Chair of Algorithms for Intelligent systems
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Coupon collector model to determine the expected time to win a sequence dice game

Masterarbeit von

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Abstract

Sequence dice game comprises 60 chips (20 green, 20 red, and 20 blue), a 6×6 game board with each number $2-9, 12$ randomly appearing four times, two dice and a set of instructions. Players or teams take turns rolling two dice and placing the corresponding sum value on the board, with a roll of 10 considered defensive while a roll of 11 is considered wild and both require specific actions. The winner is the first player or team to create a sequence of five chips in a row, vertically, horizontally, or diagonally on the board. The sequence dice game is increasing in popularity due to its intriguing and entertaining nature. However, no research has been conducted to determine the average expected time to win the game. This study investigates the expected length of the sequence dice game. We used the coupon collector model for unequal probabilities to theoretically simplify the playing board to a one-dimensional board and analysed the expected time it would take to win the game with a restricted set of rules. Empirically, we explored the expected time it would take to win the game with a one-dimensional board when the rules are followed when the rules are not followed, and with a full playing board. We proposed different strategies for the gameplay and evaluated the effects of these strategies on the expected time it takes to win the game with a full playing board. Results showed that strategies and rules decisions combination have little effect on the expected length of the game and that it is imperative for two players to follow the rules of the sequence dice game as not adhering to the rules can lead to deadlock. Additionally, we found that, when the two players used 20 chips each, the expected length of the sequence dice game was between $32-37$ rolls, while for an unlimited number of chips, the expected length of the sequence dice game was between $41-47$ rolls.

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1 Introduction

Sequential games involve players or teams of players who do not make decisions concurrently and in which one player's decisions impact the results and decisions of subsequent players or teams [Cha16]. Examples of such games include go, backgammon, chess, tic-tac-toe, and the most recent addition, sequence dice [Ltd22]. As opposed to simultaneous games, in sequential games, the order of play does not have a tactical effect, since the opposing players or teams have knowledge of the choice or action of the first player or team.

Douglas Reuter (born March 17, 1949) developed the abstract strategy board and dice sequence game, or sequence dice game, over a four-year period in his late 20s [Rya18]. In 1982, the game was published commercially as "Sequence Five" [Rya18]. In 2017, Goliath game company acquired Jax Limited, which had obtained the exclusive rights to produce, distribute, and merchandise the game, and all associated licences. During an interview with [Rya18], Reuter described the game as "a basic match game" derived from applied mathematics. The objective of the game is for a person or team to create a sequence of five chips in a row on the board.

The Sequence Dice Game comprises 60 chips, a game board, two dice and a set of instructions. As illustrated in Figure 1.1, the chips are divided into three equal groups of 20, with each colour – red, blue and green – represented by 20 chips. The game board is usually placed on a flat surface, such as a floor or table, and is available in different sizes, with the largest measuring 32 by 27 inches. The playing board includes 36 spaces with numbers from 2 to 12, with each number appearing four times apart from 10 and 11 which are not present. The dice are used to allocate chips to spots on the board, and the instructions contain detailed rules for playing the game.

To begin the game, the playing board is kept on a flat surface, and the game chips are divided amongst teams or players, with each team or player having a unique colour distinct from the other competing players or teams. The players or teams then take turns

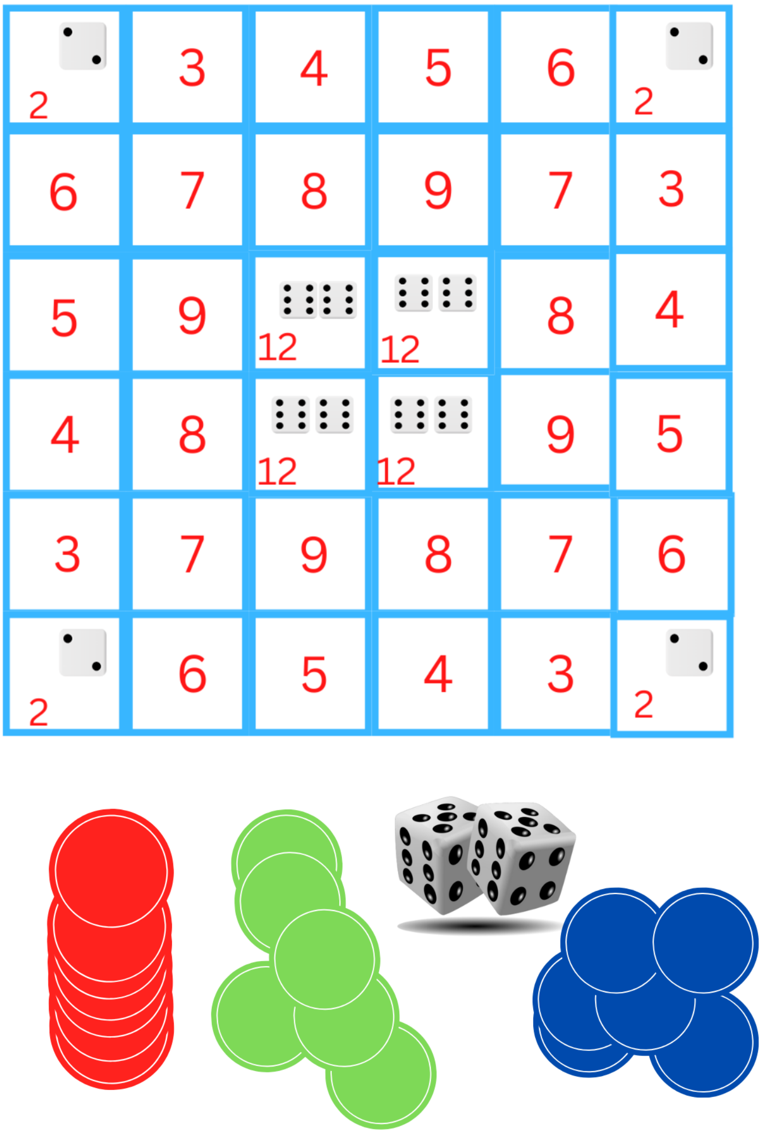


Figure 1.1: Sample representation of the sequence dice game board with chips and dice

rolling two dice to determine who takes the first turn [M17]. The player or team with the highest result of the die roll takes the first turn. As each player or team takes their turn rolling the die, their chips are placed on the corresponding number on the playing board. If all four cells of a particular number have been covered by the opponent's chips and the die roll corresponds to this number, the player replaces one of their chips with one of the opponent's chips on the board. Conversely, if all four cells of a particular number have been covered by the player's or team's chips and the die roll corresponds to this number, the player takes no action and passes the turn to the next or opposing player.

Special rolls in the game of Sequence Dice require specific actions. A roll of 10 is considered a defensive roll, and the player must remove an opponent's chip from the board unless the chip is placed on a space numbered 2 or 12. If no chips can be removed, the player's turn ends [Ltd22]. A roll of 11 is considered a wild roll and allows the player or team to place their chip on any empty space on the board. If there are no available spaces, the player can replace one of their opponent's chips with their own and their turn is over. If a 2 or 12 is rolled, the chip is placed on the corresponding space on the board and the player or team takes another turn. The winner is the first player or team to make a sequence of five chips in a row, vertically, horizontally, or diagonally.

1.1 Motivation and Research Questions

Games have been a ubiquitous part of human life throughout all civilizations worldwide. Quantitative and qualitative studies have been conducted on various games, such as ludo [AA11; SML21], snake and ladders [AKS93], and monopoly [DD18; Sta92]. Despite the increasing popularity of the sequence dice game due to its intriguing and entertaining nature, no research has been conducted to determine the average expected time to win the game.

In this study, we sought to investigate the amount of time required to win the sequence dice game. Specifically, the following questions will be addressed in this research:

- What is the expected number of die rolls to obtain a sequence of five chips i.e. to win the game?

  We intend to simplify this research question by varying the game board as well as the rules of the game. To this end, we sought to address the following simplified questions:

  - What is the expected length of the game if we simplify the playing board to be a one-dimensional board and with simplified rules set the game of sequence dice?

  - What is the expected length of the game if we simplify the playing board to be a one-dimensional board with the original rules of the sequence dice game?
  - Using a full playing board, what will be the expected number of dice rolls to win the game with simplified rules set of the sequence dice game?
  - Using a full playing board, what will be the expected number of dice rolls to win the game with the original rules of the sequence dice game?

- What are the effects of the strategies on the expected time it will take to win the game using a full playing board?

1.2 Research Objectives

In order to answer the research questions, we will use the coupon collector model for unequal probabilities to theoretically simplify the playing board to a one-dimensional playing board. To this end, we will use the coupon collector model for unequal probabilities We will then

- Analyse the expected time it will take to collect a sequence of five chips to win the game with a restricted set of rules.

  Empirically, we would explore objective 1 above, and thus;

- Analyse the expected time it will take to win the game using the one-dimensional board and the full playing board when we simplified the rules of the sequence dice game? when we strictly follow the rules of the game.
- Analyse the expected time it will take to win the game using the one-dimensional board and the full playing board when we do not follow the rules of the game.
- Propose strategies for the gameplay and perform a comparative analysis of these strategies on the expected time it will take to win the game using the full playing board.

1.3 Structure of the Thesis

This thesis is structured as follows: In Chapter 2, a comprehensive review of literature is conducted to identify established models and methods previously used to analyse different board and dice games. In Chapter 3, the coupon collector model for unequal probabilities is presented and applied to analyse the average time required to win the sequence dice game with a one-dimensional board with restricted rules. Also, the research method used to validate the measurement model was discussed. Chapter 4 explains the result of the simulations. Chapter 5 discusses the research findings. Finally, Chapter 6 concludes the thesis and recommends potential future directions.

2 Background

This chapter is divided into four sections. The first section defines and discusses key terms related to the study. The second section explains the component of the sequence dice game. The third section reviews mathematical models that have been applied to sequential games, while the last section reviews existing related works that have been conducted to evaluate the length of board and dice games.

2.1 Definition of Terms

2.1.1 Probability and Probability Theory

Probability is commonly used in everyday conversations to indicate the chance or likelihood that a particular event or set of events will take place. According to [Ash+00], it is defined as the chance that a certain event occurs or a number between zero and one indicating the concise chance that an event will occur. Probability theory, as defined by [Bil12], is the mathematical study of the chance of random events happening in order to forecast the behaviour of defined systems.

Selmant (2018) defines probability as the analysis of the likelihood of various probable outcomes arising from an experiment or other scenario with a "random" result. He argues that probability theories are the main means of conveying probabilities, as they are used to assign assumed probabilities to outcomes, which, along with other assumptions, can be used to evaluate the probability of complex events. Additionally, he suggests that probability quantifies how much can be known about something when not everything is known about it, and that probability theory is used to make predictions about the likely outcomes of events based on certain assumptions.

Blaise Pascal (1623 - 1662) and Pierre de Fermat (1601 - 1665) are two prominent French mathematicians widely credited as pioneers of probability theory. Through their correspondence in the seventeenth century, they developed ideas that are essential to the concepts of probability theory [Gri03].

It is an undeniable fact that probability and its theories and applications are pervasive in our world today. The uncertainty that arises from this stems from multiple sources, including the fallibility of humans and the diversity of phenomena, such as human technology, that often defy our favorite theories. As [Dou93] noted, “We live in an uncertain world, and probability risk assessment deals as directly with that fact as anything we do.” This indicates the use of probability in modeling uncertainty in our environment.

In his 1929 speech, Bertrand Russel (1872-1970) remarked that probability is the most important concept in modern science, adding that nobody has a clear understanding of its meaning. This sentiment was echoed by [Bor95], who argued that defining probability is not straightforward, and may require consideration of philosophical and logical aspects. [Fel68] further stressed that in order to define probability as a branch of mathematics, three aspects of the theory must be properly distinguished: formal logical substance (relations among undefined entities), intuitive background, and applications.

Thus, it is evident that probability theory has become increasingly linked to its applications, with theoretical developments paving the way for new areas of application. [Fel68] agrees that probability can be used in a variety of fields, and the most suitable tool for a given purpose depends on the utility of a general theory, which can be extended to other domains. [Guo14] identifies some of these domains as insurance, physics, transportation, commerce, game theory, and more.

2.1.2 Game Theory

Game theory is one of the key applications of probability theory [Abe18]. It has been observed that games are a major part of human life, with active participation in almost every civilisation since at least 2600 B.C. [DS04; Pri14] contend that, while the term “game” may be perceived as a leisurely and entertaining activity, the theory and applications of this concept extend further than gambling and sports, and include military tactics, education, and economics (the latter being the cornerstone of its foundation).

Game rules govern the conduct of players or teams and present a set of obstacles to be overcome in order to secure a reward, payment, or victory. Gaming has been an integral part of the development of human civilizations and mathematical approaches. As such, game theory provides a framework of basic principles that have various real-world applications [Abe18]. In the studies, *Board Games in Improving Pupils' Speaking Skills* [WY21] and *What do pupils get from games? Word puzzles vs. dice games* [MHW20], the authors concluded that board games have multiple positive effects on students' speaking performances, such as increasing speaking proficiency, motivation to speak, and interpersonal contact. Furthermore, through the use of games to design and evaluate a lab activity, it was determined that students had a greater understanding of course concepts and a higher level of participation and interest.

According to [DS04; Jen05], game theory has developed into a source of concepts and strategies for analysing a variety of domains and rivalries. As such, due to the numerous categories of games available, it is pertinent to focus on specific ones when necessary. In this paper, we will focus on the sequential game.

2.1.3 Game of Sequence

[San78] characterized sequential games as time-based adaptive decision-making processes in which players use prior experiences to inform future decisions. [Cha15] further elucidated this concept in the sixteenth chapter of his book, *Decisions in Engineering Designs*, by highlighting that in sequential games, multiple players or teams make decisions in alternating fashion, with the choices of one party influencing the outcome or decision of the other. In this type of game, it is typically assumed that the players or teams are perfectly knowledgeable or competent when making decisions.

Sequential games, guided by a time axis, encompass a variety of forms, such as decision or game trees. These game trees provide detailed information on the order in which players or teams act, the number of choices they may make, and their knowledge of the game at the point of decision-making. Rewards are granted to each player at the game tree's decision nodes [SL08]. Examples of sequential games include chess, backgammon, drop dead, ludo, go, and other board and dice games.

2.1.4 Board And Dice Game

In their article, *The Role of the Dice in the History of Board Games*, [VES15] discussed the variations in different kinds of games, noting that the board, pieces, tools, or equipment determine the settings or configurations of a game. As a result, studies from several domains have investigated the implications board games have on human thought, relationships, and society [J13; Don17]. Specifically, board games are defined as those which consist of a system of rules that specify the number of players or teams, the number of pieces on a board, the positions they can occupy, and the various actions each player or team can make [GRV04]. Furthermore, [Bar17; ZRH06] explained that board and dice games allow interactivity between players or teams via a playing surface (the board), tokens or dice, and a set of rules, which are clearly and explicitly stated in the playing materials [J13]. These traits distinguish the board and dice game from other forms, genres, or categories of games [GRV04].

Some important contributions to the design of board games were made by [SZ04]. They investigated and analysed the influence and significance of the board, publishing several articles on the design of board games, and its evolution was also studied by collecting data on various individuals who had conducted research on board games. Based on their findings, they concluded that board games have become increasingly popular over the past decade, and that board game design should be regarded as a distinct field from other design and implementation disciplines such as graphic design, etc. Additionally, they argued that playing board games can help to develop reasoning and cognitive abilities. This conclusion is similar to those of [J13; Don17]. As such, further investigation, examination, and analysis of board and dice games, particularly the sequence dice game, is warranted.

2.2 Components of the Sequence Dice Game

2.2.1 Chips

The Sequence Dice game includes 60 chips, evenly divided into three colors: 20 green, 20 blue, and 20 red. These chips are circular pieces of plastic that the players place on the cell corresponding to the sum of the dice they roll. Figure 2.1 shows the standard sequence dice game with chips on it.

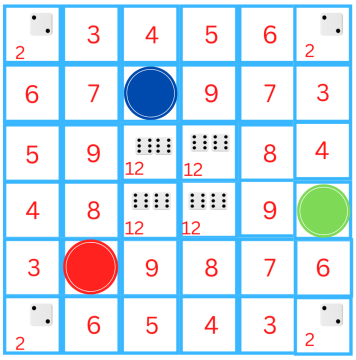


Figure 2.1: Sample representation of the sequence dice game board with chips

2.2.2 Sequence of 5 Chips in a row

This is a scenario where a player or team is able to place 5 of their chips side-by-side either vertically, horizontally, or diagonally. As an example, Figure 2.2 shows a game setting in which the green chips are placed on cells $9, 12, 12, 8, 4$ to form a sequence of 5 vertically. As a result, the player or team with the green chips is considered to be the winner, while the players or teams with the red and blue chips are considered to be the losers.

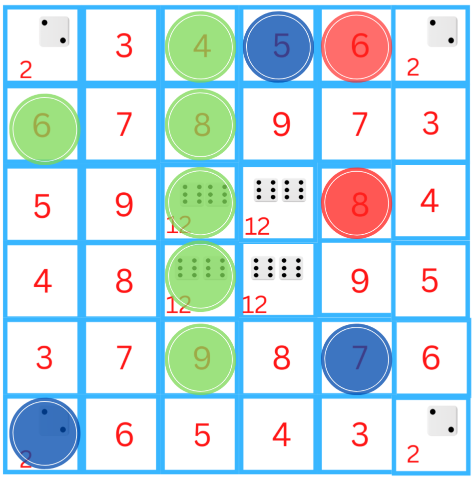


Figure 2.2: Sample representation of accumulated 5 chips in a row on a sequence dice board

2.3 Mathematical Models

There have been numerous mathematical methods employed in the analysis of games by various scholars, which are dependent upon either the type or category of the game, the purpose of the research or both. Examples of such approaches include:

2.3.1 Nash Equilibrium

Nash equilibrium is a widely adopted concept in economics, behavioural sciences, game theory, and other domains [HR04]. In game theory, Nash equilibrium identifies the optimal solution to a non-cooperative game in which each player has no incentive to change their starting strategy [OR94]. Mathematically, a Nash equilibrium [Nas51; OR94] is expressed as follows: Let S_k denote the set of strategies for the kth player, and $S = S_1, S_2, \ldots, S_n$ denote the set of strategy profiles. Thus, the elements of S comprise all possible combinations of individual strategies. If $f_k(S)$ denotes the payoff to player k when evaluated at strategy profile $s \in S$, it should be noted that the payoff to an individual player depends on the strategies of the other players as well. Accordingly, a Nash equilibrium is strictly expressed as

$$f_k(S) > f_k(S_1, S_2, \ldots, S'_k, \ldots, S_n)$$

Alternatively, it is expressed as;

$$f_k(S) \geq f_k(S_1, S_2, \ldots, S'_k, \ldots, S_n)$$

For all k, where $s'_k \in S_k$ denotes a strategy other than s_k available to player k.

2.3.2 Maximum Minimum

The minimax and maximin decision rules are central to game theory [VM47; Kor74]. In such rules, a player or team tries to minimize losses or at least limit maximum losses, while in terms of gains, the player or team seeks to maximize gains or increase minimum gains. The minimax and maximin of a game are defined as follows: Assume player p

chooses strategy s_p and the remaining players select strategy profile s_{-p}. If $u_p(S)$ denotes the utility function for player p on strategy profile s, the minimax is defined as

$$\overline{u_p} = \min_{s_{-p}} \max_{s_p} u_p(s_p, s_{-p})$$

This is the lowest value that the other players can convince player p to accept without knowing player p's strategy. The maximin is

$$\underline{u_p} = \max_{s_p} \min_{s_{-p}} u_p(s_p, s_{-p})$$

and is the largest value player p can guarantee when he is informed of the strategies of all other players. Conversely, it is the smallest value the other players can force player p to receive while knowing player p's strategy.

2.3.3 Markov Chain

In his article, Kobayashi [Kob13] explained that Markov property refers to the characteristics of a process that can make predictions for the future of the process solely on its present state, not the sequence of events that preceded it. He further outlined that the Markov Chain can be used to derive an optimal strategy for maximizing a team or player's payoff or reward. He presented the concept mathematically as follows: for any positive integer k and possible states $i_0, i_1, i_2, \ldots, i_k$ of the random variable,

$$P(T_k = i_k \mid T_{k-1} = i_{k-1}) = P(T_k = i_k \mid T_0 = i_0, T_1 = i_1, \ldots, T_{k-1} = i_{k-1})$$

2.3.4 Coupon Collector's Problem

The coupon collection problem was first introduced by Abraham de Moivre (1667-1754), a French mathematician, in 1708 [HMM84]. Since then, the coupon collector model, its variations, and applications have been discussed and accepted across various fields [FS14]. Generally, the coupon collector model, as the name implies, is a mathematical model that analyzes the collection of distinct coupons, with the focus typically aimed at determining the expected number of collections or purchases to be made to achieve a complete set of distinct coupons [FT16; AR01; FF12; Sta90; Von54; Nak08]. The manner in which these coupons arrive and the probability of their occurrence (equal

[FT16] or unequal [Von54; Nak08]) are the foundations for the variations that exist within the coupon collector model.

Suppose one person collects coupons and assumes that there is a finite number of different types of coupons (n-unique coupon) defined by a set comprising a number of unique coupons labeled $\{1, 2, 3, \ldots, n\}$. These items arrive one by one in sequence, with each successive independent event (random variable) from the set with assumed value k drawn with the probability p_k. When we have equal probabilities p_k of drawing each coupon, we have a simpler situation compared to when there are unequal chances of drawing each unique coupon. Although the latter is more realistic, particularly in the game of sequence dice, in this study, we consider a unique form of the coupon collector model, which is the connection between coupons of unequal probabilities and the single arrival of these distinct coupons into distinct sequences of subsets, each of which has unequal probabilities. Therefore, we are interested in calculating the expected number of draws required to obtain at least one of these sequences of subsets.

2.3.4.1 The Coupon Collector's Problem and Quality Control

[Luk09] applied the coupon collector's problem to certain problems in quality control sampling. He outlined how solutions to the coupon sampling problem are readily applicable to industrial sampling problems where the goal is to sample at least 1 of n specific product unit types in a well-mixed stream of products where the production unit is equally likely to be any of the n specific types. He explores the questions;

- What is the average sample size required to ensure that at least one of each of the n unit types is obtained?

  Here the sample size is examined, which will capture at least one representative from each of n distinct process stations on average. Let Y denote the random variable for the sample size. Following Luko we have:

  $$p_i = \frac{n-i}{n}$$

  $$Y_i = \frac{n}{n-i}$$

Where p_i is the probability of obtaining ith coupon and Y_i is the expected sample size for obtaining ith coupon. The expected sample size for obtaining all n distinct coupons can then be expressed as:

$$E(Y) = \sum_{i=1}^{n-i} Y_i = \sum_{i=1}^{n-i} \frac{n}{n} + \frac{n}{n-2} + \frac{n}{n-3} + \cdots + \frac{n}{1}$$

$$E(Y) = n \sum_{i=1}^{n} \frac{1}{i}$$

- What expected sample size is required to obtain $k < n$ unit in the sample?

$$E(Y) = n \sum_{i=1}^{k} \frac{1}{n-i+1}$$

- What is the expected number and variance of distinct units in a sample of fixed size k?

  Let Y denote the number of distinct units among n possible unit types that appear in a sample of size k. The expected number of distinct units in this sample can then be expressed as:

$$E(Y) = n \left(1 - \left(1 - \frac{1}{n}\right)^k\right)$$

as proved by [Luk09]

2.3.4.2 Coupon-collector's Problem With Several Parallel Collections

[FT16] applied the Coupon Collector's Problem to evaluate the expected waiting time and variance when collecting a finite number of coupons in parallel or simultaneously. The authors assumed that the coupons arrived independently, one by one, and with equal probabilities. Consider the case that m different collections are available, where the ith collection contains N_i different coupons with an equal probability of $1/N_i$ to purchase any type. Let X^i denote the random number of coupons needed to complete the ith collection, and let the number of coupons needed to complete all the m collections be $\max(X^i, \ldots, X^m)$ and the number of distinct coupons collected after n units of time is denoted by $Y_n = (Y_1^1, ..., Y_n^m)$. The sequence $\{Y_n,\ n \in \mathbb{N}\}$ is a Markov chain on the state space $\mathbb{S}$.

$$\mathbb{S} = \{(0, \ldots, 0)\} \cup \big\{(i_1, \ldots, i_m)\ \Gamma\ i_j \in \{1, \ldots, N_j\},\ j \in \{1, \ldots, m\}\big\}$$

Thus, the transition matrix with unique absorbing state $\{N_1, ..., N_m\}$ is defined as

$$\mathbb{P}[Y_{t+1} = (i_1 + \alpha_1, \ldots, i_m + \alpha_m)|Y_t = (i_1, \ldots, i_m)] = \prod_{j=1}^{m} \left(1 - \frac{i_j}{N_j}\right)^{\alpha_j} \left(\frac{i_j}{N_j}\right)^{1-\alpha_j}$$

If Q denotes the submatrix of P relative to the transient states and the fundamental matrix F of $\{Y_n,\ n \in \mathbb{N}\}$ is defined, then the expected number of coupons required to make m distinct collections is given by

$$k^{\{(N_1,\ldots,N_m)\}} = \left(\mathbb{E}_{(0,\ldots,0)}\left[D^{\{(N_1,\ldots,N_m)\}}\right], \ldots, \mathbb{E}_{(N_1,\ldots,N_{m-1})}\left[D^{\{(N_1,\ldots,N_m)\}}\right]\right)$$

$$k^{\{(N_1,\ldots,N_m)\}} = F \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Where $D^{\{(N_1,\ldots,N_m)\}} = \inf\big\{n_1 \geq 0, \ldots, n_m \geq 0\ \Gamma\ Y_n = \{(N_1, \ldots, N_m)\}\big\}$ and

$$F = \begin{pmatrix} 1 & \frac{N}{N-1} & \frac{N}{N-2} & \cdots & N \\ 0 & \frac{N}{N-1} & \frac{N}{N-2} & \cdots & N \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \cdots & \frac{N}{N-1} & N \\ 0 & \cdots & \cdots & 0 & N \end{pmatrix}$$

2.3.4.3 A Liability Allocation Game

[Han20] studied the game, where two players are expected to distribute a certain number k of counters among n boxes labeled $1, 2, \ldots, n$. At each time step $t = 1, 2, 3, \ldots$, a random box is selected, with the probability of selecting box i being p_i. A player removes one counter from the box if they have a minimum of one of the counters in the selected box; otherwise, no action is taken. The player who removes all their counters from the boxes first is the winner. In the simplest non-trivial scenario of two boxes with probabilities $(p, 1-p)$, the density function of the number of excess throws to removal for the strategy of placing g counters in box 1 and h counters in box 2 is expressed below,

$$P(X_{<g,h>} = r) = p^g(1-p)^h \left[\binom{g+h+r-1}{g-1, h+r} (1-p)^r + \binom{g+h+r-1}{g+r, h-1} p^r \right],$$

$$r = 0, 1, 2, \ldots$$

Hence, the expected time to removal is

$$E(X_{<g,h>}) = p^g(1-p)^h \Bigg[(1-p) \binom{g+h}{g-1, h+1} {}_2F_1(g+h+1, 2; h+2; 1-p) + p \binom{g+h}{g+1, h-1} {}_2F_1(g+h+1, 2; g+2; p) \Bigg]$$

where

$${}_2F_1(g; h; c; z) = \sum_{k=0}^{\infty} \frac{(g)_k (h)_k}{(c)_k} . \frac{z^k}{k!}$$

is a hypergeometric function and

$$(g)_k = g(g+1)\ldots(g+k-1)$$

is the rising factorial function.

2.3.4.4 Coupon Collecting with Quotas

[May08] analyzed a variant of the coupon collector's problem, in which the probabilities of obtaining coupons and the numbers of coupons in a collection are non-uniform. He obtained a finite expression for the generating function of the probabilities required to complete a collection. To illustrate this, consider the under-the-cap game, in which letters of a payoff word, usually the name of the manufacturer, are imprinted underneath bottlecaps and randomly distributed. This prompts consumers to buy multiple bottles of the product to collect enough letters to spell out the payoff word and win the game. Consider a payoff word containing repeated letters, such as Dr Pepper. Each of the letters D, E, P, and R must be collected a certain number of times (quota), which may be greater than one and may vary from letter to letter with non-uniform probabilities. Let L denote the set of letters of the payoff word with probability vector $\overrightarrow{p} = \langle p_\ell; \ell \in L\rangle$ and quota vector $\overrightarrow{q} = \langle q_\ell; \ell \in L\rangle$. Hence, to find the expected number of bottles $\langle T_{\overrightarrow{p},\overrightarrow{q}}\rangle$ required to spell out the payoff word, [May08] evaluated a generating function for winning the under-the-cap game as given below,

$$P'_{\overrightarrow{p},\overrightarrow{q}}(x) = \sum_{\ell \in L} \frac{p_\ell^{q_\ell}}{(q_{\ell-1})!} \int_0^\infty t^{q_{\ell-1}} e^{-t/x} \prod_{k \in L-\{\ell\}} \left(e^{p_k t} - T_{q_k}(p_k t)\right) dt$$

Therefore, $\langle T_{\overrightarrow{p},\overrightarrow{q}}\rangle = P'_{\overrightarrow{p},\overrightarrow{q}}$ (the first differential of the generating function) is given by,

$$\langle T_{\overrightarrow{p},\overrightarrow{q}}\rangle = \sum_{M \in \mathcal{P}'(L)} (-1)^{|M|+1} \sum_{\overrightarrow{r} \in Q_M^<} \frac{\binom{\Sigma_{\ell \in M} r_\ell}{\overrightarrow{r}} \left(\Pi_{\ell \in M} p_\ell^{q_\ell}\right)}{\left(\Sigma_{\ell \in M} p_\ell\right)^{1+\sum\limits_{\ell \in M} r_\ell}}$$

where $\mathcal{P}'(L)$ denote the set of non-empty subsets of letters in L, and $Q_M^<$ denote the set of finite sequences $\overrightarrow{r}$ of integers indexed by the letters in M such that $0 \le r_\ell < q_\ell$ for each $l \in M$. [May08] generalized the result of [Von54] regarding the expected number of coupons to obtain a complete collection.

2.3.4.5 A Coupon Collector's Problem With Bonuses

[NK06] explained that when there are c coupons made up of k bonus coupons and $l = c - k$ ordinary coupons, the number of days required to collect at least one of each type is denoted by $Y^{c,k}$. It is possible to collect all k coupons and one ordinary coupon on the same day if there is a stretch of luck. The distribution of the coupon collector's problem with k bonuses and l ordinary bonuses is given by

$$P(Y^{k+l,k} \leq n) = E[(1 - e^{-X_{n,l}})^k]P(Y^l) \leq n$$

for $n \geq 1$

where $X_{n,l}$ is a gamma random variable with parameter $\Gamma(n, l)$. Furthermore, the probability function is expressed as

$$p^{k+l,k}(n) = \sum_{i=0}^{k}\sum_{j=0}^{l}\binom{k}{i}\binom{l}{j}(-1)^{i+j+1}\left(\frac{i+j}{l+i}\right)\left(\frac{l-j}{l+i}\right)^{n-1}$$

for $n \geq 1$

Therefore, the expected number of days to obtain at least one of c type coupons including k bonuses is,

$$E(Y^{k+l,k}) = \frac{1}{k+l}[(k+l)H_{k+l}] + \frac{k}{k+l} \cdot 1$$

2.3.4.6 Birthday Paradox and CCP

The birthday problem and the coupon collector problem can be related to an infinite sequence of events; however, the fact that the first birthday collision or the first complete collection occurs at any fixed time n only involves finite events. [FGT92] explored the birthday paradox. Let $\mathcal{A}$ represent the dates in a year with m days and p_i denote the probability of date $a_i\colon a_i \in \mathcal{A}$. Let B_i be the time of the first collision (the time when we first encounter i distinct elements that are each repeated at least k times). The expected number of elements that need to be drawn with replacement from $\mathcal{A}$ until the first encounter of the ith different letter occurrence of a k-hit is stated as,

$$E\{B_i\} = \sum_{q=0}^{i-1} \binom{m}{q} \left[\int_0^{\infty} \left(e_{k-1}\left(\frac{t}{m}\right)\right)^{m-q} \left(e^{\frac{t}{m}} - e_{k-1}\left(\frac{t}{m}\right)\right)^{q} e^{-t} dt \right]$$

For the classical birthday paradox, where $k = 2$ and $i = 1$, the expected waiting time is expressed thus,

$$E\{B_i\} = 1 + 1!\, S_1 + 2!\, S_2 + 3!\, S_3 + \cdots + m!\, S_m$$

where S_r are symmetric functions of the probability p_i,

$$S_r = \sum_{i_1 < i_2 < \ldots i_r} p_{i_1} p_{i_2} \ldots p_{i_r}$$

2.3.5 Related Works

[AKS93] conducted a study to research the average length of the Snake and Ladder game and the effect the number of snakes and ladders has on the length of the game. They performed a simple experiment with a 10 x 10 board containing 10 snakes and 9 ladders, which were numbered from 1 to 100. A single die was used to determine the number of squares to advance, and when the player arrived at square 100, they were said to win the game. The authors simulated the game for 10 runs of 1000 games using UBASIC86 programming language and recorded an average game length of 39.1 moves. Furthermore, a Markov chain with 82 states was used to compute the expected game length, which was also performed using UBASIC86 on a Zenith microcomputer, yielding a result of 39.2. In addition, a formula derived using Gauss-Jordan reduction was used to determine the effect of adding a snake or ladder on the length of the game. The authors expected that the addition of more ladders would reduce the game length, while the addition of more snakes would increase the length of the game; however, the results obtained showed otherwise. For example, when a ladder was inserted from state 83 to state 7, the length of the game increased from 39.2 to 45.8, and when a Ladder was inserted from state 79 to state 81, the game length increased to 41.9. Conversely, introducing an extra Snake to the original game from state 29 to state 27 reduced the game length to 38.

In a research study conducted by one of the students of the University of Passau [Ism21], the average length required to play the game of snakes and ladders was evaluated. This study aimed to improve upon the work of [AKS93] by considering and changing various elements of the game such as the number and length of the snakes and ladders, as well as the distribution of die. The Mersenne Twister pseudo-random number generator was used to produce a set of random ladders and snakes, while Markov chain models were applied to determine the expected length of the game. Lastly, the simulation of the games was done using the Monte Carlo method for objective analysis. Results revealed that the dice distribution greatly influences the length of the game. For instance, in a game with less than 12 snakes and 12 ladders, a 100-sided harmonic die typically yields a lower average game length than any size of the uniform dice. Meanwhile, when the numbers of ladders and snakes are greater than 13, the 15-sided uniform dice results in a lower average game length. Additionally, the number of ladders and snakes significantly impacts the expected average length of the game. However, the average length of the game is unaffected by the number of snakes for uniform 100-sided dice. Based on the work of [Die+10] who noted that harmonic distribution spinner yields the shortest average length of $O((\log n)^2)$ for any discrete token process, [Ism21] introduced harmonic dice to the game to analyse its performance in the game. They concluded that to have a result with a minimum average game length, the number of states in the Markov chain need not be equal to the size of the dice. Additionally, their study provided evidence that the expected average length of any dice size n and $n-1$ was the same whether the dice distribution was harmonic or normal.

Moving away from the Snake and Ladder game, [AA11] proposed and analysed the performance of aggressive, fast, defensive, and random strategies to test the state-space complexity of the generic version of the Ludo game. Theoretically, it was hypothesised that it would take a player at least 65 moves to win the game of Ludo. The games were initially run 5000 times with all four players selecting the random strategy. This initial experiment was carried out to confirm that the game setup was accurate. The results revealed that each random player won $25.0 \pm 1.0\%$ more times than they lost, with no significant variations in the result at higher game runs. This proved that the 5000-game play was reliable for evaluating other strategies. The next experiment conducted was the evaluation of the performance of each basic strategy player against the other three players playing random strategy. The outcome indicated that any of these basic strategies—aggressive, defensive, or fast—were superior to the random strategy, with a winning percentage of at least 98%. The win percentages for aggressive, defensive, and fast strategies were $99.4 \pm 0.2\%$, $99.3 \pm 0.3\%$, and $99.4 \pm 0.2\%$, respectively. In another related experiment in which all four players used all the strategies, including the random strategy, it was observed that the defensive strategy had the highest winning percentage at 40%, followed by the aggressive strategy at 32% and the fast strategy at 27%. The random strategy had a 0% win rate, which was the worst performance. The defensive strategy had the highest winning percentage but also the fewest moves, with just 93, compared to the aggressive strategy's 98 and the fast strategy's 99. The researchers also developed and evaluated a "mixed strategy", a combination of the four strategies. This implies that after each roll of the dice, players can employ or switch among any of the four strategies. The combined strategy was put through tests similar to those for the basic strategies, and the results revealed that the mixed strategy is the best with a 90% win rate compared to using any of the basic strategies alone, and an average move count of 66. This is quite close to the move count of 65 that was obtained in the theoretical analysis, indicating that the mixed strategy is the most efficient one. The estimated state-space complexity of Ludo was said to be approximately 10^{22}, which is slightly larger than that of Backgammon (10^{20}) [Tes+95] and lesser than Chess (10^{50}) [All+94]. This implies that Ludo is not a trivial game and has some strategic variants.

Similarly, [Ony21] aimed to improve on the work of [AA11] by conducting a study on the length of the Ludo game. They simplified the game based on the number of players and different strategies applied. Three strategies, random-six, once-six-tokens-out, and complete-one-token-before-next, were used in the game for a single player with an n-token count, with n ranging from 1 to 10. The random-six strategy allows the player to choose whether to bring their token out from the home region of the board or move the token already on the board when the outcome of the dice roll is 6 and the player still has at least one token left in the home region. The once-six-token-out strategy requires the player to move their token from the home region to position 0 (starting point) when the outcome of the dice roll is 6 and there is still a player token on the home region. The complete-one-token-before-next strategy requires the player to concentrate on moving one token from position 0 to position 56 before another token can be taken out from the home region to position 0. The results showed that the random-six and once-six-token-out strategies outperformed the complete-one-token-before-next strategy, with the random-six and once-six-token-out strategies producing approximately 84 total turns including 70 valid turns and 14 invalid turns, and the complete-one-token-before-next strategy producing approximately 106 total turns including 68 valid turns and 39 invalid turns. This result is not significantly better than that of [AA11], who got 65 moves for the average length to play the game of Ludo. However, the author in [AA11] only did hypothetical guess to arrive at the result and as well did not explicitly state if the result is for only valid moves or for both valid and invalid moves. Based on this, [Ony21] contended that their result for the single player of the ludo game is better. For the four-player game, [Ony21] applied four pure strategies (aggressive, fast, random, and defensive) and one mixed strategy as used in [AA11] on both ludo boards with safe squares and without the safe squares variants. The random strategy requires the player to move their tokens randomly, and the aggressive strategy requires the player to move their closest token or a token that can knock out an opponent's token based on the outcome of the dice roll, or perform a random move if the move is not possible, the fast strategy requires the player to move their farthest token on the board, and the defensive strategy requires the player to defend their tokens from being knocked out, with tokens that can be knocked out being within 1-6 squares away from the opponent's tokens, and performing a random move if this is not feasible. It was discovered that when the four players used different strategies in each of the two variants, the random strategy gave the worst result with the lowest number of wins, while the aggressive strategy performed the best and had the highest number of wins. When the four strategies were also combined in 35 different ways for the four-player game, it was discovered that when the four-player in both variants of the game are all playing fast strategies, it was the worst combination of strategies for the first player to win the game. Lastly, they found out that the ludo board without safe squares had a shorter average length compared to the ludo board with safe squares version.

3 Methods

In this chapter, we evaluate the sequence dice game from the perspective of the coupon collector's problem. Specifically, we analyze the expected number of dice rolls required to win the game with a restricted set of rules and the expected number of rolls required to win the game when the rules are followed. We limit our theoretical analysis to the one-dimensional board of the sequence dice game with a single player.

3.1 Dice Sums and Probability

Suppose we have a pair of n-sided dice which has a set of probabilities $P = \{p_1, p_2, ..., p_n\}$ of being issued. Then, if the interest is recording the sum of the two outcomes that we roll, the probability that the sum is g is given by

$$\sum_{i=1}^{n} p_i p_{g-i}, \quad \text{for } g - i < 1,\ p_j = 0, \quad j = g - i \tag{3.1}$$

where $g = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$. For example, if we consider $g = 4$, then the probability of obtaining the sum 4 is $\frac{3}{36}$ from the equation. To put in a proper concept of polynomial, if the sum vector is $< s_2, s_3, ..., s_{12} >$ where s_0 denotes the sum of two dice that gives 2. The probability can be expressed as a polynomial as shown below

$$P(x) = p_1 x + p_2 x^2 + \cdots + p_n x^n \tag{3.2}$$

In general, a fair $n-$sided die can be represented by the polynomial

$$\frac{1}{n}(x + x^2 + \cdots + x^n). \tag{3.3}$$

For a single fair die, the outcomes $1 \dots 6$ are equiprobable and uniformly distributed, meanwhile for the sum of two fair dice, the outcomes $2 \dots 12$ are not equally likely. The probability of the sum of $g = 2, \dots, 12$ is given in Table 3.1. It is evident that 7 has the highest probability while 2 and 12 have the lowest.

Dice Sum	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Table 3.1: Probability of sum of two 6-sided dice

Figure 3.1 shows that the probability distribution of the sum of two fair 6-sided dice is normally distributed.

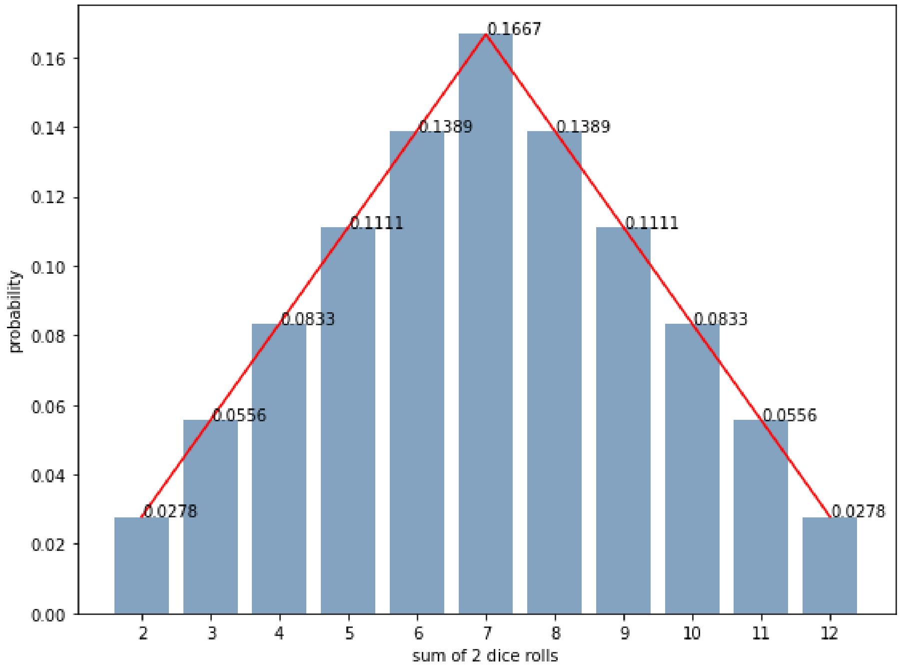


Figure 3.1: Distribution of sum of two 6-sided dice and their probabilities

3.2 Single Collection of Unique Subsets with Unequal Probabilities

We consider a scenario where there are C distinct types of coupons drawn independently, each with a different probability of appearing in m unique subsets S_e of constant size. If each unique subset does not contain more than one distinct coupon of any type, then there are C distinct coupons with probabilities P for each distinct coupon such that $p \geq 0$, $p_1 \neq p_2 \neq \ldots \neq p_c$ and $\sum_{i=1}^{n} p = 1$. Let S be the set of unique subsets of C such that $S = \{S_1, S_2, \ldots, S_m\}$, $S \subset C : \bigcup_{i=1}^{c} S_i = C$ and $\bigcap_{i=1}^{c} S_i \neq \emptyset$. Hence, given that S_e is a unique 'ordered' set and an event and/or element in set S with each S_e in set S having unequal probabilities denoted by p_e such that $p_e \geq 0$, $p_1 \neq p_2 \neq \ldots \neq p_m$ and $\sum p_e = 1$ where $e = 1, 2, \ldots, m$. Then $S_1, S_2, \ldots, S_m$ have corresponding probabilities of $p_1, p_2, \ldots, p_m$ respectively. The expected number of rolls required to complete the collection of distinct coupons of at least one of the unique subsets S_e in S is denoted by X. Let X_1 be the number of rolls required to get the first type of coupon of our collection in any unique subset S_e and X_i denotes the number of additional rolls required to collect the ith distinct coupon type in our collection. The number of rolls required to make a complete set of collections of distinct coupon types is expressed as

$$X = X_1 + X_2 + \cdots + X_{i-1} + X_i + \cdots + X_k$$
$$X = \sum_{i=1}^{k} X_i \tag{3.4}$$

Therefore, to evaluate the expected number of rolls required to make a complete collection of distinct coupons of set S_e, we apply the maximum-minimums identity presented by ([Ros09], p.345).

$$X = \max\{X_1, X_2, \ldots, X_{i-1}, X_i, \ldots, X_k\}$$
$$E[X] = E[\max\{X_1, X_2, \ldots, X_{i-1}, X_i, \ldots, X_k\}]$$

$$E[X] = \sum_{i}^{k} E[X_i] - \sum_{i<j} E[\min(X_i, X_j)] + \sum_{i<j<l} E[\min(X_i, X_j, X_l)]$$

$$- \sum_{i<j<l<n} E[\min(X_i, X_j, X_l, X_n)] + \cdots + (-1)^{k+1} E[\min(X_1, X_2, \ldots, X_{i-1}, X_i, \ldots, X_k)]$$

$$E[X] = \sum_{i}^{k} \frac{1}{p_i} - \sum_{i<j} \frac{1}{p_i + p_j} + \sum_{i<j<l} \frac{1}{p_i + p_j + p_l}$$

$$- \sum_{i<j<l<n} \frac{1}{p_i + p_j + p_l + p_n} + \cdots + (-1)^{k+1} \frac{1}{p_1 + p_2 + \cdots + p_k} \tag{3.5}$$

Note that, the last term of equation 3.5 above is equal to 1, and rewriting the expression gives,

$$E[X] = \sum_{i}^{k} \frac{1}{p_i} - \sum_{i<j} \frac{1}{p_i + p_j} + \sum_{i<j<l} \frac{1}{p_i + p_j + p_l} - \sum_{i<j<l<n} \frac{1}{p_i + p_j + p_l + p_n} + \cdots + (-1)^{k+1} \tag{3.6}$$

as proved by ([Ros09], p.345).

3.3 Parameterisation of the One-dimensional Board of the Sequence Dice Game

In this section, we parameterize the sequence dice game and explore the application of the coupon collector model. We make two assumptions to aid our objectives. First, we assume that the sequence dice game board is one-dimensional, either $C \times 1$ or $1 \times C$. This is illustrated in Figure 3.2. The probabilities of the coupons are assumed to be unequal, determined by two dice. Secondly, we assume that the game is being played by a single player, often referred to as playing against nature [Bec08; GSE18].

Therefore, we present an explicit generalization of the sequence dice game and theoretically define certain parameters. We provide a one-dimensional representation of the sequence dice game below.

Figure 3.2: One-dimensional board of the sequence dice game

$$C = \{2, 3, ..., 9, 12\} \tag{3.7}$$

Let C denote the set of all possible calibrations or distinct types of coupons that could be collected from the game of sequence dice. The elements 2, 3, $\ldots$, 9, 12 of set C are the relevant rolls or outcomes from the two dice that have a corresponding value on the sequence dice game board, each having an unequal probability of being rolled or drawn, denoted by p_h such that $p_h > 0$ and $p_1 \neq p_2 \neq \ldots \neq p_h$.

It is important to note that the occurrence or non-occurrence of coupons 10 and 11, which are not expected to be part of any sequence of collections or subset collections, substantially influences how these sequences of coupons are collected. Therefore, from Equation 3.7, if we denote each element in C such that

$$\{2, 3, \ldots, 9, 12\} = \{c_1, c_2, \ldots, c_{h-1}, c_h\}$$

Where c_1, c_2, $\ldots$, c_h are the distinct relevant coupons. The probabilities of each distinct coupon can be represented as:

$$p(c_1) = \frac{n(c_1)}{n(N)},\ p(c_2) = \frac{n(c_2)}{n(N)}\ \ldots\ p(c_h) = \frac{n(c_h)}{n(N)}$$

Generally,

$$p_c = \frac{n(h)}{n(N)}\ :\ h = c_1, c_2 \ldots c_h \tag{3.8}$$

where p_c is the probability of obtaining a distinct coupon h, and $n(N)$ is the size of the sample space for the roll of two dice.

Therefore,

$$p(C) = \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36} + \frac{5}{36} + \frac{6}{36} + \frac{5}{36} + \frac{4}{36} + \frac{1}{36} = \frac{31}{36}$$

Also, let V denote the set of irrelevant steps with elements 10 and 11 being all rolls that do not have a corresponding value on the sequence dice game board. i.e.

$$V = \{10, 11\} \tag{3.9}$$

Therefore,

$$\{10, 11\} = \{v_1, v_2\}$$

Where v_1 and v_2 are the distinct irrelevant steps. The probabilities of each irrelevant step can be represented as:

$$p(v_1) = \frac{n(v_1)}{n(N)} \quad \text{and} \quad p(v_2) = \frac{n(v_2)}{n(N)}$$

and this simplifies to

$$p(V) = \frac{n(h)}{n(N)} \quad : \quad h = v_1, v_2$$

$$p(V) = \frac{3}{36} + \frac{2}{36} = \frac{5}{36}$$

As the objective of the sequence dice game is to obtain a sequence of five chips (adjacent to one another) to win, we consider another set S, which consists of all unique subsets S_i. The elements of set S are the collections of set S_is.

$$S_1 = \begin{array}{|c|c|c|c|c|}\hline 2 & 3 & 4 & 5 & 6 \\ \hline\end{array}$$

$$S_2 = \begin{array}{|c|c|c|c|c|}\hline 3 & 4 & 5 & 6 & 7 \\ \hline\end{array}$$

$$S_3 = \boxed{4}\boxed{5}\boxed{6}\boxed{7}\boxed{8}$$

$$S_4 = \boxed{5}\boxed{6}\boxed{7}\boxed{8}\boxed{9}$$

$$S_5 = \boxed{6}\boxed{7}\boxed{8}\boxed{9}\boxed{12}$$

Therefore, the set S can be defined in the form below

$$S = \{S_1, S_2, \ldots, S_{m-1}, S_m\}$$

or

$$S = \{S_i\}_{i=1}^{m} \tag{3.10}$$

It is also important to note that,

$$\{S_1 \cup S_2 \cup \ldots \cup S_{m-1} \cup S_m\} = C$$

and

$$S_i \cap S_j \neq \emptyset : i \neq j$$

Hence, consider the set C from equation 3.7, the number of possible unique subsets of five distinct coupons that can be obtained from set C with regard to the rules of the sequence dice game is expressed thus,

$$S_1 = \{c_1, c_2, \ldots, c_{h-4}\}$$

$$S_2 = \{c_2, c_3, \ldots, c_{h-3}\}$$

$$\ldots$$

$$S_{m-1} = \{c_{h-5}, c_{h-4}, \ldots, c_{h-1}\}$$

$$S_m = \{c_{h-4}, c_{h-3}, \ldots, c_h\} \tag{3.11}$$

Therefore, h is the total number of chips on the board, m is the maximum number of unique subsets of C, and k is the maximum number of distinct coupons contained in each unique subset S_i. In the context of a one-dimensional sequence dice game, this implies that there are $m = h - k + 1$ possible unique subsets of set C to get a sequence of $k = 5$ chips.

3.4 Coupon Collection for One-dimensional Sequential Dice Game With a Restricted Set of Rules

3.4.1 Expected Time to Collect all Relevant Coupons of Equal Probability

In this section, we evaluate the expected number of rolls it would require to collect all nine (9) relevant coupons with the assumption that all 9 coupons have an equal probability of being selected. This evaluation is intended to serve as a warm-up and as baseline for comparisons. Therefore, let $N = 9$ be the number of different coupons with a probability of rolling each at any time equal to $\frac{1}{N}$. Hence, given the set of relevant steps $C = \{c_1, c_2, c_3, \ldots, c_8, c_9\}$, where $c_1 = 2$, $c_2 = 3$, $c_3 = 4$, …, $c_8 = 9$ and $c_9 = 12$ as expressed in Equation 3.7 and shown in Figure 3.2, each c_i represents relevant steps with uniform or equal probability of $\frac{1}{9}$.

Furthermore, let T represent the time or the number of rolls it would require to collect all nine coupons. We can then say that $T = T_1 + T_2 + \ldots + T_N$, where T_1 is the number of rolls to get the first coupon, T_2 the number of rolls until the second distinct coupon and T_N the number of rolls to get the last distinct coupons in our collection. Thus, for any $i = 1, 2, \ldots, N$, T_i represents the additional number of rolls required to roll or get a distinct coupon. That is, the number of rolls required to move from $i - 1$ to i different types of coupons in our collection. Therefore, the expected time it would require to collect all distinct coupons of equal probability is

$$E(T) = E(T_1) + E(T_2) + \ldots + E(T_N)$$

$$E(T) = 1 + \frac{N}{N-1} + \frac{N}{N-2} + \ldots + \frac{N}{2} + N$$

$$E(T) = N \sum_{i=1}^{N} \frac{1}{i} \tag{3.12}$$

where, $N = 9$ and $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$.

Therefore,

$$E(T) = 9\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}\right)$$

$$= 9\left(\frac{2520 + 1260 + 840 + 630 + 504 + 420 + 360 + 315 + 280}{2520}\right)$$

$$= 9\left(\frac{7129}{2520}\right)$$

$$= 9(2.829)$$

$$E(T) = 25.461$$

3.4.2 Expected time to collect all coupons with probabilities associated with two dice

In this section, we evaluate the average expected time to collect all coupons on the basis that the probabilities of the coupon are unequal and correspond to the probability of the sum of two dice. The sequence dice game with a restricted set of rules includes irrelevant rolls (10 and 11) that do not require any action but still factor into the expected rolls required to collect all nine distinct coupons. To evaluate the expected number of rolls required to collect all coupons, we conditioned the probability of the relevant rolls on the probability of irrelevant rolls.

Considering the relevant steps of set C and the irrelevant steps of set V expressed in Equations 3.7 and 3.9 respectively and let V' be the set of rolling a value other than 10 and 11, if $p(c_i)$ represents the probability of rolling a distinct relevant coupon in C, then the probability of rolling a distinct coupon given the probability of rolling values other than 10 or 11 can be expressed as

$$p(c_i) = p(c_i \mid V') \tag{3.13}$$

$$p(c_i \mid V') = \frac{p(c_i \cap V')}{p(V')} \tag{3.14}$$

where $p(V')$ denotes the probability of rolling a value other than 10 and 11 and can be evaluated as:

$$\begin{aligned} p(V') &= 1 - p(v_1) + p(v_2) \\ &= 1 - p(10) + p(11) \\ &= 1 - \left(\frac{3}{36} + \frac{2}{36}\right) \\ &= 1 - \frac{5}{36} \\ p(V') &= \frac{31}{36} \end{aligned}$$

From Equation 3.14, $p(c_i \mid V')$ we have

$$\begin{aligned} p(2) = p(2 \mid V') &= \frac{\frac{1}{36}}{\frac{31}{36}} = \frac{1}{36} \times \frac{36}{31} = \frac{1}{31} \\ p(3) = p(3 \mid V') &= \frac{\frac{2}{36}}{\frac{31}{36}} = \frac{2}{36} \times \frac{36}{31} = \frac{2}{31} \\ p(4) = p(4 \mid V') &= \frac{\frac{3}{36}}{\frac{31}{36}} = \frac{3}{36} \times \frac{36}{31} = \frac{3}{31} \\ p(5) = p(5 \mid V') &= \frac{\frac{4}{36}}{\frac{31}{36}} = \frac{4}{36} \times \frac{36}{31} = \frac{4}{31} \\ p(6) = p(6 \mid V') &= \frac{\frac{5}{36}}{\frac{31}{36}} = \frac{5}{36} \times \frac{36}{31} = \frac{5}{31} \\ p(7) = p(7 \mid V') &= \frac{\frac{6}{36}}{\frac{31}{36}} = \frac{6}{36} \times \frac{36}{31} = \frac{6}{31} \\ p(8) = p(8 \mid V') &= \frac{\frac{5}{36}}{\frac{31}{36}} = \frac{5}{36} \times \frac{36}{31} = \frac{5}{31} \\ p(9) = p(9 \mid V') &= \frac{\frac{4}{36}}{\frac{31}{36}} = \frac{4}{36} \times \frac{36}{31} = \frac{4}{31} \\ p(12) = p(12 \mid V') &= \frac{\frac{1}{36}}{\frac{31}{36}} = \frac{1}{36} \times \frac{36}{31} = \frac{1}{31} \end{aligned}$$

which also generates Table 3.2.

Relevant Steps	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**12**
Conditional Probability	$\frac{1}{31}$	$\frac{2}{31}$	$\frac{3}{31}$	$\frac{4}{31}$	$\frac{5}{31}$	$\frac{6}{31}$	$\frac{5}{31}$	$\frac{4}{31}$	$\frac{1}{31}$
$p(c_i)$	$p(c_1)$	$p(c_2)$	$p(c_3)$	$p(c_4)$	$p(c_4)$	$p(c_5)$	$p(c_6)$	$p(c_8)$	$p(c_9)$

Table 3.2: Conditional probability of relevant steps

To evaluate the expected number of rolls to get all coupons $E(T)$, we considered the expected waiting time for a relevant step in C and the expected number of relevant steps to get all coupons. Then we have

$$E(T) = E(T_W \mid V') \times E(T_R \mid V') \tag{3.15}$$

where $E(T_W \mid V')$ is the expected waiting time for a relevant step in C or a value other than 10 or 11 and $E(T_R \mid V')$ is the expected number of relevant steps to get all coupons. The expected waiting time can be expressed as

$$E(T_W \mid V') = \frac{1}{p(V')} \tag{3.16}$$

Therefore,

$$E(T_W \mid V') = \frac{1}{\frac{31}{36}} = 1 \times \frac{36}{31} = \frac{36}{31} = 1.1613$$

To calculate the expected number of relevant steps to get all coupons $E(T_R \mid V')$, recall Equation 3.6, which is presented below

$$E[X] = \sum_{i}^{k} \frac{1}{p_i} - \sum_{i<j} \frac{1}{p_i + p_j} + \sum_{i<j<l} \frac{1}{p_i + p_j + p_l} - \sum_{i<j<l<n} \frac{1}{p_i + p_j + p_l + p_n} + \cdots + (-1)^{k+1}$$

The equation is conceptually straightforward but a bit computationally elaborate as it requires essentially performing inclusion and exclusion of the probability of each relevant step up to the N relevant coupons. However, we converted the formula to a python code that calculates the expected number of relevant rolls to get all coupons, which is a more efficient way. Which is given below.

```
p = [1/31, 2/31, 3/31, 4/31, 5/31, 6/31, 5/31, 4/31, 1/31]

E = 0
for i in range(len(p)):
    E += 1/p[i]
for i in range(len(p)):
    for j in range(i+1, len(p)):
        E -= 1/(p[i] + p[j])
for i in range(len(p)):
    for j in range(i+1, len(p)):
        for k in range(j+1, len(p)):
            E += 1/(p[i] + p[j] + p[k])
for i in range(len(p)):
    for j in range(i+1, len(p)):
        for k in range(j+1, len(p)):
            for l in range(k+1, len(p)):
                E -= 1/(p[i] + p[j] + p[k] + p[l])
for i in range(len(p)):
    for j in range(i+1, len(p)):
        for k in range(j+1, len(p)):
            for l in range(k+1, len(p)):
                for m in range(l+1, len(p)):
                    E += 1/(p[i] + p[j] + p[k] + p[l]+ p[m])
for i in range(len(p)):
    for j in range(i+1, len(p)):
        for k in range(j+1, len(p)):
            for l in range(k+1, len(p)):
                for m in range(l+1, len(p)):
                    for n in range(m+1, len(p)):
                        E -= 1/(p[i] + p[j] + p[k] + p[l]+ p[m]+ p[n])
for i in range(len(p)):
    for j in range(i+1, len(p)):
        for k in range(j+1, len(p)):
            for l in range(k+1, len(p)):
                for m in range(l+1, len(p)):
                    for n in range(m+1, len(p)):
                        for o in range(n+1, len(p)):
                            E += 1/(p[i] + p[j] + p[k] + p[l]+ p[m]+ p[
    n]+ p[o])
for i in range(len(p)):
```

```
    for j in range(i+1, len(p)):
        for k in range(j+1, len(p)):
            for l in range(k+1, len(p)):
                for m in range(l+1, len(p)):
                    for n in range(m+1, len(p)):
                        for o in range(n+1, len(p)):
                            for q in range(o+1, len(p)):
                                E -= 1/(p[i] + p[j] + p[k] + p[l]+ p[m
    ]+ p[n]+ p[o]+p[q])
E += (-1)**(len(p)+1)
print('The expected number of relevant rolls to get all coupons is ', E
    )
```

Listing 3.1: Python code to calculate the expected number of relevant rolls to get all coupons

The expected number of relevant rolls to get all coupons is 50.58586255970185.

Hence from equation 3.15, the expected number of rolls to get all coupons is

$$E(T) = E(T_W \mid V') \times E(T_R \mid V')$$

$$E(T) = 1.1613 \times 50.58586255970185$$

$$E(T) = 58.7453621905818$$

3.4.3 Expected time to collect any subsets of C

In this section, We aim to evaluate the expected time to collect any subsets of C and identify the subset that is most easily attainable. Consider the set C partitioned into m subsets S, each containing five elements. The elements of each subset have unequal probabilities corresponding to the probability of the sum of two dice. To evaluate the expected time to collect any subsets S, we will evaluate the expected number of rolls required to collect each distinct subset and take the minimum among them.

Generalising the expressions in equation 3.10 such that S_i is a unique subset made up of distinct coupons w_{i+j}, where $i = 1, 2, \ldots, m$ and $j = 0, 1, 2, \ldots, k-1$, then we have

$$S_i = \{w_{i+0}, w_{i+1}, \ldots, w_{i+k-1}\}$$

$$S_i = \{w_{i+j}\}_{j=0}^{k-1} \tag{3.17}$$

where S_i is a generalized form for $\{S_1, S_2, \ldots, S_{m-1}, S_m\}$, and $\{w_{i+j}\}_{j=0}^{k-1}$ is a simplified form of $\{c_1, c_2, \ldots, c_h\}$, which are the relevant rolls with unequal probabilities p. The probability p of rolling a distinct coupon w_{i+j} of any subset S_i is obtained by conditioning relevant rolls on the probability of not obtaining irrelevant rolls. That is

$$p(w_{i+j}) = p(w_{i+j} \mid V') \tag{3.18}$$

$$p(w_{i+j} \mid V') = \frac{p(w_{i+j} \cap V')}{p(V')} \tag{3.19}$$

Let T_i be the random time to collect S_i and T be the random time to collect any set, then

$$T = \min(T_1, T_2, T_3, T_4, T_5)$$

Therefore, the expected random time to collect any subset S_i is

$$E(T) \leq \min(E(T_1), E(T_2), E(T_3), E(T_4), E(T_5)) \tag{3.20}$$

Since $T \leq T_i \ \forall i$.

Proof. Consider two arbitrary points x_1 and x_2, and two arbitrary weights w_1 and w_2 such that $w_1 + w_2 = 1$. We then define the convex combination of these two points as

$$z = w_1 x_1 + w_2 x_2$$

Now, if we apply the minimum function to both sides of the equation, we get:

$$\min(z) = \min(w_1 x_1 + w_2 x_2)$$

We then use Jensen's inequality [Jen06] on the right-hand side of the equation, which states that if f is a concave function, then $E(f(X)) \leq f(E(X))$. In this case, the function f is the minimum function, so $E(f(X))$ is equal to $\min(x_1, x_2)$. Applying Jensen's inequality, we get:

$$\min(z) = \min(w_1 x_1 + w_2 x_2) \leq \min(x_1, x_2)$$

This means that $\min(z)$ is less than or equal to the convex combination of x_1 and x_2 (which is exactly what it means for the minimum function to be concave).

Therefore, the function $\min(T_1, T_2, T_3, T_4, T_5)$ is indeed a concave. Hence,

$$E(\min(T_1, T_2, T_3, T_4, T_5)) \leq \min(E(T_1), E(T_2), E(T_3), E(T_4), E(T_5))$$

$\square$

Therefore, the expected number of rolls $E(T_i)$ required to make a complete collection of each distinct subset S_i is expressed below.

$$E(T_i) = E(T_{W_i} \mid V') \times E(T_{R_i} \mid V') \tag{3.21}$$

Where $E(T_{W_i} \mid V')$ is the expected waiting time of relevant steps for any given subset S_i of C and $E(T_{R_i} \mid V')$ denotes the expected number of relevant steps to obtain all coupons of any given subset. The expected waiting time of relevant steps for any given subset S_i is given below

$$E(T_{W_i} \mid V') = \frac{1}{p(w_{i+j} \mid V')} \tag{3.22}$$

and the expected number of relevant steps to obtain all coupons of any given subset S_i can be evaluated using equation 3.6.

Therefore to evaluate the random time T_1 to collect S_1, we have

$$S_1 = \{c_1, c_2, c_3, c_4, c_5\} = \{2, 3, 4, 5, 6\}$$

Using the conditional probability from Table 3.2, the expected waiting time $E(T_{W_i} \mid V')$ for subset S_1 is

$$E(T_{W_1} \mid V') = \frac{1}{\frac{1}{31} + \frac{2}{31} + \frac{3}{31} + \frac{4}{31} + \frac{5}{31}} = \frac{1}{\frac{15}{31}} = 1 \times \frac{31}{15} = \frac{31}{15} = 2.0667$$

To calculate the expected number of relevant steps to get all coupons $E(T_{R_1} \mid V')$ for subset S_1, we applied equation 3.6. As stated in section 3.4.2, the formula is converted to a python code below.

```python
p = [1/31, 2/31, 3/31, 4/31, 5/31]

E = 0
for i in range(len(p)):
    E += 1/p[i]
for i in range(len(p)):
    for j in range(i+1, len(p)):
        E -= 1/(p[i] + p[j])
for i in range(len(p)):
    for j in range(i+1, len(p)):
        for k in range(j+1, len(p)):
            E += 1/(p[i] + p[j] + p[k])
for i in range(len(p)):
    for j in range(i+1, len(p)):
        for k in range(j+1, len(p)):
            for l in range(k+1, len(p)):
                E -= 1/(p[i] + p[j] + p[k] + p[l])
E += (-1)**(len(p)+1)
print('The expected number of relevant rolls to get all coupons of S1
    is ', E)
```

Listing 3.2: Python code to calculate the expected number of relevant rolls to get all coupons of Subset S_1

The expected number of relevant rolls to get all coupons of S_1 is 37.52530525030525. Hence, from equation 3.21, we evaluate the expected number of rolls $E(T_i)$ to get all coupons of subset S_1 as

$$E(T_1) = E(T_{W_1} \mid V') \times E(T_{R_1} \mid V')$$

$$E(T_1) = 2.0667 \times 37.52530525030525$$

$$E(T_1) = 77.5535483608059$$

Using the above steps we generate the values in Table 3.3.

Subsets	S_1	S_2	S_3	S_4	S_5
elements w_{i+j}	$\{c_1,c_2,c_3,c_4,c_5\}$	$\{c_2,c_3,c_4,c_5,c_6\}$	$\{c_3,c_4,c_5,c_6,c_7\}$	$\{c_4,c_5,c_6,c_7,c_8\}$	$\{c_5,c_6,c_7,c_8,c_9\}$
element values	{2,3,4,5,6}	{3,4,5,6,7}	{4,5,6,7,8}	{5,6,7,8,9}	{6,7,8,9,12}
$p(w_{i+j})$	$\frac{1}{31}, \frac{2}{31}, \frac{3}{31}, \frac{4}{31}, \frac{5}{31}$	$\frac{2}{31}, \frac{3}{31}, \frac{4}{31}, \frac{5}{31}, \frac{6}{31}$	$\frac{3}{31}, \frac{4}{31}, \frac{5}{31}, \frac{6}{31}, \frac{5}{31}$	$\frac{4}{31}, \frac{5}{31}, \frac{6}{31}, \frac{5}{31}, \frac{4}{31}$	$\frac{5}{31}, \frac{6}{31}, \frac{5}{31}, \frac{4}{31}, \frac{1}{31}$
$E(T_{W_i} \mid V')$	2.0667	1.55	1.3478	1.2916	1.4762
$E(T_{R_i} \mid V')$	37.5253	21.8026	16.5018	15.0084	33.6025
$E(T_i)$	77.5535	33.7940	22.2411	19.3848	49.6040

Table 3.3: Expected time to get all coupons of all subsets

Hence, from the value of $E(T_i)$ in table 3.3, we evaluate the expected random time $E(T)$ to collect any subset S_i using the formula in equation 3.20

$$E(T) \leq \min(E(T_1), E(T_2), E(T_3), E(T_4), E(T_5))$$

$$E(T) \leq \min(77.5535, 33.7940, 22.2411, 19.3848, 49.6040)$$

$$E(T) \leq 19.3848$$

3.5 Simulation

In this section, we discuss the methods used to carry out the simulation for the one-player game with simplified rules and with original rules, and the two-player game with simplified rules and with original rules. Python version 3.10 was used to carry out the simulation. The code was written in Jupyter Notebook, a web-based environment for writing Python programs. The following standard Python 3.10 libraries were used.

- Pygame: This library is an open-source module developed for the Python programming language, with the aim of providing the tools necessary to create games and other multimedia applications. The pygame module was used to control some of the logic and the whole graphics of the sequence dice game.

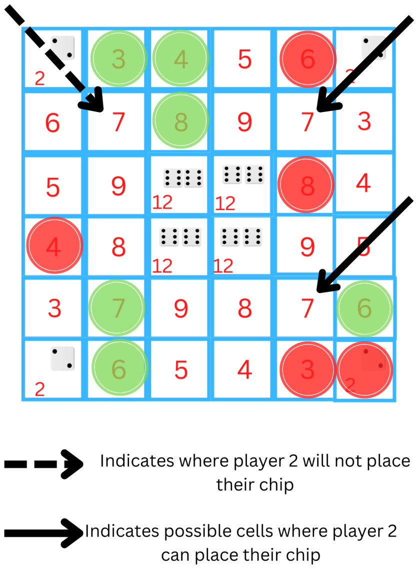
- Numpy: This python library was used to access and manipulate the sequence dice game, which was represented as a 1×9 one-dimensional board and a 6×6 full board.
- Random: This library was used to simulate the randomisation of the two dice rolls and the placement of chips randomly on the board by players.
- Panda: It was used to store the data generated from the simulation in a table and as well used for the analysis of the result.
- Itertools: This is a standard python library for creating iterations for efficient looping. The product function of the library was used to generate different combinations of strategies in a two-player game with no rules and to generate combinations of strategies and decisions in a two-player game with rules.
- Matplotlib.pyplot: This library was used for the visualisation of the results of the 1000 iterations in form of graphs and charts.
- Scipy.stats: This library contains numerous statistical functions. These functions are used for statistical analysis of the result. We used the norm function to verify the distribution of the result.

3.5.1 Strategy

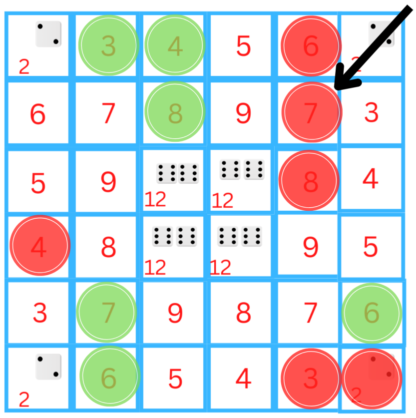
In this study, we proposed three strategies for the full board version of the sequence dice game. These strategies aimed to determine which of the four cells to place chips in when the dice are rolled. Since each number only appears once on the one-dimensional version of the board and only one placement is possible for each roll of the dice, no strategies are applicable to this version of the game. We explain the proposed three strategies for the full board version of the sequence dice game below.

3.5.1.1 Offensive Strategy

This strategy is formulated to enable a player to focus on building a sequence of 5 chips in a row in order to win the game, without considering the opponent's moves. When a player rolls the dice, they attempt to place their chip on a row, column, or diagonal of the board that contains most of their chips, as well as the value of the dice rolled that is not covered by a chip. If this is not feasible, the player then attempts to place their chip on a row, column, or diagonal that contains two or more of their chips and the value of the dice rolled uncovered by a chip. If this is also not possible, the player places their chip randomly.



(a) State of the board before dice roll



(b) State of the board after placing the chip

Figure 3.3: Offensive Strategy

Figure 3.3a shows the state of the board when player two is using a red chip to play an offensive strategy. As player two has rolled a 7, they have the option to place their chip on any of the three 7s uncovered by a chip on the board, however, they may only place their chip in the fourth column, containing the most number of red chips (3) and two 7s that are not covered by a chip. As such, player two can only place their chip on any of the 7 spaces indicated by a solid arrow and not the space indicated by a broken arrow. Figure 3.3b shows where player two finally places their chip using offensive strategy. Algorithm 1 below shows the proposed offensive strategy

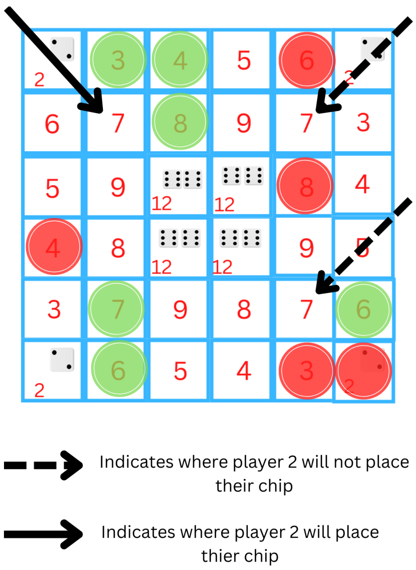
Algorithm 1: Offensive Strategy

```
for row, column, diagonal in board do
    if row.count(own_chips) ≥ column.count(own_chips) and row.count(own_chips) ≥
      diagonal.count(own_chips) and dice_roll in row then
        row[dice_roll] ← own_chip ;       /* Player places chip in the row with the most
          player chips */
        break
    else if column.count(own_chips) ≥ row.count(own_chips) and column.count(own_chips) ≥
      diagonal.count(own_chips) and dice_roll in column then
        column[dice_roll] ← own_chip ;      /* Player places chip in the column with the
          most player chips */
        break
    else if diagonal.count(own_chips) ≥ row.count(own_chips) and diagonal.count(own_chips) ≥
      column.count(own_chips) and dice_roll in diagonal then
        diagonal[dice_roll] ← own_chip ;  /* Player places chip in the diagonal with the
          most player chips */
        break
    else if row.count(own_chips) ≥ 2 and dice_roll in row then
        row[dice_roll] ← own_chip ;      /* Player places chip in the row that has two or
          more player chips */
        break
    else if column.count(own_chips) ≥ 2 and dice_roll in column then
        column[dice_roll] ← own_chip ; /* Player places chip in the column that has two
          or more player chips */
        break
    else if diagonal.count(own_chips) ≥ 2 and dice_roll in diagonal then
        diagonal[dice_roll] ← own_chip ;  /* Player places chip in the diagonal that has
          two or more player chips */
        break
    else
        random_strategy();
        break
    end
end
```

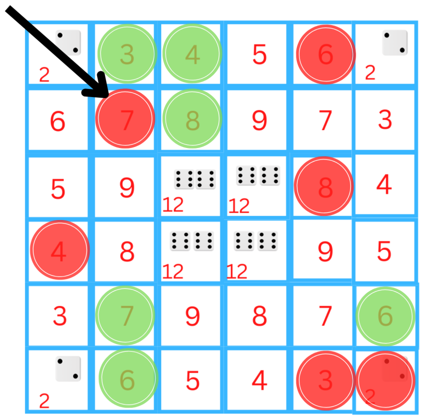
3.5.1.2 Defensive Strategy

In this strategy, the player focuses on preventing the opponent from achieving a sequence of five chips in a row to win the game, while hoping to win the game with the same strategy. Upon rolling the dice, the player attempts to place their chip on the row, column, or diagonal of the board containing the most opponent chips that also contain the value of the dice rolled, uncovered by a chip. If this is not possible, the player attempts to place their chip on any row, column, or diagonal with two or more opponent chips that also contain the value of the dice rolled, uncovered by a chip. If this is not possible, the chip is placed randomly.

Figure 3.4a illustrates the board state when player two is employing a defensive strategy with red chips. Assuming a roll of 7, player two has the potential to place their chip on any of the three 7s uncovered by a chip on the board, but due to their chosen strategy, they must place it in the 2nd column, which contains the most number of green chips (3) and one 7 that is not yet covered. As shown in figure 3.4a, the chip can only be placed on the 7 spaces indicated by a solid arrow, and not the spaces indicated by a broken arrow. Figure 3.4b shows the final placement of the chip using the defensive strategy. Algorithm 2 outlines the proposed defensive strategy.



(a) State of the board before dice roll



(b) State of the board after placing the chip

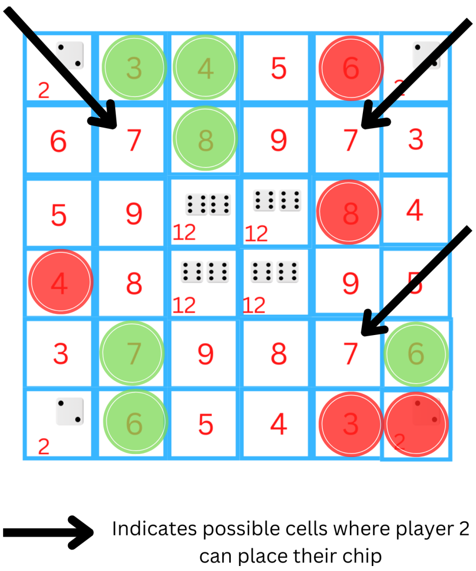
Figure 3.4: Defensive Strategy

Algorithm 2: Defensive Strategy

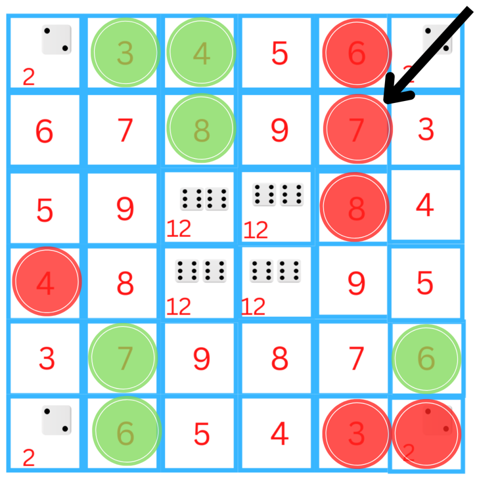
```
for row, column, diagonal in board do
    if row.count(opponent_chips) ≥ column.count(opponent˙chips) and
     row.count(opponent_chips) ≥ diagonal.count(opponent_chips) and dice_roll in
     row then
        row[dice_roll] ← own_chip ;     /* Player places chip in the row with
         the most opponent's chips */
        break
    else if column.count(opponent_chips) ≥ row.count(opponent_chips) and
     column.count(opponent_chips) ≥ diagonal.count(opponent_chips) and dice_roll
     in column then
        column[dice_roll] ← own_chip ;    /* Player places chip in the column
         with the most opponent's chips */
        break
    else if diagonal.count(opponent_chips) ≥ row.count(opponent_chips) and
     diagonal.count(opponent_chips) ≥ column.count(opponent_chips) and dice_roll
     in diagonal then
        diagonal[dice_roll] ← own_chip ;          /* Player places chip in the
         diagonal with the most opponent's chips */
        break
    else if row.count(opponent_chips) ≥ 2 and dice_roll in row then
        row[dice_roll] ← own_chip ;     /* Player places chip in the row that
         has two or more opponent's chips */
        break
    else if column.count(opponent_chips) ≥ 2 and dice_roll in column then
        column[dice_roll] ← own_chip ;    /* Player places chip in the column
         that has two or more opponent's chips */
        break
    else if diagonal.count(opponent_chips) ≥ 2 and dice_roll in diagonal then
        diagonal[dice_roll] ← own_chip ;          /* Player places chip in the
         diagonal that has two or more opponent's chips */
        break
    else
        random_strategy();
        break
    end
end
```

3.5.1.3 Random Strategy

In this strategy, players randomly place their chip in any row, column, or diagonal of the board containing the dice roll. As an example, Figure 3.5a shows the state of the board when player two has used red chips to play a random strategy. If player two rolled a 7, they could place their chip on any of the three 7s on the board, indicated by a solid arrow. Figure 3.5b shows where player two finally places their chip using a random strategy.



(a) State of the board before dice roll



(b) State of the board after placing the chip

Figure 3.5: Random Strategy

ALgorithm alg:three outlines the proposed random strategy

Algorithm 3: Random Strategy

---

if $dice\_roll \notin \{10, 11\}$ **then**

  choose *index* randomly from *dice_roll_in_board* ;

  place *own_chip* in *board*[*index*] ; `/* Player places their chip randomly on the board */`

end

3.5.2 Decision for Rules 10 and 11

When playing the sequence dice game, players must abide by the rules stated in Chapter 1. This includes removing one of their opponent's chips from the board, except for cells 2 and 12, upon rolling a 10. Additionally, a roll of 11 requires the player to place their chip on the board. In order to aid players in making decisions regarding these rules, we introduced three different types of decisions for rules 10 and 11, applied to both one-dimensional and full board variations of the game.

3.5.2.1 Lowest Probability Decision for Rule 10

For a decision of the type rule 10, a player opts to remove their opponent's chip from the cell with the lowest probability. As illustrated in Figure 3.6, when player two is using the red chips and rolls a 10, they cannot remove the green chip in cell 2 which has the lowest probability, in accordance with the rules. Instead, they must remove the green chip from cell 3, which has the next lowest probability. Algorithm 4 explained the

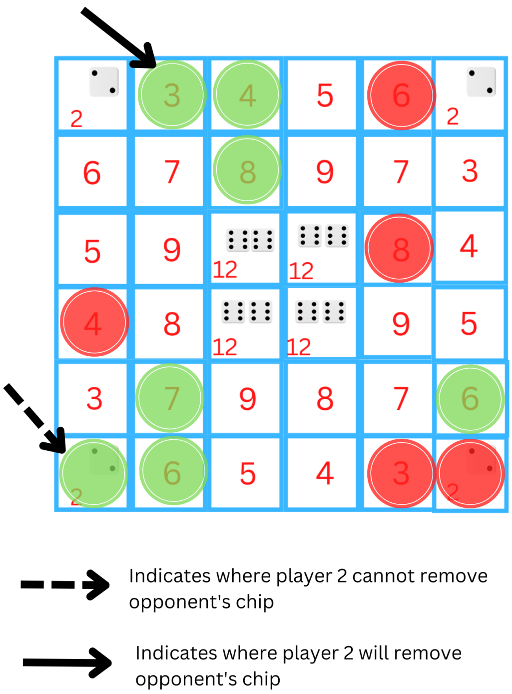


Figure 3.6: Lowest Probability Decision for Rule 10

proposed lowest probability decision for rule 10.

Algorithm 4: Lowest Probability Decision for Rule of 10

if *dice_roll* $= 10$ *and opponente_chips on board* **then**
  for *each opponent_chip p on board* **do**
    $p \leftarrow prob(p)$;
  end
  Except where $p = 2$ and $p = 12$;
  $index \leftarrow min(p)$;
  $board[index].remove(chip)$;
  `/* Remove opponent's chip from the cell with the lowest probability */`
end

3.5.2.2 Highest Probability Decision for Rule 10

For this type of rule 10 decision, a player opts to remove their opponent's chip from the cell that has the highest probability. In particular, figure 3.7 shows the state of the board when player two, using the red chips, rolls a 10. The opponent's (green) chips cover cells with a range of probabilities, but cell 6 has the highest probability. Therefore, player two will remove the opponent's chip from one of the cells 6. Algorithm 5 describes the

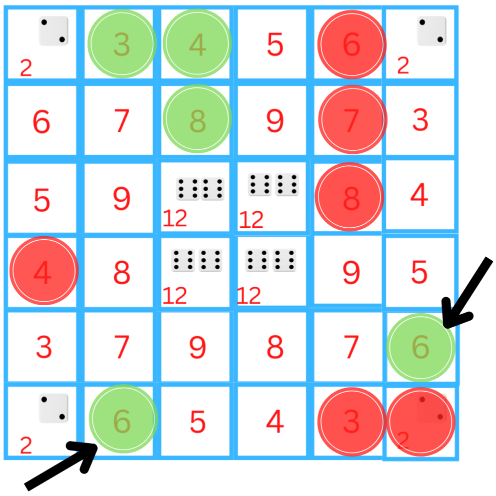


Figure 3.7: Highest Probability Decision for Rule 10

proposed highest probability decision for rule 10.

Algorithm 5: Highest Probability Decision for Rule of 10

if *dice_roll* $= 10$ *and opponent_chip on board* **then**
  for *each opponent_chip p on board* **do**
    $p \leftarrow prob(p)$;
  end
  Except where $p = 2$ and $p = 12$;
  $index \leftarrow max(p)$;
  $board[index].remove(chip)$;
  `/* Remove opponent's chip from the cell with the highest probability */`
end

3.5.2.3 Random Decision for Rule 10

This decision process is such that players randomly remove an opponent's chip from the board when a 10 is rolled. Figure 3.8 shows the state of the board when player two rolled a 10. In accordance with the rules, they had the possibility of removing any of the opponent's chips from the board, with the exception of cell 2. Consequently, player two chose to remove their opponent's chip from cell 4.

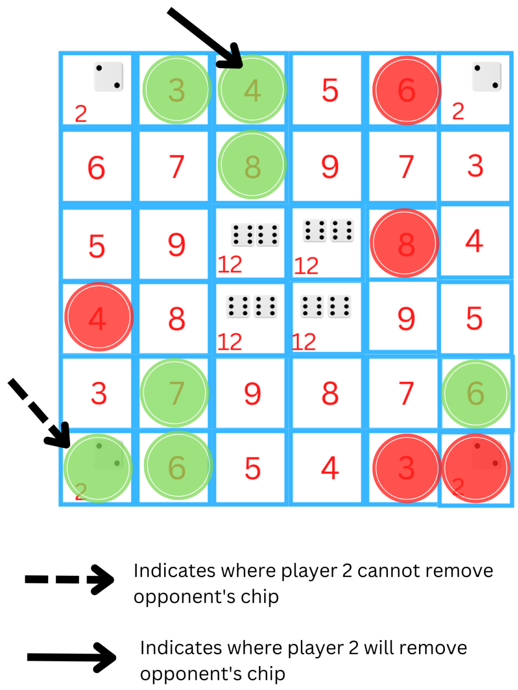


Figure 3.8: Random Decision for Rule 10

Algorithm 6 expounds on the proposed random decision for rule 10.

Algorithm 6: Random Decision for Rule of 10

```
if dice_roll = 10 then
    if opponent_chip p on board then
        Except where p = 2 and p = 12;
        index ← random.choice(opponent_chips);
        board[index].remove(chip) ;        /* Player removes opponent's chip
         randomly from the board */
    end
end
```

3.5.2.4 Lowest Probability Decision for Rule 11

For this type of rule 11 decision, the player opts to place their chip on the cell with the lowest probability, as illustrated in Figure 3.9. Here, player two has chosen to place their chip on cell 2, which has the lowest probability, indicated by the solid arrow. Algorithm 7 explained the proposed lowest probability decision for rule 11.

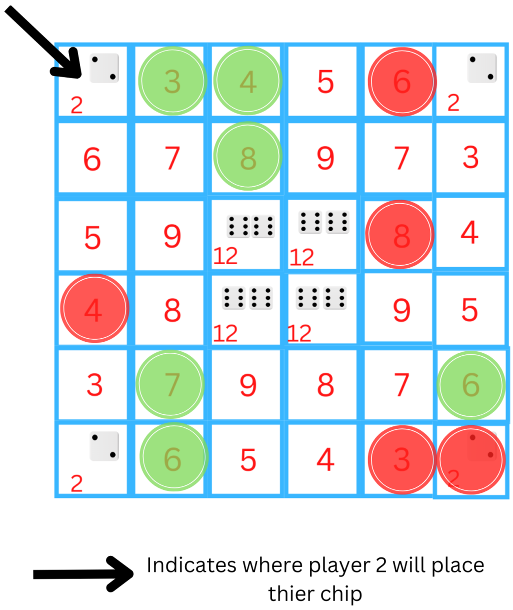


Figure 3.9: Lowest Probability Decision for Rule 11

Algorithm 7: Lowest Probability Decision for Rule of 11

if *dice_roll = 11 and cell is not covered by chip on board* **then**
  for *each uncovered cell on board* **do**
    Compute probabilities of each uncovered cell on the board ;
    $index \leftarrow min(probability\_of\_uncovered\_cell\_on\_board)$;
  end
  $board[index] \leftarrow own\_chip$; `/* Player places their chip on a cell with the lowest probability */`
end

3.5.2.5 Highest Probability Decision for Rule 11

In this instance of a rule 11 decision, the player chose to place their chip on the cell with the highest probability. Figure 3.10 shows the board state when player two, using the red chips, rolled an 11. Player two placed their chip on cell 7, which had the highest probability. This is indicated by the solid arrow. Algorithm 8 explained the proposed

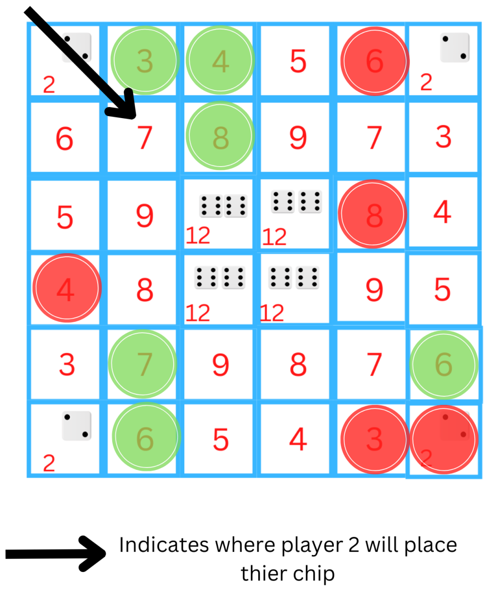


Figure 3.10: Highest Probability Decision for Rule 11

highest probability decision for rule 11.

Algorithm 8: Highest Probability Decision for Rule of 11

if *dice_roll = 11 and cell is not covered by chip on board* **then**
  for *each uncovered cell on board* **do**
    Compute probabilities of each uncovered cell on the board ;
    $index \leftarrow max(probability\_of\_uncovered\_cell\_on\_board)$;
  end
  $board[index] \leftarrow own\_chip$; `/* Player places their chip on a cell with the highest probability */`
end

3.5.2.6 Random Decision for Rule 11

For this type of rule 11 decision, a player chooses to place their chip randomly on the board. Figure 3.11 shows the state of the board when player two is using the red chips and rolled an 11. Player two chose to place their chips randomly on cell 9. This is indicated with the solid arrow.

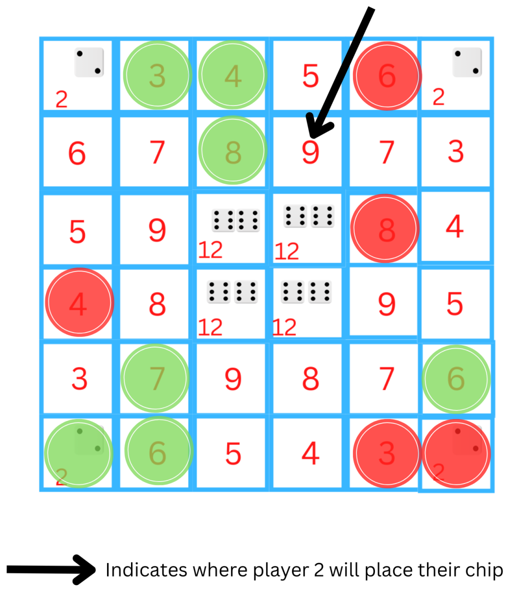


Figure 3.11: Random Decision for Rule 11

Algorithm 9 explained the proposed random decision for rule 11.

Algorithm 9: Random Decision for rule of 11

```
if dice_roll = 11 and cell is not covered by chip on board then
    index ← random.choice(uncovered_cell_on_board);
    board[index] ← own_chip ;  /* Player places their chip randomly on the
     board */
end
```

3.5.3 One-Dimension Sequence Dice Game Board

In order to investigate research questions 1 and 2, we designed the variations of the sequence dice game board as a one-dimension. The major specifications and assumptions considered for this variation are:

- The one-dimensional sequence dice game board was examined, with its discrete calibrations ranging from 2 to 9 and 12, as depicted in Figure 3.2. The numbers 2-9 and 12 are the sums of possible dice faces represented by the game board, while 10 and 11 denote specific actions related to the game.
- We proposed that the game would result in a draw for this variation if one of the two players had exhausted all their chips, as it is not specified in the rules how the game should end in this scenario.
- The number of simulations is 1000.
- Players play with 20 chips each.
- No chip placement strategies were implemented due to each number appearing only once on the board.
- The expected number of rolls to win the game is obtained by evaluating the average count of the number of dice rolls till the game is won.

Thus, we explored the variations of the one-dimensional board game below:

3.5.3.1 One Player With Simplified Rules

The one-dimensional dice game was simulated for a single-player scenario, wherein the rules of the sequence dice game were not observed. Subsequently, the average number of dice rolls needed to secure a win was evaluated.

3.5.3.2 One Player With Original Rules

A single-player simulation of the sequence dice game was conducted, wherein the rules of the game were strictly adhered to. Since the fact that each number appears only once on the one-dimensional board, no placement strategies were implemented; however, decisions for rules 10 and 11 were applied.

3.5.3.3 Two Players With Simplified Rules

In this variation of the game, two players played the game with no strategies and without following the rules of the game. Subsequently, we then evaluated the average length to win the game.

3.5.3.4 Two Players With Original Rules

We also simulated the game when two players played the game and followed the rules of the game. Hence, the decisions for rules 10 and 11 were applied. We then evaluated the average number of dice rolls required to win the game in this scenario.

3.5.4 Standard or Full Board Sequence Dice Game

To investigate research questions 3 and 4, we explored the sequence dice game board variations with a full board. Below are the general specifications and assumptions of the variations:

- The game board is the standard sequence dice game dimension, represented as a 6×6 array with discrete calibrations from 2 to 9, then 12, excluding 10 and 11 as shown in figure 1.1 Where numbers $2 \ldots 9$ and 12 are the sums of possible dice faces represented by the game board. Each number appears four times on the board randomly. Die faces 10 and 11 represent certain actions to be carried out as part of the rules of the game.
- We proposed that the sequence dice game would result in a draw for this variation if one of the two players had exhausted all their chips, as it is not specified in the rules how the game should end in this scenario.

- The number of simulations is 1000.
- Players played with 20 chips each.
- Players used offensive, defensive and random strategies for chip placement.
- The expected number of rolls to win the game is obtained by evaluating the average count of the number of dice rolls till the game is won.

Thus, we explored the variations of the number of players and full board below:

3.5.4.1 One Player With Simplified Rules

This is a variation where only one player plays the game with a full board without adhering to the rules of the sequence dice game.

3.5.4.2 One Player With Original Rules

We simulated the game where one player plays with the full board by following the rules of the sequence dice game.

3.5.4.3 Two Player With Simplified Rules

We also simulated the game to evaluate the length of the game when two players play with a full board without following the rules of the sequence dice game.

3.5.4.4 Two Player With Original Rules

This game variation is considered the exact simulation of the official sequence dice game, as it involves two players playing with a full board while adhering to the game's rules.

4 Results

This chapter presents the findings of the experiment of the one-dimensional board game and the standard full board of the game, with variations in the number of players and application of the rules of the game. The results of the experiment will be discussed in detail.

4.1 One-Dimensional Board Version

This section describes the results of the one-dimensional board version, including any patterns observed and any differences between the one-player and two-player versions when the rules of the game are applied and when the rules are not applied.

4.1.1 One Player With Simplified Rules

This experiment was designed such that one player played the one-dimensional board version of the sequence dice game without adhering to the established rules. Specifically, when the player rolled a 10 or 11, they did not take any action and when a 2 or 12 was rolled, no additional roll was taken. All other rules of the game were also disregarded. The simulation was run for 1000 iterations, and the results were recorded.

	Metrics	Value
1	Average Dice Rolls	14.29
2	Average Skipped Turns	7.87
3	Number of Wins	1000.00
4	Game Overs	0.00

Figure 4.1: Summary result of the one-player with simplified rules on a one-dimensional board

Table 4.1 provides information on the result of the experiment, which includes the average game length, average skipped turn, number of wins, and number of game overs. For 1000 games played, the average game length is 14.29, indicating that the game typically takes just over 14 rolls on average to complete. The average skipped turn is 7.87, meaning that players typically skip around 8 turns in each game.

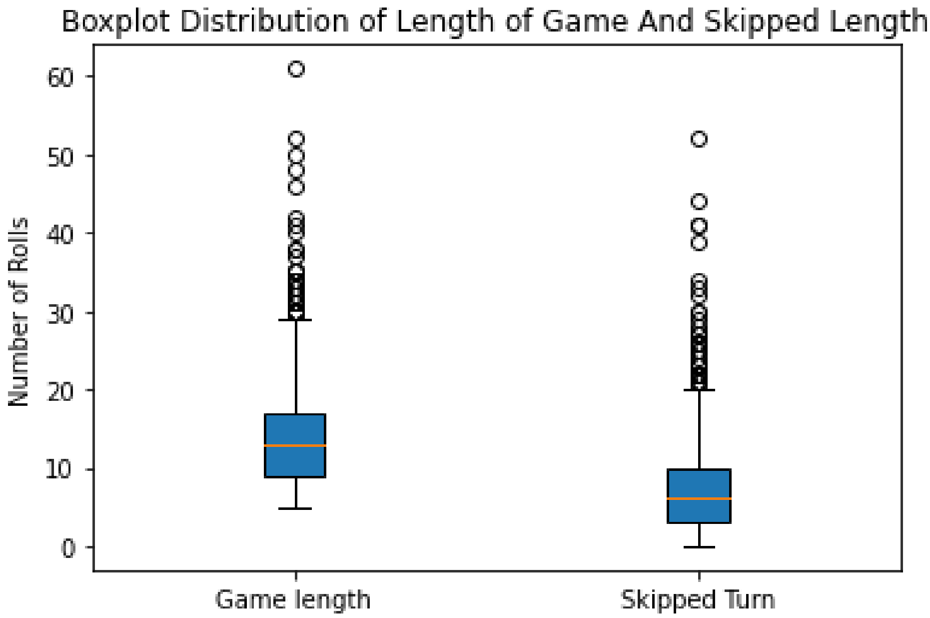
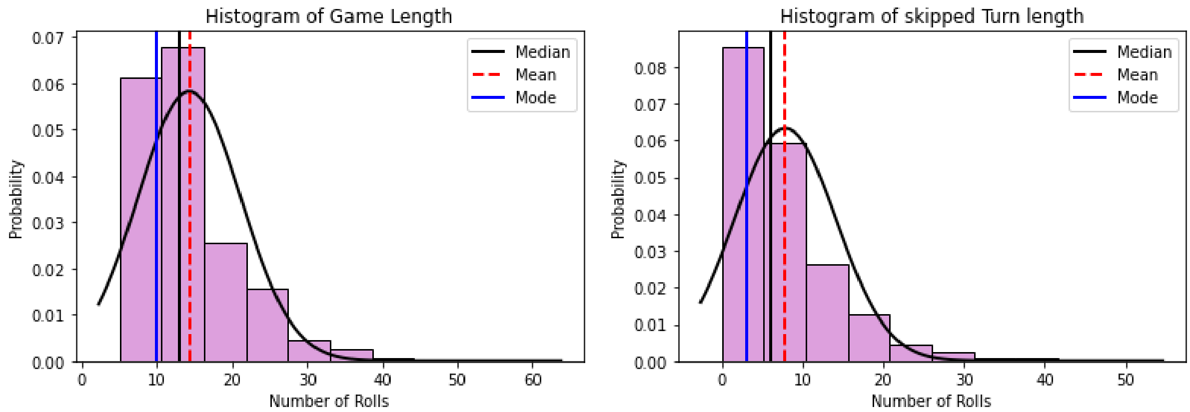


Figure 4.2: Distribution of length of games and skipped turns length

The boxplots in Figure 4.2 display the distribution of game length and skipped turns. It appears that both boxplots are positively skewed. The most common game lengths are concentrated around 9 and 18 rolls, which are the first and third quartiles respectively. It is evident from the box plot that the minimum number of dice rolls required to win this particular variation of the game is 5, and the maximum number of rolls is 61. The skipped turns are mostly concentrated around 2 and 8, which are the lower and upper quartiles respectively. It is also observed from the plot that there can be no skipped turns in a game, leading to the minimum amount of rolls required to obtain a sequence of at least five chips. Moreover, the maximum number of skipped turns is 51, resulting in the maximum number of dice rolls required to win the game. This result shows that the number of skipped turns had a direct effect on the duration of the game.



(a) Histogram of average game length

(b) Histogram of average skipped turns

Figure 4.3: One-Player With Simplified Rules - One Dimensional Board

The histograms in Figure 4.3 also illustrate the distribution of the game length and the skipped turns for the 1000 simulations conducted. The mode for game length was observed to be 10 rolls, with a median and mean between 13 and 14 rolls. The average skipped turn length had a mean and median between 6 and 8, with the mode around 5. Thus, it is likely that in an average game, there will be approximately 8 turns with no action taken on the game board. The result indicates that the data is not normally distributed as it is widely spread out. The mean and median are quite close, however, indicating that the data is symmetrical. The large number of 10s in the data set also suggests that the data is skewed toward the higher values.

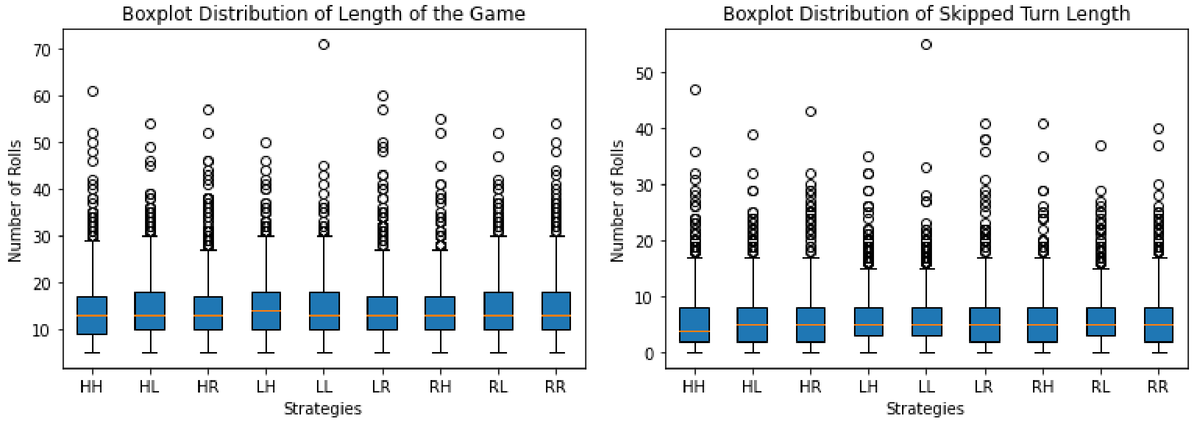
4.1.2 One Player With Original Rules

This experiment was designed to evaluate the effects of some of the established rules when one player is playing with the one-dimensional board version of the sequence dice game. When a 10 was rolled, the player removed their chip from the board according to the decision rules in play except from cells 2 and 12. When an 11 was rolled, the player placed their chip on the board according to the decision rules in play. A roll of 2 or 12 resulted in no action being taken since these rolls only affect the outcome when played by two players. All other rules of the game were disregarded. The simulation was run for 1000 iterations, and the results were recorded.

	Decisions	Average Dice Rolls	Average Skipped Turns	Number of Wins	Game Overs
1	Highest-Highest	14.29	5.86	1000	0
2	Highest-Lowest	14.30	5.90	999	1
3	Highest-Random	14.30	5.85	1000	0
4	Lowest-Highest	14.43	6.23	1000	0
5	Lowest-Lowest	14.47	6.07	1000	0
6	Lowest-Random	14.43	5.78	1000	0
7	Random-Highest	14.37	5.66	1000	0
8	Random-Lowest	14.37	5.90	1000	0
9	Random-Random	14.39	6.02	999	1

Figure 4.4: Summary result of one player with rules on a one-dimensional board.

Table 4.4 shows the result of the simulation when one player played with different decision combinations for rules 10 and 11. The decisions are divided into three categories: Highest, Lowest, and Random. The results show that the average game length, average skipped turn, number of wins, and game overs varied depending on the decisions adopted by the player. The average game length for each decision combination was quite similar, ranging from 14.29 to 14.47 rolls. The average number of skipped turns also saw a relatively small difference, ranging from 5.66 to 6.23 turns. The game ended in a win most of the time, which means the player didn't exhaust their chips before they recorded a win. Overall, it appears that using the Highest decision for both rules 10 and 11 results in the shortest game length, the least amount of skipped turns, and a relatively high number of wins. The Highest-Highest decision had an average game length of 14.29 rolls and an average skipped turn of 5.86. The Lowest-Lowest strategy was the least successful, as it had the highest number of average game lengths (14.47 rolls). This implies that the kind of decisions deployed by the player for both rules 10 and 11 have an impact on the length of the game.



(a) Boxplot of the average game length (b) Boxplot of skipped turns length

Figure 4.5: Distribution of length of games and skipped turns of all the rule decisions

Figure 4.15 above displays the distribution of the number of rolls and skipped turns when different decision rules are applied. In general, it can be concluded that the most common game lengths for all decision combinations were between 10 and 19 rolls, the most common skipped turns were between 2 and 8 and the minimum game length is 5 rolls with the corresponding minimum skipped turns of 0. The maximum game lengths and maximum skipped turns varied depending on the decision combination, with the Highest-Highest decision combination having the highest maximum game length and skipped turns, while the Lowest-Lowest decision combination had the lowest maximum game length and skipped turns.

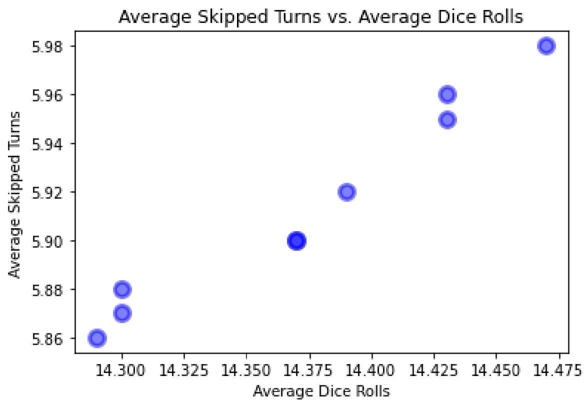


Figure 4.6: Relationship between the average game length and the average skipped turns

The scatter plot in Figure 4.6 shows the relationship between the average game length and the average skipped turns with respect to the different decision types. It appears that there is a slight positive correlation between the two variables: as the average game length increases, the average skipped turns also increase. However, the correlation is not very strong, as there are points on the graph with similar average game lengths that have different average skipped turns.

The histograms in Figure 4.7 present the distribution of game length and skipped turns across all nine combinations of decision rules for 1000 simulations. The mode of game length across all combinations is 20 rolls and the mean rolls is 13. The median game length is found to be between 12 and 13, indicating a symmetric distribution of the data around the mean and median values.

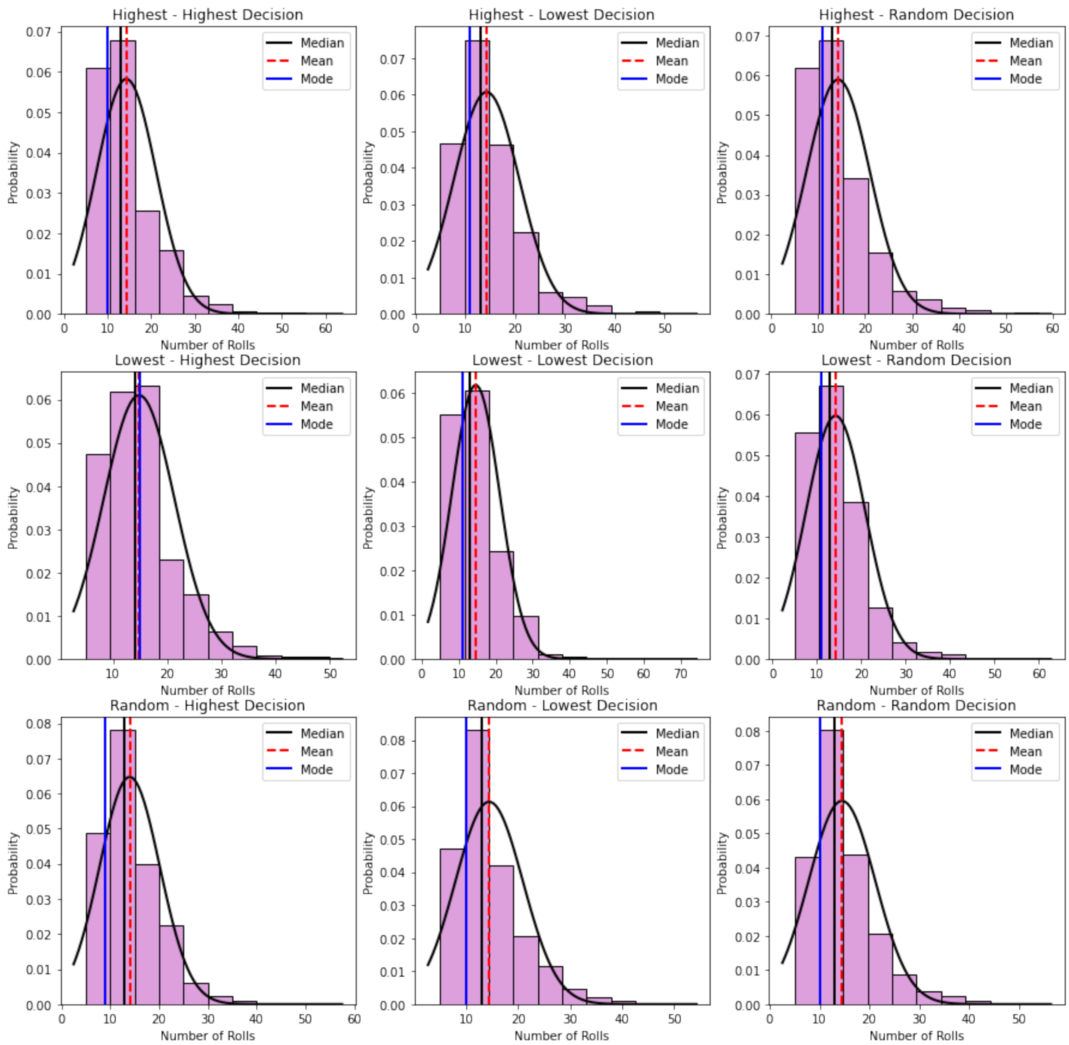


Figure 4.7: Distribution of length of games

4.1.3 Two Players With Simplified Rules

One of the objectives of this study is to evaluate the number of turns required to win when two players play the one-dimensional board of the sequence dice game with simplified rules of the game. To fulfil this objective, the game was simulated 1000 times. However, upon execution, it was observed that the game ended in a deadlock. This occurred as the players did not apply the original rules of the game and thus could not remove their opponent's chips when they rolled a 10 or a number already covered by the opponent's chips (they simply skipped turns). Additionally, since each number appears once on the one-dimensional board, players could not place their chips elsewhere when they rolled the same number. This deadlock indicates that two players cannot play the one-dimensional board of the Sequence Dice game without following the game rules, highlighting the importance of the game rules.

4.1.4 Two Players With Original Rules

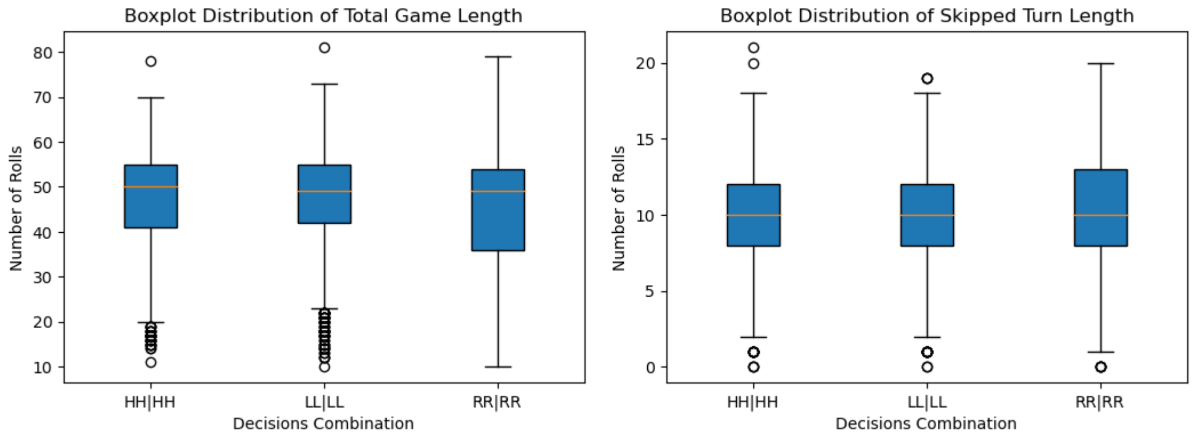
This experiment was designed to assess the effects of different rule decisions for rules 10 and 11 when two players compete in the one-dimensional board version of the sequence dice game. Whenever a 10 was rolled, the player removed their opponent's chip from the board based on the applicable rule decision, with the exceptions of cells 2 and 12. Conversely, when an 11 was rolled, the player placed their chip on the board in accordance with the rule decision. The rule decisions varied from highest probability, lowest probability, and random. When a 2 or 12 was rolled, the player had to take another turn before passing the turn to the opponent. After running the experiment for 1000 iterations, the results were recorded.

- **Results when both players used the same decisions for rules 10 and 11**:

	Decisions	Average game length	Average player one game length	Average player two game length	Player one win length	Player one % of wins	Player two win length	Player two % of wins	Player one skipped turn length	Player two skipped turn length	Average skipped turn	Draw length	% of draws
1	Highest - Highest|Highest - Highest	46.74	23.66	23.08	191	19.1	185	18.5	7050	6858	13.91	624	62.4
2	Lowest - Lowest|Lowest - Lowest	46.44	23.49	22.95	167	16.7	193	19.3	7103	6901	14.00	640	64.0
3	Random - Random|Random - Random	46.43	23.49	22.94	216	21.6	216	21.6	6760	6661	13.42	568	56.8

Figure 4.8: Summary when both players used the same decisions for rule 10 and different decisions for rule 11

The Highest-Highest|Highest-Highest and Lowest-Lowest|Lowest-Lowest gameplays had similar average game lengths, with both players having an average skipped turn length of 13.91 and 14, respectively. The win percentages for both players in both game types were also similar. For the Random-Random|Random-Random, the average game length was slightly lower, and the win percentages were higher, but the average skipped turn length was lower.



(a) Boxplot of the average game length (b) Boxplot of skipped turns length

Figure 4.9: Distribution of length of games and skipped turns when both players used the same decisions for rules 10 and 11

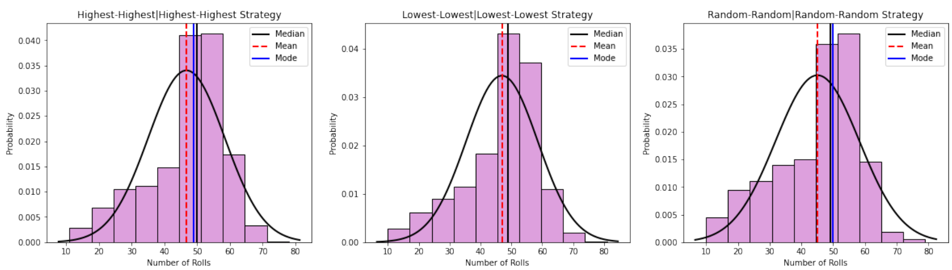


Figure 4.10: Histogram showing the distribution of the game length when both players used the same decisions for rules 10 and 11

Figure 4.9a demonstrates that the distributions of Highest-Highest|Highest-Highest and Random-Random|Random-Random gameplays are positively skewed, while the distribution of Lowest-Lowest|Lowest-Lowest is symmetrically skewed, with the minimum roll being 10 and the maximum roll being 81. The common game length was found to be between 41 and 55 rolls, with the median roll being 50.

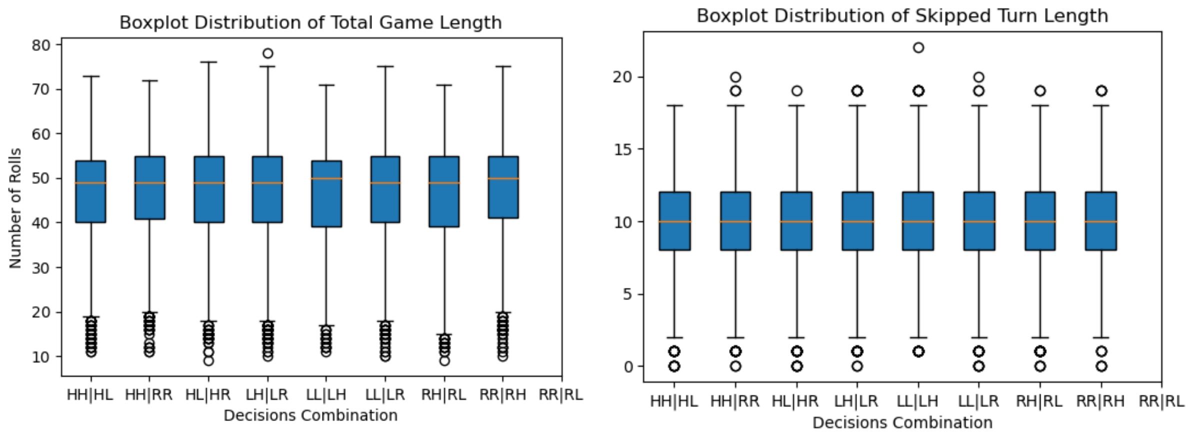
Additionally, Figure 4.9b shows that the distribution of skipped turns was similar to that of game length, indicating a positive relationship between game length and skipped turns for all gameplays.

- **Results when both players used the same decisions for rule 10 and different decisions for rule 11**:

Figure 4.11: Summary result when both players used the same decisions for rule 10 and different decisions for rule 11

	Decisions	Average game length	Average player one game length	Average player two game length	Player one win length	Player one % of wins	Player two win length	Player two % of wins	Player one skipped turn length	Player two skipped turn length	Average skipped turn	Draw length	% of draws
1	Highest - Highest|Highest - Lowest	46.35	23.47	22.89	204	20.4	205	20.5	6913	6753	13.67	591	59.1
2	Highest - Highest|Highest - Random	46.38	23.48	22.90	181	18.1	210	21.0	7068	6717	13.79	609	60.9
3	Highest - Lowest|Highest - Random	46.49	23.52	22.97	186	18.6	209	20.9	6883	6718	13.60	605	60.5
4	Lowest - Highest|Lowest - Random	46.42	23.48	22.94	212	21.2	203	20.3	7036	6776	13.81	585	58.5
5	Lowest - Lowest|Lowest - Highest	46.42	23.48	22.94	208	20.8	205	20.5	6969	6841	13.81	587	58.7
6	Lowest - Lowest|Lowest - Random	46.43	23.48	22.95	196	19.6	191	19.1	6887	6750	13.64	613	61.3
7	Random - Highest|Random - Lowest	46.46	23.50	22.96	179	17.9	186	18.6	7079	6916	13.99	635	63.5
8	Random - Random|Random - Highest	46.44	23.49	22.95	198	19.8	193	19.3	6951	6685	13.64	609	60.9
9	Random - Random|Random - Lowest	46.45	23.50	22.95	203	20.3	169	16.9	7218	7079	14.30	628	62.8

The results presented in Figure 4.11 indicate that the average game length is similar across all scenarios. However, when both players employed the lowest probability decisions for rule 10, the average game length was 46.42, and the win percentages of both players were the same regardless of the rule 11 decision used by both players.



(a) Boxplot of the average game length

(b) Boxplot of skipped turns length

Figure 4.12: Distribution of length of games and skipped turns when both players used the same decisions for rule 10 and different decisions for rule 11

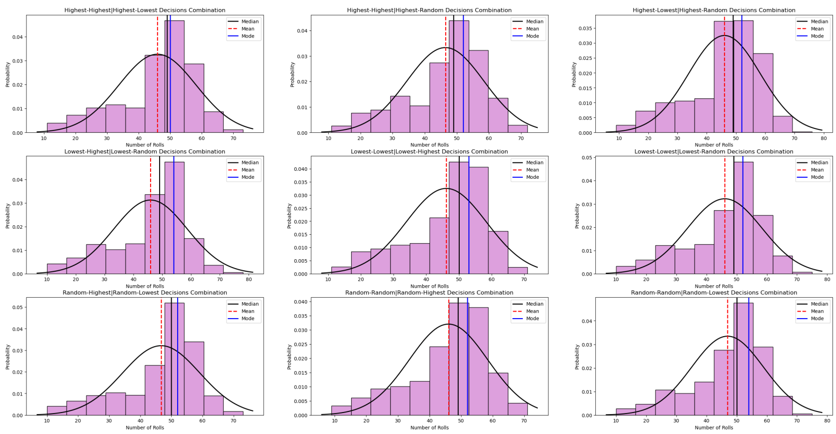


Figure 4.13: Histogram showing the distribution of the game length when both players used the same decisions for rule 10 and different decisions for rule 11

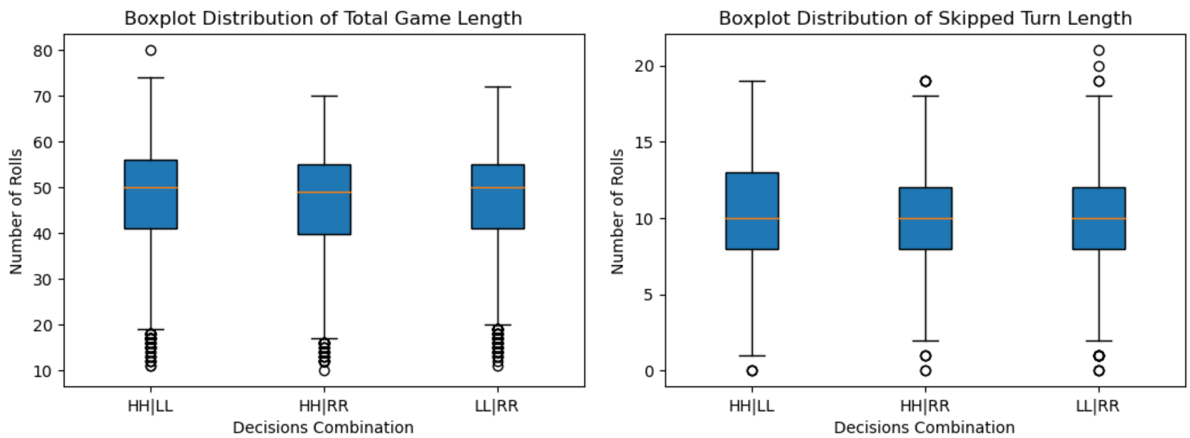
Figure 4.12a and 4.13 indicate that the game length distributions are right-skewed, with a minimum of 9 rolls and a maximum of 80 rolls. The median game length was found to be 49, except for the Lowest-Lowest|Lowest-Highest and Lowest-Lowest|Lowest-Random rules which had a median of 50. The most frequent game length occurred between 40 and 52 rolls.

- **Results when a player used the same decisions for rules 10 and 11 while the opponent used a different but same decision for rules 10 and 11**:

Figure 4.14: Summary result when a player used the same decisions for rules 10 and 11 while the opponent used a different but same decision for rules 10 and 11

	Decisions	Average game length	Average player one game length	Average player two game length	Player one win length	Player one % of wins	Player two win length	Player two % of wins	Player one skipped turn length	Player two skipped turn length	Average skipped turn	Draw length	% of draws
1	Highest - Highest|Lowest - Lowest	46.55	23.57	22.99	189	18.9	180	18.0	7195	6995	14.19	631	63.1
2	Highest - Highest|Random - Random	46.55	23.57	22.98	188	18.8	198	19.8	6918	6802	13.72	614	61.4
3	Lowest - Lowest|Random - Random	46.42	23.48	22.94	204	20.4	178	17.8	7098	6887	13.98	618	61.8

The Highest-Highest|Lowest-Lowest and Highest-Highest|Random-Random gameplays had the same average game lengths of 46.55, with both players having an average skipped turn length of 13.72 and 14.19, respectively. The win percentages for both players in both game types were also similar. For the Lowest-Lowest|Random-Random, the average game length was slightly lower and there is a little difference between player one and two win percentage.



(a) Boxplot of the average game length

(b) Boxplot of skipped turns length

Figure 4.15: Distribution of length of games and skipped turns when a player used the same decisions for rules 10 and 11 while the opponent used a different but same decision for rules 10 and 11

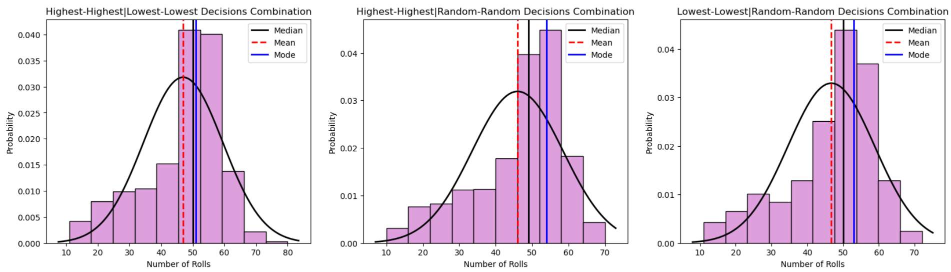


Figure 4.16: Histogram showing the distribution of the game length when a player used the same decisions for rules 10 and 11 while the opponent used a different but same decision for rules 10 and 11

Figure 4.15a and 4.16 indicate that the game length distributions are right-skewed, with the minimum being 10 rolls and the maximum being 80 rolls. The most frequent game length was found to be between 41 and 55 rolls. Figure 4.15b further illustrates that the common skipped turns ranged from 8 to 12, with the median being 10.

Overall, we tested 81 decisions combination for this variation, but only relevant results were reported since all decisions combination gave similar results.

4.1.5 One Player With Simplified Rules

This variation of the sequence dice game involves a single player playing with a full board without adhering to the rules. An offensive and random strategy were implemented, as a defensive strategy was not necessary due to the single-player nature of the game. The evaluation of the number of rolls required for the player to win was then conducted.

	Strategies	Average Dice Rolls	Number of Wins	% of Wins	Game Overs	% of Game Overs	Skipped Turns Count	Average Skipped Turns
1	Offensive	21.60	796	79.6	204	20.4	4590	4.59
2	Random	21.53	766	76.6	234	23.4	4449	4.45

Figure 4.17: Summary result of one player without rules with full board.

The Table in Figure 4.17 reveals the performance of the two strategies; offensive and random, in a 1000 simulations game. The offensive strategy had an average game length of 21.60 and an average skipped turn of 4.59, indicating that on average it took approximately 22 dice rolls and approximately 5 turns were skipped. It had 796 wins out of 1000 simulations and 4590 skipped turns and 204 game overs. The random strategy had an average game length of 21.53 and an average skipped turn of 4.45, indicating that on average it took approximately 22 dice rolls and approximately 5 turns were skipped. It had 766 wins out of 1000 simulations and 4449 skipped turns and 234 game overs. Our findings suggest that the Average Game Length and Average Skipped Turn for both strategies were very similar, the offensive strategy is more advantageous than the random strategy as it yields more wins.

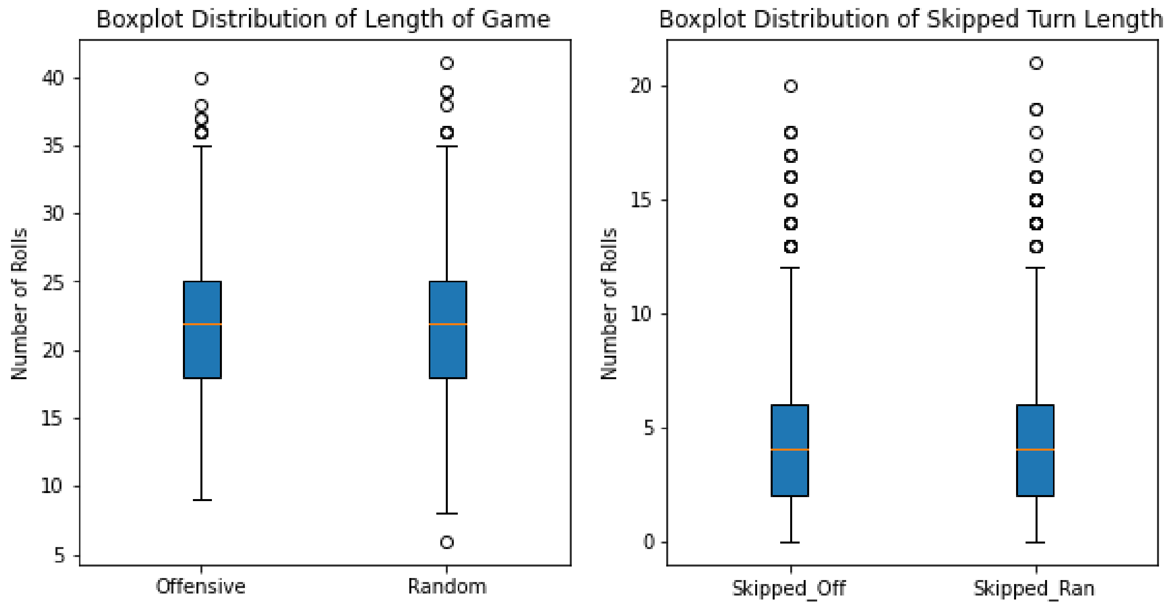


Figure 4.18: Boxplot showing the distribution of the average dice rolls and average skipped turns

The boxplot in Figure 4.18 illustrates the distribution of the length of the game and skipped turns for the offensive and random strategies. It is evident from the figure that the game length for the offensive strategy is concentrated between 17 and 25 rolls, with a mean of approximately 21 rolls. The offensive strategy had a minimum game length of 9 rolls and a maximum of 40 rolls. The game length for the random strategy is also concentrated between 17 and 25 rolls, with a mean of approximately 22 rolls. The minimum game length for the random strategy is 5 rolls and the maximum is 42 rolls. Additionally, the skipped turn for the offensive strategy is heavily concentrated between 2 and 7 turns with a mean of approximately 4 turns and a maximum of 20 turns. For the random strategy, the skipped turn is also concentrated between 2 and 6 turns, with a mean of approximately 4 turns and a maximum of 22 turns.

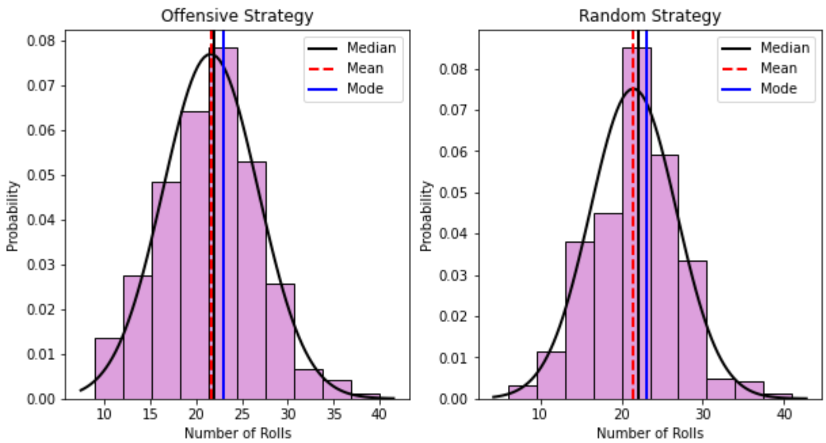


Figure 4.19: Histogram showing the distribution of the average game length for both offensive and random strategies

The histograms in Figure 4.19 show that the average game length for both the offensive and random strategies is approximately 22 rolls, respectively, with a median of approximately 22 rolls for both strategies. The offensive strategy has a mode of 23 rolls while the random strategy has a mode of approximately 22 rolls for both strategies.

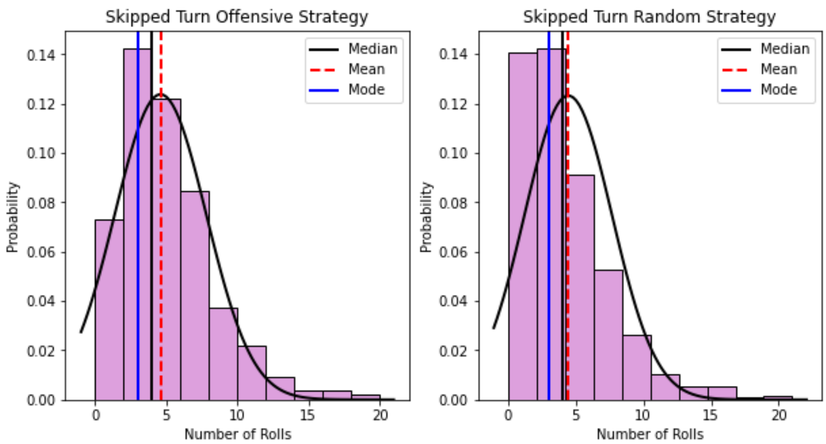


Figure 4.20: Boxplot showing the Distribution of the average skipped turns for both strategies

The average skipped turn length for both strategies was approximately 5. The offensive strategy had a mode of 3 turns and median of 4 turns while the random strategy has a mode of 3 turns and medians of 5 turns. This reflects in the peak of the corresponding distribution curves.

4.1.6 One Player With Original Rules

In this variation we examined the average number of rolls needed to obtain a sequence of five chips when playing the game of sequence dice with one player. The game is simulated 100 times, and the rule of 10 or 11 is applied while the rule of 2 and 12 is disregarded since the game is played with one player.

	Strategies	Average Game Length	Number of Wins	% of Wins	Skipped Count	Average Skipped Turn	Game Overs	% of Game Overs
1	Offensive-highest-highest	18.80	532	53.2	817	0.82	468	46.8
2	Offensive-highest-lowest	18.76	534	53.4	777	0.78	466	46.6
3	Offensive-highest-random	18.84	509	50.9	881	0.88	491	49.1
4	Offensive-lowest-highest	18.81	608	60.8	908	0.91	392	39.2
5	Offensive-lowest-lowest	18.82	577	57.7	1027	1.03	423	42.3
6	Offensive-lowest-random	18.86	556	55.6	983	0.98	444	44.4
7	Offensive-random-highest	18.86	556	55.6	889	0.89	444	44.4
8	Offensive-random-lowest	18.86	569	56.9	863	0.86	431	43.1
9	Offensive-random-random	18.87	531	53.1	902	0.90	469	46.9
10	Random-highest-highest	18.93	411	41.1	817	0.82	589	58.9
11	Random-highest-lowest	18.98	405	40.5	836	0.84	595	59.5
12	Random-highest-random	19.02	437	43.7	965	0.96	563	56.3
13	Random-lowest-highest	19.06	500	50.0	1123	1.12	500	50.0
14	Random-lowest-lowest	19.09	507	50.7	1049	1.05	493	49.3
15	Random-lowest-random	19.09	536	53.6	1066	1.07	464	46.4
16	Random-random-highest	19.11	459	45.9	921	0.92	541	54.1
17	Random-random-lowest	19.13	469	46.9	883	0.88	531	53.1
18	Random-random-random	19.14	465	46.5	848	0.85	535	53.5

Figure 4.21: Summary result of all strategies and decisions combination for the full board one player with rules variation

The Table in Figure 4.21 illustrates the results of offensive and random strategies when combined with different rules 10 and 11 decisions, such as highest, lowest, and random.

Overall, the results reveal that the average game length for all the strategies is between 18.76 and 19.14. The offensive-highest-lowest had the shortest game length with an average of 18.76 rolls. The highest number of wins was achieved by the offensive-lowest-highest strategy and decisions combination with 60.8% wins while the highest number of game overs was recorded by the Random-highest-highest strategy with 58.9% game overs. The average skipped turn for all strategies was between 0.78 and 1.12. The highest skipped count was achieved by the offensive-lowest-lowest strategy decisions combination with 1027 skipped turns. The small number of skipped turns achieved in this variation, as compared to other variations implemented, can be attributed to three factors: the board dimension, the number of players (one player in this scenario), and adherence to the rules of 10 and 11.

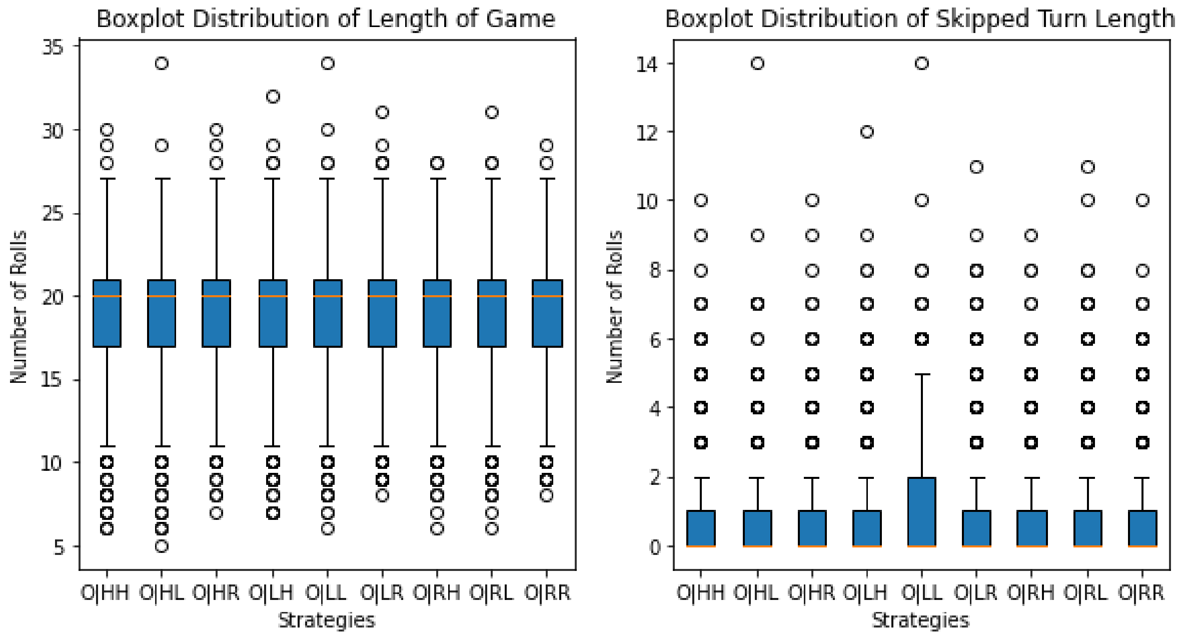


Figure 4.22: Boxplot showing the distribution of the average game length and skipped turns for the offensive strategy in combination with all the rules 10 and 11 decisions

The Boxplots in Figure 4.22 depict the distribution of the game length and skipped turn length for the offensive strategy when combined with rules 10 and 11 decisions. The distribution of skipped turns, as shown, generally lies between 0 and 1, creating a right-skewed distribution; however, there are some instances where skipped turns reach up to 14. The game length distribution is centered around 17 to 21 rolls, with a few games being as short as 5 rolls and some extending up to 35 rolls. This distribution is left-skewed.

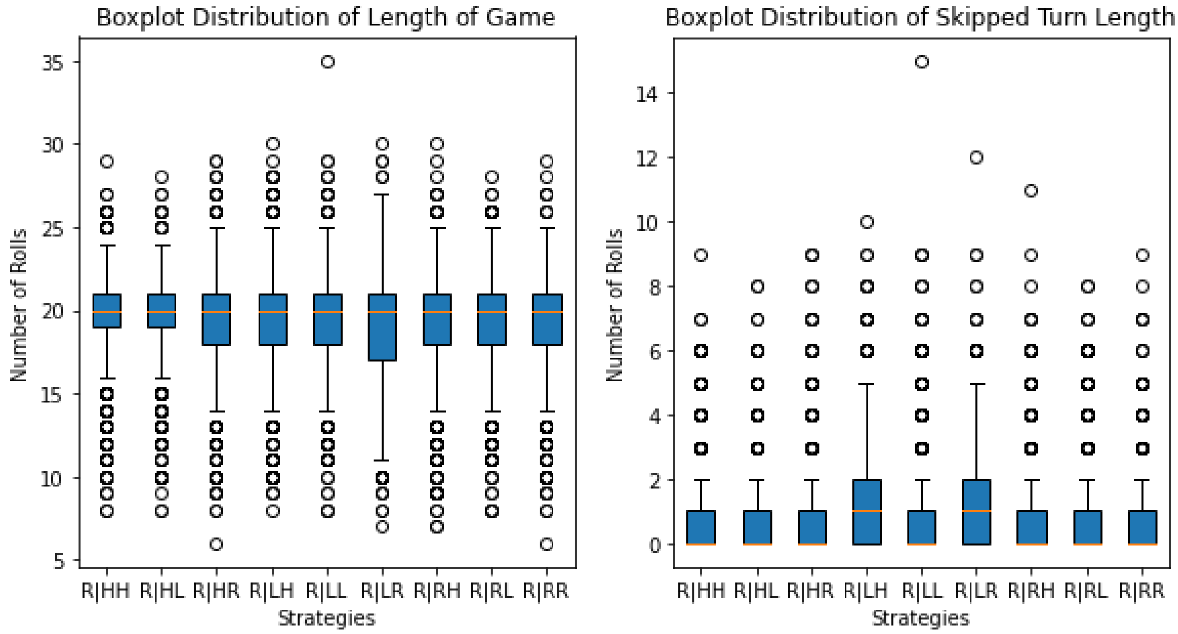


Figure 4.23: Boxplot showing the distribution of the average game length and skipped turns for the offensive strategy in combination with all the rules 10 and 11 decisions

The boxplots above illustrate the distribution of game length and skipped turn length for the random strategy when combined with rules 10 and 11 decisions. It can be seen that the average game roll for all possible combinations is concentrated between 17 and 21 rolls, with skipped turns per game concentrated between 0 and 1 roll for all possible combinations, except for the Random-lowest-highest and Random-lowest-random combination, where skipped turns are concentrated between 0 and 2. The random strategy boxplot distribution for the game length and skipped turns are similar to that of the offensive strategy.

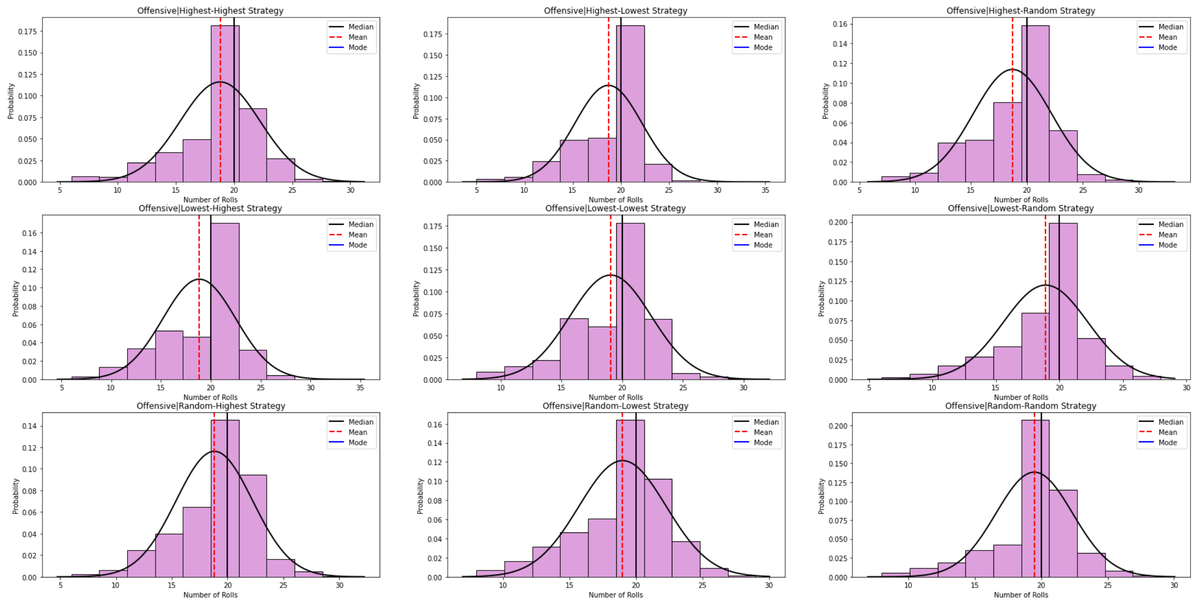


Figure 4.24: Histogram showing the distribution of the average game length and skipped turns for the offensive strategy in combination with all the rules 10 and 11 decisions

The histogram plots in Figure 4.24 show that the average game roll when the player used the offensive strategy in combination with different decisions is 19 rolls. The plots show that the mode and median of these strategies are 20 across the different possible strategies above, which shows that the game ended in a draw for most of the time.

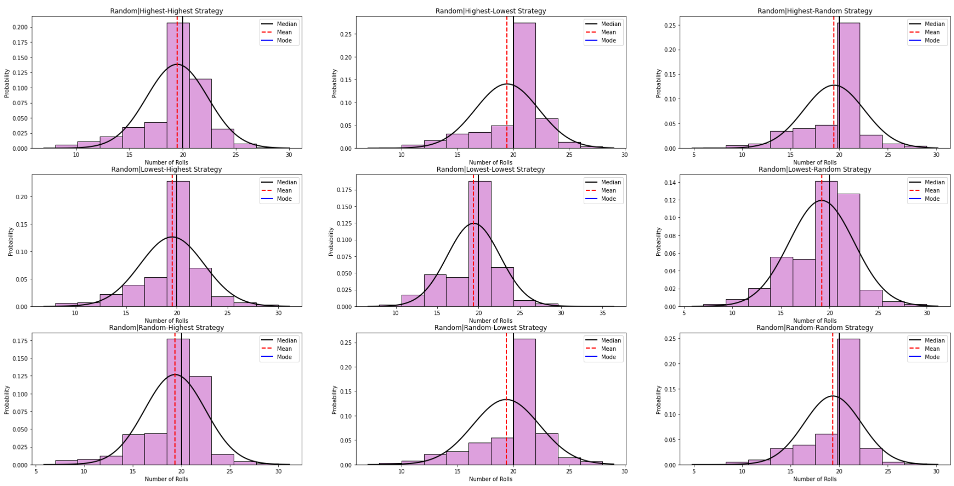


Figure 4.25: Histogram showing the distribution of the average game length and skipped turns for the random strategy in combination with all the rules 10 and 11 decisions

From the plots in Figure 4.25, most game length is approximately 19 rolls when the player used the random strategy combination with all the rules 10 and 11 decisions. The mode and median roll are 20 rolls, which also shows that the game ended in a draw most of the time.

4.1.7 Two Players With Simplified Rules

The aim is to evaluate the number of turns required to win when two players play the full board of the Sequence Dice game without following the rules of the game. To do this, the game was simulated 1000 times. However, it was observed that the game ended in a deadlock. This was due to the players not following the rules of the game and, despite each number (except 10 and 11) appearing four times on the board, players still reached a point where they could not place their chips on the board when they rolled a 10 or a number already covered by the opponent's chips (they simply skipped turns). This deadlock indicates that two players cannot play the full board of the Sequence Dice game without adhering to the game rules, which highlights the importance of the game rules. This finding is in line with the one derived from a previous simulation in which two players played using the one-dimensional board without following the rules of the sequence dice game.

4.1.8 Two Players With Original Rules

We conducted a simulation of the standard sequence dice game, involving two players playing against each other and adhering to the rules of the game. The rules include instructions for what to do when a number is covered either partially or completely with the opponent's chips. We simulated the game 1000 times to evaluate the expected number of turns required to win when two players play the full board. The results of the simulation are shown below.

- **Results when both players used different placement strategies but the same decisions for rules 10 and 11**:

  The results in figure 4.26 show that strategies and decisions have little impact on the average game length, the number of wins for each player, the number of draws, the number of skipped turns, and the percentage of wins for each player, as the

	Strategies and decisions	Average game length	Player one win count	Player one % of wins	Player two win count	Player two % of wins	Number of draws	% of draws	Average skipped turns	Player one skipped turn count	Player two skipped turn count
1	offensive - highest - highest|defensive - highest - highest	35	235	23.5	185	18.5	580	58.0	0.73	431	304
2	offensive - highest - highest|random - highest - highest	34	290	29.0	155	15.5	555	55.5	0.70	441	260
3	defensive - highest - highest|random - highest - highest	36	216	21.6	140	14.0	644	64.4	0.74	434	303
4	offensive - lowest - lowest|defensive - lowest - lowest	35	247	24.7	181	18.1	572	57.2	0.77	445	324
5	offensive - lowest - lowest|random - lowest - lowest	34	344	34.4	143	14.3	513	51.3	0.66	408	253
6	defensive - lowest - lowest|random - lowest - lowest	35	270	27.0	135	13.5	595	59.5	0.74	440	301
7	offensive - random - random|defensive - random - random	35	226	22.6	208	20.8	566	56.6	0.60	353	247
8	offensive - random - random|random - random - random	34	299	29.9	148	14.8	553	55.3	0.63	386	241
9	defensive - random - random|random - random - random	36	235	23.5	145	14.5	620	62.0	0.60	356	246

Figure 4.26: Summary result of when both players used different placement strategies but the same decisions for rules 10 and 11

results are similar. Specifically, when one player employed an offensive strategy and the other employed a random strategy with highest, lowest, and random for both decisions for rules 10 and 11, the average game length was the shortest, with 34 rolls. The longest game length and draw were recorded when one player employed a defensive strategy and the other employed a random strategy. Moreover, player one achieved the highest win rate of 34.4% when they used an offensive-lowest-lowest strategy against a random-lowest-lowest strategy.

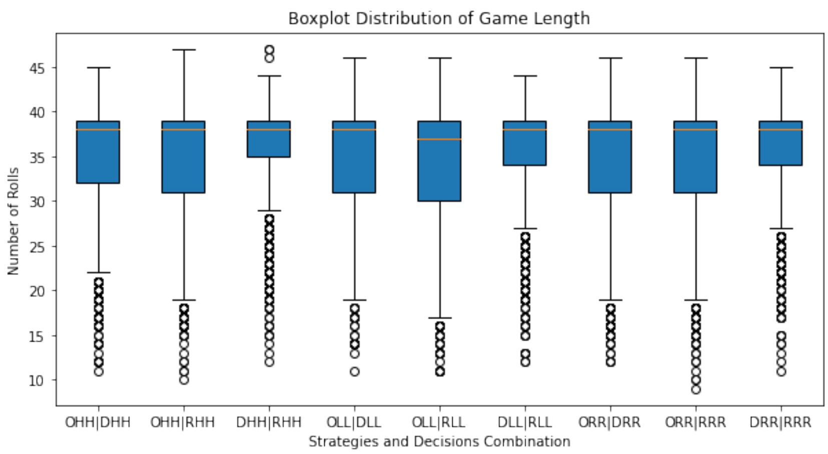


Figure 4.27: Boxplots showing the game length distribution when both players used the same placement strategy but different fixed decisions for rules 10 and 11

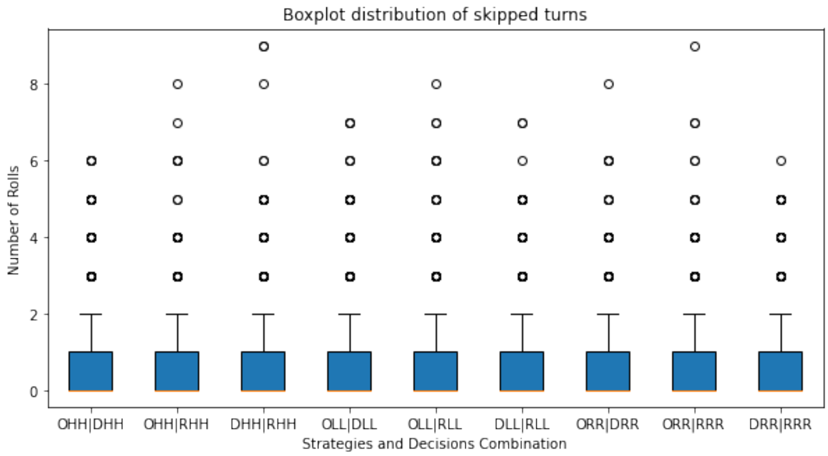


Figure 4.28: Boxplots showing the skipped turns distribution when both players used the same placement strategy but different fixed decisions for rules 10 and 11

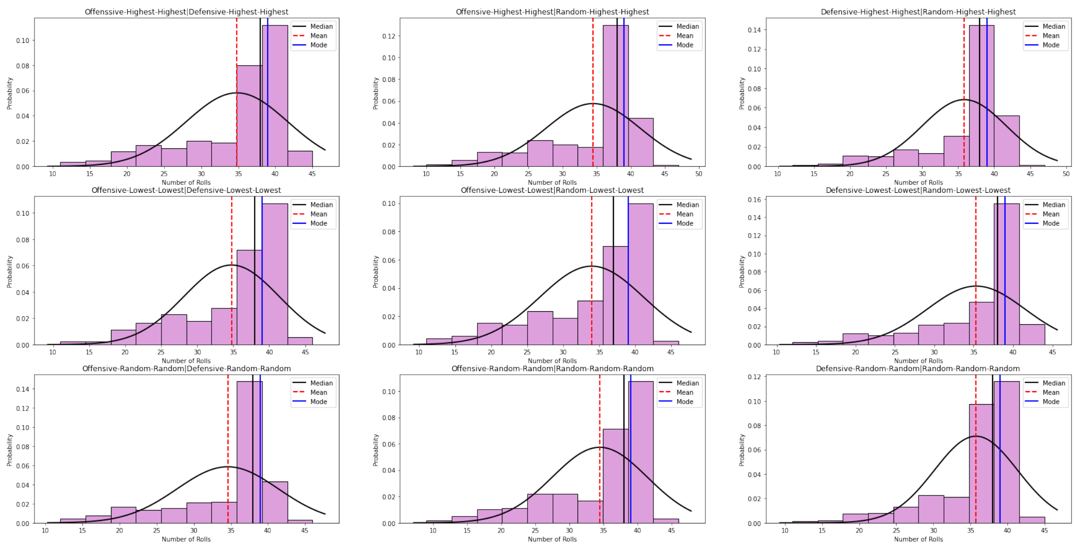


Figure 4.29: Histogram showing the game length distribution when both players used the same placement strategy but different fixed decisions for rules 10 and 11

- **Results when both players used the same placement strategy but different fixed decisions for rules 10 and 11**:

	Strategies and decisions	Average game length	Player one win count	Player one % of wins	Player two win count	Player two % of wins	Number of draws	% of draws	Average skipped turns	Player one skipped turn count	Player two skipped turn count
1	offensive - highest - highest|offensive - lowest - lowest	32	335	33.5	242	24.2	423	42.3	0.65	467	183
2	offensive - highest - highest|offensive - random - random	32	369	36.9	198	19.8	433	43.3	0.62	389	230
3	offensive - lowest - lowest|offensive - random - random	33	291	29.1	227	22.7	482	48.2	0.54	312	228
4	defensive - highest - highest|defensive - lowest - lowest	36	232	23.2	137	13.7	631	63.1	0.77	513	257
5	defensive - highest - highest|defensive - random - random	36	242	24.2	136	13.6	622	62.2	0.66	417	239
6	defensive - lowest - lowest|defensive - random - random	36	185	18.5	171	17.1	644	64.4	0.73	388	342
7	random - highest - highest|random - lowest - lowest	36	246	24.6	109	10.9	645	64.5	0.78	529	254
8	random - highest - highest|random - random - random	36	219	21.9	138	13.8	643	64.3	0.68	434	243
9	random - lowest - lowest|random - random - random	36	161	16.1	165	16.5	674	67.4	0.75	404	345

Figure 4.30: Summary result of when both players used the same placement strategy but different fixed decisions for rules 10 and 11

The results from the Table in Figure 4.30 suggest that defensive and random strategies typically lead to longer game lengths when compared to offensive strategies, with an average game length of 36 for defensive strategies and 32 for offensive strategies. Furthermore, the win count difference between players is greater when both players use an offensive strategy and smaller when both players use defensive and random strategies. The number of draws is also generally higher when defensive and random strategies are used by both players, as opposed to when offensive strategies are used. Lastly, the average number of skipped turns per game is higher when defensive and random strategies are used, with an average of 0.77 and 0.78 skipped turns per game respectively, compared to an average of 0.65 skipped turns per game when offensive strategies are employed.

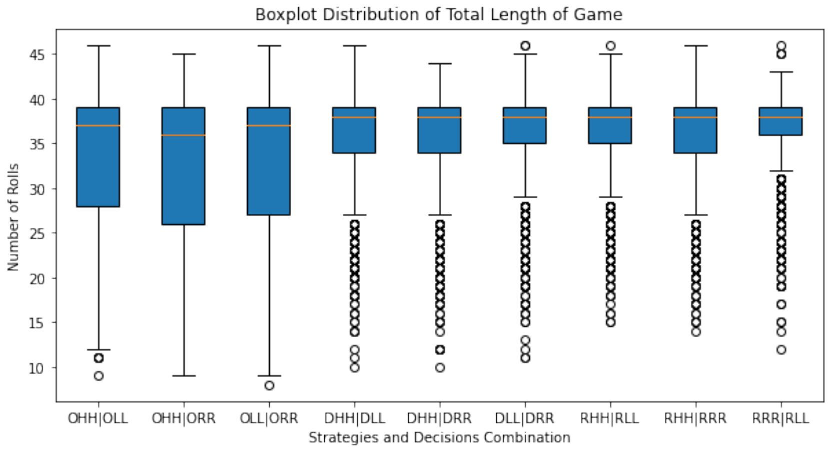


Figure 4.31: Boxplots showing the game length distribution when both players used the same placement strategy but different fixed decisions for rules 10 and 11

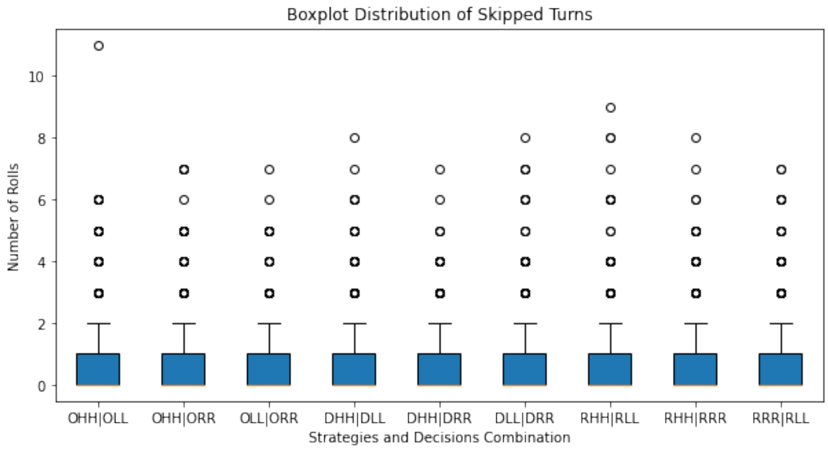


Figure 4.32: Boxplots showing the skipped turns distribution when both players used the same placement strategy but different fixed decisions for rules 10 and 11

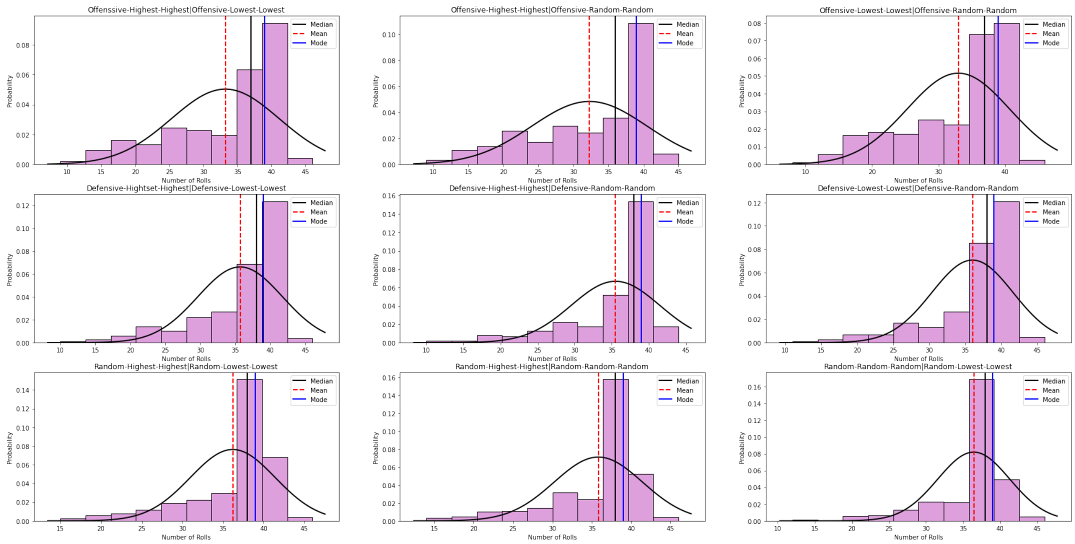


Figure 4.33: Histograms showing the game length distribution when both players used the same placement strategy but different fixed decisions for rules 10 and 11

From Figures 4.26 and 4.30, it is evident that the offensive strategy outperformed the other strategies regardless of the combination of decisions for rules 10 and 11, as it had the shortest average game length. Consequently, we then analyzed the performance of the offensive strategy when both players used different decisions for rules 10 and 11.

- **Results when both players used offensive strategy but different decisions for rules 10 and 11**:

	Strategies and decisions	Average game length	Player one win count	Player one % of wins	Player two win count	Player two % of wins	Number of draws	% of draws	Average skipped turns	Player one skipped turn count	Player two skipped turn count
1	offensive - highest - lowest|offensive - lowest - highest	33	312	31.2	236	23.6	452	45.2	0.63	374	256
2	offensive - highest - random|offensive - random - highest	33	275	27.5	244	24.4	481	48.1	0.55	323	227
3	offensive - lowest - random|offensive - random - lowest	34	248	24.8	237	23.7	515	51.5	0.59	361	227

Figure 4.34: Summary result of when both players used offensive strategy but different decisions for rules 10 and 11

Figure 4.34 shows that when player one employed the highest decision for rule 10 and the lowest decision for rule 11 and player two employed the lowest decision for rule 10 and the highest decision for rule 11, the average game length was the shortest and the draw rate was the lowest, at 45.2%, with the highest winning rate of 31.2% for player one. Conversely, the highest average game length and the number of draws were recorded when player one used the lowest decision for rule 10 and a random decision for rule 11, while player two employed a random decision for rule 10 and the lowest decision for rule 11.

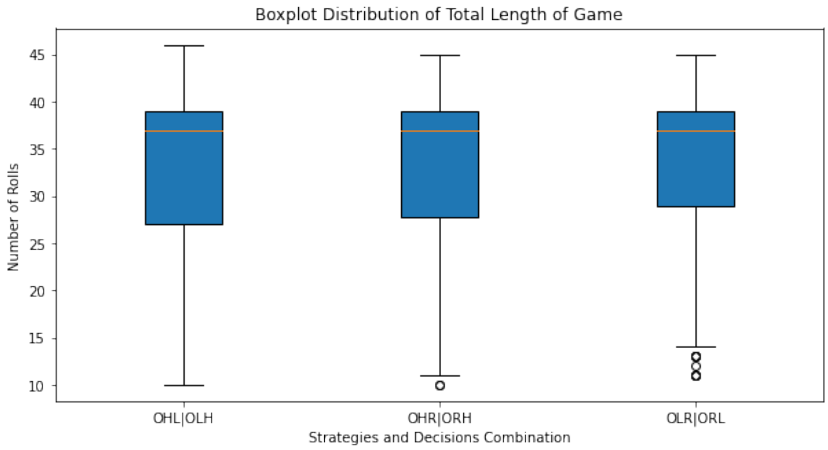


Figure 4.35: Boxplots showing the game length distribution when both players used offensive strategy but different decisions for rules 10 and 11

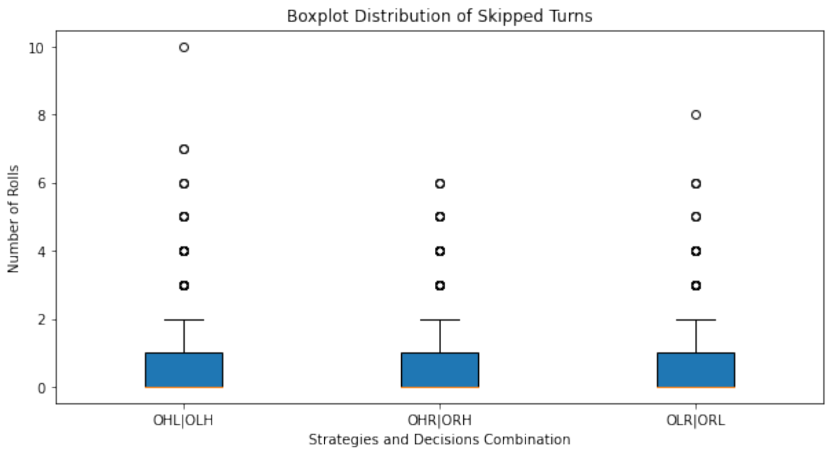


Figure 4.36: Boxplots showing the skipped turns distribution when both players used offensive strategy but different decisions for rules 10 and 11

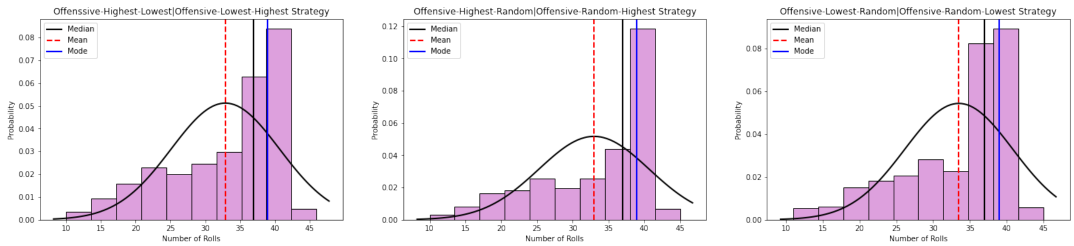


Figure 4.37: Histograms showing the game length distribution when both players used offensive strategy but different decisions for rules 10 and 11

We also analysed a scenario in which player one used an offensive strategy combined with different decisions of rules 10 and 11 to play against player two, who employed a defensive strategy and also utilised different choices of decisions for rules 10 and 11. The result is shown below:

- **Results when player one used an offensive strategy with different decisions for rules 10 and 11 and when player two used a defensive strategy with different decisions for rules 10 and 11**:

	Strategies and decisions	Average game length	Player one win count	Player one % of wins	Player two win count	Player two % of wins	Number of draws	% of draws	Average skipped turns	Player one skipped turn count	Player two skipped turn count
1	offensive - lowest - highest|defensive - highest - lowest	35	225	22.5	185	18.5	590	59.0	0.71	400	306
2	offensive - random - highest|defensive - highest - random	35	213	21.3	192	19.2	595	59.5	0.63	375	257
3	offensive - random - lowest|defensive - lowest - random	34	248	24.8	188	18.8	564	56.4	0.66	403	259
4	offensive - highest - lowest|defensive - lowest - highest	35	225	22.5	192	19.2	583	58.3	0.63	373	258
5	offensive - highest - random|defensive - random - highest	35	204	20.4	162	16.2	634	63.4	0.65	373	276
6	offensive - lowest - random|defensive - random - lowest	35	239	23.9	172	17.2	589	58.9	0.67	385	287

Figure 4.38: Summary result of when player one used an offensive strategy but different decisions for rules 10 and 11 and when player two used a defensive strategy with different decisions for rules 10 and 11

Figure 4.38 shows that the strategy of using an offensive strategy with random, highest, and lowest decisions can result in a higher chance of winning against a defensive approach with the highest, lowest, and random decisions. Specifically, players who use an offensive strategy with "Highest - Lowest" or "Random - Lowest" have the highest winning percentage, while players who use a defensive strategy of "Lowest - Highest" or "Random - Highest" have the lowest winning percentage. Additionally, the average game length tends to be 35 rolls, for most of the strategies employed, and draws account for a significant portion of the results.

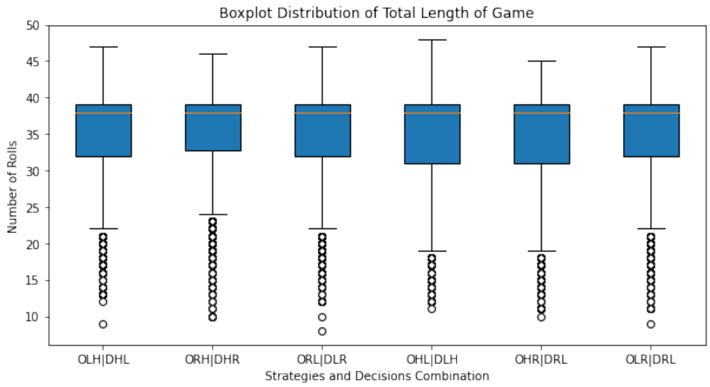


Figure 4.39: Boxplots showing the game length distribution when player one used an offensive strategy but different decisions for rules 10 and 11 and when player two used a defensive strategy with different decisions for rules 10 and 11

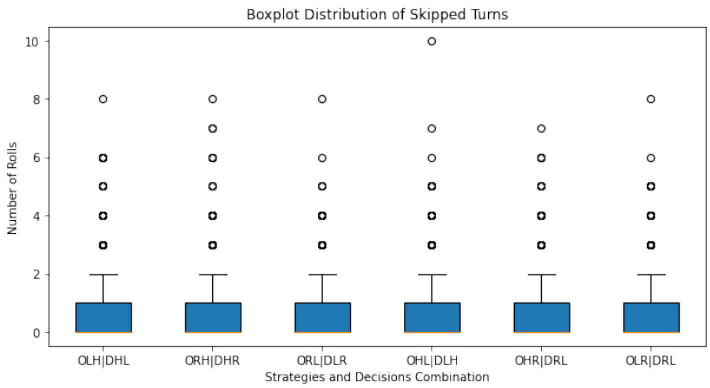


Figure 4.40: Boxplots showing the skipped turns distribution when player one used an offensive strategy but different decisions for rules 10 and 11 and when player two used a defensive strategy with different decisions for rules 10 and 11

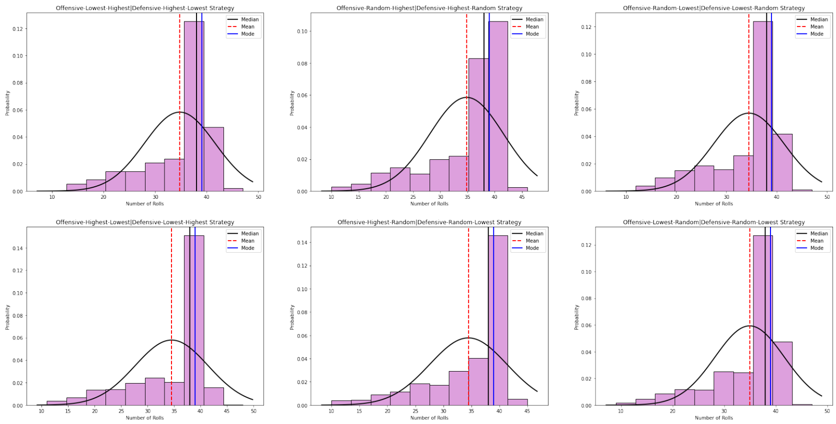


Figure 4.41: Histograms showing the game length distribution when player one used an offensive strategy but different decisions for rules 10 and 11 and when player two used a defensive strategy with different decisions for rules 10 and 11

We then analysed a scenario in which player one used either an offensive or defensive strategy to play against player two who used a random strategy, with both players using the same different combination of decisions for rules 10 and 11. The results are displayed below.

- **Results when player one used either an offensive or defensive strategy to play against player two who used a random strategy, with both players using the same different combination of decisions for rules 10 and 11**:

  The results from Figure 4.42 shows that strategies and decisions have little influence on the average game length, the percentage of wins, and the percentage of draws. In general, the average game length was slightly shorter when using the offensive strategy, while it was slightly longer when using the defensive strategy. Additionally, the average number of skipped turns and draw rate are slightly higher with the defensive strategy and lower while using the offensive strategy.

	Strategies and decisions	Average game length	Player one win count	Player one % of wins	Player two win count	Player two % of wins	Number of draws	% of draws	Average skipped turns	Player one skipped turn count	Player two skipped turn count
1	offensive - lowest - highest|random - lowest - highest	35	290	29.0	148	14.8	562	56.2	0.77	468	302
2	offensive - random - highest|random - random - highest	34	307	30.7	160	16.0	533	53.3	0.64	379	257
3	offensive - random - lowest|random - random - lowest	34	353	35.3	143	14.3	504	50.4	0.56	329	227
4	offensive - highest - lowest|random - highest - lowest	34	316	31.6	149	14.9	535	53.5	0.54	323	220
5	offensive - highest - random|random - highest - random	35	263	26.3	145	14.5	592	59.2	0.63	369	262
6	offensive - random - lowest|random - random - lowest	34	353	35.3	143	14.3	504	50.4	0.56	329	227
7	defensive - lowest - highest|random - lowest - highest	36	215	21.5	143	14.3	642	64.2	0.87	496	370
8	defensive - random - highest|random - random - highest	35	241	24.1	141	14.1	618	61.8	0.73	452	276
9	defensive - random - lowest|random - random - lowest	36	250	25.0	148	14.8	602	60.2	0.62	392	229
10	defensive - highest - lowest|random - highest - lowest	36	256	25.6	135	13.5	609	60.9	0.66	386	274
11	defensive - highest - random|random - highest - random	36	200	20.0	157	15.7	643	64.3	0.62	373	252
12	defensive - lowest - random|random - lowest - random	36	230	23.0	144	14.4	626	62.6	0.75	418	335

Figure 4.42: Summary result of when player one used either an offensive or a defensive strategy to play against player two who used a random strategy, with both players using the same different combination of decisions for rules 10 and 11

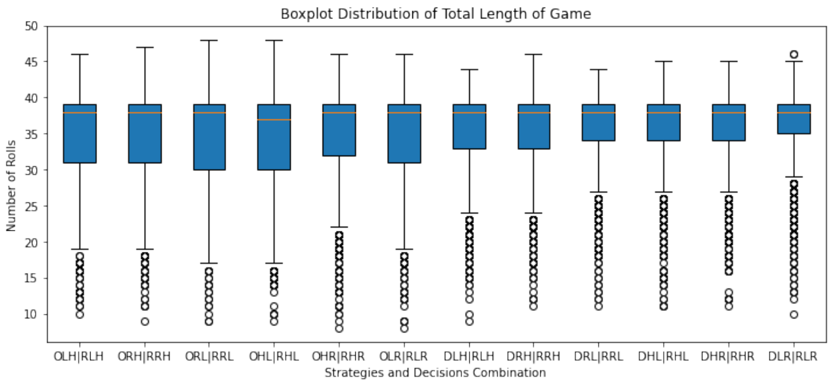


Figure 4.43: Boxplots showing the game length distribution when player one used either an offensive or a defensive strategy to play against player two who used a random strategy, with both players using the same different combination of decisions for rules 10 and 11

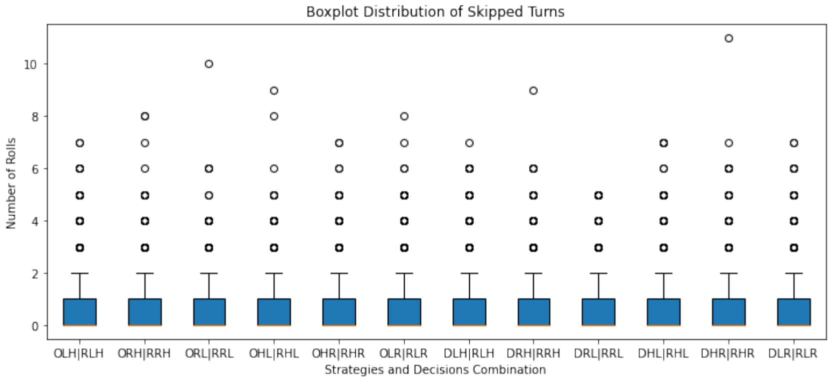


Figure 4.44: Boxplots showing the skipped turns distribution when player one used either an offensive or a defensive strategy to play against player two who used a random strategy, with both players using the same different combination of decisions for rules 10 and 11

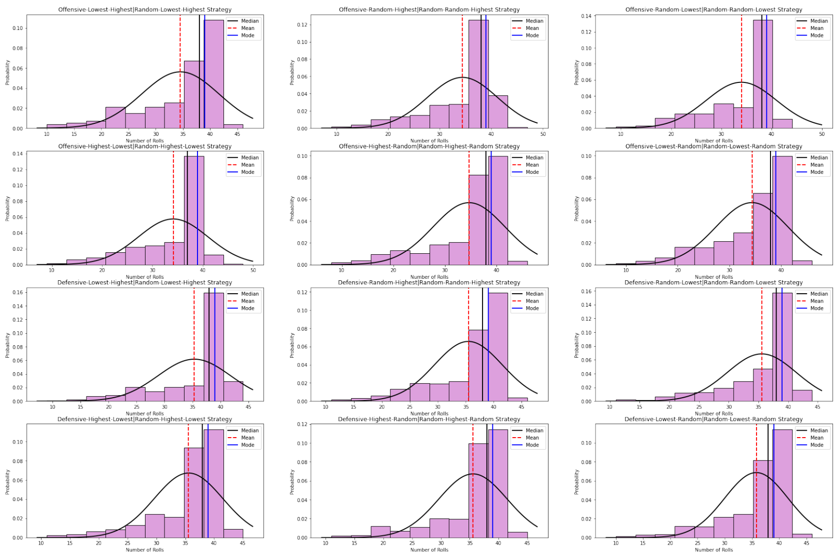


Figure 4.45: Histograms showing the game length distribution when player one used either an offensive or a defensive strategy to play against player two who used a random strategy, with both players using the same different combination of decisions for rules 10 and 11

Finally, we analysed a scenario in which player one used either an offensive or defensive strategy with different combinations of decisions for rules 10 and 11 to play against player two who used a random strategy with different combinations of decisions for rules 10 and 11.

- **Results when player one used either an offensive or defensive strategy to play against player two who used a random strategy, with both players using different combinations of decisions for rules 10 and 11**:

	Strategies and decisions	Average game length	Player one win count	Player one % of wins	Player two win count	Player two % of wins	Number of draws	% of draws	Average skipped turns	Player one skipped turn count	Player two skipped turn count
1	offensive - highest - lowest|random - lowest - highest	34	308	30.8	134	13.4	558	55.8	0.68	439	245
2	offensive - highest - random|random - random - highest	34	297	29.7	160	16.0	543	54.3	0.60	345	253
3	offensive - lowest - random|random - random - lowest	34	310	31.0	143	14.3	547	54.7	0.65	370	282
4	offensive - lowest - highest|random - highest - lowest	34	305	30.5	128	12.8	567	56.7	0.69	377	308
5	offensive - random - highest|random - highest - random	35	300	30.0	125	12.5	575	57.5	0.66	414	242
6	offensive - random - lowest|random - lowest - random	34	318	31.8	143	14.3	539	53.9	0.64	390	247
7	defensive - highest - lowest|random - lowest - highest	35	230	23.0	149	14.9	621	62.1	0.71	427	287
8	defensive - highest - random|defensive - random - highest	36	180	18.0	147	14.7	673	67.3	0.66	403	261
9	defensive - lowest - random|random - random - lowest	36	227	22.7	153	15.3	620	62.0	0.65	389	263
10	defensive - lowest - highest|random - highest - lowest	36	228	22.8	132	13.2	640	64.0	0.73	443	283
11	defensive - random - highest|random - highest - random	36	230	23.0	139	13.9	631	63.1	0.68	431	248
12	defensive - random - lowest|random - lowest - random	36	227	22.7	147	14.7	626	62.6	0.66	418	238

Figure 4.46: Summary result of when player one used either an offensive or a defensive strategy to play against player two who used a random strategy, with both players using different combinations of decisions for rules 10 and 11

The results of Figure 4.46 suggest that offensive strategies are more effective than defensive strategies in terms of achieving a lower average game length, higher win counts for player one, lower skipped turn counts for both players and fewer draws. Specifically, offensive strategies led to shorter average game lengths, higher win counts for player one, lower skipped turn counts for both players and the lowest number of draws. Conversely, defensive strategies resulted in longer average game lengths, lower win counts for player one, higher skipped turn counts for both players and the highest number of draws. These findings suggest that offensive strategies are more successful at winning games and getting low game length.

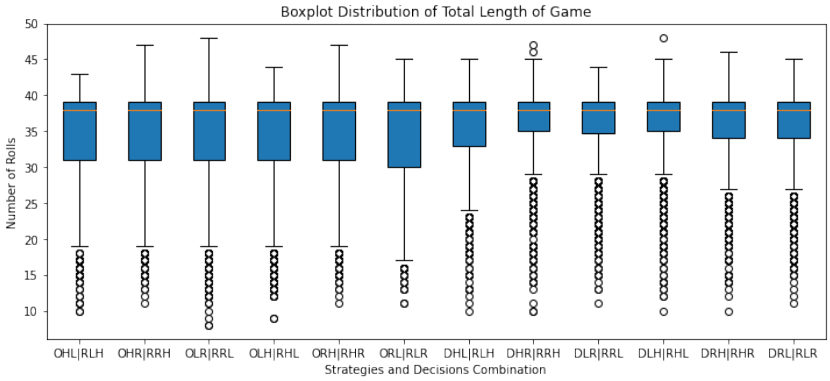


Figure 4.47: Boxplots showing the game length distribution when player one used either an offensive or a defensive strategy to play against player two who used a random strategy, with both players using different combinations of decisions for rules 10 and 11

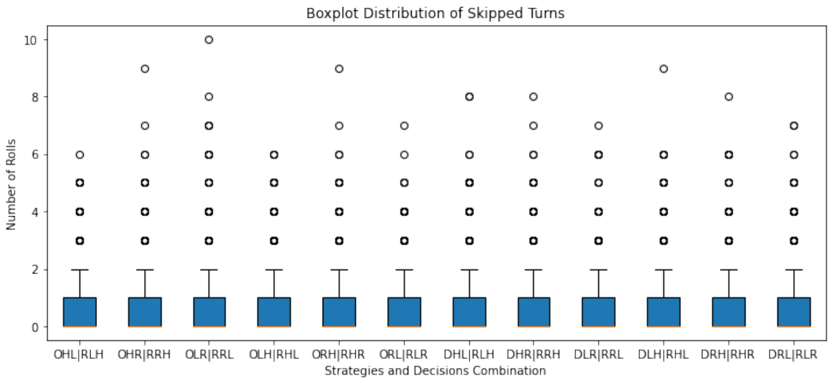


Figure 4.48: Boxplots showing the skipped turns distribution when player one used either an offensive or a defensive strategy to play against player two who used a random strategy, with both players using different combinations of decisions for rules 10 and 11

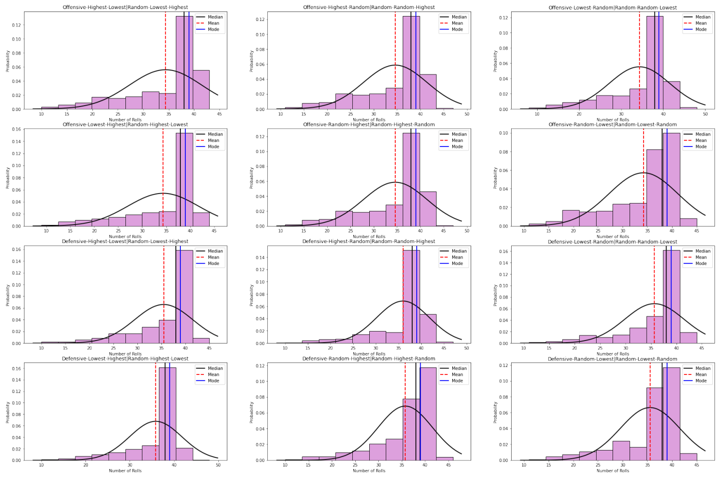


Figure 4.49: Histograms showing the game length distribution when player one used either an offensive or a defensive strategy to play against player two who used a random strategy, with both players using different combinations of decisions for rules 10 and 11

In general, we tested 729 strategies and decisions combination for this variation, but only 51 relevant results out of the 729 were reported since all decisions combination gave similar results.

4.1.9 Two Players With Original Rules - Unlimited Number of Chips

From the simulation results of the full board two-players with rules variation of the sequence dice game, the results shows that the game ended in a draw around 423 - 720 times out of 1000 simulations when players had exhausted their allocated 20 chips. This shows that having only 20 chips may not be enough to determine a winner and may not accurately reflect the game length before a winner emerged. To further analyse the game length, a variation of the full board two-players with rules of the sequence dice game was implemented with an unlimited number of chips A simulation of the game was then conducted 1000 times using the full board of the game and rules. The results of the simulation are presented below.

- **Results when both players used different placement strategies but the same decisions for rules 10 and 11 - unlimited chips**:

	Strategies and decisions	Average game length	Average player one game length	Average player two game length	Player one win count	Player one % of wins	Player two win count	Player two % of wins	Average skipped turns	Player one skipped turn count	Player two skipped turn count
1	offensive - highest - highest|defensive - highest - highest	42.18	21.39	20.79	557	55.7	443	44.3	1.51	997	979
2	offensive - highest - highest|random - highest - highest	44.34	22.46	21.88	606	60.6	394	39.4	1.66	1059	915
3	defensive - highest - highest|random - highest - highest	44.53	22.55	21.98	547	54.7	453	45.3	1.62	1166	920
4	offensive - lowest - lowest|defensive - lowest - lowest	44.32	22.45	21.87	548	54.8	452	45.2	1.63	904	781
5	offensive - lowest - lowest|random - lowest - lowest	44.34	22.46	21.88	617	61.7	383	38.3	1.62	947	790
6	defensive - lowest - lowest|random - lowest - lowest	45.53	23.05	22.48	557	55.7	443	44.3	1.69	970	818
7	offensive - random - random|defensive - random - random	44.22	22.40	21.83	541	54.1	459	45.9	1.60	830	661
8	offensive - random - random|random - random - random	44.25	22.41	21.84	589	58.9	411	41.1	1.60	781	633
9	defensive - random - random|random - random - random	46.03	23.30	22.73	513	51.3	487	48.7	1.72	952	784

Figure 4.50: Summary result of when both players used different placement strategies but the same decisions for rules 10 and 11 - unlimited number of chips

Figure 4.50 shows that strategies and decisions have less impact on game length, the number of wins for each player, the number of draws, the number of skipped turns, and the percentage of wins for each player. Specifically, when one player employed an offensive strategy and the other employed a defensive strategy with the highest probability decisions for rules 10 and 11, the average game length was the shortest, with 42 rolls. the average game length was the longest, with 46 rolls when one player employed a defensive strategy and the other employed a random strategy with random decisions for rules 10 and 11 Moreover, player one achieved the highest win rate of 61.7% when they used offensive-lowest-lowest strategy against a random-lowest-lowest strategy.

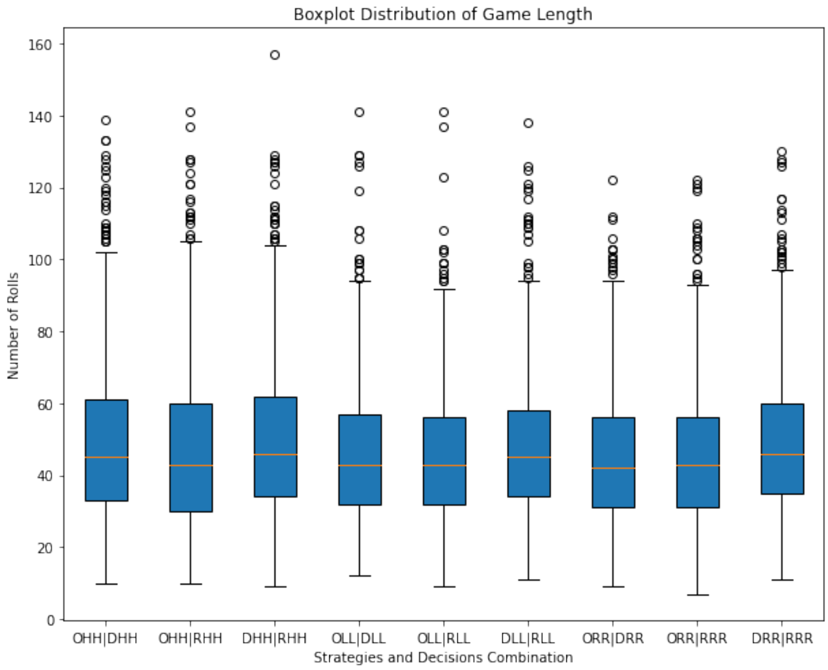


Figure 4.51: Boxplots showing the game length distribution when both players used the same placement strategy but different fixed decisions for rules 10 and 11 - unlimited number of chips

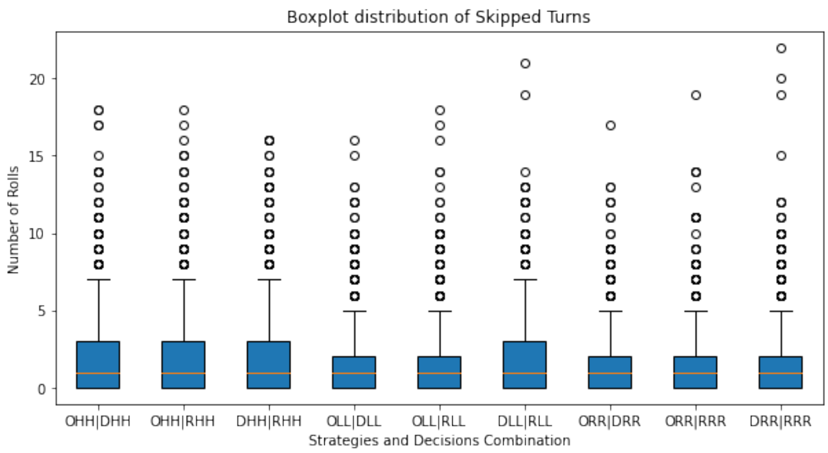


Figure 4.52: Boxplots showing the skipped turns distribution when both players used the same placement strategy but different fixed decisions for rules 10 and 11 - unlimited number of chips

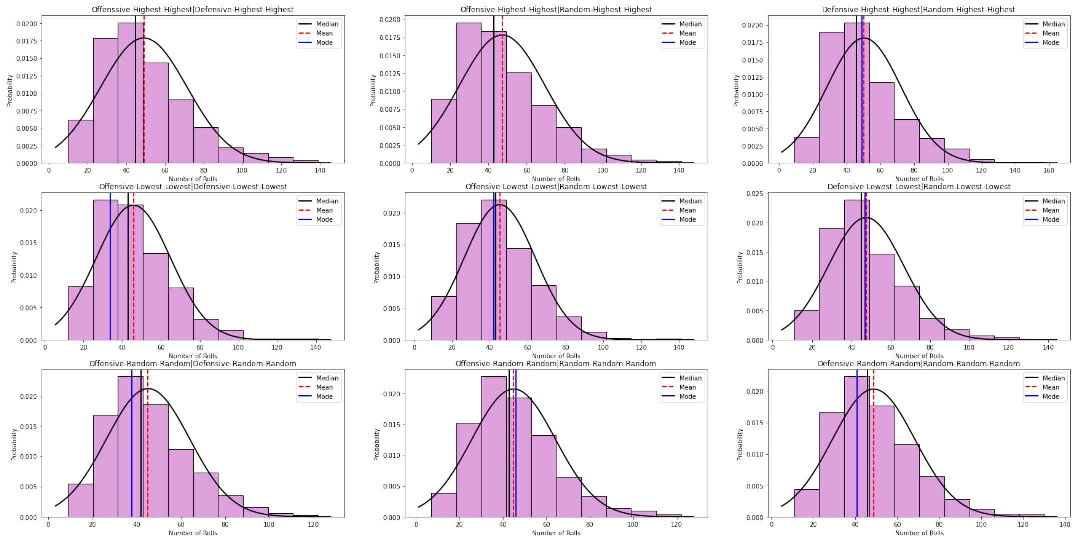


Figure 4.53: Histogram showing the game length distribution when both players used the same placement strategy but different fixed decisions for rules 10 and 11 - unlimited number of chips

- **Results when both players used the same placement strategy but different fixed decisions for rules 10 and 11 - unlimited chip**:

	Strategies and decisions	Average game length	Average player one game length	Average player two game length	Player one win count	Player one % of wins	Player two win count	Player two % of wins	Average skipped turns	Player one skipped turn count	Player two skipped turn count
1	offensive - highest - highest|offensive - lowest - lowest	41.91	21.23	20.68	575	57.5	425	42.5	1.47	894	488
2	offensive - highest - highest|offensive - random - random	41.41	21.01	20.40	603	60.3	397	39.7	1.46	844	663
3	offensive - lowest - lowest|offensive - random - random	44.30	22.44	21.86	537	53.7	463	46.3	1.62	625	706
4	defensive - highest - highest|defensive - lowest - lowest	44.43	22.50	21.93	592	59.2	408	40.8	1.62	1226	695
5	defensive - highest - highest|defensive - random - random	44.50	22.54	21.96	577	57.7	423	42.3	1.62	1089	779
6	defensive - lowest - lowest|defensive - random - random	45.50	23.04	22.46	548	54.8	452	45.2	1.69	891	933
7	random - highest - highest|random - lowest - lowest	46.13	23.35	22.78	605	60.5	395	39.5	1.73	1295	783
8	random - highest - highest|random - random - random	46.16	23.37	22.80	550	55.0	450	45.0	1.73	1084	811
9	random - lowest - lowest|random - random - random	46.55	23.56	23.00	543	54.3	457	45.7	1.76	924	926

Figure 4.54: Summary result of when both players used the same placement strategy but different fixed decisions for rules 10 and 11 - unlimited number of chips

The analysis of the data from the Table in Figure 4.54 reveals that defensive and random strategies generally lead to longer game lengths than offensive strategies, with an average game length of 45 for defensive strategies, 47 and 42 for offensive strategies. Additionally, the win count difference between players is larger when both players utilize an offensive strategy, and smaller when both players use defensive and random strategies. Furthermore, the average number of skipped turns per game is higher when defensive and random strategies are employed, with an average of 1.6 and 1.8 skipped turns per game respectively, compared to an average of 1.65 skipped turns per game when offensive strategies are employed.

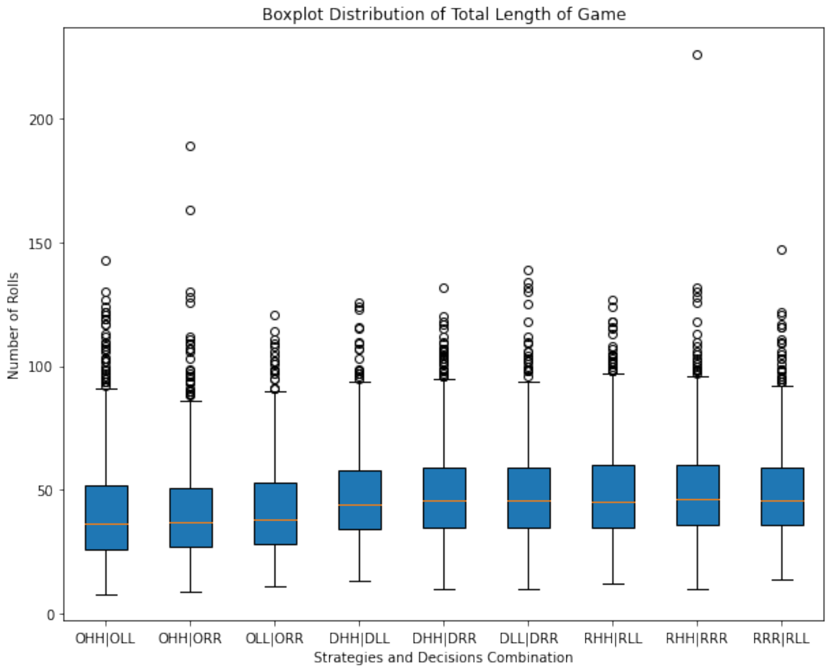


Figure 4.55: Boxplots showing the game length distribution when both players used the same placement strategy but different fixed decisions for rules 10 and 11 - unlimited number of chips

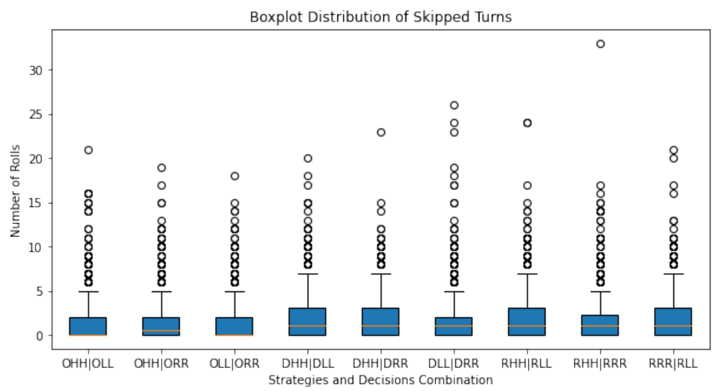


Figure 4.56: Boxplots showing the skipped turns distribution when both players used the same placement strategy but different fixed decisions for rules 10 and 11 - unlimited number of chips

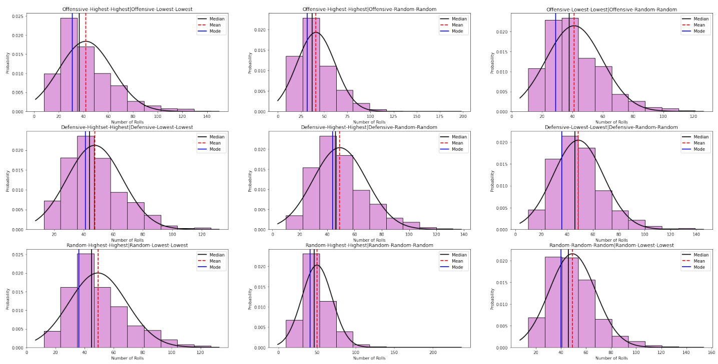


Figure 4.57: Histograms showing the game length distribution when both players used the same placement strategy but different fixed decisions for rules 10 and 11 - unlimited number of chips

From Figures 4.50 and 4.54, it is evident that the offensive strategy outperformed the other strategies regardless of the combination of decisions for rules 10 and 11, as it had the shortest average game length. Consequently, we then analyzed the performance of the offensive strategy when both players used different decisions for rules 10 and 11.

- **Results when both players used offensive strategy but different decisions for rules 10 and 11 - unlimited chips**:

	Strategies and decisions	Average game length	Average player one game length	Average player two game length	Player one win count	Player one % of wins	Player two win count	Player two % of wins	Average skipped turns	Player one skipped turn count	Player two skipped turn count
1	offensive - highest - lowest|offensive - lowest - highest	44.20	22.40	21.80	546	54.6	454	45.4	1.64	842	656
2	offensive - highest - random|offensive - random - highest	43.87	22.24	21.63	576	57.6	424	42.4	1.56	822	736
3	offensive - lowest - random|offensive - random - lowest	44.18	22.38	21.80	508	50.8	492	49.2	1.61	719	723

Figure 4.58: Summary result of when both players used offensive strategy but different decisions for rules 10 and 11 - unlimited number of chips

Figure 4.58 shows that when player one employed the highest decision for rule 10 and the random decision for rule 11, and player two employed the random decision for rule 10 and the highest decision for rule 11, the average game length was the shortest, with a winning rate of 57.6% for player one. In comparison, the highest average game length was observed when player one used the highest decision for rule 10 and the lowest decision for rule 11, while player two employed the lowest decision for rule 10 and the highest decision for rule 11, resulting in a winning rate of 50.8% for player one.

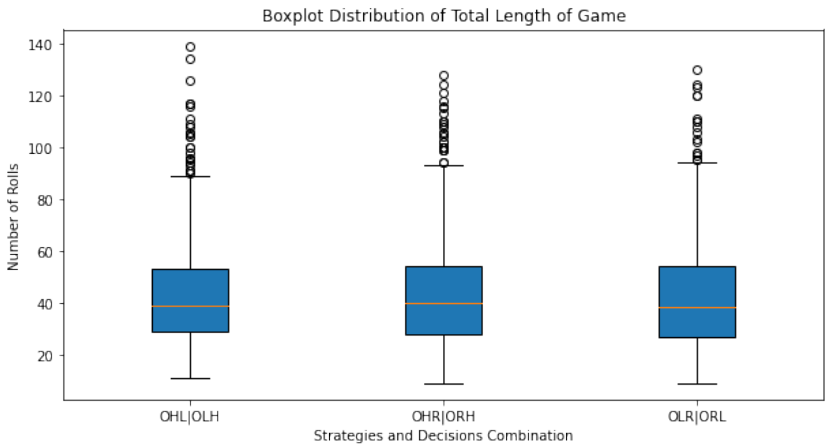


Figure 4.59: Boxplots showing the game length distribution when both players used offensive strategy but different decisions for rules 10 and 11 - unlimited number of chips

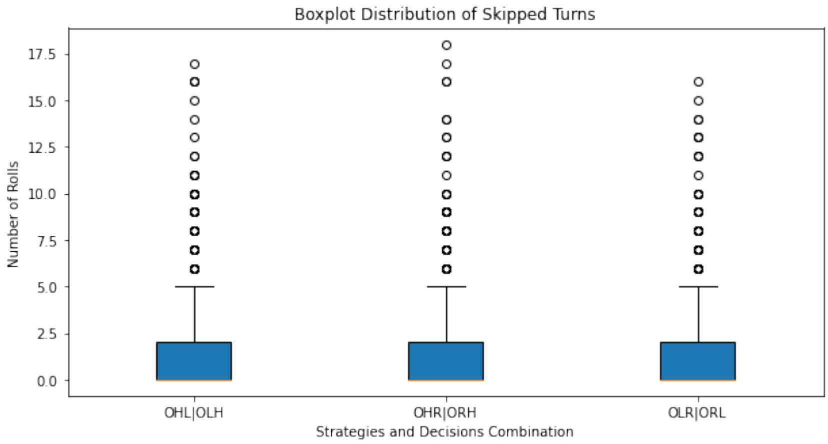


Figure 4.60: Boxplots showing the skipped turns distribution when both players used offensive strategy but different decisions for rules 10 and 11 - unlimited number of chips

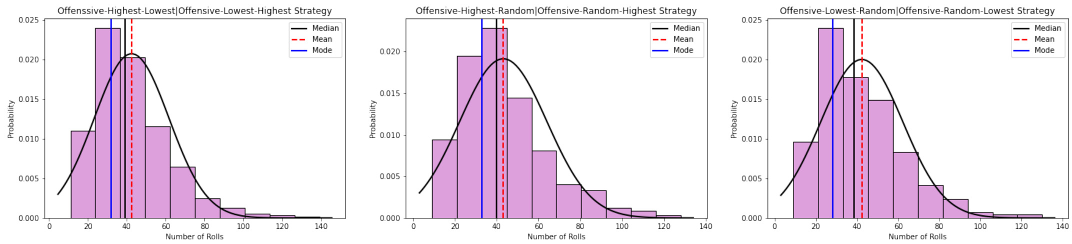


Figure 4.61: Histograms showing the game length distribution when both players used offensive strategy but different decisions for rules 10 and 11 - unlimited number of chips

We also analysed a scenario in which player one used an offensive strategy combined with different decisions of rules 10 and 11 to play against player two, who employed a defensive strategy and also utilised different choices of decisions for rules 10 and 11. The result is shown below:

- **Results when player one used an offensive strategy with different decisions for rules 10 and 11 and when player two used a defensive strategy with different decisions for rules 10 and 11 - unlimited chips**:

	Strategies and decisions	Average game length	Average player one game length	Average player two game length	Player one win count	Player one % of wins	Player two win count	Player two % of wins	Average skipped turns	Player one skipped turn count	Player two skipped turn count
1	offensive - lowest - highest|defensive - highest - lowest	44.21	22.40	21.81	456	45.6	544	54.4	1.59	1009	857
2	offensive - random - highest|defensive - highest - random	44.30	22.44	21.86	484	48.4	516	51.6	1.63	981	767
3	offensive - random - lowest|defensive - lowest - random	44.29	22.43	21.86	562	56.2	438	43.8	1.62	976	688
4	offensive - highest - lowest|defensive - lowest - highest	43.99	22.30	21.70	569	56.9	431	43.1	1.59	965	836
5	offensive - highest - random|defensive - random - highest	44.14	22.37	21.77	549	54.9	451	45.1	1.57	973	898
6	offensive - lowest - random|defensive - random - lowest	44.29	22.43	21.86	495	49.5	505	50.5	1.63	875	842

Figure 4.62: Summary result of when player one used an offensive strategy but different decisions for rules 10 and 11 and when player two used a defensive strategy with different decisions for rules 10 and 11 - unlimited chips

From Figure , it shows that the offensive-highest-lowest strategy appears to be the most successful against the defensive-lowest-highest, as player one won 56.9% of the games and average game length was 43.99. Although, the highest win rate of 54.4% was achieved by player two when playing defensive-highest-lowest against player one playing offensive-lowest-highest. This suggests that both offensive and defensive strategies are both competitive and effective in terms of both winning and time efficiency.

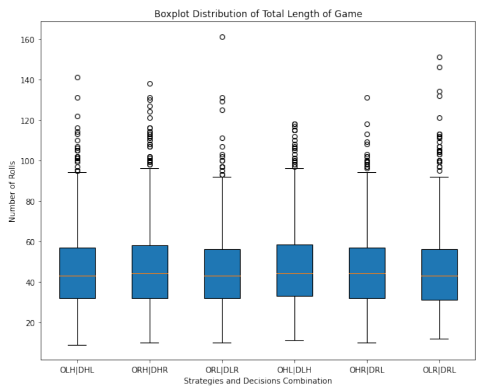


Figure 4.63: Boxplots showing the game length distribution when player one used an offensive strategy but different decisions for rules 10 and 11 and when player two used a defensive strategy with different decisions for rules 10 and 11 - unlimited number of chips

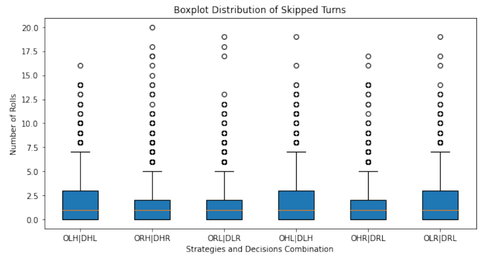


Figure 4.64: Boxplots showing the skipped turns distribution when player one used an offensive strategy but different decisions for rules 10 and 11 and when player two used a defensive strategy with different decisions for rules 10 and 11 - unlimited number of chips

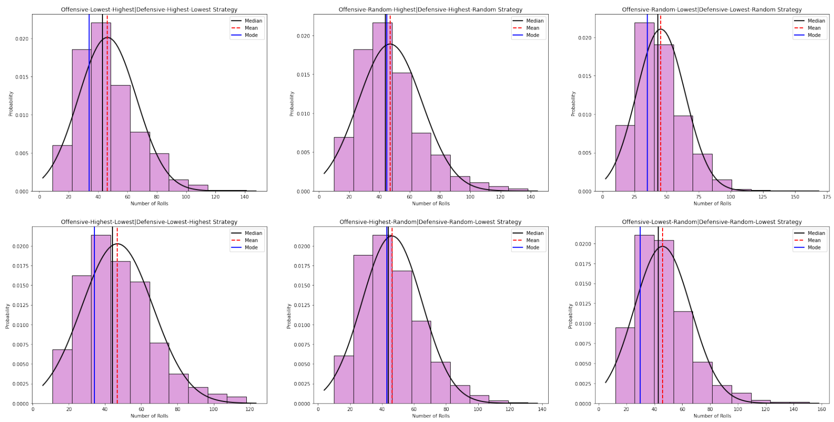


Figure 4.65: Histograms showing the game length distribution when player one used an offensive strategy but different decisions for rules 10 and 11 and when player two used a defensive strategy with different decisions for rules 10 and 11 - unlimited number of chips

We then analysed a scenario in which player one used either an offensive or defensive strategy to play against player two who used a random strategy, with both players using the same different combination of decisions for rules 10 and 11. The results are displayed below.

- **Results when player one used either an offensive or defensive strategy to play against player two who used a random strategy, with both players using the same different combination of decisions for rules 10 and 11 - unlimited chips**:

<table>
<tr><th></th><th>Strategies and Decisions</th><th>Average game length</th><th>Average player one game length</th><th>Average player two game length</th><th>Player one win count</th><th>Player one % of wins</th><th>Player two win count</th><th>Player two % of wins</th><th>Average skipped turns</th><th>Player one skipped turn count</th><th>Player two skipped turn count</th></tr>
<tr><td>1</td><td>offensive - lowest - highest|random - lowest - highest</td><td>44.53</td><td>22.56</td><td>21.97</td><td>563</td><td>56.3</td><td>437</td><td>43.7</td><td>1.63</td><td>1168</td><td>895</td></tr>
<tr><td>2</td><td>offensive - random - highest|random - random - highest</td><td>44.42</td><td>22.50</td><td>21.92</td><td>621</td><td>62.1</td><td>379</td><td>37.9</td><td>1.64</td><td>964</td><td>823</td></tr>
<tr><td>3</td><td>offensive - random - lowest|random - random - lowest</td><td>44.26</td><td>22.42</td><td>21.84</td><td>632</td><td>63.2</td><td>368</td><td>36.8</td><td>1.61</td><td>722</td><td>540</td></tr>
<tr><td>4</td><td>offensive - highest - lowest|random - highest - lowest</td><td>44.12</td><td>22.36</td><td>21.76</td><td>622</td><td>62.2</td><td>378</td><td>37.8</td><td>1.58</td><td>804</td><td>626</td></tr>
<tr><td>5</td><td>offensive - highest - random|random - random - highest</td><td>44.27</td><td>22.43</td><td>21.83</td><td>601</td><td>60.1</td><td>399</td><td>39.9</td><td>1.57</td><td>791</td><td>712</td></tr>
<tr><td>6</td><td>offensive - lowest - random|random - lowest - random</td><td>44.37</td><td>22.47</td><td>21.90</td><td>566</td><td>56.6</td><td>434</td><td>43.4</td><td>1.63</td><td>970</td><td>887</td></tr>
<tr><td>7</td><td>defensive - lowest - highest|random - lowest - highest</td><td>45.38</td><td>22.98</td><td>22.40</td><td>537</td><td>53.7</td><td>463</td><td>46.3</td><td>1.68</td><td>1377</td><td>1166</td></tr>
<tr><td>8</td><td>defensive - random - highest|random - random - highest</td><td>45.87</td><td>23.22</td><td>22.65</td><td>560</td><td>56.0</td><td>440</td><td>44.0</td><td>1.72</td><td>1102</td><td>915</td></tr>
<tr><td>9</td><td>defensive - random - lowest|random - random - lowest</td><td>45.92</td><td>23.25</td><td>22.68</td><td>590</td><td>59.0</td><td>410</td><td>41.0</td><td>1.71</td><td>764</td><td>625</td></tr>
<tr><td>10</td><td>defensive - highest - lowest|random - highest - lowest</td><td>44.81</td><td>22.69</td><td>22.11</td><td>582</td><td>58.2</td><td>418</td><td>41.8</td><td>1.63</td><td>872</td><td>711</td></tr>
<tr><td>11</td><td>defensive - highest - random|random - highest - random</td><td>45.10</td><td>22.84</td><td>22.26</td><td>543</td><td>54.3</td><td>457</td><td>45.7</td><td>1.65</td><td>933</td><td>773</td></tr>
<tr><td>12</td><td>defensive - lowest - random|random - lowest - random</td><td>45.72</td><td>23.15</td><td>22.57</td><td>536</td><td>53.6</td><td>464</td><td>46.4</td><td>1.71</td><td>1118</td><td>955</td></tr>
</table>

Figure 4.66: Summary result of when player one used either an offensive or a defensive strategy to play against player two who used a random strategy, with both players using the same different combination of decisions for rules 10 and 11 - unlimited chips

The results presented in Figure 4.66 indicate that the offensive strategy outperforms the defensive strategy when playing against a random strategy. Specifically, the average game length was shorter and the player one winning percentage was higher for the offensive strategy compared to the defensive strategy.

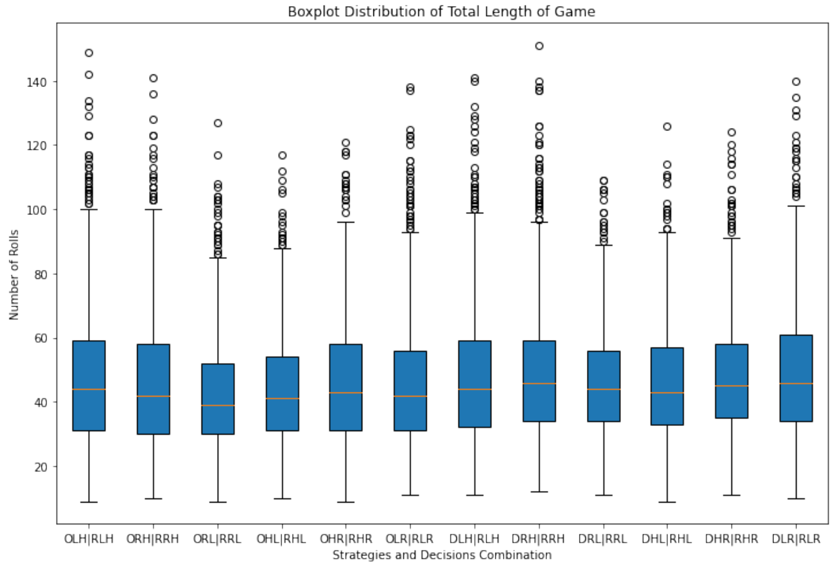


Figure 4.67: Boxplots showing the game length distribution when player one used either an offensive or a defensive strategy to play against player two who used a random strategy, with both players using the same different combination of decisions for rules 10 and 11 - unlimited number of chips

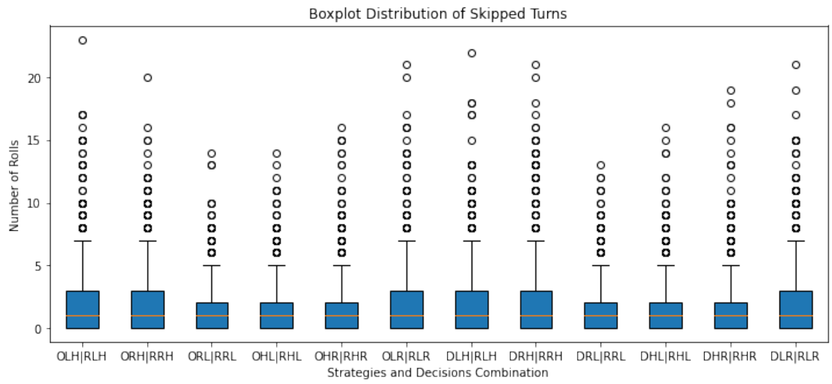


Figure 4.68: Boxplots showing the skipped turns distribution when player one used either an offensive or a defensive strategy to play against player two who used a random strategy, with both players using the same different combination of decisions for rules 10 and 11 - unlimited number of chips

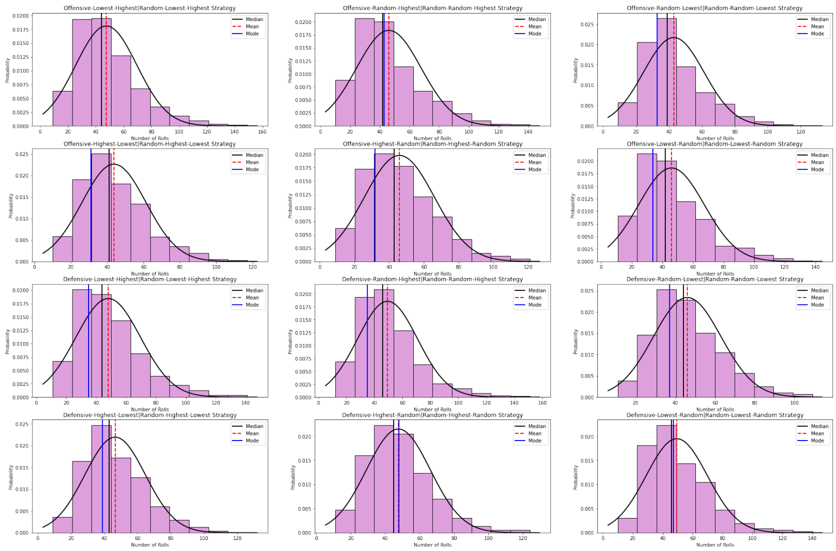


Figure 4.69: Histograms showing the game length distribution when player one used either an offensive or a defensive strategy to play against player two who used a random strategy, with both players using the same different combination of decisions for rules 10 and 11 - unlimited number of chips

Finally, we analysed a scenario in which player one used either an offensive or defensive strategy with different combinations of decisions for rules 10 and 11 to play against player two who used a random strategy with different combinations of decisions for rules 10 and 11.

- **Results when player one used either an offensive or defensive strategy to play against player two who used a random strategy, with both players using different combinations of decisions for rules 10 and 11 - unlimited chips**:

  The results seen in Figure 4.70 are comparable to those found in Figure 4.66. It was observed that an offensive strategy yielded better results than a defensive strategy when playing against a random strategy.

	Strategies and Decisions	Average game length	Average player one game length	Average player two game length	Player one win count	Player one % of wins	Player two win count	Player two % of wins	Average skipped turns	Player one skipped turn count	Player two skipped turn count
1	offensive - highest - lowest|random - lowest - highest	44.12	22.36	21.76	633	63.3	367	36.7	1.58	874	753
2	offensive - highest - random|random - random - highest	44.27	22.43	21.83	601	60.1	399	39.9	1.57	791	712
3	offensive - lowest - random|random - random - lowest	44.37	22.47	21.90	613	61.3	387	38.7	1.63	828	748
4	offensive - lowest - highest|random - highest - lowest	44.47	22.53	21.94	554	55.4	446	44.6	1.62	1026	842
5	offensive - random - highest|random - highest - random	44.41	22.49	21.92	520	52.0	480	48.0	1.64	949	755
6	offensive - random - lowest|random - lowest - random	44.27	22.42	21.85	587	58.7	413	41.3	1.61	911	654
7	defensive - highest - lowest|random - lowest - highest	44.84	22.71	22.13	619	61.9	381	38.1	1.63	1083	993
8	defensive - highest - random|random - random - highest	45.15	22.86	22.28	570	57.0	430	43.0	1.65	958	949
9	defensive - lowest - random|random - random - lowest	45.74	23.15	22.58	527	52.7	473	47.3	1.71	899	831
10	defensive - lowest - highest|random - highest - lowest	45.35	22.96	22.39	511	51.1	489	48.9	1.67	1037	963
11	defensive - random - highest|random - highest - random	45.84	23.21	22.63	509	50.9	491	49.1	1.72	1005	785
12	defensive - random - lowest|random - lowest - random	45.91	23.24	22.67	555	55.5	445	44.5	1.71	948	746

Figure 4.70: Summary result of when player one used either an offensive or a defensive strategy to play against player two who used a random strategy, with both players using different combinations of decisions for rules 10 and 11 - unlimited chips

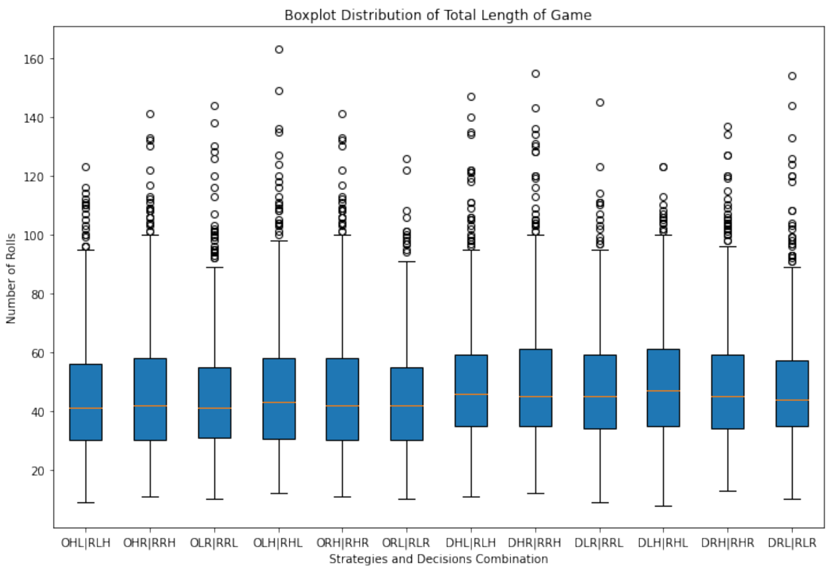


Figure 4.71: Boxplots showing the game length distribution when player one used either an offensive or a defensive strategy to play against player two who used a random strategy, with both players using different combinations of decisions for rules 10 and 11 - unlimited number of chips

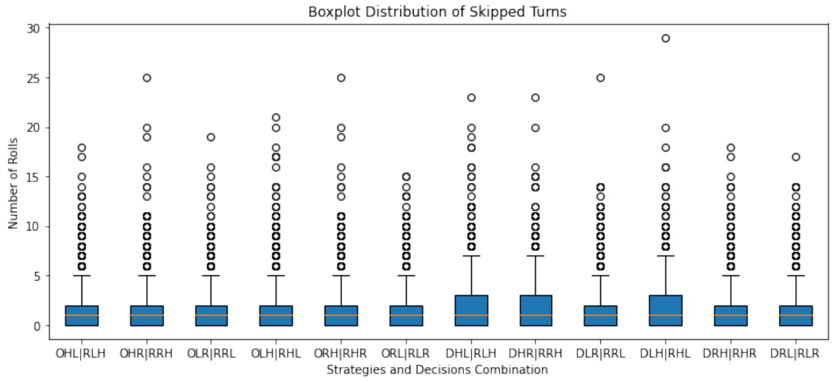


Figure 4.72: Boxplots showing the skipped turns distribution when player one used either an offensive or a defensive strategy to play against player two who used a random strategy, with both players using different combinations of decisions for rules 10 and 11 - unlimited number of chips

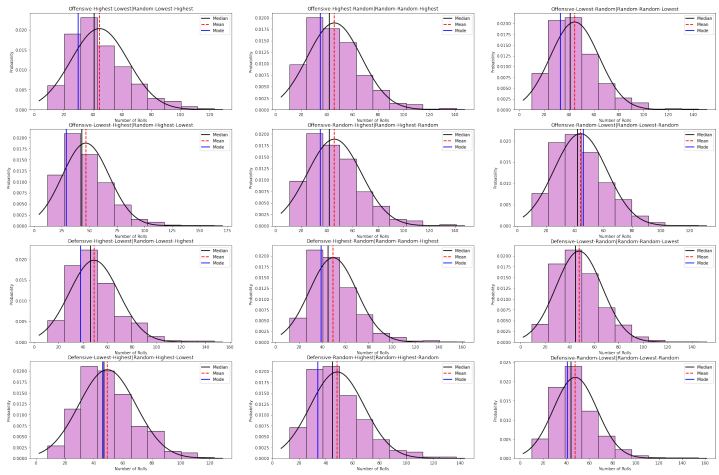


Figure 4.73: Histograms showing the game length distribution when player one used either an offensive or a defensive strategy to play against player two who used a random strategy, with both players using different combinations of decisions for rules 10 and 11 - unlimited number of chips

For this variation, we evaluated 729 strategies and decision combinations, but only 51 relevant results were reported, as all decision combinations yielded similar outcomes.

5 Discussion

5.1 Comparison between simulation and theoretical results

Our theoretical analysis was limited to the one-dimensional board of the sequence dice game with one player playing with a restricted set of rules, due to the complexity associated with other variations and the time constraints when carrying out this research. The theoretical results revealed that the shortest and easiest achievable sequence of 5 is the sequence of subset $S_4 = 5, 6, 7, 8, 9$, which requires an expected number of 19 rolls - the minimum among the 5 subsets. This result significantly differs from the average game length obtained through simulation, which was 14 rolls. Recall that $E(T) \leq \min(E(T_1), E(T_2), E(T_3), E(T_4), E(T_5))$. This implies that the expected time of the simulation is less than the minimum expected time of the theoretical result.

Similar findings were reported by AlAzzawi et al. [AK18], who used Monte Carlo simulation to model the one-player gameplay of Tic-Tac-Toe and by Gasser et al. [Gas+15], who found that a computer simulation result is less than the theoretical result in a game of Backgammon.

5.2 Effects of strategies and rules decisions on the game length

In this research, the one-dimensional board variation of the sequence dice game was evaluated. Results showed that when a single player followed the decision rules implemented for this version, there was no significant effect on the average game length. Specifically, when the player used the highest probability decision for rules 10 and 11, the average game length was 14.29, which is the shortest and similar to the result obtained when the simplified rules were applied. Additionally, the range of results for other decisions combination was approximately 14 rolls, which was also similar to the result when the rules were not applied. However, there was a slight difference in the number of skipped turns: on average, 7.87 turns were skipped when the rules were not followed, compared to 5.66 - 6.23 when the rules were followed. This similarity in the average game length is likely due to the fact that only rules 10 and 11 could feasibly be implemented for this variation involving one player, whereas rules that involved the opponent, such as rules 2 and 12, and rules that involved removing the opponent's chips when the dice are rolled, could not be implemented.

Also, when two players competed against each other on a one-dimensional board version of the sequence dice game and applied the original rules, we had 81 gameplays where two players used different decisions combination to play against each other. Our findings revealed that the rule decisions had minimal impact on the game length. Interestingly, all decision combinations yielded similar results, with the average number of dice rolls required to win this variation ranging from 46 to 47 regardless of the decision combination used by both players. The draw length for this variation ranging from 568 to 650, with an average skipped turn of 13.42 to 14.42, indicating that the game is not overly biased against either player. This suggests that players may not need to be too strategic with their decision-making when competing against an opponent on a one-dimensional board game. However, the average game length needed to win by a player in a two-player setting of the one-dimensional board when the original rules were applied was more than the average game length when only one player plays with the one-dimensional board and applied the original rules of the game.

Additionally, the results of the full board one-player game without rules showed that the average game length when offensive and random strategies were used was roughly the same, with the average skipped turn and the number of wins being slightly higher for the offensive strategy compared to the random strategy. However, the average game length of 22 rolls for the full-board one-player game with simplified rules was higher than that of the full-board one-player game with original rules, which is approximately 19 rolls.

In the full board one-player game with rules, there were 18 gameplays where the player used different strategies and decisions combination, our findings suggest that the offensive-lowest-highest strategy outperformed all other strategies and decision combinations, with an average game length of 18.81 and a win rate of 60.8%. The random-random-random strategy performed the worst, with a game length of 19.14 and a win rate of 46.5%. Overall, the offensive strategy combined with any decisions for rules 10 and 11 outperformed the random strategy combined with any decisions for rules 10 and 11. These results indicate that strategies and rules decisions combination have little impact on the length of the game in a full board one player with original rules setting.

Furthermore, in another simulation, our finding is that both in a one-dimensional board and in a full-board version of the game, two players cannot play the game without adhering to the game rules. This was observed in both board versions of the game, with both resulting in a deadlock due to players being unable to move their chips when they rolled a 10 or a number already covered by the opponent's chips. This finding emphasizes the importance of following the rules of the game in order to properly play the game and have a fair and enjoyable game experience.

Lastly, 729 gameplays where two players played against each other using different strategies and decisions combination were tested for the two-players full board with original rules. Of these, only the relevant results of 51 combinations were reported in this research as the results of all the 729 combinations were found to be similar. The table displaying the results of the additional combinations can be accessed in the here[1]. From the 51 combinations analysed, our findings showed that strategies and decisions combination had an effect on the expected game length when two players played using the full board and followed the rules. Specifically, when used in combination with different decisions, the offensive strategy outperformed the defensive strategy, with an average game length between 32-34 and 34-36 respectively, while the random strategy had an average game length between 35-37. Overall, the average game length when two players played against each other using the full board and applied the original rules was between 32-37. Moreover, the player using the offensive strategy usually had a higher win rate than the player using other strategies.

[1]https://github.com/olalekanlasisi/sequence-dice-game/blob/main/full-board-two-players-with-rules-result1.xlsx

5.3 Effects of number of chips on the game length

Results from the simulation of the full board two-player sequence dice game with rules variation show that when players had a limit of 20 chips, the game ended in a draw between 423 - 720 times out of 1000 simulations. This suggests that a limited number of chips may not be enough to determine a winner and could not accurately reflect the game length before a winner emerged. To further investigate, we examined the expected game length when two players used an unlimited number of chips and followed the rules. We ran 1000 simulations for each of the 729 different strategies and rules decisions combination and reported the results from 51 of them. The full results of the simulations can be found here[2]

Our findings indicate number of chips have significant impacts on the length of the sequence dice game. For instance, when two players used 20 chips, the average game length was approximately 32 -37 rolls with a number of draws between 423 - 720, while when two players played using unlimited number of chips, the average game length was approximately 41 - 47 rolls.

[2] https://github.com/olalekanlasisi/sequence-dice-game/blob/main/full-board-two-players-with-rules-unlimited.xlsx

6 Conclusion

This research work evaluated the expected length of a sequence dice game on a simplified one-dimensional board and on the full board version when players applied the original rules and when the simplified rules are applied. We used the coupon collector model for unequal probabilities to theoretically simplify the playing board to a one-dimensional board and analysed the expected time it would take to win the game with a restricted set of rules. We also simulated this same variation and compare with the theoretical result, we found out that the simulation result is better than the theoretical result.

Additionally, we proposed and implemented three different kinds of decisions for rules 10 and 11: highest probability, lowest probability, and random for both the one-dimensional board and the full board version of the game. For the full board version of the game with original and simplified rules for two players, we proposed three different strategies: offensive, defensive, and random. Furthermore, for the full board one player with simplified and original rules, we proposed offensive and random strategies. We then evaluated the effects these rules decisions can have alone on the expected length of the one-dimensional board version of the sequence dice game and also evaluated the effect of the strategies in combination with the rules decisions on the length of the full board version of the sequence dice game.

We found out that when two players played against each other using the full board and applied the original rules, the offensive strategy outperformed the defensive strategy and that the random strategy has the worst performance. We also discovered that strategies and rules decisions combination have less effect on the expected length of the game when two players played using the full board and applied the simplified rules. Moreover, we found out that strategies and rules decisions combination can affect the length of the game when one player plays on the full board version of the game, as the length of the game was shorter when they applied different strategies with different rules decisions combination compared to when they applied the simplified rules. Furthermore, our findings show that the expected length of the one-dimensional board version of the game when two players played against each other and applied the original rules is higher than the expected length of the full board version of the game when two players played against each other and applied the original rules. Our findings also show that it's imperative for two players to follow the rules of the sequence dice game as not adhering to the rules will lead to deadlock.

Finally, our findings revealed that when two players compete against each other with a full board and 20 chips and applied the original rules, the average game length is approximately between 32 -37 rolls, meanwhile, in a variation where the number of chips is unlimited, the expected length is between 42 to 47 rolls.

These findings provide valuable insights into the expected length of the sequence dice game and can help players in determining the best strategies and rules decisions to use when playing the sequence dice game, as well as inform them of the expected length of the game.

6.1 Future Work

This thesis provides a theoretical analysis of the expected game length of the one-dimensional board of the sequence dice game for one player when the rules are restricted using the coupon collector model. To further explore this topic, future research can be conducted on the theoretical analysis to examine the expected length of the sequence dice game in different variations proposed and implemented in this research work. Additionally, research could also be conducted to investigate the expected length of the game in both multiplayer (more than 2 players) and team settings. Lastly, research could be conducted to determine the actual number of chips that can be used by two players for the game to not end in a draw.

All the works in this thesis can be found in Github, visit this link https://github.com/olalekanlasisi/sequ
dice-game

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