Sudoku: how many grids?

(Jacques Guyot 2020 – licence **CC-BY-SA**)

A classic problem with its solution.

2				1				7
		8	3		5	6		
	5						1	
				8			9	
	9		4				7	1
	3							
							6	
	1	7		4	6		8	
5				7				2

2	6	9	8	1	4	3	5	7
1	7	8	3	2	5	6	4	9
4	5	3	7	6	9	2	1	8
7	2	4	6	8	1	5	9	3
6	9	5	4	3	2	8	7	1
8	3	1	5	9	7	4	2	6
9	8	2	1	5	3	7	6	4
3	1	7	2	4	6	9	8	5
5	4	6	9	7	8	1	3	2

We are going to be interested in puzzles, regardless of the problems and the question is: how many different sudoku puzzles exist.

By forgetting the constraints, there are 81 boxes and 9 possible values, 9^{81} is the maximum number of possible grids without constraint (i.e. 1.97E + 77)

But we understand that the number is smaller. The grid is made up of 9 blocks. Each block contains one and only once the digits from 1 to 9. The number of different blocks is equal to 9 factorial
Let 9! = 362880

The permutations for the 9-block grid give 3628809 (i.e. 1.09E + 50)

B1	B2	B3
B4	B5	B6
B7	B8	B9

Grid manipulations.

What are the operations that leave a sudoku unchanged and that give a new grid.

We see that we can easily derive from the new Sudokus, by swapping:

1) blocks (3x3) by rows and by columns
2) complete lines within the same block

3) complete columns within the same block

2	6	9	8	1	4	3	5	7
1	7	8	3	2	5	6	4	9
4	5	3	7	6	9	2	1	8
7	2	4	6	8	1	5	9	3
6	9	5	4	3	2	8	7	1
8	3	1	5	9	7	4	2	6
9	8	2	1	5	3	7	6	4
3	1	7	2	4	6	9	8	5
5	4	6	9	7	8	1	3	2

2	6	9	8	1	4	3	5	7
1	7	8	3	2	5	6	4	9
4	5	3	7	6	9	2	1	8
7	2	4	6	8	1	5	9	3
6	9	5	4	3	2	8	7	1
8	3	1	5	9	7	4	2	6
9	8	2	1	5	3	7	6	4
3	1	7	2	4	6	9	8	5
5	4	6	9	7	8	1	3	2

We see that by successive permutation of lines, of blocks, it is possible to transform sudoku. But basically it remains the same.

You can also permute two digits on the whole grid, for example permutation of 1 and 2

2	6	9	8	1	4	3	5	7
1	7	8	3	2	5	6	4	9
4	5	3	7	6	9	2	1	8
7	2	4	6	8	1	5	9	3
6	9	5	4	3	2	8	7	1
8	3	1	5	9	7	4	2	6
9	8	2	1	5	3	7	6	4
3	1	7	2	4	6	9	8	5
5	4	6	9	7	8	1	3	2

1	6	9	8	2	4	3	5	7
2	7	8	3	1	5	6	4	9
4	5	3	7	6	9	1	2	8
7	1	4	6	8	2	5	9	3
6	9	5	4	3	1	8	7	2
8	3	2	5	9	7	4	1	6
9	8	1	2	5	3	7	6	4
3	2	7	1	4	6	9	8	5
5	4	6	9	7	8	2	3	1

Grid normalization

The idea of grid standardization is to reduce the number of grids to be tested.

Normalization of B1

By permuting the numbers, it is always possible to have the following B1:

B1 =

1	2	3
4	5	6
7	8	9

This normalization reduces the search space by 9! = 362880.

Normalization of B5

By permutation of rows and columns, it is possible to constrain B5 such that the “1” is in the upper left corner and that a <b and c <d.

B5 =

1	**a**	**b**
c		
d		

This normalization reduces the search space by 36.

Normalization of B9

By permutation of rows and columns, it is possible to constrain B9 such that the "1" is in the upper left corner and that k <l and m <p.

B9 =

1	**k**	**l**
m		
p		

This normalization reduces the search space by 36.

Normalization of B5 <= B9

A final normalization, using the permutation of the blocks will allow to order B5 and B9 so that the number 1abcdefgh <= 1klmnopqr.

This latest standardization reduces the search space by almost 2.

1	2	3						
4	5	6						
7	8	9						
			1	a	b			
			c	d	e			
			f	g	h			
						1	k	l
						m	n	o
						p	q	r

There are 10,080 possible blocks for B5 and B9.

The couples B5 <= B9 is 10080 * (10080 + 1) / 2 or 50'808'240 different normalizations of the diagonal.

The program

We have developed a program to find the sudoku grid number for a given diagonal, so B1, B5 and B9 are set for each assessment.

Consider the following example:

1	2	3						
4	5	6		448			448	
7	8	9						
			1	2	3			
			4	5	6			
			7	8	9			
						1	2	3
						4	5	6
						7	8	9

By fixing B1, B5 and B9, the possibilities of grids B3 and B4 are greatly reduced.

B2 must be compatible horizontally with B1 and vertically with B5. In fact, there are only 448 possible grids.

In our example, B3 must be compatible horizontally with B1 and vertically with B9. In fact, there are also 448 possible grids.

But B2 and B3 must also be compatible. In fact, there are only 3584 combinations of B2 and B3. So quite a few situations to explore.

By fixing B2 and B3, the other blocks are strongly constrained. On average, there are 40 to 80 grids as soon as B1, B2, B3, B5 and B9 are fixed.

Runing

This program rechecked the result of 2005, Bertram Felgenhauer and Frazer Jarvis found:

6’670’903’752’021’072’936’960

This program rechecks this number, the fastest version is:

ComputeAllSolution_HALF.java

The calculation was performed on a single CPU (i7-7820x). The program was run 15 times to perform the calculation in parallel. The set lasted about 9 days.

This program built the files that count the solutions of the subgrids. (see Results folder).

The Validate_HALF program is used to sum the partial results.

```
fact9: 362880
perm2x3: 36
permB5B9: 2
sumdiag           : 1748364208
```

```
sumbig            : 7093166782680
nbsymetries       : 470292480
permB5B9          : 2
nbsudokuJG        : 6670903752021072936960
totsdkto be found: 6670903752021072936960
```