Algorithm 708
Significant Digit Computation of the Incomplete Beta Function Ratios

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An algorithm is given for evaluating the incomplete beta function ratio $I_x(a, b)$ and its complement $1 - I_x(a, b)$ A new continued fraction and a new asymptotic series are used with classical results A transportable Fortran subroutine based on this algorithm is currently in use. It is accurate to 14 significant digits when precision is not restricted by inherent error.

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1. INTRODUCTION

An algorithm is described for computing the incomplete beta function $I_x(a, b)$ and its *complement* $1 - I_x$, where I_x is defined by

$$I_x(a,b) = G(a,b)\int_0^x t^{a-1}(1-t)^{b-1}\,dt,$$

$$a > 0, \quad b > 0, \quad 0 \le x \le 1, \tag{1}$$

$$B(a,b) = 1/G(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt = \Gamma(a)\Gamma(b)/\Gamma(a+b), \tag{2}$$

and the gamma function $\Gamma(u)$ is given by

$$\Gamma(u) = \int_0^\infty e^{-t}t^{u-1}\,dt, \qquad u > 0. \tag{3}$$

Thus $I_1(a, b) = 1$. In addition, if $0 < x < 1$, then $I_x(0, b) = 1$ and $I_x(a, 0) = 0$. Use will also be made of the relation

$$1 - I_x(a,b) = I_y(b,a), \qquad y = 1 - x, \tag{4}$$

which follows by using $u = 1 - t$ in (1).

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A transportable FORTRAN subroutine named BRATIO has been written which uses the algorithm. BRATIO is designed for use on computers having k-digit single-precision floating arithmetics where $6 \le k \le 14$. On the CDC 6000–7000 series computers, BRATIO yields results accurate up to 14 significant digits for I_x and $1 - I_x$. BRATIO is available for general use in the NSWC mathematics subroutine library, [7].

A primary region of difficulty for computing $I_x(a, b)$ and $1 - I_x(a, b)$ has been when a or b is large and $x \approx a/(a + b)$. In this region, I_x changes rapidly from 0 to 1. A new continued fraction and a new asymptotic expansion are used to treat this region satisfactorily.

Section 2 contains the basic equations and algorithms used for BRATIO. Section 3 describes in a, b, x space the regions of use for these basic relations. Section 4 describes a number of specialized algorithms required in order to use the basic relations effectively. Section 5 briefly summarizes the accuracy and efficiency of BRATIO.

The function I_x occurs in many branches of science, including atomic physics, fluid dynamics, transmission theory, lattice theory, and operations research. It is perhaps best known for its extensive applications in statistics. In particular, the well-known central F-distribution, the Student's t-distribution, and the binomial distribution can all be expressed in terms of I_x, [1, p. 945].

2. BASIC RELATIONS

Let

$$p = a/(a + b) \qquad \text{and} \qquad q = 1 - p = b/(a + b). \tag{5}$$

Since I_x and $1 - I_x$ are to be computed to the greatest possible accuracy,

$$y = 1 - x \tag{6}$$

is required as input in addition to a, b, and x. In BRATIO, relations (7)–(11) are used to compute either I_x or its complement. The domains of application for (7)–(11) are given in Section 3. In this section, ϵ will denote an arbitrary relative tolerance.

BPSER(a, b, x, ϵ).

$$I_x(a, b) = G(a, b)\frac{x^a}{a}\left(1 + a \sum_{j=1}^{\infty} \frac{(1 - b)(2 - b) \cdots (j - b)}{j!(a + j)} x^j\right). \tag{7}$$

This well-known series is obtained from (1) by replacing the second factor in the integrand with its binomial expansion. Relation (7) is computed by the FORTRAN function BPSER(a, b, x, ϵ).

BUP$(a, b, x, y, n, \epsilon)$.

$$I_x(a, b) - I_x(a + n, b) = x^a y^b \sum_{j=1}^{n} \frac{\Gamma(a + b + j - 1)}{\Gamma(b)\Gamma(a + j)} x^{j-1}, \qquad (n \ge 1). \tag{8}$$

Relation (8) is computed by the function BUP(a, b, x, y, n, ϵ). BUP is used only with BPSER or the next subroutine to be described, BGRAT. The relation follows from $I_x(a+1, b) = I_x(a, b) - x^a y^b G(a, b)/a$, which can be obtained by substituting

$$(a+b)t^a(1-t)^{b-1} = at^{a-1}(1-t)^{b-1} - \frac{d}{dt}\left[t^a(1-t)^b\right] \tag{8.1}$$

in

$$I_x(a+1, b) = \frac{a+b}{aB(a,b)} \int_0^x t^a (1-t)^{b-1}\, dt.$$

(8) can be rewritten in the form

$$I_x(a, b) - I_x(a+n, b) = \frac{x^a y^b}{aB(a,b)} \sum_{i=0}^{n-1} d_i x^i,$$

$$d_{i+1} = \frac{a+b+i}{a+1+i} d_i, \qquad d_0 = 1. \tag{8.2}$$

If $b \le 1$, then for $i \ge 0$, $d_{i+1} \le d_i$ and the sequence $h_i \equiv d_i x^i$ is monotonically decreasing. Thus the computation of the sum Σh_i can be terminated when a term $h_m = d_m x^m$ is reached that satisfies $h_m \le \epsilon \Sigma_{i=0}^m h_i$, or $m = n - 1$.

When $b > 1$, then $h_i \ge h_{i+1}$ if and only if $x \le (a+1+i)/(a+b+i) \equiv r_i$. Also $r_i < r_j$ when $i < j$. Thus if k is the largest integer such that $x \ge r_{k-1}$, then

$$h_0 \le h_1 \le \cdots \le h_k \ge h_{k+1} \ge \cdots \ge h_{n-1}.$$

Therefore, if k is the largest integer for which $k \le (b-1)x/y - a$, then the computation of the sum Σh_i can be terminated when a term $h_m (m > k)$ is met that satisfies $h_m \le \epsilon \Sigma_{i=0}^m h_i$, or $m = n - 1$.

BGRAT(a, b, x, y, w, ϵ, *IERR*).

$$I_x(a, b) \approx M \sum_{n=0}^{\infty} p_n J_n(b, u), \qquad a > b, \tag{9}$$

$$T = a + \frac{b-1}{2}, \qquad u = -T \ln x, \tag{9.1}$$

$$H(c, u) = e^{-u} u^c / \Gamma(c), \qquad M = H(b, u)\Gamma(a+b)/[\Gamma(a) T^b], \tag{9.2}$$

$$p_n = (b-1)/(2n+1)! + \frac{1}{n} \sum_{m=1}^{n-1} [(mb - n)/(2m+1)!] p_{n-m},$$

$$p_0 = 1, \tag{9.3}$$

$$J_n(b, u) = \left(\frac{u}{2T}\right)^{2n} \frac{Q(b+2n, u)}{H(b+2n, u)} = \left(\frac{-\ln x}{2}\right)^{2n} \frac{Q(b+2n, u)}{H(b+2n, u)}, \tag{9.4}$$

where

$$Q(b,u) = \int_u^\infty \frac{e^{-t}t^{b-1}}{\Gamma(b)}\,dt \text{ (incomplete gamma function)}$$

$$[1, \text{p. } 260; 4]. \qquad (9.5)$$

The quantity p_n is the nth nonzero coefficient of the Maclaurin series for $(\sinh z/z)^{b-1}$, [8]. J_n is computed recursively, using

$$J_{n+1}(b,u) = \frac{(b+2n)(b+2n+1)}{4T^2}J_n(b,u) + \frac{u+b+2n+1}{4T^2}\left(\frac{\ln x}{2}\right)^{2n},$$

$$J_0(b,u) = \frac{Q(b,u)}{H(b,u)}. \qquad (9.6)$$

The asymptotic expansion (9) was derived by Wise [11].

Relation (9) is computed by the subroutine BGRAT(a, b, x, y, w, ϵ, IERR) where w and IERR are variables. Given an initial value w_0 for w, then BGRAT assigns w the value $w_0 + I_x(a,b)$. IERR indicates when underflow forces the computation of (9) to end prematurely. No accuracy is lost when this occurs since $w_0 \gg I_x(a,b)$.

BFRAC($a, b, x, y, \lambda, \epsilon$).

$$I_x(a,b) = \frac{x^a y^b}{B(a,b)}\left(\frac{\alpha_1}{\beta_1 +}\,\frac{\alpha_2}{\beta_2 +}\cdots\right), \qquad x \le p \equiv \frac{a}{a+b}, \qquad (10)$$

$$\alpha_1 = 1, \qquad \beta_1 = \frac{a}{a+1}(\lambda+1),$$

$$\lambda = a - (a+b)x = (a+b)(p-x), \qquad (10.1)$$

$$\alpha_{n+1} = \frac{(a+n-1)(a+b+n-1)}{(a+2n-1)^2}n(b-n)x^2, \quad n \ge 1, \qquad (10.2)$$

$$\beta_{n+1} = n + \frac{n(b-n)x}{a+2n-1} + \frac{a+n}{a+2n+1}$$
$$[\lambda + 1 + n(1+y)], \quad n \ge 0. \qquad (10.3)$$

The continued fraction (10) is new and is used over a very large part of the domain. It is computed by the function BFRAC($a, b, x, y, \lambda, \epsilon$). The weighting factors c_n given below are used to control overflow and underflow.

The continued fraction is motivated by considering the classical expansion

$$I_x(a,b) = \frac{x^a y^b}{aB(a,b)}\left(\frac{1}{1+}\,\frac{d_1}{1+}\,\frac{d_2}{1+}\cdots\right), \tag{10.4}$$

$$d_{2n} = \frac{n(b-n)}{(a+2n-1)(a+2n)}x, \qquad n > 0, \tag{10.5}$$

$$d_{2n+1} = -\frac{(a+n)(a+b+n)}{(a+2n)(a+2n+1)}x, \quad n \ge 0.$$

$$[1;26.5.8],[3,9]. \tag{10.6}$$

For large a where $a \gg b$, we note that $d_{2n} \approx 0$ and $d_{2n+1} \approx (1 + b/a)x$. Thus, if R_n is the iterate

$$R_n = \frac{1}{1+}\,\frac{d_1}{1+}\,\frac{d_2}{1+}\cdots\frac{d_{n-2}}{1+d_{n-1}},$$

then most of the change of values of these iterates occurs in every other iterate. Consequently, the associated continued fraction

$$\frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots$$

is considered, where

$$\begin{aligned} a_1 &= 1, & a_{n+1} &= -d_{2n-1}d_{2n} \\ b_1 &= 1 + d_1, & b_{n+1} &= 1 + d_{2n} + d_{2n+1}, \qquad n \ge 1. \end{aligned}$$

$$[10, \text{p. } 20].$$

The iterates R_{2n} are the iterates for this expansion, as already pointed out by Aroian in [3] (with some corrections given in [9]). Hence,

$$I_x(a,b) = \frac{x^a y^b}{aB(a,b)}\left(\frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\right).$$

Now, for large a, where $a \gg b$, it is clear that $d_{2n} \approx 0$ forces $a_n \approx 0$. However, for $x \approx p$, we also find that $d_{2n} \approx 0$ forces $b_n \approx 0$. This can cause division of 0 by 0 when the iterates of this continued fraction are computed, or it can cause division to overflow. Thus, this continued fraction is also not considered appropriate for computational purposes.

In order to eliminate the problems arising from $d_{2n} \approx 0$, we rescale the coefficients a_n and b_n with weighting factors c_n such that

$$\bar{\alpha}_1 = c_1 a_1, \qquad \alpha_n = c_{n-1}c_n a_n \quad (n \ge 2), \qquad \beta_n = c_n b_n \quad (n \ge 1), \tag{10.7}$$

where

$$c_n = a + 2(n-1), \qquad n \ge 1. \tag{10.8}$$

Then the iterates of

$$\frac{\bar{\alpha}_2}{\beta_1 +} \frac{\alpha_2}{\beta_2 +} \cdots$$

are the scaled iterates for

$$\frac{a_1}{b_1 +} \frac{a_2}{b_2} \cdots$$

and we obtain (10).

If $n \le b$, then $\alpha_n \ge 0$ and β_n is a positive value not near 0. To insure that the maximum value of n never exceeds b, (10) is applied only when $b \ge 40$. In this case, x must also be a sufficient distance from p when $a > 100$.

From Theorem 2 in [1, p. 19], (10) is computed using

$$\lim_{n \to \infty}(A_n/B_n) = \left(\frac{\alpha_1}{\beta_1 +} \frac{\alpha_2}{\beta_2 +} \cdots,\right)$$

where

$$\begin{aligned} A_{n+1} &= \beta_{n+1}A_n + \alpha_{n+1}A_{n-1}, \qquad A_0 = 0, \qquad A_1 = \alpha_1 \\ B_{n+1} &= \beta_{n+1}B_n + \alpha_{n+1}B_{n-1}, \qquad B_0 = 1, \qquad B_1 = \beta_1. \end{aligned} \tag{10.9}$$

Then

$$\frac{A_n}{B_n} = \frac{\alpha_1}{\beta_1 +} \frac{\alpha_2}{\beta_2 +} \cdots \frac{\alpha_n}{\beta_n}.$$

BASYM$(a, b, x, y, \lambda, \epsilon)$.

If a and b are large and $x \approx p$, then a new asymptotic expansion is used which is given by

$$I_x(a,b) \approx \frac{2}{\sqrt{\pi}} U e^{-z^2} \sum_{n=0}^{\infty} e_n L_n(z)(\beta\gamma)^n, \qquad x \le p, \tag{11}$$

where

$$U = \frac{p^a q^b}{B(a,b)} \sqrt{\frac{2\pi(a+b)}{ab}} \tag{11.1}$$

$$z = \sqrt{\varphi(x)},$$

$$\varphi(t) = -\left(a \ln\frac{t}{p} + b \ln\frac{1-t}{q}\right) \ge 0, \quad 0 < t < 1; \tag{11.2}$$

$$\beta\gamma = \sqrt{q/a}, \qquad a \le b, \tag{11.3}$$

$$\beta\gamma = \sqrt{p/b}, \qquad a \ge b;$$

$$a_n = \frac{2}{n+2} q\left[1 + (-1)^n (a/b)^{n+1}\right] \ge 0, \quad a \le b, \tag{11.4}$$

$$a_n = \frac{2}{n+2} p\left[(-1)^n + (b/a)^{n+1}\right], \quad a > b, \quad n \ge 0; \tag{11.5}$$

$$b_0^{(r)} = 1, \qquad b_1^{(r)} = ra_1, \qquad r \neq 0,$$

$$b_n^{(r)} = ra_n + \frac{1}{n}\sum_{i=1}^{n-1}[(n-i)r - i]b_i^{(r)}a_{n-i}, \qquad n = 2,3,\ldots; \quad [8]$$

$$c_n = \frac{1}{n}b_{n-1}^{(-n/2)}, \qquad (n \geq 1), \tag{11.6}$$

$$e_0 = 1, \qquad e_n = -\sum_{i=0}^{n-1} e_i c_{n-i+1}; \tag{11.7}$$

$$L_n(z) = 2^{(n/2)-1}e^{z^2}\int_z^\infty e^{-u^2}u^n\,du. \tag{11.8}$$

The expansion is computed by the function BASYM$(a, b, x, y, \lambda, \epsilon)$ where $\lambda = (a+b)(p-x)$.

The expansion is motivated by writing (1) in the form

$$I_x(a,b) = \frac{p^a q^b}{B(a,b)}\int_0^x \left(\frac{t}{p}\right)^a \left(\frac{1-t}{q}\right)^b \frac{dt}{t(1-t)}, \qquad 0 < x < 1. \tag{11.9}$$

Then a change of variables $u = \sqrt{\varphi(t)}$ (see 11.2) is made for $0 < t \leq x \leq p$, yielding

$$I_x(a,b) = \sqrt{\frac{2}{\pi}}\,U\beta\int_z^\infty e^{-u^2}\frac{u}{p-t}\,du, \tag{11.10}$$

where U is given by (11.1), and

$$\beta = \sqrt{\frac{ab}{(a+b)^3}}, \qquad z = \sqrt{\varphi(x)}. \tag{11.11}$$

The quantity z is computed from

$$z = \sqrt{a\phi\left(\frac{x}{p}\right) + b\phi\left(\frac{y}{q}\right)}, \qquad \phi(w) \equiv w - 1 - \ln w \qquad (w > 0), \tag{11.12}$$

rather than (11.2), in order to reduce the loss of accuracy when x is near p.

In order to use (11.10), $u/(p-t)$ is replaced by its Maclaurin series in u. From (11.2) and

$$a\ln\frac{t}{p} = a\ln\left(1 - \frac{p-t}{p}\right) = -a\sum_{n=1}^\infty \frac{1}{n}\left(\frac{p-t}{p}\right)^n$$

$$b\ln\frac{1-t}{q} = b\ln\left(1 + \frac{p-t}{q}\right) = -b\sum_{n=1}^\infty \frac{1}{n}\left(\frac{t-p}{q}\right)^n,$$

one obtains

$$u^2 = \varphi(t) = \frac{1}{2\beta^2}(p-t)^2\sum_{n\geq 0}\lambda_n(p-t)^n, \qquad |p-t| < \min\{p,q\}, \tag{11.13}$$

where

$$\lambda_n = \frac{2}{n+2}[qp^{-n} + (-1)^n pq^{-n}]. \tag{11.14}$$

Let γ denote a positive scaling factor. Then, from (11.13), one obtains

$$(\sqrt{2}\,\beta\gamma u)^2 = s^2 A \tag{11.15}$$

where

$$A \equiv \sum_{n \geq 0} a_n s^n \geq 0, \qquad a_n \equiv \lambda_n / \gamma^n, \qquad s = \gamma(p - t).$$

Choosing

$$\gamma = 1/p \quad \text{for } a \leq b \quad \text{and} \quad \gamma = 1/q \quad \text{for } a > b, \tag{11.16}$$

yields (11.3) and (11.4). Also the series above for A converges for $|s| < 1$, and

$$\sqrt{2}\,\beta\frac{u}{p - t} = \sqrt{A} \tag{11.17}$$

for $0 \leq s < 1$.

Now, if $v = P(s) = s\sqrt{A}$ for any complex $|s| < 1$, since P is analytic at 0 and $P(0) = 0$, by the Lagrange–Bürmann theorem [6, p. 58] the inverse $s = P^{-1}(v)$ of P is given by

$$s = \sum_{n \geq 1} c_n v^n, \tag{11.18}$$

with

$$c_n = \frac{1}{n}\mathbf{res}(P^{-n})$$

where $\mathbf{res}(P^{-n})$ is the residue of the series $P^{-n} = s^{-n}A^{-n/2}$. Thus, if $A^r = \sum_{k \geq 0} b_k^{(r)} s^k$ for $r \neq 0$, then the coefficients c_n can be obtained from (11.6) using (11.5). Also

$$\sqrt{A} = \frac{v}{s} = 1 \Big/ \sum_{n \geq 1} c_n v^{n-1} = \sum_{n \geq 0} e_n v^n$$

where e_n is given by (11.7). Consequently, from (11.17), we note that

$$\sqrt{2}\,\beta\frac{u}{p - t} = \frac{v}{s} = \sum_{n \geq 0} e_n v^n \tag{11.19}$$

when $0 \leq s < 1$. This along with (11.10) suggests that (11) may be an asymptotic expansion. In fact, it is proved in [5] that for $a \leq b$ this is the case. If $a > b$, it has not been shown that (11) is asymptotic. Nevertheless, the use of (11) has been established by extensive computer testing for $y < 1.05q$ when $b \geq 100$.

Finally, we observe that the definitions of U and L_n given in (11.1) and (11.8) are not suitable for computational purposes. U and L_n can be accurately evaluated using

$$\Delta(a) = \ln \Gamma(a) - (a - \tfrac{1}{2})\ln a + a - \tfrac{1}{2}\ln(2\pi) \qquad (11.20)$$
$$\ln U = \Delta(a+b) - \Delta(a) - \Delta(b)$$

and

$$L_0(z) = (\sqrt{\pi}/4)e^{z^2}\operatorname{erfc}(z), \qquad L_1(z) = 2^{-3/2} \qquad (11.21)$$
$$L_n(z) = 2^{-3/2}(\sqrt{2}\,z)^{n-1} + (n-1)L_{n-2}(z), \qquad n = 2,3,\ldots.$$

3. DOMAINS FOR THE SUBPROGRAMS

In BRATIO, frequently the smaller of $I_x(a,b)$ and $I_y(b,a)$ is computed and the identity $I_x(a,b) + I_y(b,a) = 1$ applied. However, at times we simply ensure that the quantity computed ($I_x(a,b)$ or $I_y(b,a)$) does not exceed .9.

Let J be the quantity $I_x(a,b)$ or $I_y(b,a)$ to be computed. Since J is to be computed to the greatest possible accuracy, the relative tolerance to be satisfied is $\epsilon = \max\{\epsilon_0, 10^{-15}\}$ where ϵ_0 is the smallest number for which $1 + \epsilon_0 > 1$ for the floating arithmetic being used. The restriction $\epsilon \geq 10^{-15}$ is necessary since many of the supporting subprograms are accurate to a maximum of 14–15 significant digits (see Section 4).

There are two main domains to consider, namely $\min(a,b) \leq 1$ and $\min(a,b) > 1$. It should be noted that if a and b, and x and y are interchanged in BPSER, BRAT, BFRAC, or BASYM, then the subprogram provides the value $I_y(b,a) = 1 - I_x(a,b)$. Also, BUP gives on the interchange the value $I_y(b,a) - I_y(b+n,a)$.

$\underline{\min(a,b) \leq 1}$

If $x > 1/2$, then a and b, and x and y are interchanged. For $x \leq 1/2$, (12) is used for computing $J = I_x(a,b)$ and (13)–(15) are used for computing $J = I_y(b,a)$. In (14) and (15), w_0 is the initial value of w, and J the final value of w.

BPSER(a, b, x, ϵ) $\quad \max(a,b) \leq 1,\ a \geq \min(0.2, b)$ (12)
$\max(a,b) \leq 1,\ a < \min(0.2, b),\ x^a \leq 0.9$
$\max(a,b) > 1,\ b \leq 1$
$\max(a,b) > 1,\ b > 1,\ x < 0.1,\ (bx)^a \leq 0.7$

BPSER(b, a, y, ϵ) $\quad \max(a,b) \leq 1,\ a < \min(0.2, b),\ x^a > 0.9,\ x \geq 0.3$
$\max(a,b) > 1,\ b > 1,\ x \geq 0.3$ (13)

BGRAT($b, a, y, x, w, 15\epsilon$, IERR), $w_0 = 0$ (14)
$\max(a,b) > 1,\ b > 15,\ 0.1 \leq x < 0.3$
$\max(a,b) > 1,\ b > 15,\ x < 0.1,\ (bx)^a > 0.7$

BGRAT($b + n, a, y, x, w, 15\epsilon$, IERR), $w_0 =$ BUP(b, a, y, x, n, ϵ),
$n = 20$ (15)

$$
\begin{gathered}
\max(a, b) > 1,\ b > 1,\ 0.1 \le x < 0.3,\ b \le 15 \\
\max(a, b) > 1,\ b > 1,\ x < 0.1,\ (bx)^a > 0.7,\ b \le 15 \\
\max(a, b) \le 1,\ a < \min(0.2, b),\ x^a > 0.9,\ x < 0.3 \\
\underline{\min(a, b) > 1}
\end{gathered}
$$

If $x > p$, then a and b, and x and y are interchanged. For $x \le p$, (16)–(21) are used for computing $J = I_x(a, b)$. In (17)–(19), n is the largest integer less than b and $\bar{b} = b - n$. Also, in (18) and (19), w_0 is the initial value of w and J the final value of w.

$$
\begin{array}{llllr}
\mathrm{BPSER}(a, b, x, \epsilon), & b < 40, & bx \le .7, & & (16) \\
\mathrm{BUP}(\bar{b}, a, y, x, n, \epsilon) + \mathrm{BPSER}(a, \bar{b}, x, \epsilon) & & & & (17) \\
 & b < 40, & bx > .7, & x \le .7, & \\
\mathrm{BGRAT}(a, \bar{b}, x, y, w, 15\epsilon, \mathrm{IERR}),\ w_0 = \mathrm{BUP}(\bar{b}, a, y, x, n, \epsilon) & & & & (18) \\
 & b < 40, & x > .7, & a > 15, & \\
\mathrm{BGRAT}(a + m, \bar{b}, x, y, w, 15\epsilon, \mathrm{IERR}),\ m = 20 & & & & (19) \\
\quad w_0 = \mathrm{BUP}(\bar{b}, a, y, x, n, \epsilon) + \mathrm{BUP}(a, \bar{b}, x, y, m, \epsilon) & & & & \\
 & b < 40, & x > .7, & a \le 15, & \\
\mathrm{BFRAC}(a, b, x, y, \lambda, 15\epsilon) & b \ge 40, & a \le \mathrm{b}, & a \le 100 & (20) \\
 & b \ge 40, & 100 < a \le b, & x < .97p & \\
 & b \ge 40, & a > b, & b \le 100 & \\
 & b \ge 40, & 100 < b < a, & y > 1.03q, & \\
\mathrm{BASYM}(a, b, x, y, \lambda, 100\epsilon) & b \ge 40, & 100 < a \le b, & x \ge .97p & (21) \\
 & b \ge 40, & 100 < b < a, & y \le 1.03q. &
\end{array}
$$

In [5, pp. 13–16], it is verified that $J \le .9$ is always satisfied. For example, if $a \ge 1$ and $b \ge 1$, then a proof by James Perry, [5, Appendix A] shows that if $x \le p$, then

$$
I_x(a, b) \le \begin{cases} 1 - 1/e & a < b \\ 1/2 & a \ge b. \end{cases}
$$

where $e = 2.71828 \cdots$. Consequently, J can be set to $I_x(a, b)$ when $x \le p$, and to $I_y(b, a)$ when $x > p$.

4. AUXILIARY FUNCTIONS

In order to compute $I_x(a, b)$ and $1 - I_x(a, b)$, procedures are needed for evaluating $\Gamma(a)$, $\ln \Gamma(a)$, the error function erf x, $\exp(x^2)$erfc x, the incomplete gamma function $Q(a, x)$, (9.5), for $a \le 1$, and the functions

$$
\begin{array}{lll}
e^x - 1 & & (22) \\
\ln(1 + x) & (|x| \le .375) & \\
\ln \Gamma(1 + x) & (-0.2 \le a \le 1.25) & \\
1/\Gamma(1 + a) - 1 & (-0.5 \le a \le 1.5) & \\
e^{-x} x^a / \Gamma(a) & (a > 0, \quad x \ge 0) & \\
\phi(x) = x - 1 - \ln x & (x > 0). &
\end{array}
$$

These functions are discussed in [4]. Also, procedures are needed for computing

$$
\begin{array}{lll}
\Delta(a) = \ln \Gamma(a) - (a - .5)\ln a + a - .5\ln(2\pi) & (a \geq 8) & (23)\\
\mathrm{ALGDIV}(a, b) = \ln[\Gamma(b)/\Gamma(a+b)], & (a \geq 0, b \geq 8) & \\
\mathrm{BCORR}(a, b) = \Delta(a) + \Delta(b) - \Delta(a+b) & (a, b \geq 8) & \\
\mathrm{BETALN}(a, b) = \ln B(a, b) & (a, b > 0) & \\
\mathrm{BRCOMP}(a, b, x, y) = x^a y^b / B(a, b) & (a, b > 0, & \\
 & \quad 0 < x < 1, & \\
 & \quad y = 1 - x). &
\end{array}
$$

Rational minimax approximations are used for the functions given in (22). Experience indicates that such approximations normally generate less error and can be considerably more efficient than the standard expansions. However, minimax approximations have the disadvantage of being limited to a fixed maximum precision. The minimax approximation used are designed to achieve a maximum precision of 14–15 significant digits.

If $\Delta(a)$ is needed only for $a \geq 20$, then the sum

$$1/(12a) - 1/(360a^3) + \cdots$$

in the asymptotic expansion of $\ln \Gamma(a)$ [1, 6.1.41] may be used. If $a \geq 15$, then the minimax approximation

$$\Delta(a) = \sum_{n=0}^{4} c_n / a^{2n+1} \tag{24}$$

$$
\begin{aligned}
c_0 &= .83333\ 33333\ 33333E-01\\
c_1 &= -.27777\ 77777\ 70481E-02\\
c_2 &= .79365\ 06631\ 83693E-03\\
c_3 &= -.59515\ 63364\ 28591E-03\\
c_4 &= .82075\ 63703\ 53826E-03
\end{aligned}
$$

can be applied, and if $a \geq 8$, the minimax approximation

$$\Delta(a) = \sum_{n=0}^{5} d_n / a^{2n+1} \tag{25}$$

$$
\begin{aligned}
d_0 &= .83333\ 33333\ 33333E-01\\
d_1 &= -.27777\ 77777\ 60991E-02\\
d_2 &= .79365\ 06668\ 25390E-03\\
d_3 &= -.59520\ 29313\ 51870E-03\\
d_4 &= .83730\ 80340\ 31215E-03\\
d_5 &= -.12532\ 29627\ 80713E-02
\end{aligned}
$$

can be used. These approximations were obtained by Morris [7]. On the CDC 6000–7000 series computers they are accurate to within 1 unit of the 14th significant digit.

Expansions for ALGDIV(a, b) and BCORR(a, b) use (25). From the definition of Δ

$$\mathrm{ALGDIV}(a, b) = w - (a + b - .5)\ln(1 + a/b) - a(\ln b - 1) \qquad (26)$$
$$w = \Delta(b) - \Delta(a + b).$$

Let

$$p = a/(a + b), \quad q = b/(a + b), \quad S_m = 1 + q + \cdots + q^{m-1} \qquad (m \geq 1).$$

Then,

$$1 - q^m = (1 - q)S_m = pS_m, \qquad \frac{pS_m}{b^m} = \frac{1}{b^m} - \frac{1}{(a + b)^m}. \qquad (27)$$

Thus, from (25) we obtain

$$w = \frac{p}{b}\sum_{n=0}^{5} d_n \frac{S_{2n+1}}{b^{2n}}, \qquad (28)$$

which completes the algorithm for ALGDIV(a, b). Also

$$\mathrm{BCORR}(a, b) = \Delta(a_0) + [\Delta(b_0) - \Delta(a_0 + b_0)]$$
$$a_0 = \min\{a, b\}, \qquad b_0 = \max\{a, b\},$$

where (25) and (28) are applied.

If $a \leq b$, then BETALN(a, b) can be accurately computed when $a \geq 1$. If $a \geq 8$, then

$$\mathrm{BETALN}(a, b) = (.5\ln(2\pi) - .5\ln b) + \mathrm{BCORR}(a, b) - u - v$$
$$u = -(a - .5)\ln[a/(a + b)], \qquad v = b\ln(1 + a/b)$$

is applied. If $2 < a < 8$, then a is reduced to the interval $[1, 2]$ by

$$B(a, b) = \frac{a - 1}{a + b - 1}B(a - 1, b).$$

Consequently, it can be assumed that $a \leq 2$.

If $b \geq 8$ then

$$\ln B(a, b) = \ln\Gamma(a) + \mathrm{ALGDIV}(a, b)$$

is applied. If $2 < b < 8$ then b is also reduced to the interval $[1, 2]$ when $a \geq 1$. Thus, we need only consider the cases: $1 \leq a \leq 2$, $1 \leq b \leq 2$, or $a < 1$ and $b < 8$. If $a \geq 1$ then

$$\ln B(a, b) = \ln\Gamma(a) + \ln\Gamma(b) - \ln\Gamma(a + b)$$

is appropriate. No loss of accuracy due to subtraction can occur since $\ln\Gamma(a)$, $\ln\Gamma(b)$, and $-\ln\Gamma(a + b)$ are nonpositive. However, subtraction does occur when $a < 1$. It currently is not clear how loss of accuracy due to subtraction can be avoided when $a < 1$. Therefore, BETALN is not used in this case.

If $\min\{a, b\} < 8$, then BRCOMP(a, b, x, y) can be computed directly from its definition. Otherwise,

$$\text{BRCOMP}(a, b, x, y) = \sqrt{\frac{ab}{2\pi(a+b)}}\, e^{-z}$$

$$z = [a\phi(1 - \lambda/a) + b\phi(1 + \lambda/b)] + \text{BCORR}(a, b)$$

is used, where λ is given in (10.1).

5. CONCLUDING REMARKS

Formulas (7), (9), (10), and (11) for $I_x(a, b)$ are of the form ζE, where E is an expansion. For example, in (10) $\zeta = x^a y^b / B(a, b)$ and E is a continued fraction. On the CDC 6000–7000 series computers, little error is generated in computing E over the domains specified in Section 3. The expansion is normally accurate to within 1 or 2 units of the 14th significant digit. However, the precision of the factor ζ is restricted by the inherent error of $I_x(a, b)$. Extensive testing on the CDC 6000–7000 series computers, comparing the results obtained by BRATIO with results from double precision code, indicates that the precisions of the values obtained for $I_x(a, b)$ and $1 - I_x(a, b)$ by BRATIO approximate the inherent errors of these functions up to a maximum of 14 significant digits. On any computer, accuracy is restricted to 14 digits because of the algorithms used for the auxiliary functions in Section 4.

On the CDC 6000–7000 series computers, a maximum of 7 terms of the series (9) for BGRAT, and a maximum of 11 terms of (11) for BASYM were observed for the domains specified in Section 3. Frequently, 40 or fewer terms of (7) for BPSER suffice, but a maximum of 92 terms has been observed when a is small, b is large, and $x \approx .3$. Also, 40 or fewer terms generally suffice for the continued fraction (10), BFRAC, but a maximum of 58 terms has been observed when a or b is exceedingly large and $x \approx a/(a + b)$.

During the last several years, BRATIO has been found to be a reliable and efficient subroutine. It is currently being used on a variety of computers, ranging from supercomputers to personal computers such as the IBM PC. The algorithm used in BRATIO supercedes earlier algorithms (including [2] and [9]).

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