Classic time series models

(AR, MA, ARMA, ARIMA, VARMAX, etc.)

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Stationarity: Example

- ***Moving Average*** MA(1) process *(the next observation is the mean of all past observations)*

$$y_t = \varepsilon_t + \theta\varepsilon_{t-1}, \quad \{\varepsilon_t\}_{t\geq 1} \sim WN(0, \sigma^2)$$

We have $\mathbb{E}[y_t] = 0$ and

$$\begin{aligned}\gamma_y(t+h, t) &= \mathbb{E}(y_{t+h}y_t) \\ &= \mathbb{E}[(\varepsilon_{t+h} + \theta\varepsilon_{t+h-1})(\varepsilon_t + \theta\varepsilon_{t-1})] \\ &= \begin{cases} \sigma^2(1+\theta^2) \ \text{ if } \ h = 0, \\ \sigma^2\theta \ \text{ if } \ h = \pm 1, \\ 0 \ \text{ otherwise} \end{cases}\end{aligned}$$

- Thus, $\{y_t\}_{t\geq 1}$ is stationary

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Stationarity: Example

- ***Autoregression*** AR(1) process

$$y_t = \varphi y_{t-1} + \varepsilon_t, \quad \{\varepsilon_t\}_{t\geq 1} \sim WN(0, \sigma^2)$$

Assume that y_t is stationary (i.e. $|\varphi| < 1$), then

$$\mathbb{E}[y_t] = \varphi \mathbb{E}[y_{t-1}] = 0 \quad \text{(first-order stationarity)}$$

$$\mathbb{E}[y_t^2] = \varphi^2 \mathbb{E}[y_{t-1}^2] + \sigma^2 = \frac{\sigma^2}{1-\varphi^2} \quad \text{(second-order stationarity)}$$

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Stationarity: Example

Assume that y_t is stationary (i.e. $|\varphi| < 1$), then

$$\mathbb{E}[y_t] = 0, \;\; \mathbb{E}[y_t^2] = \frac{\sigma^2}{1-\varphi^2}$$

$$\begin{aligned}
\gamma_y(h) &= \mathrm{Cov}\left(\varphi y_{t+h-1} + \varepsilon_{t+h}, y_t\right) = \\
&= \varphi \mathrm{Cov}(y_{t+h-1}, y_t) \\
&= \varphi \gamma_y(h-1) \\
&= \varphi^{|h|}\gamma_y(0) \;\; \text{(check for } h>0 \text{ and } h<0\text{)} \\
&= \frac{\varphi^{|h|}\sigma^2}{1-\varphi^2}
\end{aligned}$$

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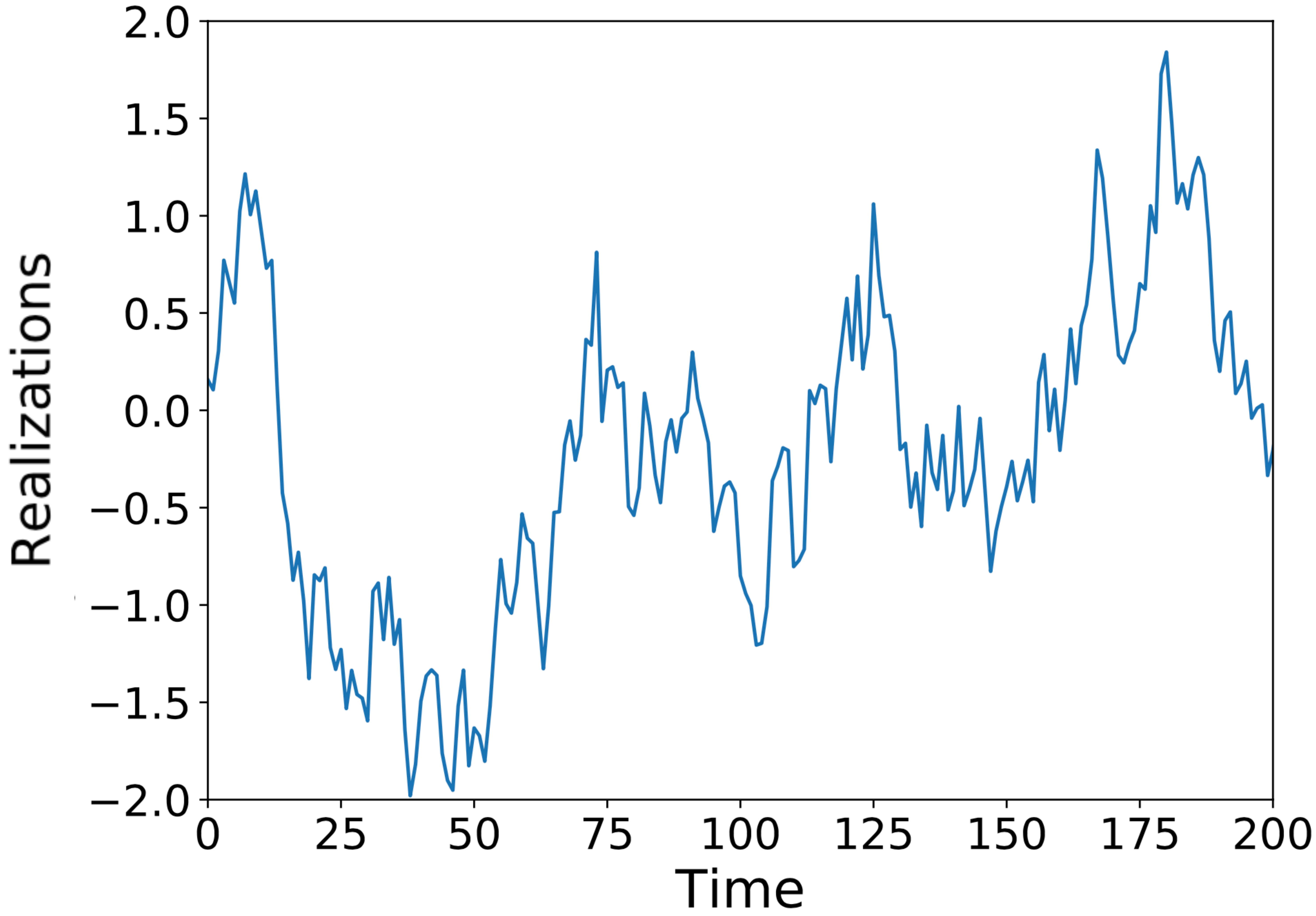
AR(1): $\varphi = 0.95$



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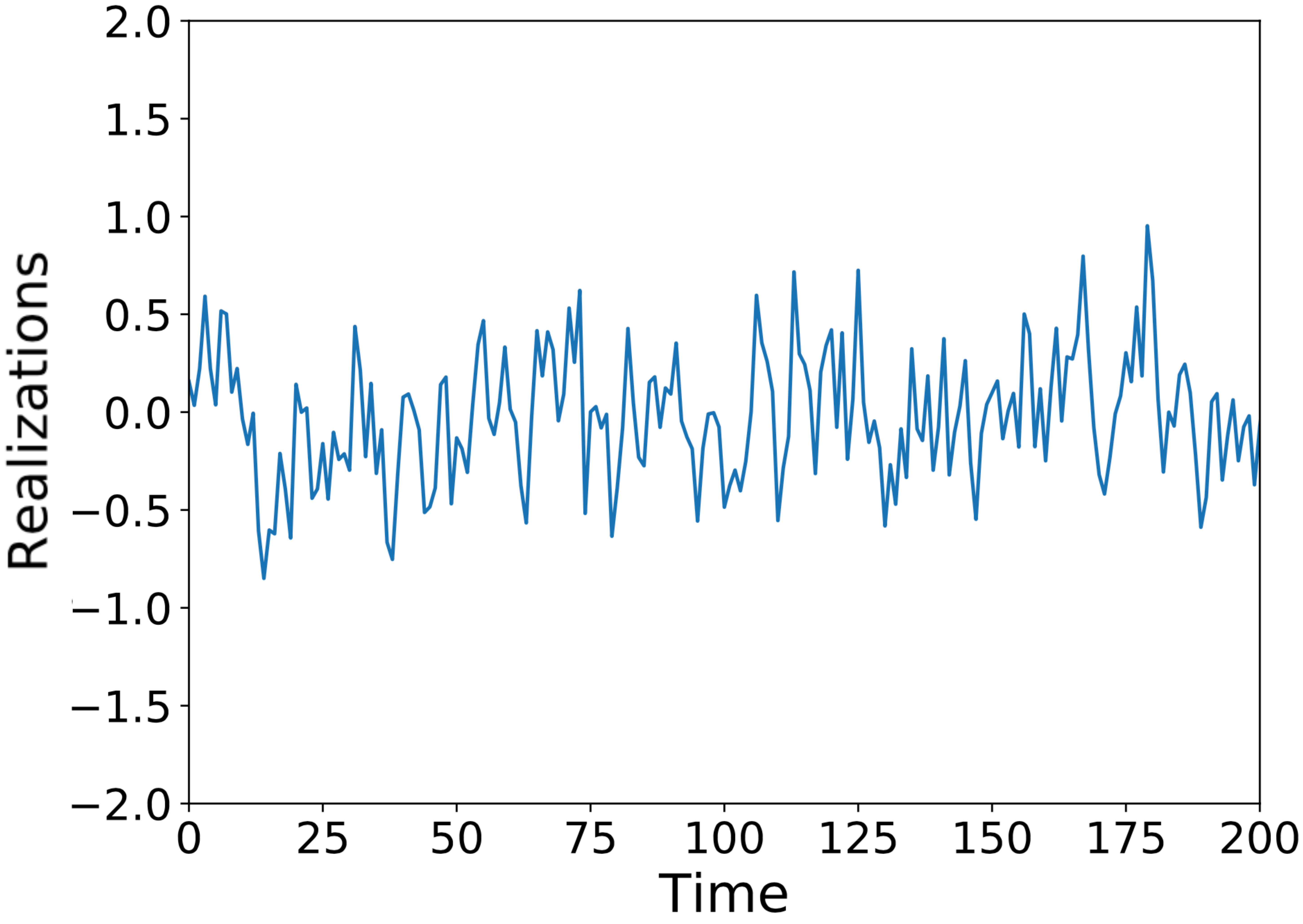
AR(1): $\varphi = 0.5$



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Problem: effective realization of AR(k) process

- How to effectively write a realization of AR(k) process in python?
- The **autocorrelation function (ACF)** of $\{y_t\}_{t\geq 1}$ is defined as

$$\begin{aligned}\rho_y(h) &= \frac{\gamma_y(h)}{\gamma_y(0)} \\ &= \frac{\mathrm{Cov}(y_{t+h}, y_t)}{\mathrm{Cov}(y_t, y_t)} \\ &= \mathrm{Corr}(y_{t+h}, y_t)\end{aligned}$$

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Sample ACF

- Sample autocovariance function

$$\widehat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-|h|} (y_{t+|h|} - \overline{y})(y_t - \overline{y})$$

- T≈ sample covariance of $(y_1, y_{h+1}), \ldots, (y_{n-h}, y_n)$, except that
  - we normalize by n instead of $n - h$, and
  - we subtract the full sample mean

- We estimate sample variance and obtain sample autocorrelation function

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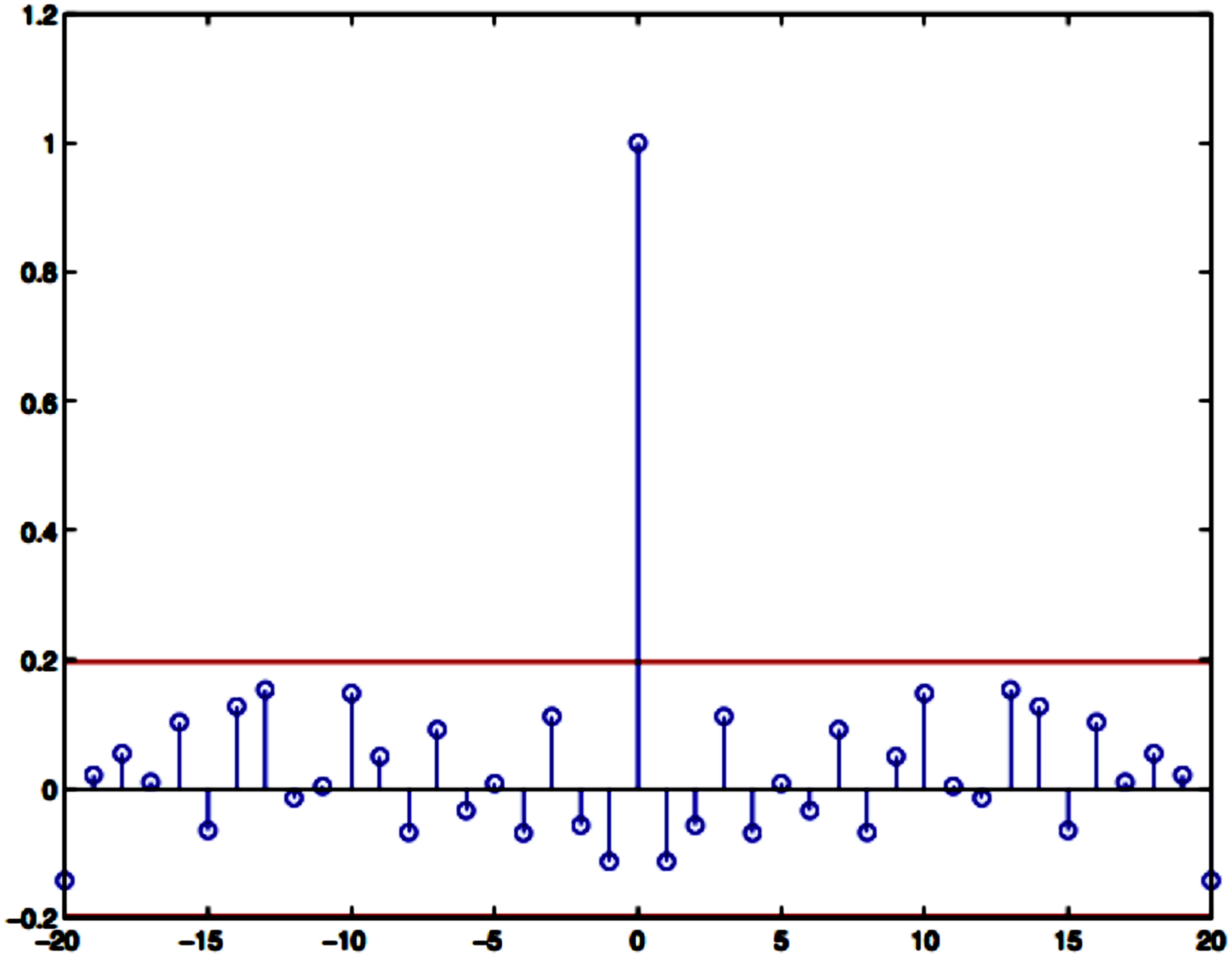
Sample ACF for Gaussian noise



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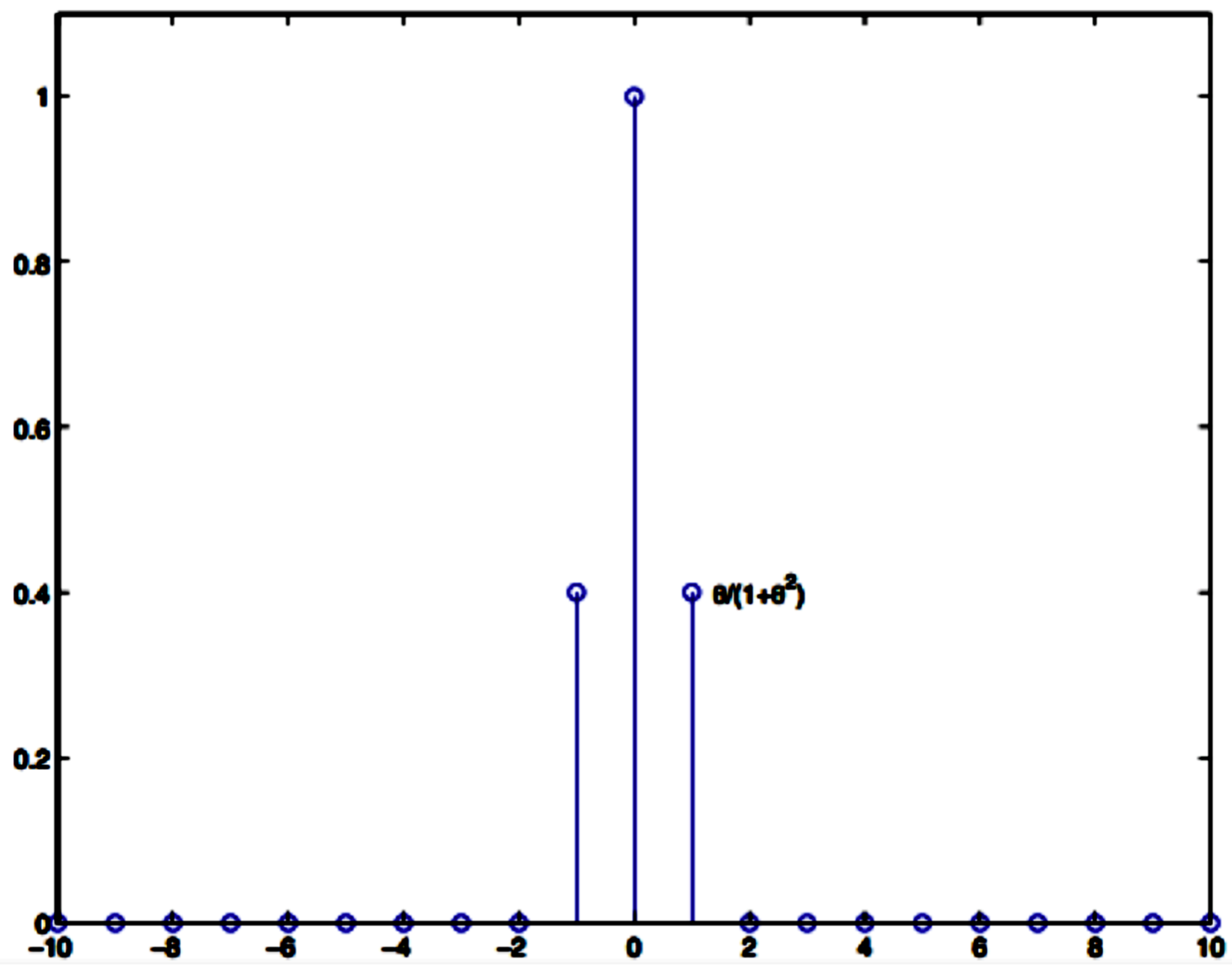
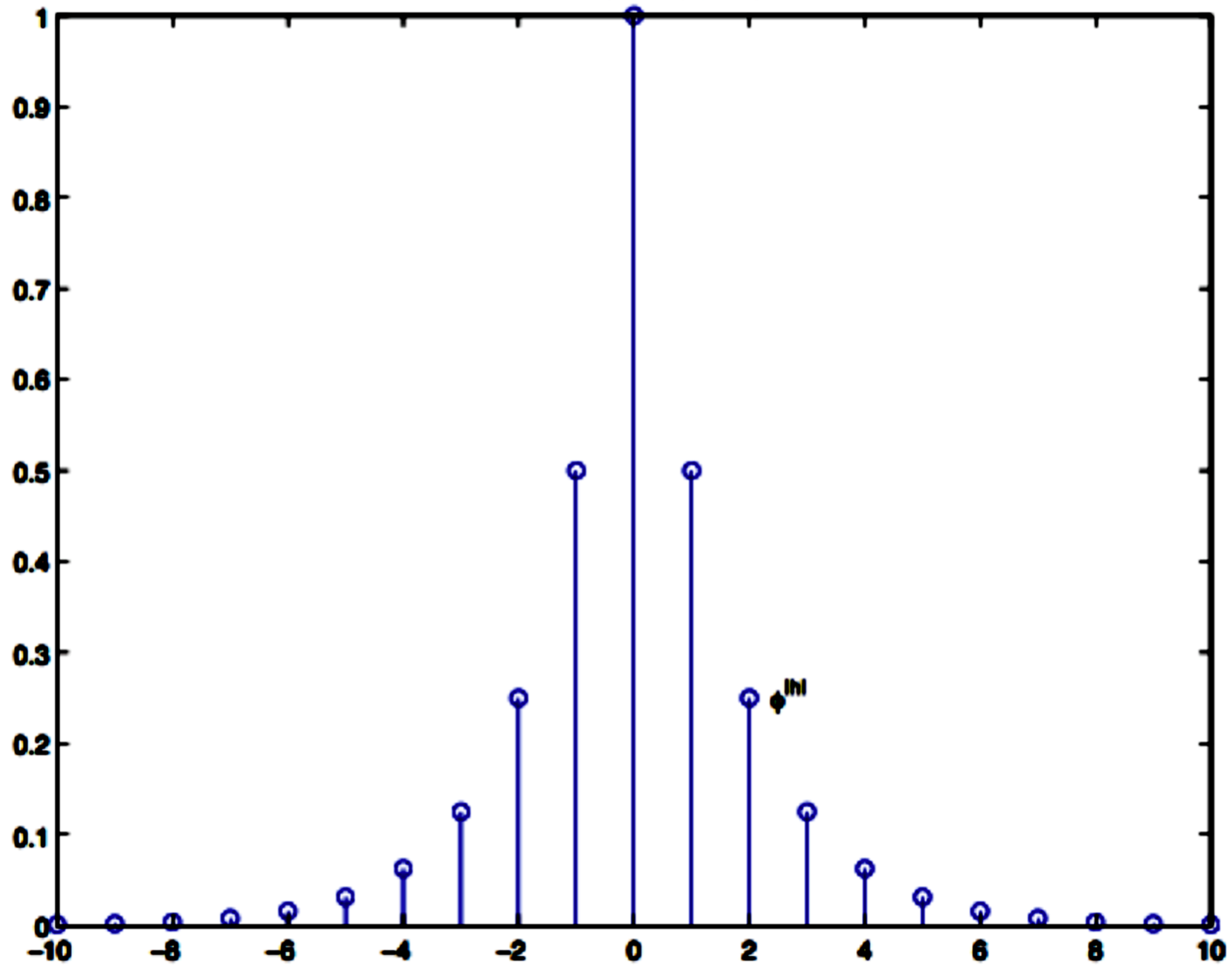
ACF: MA(1) vs AR

$$MA(1): X_t = Z_t + \theta Z_{t-1}$$



$$AR(1): X_t = \phi X_{t-1} + Z_t$$



- *Moving average*: finite range for the correlation function
- *Autoregression:* the correlation decreases, but it is never zero

Summary for sample ACF

We can recognize the sample autocorrelation functions of many non-white (even non-stationary) TS

- **Time series:**
  - White
  - Trend
  - Periodic
  - MA(q)
  - AR(p)

- **Sample ACF:**
  - $\Rightarrow$ Zero
  - $\Rightarrow$ Slow decay
  - $\Rightarrow$ Periodic
  - $\Rightarrow$ Zero for $|h| > q$
  - $\Rightarrow$ Decays to zero exponentially

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Autoregression moving average: ARMA

- An **ARMA(p,q)** process is a stationary process that satisfies

$$y_t - \varphi_1 y_{t-1} - \ldots - \varphi_p y_{t-p} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots \theta_q \varepsilon_{t-q},$$

  where $\{\varepsilon_t\}_{t \geq 1} \sim WN(0, \sigma^2)$.

- Given n observations, in case of **AR(p) process** the parameters can be estimated by least-squares

$$\widehat{\varphi} = \arg\min_{\varphi} \sum_{t=p+1}^{n} \left[y_t - \varphi_1 y_{t-1} - \ldots - \varphi_p y_{t-p}\right]^2$$

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Differencing to get ARIMA

- Let us denote $\nabla^{(1)} y_t = \nabla y_t = y_t - y_{t-1}$,
  $\nabla^{(2)} y_t = \nabla^{(1)} y_t - \nabla^{(1)} y_{t-1}$,
  $\nabla^{(k)} y_t = \nabla^{(k-1)} y_t - \nabla^{(k-1)} y_{t-1}$,
- For **ARMA(p, q)** process for $\nabla^{(d)} y_t$ we get **Autoregression Integrated Moving Average ARIMA(p, d, q)** process
- If y_t has linear trend, $\nabla^{(1)} y_t$ has no trend
- If y_t has linear trend, $\nabla^{(2)} y_t$ has no trend

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***ARIMA** models the next step in the sequence as a linear function of the differenced observations and residual errors at prior time steps*

Zoo of models

- ***Seasonal Autoregressive Integrated Moving-Average (SARIMA)*** *models the next step in the sequence as a linear function of the differenced observations, errors, differenced seasonal observations, and seasonal errors at prior time steps*

- ***Vector Autoregression (VAR)*** *models the next step in each time series using an AR model. It is the generalization of AR to multiple parallel time series, e.g. multivariate time series*

- ***Vector Autoregression Moving-Average with Exogenous Regressors (VARMAX)*** *— an extension of the VARMA model that also includes the modelling of exogenous variables. It is a multivariate version of the ARMAX method.*

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GARCH

- ***GARCH -*** Generalized autoregressive conditional heteroskedasticity:

$$
\begin{aligned}
y_t &= \varphi_1 y_{t-1} + \ldots + \varphi_p y_{t-p} + \varepsilon_t \\
\varepsilon_t | H_{t-1} &\sim \mathcal{N}(0, \sigma_t^2) \\
\sigma_t^2 &= \omega + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_q \varepsilon_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2 = \\
&= \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2,
\end{aligned}
$$

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