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Last digit is 1.

Problem 1. Determine the value of the functional

$$I[f(x)] = \int_0^{ID+1} ((y')^2 + y')dx$$

for the following functions.

(1) $f(x) = \ln(\cos x)$

$f'(x) = (\ln(\cos x)) = -tg\, x$

$$I[f(x)] = \int_0^2 (tg^2 x - tg\, x)dx = -\ln(|\cos x|)\Big|_0^2 + tg\, x\Big|_0^2 - x\Big|_0^2 = \ln(-\cos(2)) + tg\, 2 - 2$$

(2) $f(x) = xe^x$

$f'(x) = (xe^x)' = e^x + xe^x$

$$I[f(x)] = \int_0^2 ((e^x + xe^x)^2 + e^x + xe^x)dx = \int_0^2 (e^{2x} + 2e^x e^x x + (xe^x)^2 + e^x + xe^x) =$$

$$= \frac{e^x((2x^2+2x+1)e^x + 4x)}{4}\Bigg|_0^2 = \frac{13e^4 + 8e^2 - 1}{4}$$

(3) $f(x) = \sin x + \ln(\cos x)$

$f'(x) = \frac{\cos^2 x - \sin x}{\cos x}$

$$I[f(x)] = \int_0^2 \left(\left(\frac{\cos^2 x - \sin x}{\cos x}\right)^2 + \frac{\cos^2 x - \sin x}{\cos x}\right)dx = \ln(\cos x) + \frac{\sin 2x - 2x}{4} + tg\, x +$$

$$+ \sin x + 2\cos x$$

Problem 2.

$X = C^1[0; \frac{\pi}{6}]$

(1) $f(x) = 2x^2$

$f'(x) = 4x$

$$I[f(x)] = \int_0^{\frac{\pi}{6}} \sqrt{1 + (4x)^2}\,dx = \ln(\sqrt{16x^2+1} + 4x) = \frac{x\sqrt{16x^2+1}}{2} = \frac{9\,ar\sinh(\frac{2\pi}{3}) + 2\pi\sqrt{4\pi^2+9}}{72}$$

(2) $f(x) = 2\ln 3x$

$f'(x) = 2\ln 3$

$$I[f(x)] = \sqrt{4\ln^2(3) + 1}\; x\Big|_0^{\frac{\pi}{6}} = \frac{\pi\sqrt{4\ln^3(3) + 1}}{6}$$

(3) $f(x) = \ln\left(\frac{1}{\sin x}\right)$

$f'(x) = ctg\, x$

$$I[f(x)] = \int_0^{\frac{\pi}{6}} \sqrt{1 + ctg^2 x}\,dx = -\ln\left(\frac{1}{\sin x} + ctg\, x\right) = -\ln(2 + \sqrt{3})$$

Problem 3.

(1) $y(x)=0$ $\quad \bar{y}(x)=\frac{\sin 2nx}{n^2}$ $\quad [0,\pi]$

Solution: $|y(x)-\bar{y}(x)| = \left|0-\frac{\sin 2nx}{n^2}\right| = \left|-\frac{\sin 2nx}{n^2}\right| = \frac{|\sin 2nx|}{|n^2|} = \frac{1}{n^2}$

$|y'(x)-\bar{y}'(x)| = \left|0-\frac{2\cos(2nx)}{n}\right| = \left|-\frac{2\cos(2nx)}{n}\right| = \frac{|2||\cos(2nx)|}{|n|} = \frac{2}{n}$

$\frac{2}{n} > \frac{1}{n^2}$ $\quad$ 0th order

(2) $y(x)=0$ $\quad \bar{y}(x)=\frac{tg(2nx)}{n}$

0th order

$|y(x)-\bar{y}(x)| = \left|0-\frac{tg(2nx)}{n}\right| = \left|-\frac{tg\,2nx}{n}\right| = \frac{|tg\,2nx|}{|n|} = \frac{1}{n}$

$|y'(x)-\bar{y}'(x)| = |2\sec^2(2nx)| = \left|\frac{1}{n}\cdot\frac{2n}{\cos^2 2nx}\right| = \left|\frac{1}{n}\right|\cdot\left|\frac{2n}{\cos^2 2nx}\right| = \frac{2n}{n}$

$\frac{1}{n} < \frac{2n}{n}$ $\quad$ 1st order

Problem 4.

(1) for the 1st order $y(x)=\ln 2x$ $\quad \bar{y}(x)=2x$ on $[e^{-2};e]$

$f(x)=y-\bar{y}=\ln(2x)-2x$ $\qquad \frac{1}{x}-2=0$

$f'(x)=\frac{1}{x}-2$ $\qquad x=\frac{1}{2}$

$f''(x)=-\frac{1}{x^2}$

$f''(\frac{1}{2})=-\frac{1}{\frac{1}{4}}=-1\cdot 4=-4$

$\rho_0 = \max_{[e^{-2};e]}|f(x)| = (\ln(2x)-2x)\big|_{x=-4} = 8 \Rightarrow$ 0th tistance

$y'(x)=\frac{1}{x}$ $\quad \bar{y}'(x)=2$

$\left(\frac{1}{x}-2\right)' = -\frac{1}{x^2}$ $\qquad -\frac{1}{x^2}=0$, $x \neq \emptyset$

$\left|\frac{1}{x}-2\right|\Big|_{x=e^{-2}} = \frac{1}{e^{-2}}-2 = e^2-2$

$\left|\frac{1}{x}-2\right|\Big|_{x=e} = \frac{1}{e}-2$

$\rho_1 = \max\{\max_{e^{-2}\le x\le e}|y-\bar{y}|, \max_{e^{-2}\le x\le e}|y'-\bar{y}'|\} = \max[8; e^2-2] = 8$

(2) for the second order $y(x)=x$, $\bar{y}(x)=-\cos x$ on $[0,\frac{\pi}{2}]$

$f(x)=x+\cos x$ $\qquad f(x)\geq 0$

$f'(x)=1-\sin x$ $\qquad \rho_0 = \max_{[0,\pi]}|f(x)| = (x+\cos x)\big|_0 = 0+1=1$ 0th order

$1-\sin x=0$

$\sin x = 1$

$x=\frac{\pi}{2}$ $\qquad \rho_1 = \max|f(x)| = (1-\sin x)\big|_{\frac{\pi}{2}} = 1-1=0$

$f''(x)=-\cos x$ $\qquad$ 1st order

$f''(\frac{\pi}{2}) = -\cos\frac{\pi}{2} = 0$ $\qquad$ ~~1st order~~ $f'(x)=1-\sin x$

$f''(x)=-\cos x$

1st order $\qquad f'''(x)=\sin x = 0$

$y'(x)=1$ $\qquad \bar{y}(x)=-\sin x$ $\qquad x=0$

$y''(x)=-\cos x$ $\qquad \rho_0 = \max|-\cos x|\big|_0 = -1$

$-\cos x = 0$ $\qquad x=0$

$x=\frac{\pi}{2}$ $\qquad (-\cos x)\big|_0 = 1$ $\quad \rho_2 = \max\{\max|y-\bar{y}|, \max|y'-\bar{y}'|, \max|y''-\bar{y}''|\} =$

$x=\pi$

$(-\cos x)\big|_\pi = -1$ $\quad = \max\{1, 0, -1\} = 1.$

Problem 5.

Find the extremum function of $I[y]$ with Boundary Conditions.

(1) $J[y]=\int_0^1 (y'^2-y^2-y)e^{2x}dx \quad y(0)=0 \quad y(1)=e^{-1}$

$F(x,y,y')=y'^2-y^2-y$

$\frac{\partial F}{\partial y}=-2y-1 \quad \frac{\partial f}{\partial y'}=2y'$

$\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}=0$

$-2y-1-\frac{d}{dx}(2y')=-2y-1-2y''=0$

$2-2\lambda^2=0$

$-2(\lambda-1)(\lambda+1)=0$

$\lambda_1=1 \quad \lambda_2=-1$

$\bar{y}=C_1e^x+C_2e^{-x}$

$y_0=A \quad y_0''=0$

$2A=1$

$A=\frac{1}{2}$

$y=C_1e^x+\frac{C_2}{e^x}+\frac{1}{2}$

$y(x)=\left(-\frac{1}{2}-\frac{2+e^2-e}{-2e^2+2}\right)e^x+\left(\frac{2+e^2-e}{-2e^2+2}\right)e^{-x}+\frac{1}{2}$

$y(0)=0 \Rightarrow C_1e^0+\frac{C_2}{e^0}+\frac{1}{2}=0$

$y(1)=e^{-1} \Rightarrow C_1e+\frac{C_2}{e}+\frac{1}{2}=e^{-1}$

$\begin{cases} C_1+C_2+\frac{1}{2}=0 \\ C_1e+\frac{C_2}{e}+\frac{1}{2}=e^{-1} \end{cases} \Rightarrow$

$C_1=-\frac{1}{2}-C_2$

$\left(-\frac{1}{2}-C_2\right)e+\frac{C_2}{e}+\frac{1}{2}=e^{-1}$

$-\frac{1}{2}e-C_2e+\frac{C_2}{e}+\frac{1}{2}=e^{-1}$

$-C_2e+\frac{C_2}{e}=e^{-1}+\frac{1}{2}e-\frac{1}{2}$

$C_2\left(-e+\frac{1}{e}\right)=\frac{1}{e}+\frac{e}{2}-\frac{1}{2}$

$C_2=\frac{2+e^2-e}{-2e^2+2}$

$C_1=-\frac{1}{2}-\frac{2+e^2-e}{-2e^2+2}$

(2) $J[y]=\int_1^e (xy'^2+yy')dx \quad y(1)=0 \quad y(e)=1$

$F(x,y,y')=xy'^2+yy'$

$\frac{\partial F}{\partial y}=y' \quad \frac{\partial F}{\partial y'}=x2y'+y$

$\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}=0$

$y'-\frac{d}{dx}(x2y'+y)=0$

$y'-x2y''+y'=0$

$2y'-x2y''=0 \,/\cdot\frac{1}{2}$

$\cancel{\lambda_1=0 \quad \lambda_2=\frac{2}{x}}$

$y'-xy''=0$

$-xy'+2y=C$

$-xy'=C-2y$

$y'=\frac{C-2y}{x}$

$\frac{dy}{2y-C}=\frac{dx}{x}$

$\frac{\ln(2y-C_1)}{2}=\ln x+C_2$

$y=C_2x^2+C_3$

$y(1)=C_2+C_1=0$

$y(e)=C_2e^2+C_1=1$

$\begin{cases} C_2+C_1=0 \\ C_2e^2+C_1=1 \end{cases}$

$C_1=-C_2$

$C_2e^2-C_2=1$

$C_2(e^2-1)=1$

$C_2=\frac{1}{e^2-1}$

$C_1=-\frac{1}{e^2-1}$

$y(x)=\frac{1}{e^2-1}x^2-\frac{1}{e^2-1}$

(3) $J[y] = \int_0^1 \sqrt{y(1+y')^2}\,dx \qquad y(0) = y(1) = \frac{1}{\sqrt{2}}$

$F(x,y,y') = \sqrt{y(1+y')^2} = \sqrt{y(1+2y'+y'^2)} = \sqrt{y+2yy'+yy'^2}$

$\frac{\partial F}{\partial y} = \frac{y'+2y'+1}{2|\sqrt{y}+\sqrt{y}y'|} \qquad \frac{\partial F}{\partial y'} = \frac{y+y'y}{|\sqrt{y}+\sqrt{y}y'|}$

$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$

$\frac{y'+2y'+1}{2|\sqrt{y}+\sqrt{y}y|} + \frac{d}{dt}\left(\frac{y+y'y}{\sqrt{y}+\sqrt{y}y'}\right) = 0$

$\frac{y'+2y'+1}{2(\sqrt{y}+\sqrt{y}y)} + \frac{y'+y''y'}{\sqrt{y'}+\sqrt{y'}y''} = 0$

$y' = u \quad y'' = uu'$

$\frac{y(2uu'y-u^2-1)}{(u^2y+1)^{\frac{3}{2}}} = 0$

$y(2uu'y-u^2-1) = 0$

$2uu'y - u^2 - 1 = 0$

$2uu'y = u^2+1$

$uu' = \frac{u^2+1}{2y}$

$u\frac{du}{dy} = \frac{u^2+1}{2y}$

$\frac{u\,du}{u^2+1} = \frac{dy}{2y}$

$\ln(u^2+1) = \ln y + C$

$u^2+1 = e^C y$

$y'^2 = Cy - 1 \qquad y = \frac{p^2}{C} + \frac{1}{C}$

$dy = 2p\,dp/C$

$dy = p\,dx = \frac{2p\,dp}{C} \rightarrow$

$dx = \frac{2\,dp}{C}$

$p = C_1x - C_1C_2$

$y = \frac{C_1^2x^2 - 2C_1^2C_2^2x + C_1^2C_2^2 + 4}{4C_1}$

$y = C_1\left(\frac{x^2}{4} - \frac{C_2x}{2} + \frac{C_2^2}{4}\right) + \frac{1}{C_1}$

$y(0) = \frac{1}{\sqrt{2}}$

$y(1) = \frac{1}{\sqrt{2}}$

4) $J[y] = \int_0^1 yy'^2\,dx \qquad y(0) = 1 \quad y(1) = \sqrt[3]{4}$

$F(x,y,y') = yy'^2$

$\frac{\partial F}{\partial y} = y'^2 \qquad \frac{\partial F}{\partial y'} = 2yy'$

$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$

$y'^2 - 2y'y'' = 0$

$-2uu'y + u^2 = 0 \;/u$

$-2u'y + u = 0$

$-2u'y = -u$

$u' = \frac{u}{2y}$

$\int \frac{1}{u}\,du = \int \frac{1}{2y}\,dy$

$u = e^C\sqrt{y}$

$u = C\sqrt{y}$

$\int \frac{1}{\sqrt{y}}\,dy = \int C\,dx$

$y = (C_1x + C_2)^2$

$y = (C_1 0 + C_2)^2 = 1$

$y = C_2^2 = 1$

$C_2 = \pm 1 \qquad C_1 = 0$

$y(x) = 1.$

5) $J[y(x)] = \int_a^b (y^2 + y'^2 + 2ye^x)\,dx \qquad y(0) = 2 \quad y(\frac{\pi}{2}) = \frac{a}{2}$

$F(x,y,y') = y^2 + y'^2 + 2ye^x$

$\frac{\partial F}{\partial y} = 2y + 2e^x \qquad \frac{\partial F}{\partial y'} = 2y'$

$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$

$2y + 2e^x - 2y'' = 0$

$-(\lambda-1)(\lambda+1) = 0$

$\lambda = 1 \quad \lambda = -1$

$y_p = Ce^x + \frac{C_1}{e^x}$

$y_0'' = (Ax + 2A)e^x$

$-2Ae^x = -e^x$

$A = \frac{1}{2}$

$y_c = \frac{xe^x}{2}$

$y(x) = C_1e^x + \frac{C_2}{e^x} + \frac{xe^x}{2}$

$\begin{cases} C_1 + C_2 = 2 \\ C_1e^{\frac{\pi}{2}} + \frac{C_2}{e^{\frac{\pi}{2}}} + \frac{\frac{\pi}{2}e^{\frac{\pi}{2}}}{2} = 2ch\frac{\pi}{2} \end{cases}$

$C_1 = 2 - \frac{4e^\pi + 4 - e^\pi\pi}{4e^\pi + 4}$

$C_2 = \frac{-4e^\pi + 4 - e^\pi\pi}{-4e^\pi + 4}$

$y(x) = \left(2 - \frac{4e^\pi + 4 - e^\pi\pi}{4e^\pi + 4}\right)e^x + \left(\frac{4e^\pi + 4 - e^\pi\pi}{4e^\pi + 4}\right)e^{-x} + \frac{xe^x}{2}$

Problem 5

6) $J[y(x)] = \int_0^{\frac{\pi}{2}} (y^2 - y'^2 - 8y\,ch x)\,dx \qquad y(0) = 2 \qquad y(\frac{\pi}{2}) = 2ch\frac{\pi}{2}$

$F(x, y, y') = y^2 - y'^2 - 8y\,ch x$

$\frac{\partial F}{\partial y} = 2y - 8ch x \qquad \frac{\partial F}{\partial y'} = 2y'$

$2y - 8ch x - 2y'' = 0$

$y - 4ch x - y'' = 0$

$-(\lambda - 1)(\lambda + 1) = 0$

$\lambda_1 = 1 \qquad \lambda_2 = -1$

$y_P = C_1 e^x + \frac{C_2}{e^x}$

$\begin{cases} C_1 y_1 + C_1' y_2 = 0 \\ C' y_1 + C_1' y_2' = \frac{f(x)}{a_0} \end{cases}$

$y_1 = e^x \qquad y_2 = \frac{1}{e^x}$

$y_1' = e^x \qquad y_2' = -\frac{1}{e^x}$

$\begin{cases} e^x C'(x) - \frac{C_1'(x)}{e^x} = 0 \\ e^x C'(x) - C_1'(x) = \frac{4ch(x)}{-1} \end{cases}$

$W = \begin{vmatrix} e^x & \frac{1}{e^x} \\ e^x & -\frac{1}{e^x} \end{vmatrix} = -2$

$W_1 = \begin{vmatrix} 0 & \frac{1}{e^x} \\ -4ch x & -\frac{1}{e^x} \end{vmatrix} = \frac{4ch x}{e^x}$

$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & -4ch\,x \end{vmatrix} = 4e^x ch x$

$C_1' = 2e^x ch x = \frac{W_2}{W}$

$C' = \frac{W_1}{W} = \frac{2ch x}{e^x}$

$C(x) = \int C'(x)\,dx = \frac{1}{2e^{2x}} - x + C_2$

$C_1(x) = \int C_1'(x)\,dx = \frac{e^{2x}}{2} + x + C_3$

$y = xe^x + C_2 e^x + \frac{e^x}{2} + \frac{x - C_3}{e^x} + \frac{1}{2e^x}$

$\begin{cases} C_2 + \frac{1}{2} - \frac{C_3}{1} + \frac{1}{2} = 2 \\ \frac{\pi}{2} e^{\frac{\pi}{2}} + C_2 e^{\frac{\pi}{2}} + \frac{e^{\frac{\pi}{2}}}{2} + \frac{\frac{\pi}{2} - C_3}{e^{\frac{\pi}{2}}} + \frac{1}{2e^{\frac{\pi}{2}}} = 2ch\frac{\pi}{2} \end{cases}$

$\begin{cases} C_2 + C_3 = 1 \\ \frac{\pi}{2} e^{\frac{\pi}{2}} + C_2 e^{\frac{\pi}{2}} + \frac{e^{\frac{\pi}{2}}}{2} + \frac{\frac{\pi}{2} - C_3}{e^{\frac{\pi}{2}}} + \frac{1}{2e^{\frac{\pi}{2}}} = 2ch\frac{\pi}{2} \end{cases}$

$C_2 = 1 + \frac{1 - e^{\pi}\pi - e^{\pi} - \pi}{2e^{\pi} - 2}$

$C_3 = \frac{1 - e^{\pi}\pi - e^{\pi} - \pi}{2e^{\pi} - 2}$

$y = xe^x + \frac{1 - e^{\pi}\pi - e^{\pi} - \pi}{2e^{\pi} - 2} C_2 + \frac{e^x}{2} +$

$+ \; x - \frac{1 - e^{\pi}\pi - e^{\pi} - \pi}{2e^{\pi} - 2} + \frac{1}{2e^x}$

Problem 4

3) $y = e^{3x} \qquad \bar{y} = 3x \qquad$ en $[0, 3]$

0th order

$3e^{3x} - 3 > 0$

$e^{3x} = 1$

$x = 0$

$y(0) = 1.$

1st order

$\max_{[0,3]} [3e^{3x} - 3]$

$y(0) = 1$

$y(3) = (e^3 - 1)$

2nd order

$\max_{[0,3]} [9e^{3x}] = 9e^{3x}$

$y(0) = 1$

$y(3) = 9e^9$

$\rho_{1001} = \max_{0 \le x \le 1001} \{1, 3(e^9 - 1), 9e^9, 27e^9, 3^{1001} e^{9}\}$