Features of time series analysis

(autocorrelation, stationarity, cointegration, etc.)

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How to predict the position of a ball at time (T + 1)?

Sequential data is essential for some prediction problems

Simplest families of finite-dimensional distributions

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A family of finite-dimensional distributions

- A time series model specifies the joint distribution of a sequence of random variables $\{y_t\}_t = 1,2,...$

$$\mathbb{P}[y_{t_1} \leq z_1, \ldots, y_{t_k} \leq z_k] \text{ for any } k \text{ and } z_1, \ldots, z_k$$

- We mostly restrict our attention to **second-order** properties only:

$$\mathbb{E}y_t,\ \mathbb{E}(y_{t_1}, y_{t_2})$$

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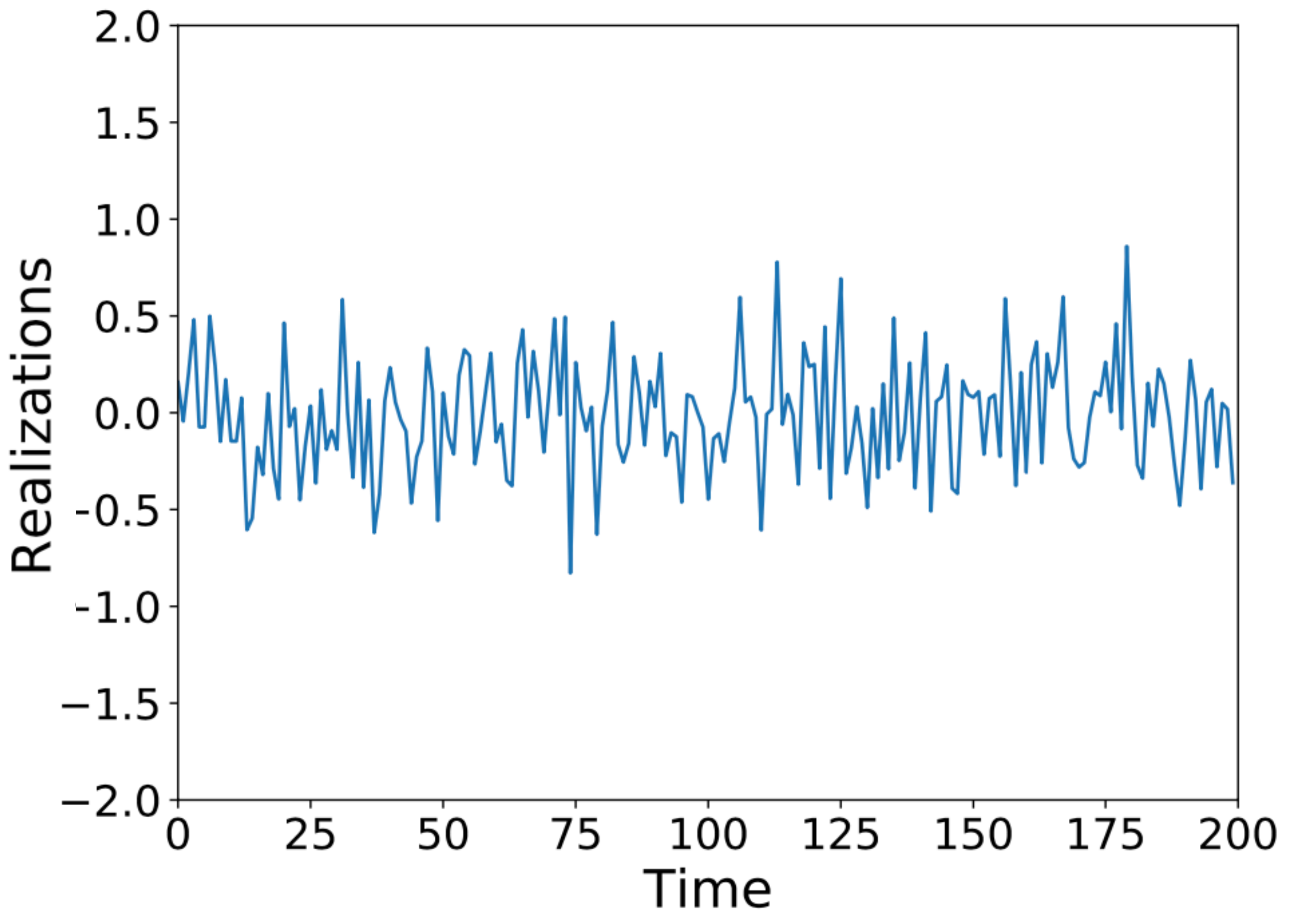
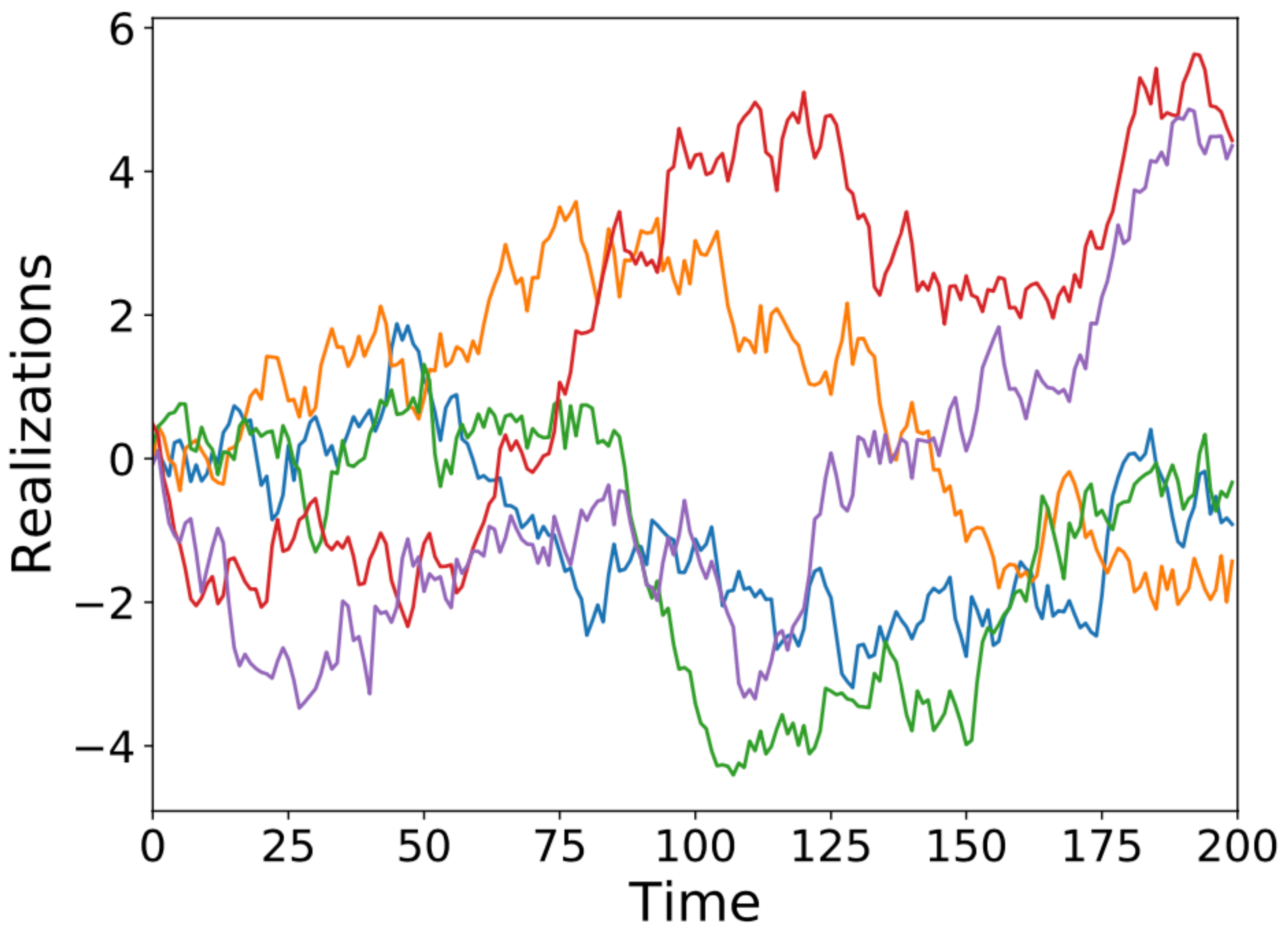
Simplest families of finite-dimensional distributions

- We consider *discrete time* for our random processes

White-noise
example



Random walk
example



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(Gaussian) White noise model

- $\{y_t\}_{t\geq 1}$ are i.i.d. with zero mean and no correlations
- In this case:

$$\mathbb{P}[y_{t_1} \leq z_1, \ldots, y_{t_k} \leq z_k] = \prod_{s=1}^{k} \mathbb{P}[y_{t_s} \leq z_s]$$

- Not interesting for forecasting:

$$\mathbb{P}[y_t \leq z_t \,|\, y_1 \leq z_1, \ldots, y_{t-1} \leq z_{t-1}] = \mathbb{P}[y_t \leq z_t]$$

- Example: **Gaussian** White Noise
  $y_t \sim WN(0,\sigma^2)$, $\mathbb{E}y_t = 0$, $Var(y_t) = \sigma^2$

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Gaussian random walk

- $y_t = \sum_{i=1}^{t} \varepsilon_i,\ \varepsilon_i \sim N(0,1)$

Examples:

- Stock prices on successive days
- The path traced by a molecule as it travels in a liquid or a gas
- The search path of a foraging animal

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Time Series and Stationarity

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(Strict) Stationarity

All statistical characteristics of that series are unchanged by shifts in time

A time series is stationary

- *Strict stationarity*: joint distribution of $(y_t, \ldots, y_{t-h})$ depends only on the lag h, and not on the time period t

- We can consider second-order properties only

Mean and Autocovariance

- Suppose that $\{y_t\}_{t\geq 1}$ is a TS with $\mathbb{E}[y_t^2] < \infty$
- Its **mean function** is

$$\mu_t = \mathbb{E}[y_t]$$

- Its autocovariance function is

$$\gamma_y(s,t) = Cov(y_s, y_t) = \mathbb{E}[(y_s - \mu_s)(y_t - \mu_t)]$$

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Weak (covariance) stationarity

- We say that $\{y_t\}_{t \geq 1}$ is (weakly) stationary if

  1. μ_t is independent of t, and
  2. For each h, $\gamma_y(t+h, t)$ is independent of t

- In that case, we write

$$\gamma_y(h) = \gamma_y(h,0)$$

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Stationarity: Example

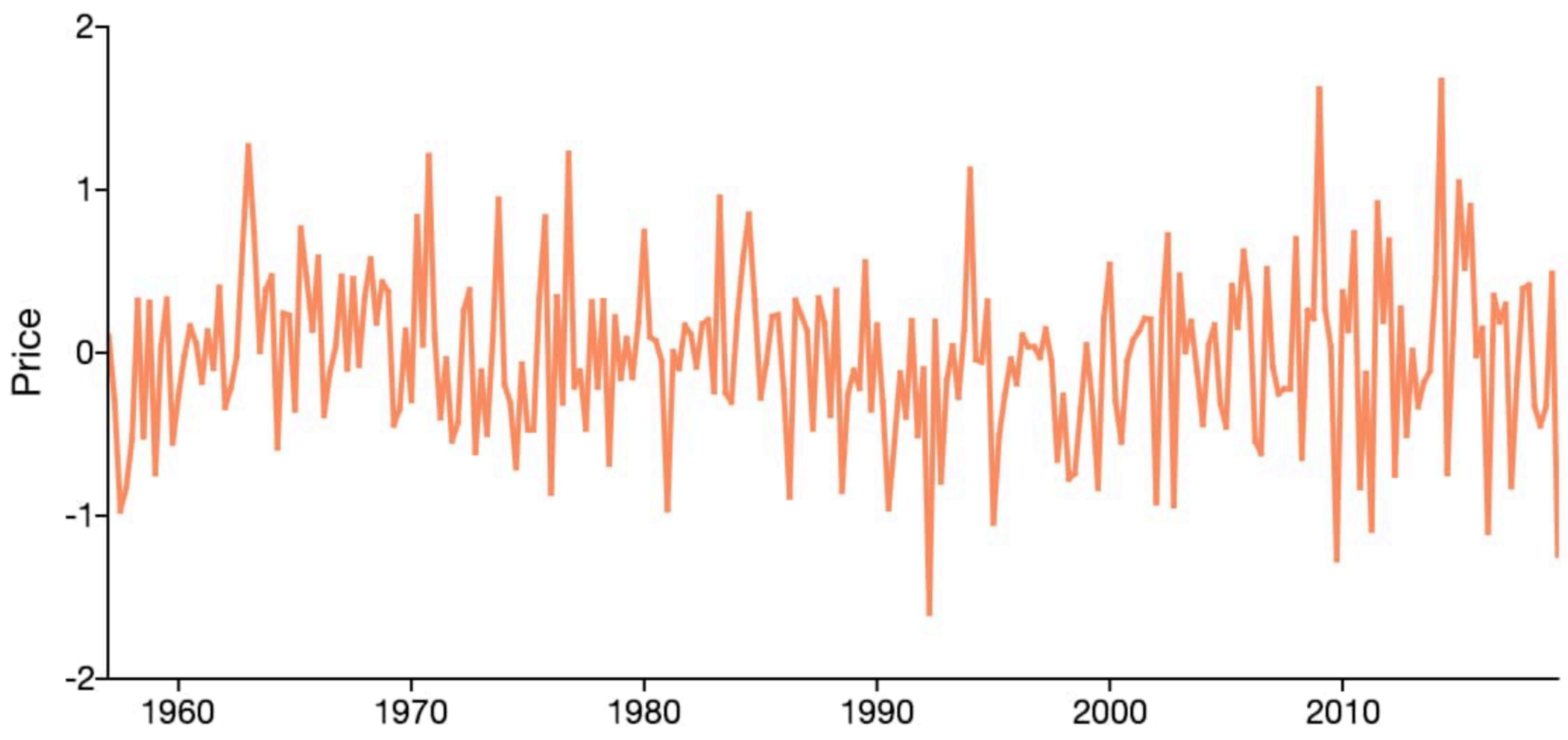
- Gaussian white noise, $\mathbb{E}[y_t] = 0$, $\mathbb{E}[y_t^2] = \sigma^2$. We get that

$$\gamma_t(t+h, t) = \begin{cases} \sigma^2, \ if \ h = 0 \\ 0, \ otherwise \end{cases}$$

- Thus,

1. $\mu_t = 0$ is independent of t, and
2. $\gamma_y(t+h, t) = \gamma_y(h, 0)$ for all t

So, $\{y_t\}_{t \geq 1}$is stationary



Source: https://cdn.aptech.com

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Nonstationarity: Example

- Random walk, $y_t = \sum_{i=1}^{t} \varepsilon_i$ for i.i.d., mean zero $\{\varepsilon_t\}_{t\geq 1}$. We get that

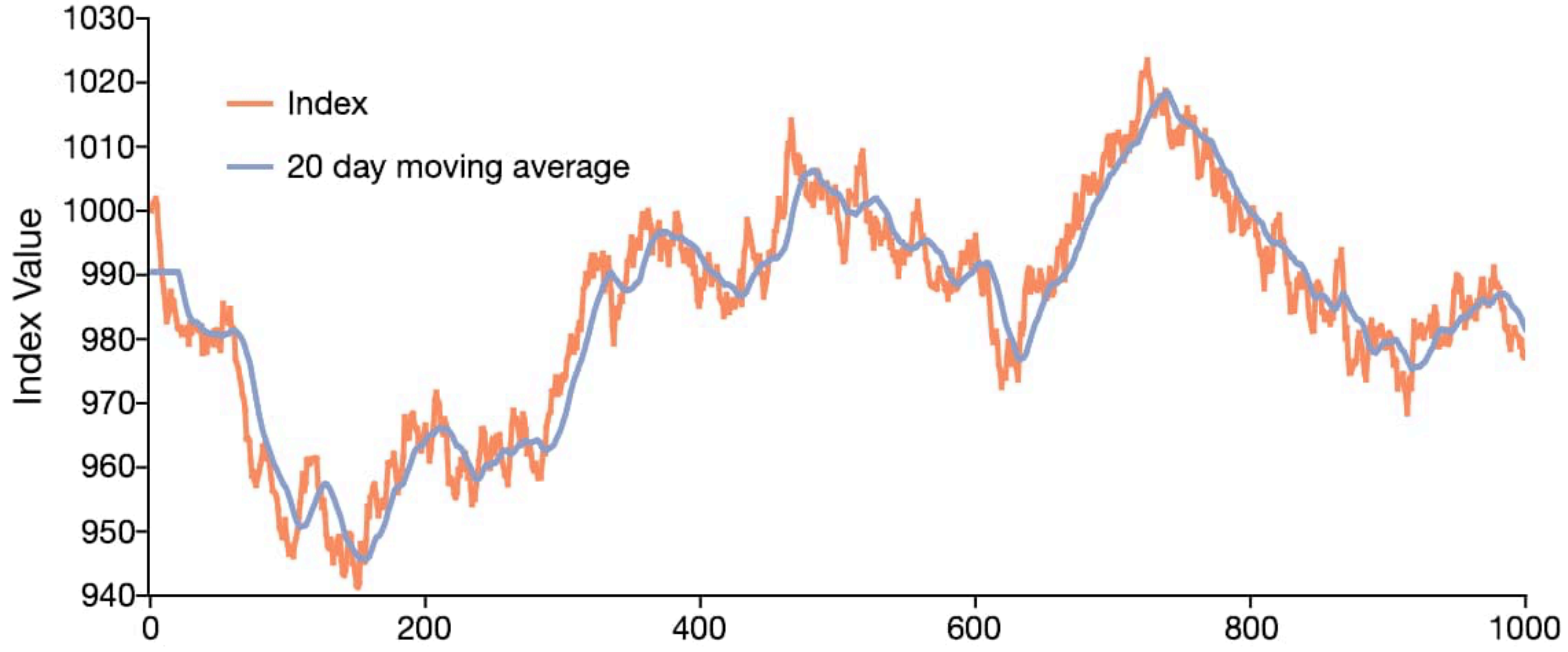
$$\mathbb{E}[y_t] = 0,\ \mathbb{E}[y_t^2] = t\sigma^2, \text{ and}$$

$$\gamma_y(t+h,t) = Cov(y_{t+h}, y_t) = Cov(y_t + \sum_{s=1}^{h} \varepsilon_{t+s}, y_t) = Cov(y_t, y_t) = t\sigma^2$$

- Thus,

1. $\mu_t = 0$ is independent of t
2. $\gamma_y(t+h,t)$ is not

So, $\{y_t\}_{t\geq 1}$ is not stationary



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Trend and Seasonality

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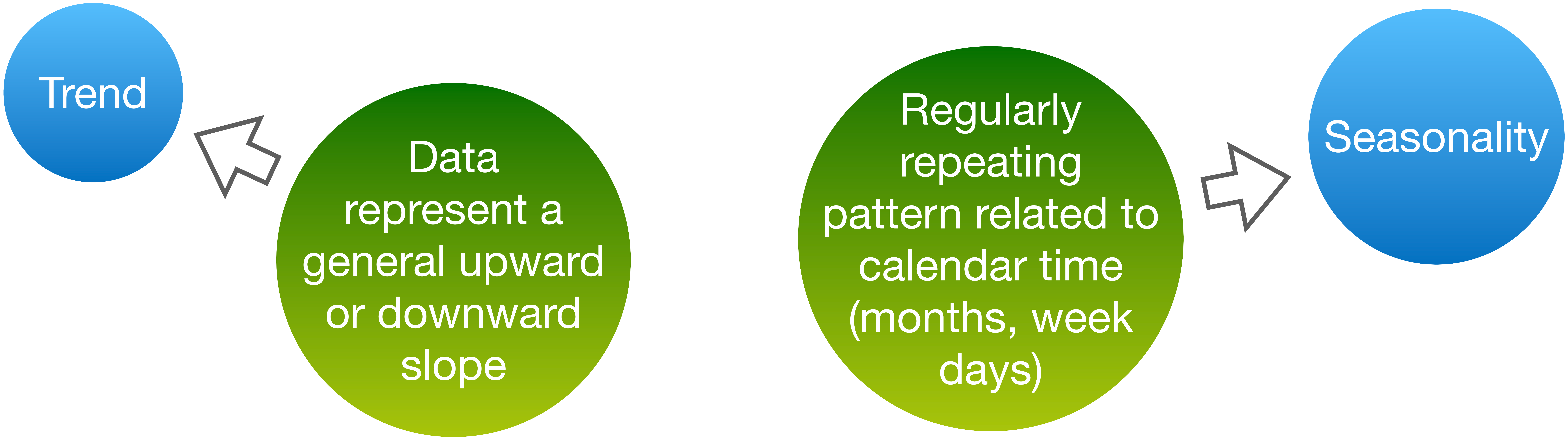
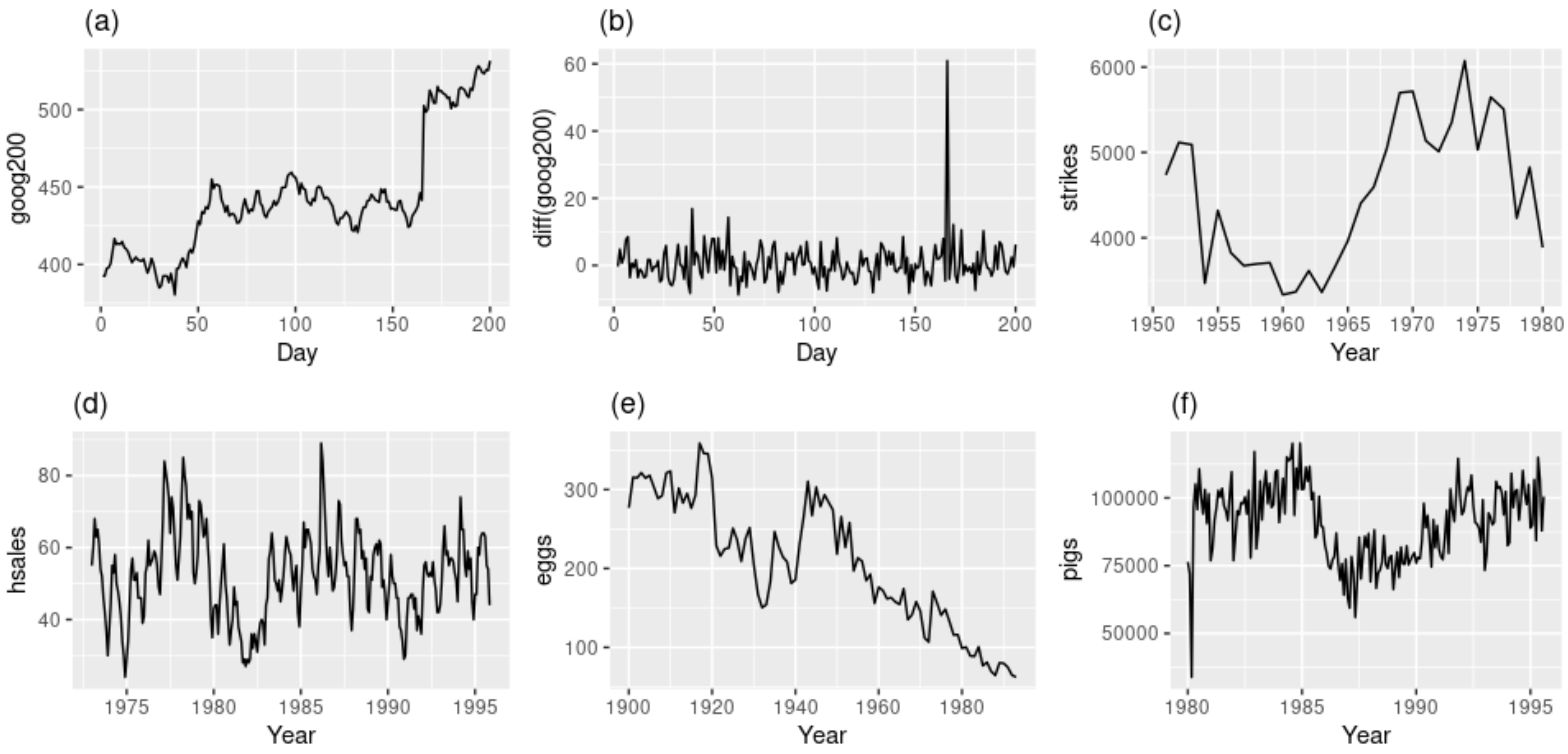
Trend and Seasonality

- A stationary TS is **one whose properties do not depend on the time at which the series is observed**.
- Thus, TS with trends, or with seasonality, are not stationary — the *trend* and *seasonality* will affect the value of the TS at different times.



Which of these series are stationary?



Source: https://otexts.com/fpp2/stationarity.html

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Time Series and Cointegration

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Dealing with nonstationarity

- ***Cointegration*** forms a synthetic stationary series from a linear combination of two or more non-stationary TS

- Let's consider a group of time series, Y_t, which is composed of 3 separate nonstationary TS:

$$y_1 = (y_{11}, y_{12}, \ldots, y_{1t})$$
$$y_2 = (y_{21}, y_{22}, \ldots, y_{2t})$$
$$y_3 = (y_{31}, y_{32}, \ldots, y_{3t})$$

- Y_t can be combined in a way that the linear combination $\beta Y_t = \beta_1 y_{1t} + \beta_2 y_{2t} + \beta_3 y_{3t}$ is stationary

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Time Series and Autocorrelation

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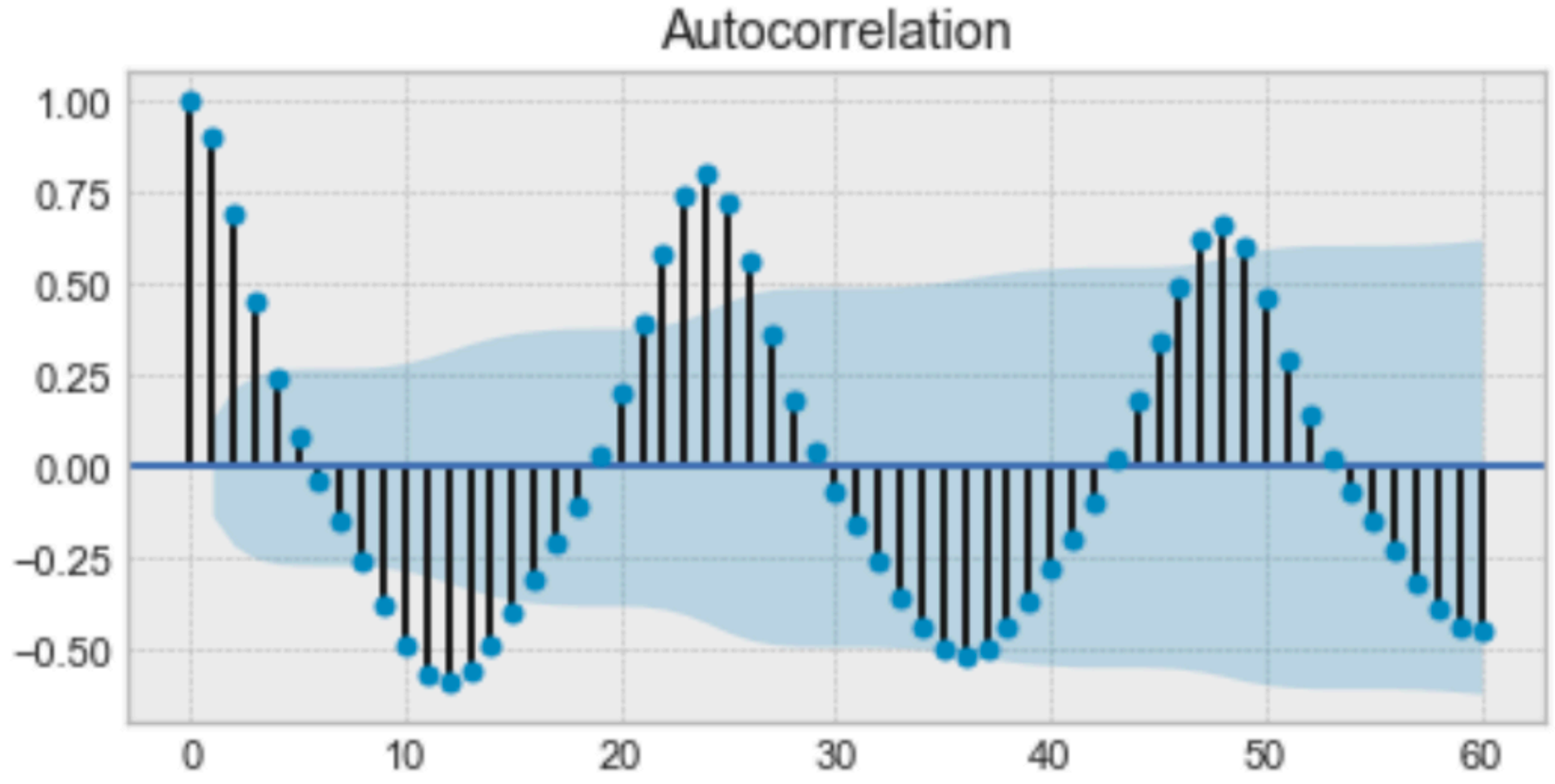
What is Autocorrelation?

Correlation between observations of the same dataset at different points in time

Autocorrelation

$$r_k = \frac{\sum_{t=h+1}^{T} (y_t - \mu_t)(y_{t-h} - \mu_t)}{\sum_{t=1}^{T} (y_t - \mu_t)^2}$$



Source: towardsdatascience.com

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statsmodels

Time Series decomposition

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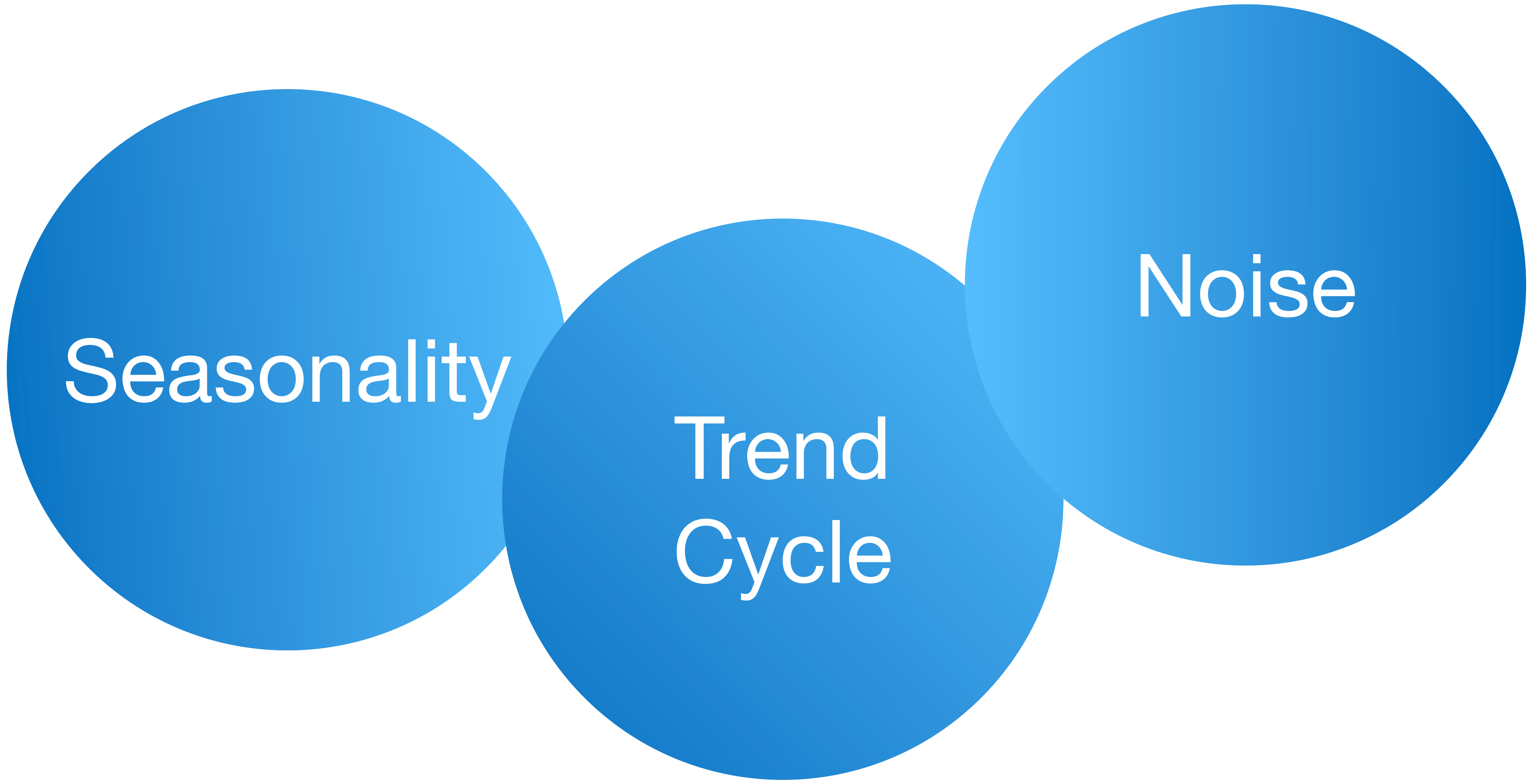
Time Series components

3 core components of TS distribution



what are the outliers or missing values in the data?

- ***Decomposition*** — deconstructing a TS into these components

$y_t = S_t + T_t + R_t,$ where

S_t - seasonal component,

T_t - trend-cycle component,

R_t - remainder component

Moving averages

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Moving average smoothing

- Eliminates some of the data randomness, leaving a smooth trend-cycle component

$$\hat{T}_t = \frac{1}{m} \sum_{h=-k}^{k} y_{t+h}, \text{ where}$$

$$m = 2k + 1$$

Trend Cycle

Source: https://otexts.com/fpp2/stationarity.html

Classical decomposition

Additive decomposition

$$\boxed{y_t = S_t + T_t + R_t}$$

- ***Step I***: compute the trend-cycle component $\hat{T}_t$
- ***Step II***: calculate the detrended series $y_t - \hat{T}_t$
- ***Step III***: to estimate the seasonal component for each season, simply average the detrended values for that season
- ***Step IV***: $\hat{R}_t = y_t - \hat{T}_t - \hat{S}_t$

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Multiplicative decomposition

$$\boxed{y_t = S_t \times T_t \times R_t}$$

- ***Step I***: compute the trend-cycle component $\hat{T}_t$
- ***Step II***: calculate the detrended series $\dfrac{y_t}{\hat{T}_t}$
- ***Step III***: to estimate the seasonal component for each season, simply average the detrended values for that season
- ***Step IV***: $\hat{R}_t = \dfrac{y_t}{\hat{T}_t \times \hat{S}_t}$

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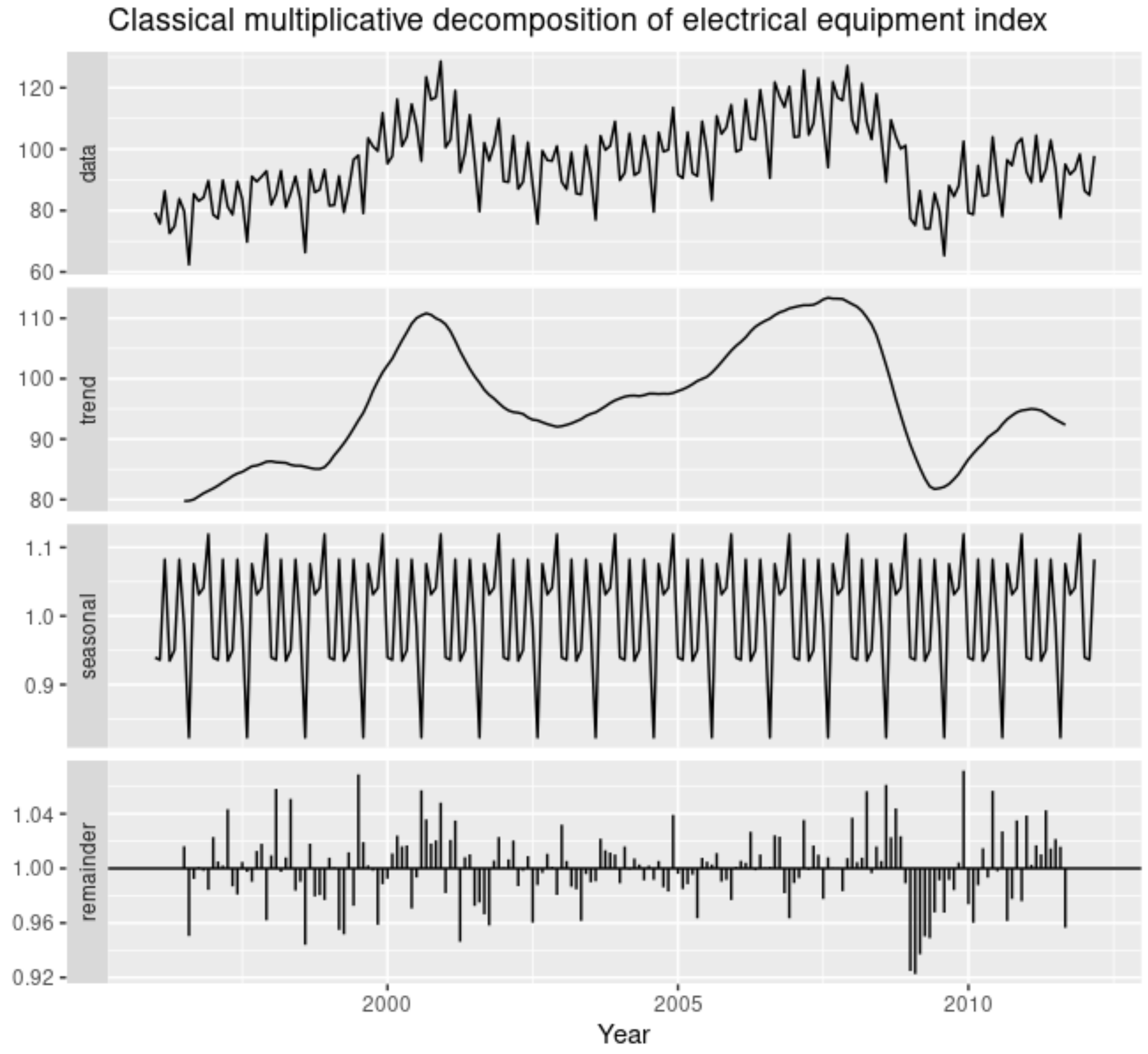
Drawbacks of classical decomposition



Source: https://otexts.com/fpp2/stationarity.html

- Estimation of $\hat{T}_t$ is unavailable for the first few and last few observations
- $\hat{T}_t$ tends to over-smooth rapid rises and falls in the data
- The classical decomposition methods are unable to capture these seasonal changes over time

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Alternatives to the Classical Approach

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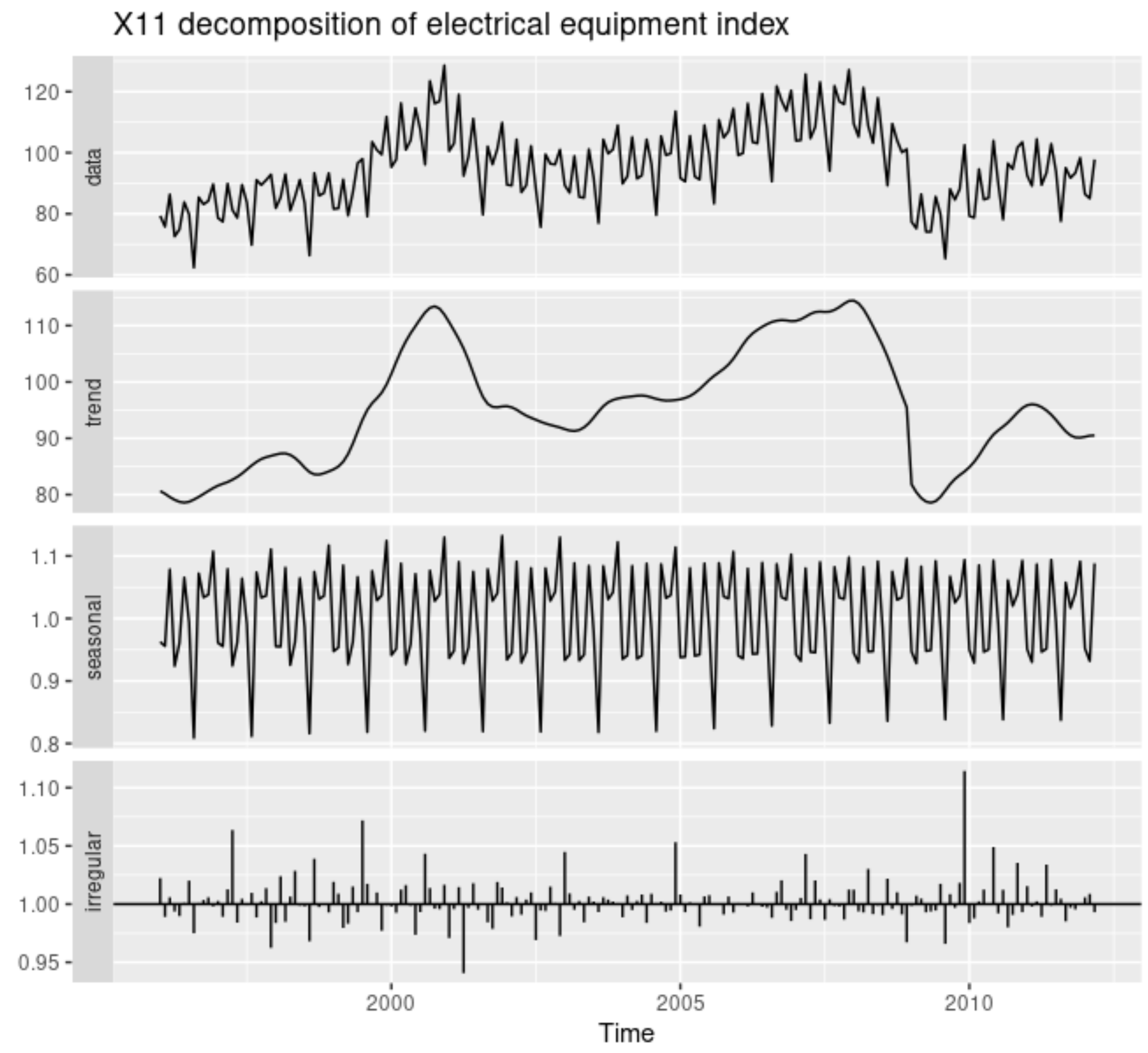
X11 decomposition



Source: https://otexts.com/fpp2/stationarity.html

- Based on classical decomposition
- Method for decomposing quarterly and monthly data
- Trend-cycle estimates are available for all observations including the end points
- The seasonal component is allowed to vary slowly over time

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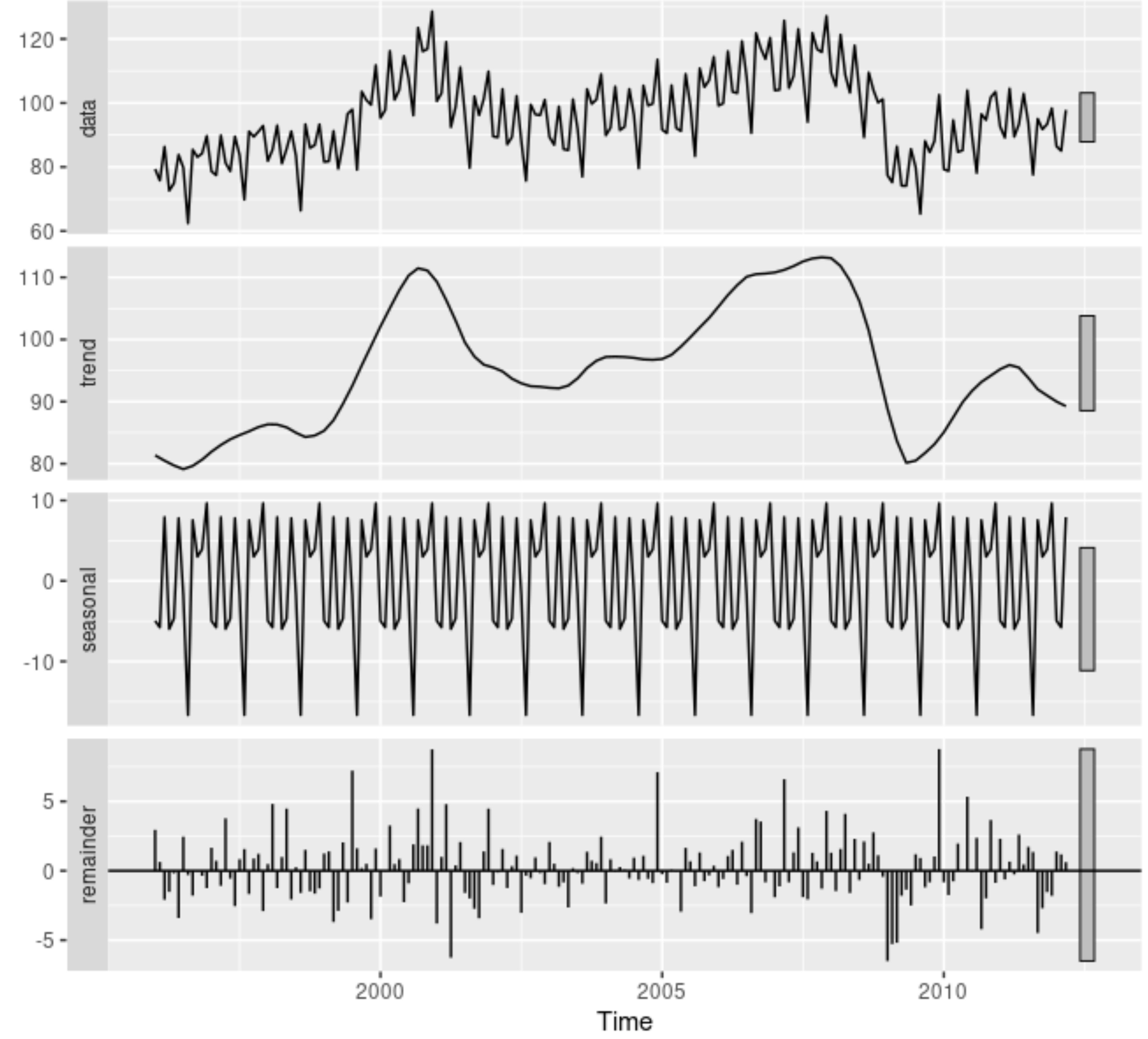
Seasonal and Trend decomposition using Loess



Source: https://otexts.com/fpp2/stationarity.html

- *Loess* - a method for estimating nonlinear relationships
- STL handle any type of seasonality, not only monthly and quarterly data
- The seasonal component is allowed to vary over time
- The smoothness of the trend-cycle can also be controlled by the user

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Forecasting with decomposition

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$y_t = \hat{S}_t + \hat{A}_t$, where

$\hat{A}_t = \hat{T}_t + \hat{R}_t$

⇨ *forecast the seasonal component $\hat{S}_t$ and the seasonally adjusted component $\hat{A}_t$ separately*

- It is usually assumed that $\hat{S}_t$ is unchanging, or changing extremely slowly, so it is forecast by simply taking the last year of the estimated component
- To forecast the seasonally adjusted component, any non-seasonal forecasting method may be used

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Conclusion

- Sequential data processing is an important topic
- In practice, most of TS are nonstationary
- Decomposition and autocorrelation analysis techniques help to find relationship between multiple TS

Next lecture: Modelling TS

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