Over all real numbers, find the minimum value of a positive real number y, such that:

$$y = \sqrt{(x+6)^2+25} + \sqrt{(x-6)^2+121} \tag{1}$$

Solution

METHOD 1, Defining the complex numbers

$$Z_1 = 5 + (x+6)i = |Z_1| = \sqrt{5^2 + (x-6)^2} \tag{2}$$

$$Z_2 = 11 + (x-6)i = |Z_2| = \sqrt{11^2 + (x-6)^2} \tag{3}$$

Applying the triangle inequality law

$$|Z_1| + |Z_2| \geqslant |Z_1 + Z_2| = |5 + (x+6)i + 11 - (x-6)i| \tag{4}$$

$$|Z_1| + |Z_2| \geqslant |5 + 11 + 6i + 6i + xi - xi| \tag{5}$$

$$|Z_1| + |Z_2| \geqslant |16 + 12i| \tag{6}$$

$$|Z_1| + |Z_2| \geqslant \sqrt{16^2 + 12^2} \tag{7}$$

$$|Z_1| + |Z_2| \geqslant \sqrt{256 + 144} \tag{8}$$

$$|Z_1| + |Z_2| \geqslant \sqrt{400} \tag{9}$$

$$|Z_1| + |Z_2| \geqslant 20 \tag{10}$$

Therefore, the minimum value is 20.

METHOD 2, Minkowski Inequality

$$\sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} \geqslant \sqrt{(x_1+x_2)^2 + (y_1+y_2)^2} \tag{11}$$

$$\sqrt{(x+6)^2 + 5^2} + \sqrt{(x-6)^2 + 11^2} = \sqrt{(x+6)^2 + 5^2} + \sqrt{(6-x)^2 + 11^2} \tag{12}$$

$$\sqrt{(x+6+6-x)^2+(5+11)^2} = \sqrt{12^2+16^2} \tag{13}$$

$$\sqrt{12^2+16^2} = \sqrt{144+256} = 20 \tag{14}$$

Thus, the minimum value is 20