Optimalisering og regulering TTK4135 - Assignment 2

Ola Kristoffer Hoff

2^{nd} February, 2023

Problem 1: The Mean Value Theorem

a Based on the Example A.2 (page 629) in the textbook, show that there exists one or more $\alpha \in (0,1)$, given $x = [0,0]^T$ and $p = [2,1]^T$.

Given the same function:

$$f(x) = x_1^3 + 3x_1x_2^2 \tag{1}$$

Just as in the example we see that:

$$\nabla f(x+\alpha p) = \begin{bmatrix} 3(x_1+\alpha p_1)^2 + 3(x_2+\alpha p_2)^2 \\ 6(x_1+\alpha p_1)(x_2+\alpha p_2) \end{bmatrix} \tag{2}$$

$$\nabla f(x+\alpha p) = \begin{bmatrix} 15\alpha^2 \\ 12\alpha^2 \end{bmatrix} \tag{3}$$

$$\nabla f(x+\alpha p)^T p = \begin{bmatrix} 15\alpha^2 & 12\alpha^2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 42\alpha^2 \tag{4}$$

$$\tag{5}$$

$$f(x+p) = f(x) + \nabla f(x+\alpha p)^T p \tag{6}$$

$$f(x+p) - f(x) = \nabla f(x+\alpha p)^T p = 42\alpha^2 \tag{7}$$

$$f(x+p) - f(x) = 42\alpha^2 \tag{8}$$

$$p_1^3 + 3p_1p_2^2 - 0 = 42\alpha^2 \tag{9}$$

$$8 + 6 = 42\alpha^2 \tag{10}$$

$$\alpha = \pm\sqrt{\frac{14}{42}} = \pm\frac{\sqrt{3}}{3} \Rightarrow \alpha = \frac{\sqrt{3}}{3} \tag{11}$$

Here we found exactly one α-value, $\frac{\sqrt{3}}{3}$.

b $f(x) = x^{\frac{1}{2}}$ is a continuous function. Explain why it is not Lipschitz continuous at $x = 0$. (See page 624 in the textbook for an explanation of Lipschitz continuity.)

If we have a function $f : \mathcal{D} \to \mathbb{R}^m$ where $\mathcal{D} \subset \mathbb{R}^n$, and we have: $\mathcal{N} \subset \mathcal{D}$, the function is Lipschitz continues if a we have a $L > 0$ such that:

$$||f(x_1) - f(x_0)|| \leq L||x_1 - x_0||, \quad \forall x_0, x_1 \in \mathcal{N} \tag{12}$$

Let's examine our function. We see that $x^{\frac{1}{2}} = \sqrt{x}$. This means that for any $x < 0$ the function is undefined. This means that at $x = 0$ the neighbourhood, $\mathcal{N}$, will contain negative x-values, so the criteria breaks down. Hence, our function is not locally Lipschitz continues at $x = 0$.

Problem 2: LP and KKT-conditions

The following linear program is in standard form:

$$\min_x c^T x \quad s.t. \quad Ax = b, \quad x \geq 0,\ c \in \mathbb{R}^n,\ x \in \mathbb{R}^n,\ b \in \mathbb{R}^m \tag{13}$$

Derive the KKT conditions (13).

We have the Lagrangian:

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x) \tag{14}$$

On standard form we split the Lagrange multipliers for the constraints, c, into two vectors: λ and s, for equalities and inequalities respectively. For the standard form the Lagrangian then becomes:

$$\mathcal{L}(x, \lambda, s) = c^T x - \lambda^T (Ax - b) - s^T x \tag{15}$$

The KKT conditions are:

$$\begin{aligned}
\nabla_x \mathcal{L}(x^*, \lambda^*)) &= 0, && (16)\\
c_i(x^*) &= 0, \quad for\ all\ i \in \mathcal{E}, && (17)\\
c_i(x^*) &\geq 0, \quad for\ all\ i \in \mathcal{I}, && (18)\\
\lambda_i^* &\geq 0, \quad for\ all\ i \in \mathcal{I}, && (19)\\
\lambda_i^* c_i(x^*) &= 0, \quad for\ all\ i \in \mathcal{E} \cup \mathcal{I}. && (20)
\end{aligned}$$

Let's convert the KKT conditions to the standard form:

$$\begin{aligned}
\nabla \mathcal{L}(x^*, \lambda^*, s^*) &= c - A^T \lambda^* - s^* = 0 && (21)\\
Ax^* - b &= 0, && (22)\\
x^* &\geq 0 && (23)\\
s^* &\geq 0 && (24)\\
{s^*}^T x &= 0 && (25)
\end{aligned}$$

The first condition is simply the gradient of the Lagrangian being equal to zero. The second is that all equality constraints must be equal to zero, hence we set the linear system equal to zero. The third states that all inequalities must be greater than or equal to zero, hence the x vector must be greater than or equal to zero. Fourth, the Lagrangian multipliers for the inequalities must be greater than or equal to zero, this is s. Fifth and lastly, the Lagrangian multipliers times the constraints, both equalities and inequalities must be equal to zero. Since the equalities are, from constraint two, defined to be zero we can omit writing them down. Hence we state that the inequalities must be equal to zero.

Problem 3: Linear Programming

In a plant three products R, S, and T are made in two process stages A and B. To make a product the following time in each process stage is required:

- 1 tonne of R: 3 hours in stage A plus 2 hours in stage B.
- 1 tonne of S: 2 hours in stage A and 2 hours in stage B.
- 1 tonne of T: 1 hour in stage A and 3 hours in stage B.

During one year, stage A has 7200 hours and stage B has 6000 hours available production time. The rest of the time is needed for maintenance. It is *required* that the available production time should be *fully utilized* in both stages. The profit from the sale of the products is:

- R: 100 NOK per tonne.
- S: 75 NOK per tonne.
- T: 55 NOK per tonne.

We wish to maximize the yearly profit.

a Formulate this as an LP problem.

The objective function must be the profit, which is the price of each product times the amount sold.

$$f(x) = (R_{prod} * R_{price} + S_{prod} * S_{price} + T_{prod} * T_{price}) \tag{26}$$
$$f(x) = x_1 * 100 + x_2 * 75 + x_3 * 55 \tag{27}$$
$$f(x) = 100x_1 + 75x_2 + 55x_3 \tag{28}$$
$$f(x) = c^T x, \text{ convert to standard form} \tag{29}$$
$$c^T x = 100x_1 + 75x_2 + 55x_3 = \begin{bmatrix} 100 & 75 & 55 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{30}$$
$$-c^T x = \begin{bmatrix} -100 & -75 & -55 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ negate the function to get a minimisation problem} \tag{31}$$

We have multiple constraints. The most obvious is that the hours of production in A and B must be fully utilised. In other word the amount of product from R, S and T times the respective time in A and B must equal the production hours.

$$c_1(x) = R_{hours\ in\ A} * R_{prod} + S_{hours\ in\ A} * S_{prod} + T_{hours\ in\ A} * T_{prod} = 7200 \tag{32}$$
$$c_1(x) = 3x_1 + 2x_2 + 1x_3 = 7200 \tag{33}$$

Then we do the same for B and get:

$$c_2(x) = 2x_1 + 2x_2 + 3x_3 = 6000 \tag{34}$$

The less obvious constraint is that the profit must be non-negative. This gives us:

$$c_3(x) = x \geq 0 \tag{35}$$

Now we need to put the constraints into the standard form as well. The equalities from equation 33 and 34 will be represented in the form: $Ax = b$. Each constraint being a row in the matrix A.

$$Ax = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b \tag{36}$$
$$Ax = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7200 \\ 6000 \end{bmatrix} = b \tag{37}$$

The problem can then be formulated as an LP on standard form like this:

$$\underset{x \in \mathbb{R}^3}{min}\ c^T x \quad \text{s.t.} \quad Ax = b, x \geq 0 \tag{38}$$

b Which basic feasible points exist?

First we note that A is full row rank.

"A vector x is a basic feasible point if it is feasible and if there exists a subset $\mathcal{B}$ of the index set $\{1, 2, ..., n\}$ such that

- $\mathcal{B}$ contains exactly m indices;
- $i \notin \mathcal{B} \Rightarrow x_i = 0$ (that i, the bound $x_i \geq 0$ can be inactive only if $i \in \mathcal{B}$);
- The $m \times m$ matrix B defined by

$$B = [A_i]_{i \in \mathcal{B}} \tag{39}$$

  is non-singular, where A_i is the ith column of A."(N&W, page 362)

In our case we have that $n = 3$ (variables, number of elements in x) and $m = 2$ (equality constraints, number of elements in b). The subset $\mathcal{B}$ has m elements, and any combination is valid, hence we have three possible choices.

If we choose the first two indices: $\mathcal{B} = \{1, 2\}$ that means that $x_3 = 0$. B is constructed of the columns of A that are from the set of $\mathcal{B}$.

$$B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \tag{40}$$

Using the *basis matrix*, B, in the linear equation in stead of A we get:

$$B\hat{x} = b \tag{41}$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7200 \\ 6000 \end{bmatrix} \tag{42}$$

$$\hat{x} = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 7200 \\ 6000 \end{bmatrix} \tag{43}$$

$$\hat{x} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 7200 \\ 6000 \end{bmatrix} \tag{44}$$

$$\hat{x} = \begin{bmatrix} 1200 \\ 1800 \end{bmatrix} \tag{45}$$

$$\hat{x} \Rightarrow x = \begin{bmatrix} 1200 \\ 1800 \\ 0 \end{bmatrix} \tag{46}$$

We have now found a BFP: $x = \begin{bmatrix} 1200 \\ 1800 \\ 0 \end{bmatrix}$.

We can now do the exact same step for $\mathcal{B} = \{1, 3\}$ and $\mathcal{B} = \{2, 3\}$. Then we get $\hat{x} = \begin{bmatrix} \frac{15600}{7} \\ \frac{3600}{7} \end{bmatrix}$ and $\hat{x} = \begin{bmatrix} 3900 \\ -600 \end{bmatrix}$ respectively. Note that the last point has $x_3 = -600$ which breaks our inequality constraint, and it is hence not feasible. Both the other points are valid BFP.

c Find the solution by checking the KKT conditions at all the feasible points found in b).

We have the Lagrangian:

$$\mathcal{L}(x, \lambda, s) = c^T x - \lambda^T (Ax - b) - s^T x \tag{47}$$

And the KKT conditions:

$$\nabla\mathcal{L}(x^*, \lambda^*, s^*) = c - A^T\lambda^* - s^* = 0 \tag{48}$$
$$Ax^* - b = 0 \tag{49}$$
$$x^* \geq 0 \tag{50}$$
$$s^* \geq 0 \tag{51}$$
$${s^*}^T x^* = 0 \tag{52}$$

We have found two x^* so that they are positive, hence feasible, the first two points.

Let's check the first one, $x = \begin{bmatrix}1200\\1800\\0\end{bmatrix}$:

The second condition is fulfilled since it's what we found our x^* through (we just removed the "null" column in A and row in x and calculated it as $B\hat{x} = b$). The third condition is also fulfilled, as stated above. We can solve for s^* to check the rest:

$$c - A^T\lambda^* - s^* = 0 \tag{53}$$
$$s^* = c - A^T\lambda^* \tag{54}$$
$$\begin{bmatrix}s_1\\s_2\\s_3\end{bmatrix} = \begin{bmatrix}-100\\-75\\-55\end{bmatrix} - \begin{bmatrix}3 & 2\\2 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix}, \quad s_1 = s_2 = 0 \tag{55}$$
$$\begin{bmatrix}0\\0\\s_3\end{bmatrix} = \begin{bmatrix}-100 - 3\lambda_1 - 2\lambda_2\\-75 - 2\lambda_1 - 2\lambda_2\\-55 - \lambda_1 - 3\lambda_2\end{bmatrix} \tag{56}$$
$$\Rightarrow \lambda = \begin{bmatrix}-25\\-\frac{25}{2}\end{bmatrix} \tag{57}$$
$$\Rightarrow s_3 = -55 + 25 + \frac{75}{2} \tag{58}$$
$$s_3 = \frac{15}{2} \tag{59}$$

Condition four is satisfied as $s^* \geq 0$. So is condition five and one as we just showed above. The restriction $s_1 = s_2 = 0$ was to fulfil condition five and since the system had a solution condition one was satisfied.

We know that the point, $x = \begin{bmatrix}1200\\1800\\0\end{bmatrix}$, is a solution since the KKT conditions are satisfied and it's a linear problem, meaning they are sufficent as well.

Let's check the last point, $x = \begin{bmatrix}\frac{15600}{7}\\0\\\frac{3600}{7}\end{bmatrix}$, too anyways:

The same logic as for the last point applies here, so we can skip right to the solving for s^*:

$$c - A^T\lambda^* - s^* = 0 \tag{60}$$

$$s^* = c - A^T\lambda^* \tag{61}$$

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} -100 \\ -75 \\ -55 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}, \quad s_1 = s_3 = 0 \tag{62}$$

$$\begin{bmatrix} 0 \\ s_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -100 - 3\lambda_1 - 2\lambda_2 \\ -75 - 2\lambda_1 - 2\lambda_2 \\ -55 - \lambda_1 - 3\lambda_2 \end{bmatrix} \tag{63}$$

$$\Rightarrow \lambda = \begin{bmatrix} -\frac{190}{7} \\ -\frac{65}{7} \end{bmatrix} \tag{64}$$

$$\Rightarrow s_2 = -75 + \frac{2 * 190}{7} + \frac{2 * 65}{7} \tag{65}$$

$$s_2 = -\frac{15}{7} \tag{66}$$

As we can see, $s_3 < 0$ and hence not satisfying the fourth KKT condition.

The profit and solution in the point $x = \begin{bmatrix} 1200 \\ 1800 \\ 0 \end{bmatrix}$ is:

$$c^T x = \begin{bmatrix} -100 & -75 & -55 \end{bmatrix} \begin{bmatrix} 1200 \\ 1800 \\ 0 \end{bmatrix} = -2,55 * 10^5 \tag{67}$$

So we have optimal profits of about a quarter of a million NOK (we just negate the answer since we did that at the start to solve a minimisation).

d Formulate the dual problem for the LP in a).

"Given the data c, b, and A, which defines the problem (13.1), we can define another, closely related, problem as follows:

$$max\ b^T\lambda, \quad s.t. \quad A^T\lambda \le c \tag{68}$$

This problem is called the dual problem for (13.1). In contrast, (13.1) is often referred to as the primal. We can restate (13.7) in a slightly different form by introducing a vector of dual slack variables s, and writing

$$max\ b^T\lambda, \quad s.t. \quad A^T\lambda + s = c, s \ge 0 \tag{69}$$

The variables (λ, s) in this problem are sometimes jointly referred to collectively as dual variables."(N&W, page 359).

This is directly translatable to our LP. We can check that our solution from earlier works:

$$A^T\lambda \le c \tag{70}$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \le \begin{bmatrix} -100 \\ -75 \\ -55 \end{bmatrix} \tag{71}$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -25 \\ -\frac{25}{2} \end{bmatrix} \le \begin{bmatrix} -100 \\ -75 \\ -55 \end{bmatrix} \tag{72}$$

$$\begin{bmatrix} -100 \\ -75 \\ -62.5 \end{bmatrix} \le \begin{bmatrix} -100 \\ -75 \\ -55 \end{bmatrix} \tag{73}$$

$$\tag{74}$$

The condition holds. Let's check if we get the max value.

$$b^T\lambda = \begin{bmatrix}7200 & 6000\end{bmatrix}\begin{bmatrix}-25 \\ -\frac{25}{2}\end{bmatrix} = -2.55 * 10^5 \tag{75}$$

The optimal solution is correct in the duality as well.

e Show that the optimal objective function value for the LP in a) equals the optimal objective function value for the dual problem in d) by showing that $c^T x^* = b^T\lambda^*$.

I have already shown that they equal the same value. However, we can look at it more formally.

"Defining $s = c - A^T\lambda$ (as in (13.8)), we find that the conditions (13.9) and (13.4) are identical! The optimal Lagrange multipliers λ in the primal problem are the optimal variables in the dual problem, while the optimal Lagrange multipliers x in the dual problem are the optimal variables in the primal problem."(N&W, page 360).

So we have from **c)** that:

$$c - A^T\lambda^* - s^* = 0 \tag{76}$$
$$c = s^* + A^T\lambda^* \tag{77}$$

Since λ and x are the same in both problems we can keep them and express s as an equation of λ and x. We use the first, second and fifth KKT conditions:

$$Ax^* - b = {s^*}^T x^* \tag{78}$$
$$Ax^* - b = (c - A^T\lambda^*)^T x^* \tag{79}$$
$$Ax^* - b = c^T x^* - {\lambda^*}^T Ax^* \tag{80}$$
$$Ax^* - b + {\lambda^*}^T Ax^* = c^T x^* \tag{81}$$
$$b - b + {\lambda^*}^T b = c^T x^* \tag{82}$$
$$b^T\lambda^* = c^T x^* \tag{83}$$
$$c^T x^* = b^T\lambda^* \tag{84}$$

f If you can make either stage A or stage B more available (i.e., more production hours available because of more efficient maintenance), which of the production stages A or B would you choose to improve? Why? Check your answer by first increasing the capacity of A by 1 hour (i.e., to 7201 hours), and then by increasing B by 1 hour.

Well since R is the most value per hour and it uses 50% more time in A than in B and the time available in A and B is only a difference of: $\frac{7200}{6000} = 20\%$, I would assume that increasing the production time available for A would allow us to produce the most profit.

The calculations will be the exact same as earlier, only with the adjusted values from A and B, therefore I will skip right to the results. If we increase A we get a total profit of: 255025 NOK; and for B: 255012, 5 NOK. Hence, it was, as assumed, more beneficial to increase the production time for A.