Covid, humidity, and a curious coincidence of length scales

by Olaf Dreyer[1]

1 Introduction

With the arrival of fall, the number of COVID cases in Germany has increased sharply (see image 1). This note discusses a possible reason for this increase. With the colder temperatures of fall comes an increased use of indoor heating and with it a decrease in humidity. We suggest that it is this change in humidity that is responsible for the increase in COVID cases.

The deeper reason for why this is even a possibility is a curious coincidence of two length scales. One scale is the size r_M of a droplet that just makes it to the ground before evaporating after having been released from a height of roughly two meters. The other scale is the likely size r_d of a droplet that is produced by a person while talking. These two scales happen to be roughly equal to 40 μm. We stress again that this is a pure coincidence.

Because these scales are roughly equal it is possible that most liquid droplets evaporate while falling to the ground when the air is dry while very few droplets evaporate when the air is humid. This is important for the spread of the virus because droplets that do not hit the ground become airborne carriers of the virus. Once airborne, the virus is no longer kept in check by measures like social distancing. An efficient way to get rid of the virus is to let the droplets that carry the virus fall to the ground where they can no longer do any damage. What this paper argues is that with the change of humidity due to the use of indoor heating more droplets do not make it to the ground and instead become airborne.

Before we make a detailed calculation in section 2 we give a rough sketch of the argument. We start by describing the evaporation of small droplet of liquid. The basic insight goes back to Maxwell (see [2] and the discussion in [3, chapter 1]). He found that the change of mass of a droplet is proportional to its radius:

$$\frac{dm}{dt} = -C_e r, \tag{1}$$

for some constant C_e. Using

$$m = \frac{4}{3}\pi\rho_d r^3, \tag{2}$$

[1]Oppenheimer Landstr. 19, 60594 Frankfurt, Email: olaf.dreyer@gmail.com

where ρ_d is the density of the droplet, we find the following equation describing the evolution of the size of the droplet:

$$r^2(t) = r_0^2 - \frac{C_e}{2\pi\rho_d} t \tag{3}$$

We see that after the time

$$t_e = \frac{2\pi\rho_d}{C_e} r_0^2 \tag{4}$$

the droplet will have evaporated completely.

We now look at how the droplet falls. In this simplified first look we just include gravity and the drag due to air resistance as described by Stoke's law. This gives

$$m\frac{dv}{dt} = mg - 6\pi\eta r(t)v. \tag{5}$$

Here η is the viscosity of air. Looking at equation (5) we see that there is a limit velocity v_s where the acceleration vanishes:

$$v_s = \frac{mg}{6\pi\eta r_0} \tag{6}$$

As the droplet falls it will evaporate and thus change its radius. We now assume that it always has the velocity v_s corresponding to the current radius $r(t)$ of the droplet. With this assumption we can find the position $x(t)$ of the droplet:

$$x(t) = v_0 t \left(1 - \frac{t}{2t_e}\right) \tag{7}$$

Here v_0 denotes the initial velocity v_s of the droplet

Let us now look at droplets that fall from a given height H. The fate of the droplet depends on its initial radius r_0. If the radius is too small then the droplet will evaporate before it reaches the ground. If, on the other hand, the droplet is large it will reach the ground intact in a time $t_H < t_e$ given by

$$H = x(t_H). \tag{8}$$

Solving for t_H gives [2]

$$t_H = t_e \left(1 - \sqrt{1 - \frac{2H}{v_0 t_e}}\right). \tag{9}$$

It was the idea of Wells [4] to combine these two regimes in a single figure by plotting the time t_e to evaporation for the droplets that evaporate together with the time t_H

[2] The fact that $v_0 t_e$ spells 'vote' is a coincidence. But still, vote.

for the droplets that do make it to the ground. Both of these times are plotted as a function of the initial radius r_0 of the droplet. For our simplified model discussed here in the introduction we obtain figure 2.

There will be one radius r_M for which the droplet completely evaporates just as it hits the ground. For this radius we have

$$t_H(r_M) = t_e(r_M) \tag{10}$$

In figure 2 the radius r_M shows up as the position of the kink in the Wells curve. As we change the constant of evaporation C_e the radius r_M changes. It is this change of r_M that is central to the argument in this note. From figure 2 we see that as we increase the constant of evaporation C_e (this would correspond to a *lowering* of the humidity) we *increase* the radius r_M. A faster evaporation requires a larger droplet if the ground is still to be reached.

These considerations are important because the typical size of a droplet exhaled by a human is roughly the same size as r_M. The proportion of droplets that fall to the ground thus depends delicately on the humidity. For a high humidity most droplets will fall to the ground while for a low humidity many droplets will evaporate instead. It is these evaporating droplets that can give rise to airborne viruses.

With the onset of fall many Germans turned on their indoor heating. This happened in the beginning of October. The indoor heating reduced the humidity in indoor spaces and the reduced humidity in turn increased r_M. Many more droplets that would have fallen to the ground, instead ended up producing airborne viruses. We argue that it is this mechanism that lead to the increase we see in the infection data.

In the next section we briefly review the calculation and then discuss the results.

2 The calculation

To derive the behavior of falling droplets we follow [5]. The results here largely coincide with the results obtained in [6] and [7]. We need two equations to describe the fall of a droplet. One equation describes the evaporation process while the other describes the fall itself. These two equations are:

$$\frac{dr}{dt} = -\frac{B}{r}\left(1 + 0.276\,\mathrm{Re}^{1/2}(v, r)\mathrm{Sc}^{1/3}\right) \tag{11}$$

$$\frac{dv}{dt} = mg\left(1 - \frac{\rho_\mathrm{g}}{\rho_\mathrm{d}}\right) - 6\pi\eta r v \tag{12}$$

where

$$B = -\frac{M_v D_\infty C p}{\rho_\mathrm{d} R T_\infty} \ln\left(\frac{1 - p_{v\mathrm{a}}/p}{1 - p_{v\infty}/p}\right) \tag{13}$$

We have used the notation from [5]. In particular, we used the appendix C in [5] that contains the numerical values of the constants used.

We have solved the differential equations described in the previous section 2 numerically using a fourth order Runge-Kutta integration scheme. The results are the Wells curves shown in figure 3.

3 Results

The most important feature of the curves in figure 3 is the behavior of the radius r_M at which the droplet just makes it to the ground as a function of the humidity ϕ. When we lower the humidity the value of the radius r_M increases. A droplet that makes it to the ground at one humidity might completely evaporate when the humidity is lowered. This can be seen more clearly in figure 4 where we show how the radius r_M varies with ϕ.

While talking, humans exhale droplets with a certain distribution of radii. Figure 5 shows the cumulative distribution function (CDF) for the radii of the exhaled droplets (from [9]), where

$$\mathrm{CDF}(r) = \int_{-\infty}^{r} dx\ \rho_{\mathrm{droplet}}(x) \tag{14}$$

and $\rho_{\mathrm{droplet}}(x)$ is the probability distribution function for the droplet radii. $\mathrm{CDF}(x)$ thus gives the probability of a human exhaling a droplet with radius x or smaller. There are very few droplets with radii smaller than 20μm or larger than 75μm. This is exactly the range of values we found for the radius r_M as the humidity ranges from zero to unity (see figure 4). We have added the curve $r_M(\phi)$ to figure 5 to show this overlap. For very dry air ($\phi = 0$) we see that most of the droplets are smaller than r_M. This means that most droplets exhaled by humans in dry conditions will evaporate before they reach the ground. In very humid conditions on the other hand a much smaller percentage of the droplets will evaporate.

This is the central message of this note. The humidity of air determines a crucially important aspect of a droplet's fate after it has been exhaled by a human. In humid conditions it is likely to reach the ground while in dry conditions it is more likely to evaporate. That this is so has to do with a curious coincidence of length scales. The typical radius of an exhaled droplet is comparable to the radius that a droplet needs to have to just make it to the ground.

4 Discussion

Looking at figure 1 it feels like someone flicked a switch sometime in early October. This note argues that it wasn't one switch but millions of switches that are responsible for the dramatic increase of new cases. When the temperatures began to fall in early October people in Germany turned on their indoor heating and with it reduced the humidity indoors. The effect of this has been an increase of airborne viruses because more droplets exhaled by Germans did not fall to the ground but evaporated instead. It is this increase in airborne viruses that is responsible for the sharp increase in new cases.

If this analysis is correct, some measures to counteract the virus need to be reviewed. Social distancing is less effective when the virus does not make it to the ground. Indoor spaces like homes, schools, offices, churches, restaurants, and shopping centers all become very dangerous if airborne viruses are allowed to accumulate.

What can be done? In the winter months we will spend more time indoors. Indoor spaces might now be fundamentally compromised. There are few things that can be done. The most drastic is to not visit shared indoor spaces. This corresponds to a second lock-down. Milder measures include the wide spread use of humidifiers and the frequent ventilation of indoor spaces. Masks need to be worn consistently indoors even if social distancing is practiced because masks prevent the droplets from getting into the air in the first place.

References

[1] Coronvirus data tracker of the Financial Times available at ft.com/covid19. The data used here was updated on October 20 2020 12.08pm BST.

[2] Maxwell J.C., Collected Scientific Papers, Cambridge **11** 625 (1889).

[3] Fuchs N.A., *Evaporation and droplet Growth in Gaseous Media*, translated by Pratt J.M., Pergamon Press (1959).

[4] Wells W. F., On air-borne infection. Study II. Droplets and droplet nuclei, American Journal of Epidemiology, **20** 3, 611-618 (1934).

[5] Kukkonen, J., Vesala, T., Kulmala, M., The interdependence of evaporation and settling for airborne freely falling droplets Journal of Aerosol Science **20**(7), 749-763 (1989).

[6] Barrett, J., Clement, C., A soluble one-dimensional problem for coupled heat conduction and mass diffusion with aerosol formation in a vapour-gas mixture, Journal of Aerosol Science **17**(2), 129-143 (1986).

[7] Clement, C., Aerosol formation from heat and mass transfer in vapourâĂŞgas mixtures Proceedings of the Royal Society A **398** 307-339 (1985).

[8] The JAVA code for the implementation of the calculation is available at `https://github.com/olafdreyer/covid`.

[9] Xie, X., Li, Y., Sun, H., Liu, L., Exhaled droplets due to talking and coughing, Journal of The Royal Society Interface **6** (suppl. 6), 703-714 (2009).

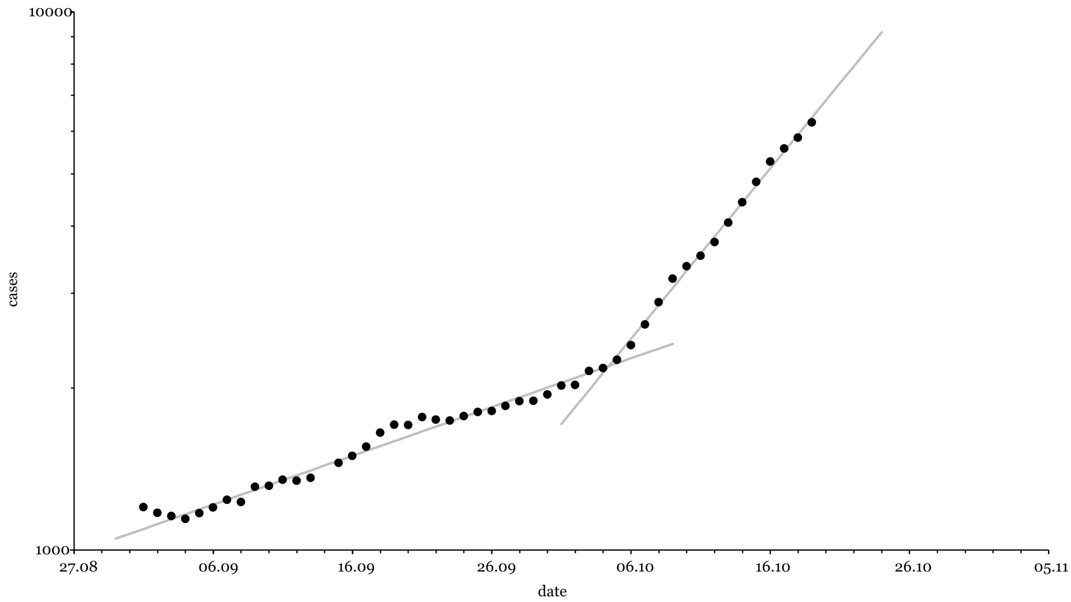

Figure 1: At the beginning of October the number of daily new COVID cases in Germany increased sharply. The solid regression lines correspond to doubling times of 33.2 days and 9.5 days. The doubling time thus decreased by a factor of three. The data is taken from [1].

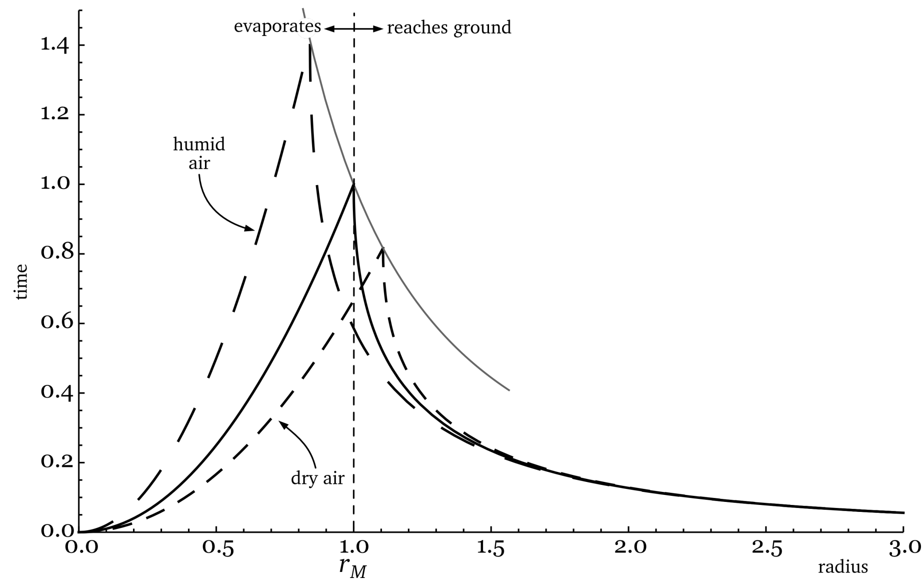

Figure 2: This figure shows the Wells curve for three different humidities. A falling droplet can reach the ground or it can evaporate on the way. If the droplet reaches the ground we plot the time t_H it requires to do so. If it evaporates we note the time t_e it takes to evaporate. We plot these times as functions of the initial radius r_0 of the droplet. There is one radius r_M where the droplet completely evaporates just as it reaches the ground. The dependence of this radius r_M on the humidity will be central to the argument of this note. If we lower the humidity the radius of the droplet that just makes it to the ground increases because it evaporates faster. The droplet thus needs to be bigger to reach the ground before it evaporates completely. For higher humidity the radius r_M decreases. If we scale the evaporation constant C_e with a parameter λ we find that r_M scales like $\lambda^{1/4}$ and that t_M scales like $\lambda^{-1/2}$. The thin grey line shows the position of $(r_M(\lambda), t_e(\lambda))$ as we vary λ.

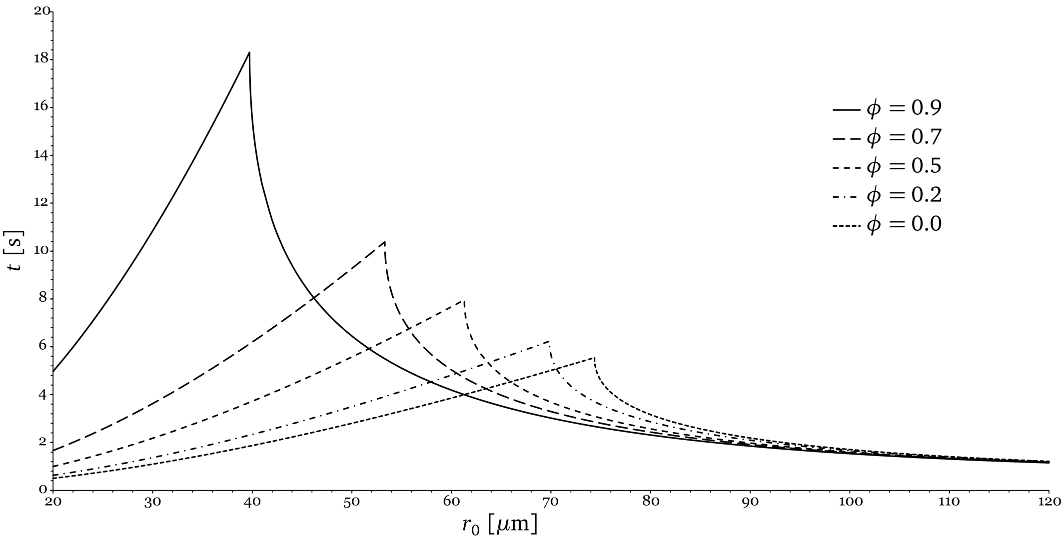

Figure 3: As a result of our calculation we obtain the Wells curves for different humidities ϕ. The temperature of air was set to 18°C or 293.15°K. The droplets were released from a height of 1.7m (the average height of a German) and they started from rest. The main feature of this graph is that the radius r_M at which the droplet just makes it to the ground before it completely evaporates (indicated by the kink in the graph) increases when one lowers the humidity. For $\phi = 0$ the value is $r_M = 74\mu$m while for $\phi = 0.9$ we find $r_M = 40\mu$m.

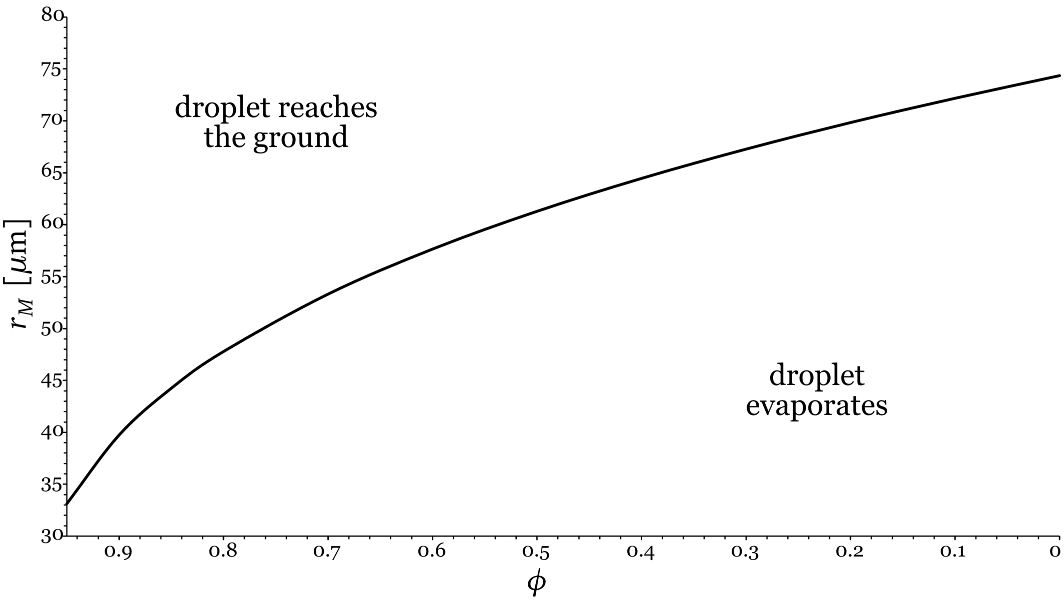

Figure 4: This figure shows the dependence of r_M on the humidity ϕ. As the humidity is lowered the value of r_M increases. A droplet that has an initial radius r_0 that is smaller than r_M will evaporate before it reaches the ground while a droplet with a radius r_0 that is larger than r_M will make it to the ground. Note that the values of r_M in this graph are comparable to typical radius of a droplet exhaled by a talking human.

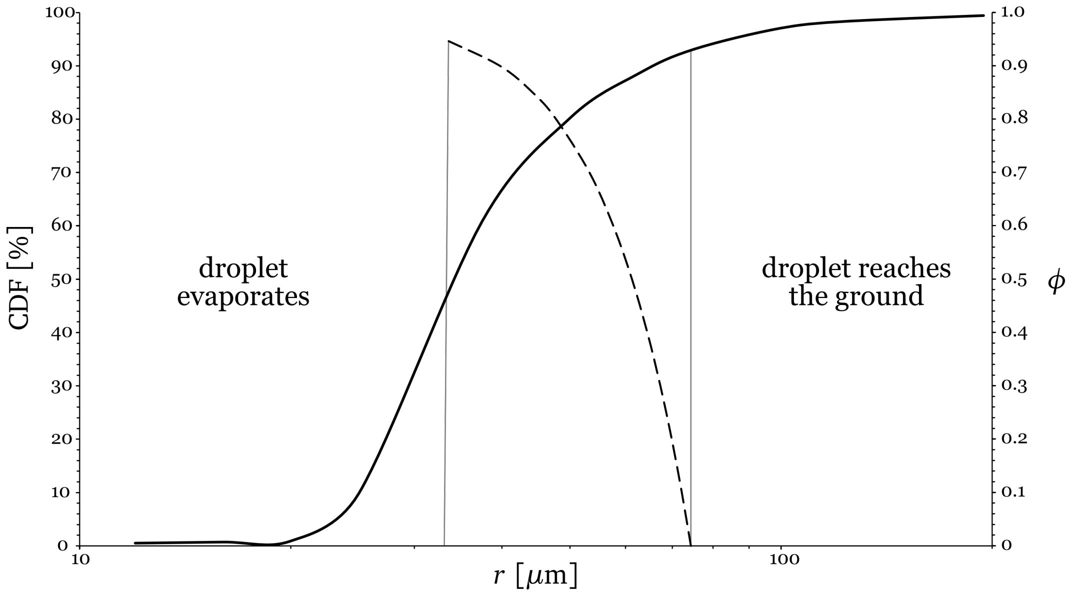

Figure 5: This graph shows the cumulative distribution function (CDF) for the radii of droplets exhausted by humans while talking (solid black line, taken from [9]). Most of the radii are in the range from 25μm to 75μm (note the logarithmic scale here). This is exactly the range of radii we have found for r_M as the humidity ϕ ranges from zero to unity. The dashed line is the function $r_M(\phi)$ from figure 4. It shows that for $\phi = 0$ almost all of the droplets will evaporate while for $\phi = 0.95$ less than half of the droplets will do so.