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A PRECISION APPROXIMATION OF THE GAMMA FUNCTION*

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The gamma function is one of the most interesting transcendentals of higher mathematics. It is an analytic function of the complex variable z which in any finite domain has no singularities other than simple poles, situated at the points $x = 0, -1, -2, -3, \cdots$. For integer values the gamma function coincides with the ordinary factorials, according to the relation

$$\Gamma(n + 1) = n! \tag{1}$$

The normalization of the gamma function to $\Gamma(n + 1)$ instead of $\Gamma(n)$ is due to Legendre and void of any rationality. This unfortunate circumstance compels us to utilize the notation $z!$ instead of $\Gamma(z + 1)$, although this notation is obviously highly unsatisfactory from the operational point of view.

Euler gave the well-known integral representation of the factorial function:

$$z! = \Gamma(z + 1) = \int_0^\infty t^z e^{-t}\, dt, \tag{2}$$

which defines $z!$ as an analytical function of z for all z whose real part is greater than -1. We will restrict ourselves to the right complex half plane by the condition

$$\operatorname{Re} z \geqq 0 \tag{3}$$

because the reflexion theorem

$$(-z)!\, z! = \frac{\pi z}{\sin \pi z} \tag{4}$$

reduces the definition of $(-z)!$ to the definition of $z!$. Our following discussions will be based on Euler's integral, which we will transform into a form that is particularly well suited for approximation purposes.

Replacing t by αt we obtain

$$z! = \alpha^{z+1} \int_0^\infty t^z e^{-\alpha t}\, dt. \tag{5}$$

Since z is a constant with respect to the process of integration, it is permissible to put

$$\alpha = 1 + \rho z, \tag{6}$$

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where we will consider ρ as some positive constant. This yields

$$(7) \qquad z! = (1 + \rho z)^{z+1} \int_0^{\infty} (te^{-\rho t})^z e^{-t}\, dt.$$

Furthermore, we put the constant factor $(e\rho)^{-z}$ in front and its reciprocal behind the integral sign, thus changing (7) to

$$(8) \qquad z! = (1 + \rho z)^{z+1}(e\rho)^{-z} \int_0^{\infty} (\rho t e^{1-\rho t})^z e^{-t}\, dt,$$

and now we put

$$e^{1-\rho t} = v.$$

With this change of variable our definition of $z!$ becomes

$$(9) \qquad \begin{aligned} z! &= (1 + \rho z)^{z+1}\rho^{-(z+1)}e^{-z} \int_0^{e} [v(1 - \log v)]^z \left(\frac{v}{e}\right)^{1/\rho} \frac{dv}{v} \\ &= \left(\frac{1}{\rho} + z\right)^{z+1} e^{-z-1/\rho} \int_0^{e} [v(1 - \log v)]^z v^{1/\rho - 1}\, dv. \end{aligned}$$

Finally we replace ρ by a new constant γ, defined by

$$1/\rho = \gamma + 1.$$

Hence the final form of the definite integral, on which we want to base the investigation of $z!$, becomes

$$(10) \qquad z! = (z + \gamma + 1)^{z+1}e^{-(z+\gamma+1)} \int_0^{e} [v(1 - \log v)]^z v^{\gamma}\, dv.$$

The constant γ, which we have at our disposal, will turn out later to be of great value for the precision of our approximation. For the time being we carry it along as a given positive (or zero) constant.

Replacing z by $z - \frac{1}{2}$ we can equally put

$$(11) \quad (z - \tfrac{1}{2})! = (z + \gamma + \tfrac{1}{2})^{z+\frac{1}{2}}e^{-(z+\gamma+\frac{1}{2})} \int_0^{e} [v(1 - \log v)]^{z-\frac{1}{2}} v^{\gamma}\, dv.$$

Our aim will be to study more closely the integral transform

$$(12) \qquad F(z) = \int_0^{e} [v(1 - \log v)]^{z-\frac{1}{2}} v^{\gamma}\, dv.$$

The function $v(1 - \log v)$ has the following course. It starts at $v = 0$ with zero value and returns to zero at $v = e$. It reaches its maximum value 1 at $v = 1$. (See Fig. 1.)

We will now transform the integration variable v into a new variable x

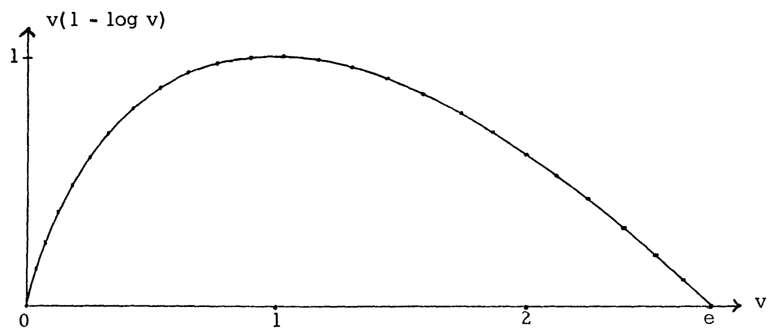

FIGURE 1

by the implicit transcendental equation

$$v(1 - \log v) = 1 - x^2, \tag{13}$$

with the boundary conditions:

$$v = 0 \quad \text{corresponds to} \quad x = -1,$$
$$v = 1 \quad \text{corresponds to} \quad x = 0,$$
$$v = e \quad \text{corresponds to} \quad x = 1.$$

Then $F(z)$ appears in the form

$$F(z) = \int_{-1}^{+1} (1 - x^2)^{z-\frac{1}{2}} v^\gamma \frac{dv}{dx}\, dx, \tag{14}$$

and if we change x to the angle variable θ by putting

$$x = \sin\theta, \tag{15}$$

we obtain the integral transform

$$F(z) = \int_{-\pi/2}^{+\pi/2} \cos^{2z}\theta\, f(\theta)\, d\theta, \tag{16}$$

where we have put

$$f(\theta) = v^\gamma \frac{dv}{dx} = v^\gamma \frac{dv}{\cos\theta\, d\theta}. \tag{17}$$

Now the implicit equation (13) can be changed to the differential equation

$$v' \log v = 2x,$$

or, substituting $\log v$ from (13), we obtain for $v(x)$ the nonlinear differential equation of first order

$$\frac{1}{2}(v^2)' - (1 - x^2)v' - 2xv = 0, \tag{18}$$

with the boundary condition

$$v(0) = 1.$$

The study of this differential equation shows that $v(x)$ is a function of the complex variable x which is analytical inside the unit circle $|x| < 1$, the only singular point occurring at $x = -1$, where $v(x)$ has a logarithmic singularity. Although $v(-1)$ goes to zero, and even

$$v'(x) = \frac{2x}{\log v}$$

goes to zero, yet $v''(-1)$ goes to infinity.

We thus see that $v(x)$ can be expanded in a Taylor series around $x = 0$ which converges uniformly for all $|x| < 1$ but the convergence becomes infinitely slow in the vicinity of the point $x = -1$. By putting the formal expansion

$$v(x) = 1 + a_1 x + a_2 x^2 + \cdots,$$

in the differential equation (18), we obtain recurrence relations for the a_k which can be solved in succession, obtaining

$$a_1 = \sqrt{2}, \qquad a_3 = -\frac{\sqrt{2}}{36}, \qquad a_5 = -\frac{23}{27}\frac{\sqrt{2}}{160}, \cdots,$$

$$a_2 = \frac{1}{3}, \qquad a_4 = \frac{2}{135}, \qquad a_6 = \frac{38}{8505}, \cdots,$$

and thus

$$\frac{dv}{dx} = a_1 + 3a_3 x^2 + 5a_5 x^4 + \cdots + 2a_2 x + 4a_4 x^3 + \cdots.$$

If for the time being we put $\gamma = 0$ and substitute this expansion in (16) for $f(\theta)$, all the odd powers vanish on account of the integration and we obtain

$$F(z) = \sqrt{2} \int_{-\pi/2}^{+\pi/2} \cos^{2z}\theta \left(1 - \frac{3}{36}\sin^2\theta - \frac{23}{27}\frac{5}{160}\sin^4\theta - \frac{11237}{5443200}\, 7\sin^6\theta - \cdots\right) d\theta. \tag{19}$$

Hence we need an infinity of definite integrals which are, however, available in closed form:

$$\int_{-\pi/2}^{+\pi/2} \cos^{2z}\theta \sin^{2k}\theta \, d\theta = \frac{(z - \frac{1}{2})!\,(k - \frac{1}{2})!}{(z + k)!}. \tag{20}$$

Hence

$$F(z) = \sqrt{2\pi}\,\frac{(z-\frac{1}{2})!}{z!}\left[1 - \frac{1}{24}\frac{1}{z+1} - \frac{23}{1152}\frac{1}{(z+1)(z+2)} - \frac{11237}{414720}\frac{1}{(z+1)(z+2)(z+3)} - \cdots\right],$$

and substituting in (11) we obtain the following convergent expansion for $z!$:

$$z! = \sqrt{2\pi}(z+\tfrac{1}{2})^{z+\frac{1}{2}}e^{-(z+\frac{1}{2})}\left[1 - \frac{1}{24}\frac{1}{z+1} - \frac{23}{1152}\frac{1}{(z+1)(z+2)} - \frac{11237}{414720}\frac{1}{(z+1)(z+2)(z+3)} - \cdots\right]. \tag{21}$$

It is of interest to compare this formula with the celebrated formula of Stirling:

$$z! = \sqrt{2\pi}z^{z+\frac{1}{2}}e^{-z}\left[1 + \frac{1}{12z} + \frac{1}{288z^2} + \frac{139}{51840}\frac{1}{z^3}\cdots\right].$$

The latter formula is divergent for all values of z, but can be used in the sense of an asymptotic expansion by terminating the series at the proper point. On the other hand, the expansion (21) converges for all values of z in the right half plane, but the convergence is too slow to be of practical value.

Our next aim will be to discover some means by which the convergence of our expansion can be improved. This can be done on the basis of the following consideration. The Taylor series is an extrapolating series and thus naturally of slow convergence. We can hope for much better results if we operate with an orthogonal set of functions which within its domain of orthogonality interpolates rather than extrapolates the given function. We have considered (16) as an integral transform of the function $f(\theta)$ to $F(z)$. In the integrand of (16) $f(\theta)$ is multiplied by an even function of θ. If we write $f(\theta)$ as a sum of its even and odd part:

$$f(\theta) = \tfrac{1}{2}[f(\theta) + f(-\theta)] + \tfrac{1}{2}[f(\theta) - f(-\theta)]$$

we notice that the contribution of the second part to the definite integral (16) vanishes, since the integration is taken between equal $\pm$ limits. Hence $f(\theta)$ can be replaced (for $\gamma = 0$) by

$$\tfrac{1}{2}[f(\theta) + f(-\theta)] = \frac{1}{2\cos\theta}\left[\frac{dv}{d\theta}(\theta) + \frac{dv}{d\theta}(-\theta)\right] = a_1 + 3a_3\sin^2\theta + 5a_5\sin^4\theta + \cdots \tag{22}$$

With this replacement, $f(\theta)$ is an *even* and *periodic* function of θ, of period π. Such a function can be expanded into a convergent Fourier series, of the form

$$f(\theta) = \tfrac{1}{2}c_0 + c_1\cos 2\theta + c_2\cos 4\theta + \cdots, \tag{23}$$

with

$$c_k = \frac{2}{\pi}\int_{-\pi/2}^{+\pi/2} f(\theta)\cos 2k\theta\, d\theta. \tag{24}$$

We might think that it will be difficult to evaluate these integrals, in view of the fact that $f(\theta)$ is a complicated function. In fact, they are explicitly available, with the help of the integral tranform $F(z)$. For this purpose we express $\cos 2k\theta$ in terms of $\cos^{2\alpha}\theta$, making use of the coefficients of the Chebyshev polynomials:

$$T_{2k}(\xi) = \sum_{\alpha=0}^{k} C_{2\alpha}^{2k}\,\xi^{2\alpha},$$

$$\cos 2k\theta = \sum_{\alpha=0}^{k} C_{2\alpha}^{2k}\cos^{2\alpha}\theta.$$

Hence

$$c_k = \frac{2}{\pi}\int_{-\pi/2}^{+\pi/2} f(\theta)\sum_{\alpha=0}^{k} C_{2\alpha}^{2k}\cos^{2\alpha}\theta\, d\theta = \frac{2}{\pi}\sum_{\alpha=0}^{k} C_{2\alpha}^{2k}F(\alpha). \tag{25}$$

We see that by *weighting the function values* $F(0), F(1), \cdots, F(k)$ *by the coefficients of the Chebyshev polynomials, we obtain the numerical values of the Fourier coefficients* c_k *in explicit form.*

Having obtained the c_k we now substitute the expansion (23) in (16), obtaining

$$F(z) = \int_{-\pi/2}^{+\pi/2}\cos^{2z}\theta\left(\tfrac{1}{2}c_0 + c_1\cos 2\theta + c_2\cos 4\theta + \cdots\right)d\theta. \tag{26}$$

The definite integrals demanded by this formula are once more available in closed form:

$$\int_{-\pi/2}^{+\pi/2}\cos^{2z}\theta\cos 2k\theta\, d\theta = \sqrt{\pi}\,\frac{(z-\frac{1}{2})!}{(z+k)!}\,\frac{z!}{(z-k)!},$$

which yields

$$F(z) = \sqrt{\pi}\,\frac{(z-\frac{1}{2})!}{z!}\left[\frac{1}{2}c_0 + c_1\frac{z}{z+1} + c_2\frac{z(z-1)}{(z+1)(z+2)} + \cdots\right].$$

Finally it will be advantageous to take the numerical factor $\sqrt{2}$ in front and write our formula in the form

$$F(z) = \sqrt{2\pi}\,\frac{(z-\frac{1}{2})!}{z!}\left[\frac{1}{2}c_0' + c_1'\frac{z}{z+1} + c_2'\frac{z(z-1)}{(z+1)(z+2)} + \cdots\right], \tag{27}$$

with

$$c_k' = \frac{\sqrt{2}}{\pi}\sum_{\alpha=0}^{k} C_{2\alpha}^{2k}\,F(\alpha), \tag{28}$$

where

$$F(\alpha) = (\alpha - \tfrac{1}{2})!\,(\alpha+\tfrac{1}{2})^{-(\alpha+\frac{1}{2})}\,e^{\alpha+\frac{1}{2}}, \tag{29}$$

and

$$z! = \sqrt{2\pi}(z+\tfrac{1}{2})^{z+\frac{1}{2}}e^{-(z+\frac{1}{2})}\left[\frac{1}{2}c_0' + c_1'\frac{z}{z+1} + \cdots\right]. \tag{30}$$

This expansion is not yet sufficient for an effective approximation of the factorial function, although it is of interest that the two-term formula

$$z! = \sqrt{2\pi}(z+\tfrac{1}{2})^{z+\frac{1}{2}}e^{-(z+\frac{1}{2})}\left(1.0163 - \frac{0.0861}{z+1} + \epsilon\right), \qquad |\,\epsilon\,| < 0.02, \tag{31}$$

is already accurate to a relative error not exceeding 2% at any point of the right complex half plane.

The reason $v(\theta)$ has a slowly convergent Fourier series is that the function dv/dx, although going to zero at $x = -1$, has an infinitely large tangent at that point. Hence there is a relatively mild "kink" at the end of the range which is of small extension. We can imagine that by taking the square, or cube, or some still higher power of this kink, the influence of the singularity will be greatly reduced and thus the convergence of the Fourier series greatly increased. This now can be achieved by making use of the constant γ which is equivalent to replacing v by $v^{\gamma+1}/(\gamma+1)$, as the formula (17) shows. For $\gamma = 0$ the accuracy of our approximation does not improve much by taking in more terms of the expansion, on account of the slow convergence of the coefficients c_k'. But let us now choose $\gamma = 1$. Here the successive coefficients diminish much more rapidly and when we reach the point beyond which the convergence becomes slow, the coefficients are already very small. Generally, the higher γ becomes, the smaller will be the value of the coefficients at which the convergence begins to slow down. At the same time, however, we have to wait longer, before the asymptotic stage is reached. Hence a large value of γ is advocated if very high accuracy is demanded, but then the required number of terms will also be larger.

We now have

$$z! = \sqrt{2\pi}(z + \gamma + \tfrac{1}{2})^{z+\frac{1}{2}} e^{-(z+\gamma+\frac{1}{2})} A_\gamma(z), \tag{32}$$

where

$$A_\gamma(z) = \tfrac{1}{2}\rho_0 + \rho_1 \frac{z}{z+1} + \rho_2 \frac{z(z-1)}{(z+1)(z+2)} + \cdots, \tag{33}$$

with

$$\rho_k = \sum_{\alpha=0}^{k} C_{2\alpha}^{2k} F(\alpha), \tag{34}$$

where

$$F(\alpha) = \frac{\sqrt{2}}{\pi} (\alpha - \tfrac{1}{2})!\, (\alpha + \gamma + \tfrac{1}{2})^{-(\alpha+\frac{1}{2})} e^{\alpha+\gamma+\frac{1}{2}}. \tag{35}$$

Table 1 gives a good idea of the manner in which the ρ_k coefficients decrease with increasing values of γ.

TABLE 1

	$\gamma = 1$	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 3$
$\frac{1}{2}\rho_0$	1.459843	2.0844142	3.07380467	7.06165881
ρ_1	−0.460642	−1.0846349	−2.11237574	−6.59935794
ρ_2	0.001054	0.0001207	0.03862116	0.53965224
ρ_3	−0.000338	0.0001145	−0.00005100	−0.00195197
ρ_4	0.000118	−0.0000171	0.00000048	−0.00000133
ρ_5	−0.000051	0.0000018	0.00000067	0.00000022

A good check on the accuracy of the truncated Fourier series is provided by evaluating the approximate value of $f_\gamma(\theta)$ at $\theta = -\pi/2$, that is by forming the alternate sum

$$\tfrac{1}{2}\rho_0 - \rho_1 + \rho_2 - \rho_3 + \cdots = f_\gamma\left(-\frac{\pi}{2}\right) = \frac{e^\gamma}{\sqrt{2}}.$$

We know from the theory of the Fourier series that the maximum local error (after reaching the asymptotic stage) can be expected near to the point of singularity. Let this error be η. Then a simple estimation shows that the influence of this error on the integral transform (16) (for values of z which stay within the right complex half plane), cannot be greater than $(\pi/2)\eta$. Thus we can give a definite error bound for the approximation obtained.

Finally, it is convenient to resolve the rational functions which appear in (34), into their constituent partial fractions:

$$\frac{z}{z+1} = 1 - \frac{1}{z+1}, \qquad \frac{z(z-1)}{(z+1)(z+2)} = 1 + \frac{2}{z+1} - \frac{6}{z+3},$$

$$\frac{z(z-1)(z-2)}{(z+1)(z+2)(z+3)} = 1 - \frac{3}{z+1} + \frac{24}{z+2} - \frac{30}{z+3},$$

$$\frac{z(z-1)(z-2)(z-3)}{(z+1)(z+2)(z+3)(z+4)} = 1 + \frac{4}{z+1} - \frac{60}{z+2} + \frac{180}{z+3} - \frac{140}{z+4}.$$

The final results can be tabulated as follows:

$$z! = (z + \gamma + \tfrac{1}{2})^{z+\frac{1}{2}} e^{-(z+\gamma+\frac{1}{2})} \sqrt{2\pi}[A_\gamma(z) + \epsilon].$$

$$\underline{\gamma = 1}$$

$$A_1(z) = 0.9992 + \frac{0.46064}{z+1}, \qquad |\epsilon| < 0.001,$$

$$\underline{\gamma = 1.5}$$

$$A_{1.5}(z) = 0.999779 + \frac{1.084635}{z+1}, \qquad |\epsilon| < 0.00024,$$

$$\underline{\gamma = 2}$$

$$A_2(z) = 1.0000509 + \frac{2.18961806}{z+1} - \frac{0.23172696}{z+2}, \qquad |\epsilon| < 5.1 \cdot 10^{-5},$$

$$A_2(z) = 0.99999909 + \frac{2.18977107}{z+1} - \frac{0.23295108}{z+2} + \frac{0.00153015}{z+3}, \qquad |\epsilon| < 1.5 \cdot 10^{-6},$$

$$\underline{\gamma = 3}$$

$$A_3(z) = 1.00000114 + \frac{7.68451833}{z+1} - \frac{3.28476072}{z+2} + \frac{0.05855910}{z+3}, \qquad |\epsilon| < 1.4 \cdot 10^{-6},$$

$$\underline{\gamma = 4}$$

$$A_4(z) = 0.9999999469 + \frac{24.7158060592}{z+1} - \frac{19.2112843044}{z+2} + \frac{2.4635062800}{z+3} - \frac{0.0096933620}{z+4}, \qquad |\epsilon| < 5 \cdot 10^{-8},$$

$\underline{\gamma = 5}$

$$A_5(z) = 1.000000000178 + \frac{76.180091729406}{z+1} - \frac{86.505320327112}{z+2} + \frac{24.014098222230}{z+3} - \frac{1.231739516140}{z+4} + \frac{0.001208580030}{z+5} - \frac{0.000005363820}{z+6}, \qquad |\epsilon| < 2 \cdot 10^{-10}.$$

Particularly remarkable is the approximation of only two terms ($\gamma = 1.5$):

$$z! = (z+2)^{z+\frac{1}{2}} e^{-(z+2)} \sqrt{2\pi} \left(0.999779 + \frac{1.084635}{z+1}\right),$$

which is correct up to a relative error of $2.4 \cdot 10^{-4}$ everywhere in the right complex half plane.

The error of the truncated series would rapidly increase if z moved over to the negative half plane. It is of interest to observe, however, that the convergence of the infinite expansion extends even to the negative realm and is in fact limited by the straight line

$$\operatorname{Re} z = -(\gamma + \tfrac{1}{2}).$$

The larger γ becomes, the more the domain shrinks in which the series diverges. If γ grows to infinity, we obtain a representation of the factorial function which holds *everywhere* in the complex plane. In this case we are able to give the coefficients of the series (34) in explicit form, due to the extreme nature of the function v^{γ}. We thus obtain the following limit relation:

$$z! = 2 \lim_{\gamma \to \infty} \gamma^z \left[\frac{1}{2} - e^{-1/\gamma} \frac{z}{z+1} + e^{-4/\gamma} \frac{z(z-1)}{(z+1)(z+2)} - \cdots\right]$$
$$= 2 \lim_{\gamma \to \infty} \gamma^z \sum_{k=0}^{\infty}{}' (-1)^k e^{-k^2/\gamma} \frac{\binom{z}{k}}{\binom{z+k}{k}}$$

valid for all values of z. The proper interpretation of this peculiar limit law and its possible relation to the same series but with finite values of γ, requires further investigation.

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