Computation of the Incomplete Gamma Function Ratios and their Inverse

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An algorithm is given for computing the incomplete gamma function ratios $P(a, x)$ and $Q(a, x)$ for $a \geq 0$, $x \geq 0$, $a + x \neq 0$. Temme's uniform asymptotic expansions are used. The algorithm is robust; results accurate to 14 significant digits can be obtained. An extensive set of coefficients for the Temme expansions is included.

An algorithm, employing third-order Schröder iteration supported by Newton–Raphson iteration, is provided for computing x when a, $P(a, x)$, and $Q(a, x)$ are given. Three iterations at most are required to obtain 10 significant digit accuracy for x.

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1. INTRODUCTION

Let $\Gamma(a)$ denote the complete gamma function. In this paper an algorithm is given for computing the incomplete gamma function ratios

$$P(a, x) = \frac{1}{\Gamma(a)} \int_0^x e^{-t} t^{a-1}\, dt, \tag{1}$$

$$Q(a, x) = 1 - P(a, x) = \frac{1}{\Gamma(a)} \int_x^\infty e^{-t} t^{a-1}\, dt, \tag{2}$$

for $a \geq 0$, $x \geq 0$, $a + x \neq 0$. An algorithm is also described for the inverse problem of computing x when a, $P(a, x)$, and $Q(a, x)$ are given. The algorithms are robust, yielding results accurate up to 14 significant digits for $P(a, x)$ and $Q(a, x)$, and 10 significant digits for x.

Section 2 contains a discussion of the auxiliary functions needed for the algorithms.

The primary region of difficulty for computing P and Q has been when a is large and $x \approx a$. Temme's uniform asymptotic expansions [14] apply to this region and permit improvement over previous algorithms [4, 9, 10]. Section 3 contains the algorithm for computing P and Q. A transportable FORTRAN

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subroutine and BASIC program, both named GRATIO, have been written which employ the algorithm.

The procedure for computing x when a, $P(a, x)$, and $Q(a, x)$ are given is described in Section 4. The algorithm supersedes previous work [5]. Third-order Schröder iteration is employed [11, p. 529]. A transportable FORTRAN subroutine and BASIC program, both named GAMINV, have been written which use the algorithm. GAMINV computes x to at least 10 significant digit accuracy in 3 or fewer iterations.

Hereafter the discussion concerning GRATIO and GAMINV will refer only to the FORTRAN subroutines. GRATIO and GAMINV are single precision routines. Extensive testing was performed on the CDC 6000–7000 series computers using double precision codes to check the single precision results. All given accuracy results are for the CDC 14-digit floating-point arithmetics.

2. AUXILIARY FUNCTIONS

In order to compute P, Q, and their inverse x, procedures are needed for evaluating $\Gamma(a)$, $\ln \Gamma(a)$, the error function erf x, $\exp(x^2)$erfc x, and the functions

$$
\begin{aligned}
&\ln(1 + a) && (|a| \le .375) \\
\Delta(a) &= \ln \Gamma(a) - \left(a - \frac{1}{2}\right)\ln a + a - \frac{1}{2}\ln 2\pi && (a \ge 15) \\
R(a, x) &= \frac{e^{-x}x^a}{\Gamma(a)} && (a > 0,\ x \ge 0) \\
\phi(\lambda) &= \lambda - 1 - \ln \lambda && (\lambda > 0) \\
L(x) &= \exp(x) - 1 && \\
H(a) &= \frac{1}{\Gamma(a + 1)} - 1 && (-.5 \le a \le 1.5).
\end{aligned}
\tag{3}
$$

Rational minimax approximations, designed by Morris [13], are used for computing these functions. Experience indicates that minimax approximations normally generate less error and can be considerably more efficient than the classical expansions. However, minimax approximations have the disadvantage of being limited to a fixed maximum precision.

For maximum accuracy, $\Gamma(a)$ employs the algorithm in [13, pp. 33–34]. For $a \ge 15$, $\Delta(a)$ is computed using the minimax approximation in [6, pp. E14–15]. When $\Delta(a)$ is needed only for $a \ge 20$, then the sum $1/12a - 1/360a^3 + \cdots$ in the classical asymptotic expansion for $\ln \Gamma(a)$ [1, 6.1.41] is used. Erf x and $\exp(x^2)$erfc x are computed using Cody's minimax approximations [3].

For $|a| \le .375$, in order to avoid loss of accuracy because of the sum $1 + a$, $\ln(1 + a)$ is computed by the minimax approximation in Appendix A. This approximation is accurate to within 2 units of the 14th significant digit.

R is computed directly from its definition in (3) when $a < 20$. For $a \ge 20$, underflow, overflow, and generated error are minimized by using

$$R(a, x) = \sqrt{a/(2\pi)}\exp[-a\phi(\lambda) - \Delta(a)], \quad \lambda = x/a. \tag{4}$$

In order to avoid cancellation error when $L(x)$ and $H(a)$ are near zero, minimax approximations are employed for $L(x)$ and $H(a)$ for $|x| \le .15$ and $-.5 \le a \le 1.5$.

The approximations, which are accurate to within 2 units of the 14th significant digit, are given in Appendices B and C.

For $a \geq 15$, $\ln \Gamma(a)$ is computed using $\Delta(a)$. Otherwise, if $a < 15$, then the argument of $\ln \Gamma(a)$ is reduced to the interval [.8, 2.25]. Since $\ln \Gamma(1) = \ln \Gamma(2) = 0$, approximations are needed for $\ln \Gamma(1 + x)$ and $\ln \Gamma(2 + x)$ to ensure that $\ln \Gamma(a)$ can be computed to full accuracy on this interval. The minimax approximations used are given in Appendix D. The approximations are accurate to within 1.5 units in the 14th significant digit.

$\phi(\lambda)$ is computed directly from its definition in (3) except when $.61 \leq \lambda \leq 1.57$. Cancellation error is minimized when $\lambda \approx 1$ by using

$$\begin{aligned} \phi(\lambda) &= 2r^2\left[\frac{1}{1-r} - r\phi_1(r)\right], \quad r = \frac{\lambda - 1}{\lambda + 1} \\ \phi_1(r) &= \sum_{n\geq 0} \frac{r^{2n}}{2n + 3}, \end{aligned} \tag{5}$$

where $.82 \leq \lambda \leq 1.18$ $(-.18/1.82 \leq r \leq .18/2.18)$. $\phi_1(r)$, which is obtained from the Maclaurin expansion of $\ln \lambda(r)$ [1, 4.1.27], is evaluated by the minimax approximation given in Appendix E. For maximum accuracy, the use of (5) is extended to the interval $.61 \leq \lambda \leq 1.57$ by the reduction formula

$$\phi(\lambda) = \left[\phi(c) + \frac{(\lambda - c)(c - 1)}{c}\right] + \phi(\lambda/c). \tag{6}$$

This formula transforms the argument of ϕ to the interval [.82, 1.18]. For $.61 \leq \lambda < .82$, c is assigned the value .7; for $1.18 < \lambda \leq 1.57$, c is set to 4/3. $\phi(\lambda)$ is computed correctly to within 3 units of the 14th significant digit for all $\lambda > 0$.

3. EVALUATION OF $P(a, x)$ AND $Q(a, x)$

In GRATIO, normally the smaller of P and Q is computed and the identity $P + Q = 1$ applied. However, occasionally we simply ensure that the quantity computed (P or Q) does not exceed .9. The input argument IND is provided, allowing the user to specify whether full or limited accuracy is desired. IND may be any integer. If IND = 0, then it is assumed that the user is requesting as much accuracy as possible (up to 14 significant digits). Otherwise, if IND = 1, then it is assumed that P and Q are needed to only 6 significant digits, and if IND $\neq$ 0, 1, then it is assumed that 3 digit accuracy suffices. The 3 digit case frequently runs twice as fast as the IND = 0 case.

Internally three parameters (BIG, x_0, e_0) are used, which depend on the accuracy requested by the user. These parameters have the following values:

IND	BIG	x_0	e_0
0	20	31	.25E-3
1	14	17	.25E-1
2	10	9.7	.14

(7)

Different strategies are used depending on whether $a < 1$, $1 \le a <$ BIG, or $a \ge$ BIG.

For $a < 1$, the following relations are used:

$$\begin{aligned} P(1/2, x) &= \operatorname{erf}\sqrt{x} \qquad x < 1/4 \\ Q(1/2, x) &= \operatorname{erfc}\sqrt{x} \qquad x \ge 1/4 \end{aligned} \tag{8}$$

$$P(a, x) = \frac{x^a(1-J)}{\Gamma(a+1)} \qquad J = -a \sum_{n \ge 1} \frac{(-x)^n}{(a+n)n!} \tag{9}$$

$$Q(a, x) = \frac{x^a J - L(a \ln x)}{\Gamma(a+1)} - H(a) \tag{10}$$

$$Q(a, x) = R(a, x)\left[\frac{1}{x+}\,\frac{1-a}{1+}\,\frac{1}{x+}\,\frac{2-a}{1+}\,\frac{2}{x+}\cdots\right] \quad \text{(Continued fraction).} \tag{11}$$

Expansion (9), the Taylor series in x for $x^{-a}P(a, x)$, results from term-by-term integration in (1), after replacing $\exp(-x)$ with its Maclaurin series. Expression (10) is obtained from (9), using $x^a = 1 + L(a \ln x)$ and $1/\Gamma(a + 1) = 1 + H(a)$. The continued fraction (11) is derived in [16, p. 356].

For $x < 1.1$ (when $a < 1$), let

$$\alpha(x) = \begin{cases} \ln\sqrt{.765}/\ln(x) & \text{if} \quad x < \dfrac{1}{2} \\ x/2.59 & \text{if} \quad x \ge \dfrac{1}{2} \end{cases} \tag{12}$$

Since P is a decreasing function of a, (9) is used if $a \ge \alpha(x)$, and (10) is used if $a < \alpha(x)$. $P(\alpha(x), x)$ is near .9, but is less than .9 for all x. In [9], $\alpha(x)$ is defined so that the choice between computing P or Q is made at $P \approx .5$ rather than .9. However, we prefer using (9) whenever possible since it is better behaved numerically than (10).

For $x \ge 1.1$, the continued fraction (11) is used. Recurrence formulas for its evaluation are given by

$$\begin{aligned} A_1 &= 1 & A_2 &= 1 \\ B_1 &= x & B_2 &= x + 1 - a \\ A_{2n+1} &= xA_{2n} + nA_{2n-1} & A_{2n} &= A_{2n-1} + (n-a)A_{2n-2} \\ B_{2n+1} &= xB_{2n} + nB_{2n-1} & B_{2n} &= B_{2n-1} + (n-a)B_{2n-2}, \end{aligned} \tag{13}$$

where $Q/R = \lim_{n\to\infty} A_n/B_n$. These relations follow from Theorem (2) [1, p. 19].

For $1 \le a <$ BIG, (11) and the following relations are used:

$$\begin{aligned} Q(a, x) &= e^{-x} \sum_{n=0}^{a-1} \frac{x^n}{n!}, \quad a = 1, 2, \ldots \\ Q(a, x) &= \operatorname{erfc}\sqrt{x} + \frac{e^{-x}}{\sqrt{\pi x}} \sum_{n=1}^{i} \frac{x^n}{(1 - 1/2)\cdots(n - 1/2)}, \\ & \qquad a = i + 1/2 \quad (i = 1, 2, \ldots) \end{aligned} \tag{14}$$

$$P(a, x) = \frac{R(a, x)}{a}\left[1 + \sum_{n\ge 1} \frac{x^n}{(a+1)\cdots(a+n)}\right] \tag{15}$$

$$Q(a, x) = \frac{R(a, x)}{x}\left[1 + \sum_{n=1}^{N-1} \frac{a_n}{x^n} + \frac{\theta_N a_N}{x^N}\right] \quad \text{(Asymptotic expansion)}$$

$$a_n = (a-1)(a-2)\cdots(a-n), \quad (n \ge 1)$$
$$\theta_N = e^x x^{N+1-a} \int_x^\infty e^{-t} t^{a-N-1}\, dx, \quad (N > 1). \tag{16}$$

Equations (14) follow from $Q(1, x) = \exp(-x)$, $Q(1/2, x) = \operatorname{erfc} \sqrt{x}$, and $Q(a+1, x) = Q(a, x) + R(a, x)/a$, where the latter results from an integration by parts on (2). Expressions (15) and (16) follow from successive integration by parts of (1) and (2). For $x > a - N$, θ_N can be bounded by integrating with respect to $u = t - (a - N)\ln t$. One obtains

$$\theta_N = e^x x^{N+1-a} \int_{u_0}^\infty \frac{e^{-u}}{t - (a-N)}\, du < \frac{x}{x - (a-N)},$$

where $u_0 = x - (a - N)\ln x$. This bound was supplied by a referee. The parameter x_0 in (7) is the smallest value of x for which (16) is applied. The bound for θ_N may be used to obtain values for x_0. The values in (7) were selected by experimentation.

The finite sums (14) are used when $a \le x < x_0$ and $a = n/2$ for n an integer ≥ 2. Otherwise, for $1 \le a <$ BIG:

if	$x \le \max[a, \ln 10]$,	then (15) is applied; else
if	$x < x_0$,	then (11) is used, and
if	$x \ge x_0$,	then (16) is used.

For $x < \ln 10$, (15) is used since it is more efficient than (11) and $P(a, x) < .9$.

For $a \ge$ BIG, (11), (15), (16), and the following expansions are used:

$$P(a, x) = \frac{1}{2}\operatorname{erfc}\sqrt{y} - \frac{e^{-y}}{\sqrt{2\pi a}}\, T(a, \lambda) \quad (\lambda \le 1)$$
$$Q(a, x) = \frac{1}{2}\operatorname{erfc}\sqrt{y} + \frac{e^{-y}}{\sqrt{2\pi a}}\, T(a, \lambda) \quad (\lambda > 1) \tag{17}$$
$$\lambda = x/a, \qquad y = a\phi(\lambda) \quad (\text{see } (3)).$$

$$T(a, \lambda) = \sum_{k=0}^{N} c_k(z) a^{-k}$$
$$c_k(z) = \sum_{n=0}^{L(k)} D_k(n) z^n \tag{18}$$
$$z = \begin{cases} \sqrt{2\phi(\lambda)} & (\lambda \ge 1) \\ -\sqrt{2\phi(\lambda)} & (\lambda < 1). \end{cases}$$

These formulas were derived by Temme [14, 15]. An extensive set of the $D_k(n)$ coefficients is given in Appendix F. The coefficients were computed from the recurrence relations in [15, p. 762] using Brent's multiple precision arithmetic (set at 50 digits) [2].

IND	λ	N	$L(0)$	$L(1)$	$L(2)$	$L(3)$	$L(4)$	$L(5)$	$L(6)$	$L(7)$
0	$.001 < \sigma \leq .4$	7	13	12	10	8	6	4	2	0
	$\sigma \leq .001$	7	7	6	5	4	2	2	1	0
1	$e_0/\sqrt{a} < \sigma \leq .4$	2	6	4	1	—	—	—	—	—
	$\sigma \leq e_0/\sqrt{a}$	2	2	1	1	—	—	—	—	—
2	$e_0/\sqrt{a} < \sigma \leq .4$	0	3	—	—	—	—	—	—	—
	$\sigma \leq e_0/\sqrt{a}$	0	1	—	—	—	—	—	—	—

$\sigma = |1 - \lambda|$

Fig. 1. Values used for N and $L(k)$ in (18).

Equations (17) and (18) treat the region of difficulty mentioned in the Introduction, namely that region of the ax-plane where $x \approx a$, with a large. In [14], Temme shows that for $a \to \infty$ and $y \to 0$, formulas (17) are uniform asymptotic expansions converging uniformly to $P = Q = 1/2$. For $y \approx 0$, the following simplified form of (17) is also used.

$$\begin{aligned} P(a, x) &= E(y) - \frac{1-y}{\sqrt{2\pi a}}\, T(a, \lambda) \quad \lambda \leq 1 \\ Q(a, x) &= E(y) + \frac{1-y}{\sqrt{2\pi a}}\, T(a, \lambda) \quad \lambda > 1 \\ E(y) &= \frac{1}{2} - \left(1 - \frac{y}{3}\right)\sqrt{y/\pi}. \end{aligned} \tag{19}$$

Let $\sigma = |1 - \lambda|$ and e_0 be the value given in (7). If $\sigma \leq .4$, then the general Temme expansion (17) is used except when $\sigma \leq e_0/\sqrt{a}$, in which case (19) is employed. For $\sigma > .4$, (15) is applied when $\lambda < .6$, (11) is applied when $\lambda > 1.4$ and $x < x_0$, and (16) is applied when $\lambda > 1.4$ and $x \geq x_0$.

The table in Figure 1 gives the values of N and $L(k)$ used in (18). The flowchart in Figure 2 summarizes the use of (8)–(19) in GRATIO.

When $a \to \infty$ and $x/a \to 1$, the inherent error of P and Q increases until it becomes sufficiently large so as to make P and Q computationally indeterminant. Throughout the remainder of this paper, let ϵ be the smallest number for which $1 + \epsilon > 1$ in the floating-point arithmetic being used. Then, if the conditions

$$\begin{aligned} \sigma &\leq 2\epsilon \\ a\epsilon^2 &> 3.28E\text{-}3 \end{aligned} \tag{20}$$

are both satisfied, a relative error analysis shows that neither $P(a, x)$ nor $Q(a, x)$ can be determined with certainty to 1 correct digit [6, Appendix B]. When conditions (20) hold, GRATIO performs no computation. Instead, the indeterminancy is reported to the user.

In practice, GRATIO has been found to be a reliable subroutine. On the CDC 6000–7000 series computers, the requested precision is normally achieved except when underflow forces P and Q to be assigned the values 0 and 1, or when the inherent error of P and Q forces a lower precision. Computational indeterminancy occurs on the CDC when $\sigma \leq 1.4E\text{-}14$ and $a \geq 6.6E25$.

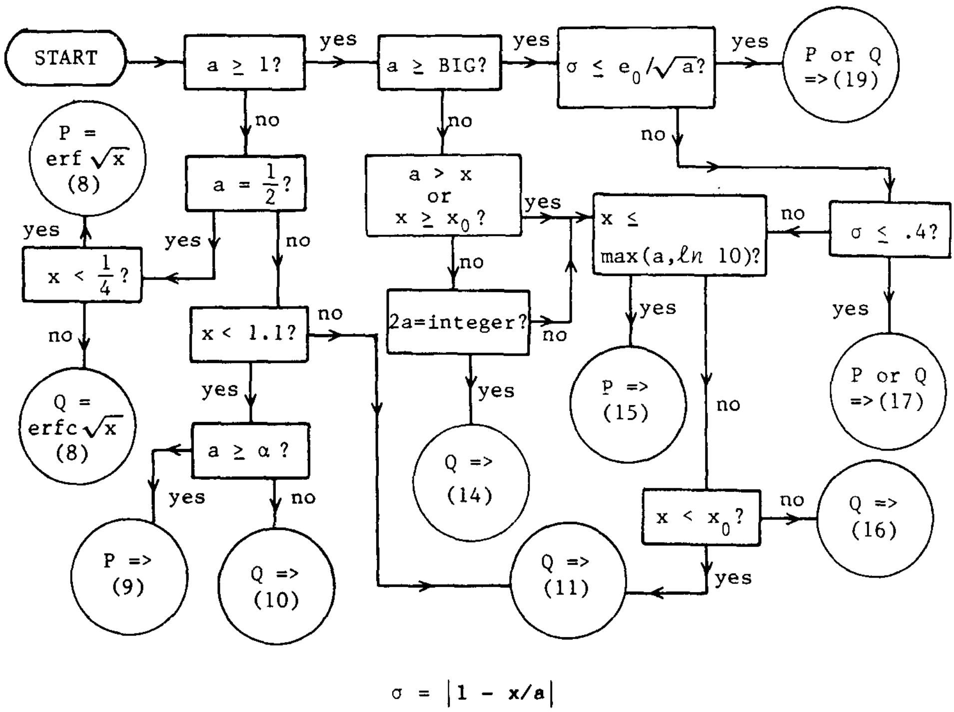

Fig. 2. Flowchart of the algorithm for GRATIO. *Note.* BIG, x_0, e_0 are given in (7) and α in (12.)

4. EVALUATION OF THE INVERSE

The computation of x when a, $P(a, x)$, and $Q(a, x)$ are given is of importance in statistics because of its relationship to the percentage points of the chi-square distribution [5]. In most cases iteration is required to obtain x. Third-order Schröder iteration normally suffices. However, when Schröder iterates cannot be used, then Newton–Raphson iterates are generated. These iterative procedures are appropriate when a high accuracy routine such as GRATIO is available and the required derivatives can be efficiently generated. The derivatives involved present no problems, since they depend only on $R(a, x)$ (see Section 2). For efficiency and convergence, a good initial approximation x_0 is needed for x. Consider the selection of x_0. It is assumed that $a \neq 1$, since $x = -\ln Q$ when $a = 1$.

The following approximations x_0 are used for $a < 1$. Here $B = Q\Gamma(a)$ and $c = .57721\ldots$ (Euler's constant).

$$
\begin{aligned}
x_0 &= \frac{u}{1 - u/(a+1)} && B > .6 \text{ or } B \geq .45 \text{ and } a \geq .3 \\
u &= [P\Gamma(a+1)]^{1/a} && \text{if } \; BQ > 10^{-8} \\
u &= \exp(-Q/a - c) && \text{otherwise}
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
x_0 &= t \exp u && a < .3 \text{ and } .35 \leq B \leq .6 \\
t &= \exp(-c - B) \\
u &= t \exp t
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
&x_0 = y - (1-a)\ln v - \ln\left[1 + \frac{1-a}{1+v}\right] && .15 \le B < .35 \\
&y = -\ln B && \text{or} \qquad (23)\\
&v = y - (1-a)\ln y && .15 \le B < .45 \text{ and } a \ge .3
\end{aligned}
$$

$$
x_0 = y - (1-a)\ln v - \ln\left[\frac{v^2 + 2(3-a)v + (2-a)(3-a)}{v^2 + (5-a)v + 2}\right] \qquad (24)
$$

$$
.01 < B < .15
$$

$$
\begin{aligned}
&y = -\ln B \\
&v = y - (1-a)\ln y \\
&x_0 = y + c_1 + c_2/y + \cdots + c_5/y^4 && B \le .01 \\
&y = -\ln B \\
&c_1 = (a-1)\ln y \\
&c_2 = (a-1)(1+c_1) \\
&c_3 = (a-1)\left[-\frac{1}{2}c_1^2 + (a-2)c_1 + \frac{3a-5}{2}\right] && (25)\\
&c_4 = (a-1)\left[\frac{1}{3}c_1^3 - \frac{3a-5}{2}c_1^2 + (a^2-6a+7)c_1 + \frac{11a^2-46a+47}{6}\right] \\
&c_5 = (a-1)\left[-\frac{1}{4}c_1^4 + \frac{11a-17}{6}c_1^3 + (-3a^2+13a-13)c_1^2 \right.\\
&\qquad \left. + \frac{2a^3-25a^2+72a-61}{2}c_1 + \frac{25a^3-195a^2+477a-379}{12}\right]
\end{aligned}
$$

The ranges for these formulas were established by computer testing. If $B \le 10^{-28}$, then x is assigned the value x_0 given by (25), which is accurate to at least 10 significant digits. (25) was derived by Fettis [7]. The remaining formulas are motivated by the following heuristic reasoning.

Formulas (21). Since $a < 1$, for $x \ge 1$ it follows that $B \le e^{-x} \le e^{-1}$. Consequently, $x < 1$ when $B \ge .45$, and from (9):

$$
x = [P\Gamma(a+1)]^{1/a}(1-J)^{-1/a}. \qquad (26)
$$

If $u = [P\Gamma(a+1)]^{1/a}$ is selected as the first approximation for x, then from (26) we obtain the approximation

$$
x_0 = [P\Gamma(a+1)]^{1/a}\left[1 - \frac{au}{a+1}\right]^{-1/a} = u\left[1 - \frac{au}{a+1}\right]^{-1/a} \approx \frac{u}{1 - u/(a+1)}.
$$

For $BQ \approx 0$, $Q \approx 0$ since $B \ge .45$, $a \le Q/B \approx 0$, $B = Q\Gamma(a+1)/a \approx Q/a$, and

$$
1/a \ln P = 1/a \ln(1-Q) = 1/a(-Q - Q^2/2 - \cdots) \approx -Q/a.
$$

Moreover, by the Taylor expansion for $1/\Gamma(a+1)$ [1, 6.1.34] we have $1/a \ln \Gamma(a+1) \approx -c$. Consequently $\ln u \approx -Q/a - c$.

Formula (22). For small a, $B \approx \int_x^\infty e^{-t}/t\,dt \approx -c - \ln x + x$. Consequently,

$$
x \approx t \exp x,
$$

where $t = \exp(-c - B)$. Letting t denote the first approximation for x, we then obtain the approximation $u = t \exp t$. This in turn induces the approximation $x_0 = t \exp u$.

Formulas (23)–(24). For sufficiently small B, (16) can be used to obtain approximations for x. Considering only the first term of (16), $Q \approx R/x$ yields

$$x \approx -\ln B - (1 - a)\ln x. \tag{27}$$

Since $x \gg (1 - a)\ln x$ for sufficiently large x, $y = -\ln B$ is selected as the initial approximation for x. Then, from (27), we obtain the approximation

$$v = -\ln B - (1 - a)\ln y.$$

This result can be improved by using the approximation

$$Q(a, x) \approx \frac{R(a, x)}{x}\left[\frac{x + 1}{x + 2 - a}\right]. \quad [12, \text{p. } 201] \tag{28}$$

Rewriting this approximation in the form

$$x \approx -\ln B - (1 - a)\ln x - \ln\left[1 + \frac{1 - a}{1 + x}\right] \tag{29}$$

and applying it to v gives (23). Similarly, v can be improved by

$$Q(a, x) \approx \frac{R(a, x)}{x}\left[\frac{x^2 + (5 - a)x + 2}{x^2 + 2(3 - a)x + (2 - a)(3 - a)}\right]. \quad [12, \text{p. } 201] \tag{30}$$

In this case (24) is obtained.

For $a > 1$, let w denote the Cornish–Fisher 6-term approximation for x [8]; that is,

$$w = a + s\sqrt{a} + \frac{s^2 - 1}{3} + \frac{s^3 - 7s}{36\sqrt{a}} - \frac{3s^4 + 7s^2 - 16}{810a} + \frac{9s^5 + 256s^3 - 433s}{38880a\sqrt{a}}, \tag{31}$$

where $Q(a, x) = 1/2 \operatorname{erfc}(s/\sqrt{2})$. The value of s is obtained from the minimax approximation

$$s = (-1)^m\left[t - \frac{a_0 + a_1t + a_2t^2 + a_3t^3}{1 + b_1t + b_2t^2 + b_3t^3 + b_4t^4}\right] \quad [5] \tag{32}$$

where $t = \sqrt{-2\ln\tau}$ and

$$m = \begin{cases} 1 & \text{if } 0 < P < 1/2 \\ 0 & \text{if } 1/2 \le P < 1 \end{cases} \qquad \tau = \begin{cases} P & \text{if } 0 < P < 1/2 \\ Q & \text{if } 1/2 \le P < 1 \end{cases}$$

$$\begin{aligned} a_0 &= 3.31125922108741 & b_1 &= 6.61053765625462 \\ a_1 &= 11.6616720288968 & b_2 &= 6.40691597760039 \\ a_2 &= 4.28342155967104 & b_3 &= 1.27364489782223 \\ a_3 &= .213623493715853 & b_4 &= .361170810188420E\text{-}1. \end{aligned}$$

When $a \geq 500$ and $|1 - w/a| < 10^{-6}$, x is assigned the value w, which is accurate to at least 10 significant digits. In this case iteration is not needed. When a is sufficiently large and w/a sufficiently near 1, w is accurate to 12 significant digits. In contrast, the inherent error of P and Q forces P and Q to be computable to decreasing precision when $a \to \infty$ and $x/a \to 1$. Indeed, P and Q become computationally indeterminant when conditions (20) hold. Thus iteration is not appropriate for a considerable portion of the region $a \geq 500$ and $|1 - w/a| < 10^{-6}$.

Assume now that $a < 500$ or $|1 - w/a| > 10^{-6}$. For $P > 1/2$, (25) and the following approximation are used.

$$\begin{aligned} u &= T(w) = -\ln B + (a-1)\ln w - \ln\left[1 + \frac{1-a}{1+w}\right] \\ x_0 &= T(u). \end{aligned} \tag{33}$$

This approximation is motivated by applying (29) to improve w. If $w < 3a$, then x_0 is assigned the value w. Otherwise, if $w \geq 3a$, let $D = \max[2, a(a-1)]$. Then (25) is used when $B \leq 10^{-D}$ and (33) is used when $B > 10^{-D}$.

For $P \leq 1/2$, we first note from (15) that $x \approx F_n(x)$ where

$$\begin{aligned} F_n(x) &= \exp\{[v + x - \ln S_n(x)]/a\} \\ v &= \ln[P\Gamma(a+1)] \\ S_0 &= 1 \\ S_n &= 1 + \sum_{i=1}^{n} x^i/[(a+1)\cdots(a+i)] \quad (n = 2, 3, \ldots). \end{aligned} \tag{34}$$

If $w > .15(a+1)$ let $z = w$. Otherwise, if $w \leq .15(a+1)$ define z as follows:

$$\begin{aligned} u_1 &= F_0(w) \\ u_2 &= F_2(u_1) \\ u_3 &= F_2(u_2) \\ z &= F_3(u_3) \end{aligned} \tag{35}$$

If $z \leq .002(a+1)$, then x is assigned the value z, which is accurate to at least 10 significant digits. In this case iteration is not needed. Otherwise, if $.002(a+1) < z \leq .01(a+1)$ or $z > .7(a+1)$, then x_0 is assigned the value z. If $.01(a+1) < z \leq .7(a+1)$, then x_0 is defined by

$$\begin{aligned} \bar{z} &= F_N(z) \\ x_0 &= \bar{z}\left[1 - \frac{a \ln \bar{z} - \bar{z} - v + \ln S_N(z)}{a - \bar{z}}\right] \end{aligned} \tag{36}$$

where N is the smallest integer such that $z^N/[(a+1)\cdots(a+N)] \leq 10^{-4}$. The latter approximation in (36) is motivated by considering (15) in the form $G(x) \approx 0$, where

$$G(x) = v + x - a \ln x - \ln S_n(x).$$

Then the approximation for x_0 follows by applying a Newton–Raphson correction

$$x = \bar{z} - \frac{G(\bar{z})}{G'(\bar{z})},$$

where $G'(\bar{z})$ is approximated by $1 - a/\bar{z}$ and $S_n(\bar{z})$ is approximated by $S_n(z)$.

In GAMINV, the argument $X0$ is provided to let the user specify an initial approximation x_0 for x. If $X0 \le 0$, then the initial approximation given in this section is used. Given a, $P(a, x)$, $Q(a, x)$, and x_0, iterates $x_1, x_2, \ldots$ are generated by

$$\begin{aligned} x_{n+1} &= x_n(1 - h_n) && \text{Schröder [11, pp. 529–531]} \\ h_n &= t_n + w_n t_n^2 && \\ w_n &= (a - 1 - x_n)/2 && \\ t_n &= \frac{P(a, x_n) - P(a, x)}{R(a, x_n)} && \text{if} \quad P(a, x) \le \frac{1}{2} \\ t_n &= \frac{Q(a, x) - Q(a, x_n)}{R(a, x_n)} && \text{if} \quad P(a, x) > \frac{1}{2} \end{aligned} \tag{37}$$

when $|t_n| \le .1$ and $|w_n t_n| \le .1$. If either of these conditions is violated, then

$$\begin{aligned} x_{n+1} &= x_n(1 - h_n) \quad \text{Newton–Raphson} \\ h_n &= t_n \end{aligned} \tag{38}$$

is used. If x_{n+1} is obtained by (37), $|w_n| \ge 1$, and $|w_n t_n^2| \le \bar{\epsilon}$ where $\bar{\epsilon} = 10^{-10}$, then x is assigned the value x_{n+1} and the routine terminates. Otherwise, x is assigned the value x_{n+1} when $|h_n| < \bar{\epsilon}$ or

$$\begin{aligned} &|h_n| \le \tau \quad \text{and} \\ &|P(a, x_n) - P(a, x)| \le \tau P(a, x) \quad \text{if} \quad P(a, x) \le 1/2 \\ &|Q(a, x) - Q(a, x_n)| \le \tau Q(a, x) \quad \text{if} \quad P(a, x) > 1/2. \end{aligned} \tag{39}$$

The tolerance τ is set to 10^{-5}, which has been found by machine testing to give the result correct to 10 significant digits when a 10 or more digit floating-point arithmetic is used.

When iteration fails, a variable (IERR) is set reporting the failure to the user. This occurs when

(a) $P(a, x_n)$ or $Q(a, x_n)$ has the computed value 0,
(b) $x_n \le 0$,
(c) $P(a, x_n)$ or $Q(a, x_n)$ cannot be computed to at least 5 digit accuracy, or
(d) more than 20 iterations are required.

If x_0 is the initial approximation given in this section, then (a) and (b) occur when $P(a, x)$, $Q(a, x)$, or the solution x is less than $10^{10}\mu$, where μ is the smallest positive number in the floating-point arithmetic being used. Situations (c) and (d) arise only when the initial approximation is provided by the user. By reasoning similar to that in [6, Appendix B], (c) is assumed to occur when $|1 - x_n/a| \le 2\epsilon$ and $a\epsilon^2 > .4\text{E-}10$.

If a k-digit floating-point arithmetic is being used where $k < 10$, then the iteration procedure is employed with $\bar{\epsilon} = 10^{-8}$. Here it is assumed that $k \ge 6$. Also, iteration is not used in the following cases.

(a) If $a < 1$ and $B \le 10^{-13}$, then x is assigned the value x_0 given by (25).
(b) If $a \ge 100$ and $|1 - w/a| \le 10^{-4}$ where w is given by (31), then x is assigned the value w.
(c) If $a > 1$, $a < 100$ or $|1 - w/a| > 10^{-4}$, $P \le 1/2$, and $z \le .006(a + 1)$ where z is given by (35), then x is assigned the value z.

In these cases, x is accurate to at least 8 significant digits when $k \geq 9$.

In practice, GAMINV is found to be a robust and efficient routine. When an initial approximation in this section is used, convergence is normally achieved in no more than 2 iterations if $a > 1$ and in less than 4 iterations if $a < 1$.

APPENDIX A

Evaluation of $\ln(1 + a)$ for $|a| \leq .375$. Let $t = a/(a + 2)$. Then,

$$\ln(1 + a) = 2t\left[\frac{1 + p_1t^2 + p_2t^4 + p_3t^6}{1 + q_1t^2 + q_2t^4 + q_3t^6}\right]$$

where

```
P1 = -.12941 89230 21993E+01        Q1 = -.16275 22563 55323E+01
P2 =  .40530 34928 62024E+00        Q2 =  .74781 10140 37616E+00
P3 = -.17887 45460 12214E-01        Q3 = -.84510 42179 45565E-01
```

APPENDIX B

Evaluation of $L(x) = \exp(x) - 1$. *Let* $\omega = e^x$. Then,

$$L(x) = \omega - 1 \qquad x < -.15$$
$$L(x) = \omega(1 - 1/\omega) \qquad x > .15$$
$$L(x) = x\left[\frac{1 + p_1x + p_2x^2}{1 + q_1x + \cdots + q_4x^4}\right] \qquad |x| \leq .15$$

where

```
P1 =  .91404 19148 19518E-09
P2 =  .23808 23610 44469E-01

Q1 = -.49999 99990 85958E+00
Q2 =  .10714 15689 80644E+00
Q3 = -.11904 11797 60821E-01
Q4 =  .59513 08118 60248E-03
```

APPENDIX C

Evaluation of $H(a) = 1/\Gamma(a + 1) - 1$ for $-.5 \leq a \leq 1.5$. Let $t = a$ for $a \leq 1/2$ and $t = a - 1$ for $a > 1/2$. Then,

$$H(a) = a(\omega + 1) \qquad \text{if} \quad -1/2 \leq a \leq 0$$
$$H(a) = a\omega_1 \qquad \text{if} \quad 0 \leq a \leq 1/2$$
$$H(a) = \frac{t}{a}\,\omega \qquad \text{if} \quad 1/2 \leq a \leq 1$$
$$H(a) = \frac{t}{a}\,(\omega_1 - 1) \qquad \text{if} \quad 1 \leq a \leq 3/2$$

where

$$\omega = \frac{r_0 + r_1t + \cdots + r_8t^8}{s_0 + s_1t + s_2t^2}$$
$$\omega_1 = \frac{p_0 + p_1t + \cdots + p_6t^6}{q_0 + q_1t + \cdots + q_4t^4}$$

P0 =	.57721 56649 01533E+00	Q0 =	.10000 00000 00000E+01
P1 =	-.40907 81930 05776E+00	Q1 =	.42756 96130 95214E+00
P2 =	-.23097 53808 57675E+00	Q2 =	.15845 16724 30138E+00
P3 =	.59727 53304 52234E-01	Q3 =	.26113 20214 41447E-01
P4 =	.76696 81816 49490E-02	Q4 =	.42324 42978 96961E-02
P5 =	-.51488 97713 23592E-02		
P6 =	.58959 74286 11429E-03		

R0 =	-.42278 43350 98468E+00	S0 =	.10000 00000 00000E+01
R1 =	-.77133 03838 16272E+00	S1 =	.27307 61353 03957E+00
R2 =	-.24475 77652 22226E+00	S2 =	.55939 82369 57378E-01
R3 =	.11837 89898 72749E+00		
R4 =	.93035 72933 60349E-03		
R5 =	-.11829 09934 45146E-01		
R6 =	.22304 76611 58249E-02		
R7 =	.26650 59790 58923E-03		
R8 =	-.13267 49097 66242E-03		

APPENDIX D

Evaluation of $\ln \Gamma(a)$ for $.8 \le a \le 2.25$. If $1 + a_1 x + a_2 x^2 + \cdots$ is the Taylor series for $1/\Gamma(1 + x)$, let

$$\omega = \frac{1}{x}\left[\frac{1}{\Gamma(1 + x)} - 1\right] = a_1 + a_2 x + \cdots. \tag{D-1}$$

Then, $1/\Gamma(1 + x) = 1 + x\omega$, so that $-\ln \Gamma(1 + x) = \ln(1 + x\omega)$. Consider

$$g(\lambda) = \frac{\ln(1 + \lambda)}{\lambda} = 1 - \frac{\lambda}{2} + \frac{\lambda^2}{3} - \cdots. \tag{D-2}$$

Then,

$$\frac{\ln \Gamma(1 + x)}{x} = -\omega g(x\omega) \tag{D-3}$$

can be used to compute minimax approximations for $\ln \Gamma(1 + x)/x$. To obtain approximations for $\ln \Gamma(2 + x)/x$, we note that

$$\frac{1}{\Gamma(2 + x)} = \frac{1}{(x + 1)\Gamma(x + 1)} = \frac{1 + x\omega}{1 + x} = 1 + \frac{(\omega - 1)x}{1 + x}. \tag{D-4}$$

Let $r = (\omega - 1)x/(1 + x)$. Then, from (D-4), $-\ln \Gamma(2 + x) = \ln(1 + r) = rg(r)$. Hence,

$$\frac{\ln \Gamma(2 + x)}{x} = \frac{1 - \omega}{1 + x} g(r) \tag{D-5}$$

can be used to obtain minimax approximations for $\ln \Gamma(2 + x)/x$. The following approximations are generated from (D-3) and (D-5).

$$\begin{aligned} \frac{\ln \Gamma(1 + x)}{x} &= -\frac{p_0 + p_1 x + \cdots + p_6 x^6}{q_0 + q_1 x + \cdots + q_6 x^6} \qquad -.2 \le x \le .6 \\ \frac{\ln \Gamma(2 + x)}{x} &= \frac{r_0 + r_1 x + \cdots + r_5 x^5}{s_0 + s_1 x + \cdots + s_5 x^5} \qquad -.4 \le x \le .25 \end{aligned} \tag{D-6}$$

P0 =	.57721 56649 01533E+00	Q0 =	.10000 00000 00000E+01
P1 =	.84420 39221 87225E+00	Q1 =	.28874 31954 73681E+01
P2 =	-.16886 05936 46662E+00	Q2 =	.31275 50889 14843E+01
P3 =	-.78042 76155 33591E+00	Q3 =	.15687 51932 95039E+01
P4 =	-.40205 57993 10489E+00	Q4 =	.36195 19901 01499E+00
P5 =	-.67356 22143 25671E-01	Q5 =	.32503 88682 53937E-01
P6 =	-.27193 57083 22958E-02	Q6 =	.66746 56187 96164E-03

R0 =	.42278 43350 98467E+00	S0 =	.10000 00000 00000E+01
R1 =	.84804 46145 34529E+00	S1 =	.12431 33998 77507E+01
R2 =	.56522 10506 91933E+00	S2 =	.54804 21098 32463E+00
R3 =	.15651 30604 86551E+00	S3 =	.10155 21874 39830E+00
R4 =	.17050 24840 22650E-01	S4 =	.71330 96123 91000E-02
R5 =	.49795 82076 39485E-03	S5 =	.11616 54759 89616E-03

APPENDIX E

Evaluation of $\phi(\lambda) = \lambda - 1 - \ln \lambda$ for $.82 \le \lambda \le 1.18$. Let $r = (\lambda - 1)/(\lambda + 1)$. Then,

$$\phi(\lambda) = 2r^2\left[\frac{1}{1-r} - r\phi_1(r)\right]$$

where

$$\phi_1(r) = \frac{p_0 + p_1 r^2 + p_2 r^4}{q_0 + q_1 r^2 + q_2 r^4} \quad \left(-\frac{.18}{1.82} \le r \le \frac{.18}{2.18}\right)$$

P0 =	.33333 33333 33333E+00	Q0 =	.10000 00000 00000E+01
P1 =	-.22469 64131 12536E+00	Q1 =	-.12740 89239 33623E+01
P2 =	.62088 68153 75787E-02	Q2 =	.35450 87183 69557E+00 .

APPENDIX F

Temme coefficients $D_k(n)$.

n	$D_0(n)$	$D_1(n)$
0	-.3333333333 3333333333 3333333333E+00	-.1851851851 8518518518 5185185185E-02
1	.8333333333 3333333333 3333333333E-01	-.3472222222 2222222222 2222222222E-02
2	-.1481481481 4814814814 8148148148E-01	.2645502645 5026455026 4550264550E-02
3	.1157407407 4074074074 0740740741E-02	-.9902263374 4855967078 1893004115E-03
4	.3527336860 6701940035 2733686067E-03	.2057613168 7242798353 9094650206E-03
5	-.1787551440 3292181069 9588477366E-03	-.4018775720 1646090534 9794238683E-06
6	.3919263178 5224377816 9704095630E-04	-.1809855033 4489977837 0285914868E-04
7	-.2185448510 6799921614 7364295512E-05	.7649160916 0811100846 3742149809E-05
8	-.1854062210 7151599607 0179883623E-05	-.1612090089 4563446003 7752218822E-05
9	.8296711340 9530860050 1624213166E-06	.4647127802 8074343422 6135033939E-08
10	-.1766595273 6826079304 3600542457E-06	.1378633446 9157209593 1187533077E-06
11	.6707853543 4014985803 6939710030E-08	-.5752545603 5177049640 2194531835E-07
12	.1026180978 4240308042 5739573227E-07	.1195162859 9778147324 3076536700E-07
13	-.4382036018 4533531865 5297462245E-08	-.1754324171 9747647623 7547551202E-10
14	.9147699582 2367902341 8248817633E-09	-.1009154371 0600412627 4577504687E-08
15	-.2551419399 4946249766 8779537994E-10	.4162792991 8425826362 3372347220E-09
16	-.5830772132 5504250674 6408945040E-10	-.8563907026 4929806380 7431562580E-10
17	.2436194802 0667416243 6940696708E-10	.6067215101 6047586151 2701762170E-13
18	-.5027669280 1141755890 9054985926E-11	.7162498964 8114853900 7961017166E-11
19	.1100439203 1956134770 8374174497E-12	-.2933186643 7714371174 0636683616E-11
20	.3371763262 4009853788 2769884169E-12	.5996696365 6836887233 0374527569E-12

21	-.1392388722 4181620659 1936618490E-12	-.2167178652 7323314101 7100472780E-15
22	.2853489380 7047443203 9669099053E-13	-.4978339972 3692616405 2815522048E-13
23	-.5139111834 2425726189 9064580300E-15	.2029162882 3713424773 6694804326E-13
24	-.1975228829 4349442835 3962401581E-14	-.4131255713 8106100493 5108332558E-14
25	.8099521156 7045613340 7115668703E-15	.8286516239 8830964438 0188591058E-18
26	-.1652253121 6398161819 1514820265E-15	.3410030886 9333327933 6339355911E-15
27	.2530543009 7478884232 7061090060E-17	-.1385419530 2893971535 7034547426E-15
28	.1168693973 8559576588 8230876508E-16	.2812346653 2288746656 8860332727E-16

n	$D_2(n)$	$D_3(n)$
0	.4133597883 5978835978 8359788360E-02	.6494341563 7860082304 5267489712E-03
1	-.2681327160 4938271604 9382716049E-02	.2294720936 2139917695 4732510288E-03
2	.7716049382 7160493827 1604938272E-03	-.4691894943 9525571212 8140111679E-03
3	.2009387860 0823045267 4897119342E-05	.2677206320 6283885296 2309752433E-03
4	-.1073665322 6365160521 5391223622E-03	-.7561801671 8839764107 2538191880E-04
5	.5292344882 9120125416 4217127180E-04	-.2396505113 8672966519 3314027333E-06
6	-.1276063518 8618727713 3779191392E-04	.1108265411 5347302361 4770299727E-04
7	.3423578734 0961380741 9020039047E-07	-.5674952826 9915965674 9963105702E-05
8	.1372195730 9062933205 5943852926E-05	.1423090073 2435883914 5518944706E-05
9	-.6298992138 3800550229 0672234278E-06	-.2786108029 1528142240 5802158211E-10
10	.1428061420 6064241791 5846008823E-06	-.1695840409 1930277289 8641687958E-06
11	-.2047709842 1990866014 9195854409E-09	.8099464905 3880823633 5278504853E-07
12	-.1409252991 0867521053 2930244154E-07	-.1911116848 5973654060 6728140873E-07
13	.6228974084 9220220335 6394293530E-08	.2392862043 9808117968 6413514022E-11
14	-.1367048839 6617113499 2724380284E-08	.2062013181 5488798436 9925818487E-08
15	.9428356159 0146781954 7711211663E-12	-.9460496661 8551321737 5417988510E-09
16	.1287225240 0089318059 5479368873E-09	.2154104977 5774907838 0130268469E-09
17	-.5564595613 4363321146 5414765895E-10	-.1388823336 8139030460 3424682491E-13
18	.1197593554 6366981003 5898150310E-10	-.2189476168 1963939406 4123400466E-10
19	-.4168978225 1838635040 3836626692E-14	.9790998951 1716851256 8262802256E-11
20	-.1094064042 7884594409 9299008641E-11	-.2178219188 0180962115 3859472011E-11
21	.4662239946 3901357463 2620492246E-12	.6208819573 4079014258 1663616850E-16
22	-.9905105763 9069059784 4122258212E-13	.2126978363 2797369769 6702537115E-12
23	.1893187676 8373514505 6885183171E-16	-.9344688791 5174333312 7396765627E-13
24	.8859221872 5911272617 6031067029E-14	.2045367122 6782849324 9215913063E-13

n	$D_4(n)$	$D_5(n)$
0	-.8618882909 1671169860 4702719929E-03	-.3367985533 6635815030 8767592718E-03
1	.7840392217 2006662747 4034881442E-03	-.6972813758 3658577742 9398828576E-04
2	-.2990724803 0319017973 3389609933E-03	.2772753244 9593920787 3364251965E-03
3	-.1463845257 8843418178 1232535691E-05	-.1993257051 6188847700 3360405281E-03
4	.6641498215 4651221866 5853782452E-04	.6797780477 9372078388 1640176604E-04
5	-.3968365047 1794346644 3123507595E-04	.1419062920 6439670148 3392727106E-06
6	.1137572697 0678419098 0552042886E-04	-.1359404818 9768693278 4583938838E-04
7	.2507497226 2375328016 5221942390E-09	.8018470256 3342015397 1925719804E-05
8	-.1695414953 6558306014 7164356782E-05	-.2291481176 5080951703 8048790129E-05
9	.8907507532 2053096888 2898422506E-06	-.3252473551 2984539516 6230137750E-09
10	-.2292934834 0008048705 7216364891E-06	.3465284649 1085264955 9195496828E-06
11	.2956794137 5440490469 6572852500E-10	-.1844718719 1171343276 5322367375E-06
12	.2886582974 2708783629 7341274604E-07	.4824096703 7894180756 3762631739E-07
13	-.1418973943 7803219389 4774303904E-07	-.1798946672 1743515302 5754291717E-13
14	.3446358049 9464897065 9527720474E-08	-.6306194500 0135234351 7516981426E-08
15	-.2302451717 4528067132 0192735850E-12	.3162417628 7745679377 3762181541E-08
16	-.3940923302 8046405275 0697640085E-09	-.7840924253 6974292900 0839303523E-09
17	.1860233896 8504501913 4258533045E-09	.5192679165 2540407237 7621764441E-14
18	-.4356323005 0566180438 0678327446E-10	.9358944242 3067835845 9590623924E-10
19	.1278600101 6296231266 0550463350E-14	-.4513426216 1632782310 1171193130E-10
20	.4679275026 6579194620 0382739992E-11	.1079912999 3116827040 9835885076E-10

n	$D_6(n)$	$D_7(n)$
0	.5313079364 6399222316 5748542978E-03	.3443676068 9237767125 4279625109E-03
1	-.5921664373 5369388286 4836225604E-03	.5171790908 2605921933 7057843002E-04
2	.2708782096 7180448277 1279183488E-03	-.3349316108 1142236311 6635090580E-03
3	.7902353232 6603278721 2032944391E-06	.2812695154 7632370227 3722110708E-03
4	-.8153969367 5619687509 2890088465E-04	-.1097658224 4684731023 5396824501E-03
5	.5611682753 1062496500 3775619041E-04	-.1274100909 5484485379 4579954588E-06
6	-.1832911658 2843375567 3259749374E-04	.2774445151 1563644157 0715073934E-04

7	-.3079613450 6033047825 6414192547E-08	-.1826348880 5711332661 4324442682E-04
8	.3465155368 8036090867 3728529745E-05	.5787694949 7350523989 4178121071E-05
9	-.2029132739 6058603726 9527254583E-05	.4938758933 9362703998 1813418399E-09
10	.5788792863 1490037088 9997586203E-06	-.1059536701 4026042733 8098566210E-05
11	.2338630673 8266569893 3480579232E-12	.6166714376 1104074785 8836254005E-06
12	-.8828600746 3304835250 5085243179E-07	-.1756297335 9060461937 8669693914E-06
13	.4743595888 0408127803 2150770596E-07	-.1297447328 7015438707 0200245453E-11
14	-.1254541502 0710382445 7130611215E-07	.2695423606 2889659836 8920250428E-07
15	.8649648858 0102924713 4668303019E-13	-.1457835290 8731270976 8807136111E-07
16	.1684605897 9264062708 4357772386E-08	.3887645959 3861749980 7195968090E-08

n	$D_8(n)$	$D_9(n)$
0	-.6526239185 9530941892 2034919727E-03	-.5967612901 9274625012 4390067179E-03
1	.8394987206 7208727999 3357516765E-03	-.7204895416 0200105590 8571930225E-04
2	-.4382970985 4172100506 1087953051E-03	.6782308837 6673283616 1951166001E-03
3	-.6969091458 4205519713 6911097362E-06	-.6401475260 2627584510 0045652582E-03
4	.1664484664 2067547837 3845726623E-03	.2775010763 4328704499 2374518206E-03
5	-.1278351767 9769218585 3344001462E-03	.1819700838 0465151046 1686554030E-06
6	.4629953263 6913042906 1361032704E-04	-.8479507117 0685031823 9732559633E-04
7	.4557909867 9227077116 2749294232E-08	.6105192082 5015310176 4709122741E-04
8	-.1059527112 5805195471 8238500313E-04	-.2107392018 3404862408 2975255894E-04
9	.6783342904 8651666227 3073740749E-05	-.8858589014 1255993892 1724802750E-09
10	-.2107547666 6258804246 9972680229E-05	.4528453595 3805377110 8975938164E-05
11	-.1721373143 2817144999 3181614912E-10	-.2842781502 2504407938 0272675993E-05
12	.3773587741 6110979338 0344937299E-06	.8708234177 8646411676 1231237189E-06

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REFERENCES

1. ABRAMOWITZ, M., AND STEGUN, I. A. (Eds.) *Handbook of Mathematical Functions.* Dover, New York, 1965.
2. BRENT, R. P. A FORTRAN multiple-precision arithmetic package. *ACM Trans. Math. Softw. 4* (1978), 57–70.
3. CODY, W. J. Rational Chebyshev approximations for the error function. *Math. Comput. 23* (1969), 631–637.
4. DIDONATO, A. R., AND HAGEMAN, R. K. Computation of the incomplete gamma function ratios. Rep. NSWC/DL TR-3482, Naval Surface Weapons Center, Dahlgren, Va., 1976.
5. DIDONATO, A. R., AND HAGEMAN, R. K. Computation of the percentage points of the chi-square distribution. Rep. NSWC/DL TR-3569, Naval Surface Weapons Center, Dahlgren, Va., 1977.
6. DIDONATO, A. R. The incomplete gamma function ratios using Temme's asymptotic expansions. Rep. NSWC TR 84-79, Naval Surface Weapons Center, Dahlgren, Va., 1984.
7. FETTIS, H. E. An asymptotic expansion for the upper percentage points of the χ^2-distribution. *Math. Comput. 33* (1979), 1059–1064.
8. FISHER, R. A., AND CORNISH, E. A. The percentile points of distributions having known cumulants. *Technometrics 2* (1960), 209–225.
9. GAUTSCHI, W. A computational procedure for incomplete gamma functions. *ACM Trans. Math. Softw. 5* (1979), 466–481.
10. GAUTSCHI, W. Algorithm 542. Incomplete gamma functions. *ACM Trans. Math. Softw. 5* (1979), 482–489.
11. HENRICI, P. *Applied and Computational Complex Analysis, vol. 1.* Wiley, New York, 1974.
12. LUKE, Y. L. *The Special Functions and Their Approximations, vol. 2.* Academic Press, New York, 1969.

13. MORRIS, A. H. NSWC library of mathematics subroutines. Rep. NSWC TR 84-143, Naval Surface Weapons Center, Dahlgren, Va., 1984.
14. TEMME, N. M. Uniform asymptotic expansion of the incomplete gamma functions and the incomplete beta function. *Math. Comput. 29* (1975), 1109–1114.
15. TEMME, N. M. The asymptotic expansion of the incomplete gamma functions. *SIAM J. Math. Anal. 10* (1979), 757–766.
16. WALL, H. S. *Analytic Theory of Continued Fractions.* Van Nostrand, New York, 1948.

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