Completing correlation matrices

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1 Introduction

2 The completion criterium

Given an incomplete correlation matrix what is the right way to complete it? To motivate the answer we need to introduce some notation. Let

$$X = (x_1, \ldots, x_k)^T \in \mathbb{R}^k \tag{1}$$

be a k dimensional random variable with a density function given by a k dimensional normal distribution

$$\begin{aligned} f(X) &= N_k[\mu, H_X](X) && (2) \\ &= (2\pi)^{-k/2} \det(H_X)^{-1/2} \exp\left(-\frac{1}{2}(X-\mu)^T H_X^{-1}(X-\mu)\right), && (3) \end{aligned}$$

with mean μ and covariance matrix H. A crucial property of the normal distribution is that we condition on some of the x_i, $i = 1, \ldots, k$, we again obtain a normal distribution. Assume that we want to condition on the last l components of X:

$$Z = (x_{k-l+1}, \ldots, x_k)^T \in \mathbb{R}^l. \tag{4}$$

Let us partition the matrix H in a way that reflects this partition of X:

$$H_X = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \tag{5}$$

with $A \in M_{k-l}$, $C \in M_l$, and $B \in M_{k-l,l}$. Let $\bar{X}$ be the first $k-l$ components of X:

$$\bar{X} = (x_1, \ldots, x_{k-l})^T \in \mathbb{R}^{k-l} \tag{6}$$

Let us also assume that X has zero mean:

$$\mu = 0 \tag{7}$$

The density function for X conditioned on Z is then given by

$$f(X|Z) = N_{k-l}[BC^{-1}Z, H_X/C](\bar{X}), \tag{8}$$

(see [1, chapter 8] and [2]) where H_X/C is the Schur complement of C in H_X:

$$H_X/C = A - BC^{-1}B^T \tag{9}$$

(Emilie Haynsworth introduced the name and highlighted its usefulness in [3]. Schur originally made use of it in [7]. For an overview of the properties of the Schur complement see [5].) We see that we again obtain a normal distribution. We will see the Schur complement again soon.

Now let us look at a second set of random variables

$$Y = (x_{k-l+1}, \ldots, x_k, x_{k+1}, \ldots, x_n)^T \in \mathbb{R}^{n-k}, \tag{10}$$

that has an overlap Z with the random variables X. Let the density function for Y be given by

$$f(Y) = N_{n-k}[0, H_Y](Y), \tag{11}$$

with

$$H_Y = \begin{pmatrix} C & D \\ D^T & E \end{pmatrix}. \tag{12}$$

Note that the submatrix C is shared with H_X. In our context the matrix might have been obtained by a previous calibration of a model that is part of both X and Y. If we were to condition on Z we would find

$$f(Y|Z) = N_{n-k-l}[D^T C^{-1} Z, H_Y/C](\bar{Y}), \tag{13}$$

with

$$H_Y/C = E - D^T C^{-1} D \tag{14}$$

and $\bar{Y}$ is the part of Y that is not Z. If we want to describe X and Y together we are looking at the matrix

$$H = \begin{pmatrix} A & B & W \\ B^T & C & D \\ W^T & D^T & E \end{pmatrix}, \tag{15}$$

with a yet to be determined matrix $W \in M_{k-l,n-l}$. Because X and Y share the random variables in Z we can not make them independent. The next best thing that we can do is to demand that if we fix Z the remaining parts of X and Y are independent. The combined density for X and Y when we condition on Z is given by

$$f(X, Y|Z) = N_{n-l}[(B^T, D)^T C^{-1} Z, H/C](\bar{X}, \bar{Y}), \tag{16}$$

with

$$H/C = \begin{pmatrix} A - BC^{-1}B^T & W - BC^{-1}D \\ W^T - D^T C^{-1} B^T & E - D^T C^{-1} D \end{pmatrix} \tag{17}$$

$$= \begin{pmatrix} H_X/C & W - BC^{-1}D \\ W^T - D^T C^{-1} B^T & H_Y/C \end{pmatrix}. \tag{18}$$

We see that if we set

$$W = BC^{-1}D \tag{19}$$

the above matrix becomes block diagonal and the conditional densities obey

$$f(X,Y|Z) = f(X|Z)f(Y|Z) \tag{20}$$

because the blocks in the diagonal are H_X/C and H_Y/C from equations (9) and (14). Given that Z is part of both X and Y this is as much independence as we can ask for.

3 Conclusion

References

[1] C. Radharkishna Rao, Linear Statistical Inference and its Applications, 2nd edition, John Wiley & Sons, Inc, 2002.

[2] Richard W. Cottle, Manifestations of the Schur complement, Linear Algebra and its Applications **8**, 1974, 189–211.

[3] Emilie Virginia Haynsworth, Determination of the inertia of a partitioned Hermitian matrix, Linear Algebra and its Applications **1**, 1968, 73–81.

[4] J. Schur, Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind, Journal für die reine und angewandte Mathematik **147**, 205 – 232, 1917.

[5] Roger A. Horn, Basic Properties of the Schur Complement. In Fuzhen Zhang (Ed.) The Schur complement and its application (17–46). Springer, 2005.

[6] Roger A. Horn, Charles R. Johnson, *Matrix Analysis*, 2nd Edition, Cambridge University Press, 2013.

[7] J. Schur, Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind, Journal für die reine und angewandte Mathematik **147**, 205 – 232, 1917.

[8] T. Banachiewicz, Zur Berechnung der Determinanten, wie auch der Inversen, und zur darauf basierten Auflösung der Systeme linearer Gleichungen, Acta Astronom. Sér. C 3, 41–67, 1937.

[9] Frazer, R., Duncan, W., Collar, A. (1938). Elementary Matrices And Some Applications To Dynamics And Differential Equations, Cambridge University Press, 1938.

[10] Ouellette, D. (1981). Schur complements and statistics Linear Algebra and its Applications 36, 187-295. https://dx.doi.org/10.1016/0024-3795(81)90232-9

[11] Albert, A. (1969). Conditions for Positive and Nonnegative Definiteness in Terms of Pseudoinverses SIAM Journal on Applied Mathematics 17(2), 434-440. https://dx.doi.org/10.1137/0117041

[12] Smith, R. (2008). The positive definite completion problem revisited Linear Algebra and its Applications 429(7), 1442-1452. https://dx.doi.org/10.1016/j.laa.2008.04.020

[13] Penrose, R. (1955). A generalized inverse for matrices Mathematical Proceedings of the Cambridge Philosophical Society 51(3), 406-413. https://dx.doi.org/10.1017/s0305004100030401

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