Comment on Taleb's Bitcoin Paper

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1 Setup

We use the standard setup to describe a foreign exchange market. We thus have two interest rates and one exchange rate. Let

$$P_E(t,T) \tag{1}$$

be the discount factor for the base currency and let

$$P_B(t,T) \tag{2}$$

be the discount factor for the foreign currency. In our example the foreign currency will be bitcoin itself.

> **Remark 1** The introduction of a discount factor for bitcoins might seem unusual. This is standard praxis in the description of different assets classes including proper currencies, inflation, equities, and commodities. See [1], [2], and [3] for more details (in [1] this is called the 'The Foreign-Currency Analogy').

The exchange rate at time t between the base currency and bitcoins is denoted by

$$X(t). \tag{3}$$

The rate $X(t)$ give the value of one bitcoin B in the base currency E.

A complete description of the bitcoin market would now consists of models for the two discount factors P_E and P_B together with a model for the exchange rate X.

> **Example 1** The simplest model of this kind would be a model where the interest rates are non-stochastic and the exchange rate is modeled by a simple Black-Scholes model:
>
> $$\frac{dX}{X} = \mu dt + \sigma dW \tag{4}$$
>
> The price of a bitcoin option would then be given by a Garman-Kohlhagen type formula:
>
> $$V(t) = P_E(t,T)\Big(F(t,T)N_{(0,1)}[d_1] - KN_{(0,1)}[d_2]\Big), \tag{5}$$
>
> with
>
> $$d_{1/2} = \frac{\ln\left(\frac{F(t,T)}{K}\right) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \tag{6}$$

where $F(t,T)$ is the forward:

$$F(t,T) = X(t)\frac{P_B(t,T)}{P_E(t,T)} \tag{7}$$

We will see more of the forward in a bit.

We will not specify a particular model here but rely on arguments that are model-independent.

2 The forward

Let us calculate the value of the payment of one bitcoin at time T. This value is given by

$$V(t) = E\left[\frac{X(T)}{N(T)}\right], \tag{8}$$

where $N(T)$ is the standard money market account for the base currency E. We now perform two changes of numeraire. First we transform to the forward measure with the numeraire

$$M(t) = P(t,T). \tag{9}$$

This gives

$$V(t) = P_E(t,T)E^T[X(T)], \tag{10}$$

where $E^T[\cdot]$ is the expectation with respect to the forward measure. Transforming instead to the foreign forward measure using the numeraire

$$M(t) = X(t)P_B(t,T) \tag{11}$$

gives

$$\begin{aligned} V(t) &= X(t)P_B(t,T) && (12)\\ &= P_E(t,T)X(t,T), && (13) \end{aligned}$$

where we have introduced the forward exchange rate

$$X(t,T) = X(t)\frac{P_B(t,T)}{P_E(t,T)}. \tag{14}$$

Comparing equation (10) with equation (13) gives

$$X(t,T) = E^T[X(T)]. \tag{15}$$

The forward exchange rate is thus given by the expectation value of $X(T)$ in the forward measure.

3 Taleb's argument

Using the language of the previous sections we can now reformulate Taleb's argument as follows:

Usually $X(t)$ is used to calculate the forward $X(t,T)$. Taleb argues in the other direction. He uses the knowledge of the forward $X(t,T)$ to calculate $X(t)$. It follows from equations (14) and (15) that

$$X(t) = \frac{P_E(t,T)}{P_B(t,T)} X(t,T) \tag{16}$$

$$= \frac{P_E(t,T)}{P_B(t,T)} E^T[X(T)]. \tag{17}$$

Taleb argues that for T large enough all paths will have seen the exchange rate $X(T)$ dip below the absorbing boundary (see appendix A) and thus vanish, i.e.

$$\lim_{T\to\infty} E^T[X(T)] = 0. \tag{18}$$

It then follows from equation (17) that $X(t)$ (i.e. the value of bitcoin today) also vanishes.

A The absorbing boundary

A.1 The reflection principle

Let $W(t)$ be a Brownian motion. Let us denote by τ_A the first time a path of the Brownian motion had the value A. Let us introduce the set of paths $W(t)$ for which $\tau_A \leq T$ but which have a value $W(T)$ that is larger than $w > A$:

$$C_A(T,w) = \{W(t)|\tau_A \leq T \text{ and } W(T) > w\} \tag{19}$$

An example of an element of $C_A(T,w)$ is the red path shown in figure 1. Let us change this path by mirroring the part of the path after τ_A on the line given by A. The new path thus goes up when the original path goes down and down when the original path goes up. In figure 1 this is the blue path. This paths now ends up at a value that is smaller than

$$A - (w - A) = 2A - w. \tag{20}$$

Let us thus introduce the set of paths that are smaller than a given value at time T:

$$D(T,v) = \{W(t)|W(T) < v\} \tag{21}$$

Following the above argument we have shown that there is a one-to-one correspondence between $C_A(T,w)$ and $D(T,2A-w)$. For the standard Brownian motion that we are looking at it follows that the probabilities of these sets are equal:

$$P(C_A(T,w)) = P(D(T,2A-w)) \tag{22}$$

This is called the *reflection principle*.

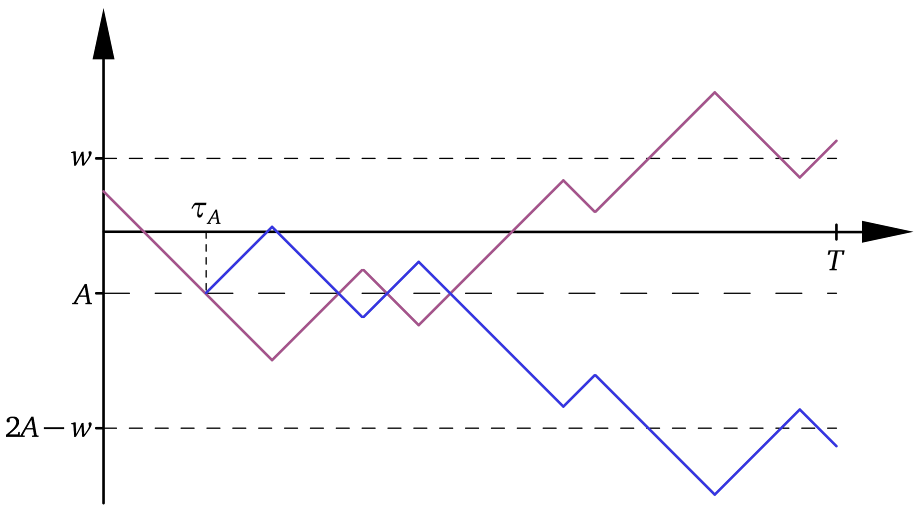

Figure 1: For every path that assumes the value A at any time before T (i.e. a path for which $\tau_A \leq T$) but that has a value $W(T) > w$ there is a path for which $W(T) < 2A-w$. In this figure the red path crosses A at time τ_A but has a value larger than w at time T. The blue path is obtained by reflecting the red path around the dashed line at A. It ends lower than $2A-w$.

A.2 The absorbing boundary

We will use the reflection principle to deal with the absorbing boundary. The Brownian motion that we were looking at in the last section is the exchange rate $X(t)$. We want to know how many paths of X have been below a certain value A at time T. A complication arises because we assume that the boundary is absorbing. If we just counted the paths that are below A at time T we would miss out on those paths that dipped below A at a time $\tau < T$. The set of paths that should be counted as having zero exchange rate at time T thus splits into two sets. There is the set of paths with $X(T) < A$ and then there is the set of paths $X(\tau) < A$ for some $\tau < T$ but with $X(T) > A$.

We can use the notation of the last section to denote these sets. The set of paths for which $X(T) > A$ is

$$D(T,A). \tag{23}$$

The other set of paths is given by

$$C_A(T,A). \tag{24}$$

Let us make the connection to the last section explicit:

$$A \simeq A \tag{25}$$

$$w \simeq A \tag{26}$$

$$\tau_A \simeq \tau \tag{27}$$

The reflection principle then says that these two sets have the same probability:

$$P(C_A(T,A)) = P(D(T,A)) \tag{28}$$

We need to count *both* of these sets. The probability that the exchange rate vanishes is thus

$$p_0 = 2P(D(T,A)). \tag{29}$$

When the width σ of the Brownian motion goes to infinity we have

$$\lim_{T\to\infty} P(D(T,A)) = \frac{1}{2}, \tag{30}$$

for any value of A. We thus have

$$\lim_{T\to\infty} p_0 = 1. \tag{31}$$

References

[1] Damiano Brigo, *Fabio Mercurio, Interest Rate Models – Theory and Practice, With Smile, Inflation and Credit*, 2nd Edition, Springer, 2006.

[2] Iain J. Clark, *Foreign exchange option pricing: a practitioner's guide*, Wiley, 2011.

[3] Iain J. Clark, *Commodity Option Pricing: a practitioner's guide*, Wiley, 2014.