{+16}{(x^2+4)^3} (3x^2 - 4)$



x		0	
y'	+	0	−
y	↗	2	↘

x		$\frac{-2}{\sqrt{3}}$		$\frac{2}{\sqrt{3}}$	
y''	+	0	−	0	+
y	∪	$\frac{3}{2}$	∩	$\frac{3}{2}$	∪

$\dfrac{8}{\frac{4}{3}+4}$

$= \dfrac{2}{\left(\frac{1}{3}+1\right)} = \dfrac{4}{3}$

$= \dfrac{2 \cdot 3}{4_2}$

(4) $y = x^3 + 6x^2 + 9x + 2$

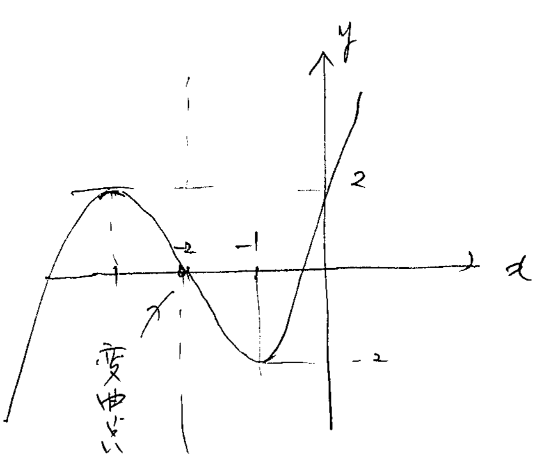
$y' = 3x^2 + 12x + 9$

$= 3(x^2 + 4x + 3)$

$= 3(x+3)(x+1)$

$y'' = 6x + 12$

$= 6(x+2)$



x		−3		−1	
y'	+	0	−	0	+
y	↗	$\boxed{\ }$	↘	+	↗

$-27 + 6 \cdot 9 - 27 + 2$

54

$= \boxed{2}$

$-1 + 6 - 9 + 2$

$= -2$

x		−2	
y''	−	0	+
y	∩	△	∪

$-8 + 24 - 18 + 2$

$= 0$

2.7 不定形とロピタルの定理

◇ 不定形

(例) $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$ ($\frac{0}{0}$ 不定形), $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x}$ ($\frac{0}{0}$ 不定形)

$\sin x \to 0$, $x \to 0$, $1-\cos x \to 0$, $x \to 0$

定理 2.10

$f(x)$, $g(x)$ は $x=a$ の近くで微分可能, $f(a)=g(a)=0$, $g'(x)\neq 0$ とする

もし $\displaystyle\lim_{x\to a}\frac{f'(x)}{g'(x)}=A$ が存在するなら $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=A$

A: 有限

(∵) コーシーの平均値の定理より

$x\neq a$ に対して

$$\frac{f(x)-\boxed{f(a)}}{g(x)-\boxed{g(a)}}=\frac{f'(c)}{g'(c)}$$

$f(a)=0$, $g(a)=0$

(c は a と x の間の実数) が成立.

$x\to a$ のとき $c\to a$ なので

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{c\to a}\frac{f'(c)}{g'(c)}=A. \quad \boxtimes$$

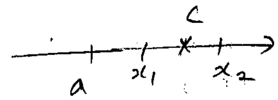
定理2.1

f(x), g(x) は $x=a$ の近くで $x=a$ を除いて微分可能

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty \text{ と仮定。} \lim_{x\to a}\frac{f'(x)}{g'(x)} = A \quad (-\infty \leqq A \leqq \infty)$$

が存在するなら $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = A.$

まず $x\to a+0$ の極限を考える.



$a < x_1 < x_2$ に対して

$$\frac{f(x_1)-f(x_2)}{g(x_1)-g(x_2)} = \frac{f'(c)}{g'(c)} \qquad (x_1 < c < x_2)$$

$$\frac{\dfrac{f(x_1)}{g(x_1)} - \dfrac{f(x_2)}{g(x_1)}}{1 - \dfrac{g(x_2)}{g(x_1)}} = \frac{f'(c)}{g'(c)}$$

$$\frac{f(x_1)}{g(x_1)} = \frac{f'(c)}{g'(c)}\left\{1 - \frac{g(x_2)}{g(x_1)}\right\} + \frac{f(x_2)}{g(x_1)}$$

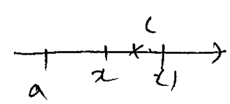
ここで $x_1 \to a+0$ とすると $g(x_1)\to\infty$ より

$$\lim_{x_1\to a+0}\frac{f(x_1)}{g(x_1)} = \underbrace{\lim_{c\to}\frac{f'(c)}{g'(c)}}_{?},$$

との論法使わないとツメが甘い　$x\to a+0$ で考える

まず $\displaystyle\lim_{x\to a+0}\frac{f'(x)}{g'(x)} = A \neq \pm\infty$ のときを考える.

$a < x < x_1$ に対して $\left|\dfrac{f'(x)}{g'(x)} - A\right| < \varepsilon.$ と出来る.



$$\frac{f(x)-f(x_1)}{g(x)-g(x_1)} = \frac{f'(c)}{g'(c)} \qquad \frac{\dfrac{f(x)}{g(x)} - \dfrac{f(x_1)}{g(x)}}{1 - \dfrac{g(x_1)}{g(x)}} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}\left\{1 - \frac{g(x_1)}{g(x)}\right\} + \frac{f(x_1)}{g(x)}$$

よって

$$\left|\frac{f(x)}{g(x)} - A\right| \leqq \left|\frac{f(x)}{g(x)} - \frac{f'(c)}{g'(c)}\right| + \left|\frac{f'(c)}{g'(c)} - A\right| \leqq \varepsilon + \varepsilon = 2\varepsilon.$$

これは $\displaystyle\lim_{x \to a+0} \frac{f(x)}{g(x)} = A$ を示す.

$\displaystyle\lim_{x \to a-0} \frac{f(x)}{g(x)} = A$ も同様

$A = \infty$ の場合は.

$\forall L,\quad a < {}^\forall x < {}^\exists x_1 \quad \dfrac{f'(x)}{g'(x)} > L$ と出来る.

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}\left\{1 - \frac{g(x_1)}{g(x)}\right\} + \frac{f(x_1)}{g(x)}$$

$a < x < \delta \quad 1 - \dfrac{g(x_1)}{g(x)} > \dfrac{1}{2}, \quad \dfrac{f(x_1)}{g(x)} > -\dfrac{1}{2}$ と出来る.

$$\frac{f(x)}{g(x)} > L\frac{1}{2} - \frac{1}{2} = \frac{L-1}{2}$$

これは $\displaystyle\lim_{x \to a+0} \frac{f(x)}{g(x)} = \infty$ を示している.

$A = -\infty$ の場合も同様

追記 2.6

(1) $0\cdot\infty$ の不定形のとき $f\cdot g = \dfrac{f}{\frac{1}{g}}$ ($\frac{0}{0}$ 不定形) $= \dfrac{g}{\frac{1}{f}}$ ($\frac{\infty}{\infty}$ 不定形)

(2) $\infty^0,\ 1^\infty,\ 0^0$ など指数の入った不定形形のときは 対数を考えるとよい

ex $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

解 $\displaystyle\log\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right) = \lim_{x\to\infty}\log\left(1+\frac{1}{x}\right)^x = \lim\frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$

$\displaystyle = \lim\frac{\frac{1}{1+\frac{1}{x}}\cdot\cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1.$

よって $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$

(3) $\infty-\infty$ 不定形のとき

$$f-g = \frac{\frac{1}{g}-\frac{1}{f}}{\frac{1}{fg}} \quad \left(\frac{0}{0}\text{ 不定形}\right)$$

注意 2.4

$\infty\cdot\infty=\infty$, $\infty+\infty=\infty$, $\dfrac{1}{+0}=\infty$, $\dfrac{1}{-0}=-\infty$, $\dfrac{1}{\infty}=0$

例2-12

(1) $\displaystyle \lim_{x\to 0}\frac{e^{x}-e^{-x}}{\sin x}$ （$\frac{0}{0}$ 不定形）

$$=\lim_{x\to 0}\frac{e^{x}+e^{-x}}{\cos x}=\frac{1+1}{1}=2$$

(2) $\displaystyle \lim_{x\to +0}\left(\frac{1}{x}\right)^{\sin x}$ の log を考える

$$\lim_{x\to +0}\log\left(\frac{1}{x}\right)^{\sin x}=\lim_{x\to +0}-\sin x\log x \quad (0\cdot\infty \text{ 不定形})$$

$$=\lim_{x\to +0}\frac{\log x}{-\frac{1}{\sin x}}=\lim\frac{\frac{1}{x}}{\frac{\cos x}{\sin^2 x}}=\lim_{x\to +0}\underbrace{\frac{\sin x}{x}}_{\to 1}\underbrace{\frac{\sin x}{\cos x}}_{\to 0}=0$$

よって $\displaystyle \lim_{x\to +0}\left(\frac{1}{x}\right)^{\sin x}=e^{0}=1$

(3) $\displaystyle \lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\log x}\right)$ （$\infty-\infty$ 不定形）

$$=\lim\frac{\log x-\frac{x-1}{x}}{\frac{1}{\frac{x}{(x-1)\log x}}}=\lim_{x\to 1}\frac{\log x-\frac{x-1}{x}}{\frac{x-1}{x}\log x} \quad \left(\frac{0}{0}\text{ 不定形}\right)$$

$$=\lim\frac{\frac{1}{x}-\frac{x-(x-1)}{x^2}}{\frac{\left(\log x+(x-1)\frac{1}{x}\right)x-(x-1)\log x}{x^2}}=\lim\frac{\frac{1}{x}-\frac{1}{x^2}}{\frac{\cancel{x\log x}+x-1-\cancel{x\log x}+\log x}{x^2}}$$

$$=\lim\frac{x-1}{x-1+\log x}=\lim_{x\to 1}\frac{1}{1+\frac{1}{x}}=\frac{1}{2}$$

問 2.12.

(1) $$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{\sin x}{2x} = \underline{\frac{1}{2}}_{\,//}$$

(2) $$\lim_{x\to 0}\frac{x-\log(1+x)}{x^2} = \lim_{x\to 0}\frac{1-\frac{1}{1+x}}{2x} = \lim_{x\to 0}\frac{\frac{x}{1+x}}{2x} = \lim_{x\to 0}\frac{\cancel{x}}{2\cancel{x}(1+x)} = \underline{\frac{1}{2}}_{\,//}$$

(3) $$\lim_{x\to \infty} x\left(e^{1/x}-1\right) \quad (\infty\cdot 0 \text{ 不定形})$$

$$= \lim_{x\to\infty}\frac{\left(e^{1/x}-1\right)'}{\left(\frac{1}{x}\right)'} = \lim_{x\to\infty}\frac{e^{1/x}\,\cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}} = e^0 = \underline{1}_{\,//}$$

(4) $$\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x$$

$$\lim_{x\to\infty} x\log\left(\frac{x+1}{x-1}\right) = \lim_{x\to\infty}\frac{\log\left(\frac{x+1}{x-1}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{x-1}{x+1}\;\frac{x-1-(x+1)}{(x-1)^2}}{\frac{-1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{-1}{x+1}\cdot\frac{-2}{x-1}\;\frac{x^2}{-1} = \underline{2}_{\,//}$$

(5) $$\lim_{x\to +0} x^x$$

$$\left(\lim_{x\to +0} x\log x = \lim\frac{\log x}{\frac{1}{x}} = \lim_{x\to +0}\frac{\frac{1}{x}}{\frac{-1}{x^2}} = 0\right)$$

$$\to = \lim_{x\to +0} e^{x\log x} = e^0 = 1$$

(6) $$\lim_{x\to\infty}\frac{x^3}{e^x} = \lim\frac{3x^2}{e^x} = \lim\frac{6x}{e^x} = \lim_{x\to\infty}\frac{6}{e^x} = 0$$

3. 不定積分と定積分

3.1 不定積分

◇ $f(x)$ に対して $F'(x)=f(x)$ となる $F(x)$ のことを原始関数という
or 不定積分

$$F(x)=\int f(x)\,dx$$ と書く.

$F(x)$ を1つの不定積分とすると他の不定積分はすべて $F(x)+C$ の形に書ける

C：積分定数という

$F(x)$ を求めることを $f(x)$ を積分するという

$$F(x) \underset{\text{積分}}{\overset{\text{微分}}{\rightleftarrows}} f(x)$$

微分公式を読みかえることで積分公式が導かれる.

◇ 公式

(1) $$\int x^{\alpha}\,dx=\frac{x^{\alpha+1}}{\alpha+1}$$

$$x^p \underset{\text{積分}}{\overset{\text{微分}}{\rightleftarrows}} px^{p-1}$$

$p\neq 0$ のとき. $\dfrac{x^p}{p} \rightleftarrows x^{p-1}$

$\alpha=p-1$ として $\alpha\neq -1$ のとき $$\frac{x^{\alpha+1}}{\alpha+1} \underset{\text{積分}}{\overset{\text{微分}}{\rightleftarrows}} x^{\alpha}$$ よって $$\int x^{\alpha}\,dx=\frac{x^{\alpha+1}}{\alpha+1}$$

(2) $$\int \frac{1}{x}\,dx=\log|x|$$

$$\log|x| \underset{\text{積分}}{\overset{\text{微分}}{\rightleftarrows}} \frac{1}{x}$$

(3) $$\int \frac{1}{x^2+1}\,dx=\tan^{-1}x$$

$$\tan^{-1}x \underset{\text{積分}}{\overset{\text{微分}}{\rightleftarrows}} \frac{1}{1+x^2}$$

おぼえる必要るい

(4) $$\int \frac{1}{x^2-1}\,dx=\int\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx=\frac{1}{2}\left(\log|x-1|-\log|x+1|\right)=\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|$$

(15) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x$

$\displaystyle \sin^{-1}x \rightleftarrows \frac{1}{\sqrt{1-x^2}}$

(16) $\displaystyle\int \frac{1}{\sqrt{x^2+a}}\,dx = \log\left|x+\sqrt{x^2+a}\right|$ ただし $(a \neq 0)$

$\displaystyle \log\left|x+\sqrt{x^2+a}\right| \underset{\text{積分}}{\overset{\text{微分}}{\rightleftarrows}} \frac{1+\frac{1}{2}(x^2+a)^{-\frac{1}{2}}\cdot 2x}{x+\sqrt{x^2+a}} = \frac{1}{\sqrt{x^2+a}}\cdot\frac{\sqrt{x^2+a}+x}{x+\sqrt{x^2+a}} = \frac{1}{\sqrt{x^2+a}}$

(17) $\displaystyle\int \sin x\,dx = -\cos x$

$\cos x \underset{\text{積分}}{\overset{\text{微分}}{\rightleftarrows}} -\sin x \iff -\cos x \underset{\text{積分}}{\rightleftarrows} \sin x$

(18) $\displaystyle\int \cos x\,dx = \sin x$

導出易 (9) $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-(\cos x)'}{\cos x}\,dx = -\log|\cos x|$

〃 (10) $\displaystyle\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{(\sin x)'}{\sin x}\,dx = \log|\sin x|$

〃 (11) $\displaystyle\int \sqrt{1-x^2}\,dx = \int x'\sqrt{1-x^2}\,dx = x\sqrt{1-x^2} - \int x\,\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\cdot(-2x)\,dx$

$\displaystyle = x\sqrt{1-x^2} + \int \frac{x^2}{\sqrt{1-x^2}}\,dx = x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx$

$\displaystyle = x\sqrt{1-x^2} - \boxed{\int \sqrt{1-x^2}\,dx} + \underbrace{\int \frac{1}{\sqrt{1-x^2}}\,dx}_{\sin^{-1}x}$

$\displaystyle \iff 2\int \sqrt{1-x^2}\,dx = \frac{x\sqrt{1-x^2}+\sin^{-1}x}{2}$

よって $\displaystyle\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left\{x\sqrt{1-x^2}+\sin^{-1}x\right\}$

(12) $\displaystyle\int \sqrt{x^2+a}\,dx = x\sqrt{x^2+a} - \boxed{\int \frac{x \cdot \cancel{2}x}{\cancel{2}\sqrt{x^2+a}}\,dx}$

$$\int \frac{x^2+a-a}{\sqrt{x^2+a}}\,dx = \int \sqrt{x^2+a} - \frac{a}{\sqrt{x^2+a}}\,dx$$

$\Rightarrow$ $\displaystyle 2\int \sqrt{x^2+a}\,dx = x\sqrt{x^2+a} + a\underbrace{\boxed{\int \frac{1}{\sqrt{x^2+a}}\,dx}}_{=\log\left|x+\sqrt{x^2+a}\right|}$

よって $\displaystyle\int \sqrt{x^2+a}\,dx = \frac{1}{2}\left\{x\sqrt{x^2+a} + a\log\left|x+\sqrt{x^2+a}\right|\right\}$

(13) $\displaystyle\int \sec^2 x\,dx = \int \frac{1}{\cos^2 x}\,dx = \tan x$

$\because$ $\tan x \underset{\text{積}}{\overset{\text{微}}{\rightleftarrows}} \dfrac{1}{\cos^2 x}$

(14) $\displaystyle\int \mathrm{cosec}^2 x\,dx = \int \frac{1}{\sin^2 x}\,dx = -\cot x$

$\because$ $\cot x \underset{\text{積}}{\overset{\text{微}}{\rightleftarrows}} \dfrac{-\sin x\sin x - \cos x\cos x}{\sin^2 x} = -\dfrac{1}{\sin^2 x}$

(15) $\displaystyle\int \sec x\,dx = \int \frac{1}{\cos x}\,dx = \int \frac{\cos x}{1-\sin^2 x}\,dx = \int \frac{(\sin x)'}{1-\sin^2 x}\,dx$

$u = \sin x$

$$= \frac{1}{2}\int \frac{1}{1-u} + \frac{1}{1+u}\,du = \frac{1}{2}\left\{-\log|u-1| + \log|u+1|\right\}$$

$$= \frac{1}{2}\log\left|\frac{u+1}{u-1}\right| = \frac{1}{2}\log\left|\frac{\sin x+1}{\sin x-1}\right|$$

$$= \frac{1}{2}\log\left|\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{1-2\sin\frac{x}{2}\cos\frac{x}{2}}\right| = \frac{1}{2}\log\left|\frac{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2}\right|$$

$$= \log\left|\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right| = \log\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| = \log\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$$

(16) $\int \operatorname{cosec} x\, dx = \log \left| \tan \frac{x}{2} \right|$

∵ $\int \frac{1}{\sin x} dx = \int \frac{\sin x}{1-\cos^2 x} dx = -\int \frac{(\cos x)'}{1-\cos^2 x} dx$ $u = \cos x$

$= -\int \frac{1}{1-u^2} dx = -\frac{1}{2} \log \left| \frac{\cos x + 1}{\cos x - 1} \right|$

$= \frac{1}{2} \log \left| \frac{1-\cos x}{1+\cos x} \right| = \frac{1}{2} \log \left| \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} \right| = \log \left| \frac{\sin x/2}{\cos x/2} \right| = \log \left| \tan \frac{x}{2} \right|$

(17) $\int e^x dx = e^x$

∵ $e^x \underset{積}{\overset{微}{\rightleftarrows}} e^x$

定理 3-1

(1) $\int k f(x)\, dx = k \int f(x)\, dx$ ∵ $\left(k \int f(x) dx\right)' = k f(x)$

定数

(2) $\int \{ f(x) \pm g(x) \} dx = \int f(x)\, dx \pm \int g(x)\, dx$

∵ $\left(\int f(x) dx \pm \int g(x) dx \right)' = f(x) + g(x)$

例3.1，問3.1は、置換・部分積分法（$\boxed{3.2}$）をやってから。

問3.2 済

$\boxed{3.2}$ 置換積分法・部分積分法・有理関数の積分法

◇

$$\int f(x)\,dx \underset{\uparrow}{=} \int f(g(t))\,g'(t)\,dt$$

$x=g(t)$

$dx=g'(t)\,dt$

∵ $\int f(x)\,dx = F(x)$ とおいて $F(g(t)) = \int f(g(t))\,g'(t)\,dt$

両辺を t で微分

$$(\text{右辺})' = \underbrace{F'(g(t))}_{f(g(t))}\,g'(t) = f(g(t))$$ よって示せた

(例) $$\int \frac{f'(x)}{f(x)}\,dx \underset{\uparrow}{=} \int \frac{du}{u} = \log|u| = \log|f(x)|$$

$u=f(x)$

$du=f'(x)\,dx$

◇ 部分積分法

定理3.3 $$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

∵ $(右辺)' = \cancel{f'(x)g(x)} + f(x)g'(x) - \cancel{f'(x)g(x)} = f(x)g'(x)$ ☒

例.3.3.

(1) $$\int x\log x\,dx = \int\left(\frac{x^2}{2}\right)'\log x\,dx = \frac{x^2}{2}\log x - \int\frac{x^2}{2}\frac{1}{x}\,dx$$

$$= \underline{\frac{x^2}{2}\log x - \frac{x^2}{4}}$$

x → $\left(\frac{x^2}{2}\right)'$ と見なす

(2) $$\int \underset{\left(\frac{x^2}{2}\right)'}{\underset{\|}{x}}\sin^{-1}x\,dx = \frac{x^2}{2}\sin^{-1}x - \boxed{\int\frac{x^2}{2}\underbrace{(\sin^{-1}x)'}_{=\frac{1}{\sqrt{1-x^2}}}\,dx}$$

$$\boxed{} = \frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\int\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{1}{2}\underbrace{\boxed{\int\sqrt{1-x^2}\,dx}}_{=\frac{1}{2}\left(x\sqrt{1-x^2}+\sin^{-1}x\right)} - \frac{1}{2}\underbrace{\int\frac{dx}{\sqrt{1-x^2}}}_{=\sin^{-1}x}$$

公式(11)

与式 $$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{4}\left(x\sqrt{1-x^2}+\sin^{-1}x\right) - \frac{1}{2}\sin^{-1}x$$

$$= \underline{\frac{x^2}{2}\sin^{-1}x - \frac{1}{4}\sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2}}$$

例題 3.2

(1) $\displaystyle\int \frac{1}{x^2-2x+5}\,dx = \int \frac{1}{4\left\{\left(\frac{x-1}{2}\right)^2+1\right\}}\,dx$

$\left(x^2-2x+5 = (x-1)^2+4 = 4\left\{\left(\frac{x-1}{2}\right)^2+1\right\}\right)$

$u=\frac{x-1}{2}$ とおくと. $du = \frac{1}{2}\,dx$ より

$\displaystyle = \int \frac{1}{4(u^2+1)}\,2\,du = \frac{1}{2}\int \frac{1}{u^2+1}\,du = \frac{1}{2}\tan^{-1}u = \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)$ //

(2) $\displaystyle\int x(x^2-3)^5\,dx$

$\left(u=x^2-3 \text{ とおくと } du = 2x\,dx\ .\ x\,dx = \frac{du}{2}\right)$

$\displaystyle = \int u^5\,\frac{du}{2} = \frac{u^6}{12} = \frac{(x^2-3)^6}{12}$ //

◇ 有理関数の積分法

多項式 → $\dfrac{f(x)}{g(x)}$ の積分
多項式 →

⇒ (I) 割り算実行して

$$f(x) = \underbrace{Q(x)}_{\text{これはすぐに積分出来る。}} + \frac{R(x)}{g(x)} \leftarrow R(x)\text{の次数} < g(x)\text{の〃}$$

⇒ (II) $\dfrac{R(x)}{g(x)}$ を部分分数に分解する。(後述) $\dfrac{R(x)}{g(x)} = \underbrace{\bigcirc + \bigcirc + \cdots + \bigcirc}_{\text{分解された}}$

⇒ 分解された項の積分は

どちらか or

$$\begin{cases} \displaystyle\int \frac{A}{(x-a)^n}\,dx & \leftarrow \text{公式(7)(12)で求まる（すぐ）} \\[2ex] \displaystyle\int \frac{Bx+C}{(x^2+px+q)^m}\,dx \quad (*) & \text{ただし } (p^2-4q<0) \end{cases}$$

正.

$$\left(x+\frac{p}{2}\right)^2 - \frac{p^2-4q}{4} = \left(\underbrace{\boxed{x+\frac{p}{2}}}_{t\text{とおく}}\right)^2 + \underbrace{q-\frac{p^2}{4}}_{a^2\text{とおく}}$$

$$(*) = \int \frac{Bt + C - \frac{pB}{2}}{(t^2+a^2)^m}\,dt = B\int \frac{t}{(t^2+a^2)^m}\,dt - \frac{pB}{2}\underbrace{\boxed{\int \frac{dt}{(t^2+a^2)^m}}}_{I_m}$$

$$\rightarrow \begin{cases} \dfrac{1}{2}\log|t^2+a^2| & (m=-1) \\[2ex] \dfrac{1}{2}\dfrac{1}{(-m+1)}(t^2+a^2)^{-m+1} & (m\neq -1) \end{cases}$$

$$I_m = \int \frac{dt}{(t^2+a^2)^m} = \int \frac{t'}{(t^2+a^2)^m}\,dt = \frac{t}{(t^2+a^2)^m} - \int \frac{t\,(-m)}{(t^2+a^2)^{m+1}}\cdot 2t\,dt$$

$$= \frac{t}{(t^2+a^2)^m} + 2m\int \frac{t^2+a^2-a^2}{(t^2+a^2)^{m+1}}\,dt$$

$$= \frac{t}{(t^2+a^2)^m} + 2m\left(I_m - a^2 I_{m+1}\right)$$

(I_m つづき)

$m \to m-1$ にして

$$I_m = \frac{1}{2(m-1)a^2}\left\{\frac{t}{(t^2+a^2)^{m-1}} + (2m-3)I_{m-1}\right\}$$ … 漸化式 $(m-1>0)$

$$I_m \to I_{m-1} \to \cdots \to I_1 = \int \frac{dt}{t^2+a^2} = \frac{1}{a}\tan^{-1}\frac{t}{a}$$ [公式(3)]

I_m は $I_1 = \frac{1}{a}\tan^{-1}\frac{t}{a}$ から $\underbrace{\text{求まる}}_{\text{漸化式を用いて}}$.

例 3-1

(1) $\displaystyle \int \frac{5}{4x^2+3}\,dx = \frac{5}{3}\int \frac{1}{\frac{4}{3}x^2+1}\,dx$

$\left(\dfrac{2x}{\sqrt{3}} = t \right.$ とおくと $\quad \dfrac{2dx}{\sqrt{3}} = dt \;,\; \left. dx = \dfrac{\sqrt{3}}{2}dt \right)$

$\displaystyle = \frac{5}{3}\int \frac{1}{t^2+1}\,\frac{\sqrt{3}}{2}dt = \frac{5}{2\sqrt{3}}\tan^{-1}t = \underline{\frac{5}{2\sqrt{3}}\tan^{-1}\left(\frac{2x}{\sqrt{3}}\right)}$

(2) $\displaystyle \int \frac{3}{\sqrt{5x^2+4}}\,dx = \frac{3}{2}\int \frac{1}{\sqrt{\frac{5}{4}x^2+1}}\,dx$

$\left(\dfrac{\sqrt{5}}{2}x = t ,\quad \dfrac{\sqrt{5}}{2}dx = dt ,\quad dx = \dfrac{2}{\sqrt{5}}dt \right)$

$\displaystyle = \frac{3}{2}\int \frac{1}{\sqrt{t^2+1}}\,\frac{2}{\sqrt{5}}dt = \frac{3}{\sqrt{5}}\int \frac{1}{\sqrt{t^2+1}}dt = \frac{3}{\sqrt{5}}\log\left|t+\sqrt{t^2+1}\right|$

$\displaystyle = \frac{3}{\sqrt{5}}\log\left|\frac{\sqrt{5}}{2}x + \sqrt{\frac{5}{4}x^2+1}\right|$

$\displaystyle = \frac{3}{\sqrt{5}}\log\left(\frac{\sqrt{5}}{2}\left|x+\sqrt{x^2+\frac{4}{5}}\right|\right)$

$\displaystyle = \frac{3}{\sqrt{5}}\log\left|x+\sqrt{x^2+\frac{4}{5}}\right| + \cancel{\frac{3}{\sqrt{5}}\log\frac{\sqrt{5}}{2}}$

定数なのでなくてもいい

(3) $\displaystyle \int \frac{1}{1-x^2}\,dx = \int \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right)dx$

$\displaystyle = \frac{1}{2}\left\{\log|1+x| - \log|1-x|\right\} = \underline{\frac{1}{2}\log\left|\frac{1+x}{1-x}\right|}$

$\displaystyle = -\frac{1}{2}\log\left(\left|\frac{1+x}{1-x}\right|^{-1}\right) = -\frac{1}{2}\log\left|\frac{1-x}{1+x}\right|$

(4) $\displaystyle\int \frac{1}{\sqrt{16-9x^2}}\,dx$

$$= \frac{1}{4}\int \frac{1}{\sqrt{1-\frac{9}{16}x^2}}\,dx = \frac{1}{4}\int \frac{1}{\sqrt{1-t^2}}\,\frac{4}{3}\,dt = \frac{1}{3}\int \frac{dt}{\sqrt{1-t^2}}$$

$\left(t = \frac{3}{4}x \qquad dt = \frac{3}{4}dx \right)$

$$= \frac{1}{3}\sin^{-1}t = \underline{\frac{1}{3}\sin^{-1}\left(\frac{3}{4}x\right)}$$

(5) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{x^2-1+1}{\sqrt{1-x^2}}\,dx$

$$= \int \frac{1}{\sqrt{1-x^2}} - \sqrt{1-x^2}\,dx$$

$$= \sin^{-1}x - \boxed{\int x'\sqrt{1-x^2}\,dx}$$

$$\boxed{} = x\sqrt{1-x^2} - \int x\,\frac{1}{2}\,\frac{2x}{\sqrt{1-x^2}}\,dx$$

$$= \sin^{-1}x - x\sqrt{1-x^2} - \int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\left\{ \sin^{-1}x - x\sqrt{1-x^2} \right\}$$

問 3-1

(1) $\int \frac{1}{2x} dx = \underline{\frac{1}{2}\log|x|}$

(2) $\int \sin \underbrace{\boxed{\frac{x}{2}}}_{t} dx = \int \sin t \cdot 2dt = 2(-\cos t) = \underline{-2\cos\frac{x}{2}}$

$\left(t = \frac{x}{2} \quad dt = \frac{dx}{2} \right)$

(3) $\int \cos \underbrace{\boxed{2x}}_{t} dx = \int \cos t \frac{dt}{2} = \frac{\sin t}{2} = \underline{\frac{\sin 2x}{2}}$

$\left(t = 2x \quad dt = 2dx \right)$

(4) $\int e^{\overbrace{\boxed{-x}}^{t}} dx = \int e^{t}(-)dt = -e^{t} = \underline{-e^{-x}}$

$\left(t = -x. \quad dt = -dx \right)$

(5) $\int \cot x + \tan x \, dx$

$= \int \frac{\cos x}{\underbrace{\boxed{\sin x}}_{t}} dx + \int \frac{\sin x}{\underbrace{\boxed{\cos x}}_{u}} dx$

$t = \sin x. \quad dt = \cos x \, dx \quad , \quad u = \cos x \quad du = -\sin x \, dx$

$= \int \frac{dt}{t} + \int \frac{-du}{u} = \log|t| - \log|u|$

$= \underline{\log\left|\frac{\sin x}{\cos x}\right|}$

(6) $\displaystyle\int \frac{4}{\sqrt{3x^2-6}}dx = \frac{4}{\sqrt{6}}\int \frac{1}{\sqrt{\frac{x^2}{2}-1}}dx = \frac{4}{\sqrt{6}}\int \frac{1}{\sqrt{t^2-1}}\sqrt{2}\,dt$

$\left(t = \frac{x}{\sqrt{2}} \right.$ とおくと $\left. dt = \frac{dx}{\sqrt{2}} \right)$

$= \displaystyle\frac{4}{\sqrt{3}}\int \frac{1}{\sqrt{t^2-1}}dt = \frac{4}{\sqrt{3}}\log\left|t+\sqrt{t^2-1}\right|$

$= \displaystyle\frac{4}{\sqrt{3}}\log\left|\frac{x}{\sqrt{2}}+\sqrt{\frac{x^2}{2}-1}\right|$

$\therefore \displaystyle\frac{4}{\sqrt{3}}\log\left|x+\sqrt{x^2-2}\right|$ + 定数

(7) $\displaystyle\int \frac{x^2}{\sqrt{x^2+4}}dx = \frac{1}{2}\int \frac{x^2}{\sqrt{\left(\frac{x}{2}\right)^2+1}}dx = \frac{1}{2}\int \frac{4t^2}{\sqrt{t^2+1}}2\,dt$

$\left(t = \left(\frac{x}{2}\right) \right.$ とおく. $\left. dt = \frac{1}{2}dx \right)$

$= 4\boxed{\displaystyle\int \frac{t^2}{\sqrt{t^2+1}}dt}$

$\left(\boxed{} = \displaystyle\int \frac{t^2+1-1}{\sqrt{t^2+1}}dt = \underbrace{\int \sqrt{t^2+1}\,dx}_{\frac{1}{2}\left\{t\sqrt{t^2+1}+\log\left|t+\sqrt{t^2+1}\right|\right.} - \underbrace{\int \frac{1}{\sqrt{t^2+1}}dt}_{\log\left|t+\sqrt{t^2+1}\right|} \right)$

(公式12)　(公式(6))

与式 $= 2\left\{t\sqrt{t^2+1}+\log\left|t+\sqrt{t^2+1}\right|\right\} - 4\log\left|t+\sqrt{t^2+1}\right|$

$= 2\,\frac{x}{2}\sqrt{\frac{x^2}{4}+1} - 2\log\left|\frac{x}{2}+\sqrt{\frac{x^2}{4}+1}\right|$

$= \frac{1}{2}x\sqrt{x^2+4} - 2\log\left|x+\sqrt{x^2+4}\right|$.

(8) $\displaystyle\int \frac{1-x^2}{1+x^2}dx = \int \frac{2-(x^2+1)}{1+x^2}dx = \int \frac{2}{1+x^2}dx - \int dx$

$= 2\tan^{-1}x - x$

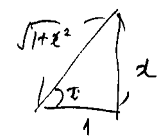
問 3.3

(1) $\displaystyle\int \frac{1}{(1+x^2)^{3/2}}\,dx = \int \frac{1}{(1+\tan^2 t)^{3/2}}\,\frac{1}{\cos^2 t}\,dt$

$\left(x = \tan t \ , \quad dx = \dfrac{1}{\cos^2 t}\,dt\right)$

$\boxed{1+\tan^2 x = \dfrac{1}{\cos^2 t}}$

$\displaystyle = \int \frac{1}{(\cos^{-2} t)^{\frac{3}{2}}}\,\frac{1}{\cos^2 t}\,dt$



$\displaystyle = \int \frac{1}{\cos^{-3} t \cdot \cos^2 t}\,dt = \int \cos t\,dt = \sin t = \underline{\frac{x}{\sqrt{1+x^2}}}$

(2) $\displaystyle\int \frac{3x}{\sqrt{1-x^4}}\,dx = \int \frac{3/2}{\sqrt{1-t^2}}\,dt = \frac{3}{2}\int \frac{1}{\sqrt{1-t^2}}\,dt = \frac{3}{2}\sin^{-1} t$

$\left(x^2 = t \ . \quad 2x\,dx = dt\right)$

$= \underline{\dfrac{3}{2}\sin^{-1} x^2}$

(3) $\displaystyle\int \frac{x^2}{x^6-1}\,dx = \int \frac{1}{t^2-1}\,\frac{dt}{3} = \frac{1}{3}\int \frac{1}{t^2-1}\,dt = \frac{1}{3}\int \left(\frac{1}{t-1} - \frac{1}{t+1}\right)dt$

$\left(x^3 = t \ , \quad 3x^2\,dx = dt\right)$

$\displaystyle = \frac{1}{3}\left\{\log|t-1| - \log|t+1|\right\} = \frac{1}{3}\log\left|\frac{t-1}{t+1}\right| = \underline{\frac{1}{3}\log\left|\frac{x^3-1}{x^3+1}\right|}$

問 3.4

(1) $\displaystyle\int x^2 \underbrace{\cos x}_{(\sin x)'}\, dx$

$\displaystyle = x^2 \sin x - \int 2x \underbrace{\sin x}_{(-\cos x)'}\, dx = x^2 \sin x - \left\{ 2x(-\cos x) - \int 2(-\cos x)\, dx \right\}$

$= \underline{x^2 \sin x + 2x\cos x - 2\sin x}$

(2) $\displaystyle\int \underbrace{1}_{x'} \log x \, dx = x \log x - \int x \frac{1}{x}\, dx = \underline{x\log x - x}$.

(3) $\displaystyle\int x \underbrace{e^x}_{(e^x)'}\, dx = x e^x - \int x' e^x\, dx = \underline{x e^x - e^x}$

(4) $\displaystyle\int x^3\sqrt{1-x^2}\, dx = \int x \cdot x^2 \sqrt{1-x^2}\, dx = \int (1-t)\sqrt{t}\, \frac{dt}{-2}$

($t = 1-x^2$, $x^2 = 1-t$ とおくと, $dt = -2x\, dx$)

$\displaystyle = \int \left(t^{\frac{1}{2}} - t^{\frac{3}{2}}\right)\frac{1}{-2}\, dt = \left(\frac{2}{3}t^{\frac{3}{2}} - \frac{2}{5}t^{\frac{5}{2}}\right)\frac{1}{-2} = \frac{1}{5}t^{\frac{5}{2}} - \frac{1}{3}t^{\frac{3}{2}}$

$\displaystyle = \underline{\frac{1}{5}(1-x^2)^{\frac{5}{2}} - \frac{1}{3}(1-x^2)^{\frac{3}{2}}}$.

$\displaystyle = \frac{3(1-x^2) - 5}{15}(1-x^2)^{\frac{3}{2}} = \underline{\frac{-3x^2-2}{15}(1-x^2)^{\frac{3}{2}}}$.

◇ 部分分数に分解.

$$\frac{P(x)}{(x-a)^n} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}$$

$$\frac{P(x)}{(x^2+px+q)^m} = \frac{B_1x+C_1}{x^2+px+q} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \cdots + \frac{B_mx+C_m}{(x^2+px+q)^m}$$

例題 3.4.

$$\frac{x^4-3x^2+3x-7}{(x+2)(x-1)^2}$$

分母 =

$$\begin{array}{r} x^2-2x+1 \\ x+2 \\ \hline x^3-2x^2+x \\ +2x^2-4x+2 \\ \hline x^3-3x+2 \end{array}$$

(1回.)

$$\begin{array}{r} x \\ x^3-3x+2 \,\big)\, x^4-3x^2+3x-7 \\ x^4-3x^2+2x \\ \hline x-7 \end{array}$$

$$x + \underbrace{\frac{x-7}{(x+2)(x-1)^2}}$$

$$\sim\sim = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\underbrace{x-7}_{①} = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$$

$$= A(x^2-2x+1) + B(x^2+x-2) + C(x+2)$$

$$= \underbrace{(A+B)x^2 + (-2A+B+C)x + (A-2B+2C)}_{②}$$

すべてのxで ①=②が成立するには

$$\begin{cases} A+B=0 \longrightarrow B=-A \\ -2A+B+C=1 \xrightarrow{\Downarrow} -3A+C=1 \\ A-2B+2C=-7 \longrightarrow 3A+2C=-7 \end{cases}$$

$\longrightarrow 3C=-6 \quad C=-2$

$A = \dfrac{-7+4}{3} = -1,\ B=1.$

よって

$$与式 = x + \frac{-1}{x+2} + \frac{1}{x-1} + \frac{-2}{(x-1)^2}$$

例 3.5

$$\int x + \frac{-1}{x+2} + \frac{1}{x-1} + \frac{-2}{(x-1)^2}\,dx$$

$$= \frac{x^2}{2} - \log|x+2| + \log|x-1| - 2\cdot(x-1)^{-1}$$

例 3.6

$$\int \frac{1}{x^3+1}\,dx$$

$$\frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} = \frac{1}{x^3+1}$$

$$\text{左辺} = A(x^2-x+1) + \underbrace{(Bx+C)(x+1)}_{Bx^2+(B+C)x+C}$$

$$= (A+B)x^2 + (-A+B+C)x + A + C = 1\ (\text{右辺})$$

$$\begin{cases} A+B=0 \longrightarrow B=-A \\ -A+B+C=0 \longrightarrow -A-A+1-A=0 \\ A+C=1 \longrightarrow C=1-A \end{cases} \qquad B=-\frac{1}{3},\quad A=\frac{1}{3},\quad C=\frac{2}{3}.$$

$$\text{与式} = \int \frac{\frac{1}{3}}{x+1} + \frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}\,dx = \frac{1}{3}\log|x+1| - \frac{1}{3}\boxed{\int\frac{x-2}{x^2-x+1}\,dx} \quad \cdots(*)$$

$$\int \frac{x-2}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx = \frac{4}{3}\int\frac{x-2}{\frac{4}{3}(x-\frac{1}{2})^2+1}\,dx$$

$$\left(t = \frac{2}{\sqrt{3}}(x-\tfrac{1}{2}) \qquad x\cancel{-\tfrac{1}{2}} = \frac{\sqrt{3}}{2}t + \frac{1}{2} \qquad dx = \frac{\sqrt{3}}{2}\,dt \right)$$

$$= \frac{\sqrt{3}}{2}\frac{4}{3}\int\frac{\frac{\sqrt{3}}{2}t+\frac{1}{2}-2}{t^2+1}\,dt = \frac{\sqrt{3}}{2}\frac{1}{\sqrt{3}}\underbrace{\boxed{\int\frac{2t}{t^2+1}\,dt}}_{\log(t^2+1)} + \frac{\sqrt{3}}{2}\frac{4}{3}\left(\frac{-3}{2}\right)\underbrace{\boxed{\int\frac{1}{t^2+1}\,dt}}_{\tan^{-1}t}$$

$$(*) = \frac{1}{3}\log|x+1| - \frac{1}{6}\log(t^2+1) + \frac{1}{\sqrt{3}}\tan^{-1}t$$

$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log\left(\frac{4}{3}(x-\tfrac{1}{2})^2+1\right) \ldots \tan^{-1}(2 \ldots$$

問35

(1)

$$\begin{array}{r} x^2+x+4 \\ x^3-4x\,\overline{)\,x^5+x^4 \qquad -8} \\ \underline{x^5-4x^3 \qquad\quad} \\ x^4+4x^3 \qquad \\ \underline{x^4-4x^2 \qquad} \\ 4x^3+4x^2 \quad \\ \underline{4x^3-16x \quad} \\ 4x^2+16x-8 \end{array}$$

$$\int x^2+x+4+\frac{4x^2+16x-8}{x^3-4x}\,dx = \frac{x^3}{3}+\frac{x^2}{2}+4x+4\boxed{\int\frac{x^2+4x-2}{x(x^2-4)}\,dx}\cdots(*)$$

$$\frac{x^2+4x-2}{x(x-2)(x+2)} = \frac{a}{x-2}+\frac{b}{x}+\frac{c}{x+2}$$

$$\begin{aligned} x^2+4x-2 &= a(x^2+2x)+b(x^2-4)+c(x^2-2x) \\ &= (a+b+c)x^2+(2a-2c)x+(-4b) \end{aligned}$$

$a+b+c=1 \rightarrow a+c=\frac{1}{2}$

$2a-2c=4 \rightarrow a-c=2$ $\qquad 2a=\frac{5}{2} \qquad \underline{a=\frac{5}{4}}$

$-4b=-2 \rightarrow \underline{b=\frac{1}{2}} \qquad \underline{c=\frac{1}{2}-\frac{5}{4}=\frac{-3}{4}}$

$$\boxed{} = \frac{5}{4}\log|x-2|+\frac{1}{2}\log|x|-\frac{3}{4}\log|x+2|$$

$$\begin{aligned} (*) &= \frac{x^3}{3}+\frac{x^2}{2}+4x+4\left(\frac{5}{4}\log|x-2|+\frac{1}{2}\log|x|-\frac{3}{4}\log|x+2|\right) \\ &= \frac{x^3}{3}+\frac{x^2}{2}+4x+5\log|x-2|+2\log|x|-3\log|x+2| \end{aligned}$$

(2) $$\frac{x}{(1+x)^2(1+x^2)} = \frac{a}{1+x} + \frac{b}{(1+x)^2} + \frac{cx+d}{1+x^2}$$

$$= \frac{a(1+x)(1+x^2)+b(1+x^2)+(cx+d)(x^2+2x+1)}{(1+x)^2(1+x^2)}$$

$$= \underline{(a+c)x^3+(a+b+2c+d)x^2+(a+c+2d)x+(a+b+d)}$$

//

$a+c=0 \longrightarrow \boxed{a=0}$

$a+b+2c+d=0 \rightarrow a+b+2c=-\frac{1}{2}$

$a+c+2d=1 \rightarrow 2d=1,\ \boxed{d=\frac{1}{2}} \rightarrow 2c=0,\ \boxed{c=0}$

$a+b+d=0 \longrightarrow a+b=-\frac{1}{2} \rightarrow \boxed{b=-\frac{1}{2}}$

$$\int 与式\, dx = \int \frac{\cancel{0}}{1+x} + \frac{-\frac{1}{2}}{(1+x)^2} + \frac{\frac{1}{2}}{1+x^2}\, dx$$

$$= -\frac{1}{2}(-1)(1+x)^{-1} + \frac{1}{2}\tan^{-1}x$$

$$= \underline{\frac{1}{2(1+x)} + \frac{\tan^{-1}x}{2}},$$

(3) $\dfrac{1}{x(1+x^2)} = \dfrac{a}{x} + \dfrac{bx+c}{1+x^2}$

$= \dfrac{a(1+x^2)+(bx+c)x}{x(1+x^2)}$

$1 = (a+b)x^2 + (c)x + a$

$\begin{cases} a+b=0 \longrightarrow b=-1. \\ \boxed{c=0} \\ \boxed{a=1} \end{cases}$

$\displaystyle\int \frac{1}{x} + \frac{-x}{1+x^2}\,dx = \log|x| - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$

$\log(1+x^2)$

$= \underline{\log|x| - \frac{1}{2}\log(1+x^2)} = \underline{\log\left|\dfrac{x}{\sqrt{1+x^2}}\right|}$

問3.6 済

3.3 三角関数，無理関数，指数関数，対数関数の積分

$f(x)$ … x の有理関数
$f(x,y)$ … x,y の 〃 } とする

三角関数

(1) $\displaystyle\int f(\sin x)\underbrace{\cos x}_{(\sin x)'}\,dx \Rightarrow t=\sin x$

(2) $\displaystyle\int f(\cos x)\underbrace{\sin x}_{(-\cos x)'}\,dx \Rightarrow t=\cos x$

(3) $\displaystyle\int f(\sin^2 x,\cos^2 x)\,dx \Rightarrow t=\tan x$

∵ $t^2=\dfrac{1-\cos^2 x}{\cos^2 x}$　　$(t^2+1)\cos^2 x=1$

$\cos^2 x=\dfrac{1}{t^2+1}$

$\sin^2 x=\dfrac{t^2}{t^2+1}$

$dt=\dfrac{1}{\cos^2 x}\,dx \Rightarrow dx=\cos^2 x\,dt$

$=\dfrac{1}{t^2+1}\,dt$

t の有理関数の積分に帰着するから

(4) $\displaystyle\int f(\sin x,\cos x)\,dx \Rightarrow t=\tan\frac{x}{2}$

∵ $\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}=2\underbrace{\tan\frac{x}{2}}_{t}\underbrace{\cos^2\frac{x}{2}}_{\frac{1}{t^2+1}}=\underline{\dfrac{2t}{t^2+1}}$

$\cos x=2\cos^2\frac{x}{2}-1=\dfrac{2}{t^2+1}-1=\dfrac{2-(t^2+1)}{t^2+1}$

$=\underline{\dfrac{1-t^2}{1+t^2}}$

$dt=\dfrac{1}{\cos^2\frac{x}{2}}\cdot\dfrac{1}{2}\,dx$

$dx=2\underbrace{\cos^2\left(\frac{x}{2}\right)}_{\frac{1}{t^2+1}}\,dt=\underline{\dfrac{2}{1+t^2}\,dt}$

t の有理関数の積分に帰着するから

無理関数

$$\int f(x, \sqrt[n]{ax+b})\,dx \Rightarrow t = \sqrt[n]{ax+b}$$

$$\because \quad t^n = ax+b$$

$$x = \frac{t^n - b}{a}$$

$$dx = \frac{n}{a} t^{n-1}\,dt$$

tの有理関数の積分に帰着

$$\int f\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx \Rightarrow t = \sqrt[n]{\frac{ax+b}{cx+d}}$$

$$\because \quad t^n = \frac{ax+b}{cx+d}$$

$$t^n(cx+d) = ax+b$$

$$(ct^n - a)x = b - dt^n$$

$$x = -\frac{b-dt^n}{a-ct^n}$$

$$\frac{dx}{dt} = \frac{dnt^{n-1}(a-ct^n) + (b-dt^n)(-cnt^{n-1})}{(a-ct^n)^2}$$

$$= \frac{(\cancel{-dnc} + \cancel{dnc})t^{2n-1} + (adn - bcn)t^{n-1}}{}$$

$$= \frac{(ad-bc)n\,t^{n-1}}{(a-ct^n)^2}$$

$$dx = \frac{ad-bc}{(a-ct^n)^2} n\,t^{n-1}\,dt$$

tの有理関数の積分に帰着

$\int f(x, \sqrt{ax^2+bx+c})\,dx$

(i) $a>0$ のとき. $\sqrt{ax^2+bx+c} = t-\sqrt{a}\,x$ とおく

(例) $\int \sqrt{x^2+A}\,dx$

$$\left(\begin{aligned} & t-x=\sqrt{x^2+A}, \quad t^2-2tx+\cancel{x^2}=\cancel{x^2}+A \\ & \qquad 2tx = t^2-A \\ & \qquad x=\frac{t^2-A}{2t} \qquad \frac{dx}{dt}=\frac{2t\cdot 2t-(t^2-A)\cdot 2}{4t^2} \\ & \qquad\qquad = \frac{4t^2-2t^2+2A}{4t^2} \\ & \qquad\qquad = \frac{2t^2+2A}{4t^2} \end{aligned}\right)$$

$$= \int \left(t-\frac{t^2-A}{2t}\right)\left(\frac{2t^2+2A}{4t^2}\right)dt$$

$$= \int \left(\underbrace{t-\frac{t}{2}}_{\frac{t}{2}}+\frac{A}{2t}\right)\left(\frac{1}{2}+\frac{A}{2t^2}\right)dt$$

$$= \int \left(\frac{t}{4}+\underbrace{\frac{A}{4t}+\frac{A}{4t}}_{\frac{A}{2t}}+\frac{A^2}{4t^3}\right)dt = \frac{t^2}{8}+\frac{A}{2}\log|t| - \frac{A^2}{8}t^{-2}$$

$$= \frac{1}{8}\left(t^2-\frac{A^2}{t^2}\right)+\frac{A}{2}\log|t|$$

$$\left(\begin{aligned} & t = x+\sqrt{x^2+A} \longrightarrow \frac{A}{t} = A\,\frac{\sqrt{x^2+A}-x}{A} \rightarrow \left(\frac{A}{t}\right)^2 = x^2+A+x^2-2x\sqrt{x^2+A} \\ & \rightarrow t^2 = x^2+x^2+A+2x\sqrt{x^2+A} \\ & \qquad = 2x^2+A+2x\sqrt{x^2+A} \\ & \sim\sim = 4x\sqrt{x^2+A} \end{aligned}\right)$$

$$= \frac{1}{2}x\sqrt{x^2+A}+\frac{A}{2}\log|x+\sqrt{x^2+A}|$$

($D<0$ のとき.  なので 実で積が出来ない)

(iii) $a<0$, $D>0$ のとき.

$$ax^2+bx+c = a(x-\alpha)(x-\beta) \qquad (\alpha<\beta)$$

$$\sqrt{\frac{x-\alpha}{\beta-x}} = t$$

x	α	β
t	0	∞

(例) $\displaystyle\int \sqrt{1-x^2}\,dx \qquad t = \sqrt{\frac{x-(-1)}{1-x}} = \sqrt{\frac{x+1}{1-x}}$

$$t^2 = \frac{x+1}{1-x} = -1+\frac{2}{1-x} \qquad t^2+1 = \frac{2}{1-x} \qquad 1-x = \frac{2}{t^2+1}$$

$$x = 1-\frac{2}{t^2+1} = \frac{t^2-1}{t^2+1}$$

$$\sqrt{1-x^2} = \sqrt{\frac{(t^2+1)^2-(t^2-1)^2}{(t^2+1)^2}} = \sqrt{\frac{4t^2}{(t^2+1)^2}} = \frac{2t}{t^2+1}$$

$$\frac{dx}{dt} = \frac{2t(t^2+1)-(t^2-1)\cdot 2t}{(t^2+1)^2} = \frac{4t}{(t^2+1)^2}$$

$$= \int \frac{2t}{t^2+1}\frac{4t}{(t^2+1)^2}\,dt = \int \frac{8t^2}{(t^2+1)^3}\,dt = 8\underbrace{\boxed{\int \frac{1}{(t^2+1)^2}\,dt}}_{I_2} - 8\underbrace{\boxed{\int \frac{1}{(t^2+1)^3}\,dt}}_{I_3} \quad \cdots (*)$$

$$\left(\frac{8t^2}{(t^2+1)^3} = 8\left(\frac{1}{(t^2+1)^2} - \frac{1}{(t^2+1)^3}\right) \right)$$

P.66 (8) より $\displaystyle I_n = \frac{1}{2(n-1)}\left\{\frac{t}{(t^2+1)^{n-1}} + (2n-3)\,I_{n-1}\right\}$

$$I_3 = \frac{1}{2\cdot 2}\left\{\frac{t}{(t^2+1)^2} + 3\,I_2\right\}, \qquad I_2 = \frac{1}{2}\left\{\frac{t}{(t^2+1)} + I_1\right\}$$

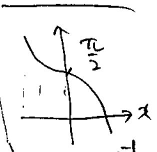
$$\frac{x}{2}\sqrt{1-x^2} + \frac{\sin^{-1}x}{2} \quad \left(\frac{\text{答}}{\,}\right)$$

$$(*) = 8I_2 - 2\frac{t}{(t^2+1)^2} - 6I_2 = 2I_2 - \frac{2t}{(t^2+1)^2} = \frac{t}{t^2+1} + I_1 - \frac{2t}{(t^2+1)^2}$$

$\Uparrow$

$$= I_1 + t\frac{t^2-1}{(t^2+1)^2} = I_1 + \sqrt{\frac{x+1}{1-x}}\,\frac{\frac{2x}{1-x}}{\frac{2^2}{(1-x)^2}} = I_1 + \sqrt{\frac{x+1}{1-x}}\,\frac{x(1-x)}{2} = \frac{x}{2}\sqrt{(x+1)(1-x)} + I_1$$

$$\left(t^2-1 = \frac{2}{1-x} - 2 = \frac{2-2+2x}{1-x} = \frac{2x}{1-x} \right)$$



$\cos^{-1}$

$$I_1 = \int \frac{dt}{t^2+1} = \tan^{-1}t = \tan^{-1}\sqrt{\frac{x+1}{1-x}} = \theta = \frac{1}{2}\sin^{-1}x$$

$$2\theta = \cos^{-1}(-x) = \frac{\pi}{2} - \cos^{-1}x$$

$$= \sin^{-1}x$$

$\boxed{メモ}$

$\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

$\uparrow \sqrt{1-x^2}$

$\sqrt{x+1}$

(8) $\int f(x, \sqrt{a^2-x^2})\,dx = \int f(a\sin\theta, a\cos\theta)\, a\cos\theta\, d\theta$ → 三角(4) 利用

$$\left(\begin{array}{l} x = a\sin\theta \ , \quad \sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = a\cos\theta \\ \left(-\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}\right) \\ \frac{dx}{d\theta} = a\cos\theta \end{array}\right)$$

(9) $\int f(x, \sqrt{x^2+a^2})\,dx = \int f\left(a\tan\theta, \frac{a}{\cos\theta}\right)\frac{a}{\cos^2\theta}\,d\theta$ → 三角(4) 利用

$$\left(\begin{array}{l} x = a\tan\theta \ , \quad \sqrt{x^2+a^2} = \sqrt{a^2(\tan^2\theta+1)} = \frac{a}{\cos\theta} \\ \frac{dx}{d\theta} = a\frac{1}{\cos^2\theta} \Rightarrow dx = \frac{a}{\cos^2\theta}\,d\theta \end{array}\right)$$

(10) $\int f(x, \sqrt{x^2-a^2})\,dx = \int f\left(a\sec\theta, \frac{a\sin\theta}{\cos\theta}\right)\frac{\sin\theta}{\cos^2\theta}\,d\theta$ → 〃

$\boxed{\text{メモ} \quad \sec\theta = \frac{1}{\cos\theta}}$

$x = a\sec\theta$, $\sqrt{x^2-a^2} = \sqrt{\frac{a^2}{\cos^2\theta}-a^2} = \frac{a\sin\theta}{\cos\theta}$

$\left(0 \leqq \theta \leqq \pi, \ \theta \neq \frac{\pi}{2}\right)$

$\frac{dx}{d\theta} = \frac{+1}{\cos^2\theta}\sin\theta \Rightarrow dx = \frac{\sin\theta}{\cos^2\theta}\,d\theta$

(11) $\int f(e^x)\, e^x\, dx = \int f(t)\, dt$

$\left(t = e^x. \quad \frac{dt}{dx} = e^x \Rightarrow dt = e^x dx \right)$

(12) $\int f(\log x)\, \frac{1}{x}\, dx = \int f(t)\, dt$

$\left(t = \log x, \quad \frac{dt}{dx} = \frac{1}{x} \quad dt = \frac{1}{x} dx \right)$

例 3.7

(1) $\displaystyle\int \frac{1}{2+\cos x}\,dx$

$t=\tan\frac{x}{2}$ $\qquad \cos(2\cdot\frac{x}{2}) = \cos^2\frac{x}{2}-\sin^2\frac{x}{2} = 2\cos^2\frac{x}{2}-1 = \frac{2}{t^2+1}-1 = \frac{1-t^2}{t^2+1}$

$t^2 = \dfrac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} \rightarrow \cos^2\frac{x}{2}\ t^2 = 1-\cos^2\frac{x}{2} \rightarrow (t^2+1)\cos^2\frac{x}{2} = 1 \qquad \cos^2\frac{x}{2} = \frac{1}{t^2+1}$

$\dfrac{dt}{dx} = \dfrac{1}{\cos^2\frac{x}{2}}\,\dfrac{1}{2} = \dfrac{1}{2\,\frac{1}{t^2+1}} \longmapsto \dfrac{2}{1+t^2}\,dt = dx$

$$= \int \frac{1}{2+\frac{1-t^2}{1+t^2}}\,\frac{2}{1+t^2}\,dt = \int \frac{2}{2+2t^2+1-t^2}\,dt = \int \frac{2}{t^2+3}\,dt$$

$$= \frac{2}{3}\int \frac{1}{(\frac{t}{\sqrt{3}})^2+1}\,dt = \frac{2}{\sqrt{3}}\int \frac{1}{(\frac{t}{\sqrt{3}})^2+1}\,d\left(\frac{t}{\sqrt{3}}\right) = \underline{\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+C}\ //$$

(2) $\displaystyle\int \frac{\sin x}{1+\sin x}\,dx$

$\left(t=\tan\frac{x}{2}, \qquad dx = \frac{2}{1+t^2}\,dt \right)$

$\sin(2\,\frac{x}{2}) = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\tan\frac{x}{2}\cos^2\frac{x}{2} = \dfrac{2t}{t^2+1}$

$$= \int \frac{\frac{2t}{t^2+1}}{1+\frac{2t}{t^2+1}}\,\frac{2}{1+t^2}\,dt = \int \frac{2t}{t^2+2t+1}\,\frac{2}{1+t^2}\,dt$$

$$= 4\int \frac{t}{(t+1)^2(t^2+1)}\,dt = 2\int \frac{1}{t^2+1} - \frac{1}{(t+1)^2}\,dt$$

$\left(\dfrac{t}{(t+1)^2(t^2+1)} = \dfrac{1}{2}\left(\dfrac{1}{t^2+1} - \dfrac{1}{(t+1)^2}\right) = \dfrac{1}{2}\left(\dfrac{2t}{\quad ''\quad}\right) \right)$

$$= 2\tan^{-1}t - \frac{2}{-1}(t+1)^{-1} = \underline{2\tan^{-1}t + 2\,\frac{1}{t+1}}$$

$$(3)\ \int \frac{\overset{(\sin x)'}{\overset{\|}{\boxed{\cos x}}}}{1+\sin^2 x}\,dx = \int \frac{dt}{1+t^2} = \underline{\tan^{-1} t}\,_{//}$$

$$\left(t = \sin x. \qquad \frac{dt}{dx} = \cos x \rightarrow dt = \cos x\, dx \right)$$

$$(4)\ \int \frac{1}{\cos^2 x + 4\sin^2 x}\,dx$$

演(3) より $t = \tan x$ とおくと

$$t^2 = \frac{\sin^2 x}{\cos^2 x} \rightarrow t^2\cos^2 x = 1-\cos^2 x \rightarrow (t^2+1)\cos^2 x = 1 \rightarrow \cos^2 x = \frac{1}{t^2+1}$$

$$\sin^2 x = \frac{t^2}{t^2+1}$$

$$\frac{dt}{dx} = \frac{1}{\cos^2 x} = t^2+1 \rightarrow \frac{dt}{t^2+1} = dx$$

$$= \int \frac{1}{\frac{1}{t^2+1} + \frac{4t^2}{t^2+1}} \frac{1}{t^2+1}\,dt = \int \frac{1}{4t^2+1}\,dt = \frac{1}{2}\int \frac{1}{(2t)^2+1}\,d(2t)$$

$$= \underline{\frac{1}{2}\tan^{-1}(2t)}\,,$$

問3.7

(1) $\int \tan^3 x \, dx$

$= \int \frac{\sin^3 x}{\cos^3 x} dx = \int \frac{1-\cos^2 x}{\cos^3 x} \sin x \, dx = \int \frac{1-t^2}{t^3} (-dt)$

$\left(t = \cos x \quad \frac{dt}{dx} = -\sin x \to -dt = \sin x \, dx \right)$

$= \int \frac{t^2-1}{t^3} dt = \int \frac{1}{t} - t^{-3} dt = \log|t| - \frac{1}{-2} t^{-2} = \underline{\log|t| + \frac{1}{2} t^{-2}}$

(2) $\int \frac{1+\sin x}{\sin x (1+\cos x)} dx$

ココ

$\left(t = \tan\frac{x}{2} \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2} \quad \frac{2dt}{t^2+1} = dx \right)$

$= \int \frac{1 + \frac{2t}{1+t^2}}{\frac{2t}{\cancel{1+t^2}} \left(1 + \frac{2}{1+t^2}\right)} \frac{1}{\cancel{t^2+1}} dt = \int \frac{t^2+1+2t}{2t(t^2+3)} dt$

$\left(\frac{t^2+2t+1}{t(t^2+3)} = \frac{a}{t} + \frac{bt+c}{t^2+3} = \frac{at^2+3a+bt^2+ct}{t(t^2+3)} = \frac{(a+b)t^2+ct+3a}{〃} \right)$

$\begin{cases} a+b=1 \to b=\frac{2}{3} \\ c=2 \\ 3a=1 \to a=\frac{1}{3} \end{cases}$

$= \frac{1}{2} \int \frac{\frac{1}{3}}{t} + \frac{\frac{2}{3}t+\frac{1}{3}}{t^2+3} dt = \frac{1}{6} \underbrace{\int \frac{1}{t} dt}_{\log|t|} + \frac{1}{6} \underbrace{\int \frac{2t}{t^2+3} dt}_{\log(t^2+3)} + \frac{1}{\sqrt{3}} \underbrace{\int \frac{1}{\left(\frac{t}{\sqrt{3}}\right)^2+1} \frac{dt}{\sqrt{3}}}_{\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)}$

$= \frac{1}{6} \log|t| + \frac{1}{6} \log(t^2+3) + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right)$

$= \int \frac{1+\frac{2t}{1+t^2}}{\frac{2t}{\cancel{1+t^2}}\left(1+\frac{1-t^2}{1+t^2}\right)} \frac{2}{\cancel{t^2+1}} dt = \int \frac{t^2+2t+1}{t \cdot 2} dt = \int \frac{t}{2} + 1 + \frac{1}{2t} dt$

$= \frac{t^2}{4} + t + \frac{1}{2} \log|t|$

(3) $\displaystyle\int \frac{1}{1+\sin x}\,dx$

$$= \int \frac{1-\sin x}{\underset{\underset{\cos^2 x}{\parallel}}{\boxed{1-\sin^2 x}}}\,dx = \int \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\,dx = \underbrace{\int \frac{1}{\cos^2 x}\,dx}_{①} + \underset{\underset{-1(\cos x)^{-1}}{\parallel}}{\boxed{\int \frac{(\cos x)'}{\cos^2 x}\,dx}} \cdots (*)$$

$$\left(\begin{array}{l} ① \text{ は } t = \tan x. \quad t^2 = \frac{1-\cos^2 x}{\cos^2 x} \quad (t^2+1)\cos^2 x = 1 \\ \qquad\qquad\qquad\qquad\qquad\qquad \cos^2 x = \frac{1}{t^2+1} \\ \frac{dt}{dx} = \frac{1}{\cos^2 x} \qquad dt = \frac{1}{\cos^2 x}\,dx \end{array} \right)$$

$$① = \int dt = t.$$

$$(*) = \tan x - \frac{1}{\cos x}$$ //

(4) $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{x}{2} + \frac{\sin 2x}{4}$$

(5) $$\int \frac{1}{2\sin x + \cos x}\,dx = \int \frac{1}{4t/1+t^2 + 1-t^2/1+t^2} \cdot \frac{2}{1+t^2}\,dt$$

$$\left(t = \tan\frac{x}{2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad dx = \frac{2}{1+t^2}\,dt \right)$$

$$= \int \frac{1}{-t^2+4t+1}\,2\,dt = -2\int \frac{1}{t^2-4t-1}\,dt$$

$$\left(\begin{array}{l} t^2-4t-1 = 0 \qquad t = 2 \pm \sqrt{4+1} = 2 \pm \sqrt{5} = \alpha, \beta \quad (\alpha < \beta) \\ \frac{1}{t^2-4t-1} = \frac{1}{\beta-\alpha}\left(\frac{1}{t-\beta} - \frac{1}{t-\alpha}\right) = \frac{1}{\beta-\alpha}\left(\frac{\beta-\alpha}{\,''\,}\right) \end{array} \right)$$

$$= -\frac{2}{2\sqrt{5}}\int \frac{1}{t-\beta} - \frac{1}{t-\alpha}\,dt = -\frac{1}{\sqrt{5}}\left\{\log|t-\beta| - \log|t-\alpha|\right\}$$

$$= -\frac{1}{\sqrt{5}}\log\left|\frac{t-2-\sqrt{5}}{t-2+\sqrt{5}}\right|$$ //

例 3.8.

(1) $\displaystyle\int \frac{1}{1+\sqrt[3]{1+x}}\,dx = \int \frac{1}{1+t}\,3t^2\,dt$

$\left(t=\sqrt[3]{1+x} \rightarrow t^3 = 1+x \rightarrow 3t^2 dt = dx \right)$

無理(5)より

$$\begin{array}{r} t-1 \\ t+1 \overline{)\, t^2 \qquad} \\ t^2+t \\ \hline -t \\ -t-1 \\ \hline 1. \end{array}$$

$\displaystyle = 3\int t-1+\frac{1}{t+1}\,dt = 3\left(\frac{t^2}{2}-t+\log|t+1|\right)$

(2) $\displaystyle\int \frac{1}{\sqrt{(x-1)(2-x)}}\,dx$

無理(7)iiより $t=\sqrt{\dfrac{x-1}{2-x}} \rightarrow t^2 = \dfrac{x-1}{2-x} \rightarrow (2-x)t^2 = x-1$

$(t^2+1)x = 2t^2+1$

$x = \dfrac{2t^2+1}{t^2+1} = 2-\dfrac{1}{t^2+1}$

$\dfrac{dx}{dt} = (t^2+1)^{-2}\cdot 2t \qquad dx = \dfrac{2t}{(t^2+1)^2}\,dt$

$\sqrt{(x-1)(2-x)} = \sqrt{t^2(2-x)^2} = t(2-x) = t\cdot\dfrac{1}{t^2+1}$

与式 $\displaystyle = \int \frac{\cancel{t^2+1}}{t}\,\frac{2\cancel{t}}{(t^2+1)^{\cancel{2}}}\,dt = 2\int\frac{dt}{t^2+1} = 2\tan^{-1}t$

(3) $$\int \frac{1}{(x-1)\sqrt{x^2-4x-2}}\,dx$$

$$t - x = \sqrt{x^2-4x-2} \quad \rightarrow \quad t^2 - 2tx + \cancel{x^2} = \cancel{x^2} - 4x - 2$$

$$(2t-4)x = t^2+2$$

$$x = \frac{t^2+2}{2(t-2)}$$

$$x-1 = \frac{t^2+2-2t+4}{2t-4} = \frac{t^2-2t+6}{〃}$$

$$\sqrt{x^2-4x-2} = t - x = t - \frac{t^2+2}{2t-4} = \frac{2t^2-4t-t^2-2}{2t-4} = \frac{t^2-4t-2}{2t-4}$$

$$\frac{dx}{dt} = \frac{2t(2t-4)-(t^2+2)\cdot 2}{(2t-4)^2} = \frac{4t^2-8t-2t^2-4}{〃} = \frac{2t^2-8t-4}{(2t-4)^2}$$

$$与式 = \int \frac{1}{\frac{t^2-2t+6}{\cancel{2t-4}}\ \frac{\cancel{t^2-4t-2}}{\cancel{2t-4}}}\ \frac{\cancel{2t^2-8t-4} \rightarrow 2}{(\cancel{2t-4})^2}\,dt$$

$$= \int \frac{2}{t^2-2t+6}\,dt.$$

$$= 2\int \frac{1}{(t-1)^2+5}\,dt$$

$$= \frac{2}{5}\int \frac{1}{\left(\frac{t-1}{\sqrt{5}}\right)^2+1}\,dt = \frac{2}{\sqrt{5}}\int \frac{1}{\left(\frac{t-1}{\sqrt{5}}\right)^2+1}\,d\left(\frac{t-1}{\sqrt{5}}\right)$$

$$= \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{t-1}{\sqrt{5}}\right)$$

×

(3) $\displaystyle\int \frac{1}{(x-1)\sqrt{x^2-4x-2}}\,dx$ 　 $\boxed{\sqrt{x^2-4x-2} = t-x}$ が"正しい"

無理(7)(i) より

$t=\sqrt{x^2-4x-2}-x \rightarrow t^2+2tx+\cancel{x^2}=\cancel{x^2}-4x-2$

$(2t+4)x=-t^2-2, \quad x=\dfrac{-t^2-2}{2t+4}$

$x-1 = \dfrac{-t^2-2-2t-4}{2t+4} = \dfrac{-t^2-2t-6}{2t+4}$

$\sqrt{x^2-4x-2} = t+x = t+\dfrac{-t^2-2}{2t+4} = \dfrac{2t^2+4t-t^2-2}{2t+4} = \dfrac{t^2+4t-2}{2t+4}$

$\dfrac{dx}{dt} = \dfrac{-2t(2t+4)-(-t^2-2)\cdot 2}{(2t+4)^2} = \dfrac{-4t^2-8t+2t^2+4}{〃} = \dfrac{-2t^2-8t+4}{〃}$

$= \dfrac{-2}{4}\,\dfrac{t^2+4t-2}{(t+2)^2}$

与式 $= \displaystyle\int \frac{1}{\frac{1}{4}\frac{-t^2-2t-6}{\cancel{2}t+\cancel{4}_2}\,\cancel{\frac{t^2+4t-2}{\cancel{2}t+\cancel{4}_2}}}\;-\frac{1}{2}\cancel{\frac{t^2+4t-2}{(t+2)^2}}\,dt$

$= \displaystyle\int \frac{+\cancel{4}2}{t^2+2t+6}\,dt = 2\int \frac{1}{(t+1)^2+5}\,dt = \frac{2}{\sqrt5}\int \frac{1}{\left(\frac{t+1}{\sqrt5}\right)^2+1}\,\frac{dt}{\sqrt5}$

$= \dfrac{2}{\sqrt5}\tan^{-1}\left(\dfrac{t+1}{\sqrt5}\right)$

問 3.8

(1) $\displaystyle\int \frac{1}{x}\sqrt{\frac{1-x}{x}}\,dx$

$t=\sqrt{\frac{1-x}{x}}$ $\quad t^2=\frac{1-x}{x}$ $\quad t^2x=1-x,\quad (t^2+1)x=1,\quad x=\frac{1}{t^2+1}$

$\frac{dx}{dt}=\frac{-1}{(t^2+1)^2}(2t) \rightarrow dx=\frac{-2t}{(t^2+1)^2}dt$

$$=\int (t^2+1)t\,\frac{-2t}{(t^2+1)^2}\,dt=\int \frac{-2t^2-2+2}{t^2+1}\,dt=\int -2+\frac{2}{t^2+1}\,dt$$

$$=\underline{-2t+2\tan^{-1}t}\,//$$

(2) $\displaystyle\int \frac{1}{\sqrt{x^2+2}}\,dx=\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(\frac{x}{\sqrt{2}}\right)^2+1}}\,dx=\int \frac{1}{\sqrt{\left(\frac{x}{\sqrt{2}}\right)^2+1}}\,d\left(\frac{x}{\sqrt{2}}\right)$

$$=\underline{\log\left|\frac{x}{\sqrt{2}}+\sqrt{\left(\frac{x}{\sqrt{2}}\right)^2+1}\right|=\log\left|x+\sqrt{x^2+2}\right|+C}\,//$$

(別)

$\sqrt{x^2+2}=t-x$ とおくと $\quad \cancel{x^2}+2=t^2-2tx+\cancel{x^2}$

$2tx=t^2-2.\quad x=\frac{t^2-2}{2t}$

$\sqrt{x^2+2}=t-\frac{t^2-2}{2t}=\frac{2t^2-t^2+2}{2t}=\frac{t^2+2}{2t}$

$\frac{dx}{dt}=\frac{2t\cdot 2t-(t^2-2)\cdot 2}{4t^2}=\frac{2t^2+4}{4t^2}=\frac{t^2+2}{2t^2}$

$$\text{与式}=\int \frac{1}{\frac{\cancel{t^2+2}}{2t}}\,\frac{\cancel{t^2+2}}{2t^2}\,dt=\int\frac{dt}{t}=\underline{\log|t|}=\underline{\log\left|x+\sqrt{x^2+2}\right|}$$

(3) $\displaystyle\int \frac{1}{(x-1)\sqrt{2+x-x^2}}\,dx$

$$-x^2+x+2 = -(x^2-x-2) = -(x+1)(x-2) \quad \begin{cases} \alpha = -1 \\ \beta = 2. \end{cases}$$

$$t = \sqrt{\frac{x-\alpha}{\beta-x}} = \sqrt{\frac{x+1}{2-x}}$$

$$t^2 = \frac{x+1}{2-x} \rightarrow t^2(2-x) = x+1 \rightarrow (1+t^2)x = 2t^2-1$$

$$x = \frac{2t^2-1}{1+t^2}$$

$$x-1 = \frac{2t^2-1-1-t^2}{1+t^2} = \frac{t^2-2}{t^2+1}$$

$$\sqrt{2+x-x^2} = \sqrt{(x+1)(2-x)} = \sqrt{t^2(2-x)^2} = t(2-x)$$

$$= t\left(\frac{2+\cancel{2t^2}-\cancel{2t^2}+1}{1+t^2}\right) = \underline{\frac{3t}{t^2+1}}$$

$$\frac{dx}{dt} = \frac{4t(1+t^2)-(2t^2-1)2t}{(1+t^2)^2} = \frac{2t(2+\cancel{2t^2}-\cancel{2t^2}+1)}{(1+t^2)^2} = \frac{6t}{(1+t^2)^2}$$

$$= \int \frac{1}{\frac{t^2-2}{\cancel{t^2+1}}\cdot\frac{\cancel{3t}}{\cancel{t^2+1}}}\cdot\frac{\cancel{6t}\,2}{\cancel{(t^2+1)^2}}\,dt = \int \frac{2}{t^2-2}\,dt$$

$$= \frac{\cancel{2}}{\cancel{2}\sqrt{2}}\int \frac{1}{t-\sqrt{2}} - \frac{1}{t+\sqrt{2}}\,dt = \frac{1}{\sqrt{2}}\log|t-\sqrt{2}| - \log|t+\sqrt{2}|$$

$$= \underline{\frac{1}{\sqrt{2}}\log\left|\frac{t-\sqrt{2}}{x+\sqrt{2}}\right|}\ //$$

(3)' $\displaystyle\int \frac{1}{(x-1)\sqrt{2-x-x^2}}\,dx$ → しまった $\displaystyle\int \frac{1}{(x-1)\sqrt{2+x-x^2}}\,dx$ だった

無理(7)(ii)より $-x^2-x+2 = -(x^2+x-2) = -(x-1)(x+2)$ $\begin{cases}\alpha=-2\\ \beta=1\end{cases}$

$t = \sqrt{\dfrac{x-\alpha}{\beta-x}} = \sqrt{\dfrac{x+2}{1-x}}$ とおくと

$t^2 = \dfrac{x+2}{1-x} \to (1-x)t^2 = x+2$ $\quad (1+t^2)x = t^2-2$, $x = \dfrac{t^2-2}{1+t^2}$

$x-1 = \dfrac{t^2-2}{t^2+1} - 1 = \dfrac{t^2-2-(t^2+1)}{t^2+1} = \dfrac{-3}{t^2+1}$

$\sqrt{2-x-x^2} = \sqrt{(1-x)(x+2)} = \sqrt{(1-x)^2t^2} = (1-x)t = \dfrac{3}{t^2+1}\cdot t$

$\dfrac{dx}{dt} = \dfrac{2t(t^2+1)-(t^2-2)\cdot 2t}{(1+t^2)^2} = \dfrac{2t}{(1+t^2)^2}\cdot 3$

与式 $= \displaystyle\int \frac{1}{\frac{-3}{\cancel{t^2+1}}\;\frac{3\cancel{t}}{\cancel{t^2+1}}}\;\frac{6\cancel{t}}{\cancel{(1+t^2)^2}}\,dt$

$= \displaystyle\int \frac{\cancel{6}2}{-\cancel{9}3}\,dt = -\frac{2}{3}t$

(4) $\displaystyle\int \frac{1}{x^2(x^2-1)^{3/2}}\,dx$

$x=\sec\theta$ とおくと　$x^2=\dfrac{1}{\cos^2\theta}$, $x^2-1=\dfrac{1-\cos^2\theta}{\cos^2\theta}=\dfrac{\sin^2\theta}{\cos^2\theta}$

$(x^2-1)^{\frac{3}{2}}=\dfrac{\sin^3\theta}{\cos^3\theta}$

$\dfrac{dx}{d\theta}=\dfrac{-1\cdot(-\sin\theta)}{\cos^2\theta}=\dfrac{\sin\theta}{\cos^2\theta}$

与式 $=\displaystyle\int \frac{1}{\cancel{\frac{1}{\cos^2\theta}}\ \frac{\sin^3\theta}{\cos^3\theta}}\ \frac{\sin\theta}{\cancel{\cos^2\theta}}\,d\theta=\int\frac{\cos^3\theta}{\sin^2\theta}\,d\theta$

$=\displaystyle\int\frac{1-\sin^2\theta}{\sin^2\theta}\underset{(\sin\theta)'}{\boxed{\cos\theta}}\,d\theta=\int\frac{1-t^2}{t^2}\,dt=\int\frac{1}{t^2}-1\,dt=\underline{\frac{-1}{t}-t}$

$t=\sin\theta$

(5) $\displaystyle\int \frac{1}{x\sqrt{4-x^2}}\,dx = \int \frac{1}{\frac{1}{t}\sqrt{4-\frac{1}{t^2}}}\,\frac{-1}{t^2}\,dt$

$\left(x = \dfrac{1}{t} \right.$ とおく. $\left. \dfrac{dx}{dt} = \dfrac{-1}{t^2}. \right)$

$\displaystyle = \int \frac{-1}{\sqrt{4t^2-1}}\,dt. = \frac{-1}{2}\int \frac{1}{\sqrt{(2t)^2-1}}\,d(2t)$

$\displaystyle = -\frac{1}{2}\log\left|2t+\sqrt{(2t)^2-1}\right| = \underline{-\frac{1}{2}\log\left|t+\sqrt{t^2-\frac{1}{4}}\right| + C}$ .//

例 3.9

(1). $\int \frac{1}{e^{2x}-2e^{x}}\,dx = \int \frac{1}{t^2-2t}\,\frac{dt}{t} = \int \frac{1}{t^2(t-2)}\,dt \quad \cdots (*)$

$\left(t = e^{x} \qquad \frac{dt}{dx} = e^{x}, \quad dx = \frac{dt}{e^{x}} = \frac{dt}{t} \right)$

$\frac{1}{t^2(t-2)} = \frac{a}{t} + \frac{b}{t^2} + \frac{c}{t-2} = \frac{at(t-2)+b(t-2)+ct^2}{t^2(t-2)}$

分子 $= (a+c)t^2 + (-2a+b)t + (-2b)$

$\begin{cases} a+c=0 \longrightarrow c = \frac{1}{4} \\ -2a+b=0 \longrightarrow a = \frac{b}{2} = -\frac{1}{4} \\ -2b=1 \to b = -\frac{1}{2} \end{cases}$

$(*) = \int \frac{-\frac{1}{4}}{t} + \frac{-\frac{1}{2}}{t^2} + \frac{\frac{1}{4}}{t-2}\,dt = -\frac{1}{4}\log|t| + \frac{1}{2}t^{-1} + \frac{1}{4}\log|t-2|$

$= \underline{\frac{1}{4}\log\left|\frac{t-2}{t}\right| + \frac{1}{2}\frac{1}{t}}$ //

(2) $\int \frac{\sqrt{1+\log x}}{x}\,dx = \int \sqrt{t}\,dt = \frac{2}{3}t^{\frac{3}{2}}$

$\left(\frac{1}{x} = (1+\log x)' \right.$ より $t = 1+\log x$ とおく $\left. \right)$

(3) $\int \frac{1}{\sqrt{e^{3x}+4}}\,dx$

$\left(\sqrt{e^{3x}+4} = t. \longrightarrow e^{3x}+4 = t^2 \longrightarrow e^{3x}\cdot 3\frac{dx}{dt} = 2t \right.$

$\left. \frac{dx}{dt} = \frac{2}{3e^{3x}}t = \frac{2}{3(t^2-4)}t \right)$

与式 $= \int \frac{1}{\cancel{t}}\,\frac{2}{3(t^2-4)}\cancel{t}\,dt = \frac{2}{3}\frac{1}{4}\int \frac{1}{t-2} - \frac{1}{t+2}\,dt$

$= \underline{\frac{1}{6}\log\left|\frac{t-2}{t+2}\right|}$ //

問3.9.

(1) $\int \frac{(\log x)^n}{x} dx \quad (n \neq -1) = \int t^n dt = \underline{\frac{t^{n+1}}{n+1}}$ //

$\left(t = \log x. \quad \frac{dt}{dx} = \frac{1}{x} \rightarrow \frac{1}{x} dx = dt \right)$

(2) $\int \frac{e^{2x}}{\sqrt[4]{e^x+1}} dx$

$\left(\begin{array}{l} t = \sqrt[4]{e^x+1} \longrightarrow t^4 = e^x + 1 \longrightarrow e^x \frac{dx}{dt} = 4t^3 \\ dx = \frac{4t^3}{e^x} dt = \frac{4t^3}{t^4-1} dt \end{array} \right)$

与式 $= \int \frac{(t^4-1)^{\cancel{2}}}{\cancel{t}} \frac{4t^{\cancel{3}2}}{\cancel{(t^4-1)}} = \int (t^4-1) \cdot 4t^2 dt$

$= \int 4t^6 - 4t^2 dt = \underline{\frac{4}{7}t^7 - \frac{4}{3}t^3} = \frac{4}{7}(e^x+1)^{\frac{7}{4}} - \frac{4}{3}(e^x+1)^{\frac{3}{4}}$

$\boxed{\text{check.}}$

微分 $\rightarrow \frac{4}{7} \cdot \frac{7}{4} (e^x+1)^{\frac{3}{4}} \cdot e^x - \frac{4}{3} \cdot \frac{3}{4} (e^x+1)^{-\frac{1}{4}} e^x$

$= (e^x+1)^{1-\frac{1}{4}} e^x - e^x (e^x+1)^{-\frac{1}{4}}$

$= \frac{e^x}{\sqrt[4]{e^x+1}} (e^x+1-1) = \frac{e^{2x}}{\sqrt[4]{e^x+1}} \qquad \boxed{\text{ok}}$

$$(3)\quad \int \frac{\overset{(e^x+e^{-x})'}{\overset{\shortparallel}{\boxed{e^x - e^{-x}}}}}{e^x + e^{-x}}\,dx$$

$$= \underline{\log\left(e^x + e^{-x}\right)}$$

$$(4)\quad \int x e^{-x^2}\,dx = \int e^{-x^2}\frac{(-x^2)'}{-2}\,dx = \underline{\frac{e^{-x^2}}{-2}}\,//$$

$$(5)\quad \int \log(1+\sqrt{x})\,dx = \int \log(1+t)\overset{(t^2)'}{2t}\,dt$$

$$\left(t = \sqrt{x},\quad \frac{dt}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2t},\quad 2t\,dt = dx\right)$$

$$= t^2\log(1+t) - \underbrace{\int t^2\frac{1}{t+1}\,dt}_{①} \quad \cdots (*)$$

$$\begin{array}{r|l} & t-1 \\ \hline t+1 & t^2 \\ & t^2+t \\ \hline & -t \\ & -t-1 \\ \hline & 1 \end{array}$$

$$① = \int t-1+\frac{1}{t+1}\,dt$$

$$= \frac{t^2}{2} - t + \log|t+1|$$

$$(*) = \underline{t^2\log(1+t) - \left(\frac{t^2}{2} - t + \log|t+1|\right)}$$

$\boxed{3.4}$ 定積分

◇ $f(x)$ がある区間で不定積分を持ち、その1つを $F(x)$ とすると、

$$(\quad F'(x) = f(x) \quad)$$

$$\underbrace{\int_a^b f(x)\,dx}_{a\text{から}b\text{までの定積分}} = F(b) - F(a) = \left[F(x)\right]_a^b$$

上端 (b)、下端 (a)

◇ 基本的な性質

(1) $$\int_a^a f(x)\,dx = F(a) - F(a) = 0$$

(1)' $$\int_a^b f(x)\,dx = F(b) - F(a) = -(F(a) - F(b)) = -\int_b^a f(x)\,dx$$

(2) $$\int_a^b f(x)\,dx = F(b) \underbrace{\;}_{-F(c)+F(c)} - F(a) = \int_c^b f(x)\,dx + \int_a^c f(x)\,dx$$

$$= \int_a^c + \int_c^b$$

線形性

(3) $$\int_a^b \{f(x) \pm g(x)\}\,dx = \left[F(x) \pm G(x)\right]_a^b = \left[F(x)\right]_a^b \pm \left[G(x)\right]_a^b$$

$$= \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

(4) $$\int_a^b k f(x)\,dx = \left[kF(x)\right]_a^b = k\left[F(x)\right]_a^b = k\int_a^b f(x)\,dx$$

定理 3.5

$f(x)$ は $[a, b]$ で連続　$x = g(t)$ は $[\alpha, \beta]$ で微分可能
$g'(t)$ は連続　$a = g(\alpha)$，$b = g(\beta)$

$$\Rightarrow \int_a^b f(x)\,dx = \int_\alpha^\beta f(g(t))\,g'(t)\,dt$$

$$\left(\begin{array}{l} x = g(t), \quad dx = g'(t)\,dt \\ \begin{array}{c|c} x & a \longrightarrow b \\ \hline t & \alpha \longrightarrow \beta \end{array} \end{array} \right)$$

定理 3.6　　$f(x), g(x)$ が C^1 級

$$\int_a^b f(x) g'(x)\,dx = \left[f(x) g(x) \right]_a^b - \int_a^b f'(x) g(x)\,dx$$

$$\because \int_a^b f(x) g'(x)\,dx = \left[f(x) g(x) - \int f'(x) g(x)\,dx \right]_a^b$$

$$= \left[f(x) g(x) \right]_a^b - \int_a^b f'(x) g(x)\,dx$$

例 3.10.

$$\int_0^{\pi/2} \frac{\sin x}{1+\sin x}\,dx = \int_0^1 \frac{\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2}} \quad \frac{2}{1+t^2}\,dt$$

$t = \tan\frac{x}{2} \qquad \sin x = \frac{2t}{1+t^2}$

x	$0 \to \pi/2$
t	$0 \to 1$

$$\frac{dt}{dx} = \frac{1}{\cos^2\frac{x}{2}}\,\frac{1}{2} = \frac{(1+t^2)}{2} \rightarrow \frac{2\,dt}{1+t^2} = dx$$

$$\underbrace{1+\tan^2\frac{x}{2}}_{1+t^2} = \frac{1}{\cos^2\frac{x}{2}}$$

$$= \int_0^1 \frac{4t}{(t^2+2t+1)(1+t^2)}\,dt = \int_0^1 2\left(\frac{1}{1+t^2} - \frac{1}{t^2+2t+1}\right)dt$$

$$= 2\left[\tan^{-1} t\right]_0^1 + 2\left[(t+1)^{-1}\right]_0^1$$

$$= 2\cdot\frac{\pi}{4} + 2\underbrace{\left\{\frac{1}{2} - 1\right\}}_{-\frac{1}{2}} = \underline{\frac{\pi}{2} - 1}$$

例 3.11

$$\int_1^e x\log x\,dx = \int_1^e \left(\frac{x^2}{2}\right)'\log x\,dx = \left[\frac{x^2}{2}\log x\right]_1^e - \int_1^e \frac{x^{\cancel{2}}}{2}\frac{1}{\cancel{x}}\,dx$$

$$= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e = \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right) = \underline{\frac{e^2}{4} + \frac{1}{4}}$$

問 3.10

(1) $$\int_2^3 \frac{1}{1-x^2}\,dx = \int_2^3 \frac{1}{2}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)dx = \left[\frac{1}{2}\underbrace{\left(-\log|1-x|+\log|x+1|\right)}_{\log\left|\frac{x+1}{x-1}\right|}\right]_2^3$$

$$= \frac{1}{2}\left\{\log\frac{4}{2} - \log\frac{3}{1}\right\} = \underline{\frac{1}{2}\log\frac{2}{3}}$$

(2) $$\int_0^{1/2}\frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{1/2}\frac{-1+x^2}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \int_0^{1/2} -\sqrt{1-x^2}+\frac{1}{\sqrt{1-x^2}}\,dx = \Big[\underbrace{-\frac{1}{2}\left(x\sqrt{1-x^2}+\sin^{-1}x\right)+\sin^{-1}x}_{-\frac{1}{2}x\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x}\Big]_0^{1/2}$$

$$= -\frac{1}{4}\sqrt{\frac{3}{4}}+\frac{1}{2}\underbrace{\sin^{-1}\frac{1}{2}}_{\frac{\pi}{6}} = \underline{\frac{\pi}{12}-\frac{\sqrt{3}}{8}} \;/\!/$$

(3) $$\int_0^{\sqrt{3}/2}\frac{5}{4x^2+3}\,dx = \frac{5}{3}\int_0^{\sqrt{3}/2}\frac{1}{\frac{4}{3}x^2+1}\,dx = \frac{5}{3}\int_0^1\frac{1}{t^2+1}\,\frac{\sqrt{3}}{2}\,dt$$

$\left(t=\frac{2}{\sqrt{3}}x \quad dt=\frac{2}{\sqrt{3}}dx \right)$

x	$0 \to \sqrt{3}/2$
t	$0 \to 1.$

$$= \frac{5\sqrt{3}}{6}\left[\tan^{-1}t\right]_0^1 = \frac{5}{6}\sqrt{3}\,\frac{\pi}{4} = \underline{\frac{5\sqrt{3}}{24}\pi}$$

(4) $$\int_0^1\frac{3}{\sqrt{5x^2+4}}\,dx = \frac{3}{2}\int_0^1\frac{1}{\sqrt{\left(\frac{\sqrt{5}}{2}x\right)^2+1}}\,dx = \frac{3}{2}\int_0^{\frac{\sqrt{5}}{2}}\frac{1}{\sqrt{t^2+1}}\,\frac{2}{\sqrt{5}}\,dt$$

$\left(t=\frac{\sqrt{5}}{2}x \;,\quad dt=\frac{\sqrt{5}}{2}dx \right)$

x	$0 \to 1$
t	$0 \to \frac{\sqrt{5}}{2}$

$$= \frac{3}{\sqrt{5}}\int_0^{\frac{\sqrt{5}}{2}}\frac{1}{\sqrt{t^2+1}}\,dt$$

$$= \frac{3}{\sqrt{5}}\left[\log\left|t+\sqrt{t^2+1}\right|\right]_0^{\sqrt{5}/2}$$

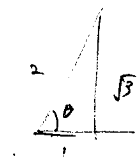
$$= \frac{3}{\sqrt{5}}\left(\log\left|\frac{\sqrt{5}}{2}+\underbrace{\sqrt{\frac{5}{4}+1}}\right| - \underbrace{\log 1}_{0}\right) = \underline{\frac{3}{\sqrt{5}}\log\left(\frac{\sqrt{5}}{2}+\frac{3}{2}\right)} \;/\!/$$

(5) $\displaystyle\int_0^{2/\sqrt{3}} \frac{1}{\sqrt{16-9x^2}}\,dx = \frac{1}{4}\int_0^{2/\sqrt{3}} \frac{1}{\sqrt{1-(\frac{3}{4}x)^2}}\,dx$

$$\left(\begin{array}{ll} t=\frac{3}{4}x & dt=\frac{3}{4}\,dx \\ \begin{array}{c|c} x & 0 \to 2/\sqrt{3} \\ \hline t & 0 \to \frac{\sqrt{3}}{2} \end{array} & \end{array}\right) = \frac{1}{4}\int_0^{\sqrt{3}/2} \frac{1}{\sqrt{1-t^2}}\,\frac{4}{3}\,dt$$

$$= \frac{1}{3}\left[\sin^{-1} t\right]_0^{\sqrt{3}/2}$$

$$= \frac{1}{3}\underbrace{\sin^{-1}\frac{\sqrt{3}}{2}}_{\theta} = \frac{1}{3}\cdot\frac{\pi}{3} = \underline{\frac{\pi}{9}}\,_{//}$$



(6) $\displaystyle\int_0^1 e^{-2x}\,dx = \left[\frac{e^{-2x}}{-2}\right]_0^1 = -\frac{1}{2}\left(e^{-2}-e^{0}\right) = \underline{\frac{1-e^{-2}}{2}}\,_{//}$

(7) $\displaystyle\int_0^1 \frac{\overset{+4-4}{x^2}}{\sqrt{x^2+4}}\,dx = \int_0^1 \sqrt{x^2+4} - \frac{4}{\sqrt{x^2+4}}\,dx$

$$= \left[\ \frac{1}{2}\left\{x\sqrt{x^2+4} + 4\log\left(x+\sqrt{x^2+4}\right)\right\} - 4\log\left|x+\sqrt{x^2+4}\right|\ \right]_0^1$$

$$= \frac{1}{2}\left\{\sqrt{5} + 4\log(1+\sqrt{5})\right\} - 4\log(1+\sqrt{5})$$

$$-\left(\frac{1}{2}\left\{ + 4\log 2\right\} - 4\log 2\right)$$

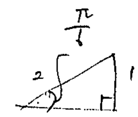
$$= \frac{\sqrt{5}}{2} - 2\log(1+\sqrt{5}) + 2\log 2$$

$$= \underline{\frac{\sqrt{5}}{2} - 2\log\frac{1+\sqrt{5}}{2}}\,_{//}$$

(8) $\displaystyle\int_0^{\cancel{2/\sqrt{3}}\ 1/\sqrt{2}} \frac{3x}{\sqrt{1-x^4}}\,dx = \frac{3}{2}\int_0^{\cancel{4/3}\ \frac{1}{2}} \frac{1}{\sqrt{1-t^2}}\,dt = \frac{3}{2}\left[\sin^{-1} t\right]_0^{\cancel{4/3}\ \frac{1}{2}} = \frac{3}{2}\cdot\frac{\pi}{6}$

$\displaystyle = \frac{\pi}{4}$ //

$$\left(\begin{array}{l} t = x^2 \qquad \dfrac{dt}{dx} = 2x. \qquad \dfrac{dt}{2} = x\,dx \\ \begin{array}{c|l} x & 0 \longrightarrow \cancel{2/\sqrt{3}}\ 1/\sqrt{2} \\ \hline t & 0 \longrightarrow \cancel{4/3}\ \frac{1}{2} \end{array} \end{array}\right) = \frac{3}{2}$$



(9) $\displaystyle\int_0^{\pi/2} x^2 \underbrace{\boxed{\cos x}}_{=(\sin x)'}\,dx = \underbrace{\left[x^2 \sin x\right]_0^{\pi/2}}_{\frac{\pi^2}{4}} - \underbrace{\int_0^{\pi/2} 2x\, \overbrace{\boxed{\sin x}}^{(-\cos x)'}\,dx}_{①}$ $\cdots(*)$

$① = \displaystyle\left[2x(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} 2(-\cos x)\,dx$

$\displaystyle = \pi(\underbrace{-\cos \pi/2}_{0}) - 2\left[\sin x\right]_0^{\frac{\pi}{2}}$

$= \cancel{\pi} - 2$

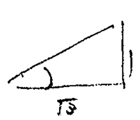
$(*) = \dfrac{\pi^2}{4} - \cancel{\pi} + 2$ // $= \dfrac{\pi^2}{4} + 2$ //

(10) $\displaystyle\int_0^{\pi/2} \frac{1}{2+\cos x}\,dx = \int_0^1 \frac{1}{2+\frac{1-t^2}{1+t^2}}\ \frac{2}{1+t^2}\,dt$

$$\left(\begin{array}{l} t = \tan\frac{x}{2} \qquad \cos x = \dfrac{1-t^2}{1+t^2} \qquad \dfrac{dx}{\cancel{dt}} = \dfrac{2}{1+t^2}\,dt \\ \begin{array}{c|l} x & 0 \longrightarrow \pi/2 \\ \hline t & 0 \longrightarrow 1. \end{array} \end{array}\right)$$

$\displaystyle = \int_0^1 \frac{2}{2t^2+2+1-t^2}\,dt = \int_0^1 \frac{2}{t^2+3}\,dt = \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$

$\displaystyle = \frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}\ \frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}$ //



(11) $\displaystyle\int_{-1}^{1} \frac{1}{(1+x^2)^2}\,dx$

$$\left(\text{P.69. 例3.6} \quad I_n = \int \frac{dx}{(x^2+a^2)^n} = \frac{1}{2(n-1)a^2}\left\{ \frac{x}{(x^2+a^2)^{n-1}} + (2n-3)I_{n-1} \right\} \right)$$

$$= \frac{1}{2}\left\{ \underbrace{\left[\frac{x}{(x^2+1)} \right]_{-1}^{1}}_{\underbrace{\frac{1}{2}-\frac{-1}{2}}_{1}} + \underbrace{\int_{-1}^{1} \frac{1}{1+x^2}\,dx}_{\underbrace{\underbrace{\tan^{-1}1}_{\frac{\pi}{4}} - \underbrace{\tan^{-1}(-1)}_{-\frac{\pi}{4}}}_{\frac{\pi}{2}}} \right\} = \underline{\frac{1}{2}\left\{ 1+\frac{\pi}{2} \right\}}$$ //

(12) $$\int_0^1 x \underset{(e^x)'}{\boxed{e^x}}\,dx = \underbrace{\left[xe^x \right]_0^1}_{e} - \underbrace{\int_0^1 e^x\,dx}_{\left[e^x\right]_0^1} = e - \{e-1\} = \underline{1}$$ //

(13) $$\int_0^1 x^3\sqrt{1-x^2}\,dx = \int_0^1 x^2\sqrt{1-x^2}\,x\,dx = \int_1^0 (1-t)\sqrt{t}\left(-\frac{1}{2}\right)dt$$

$\left(t=1-x^2 \quad \frac{dt}{dx} = -2x \right)$

x	$0 \to 1$
t	$1 \to 0$

$$= \frac{1}{2}\int_0^1 t^{\frac{1}{2}} - t^{\frac{3}{2}}\,dt$$

$$= \frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}} - \frac{2}{5}t^{\frac{5}{2}} \right]_0^1 = \frac{1}{2}\left(\frac{2}{3} - \frac{2}{5} \right)$$

$$= \underline{\frac{2}{15}}$$ //

(14) $$\int_0^1 \frac{1-x^2\boxed{-1+1}}{1+x^2}\,dx = \int_0^1 -1 + \frac{2}{1+x^2}\,dx = \left[-x + 2\tan^{-1}x \right]_0^1$$

$$= -1 + 2\tan^{-1}1 = \underline{-1+\frac{\pi}{2}}$$ //

(15) $\int_0^1 \log(1+\sqrt{x})\, dx \quad = \int_0^1 \log(1+t)\, 2t\, dt$

$\Big(t = \sqrt{x} \qquad dt = \frac{1}{2} x^{-\frac{1}{2}} dx \qquad \underset{t}{\sqrt{x}}\, 2dt = dx$

x	$0 \to 1$
t	$0 \to 1$

$= \underbrace{\left[\log(1+t)\, t^2\right]_0^1}_{\log 2} - \underbrace{\int_0^1 \underbrace{\frac{1}{1+t} \cdot t^2}_{t-1+\frac{1}{t+1}}\, dt}_{①} \quad \cdots (*)$

$$\begin{array}{r} t-1 \\ t+1 \overline{\big) t^2 \qquad} \\ t^2+t \\ \hline -t \\ -t-1 \\ \hline 1 \end{array}$$

$① = \left[\frac{t^2}{2} - t + \log|t+1|\,\right]_0^1 = \frac{1}{2} - 1 + \log 2 = -\frac{1}{2} + \log 2$

$(*) = \underline{\frac{1}{2}}$ //

(16) $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\, dx = \int_0^1 \frac{dt}{1+t^2} = \left[\tan^{-1} t\right]_0^1 = \underline{\frac{\pi}{4}}$ //

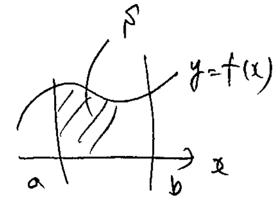
$\Big(t = \sin x \qquad \frac{dt}{dx} = \cos x . \qquad dt = \cos x\, dx$

x	$0 \to \pi/2$
t	$0 \to 1$

$\Big)$

$\boxed{3.5}$ 面積，不等式，微分と積分の関係

定理 3.7　$f(x)$ は $[a,b]$ で連続，$f(x) \geqq 0$



$$S = \int_a^b f(x)\,dx$$

定理 3.8　$f(x), g(x)$，$[a,b]$ で連続

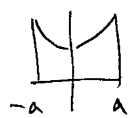
(1)　$g(x) \leqq f(x)$ ならば　$\displaystyle\int_a^b g(x)\,dx \leqq \int_a^b f(x)\,dx$

(2)　$\displaystyle\left|\int_a^b f(x)\,dx\right| \leqq \int_a^b |f(x)|\,dx$

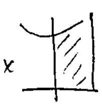
定理 3.9　$f(x)$ が $[a,b]$ で連続 $\Rightarrow$ $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

◇ 公式

(3)　$f(x)$ が偶関数 $\Rightarrow$

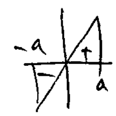


$= 2 \times$



$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)$$

(4)　$f(x)$ が奇〃 $\Rightarrow$



$= 0$

$$\int_{-a}^a f(x)dx = 0$$

(5) $$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \dfrac{n-1}{n}\,\dfrac{n-3}{n-2}\cdots\dfrac{4\cdot 2}{5\cdot 3} & n \geqq 2\ (\text{奇数}) \\ \dfrac{n-1}{n}\,\dfrac{n-3}{n-2}\cdots\dfrac{3}{4}\,\dfrac{1}{2}\times\dfrac{\pi}{2} & n \geqq 2\ (\text{偶数}) \end{cases}$$

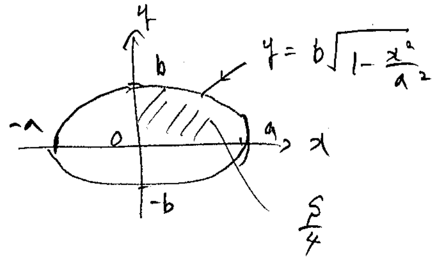
フーリエ級数で用いる

(6) $$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

(7) $$\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0$$

例 3.12

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ の囲む面積 S は?



(解) $\frac{y}{b}=\pm\sqrt{1-\frac{x^2}{a^2}}$

$$\frac{S}{4}=\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx=\int_0^1 b\sqrt{1-t^2}\,a\,dt=ab\int_0^1\sqrt{1-t^2}\,dt$$

$$\left(t=\frac{x}{a}\quad \begin{array}{c|c} x & 0\to a \\ \hline t & 0\to 1\end{array}\qquad dt=\frac{dx}{a}\right)=\frac{ab}{2}\left[t\sqrt{1-t^2}+\sin^{-1}t\right]_0^1$$

$$=\frac{ab}{2}\underbrace{\left(\sin^{-1}1\right)}_{=\frac{\pi}{2}}=\frac{ab}{4}\pi$$

$S=ab\pi$

例 3.13. $\log(1+\sqrt{2})<\int_0^1\frac{dx}{\sqrt{1+x^n}}<1$ $(n>2)$ を示せ

$n>2$ のとき $0<x<1$ で $0<x^n<x^2<1$

$$1\underset{①}{<}\sqrt{1+x^n}\underset{②}{<}\sqrt{1+x^2}$$

① $\Rightarrow$ $\frac{1}{\sqrt{1+x^n}}<1$ ② $\Rightarrow$ $\frac{1}{\sqrt{1+x^2}}<\frac{1}{\sqrt{1+x^n}}$

よって $\frac{1}{\sqrt{1+x^2}}<\frac{1}{\sqrt{1+x^n}}<1$

$$\underbrace{\int_0^1\frac{1}{\sqrt{1+x^2}}dx}<\int_0^1\frac{1}{\sqrt{1+x^n}}dx<\underbrace{\int_0^1 dx}_{=1}$$

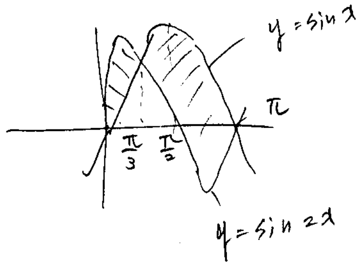
$\left[\log|x+\sqrt{1+x^2}|\right]_0^1$

$=\log(1+\sqrt{2})$

よって $\log(1+\sqrt{2})<\int_0^1\frac{dx}{\sqrt{1+x^n}}<1$

問 3.11

$y=\sin 2x$ と $y=\sin x$ が $0 \leqq x \leqq \pi$ において囲む部分の面積 S_1, S_2 の和



交点は $\sin 2x = \sin x$

$2\sin x\cos x = \sin x$

$\sin x\left(\cos x - \frac{1}{2}\right) = 0$

$\sin x = 0$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$

$$\int_0^{\frac{\pi}{3}} \sin 2x - \sin x \, dx + \int_{\frac{\pi}{3}}^{\pi} \sin x - \sin 2x \, dx \quad \cdots (*)$$

$F(x) = -\frac{\cos 2x}{2} + \cos x$ とおくと

$(*) = F\left(\frac{\pi}{3}\right) - F(0) - F(\pi) + F\left(\frac{\pi}{3}\right)$

$= 2F\left(\frac{\pi}{3}\right) - F(0) - F(\pi)$

$= 2\left(\underbrace{-\frac{\cos\frac{2}{3}\pi}{2}}_{= -\frac{1}{2}\left(-\frac{1}{2}\right) = \frac{1}{4}} + \underbrace{\cos\frac{\pi}{3}}_{\frac{1}{2}}\right) - \underbrace{\left(\frac{-1}{2} + 1\right)}_{\frac{1}{2}} - \underbrace{\left(\frac{-1}{2} - 1\right)}_{-\frac{3}{2}}$

$= \underbrace{2\times\frac{3}{4}}_{1} \underbrace{- \frac{1}{2} + \frac{3}{2}}_{1} = \frac{6-1}{2} = \frac{5}{2}$

問3.12

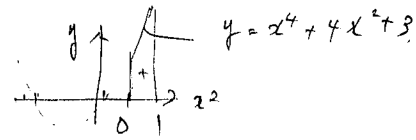
$0 < x < 1$ のとき

$$\sqrt{1-x^2} \underset{①}{\leq} \sqrt{1-x^4} \underset{②}{\leq} 2\sqrt{1+x^2} \quad (*)$$ を示し.

$$\frac{\pi}{2\sqrt{2}} < \int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx < \frac{\pi}{2}$$ を示せ.

[解] ① は $x^2 > x^4$ より $0 < 1-x^2 < 1-x^4 \Longrightarrow \sqrt{1-x^2} \underset{①}{<} \sqrt{1-x^4}$

② $\Leftrightarrow$ $1-x^4 < 4(1+x^2)$ $\Leftrightarrow$ $\underbrace{x^4+4x^2+3}_{(x^2+3)(x^2+1)} > 0$ を示せば OK.



(*) 示せた

このとき
$$\frac{1}{\sqrt{1-x^4}} < \frac{1}{\sqrt{1-x^2}} \qquad \frac{1}{2\sqrt{1+x^2}} < \frac{1}{\sqrt{1-x^4}}$$

よって
$$\int_0^1 \frac{1}{2\sqrt{1+x^2}}\,dx < \int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx < \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\frac{1}{2}\left[\log\left|x+\sqrt{1+x^2}\right|\right]_0^1 \qquad \left[\sin^{-1}x\right]_0^1$$

$$\frac{1}{2}\log(1+\sqrt{2}) \qquad = \underbrace{\sin^{-1}1}_{\frac{\pi}{2}} - \underbrace{\sin^{-1}0}_{0} = \frac{\pi}{2}$$

問題がまちがっているだろ!

$$\sqrt{1-x^2} < \sqrt{1-x^4} < \sqrt{2}\sqrt{1-x^2}$$

$$1-x^2 < (1-x^2)(1+x^2) < 2(1-x^2)$$

より

$$\sqrt{1-x^2} < \sqrt{1-x^4} < \sqrt{2}\sqrt{1-x^2}$$

$$\frac{1}{\sqrt{2}}\underbrace{\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx}_{\frac{\pi}{2}} < \int \frac{1}{\sqrt{1-x^4}}\,dx < \underbrace{\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx}_{\frac{\pi}{2}}$$ より $$\frac{\pi}{2\sqrt{2}} < \int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx < \frac{\pi}{2}$$

問 3.13

$$F(x) = \int_0^x (x-t)\cos(3t)\,dt$$

$F''(x)$ は?

$$F'(x) = \underbrace{(x-x)\cos 3x}_{0} + \underbrace{\int_0^x \cos 3t\,dt}_{\left[\frac{\sin 3t}{3}\right]_0^x} = \frac{\sin 3x}{3}$$

$$F''(x) = \frac{\cos 3x}{\cancel{3}}\cdot\cancel{3} = \underline{\cos 3x}$$

別) $F(x) = \int_0^x x\cos 3t - t\cos 3t\,dt$

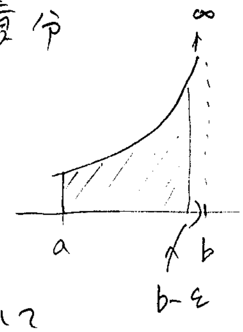
$$= x\int_0^x \cos 3t\,dt - \int_0^x t\cos 3t\,dt$$

$$F'(x) = \int_0^x \cos 3x\,dt + \cancel{x\cos 3x} - \cancel{x\cos 3x}$$

$$\underline{F''(x) = \cos 3x}$$

$\boxed{3.6}$ 広義積分

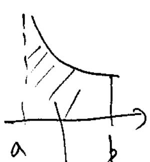
◇特異積分

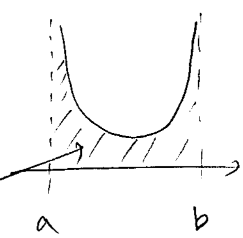


ε>0 に対して

$$\int_a^{b-\varepsilon} f(x)\,dx \text{ は存在する}$$

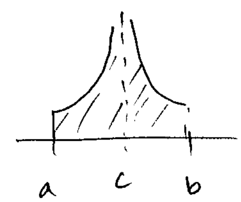
$\displaystyle\lim_{\varepsilon\to 0}\int_a^{b-\varepsilon} f(x)\,dx$ が存在するとき $\displaystyle\underbrace{\int_a^b f(x)\,dx}_{\substack{\text{特異積分}\\(\text{広義積分})}}$ という。





$$\int_a^b f(x)\,dx = \lim_{\substack{\varepsilon\to 0\\ \varepsilon'\to 0}} \int_{a+\varepsilon}^{b-\varepsilon'} f(x)\,dx$$

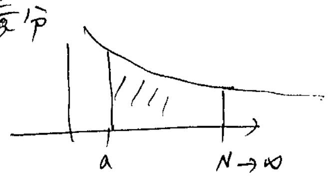
$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0}\int_{a+\varepsilon}^{b} f(x)\,dx$$



$$\int_a^b f(x)\,dx$$

$$= \int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon'}^{b} f(x)\,dx$$

◇ 無限積分



無限積分 $\int_a^\infty f(x)\,dx = \lim_{N\to\infty}\int_a^N f(x)\,dx$

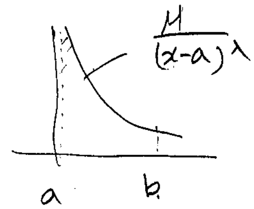
$\int_{-\infty}^a f(x)\,dx$, $\int_{-\infty}^\infty f(x)\,dx$ も同様。

定理3.10

(1) $f(x)$ は $(a,b]$ で連続

$M>0$, $\lambda<1$ に対して $|f(x)|(x-a)^\lambda \le M$ $\quad (a<x<b)$

$\Rightarrow$ $\int_a^b f(x)\,dx$ 存在.



$$0<\underbrace{\int_{a+\varepsilon}^b |f(x)|\,dx}_{\substack{\downarrow \\ \text{存在する}}} < \underbrace{\int_{a+\varepsilon}^b \frac{M}{(x-a)^\lambda}\,dx}_{\substack{\downarrow\ \varepsilon\to 0 \\ \text{存在する}}}$$

$\int_a^b |f(x)|$ が存在すれば $\int_a^b f(x)\,dx$ 存在.

(2) $f(x)$ が $[a,\infty)$ で連続 $M>0$, $\lambda>1$ に対して $x^\lambda |f(x)| \le M$

$\Rightarrow$ $\int_a^\infty f(x)\,dx$ は存在する.

例 3.14

(1) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}\,dx$

$\displaystyle= \lim_{\varepsilon\to 0}\left[\sin^{-1}x\right]_0^{1-\varepsilon} = \underbrace{\sin^{-1}(1)}_{=\frac{\pi}{2}} - \underbrace{\sin^{-1}(0)}_{=0} = \frac{\pi}{2}$

(2) $\displaystyle\underline{\int_{-1}^0 \frac{1}{1-x^2}\,dx} \cdot \int_{-1+\varepsilon}^0 \frac{1}{1-x^2}\,dx$

$\displaystyle= \int_{-1+\varepsilon}^0 \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right)dx = \frac{1}{2}\left[-\log|x-1| + \log|x+1|\right]_{-1+\varepsilon}^0$

$\displaystyle= \frac{1}{2}\left[\log\left|\frac{x+1}{x-1}\right|\right]_{-1+\varepsilon}^0 = \frac{1}{2}\left(0 - \log\frac{\varepsilon}{2}\right) \xrightarrow[(\varepsilon\to 0)]{} \underline{+\infty}$ //

(3) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+4}\,dx = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-\infty}^{\infty} = \frac{1}{2}\Big(\underbrace{\tan^{-1}\infty}_{=\frac{\pi}{2}} - \underbrace{\tan^{-1}(-\infty)}_{=-\frac{\pi}{2}}\Big) = \underline{\frac{\pi}{2}}$ //

問 3.14

(1) $\int_0^1 x\log x\,dx = \underbrace{\left[\frac{x^2}{2}\log x\right]_0^1}_{0} - \underbrace{\int_0^1 \frac{x^2}{2}\frac{1}{x}\,dx}_{\left[\frac{x^2}{4}\right]_0^1}$

$= \frac{1}{4}$

(2) $\int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}}$

X

$t = \sqrt{\frac{2-x}{x-1}}$ $\quad t^2 = \frac{2-x}{x-1}$ $\quad \sqrt{(x-1)(2-x)} = \sqrt{(x-1)^2t^2} = (x-1)t$

$2t\frac{dt}{dx} = \left(-1 + \frac{1}{x-1}\right)' = \frac{-1}{(x-1)^2} \Rightarrow -(x-1)^2 2t\,dt = dx$

x	1	2
t	∞	0

○

$t = \sqrt{\frac{x-1}{2-x}}$ $\quad t^2 = \frac{x-1}{2-x}$ $\quad x-1 = t^2(2-x)$ $\rightarrow$ $(t^2+1)x = 2t^2+1$

$x = \frac{2t^2+1}{t^2+1}$

$\sqrt{(x-1)(2-x)} = \sqrt{t^2(2-x)^2} = t(2-x)$

x	$1 \to 2$
t	$0 \to \infty$

$t^2 = \frac{x-2+1}{2-x} = -1 + \frac{1}{2-x} = -1 - \frac{1}{x-2}$

$2t\frac{dt}{dx} = -1\cdot\frac{-1}{(x-2)^2}$ $\quad (x-2)^2 2t\,dt = dx$

与式 $= \int_0^\infty \frac{(x-2)^2 \cancel{2t}}{\cancel{t}(2-x)}\,dt = \int_0^\infty (2-x)\,dt = \int_0^\infty 2 - \frac{2t^2+1}{t^2+1}\,dt$

$= \int_0^\infty \frac{\cancel{2t^2}+2-\cancel{2t^2}-1}{t^2+1}\,dt = \int_0^1 \frac{1}{t^2+1}\,dt = \left[\tan^{-1}t\right]_0^1 = \underline{\frac{\pi}{4}}$

(3) $\displaystyle\int_0^3 \frac{x}{(x^2-1)^{2/3}}\,dx$　　　$x=1$ で不連続

$$\left(t = x^2-1. \quad \frac{dt}{dx} = 2x \quad \frac{dt}{2} = x\,dx \right)$$

x	$0 \to 3$
t	$-1 \to 8$

$$= \int_{-1}^{8} \frac{dt}{2\,t^{2/3}} = \frac{1}{2}\int_{-1}^{8} t^{-2/3}\,dt = \frac{1}{2}\, 3\left[t^{\frac{1}{3}}\right]_{-1}^{8}$$

$$= \frac{3}{2}\left\{(2^3)^{\frac{1}{3}} - (-1)^{\frac{1}{3}}\right\} = \frac{3}{2}\left(2-(-1)\right) = \underline{\frac{9}{2}}\,,,$$

(4) $\displaystyle\int_1^{\infty} \frac{1}{x^{\alpha}}\,dx$　　$(\alpha>0)$

$$\cancel{= \frac{1}{1-\alpha}\left[x^{1-\alpha}\right]_1^{\infty} = \left(\lim_{x\to\infty} x^{1-\alpha}\right) - 1 \quad \begin{cases} 0<\alpha<1 & \infty \\ \alpha=1 & 0 \\ \alpha>1 & -1. \end{cases}}$$

$$\begin{cases} \alpha \neq 1 \text{ のとき} \quad \dfrac{\left[x^{1-\alpha}\right]_1^{\infty}}{1-\alpha} = \begin{cases} 0<\alpha<1 \text{ のとき} & \infty \\ 1<\alpha \text{ のとき} & \dfrac{-1}{1-\alpha} = \dfrac{1}{\alpha-1} \end{cases} \\ \alpha = 1 \text{ のとき} \quad \left[\log x\right]_1^{\infty} = \infty \end{cases}$$

(5) $\displaystyle\int_1^{\infty} \frac{1}{x(1+x^2)}\,dx = \int_1^{\infty} \frac{x}{x^2(1+x^2)}\,dx$

$$\left(t = x^2 \quad \frac{dt}{dx} = 2x \quad x\,dx = \frac{dt}{2} \right)$$

x	$1 \to \infty$
t	$1 \to \infty$

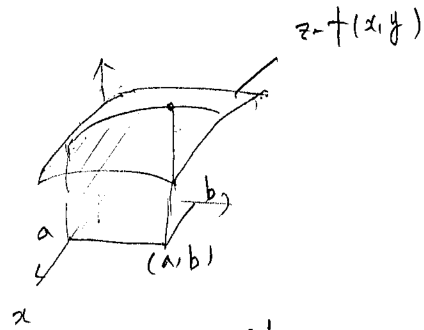
$$= \int_1^{\infty} \frac{1}{t(t+1)}\,\frac{dt}{2} = \frac{1}{2}\int_1^{\infty}\left(\frac{1}{t} - \frac{1}{t+1}\right)dt$$

$$= \frac{1}{2}\left[\log\frac{|t|}{|t+1|}\right]_1^{\infty} = \frac{1}{2}\left(\log 1 - \log\frac{1}{2}\right) = \underline{\frac{1}{2}\log 2}$$

(P. 100)

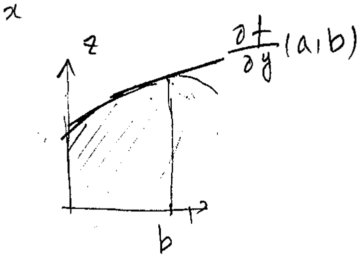
$\boxed{5.2}$ 偏導関数

◇ 偏微分係数



$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y(a,b) = \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k}$$



◇ 偏導関数　$f_x(x,y)$, $f_y(x,y)$

$$\frac{\partial f}{\partial x}(x,y) \ , \ \frac{\partial f}{\partial y}(x,y)$$

例 5.2

$f(x,y) = e^{2x}\sin y$ のとき $(x,y) = (1, \frac{\pi}{2})$ の偏微分係数を求めよ.

$$f_x(x,y) = \frac{\partial}{\partial x}(e^{2x}\sin y) = 2e^{2x}\sin y$$

$$f_y(x,y) = \frac{\partial}{\partial y}(e^{2x}\sin y) = e^{2x}\cos y$$

$$f_x(1, \tfrac{\pi}{2}) = 2e^2\sin\frac{\pi}{2} = 2e^2$$

$$f_y(1, \tfrac{\pi}{2}) = e^2\cos\frac{\pi}{2} = 0$$

問. 5.3

(1) $z = \dfrac{4x-5y}{2x+3y}$

$$z_x = \frac{\partial}{\partial x}\frac{4x-5y}{2x+3y} = \frac{4(2x+3y)-(4x-5y)2}{(2x+3y)^2} = \frac{22y}{(2x+3y)^2}$$

$$z_y = \frac{\partial}{\partial y}\frac{4x-5y}{2x+3y} = \frac{-5(2x+3y)-(4x-5y)3}{(2x+3y)^2} = \frac{-22x}{\text{〃}}$$

(2) $z = \sin^{-1}\dfrac{x}{y}$ $(x>0,\ y>0)$

$$z_x = \frac{\partial}{\partial x}\sin^{-1}\frac{x}{y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\ \frac{1}{y} = \underline{\frac{1}{\sqrt{y^2-x^2}}}\ \text{〃}$$

$$z_y = \frac{\partial}{\partial y}\sin^{-1}\frac{x}{y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\ \frac{x(-1)}{y^2} = \underline{-\frac{x}{y\sqrt{y^2-x^2}}}\ \text{〃}$$

(3) $z = x^y$ $(x>0)$

$$z_x = \frac{\partial}{\partial x}x^y = \underline{y x^{y-1}}\ \text{〃}$$

$$z_y = \frac{\partial}{\partial y}x^y = \frac{\partial}{\partial y}e^{y\log x} = e^{y\log x}\log x = \underline{x^y\log x}\ \text{〃}$$

問 12 5.4, 5.5 omit

p.140　微分方程式の解法

◇ 微分方程式

独立変数 x と関数 y と $y', y'', \cdots y^{(n)}$ の間の方程式

方程式の階数 $= n$

$y\ y' \cdots\ y^{(n)}$ について1次式のもの

線形微分方程式という　$y^{(n)} + P_1(x)y^{(n-1)} + P_2(x)y^{(n-2)} + \cdots + P_n(x)y = Q(x)$

$Q(x)=0$ のものを 同次（斉次）方程式

◇ 微分方程式の解

微分方程式をみたす関数 ＝ 解，解を求めることを「方程式を解く」という

n 階方程式の解で n 個の任意定数を含むもの ＝ 一般解

任意定数に特定の値を代入して得られる解 ＝ 特殊解

(例) $y = A\cos 2x + B\sin 2x$ ①（A, B は定数）

$y' = -2A\sin 2x + 2B\cos 2x$

$y'' = -4A\cos 2x - 4B\sin 2x = -4(A\cos 2x + B\sin 2x) = -4y$

$\therefore\ y'' = -4y$ ② ← 2階微分方程式

①は一般解

②から ①を求めることを微分方程式を解くという

◇ 1階微分方程式

I 変数分離形

$$\frac{dy}{dx} = \frac{P(x)}{Q(y)} \text{ のもの} \qquad \underbrace{Q(y)\,dy}_{y\text{だけ}} \;\vdots\; \underbrace{P(x)\,dx}_{x\text{だけ}}$$

$$\int Q(y)\,dy = \int P(x)\,dx + C$$

II 同次形

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

[解法] $u = \frac{y}{x}$ とおいて $\frac{du}{dx}$ の微分方程式に変形

例A.1

(1) $x^3 \frac{dy}{dx} + y^2 = 0$

$$x^3 \frac{dy}{dx} = -y^2 \rightarrow -\int \frac{dy}{y^2} = \int \frac{1}{x^3}\,dx$$

$$\Rightarrow \quad y^{-1} = \frac{1}{-2}x^{-2} + C = \frac{-1+2x^2C}{2x^2}$$

$$\underline{y = \frac{2x}{2x^2C-1}}.$$

(2) $2xy\frac{dy}{dx} = x^2+y^2$ $\quad x^2$で割ると. $\quad 2\frac{y}{x}\frac{dy}{dx} = 1+\left(\frac{y}{x}\right)^2$

$u = \frac{y}{x}$ とおくと $\quad \frac{du}{dx} = \frac{y'x-y}{x^2} = \frac{\frac{1+u^2}{2u}x - ux}{x^2} = \frac{\frac{1+u^2}{2u}-u}{x} = \frac{\frac{1-u^2}{2u}}{x}$

$2u\,y' = 1+u^2 \rightarrow y' = \frac{1+u^2}{2u}$ $\quad$ と $y = ux$

$$\underbrace{\int \frac{2u}{1-u^2}\,du}_{-\int \frac{(u^2-1)'}{u^2-1}du \,=\, -\log|u^2-1|} = \underbrace{\int \frac{1}{x}\,dx}_{\log|x|} \Rightarrow \log|u^2-1| = -\log|x| + \log C$$

$$|u^2-1| = \frac{C}{|x|} \quad (C>0)$$

$$u^2-1 = \frac{C}{x}$$

$$\frac{y^2}{x^2}-1 = \frac{C}{x} \qquad \underline{y^2-x^2 = Cx}.$$

VI 定数係数の2階同次線形微分方程式

$$\frac{d^2y}{dx^2} + \underbrace{a\frac{dy}{dx} + b}_{\text{定数}}y = 0 \quad \cdots (*)$$

[解法] $y = e^{tx}$ の形と仮定すると

$y' = e^{tx}t$, $y'' = e^{tx}t^2$ を

(*) に代入. $e^{tx}(t^2 + at + b) = 0$

$t^2 + at + b = 0$ … <u>特性方程式</u> という

⇒ 解を α, β とすると

(i) $\underbrace{\alpha \neq \beta}_{\text{(実数)}}$ のとき $y = \underbrace{C_1 e^{\alpha x} + C_2}_{\text{任意定数}} e^{\beta x}$

(ii) $\alpha = \beta$ のとき $y = C_1 e^{\alpha x} + C_2 \, x e^{\alpha x}$

(iii) $\alpha = p + qi$, $\beta = p - qi$ のとき

$y = C_1 e^{px}\cos qx + C_2 e^{px}\sin qx$

例 A.3.

(1) $y'' - 7y' + 12y = 0$

$t^2 - 7t + 12 = (t-4)(t-3) = 0$

$t = 3, 4$ $\qquad$ $\underline{y = C_1 e^{3x} + C_2 e^{4x}}$

(2) $y'' + 2y' + 2y = 0$

$t^2 + 2t + 2 = 0$ $\qquad$ $t = -1 \pm \sqrt{1^2 - 2} = -1 \pm i$

$\underline{y = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x}$

(3) $y'' - 2y' + y = 0$

$t^2 - 2t + 1 = 0$ $\qquad$ $t = 1$

$\underline{y = C_1 e^{x} + C_2 x e^{x}}$