Monotone Data Flow Analysis Frameworks

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Today's agenda

1. Introduction
2. Background
3. Monotone Data Flow Analysis Frameworks
4. Approaches to solving monotone frameworks
5. A Variant of Kildall's Algorithm
6. Undecidability of MOP Problem for monotone frameworks

Overview

2 Background

3 Monotone Data Flow Analysis Frameworks

4 Approaches to solving monotone frameworks

5 A Variant of Kildall's Algorithm

6 Undecidability of MOP Problem for monotone frameworks

Overview

2 Background
- Flow graph
- Semilattice
- Semilattice: ordering
- Semilattice: 0 and 1
- Semilattice: bounded chains

3 Monotone Data Flow Analysis Frameworks

4 Approaches to solving monotone frameworks

5 A Variant of Kildall's Algorithm

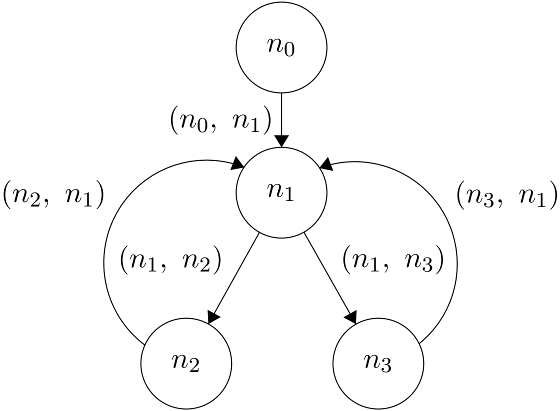
6 Undecidability of MOP Problem for monotone frameworks

Flow graph

Definition

A flow graph is a triple $G = (N, E, n_0)$.

Example:
$N = \{n_0, n_1, n_2, n_3\}$
$E = \{(n_0, n_1),$
$(n_1, n_2), (n_1, n_3),$
$(n_2, n_1), (n_3, n_1)\}$



Semilattice

Definition

A semilattice is a set L with *meet* operation $\wedge$ where the meet operation satisfies the following properties:

$$
\begin{aligned}
a \wedge a &= a && (\textit{idempotent}) \\
a \wedge b &= b \wedge a && (\textit{commutative}) \\
a \wedge (b \wedge c) &= (a \wedge b) \wedge c && (\textit{associative})
\end{aligned}
$$

Semilattice: ordering

Definition

The meet $\wedge$ defines a partial order on L

$$a \geqq b \qquad \text{iff } a \wedge b = b$$
$$a > b = b \wedge a \qquad \text{iff } a \wedge b = b \text{ and } a \neq b$$

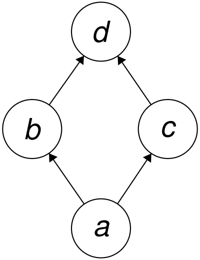


Figure: $d > b, c > a$. However, $b \not> c$ and $c \not> b$.

Definition

Element $e \in L$ is called zero, labeled 0, if

$$e \wedge x = e \qquad \forall x \in L$$

Analogous to $\perp$ from last lecture.

Definition

Element $e \in L$ is called one, labeled 1, if

$$e \wedge x = x \qquad \forall x \in L$$

Analogous to $\top$ from last lecture.

Example

$L = \{\{1,2\}, \{1\}, \{2\}, \{\}\}$
$A \wedge B = A \cap B$
Example 0: $\{\}$
Example 1: $\{1,2\}$

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Definition

A sequence $x_1, x_2, \ldots, x_n$ forms a chain if $x_i > x_{i+1}$ for $1 \leqq i < n$.

Definition

The set L is said to be bounded if for each $x \in L$ there is a constant b_x such that each chain beginning with x has length at most b_x.

Example

$L = \{\{1, 2\}, \{1\}, \{2\}, \{\}\}$
$A \wedge B = A \cap B$
Example chain: $\{1, 2\}, \{1\}, \{\}$
Any finite set is bounded. Can infinite sets be bounded?

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2 Background

3 Monotone Data Flow Analysis Frameworks
- Monotone function space
- Monotone data flow analysis framework
- Constant propagation

4 Approaches to solving monotone frameworks

5 A Variant of Kildall's Algorithm

6 Undecidability of MOP Problem for monotone frameworks

Monotone function space

Definition

Given a bounded semilattice L, a set of functions F on L is said to be a *monotone function space* associated with L if the following conditions are satisfied:

M1 Each $f \in F$ satisfies the monotonicity condition if for all $x, y \in L$,

$$f(x \wedge y) \leqq f(x) \wedge f(y)$$

M2 There exists an identity function i in F.

M3 F is closed under composition.

M4 L is equal to the closure of $\{0\}$ under the meet operation and application of functions in F (more on next slide).

Monotone function space (M4)

Definition

M4 L is equal to the closure of $\{0\}$ under the meet operation and application of functions in F.

- Read: given F, $\{0\}$ and the meet operation, we can generate all elements of L.
- Alternatively: Any element in L can be expressed as a sequence of applications of functions in F and the meet operation with $\{0\}$.

Monotone data flow analysis framework

Definition

A Monotone data flow analysis framework (from here on *framework*) is a triple $D = (L, \wedge, F)$, where

(1) L is a bounded semilattice with meet $\wedge$

(2) F is a monotone function space associated with L

Framework instance

Definition

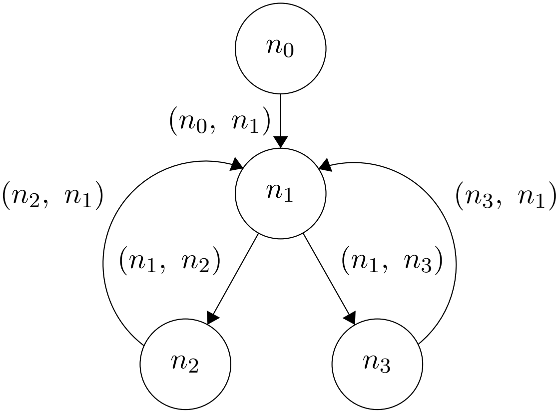
A particular instance of a framework is a pair $I = (G, M)$, where

(1) $G = (N, E, n_0)$ is a flow graph.

(2) $M : N \rightarrow F$ is a function which maps each node in N to a function in F.

Putting it together

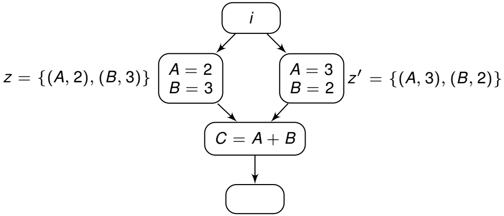
- Instance $I = (G, M)$
- Framework $D = (L, \wedge, F)$
- $M : N \rightarrow F$
- $\forall f \in F, f : L \rightarrow L$
- $\wedge : L \times L \rightarrow L$



Constant propagation

Constant propagation can be formalized as a monotone data flow analysis framework where

$$\begin{aligned} CONST &= (L, \wedge, F) && \text{framework} \\ V &= \{A_1, A_2, \ldots\} && \text{infinite set of variables} \\ L &\subset 2^{V \times \mathbb{R}} && \text{possible variable assignments} \\ \wedge &= \text{set intersection} \end{aligned}$$



Constant propagation is not distributive

Definition

For all $x, y \in L$ and $f \in F$, distributivity is satisfied when

$$f(x \wedge y) = f(x) \wedge f(y)$$

- $f(x \wedge y) = \{(A,2),(B,3)\} \cap \{(A,3),(B,2)\} = \emptyset$
- $f(x) \wedge f(y) = \{(A,2),(B,3),(C,5)\} \cap \{(A,3),(B,2),(C,5)\} = \{(C,5)\}$

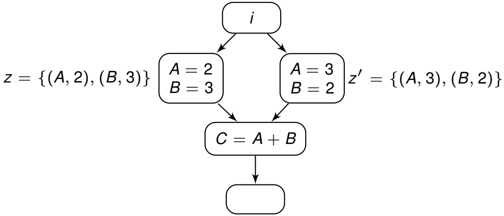


Figure: Counter example to distributivity of CONST

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- Algorithm 1 (Kildall's Algorithm)
- Properties of Algorithm 1

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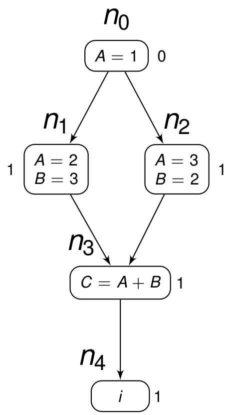
Initialization

$$A[n] = \begin{cases} 0 & \text{if } n = n_0 \\ 1 & \textit{otherwise} \end{cases}$$

- 0 - zero element of the lattice
- 1 - one element of the lattice
- i - identity function
- $L \subset 2^{V \times \mathbb{R}}$ where $V = \{A, B, C\}$



Iteration Step

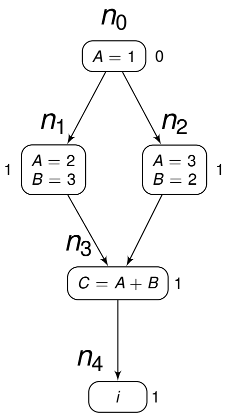
Visit nodes other than n_0 in order $v_1, v_2, ...$, for each visited node n, set

$$A[n] = \bigwedge_{p \in PRED(n)} f_p(A[p])$$

Conditions of the sequence$(v_1, v_2, ...)$:

- If a node doesn't satisfy the equation above, it should be visited again.
- If all nodes satisfy the equation, the sequence eventually end.

Example: n_2, n_1, n_3, n_2, n_3



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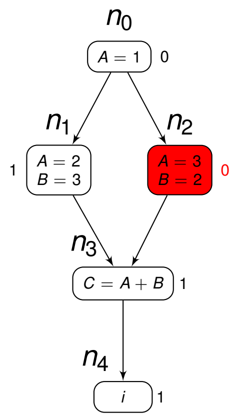

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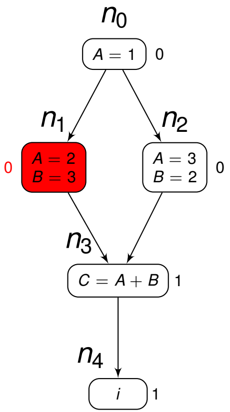

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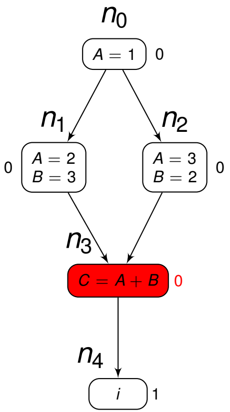

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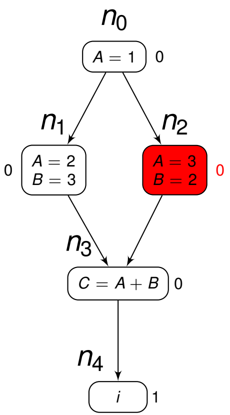

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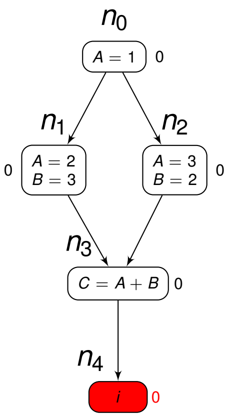

n_2, n_1, n_3, n_2, n_4

Properties of Algorithm 1

Theorem

Given an instance $I = (G, M)$ of a framework $D = (L, \wedge, F)$ as input to Algorithm 1:

(i) *Algorithm 1 will eventually halt.* *proof: A[n] decrease and L is bounded.*

(ii) $(\forall n \in N) A[n] \leq \bigwedge_{P \in PATH(n)} f_P(0)$. *proof: Induction on the length of P.*

(iii) *The result we get is unique independent of the order in which the nodes are visited.* *proof: A[n] is the maximal fix point of the set of equations.*

(iv) *There're cases where A[n] is strictly less than MOP when the data flow graph is monotone.* *proof: illustrated in the example.*

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5 A Variant of Kildall's Algorithm
- Algorithm 2
- Properties of Algorithm 2
- Algorithm 2 is not Universal

6 Undecidability of MOP Problem for monotone frameworks

Initialization

$$B[n] = \begin{cases} f_{n_0}(0) & \text{if } n = n_0 \\ 1 & \textit{otherwise} \end{cases}$$

- 0 - zero element of the lattice
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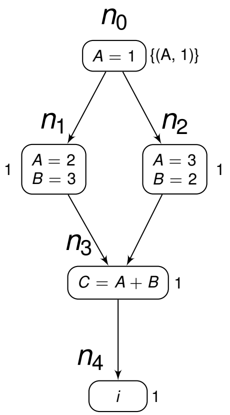

Iteration Step

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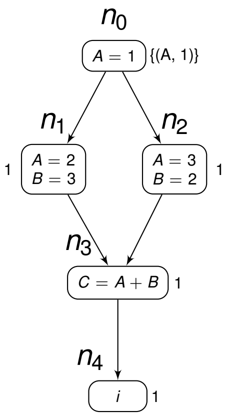
$$B[n] = \bigwedge_{p \in PRED(n)} f_n(B[p])$$

Same conditions for the sequence as in Algorithm 1.



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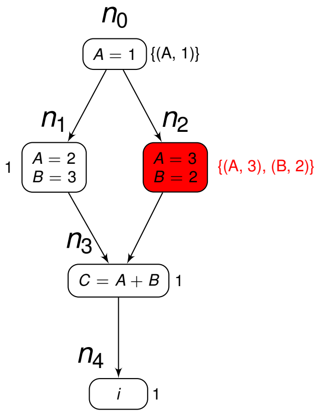

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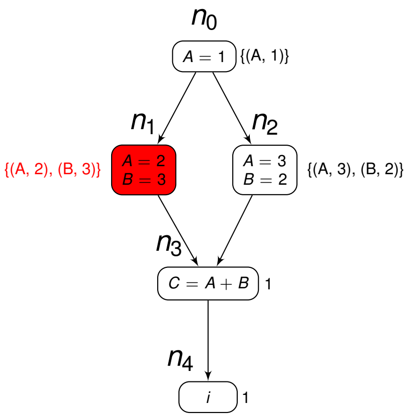

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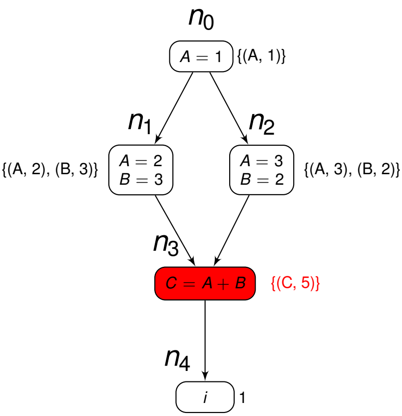

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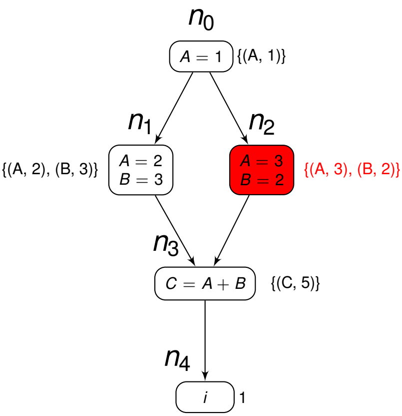

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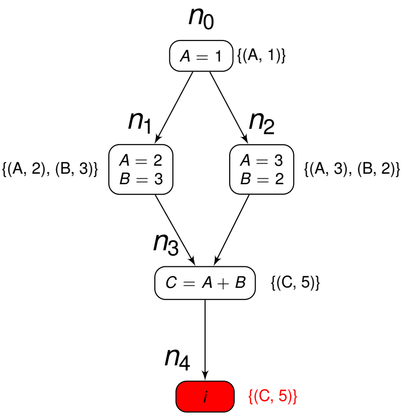
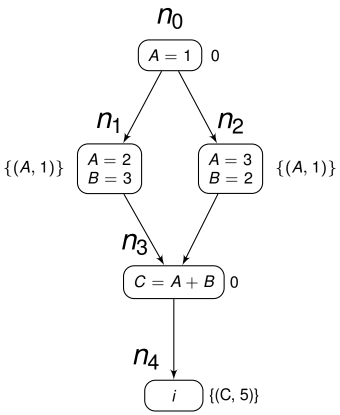


$n_2, n_1, n_3, n_2, {\color{red}n_4}$

Final Step

For each node, set

$$H[n] = \begin{cases} 0 & \text{if } n = n_0 \\ \bigwedge_{p \in PRED(n)} B[p] & \textit{otherwise} \end{cases}$$



Properties of Algorithm 2

Theorem

Given an instance $I = (G, M)$ of a framework $D = (L, \wedge, F)$ as input to Algorithm 2:

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(iii) $(\forall n \in N) H[n] \leq \bigwedge_{P \in PATH(n)} f_P(0)$. *proof: Induction on the length of P.*

(iv) *If A[n] is the result of applying Algorithm 1 to $I = (G, M)$, then $A[n] \leq H[n]$.*

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(iv) *If A[n] is the result of applying Algorithm 1 to $I = (G, M)$, then $A[n] \leqq H[n]$.* *Let's prove this!*

Property (iv) of Algorithm 2 - Proof

Proposition (*iv*)

If A[n] is the result of applying Algorithm 1 to $I = (G, M)$, then $A[n] \leqq H[n]$.

Proof by induction on steps of algorithm 2

For node n_0, the property trivially holds because $A[n_0] = H[n_0] = 0$.
For all other node $n \in N$, $A[n] \leqq H[n]$ is equivalent to: $f_n(A[n]) \leqq B[n]$ because

$$A[n] = \bigwedge_{p \in PRED(n)} f_p(A[p])$$

$$H[n] = \bigwedge_{p \in PRED(n)} B[p]$$

Base step ($m = 0$). $B^0[n] = 1$ and by definition $f_n(A[n]) \leqq 1$

Proof by induction on steps of algorithm 2 (cont.)

Induction step ($m > 0$). Our induction hypothesis (IH) is:

$$B^{m-1}[n] \geqq f_n(A[n])$$

Then

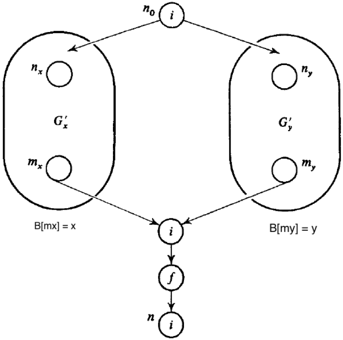
$$\begin{aligned} B^m[n] &= \bigwedge_{p \in PRED(n)} f_n(B^{m-1}[p]) \\ &\geqq \bigwedge_{p \in PRED(n)} f_n(f_p(A[p])) && \text{(IH)} \\ &\geqq f_n\Big(\bigwedge_{p \in PRED(n)} f_p(A[p])\Big) && \text{(monotonicity)} \\ &= f_n(A[n]) \end{aligned}$$

$\square$

Algorithm 2 is not Universal

An infinite number of counterexamples exist. Consider the following constructed instance of a non-distributive monotone framework:

$$MOP(n) = \bigwedge_{P \in PATH(n)} f_P(0) \geqq f(x) \wedge f(y) > f(x \wedge y) = H[n].$$



Overview

2. Background
3. Monotone Data Flow Analysis Frameworks
4. Approaches to solving monotone frameworks
5. A Variant of Kildall's Algorithm
6. Undecidability of MOP Problem for monotone frameworks
   - Previous result
   - Strengthened result

Previous result

Computing the MOP solution for a monotone framework is NP-hard.

- Dana Charmian Angluin[1]

[1] Missing reference

Strengthened result

Theorem

Given arbitrary instance $I = (G, M)$ of an arbitrary monotone framework $D = (L, \wedge, F)$, there does not exist an algorithm which computes $\bigwedge_{P \in PATH(n)} f_P(0)$ for all nodes $n \in G$.

Intuition

The *Modified Post Correspondence Problem* (MPCP), which is well known to be undecidable, can be reduced to the *MOP problem*. This is accomplished by modeling the MPCP problem as a monotone data flow framework.

MPCP example

Definition (Modified Post Correspondence Problem)

Given arbitrary lists A and B, of k strings each in $\{0,1\}^+$, say

$$A = w_1, w_2, \ldots, w_k \quad B = z_1, z_2, \ldots, z_k$$

does there exist a sequence of integers $i_1, i_2, \ldots, i_r$ such at

$$w_1 w_{i_1} w_{i_2} \ldots w_{i_r} = z_1 z_{i_1} z_{i_2} \ldots z_{i_r}$$

This example has a solution: 3, 2, 2, 4.

$$w_1 w_3 w_2 w_2 w_4 = 11{,}0111{,}1{,}1{,}10 = 1101111110$$
$$x_1 x_3 x_2 x_2 x_4 = 1{,}10{,}111{,}111{,}0 = 1101111110$$

i	w_i	x_i
1	11	1
2	1	111
3	0111	10
4	10	0

Reducing MPCP to MOP

Define

- L = 0, \$, and all strings of integers beginning with 1
- $x \wedge y = 0$ if $x \neq y$
- $h(x) = $ "1"
- $f_i(0) = 0, f_i(\$) = \$,$ $f_i(x) = concat(x, i)$
- $g(0) = 0, g(\$) = \$$
- $g(x) = 0$ if x is a solution
- $g(x) = \$$ otherwise

We have $MOP(*) = 0$ if there's a solution, otherwise $MOP(*) = \$$.

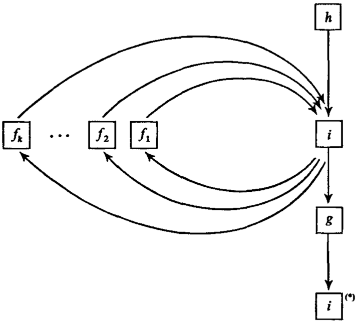


Figure: The solution to a MPCP would be one path from n_0 to n