STREETER-PHELPS BOD-DO MODEL,

THOMAS MODEL &

THE LOTKA-VOLTERRA MODEL

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Abstract

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Meaning of Abbreviations:

BOD: Biological Oxygen Demand
DO: Dissolved oxygen

Abstract

Models are abstraction of reality, they help to explain the complex relationships that exist within the environment using mathematical relationships – this is essentially typical of the models used to explain the complex interactions that takes place in aquatic ecosystems. As such, this report presents the results of three models that were implemented during the surface water laboratory classes. Specifically including the Streeter-Phelps BOD-DO model with its' improved version of Thomas Model, and the Lotka-Volterra Model. The various analyses conducted in this report revealed that, there is an inverse relationship between BOD and DO over time. It further revealed that, there are some limitations of the Streeter-Phelps BOD-DO model, which notably include lack of consideration for oxygen removal by sedimentation. This was however verified by the Thomas Model - which revealed that sedimentation coefficient is an important factor in determining oxygen concentration, as it contributes to oxygen depletion in the water through decomposition. In addition, the Lotka-Volterra Model revealed that there is a direct periodic relationship between the prey and the predator within an ecosystem, and that the population of the predator is influenced by the abundance of the prey population. The findings made in these analyses are all in-line with affirmations made in literatures regarding the models considered in this report.

1.0 Introduction

Water is a substance that consists of hydrogen and oxygen and existing in gaseous, liquid, and solid states. It is one of the most abundant and the most essential of all available compounds (Zumdahl, 2021). Surface water, however, is any body of water found on the Earth's surface, including both the saltwater in the ocean and the freshwater in rivers, streams, and lakes (National Geogrpahic, 2021). The benefits of surface water is countless – as virtually all life forms depends upon it for survival. Surface water in its various forms as either rivers, lakes, and swamps are harnessed and used in everyday life for cooking, washing, bathing, and drinking. They also offer supports through income generation from fish cultivation for irrigation and recreation and lots more (Deep Ocean Facts, 2021).

Despite the great benefits that humans derive from surface water, they are constantly being threatened - surface water resources face a host of serious threats, all of which are caused primarily by human activity (UN, 2006). They include sedimentation, pollution, climate change, deforestation, landscape changes, and urban growth. All of which consequently affects the various interactions within the aquatic ecosystems. One of the most serious threats to water resources is Pollution, which often takes place when harmful substances—often chemicals or microorganisms—contaminate a stream, river, lake, ocean, aquifer, or other body of water, thereby reducing oxygen levels in the water, degrading water quality and rendering it toxic to lives within the water, the environment, and humans. Against this backdrop, several models have been developed to manage and understand surface water quality and mechanisms. These models are critically important tools as they help to better understand the interactions between the various elements of surface water in response to pollution, ecological balance and food chain, and how to protect them. Therefore, this report is based on three important models used during the laboratory session of the course of Surface Water protection. The Streeter-Phelps BOD-DO model, Thomas Model and the Lotka-Volterra Model, the details of the models and how they were implemented are therefore discussed in subsequent sections.

1.1 Materials and Methods - The Streeter-Phelps BOD-DO model

The Streeter-Phelps BOD-DO model is often used as a water quality modelling tool in the study of water pollution (Streeter & Phelps, 1925). It describes how dissolved oxygen declines in a river along a certain distance by degradation of biological oxygen demand. According to Paweł, (2021), the model is built on the following assumptions;

- the modelled waterbody is a perfect mixing tank, with a mean depth h and no stratification,
- the water flow is time-independent (i.e. constant in time): the velocity of water is constant, the water temperature is constant,
- only two processes are taken into account: oxygen consumption for organic matter decomposition and aeration; organic matter decay is assumed to follow 1st order reactions.

The various parameters used in the model are depicted in Table 1. In addition, the equations used in the analyses are also stated (as provided in the laboratory manual).

Table 1. Variables and Parameters of the Streeter-Phelps BOD-DO model

Symbol	Units	Meaning
C	mg/l	dissolved oxygen concentration
S	mg/l	oxygen concentration at full saturation for given temperature and pressure
D	mg/l	oxygen deficit
BOD	mgO_2/l	biological oxygen demand
$\mathbf{k_1}$	1/d	organic matter decay coefficient
$\mathbf{k_2}$	1/d	aeration coefficient
$\mathbf{k_3}$	1/d	sediment sedimentation coefficient
T	◦C	water temperature
u	m/s	water flow velocity
h	m	mean water depth
$\mathbf{BOD_0}$	mgO_2/l	initial biological oxygen demand
$\mathbf{C_0}$	mg/l	initial dissolved oxygen concentration

Source: Practical manual – by Paweł Gilewski, 2021

The equations used in the Streeter-Phelps BOD-DO model

The **1st order oxygen consumption for BOD decay:**

$$\frac{dC}{dt} = -k_1 \times BOD \tag{1}$$

where: C = dissolved oxygen concentration
BOD = biological oxygen demand
k_1 = the organic material decay coefficient.
And both C and BOD are time-varying state variables,

Therefore **oxygen deficit:**

$$D = S - C \tag{2}$$

By differentiating both sides of the equation 2 we obtain:

$$\frac{dD}{dt} = -\frac{dC}{dt} \tag{3}$$

Hence, the **aeration process** was expressed as:

$$\frac{dC}{dt} = k_2 \times D \tag{4}$$

Where; k_2 is the aeration coefficient, and dependent on the temperature T, water velocity u and the mean water depth h.

$\boldsymbol{k_2}$ was expressed using the **empirical equation**:

$$k_2 = \frac{3.09u^{0.5}(1.025^{T-20})^{0.5}}{h^{0.5}} \tag{5}$$

And Consequently obtained the **state equation of the Streeter-Phelps** model:

$$\frac{dC}{dt} = -k_1 \times BOD + k_2 \times D \tag{6}$$

In order to reduce the number of variables we then expressed the **dissolved oxygen concentration using oxygen deficit** as:

$$\frac{dD}{dt} = k_1 \times BOD - k_2 D \tag{7}$$

Also, the **BOD decay equation** was used, which describes organic matter decomposition:

$$\frac{dBOD}{dt} = -k_1 \times BOD \tag{8}$$

And, **Thomas model equation for BOD** decay is expresses as:

$$\frac{dBOD}{dt} = -(k_1 - k_3) \times BOD \tag{9}$$

For this simulation, R-studio version 6.4.1.2 was used. As such, the equations highlighted in the previous sections were compiled in R-studio, by using the values shown in Table 2 with their corresponding parameters to create an initial condition. In order to implement the model in R-studio, A new directory was created for the code, and results that was generated was then saved inside the directory. Thereafter, a new R-notebook file was created, and the codes were consequently written in the notebook within individual R Chunks. The page was formatted appropriately, and library deSolve was loaded. Thereafter, the following procedures were followed for the full implementation of the model in R-studio;

- Equations eight (8) and seven (7) were first coded using the appropriate R syntax and semantics.
- Dissolved oxygen saturation was defined as a function of temperature
- The flow velocity (u), water depth (h), water temperature (T), where defined as variables
- The Aeration coefficient (k2) was defined using equation 5
- The values of the remaining parameters shown in Table 2 were declared, with the initial of value of BOD declared (and altered accordingly during the simulation)
- The Numerical solutions of the dependent variables in the model was thereafter obtained.
- Finally, the results were plotted with appropriate chat elements
- The process was repeated as required in the task.

Table 2: Parameters for Initial Conditions

S/N	Values	Parameters	Units

1	0.3	u	m/s
2	3	h	m
3	20	T	^0C
4	0.2	K_1	1/d
5	6	C_0	mg/l
6	10	BOD_0	mg/l

Source: Practical manual – by Paweł Gilewski, 2021

Where stoichiometric factor s = 3

1.1.1 Discussion of Results - The Streeter-Phelps BOD-DO model

For the initial trial, that is, when the value of BOD was taken to be 10 mg/l, the results depicted in Figure 1 and Table 3 was obtained for a period 30 days. The result shows that there is an inverse relationship between BOD and DO over time. As BOD decreases, DO though, first decrease shortly, and thereafter continue to increase up to the point (the last time step) where BOD achieves the value of 0.0247876mg/l. In contrast at that point, DO achieves the value of 9.002671mg/l. this results proves the correctness of the model as it validates the actual inverse relationship that exists between BOD and DO in aquatic ecosystems. The plot of BOD and DO which graphically illustrate the inverse relationship that exist between the two parameter is therefore depicted in Figure 1 and 2 for initial values of BOD taken to be 10 and 15 mg/l respectively.

As requested in the to-do tasks noted in the practical manual, using the code already prepared, further verification of the performance of the model was caried out. This was achieved by simply modifying some of the values assigned to the parameters. For instance It was discovered that **15 mg/l of initial BOD is required to achieve dissolved oxygen concentration of 7 mg/l after 10 days – this was achieved by using the code res[10,3],** which returns DO value of **7.10925 mg/**l (**approx. 7.1 mg/l**)for the tenth day.

In order to verify the situations where the Streeter-Phelps model provides unreliable result, several initial conditions of BOD were tried. After several attempts, negative result **(i.e -0.3656862 mg/l**) was obtained as value of dissolved oxygen at time step 3, for initial BOD value of **23 mg/l**. Negative oxygen concentration continue to be calculated over series of time steps as increased BOD values are continued to be supplied into the model. The negative values of DO calculated at some time steps by the model for BOD of 23 mg/l upwards is not practical and not reliable. Hence, higher initial BOD values gives unreliable result.

There are several limitations of the Streeter-Phelps model. For example the model does not consider;

- removal of BOD by sedimentation or adsorption and its addition along the stretch
- Removal of DO by diffusion into the benthal layer
- Addition of oxygen by photosynthetic action of plankton and plants
- Removal of oxygen by respiration
- Continuous redistribution of both BOD and DO by longitudinal dispersion

1.1.2 The Thomas Model

On the basis of these limitations, Thomas model, shown in equation 9, seek to address the sedimentation limitation within the Streeter-Phelps model. As a result, Thomas was model was simulated in this task, with the sedimentation coefficient k3 assumed to be 0.1. The New result obtained for the simulation using the Thomas model, and taken the initial value of BOD to be 10 mg/l is therefore shown in Table 4 and Figure 4. The result obtained here varies from the result previously obtained when the sedimentation coefficient was not included in the equation. The concentration of dissolved oxygen at 31st time step (as shown in Table 5) is **8.681245mg/l**, which is different from **9.002671mg/l** obtained when the sedimentation coefficient was not considered. In addition, across all the time steps, the minimum concentration of DO in Thomas model is **4.319064mg/l** (at time step 4), while that of the Streeter-Phelps is **4.940292mg/l** (also obtained at time step 3). This results obviously proves that sediments affects/reduces dissolved oxygen concentration in water. It also proves the limitation of the Streeter-Phelps model, while at the same time, validating the Thomas model, by showing that sedimentation plays important role in oxygen concentration and biochemical oxygen demand in aquatic ecosystems.

Further still, in order to show how flow velocity influences minimum concentration of oxygen, the previous steps were repeated for different mean water velocities. In order to have a robust basis for comparison, the flow velocities were taken to be 0.1, 0.5, and 0.7 m/s, against the initial 0.3m/s. The result indicated that, when the flow velocity is 0.1, the minimum oxygen concentration was 1.710242 mg/l, when the flow velocity was 0.5 m/s, the minimum oxygen concentration was 5.206823mg/l, and when the flow velocity was 0.7, the minimum oxygen concentration was 5.712048mg/l, the inference that can be derived from this is that the flow velocity greatly influence the oxygen concentration in water. Specifically, the higher the flow velocity, the more the dissolved oxygen concentration in the water. This is essentially in line with the affirmation made on the WaterWeb, (2008) that flow velocity directly affects the amount of oxygen dissolved in the water. Higher volumes of faster moving water increases the turbulent diffusion of atmospheric oxygen into the water. While, low flow conditions are much less conducive to oxygenation, thereby making oxygen concentration lower when the flow is low.

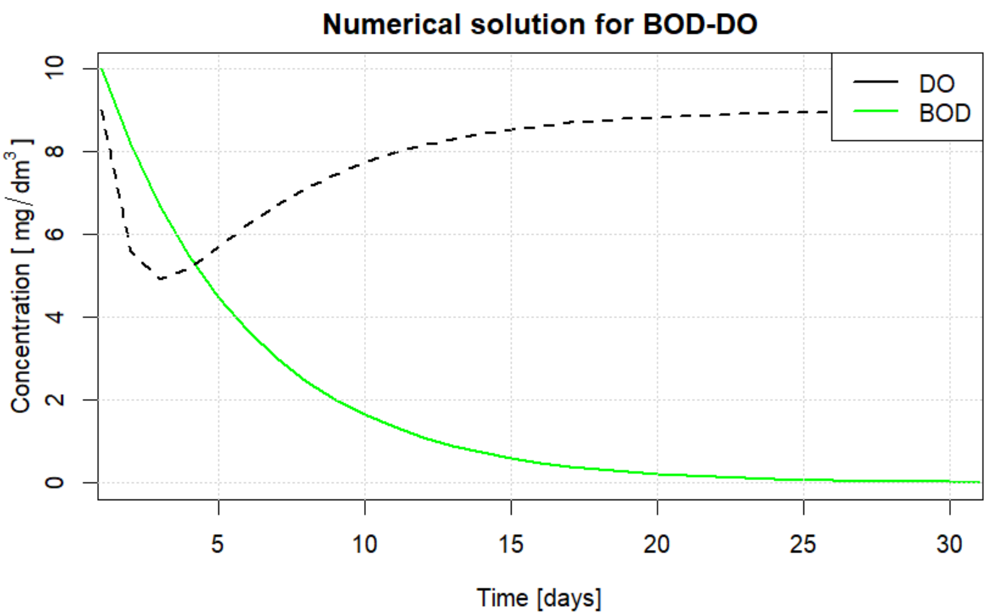

Figure 1: Showing BOD and DO Concentration over time at BOD_0 = 10 mg/l

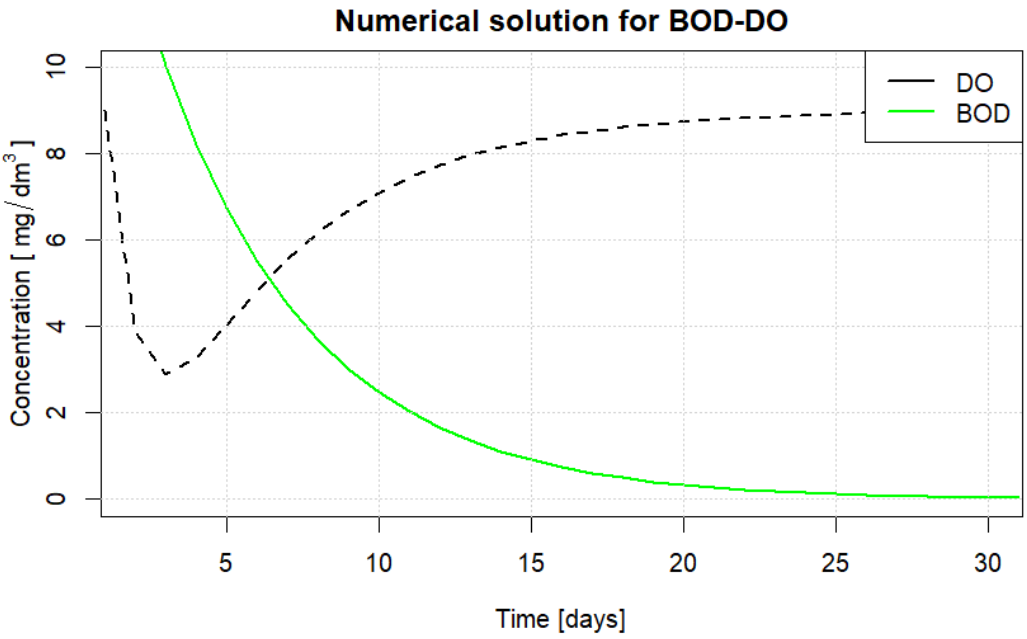

Figure 2: Showing BOD and DO Concentration over time at BOD_0 = 15 mg/l

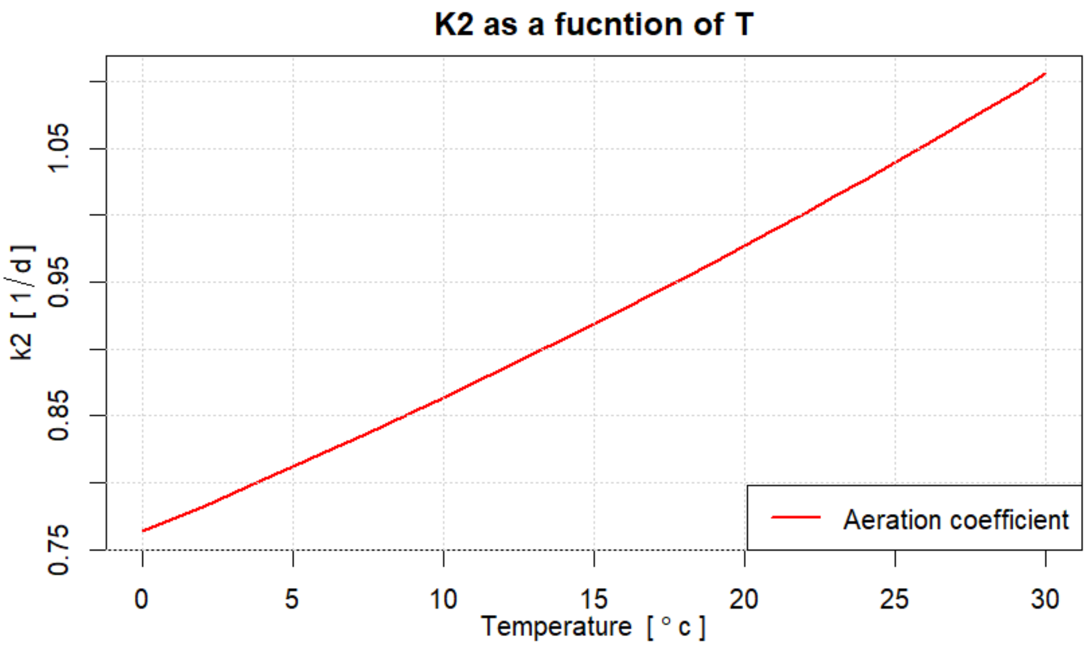

Figure 3: Showing Aeration coefficient K2 as a function of Temperature T at BOD_0 = 10 mg/l

As shown in Figure 3, there is a significant direct relationship between aeration coefficient and temperature. As temperature increases, aeration coefficient also increases. This is as a result of the inverse relation between temperature and DO. As the temperature of the water increases, more air is required to dissolve the oxygen, thereby leading to increased aeration coefficients within the system. Higher aeration values indicates lower levels of oxygen – that is, more oxygen is needed in the water.

Table 3: Results of the Numerical Solution, showing BOD and DO concentration over 30 days for BOD = 10 mg/l

Time	BOD	DO
1	10	9.021808
2	8.187308	5.606638
3	6.703201	4.940292
4	5.488117	5.196334
5	4.49329	5.707673
6	3.678795	6.239885
7	3.011942	6.718365
8	2.46597	7.126199
9	2.018965	7.46616
10	1.652989	7.746775
11	1.353353	7.977382
12	1.108031	8.16651
13	0.907179	8.321476
14	0.742735	8.448397
15	0.608099	8.552329
16	0.497871	8.637427
17	0.407622	8.707101
18	0.333733	8.764147
19	0.273237	8.810853
20	0.223708	8.849092
21	0.183157	8.880401
22	0.149956	8.906033
23	0.122774	8.92702
24	0.100518	8.944202
25	0.082297	8.95827
26	0.067379	8.969787
27	0.055165	8.979217
28	0.045165	8.986938
29	0.036978	8.993259
30	0.030275	8.998434
31	0.024787	9.002671

Source: Laboratory tutorial – Led by Paweł Gilewski, 2021

Table 4: Numerical Solution (Using Thomas Model) for BOD = 10 mg/l

Time	BOD	DO
1	10	9.021808
2	9.048374	5.406984
3	8.187308	4.390425
4	7.408182	4.319064
5	6.703201	4.573842
6	6.065307	4.924576
7	5.488116	5.287174
8	4.965853	5.632294
9	4.49329	5.950981
10	4.065697	6.241752
11	3.678794	6.505761
12	3.328711	6.744989
13	3.011943	6.961579
14	2.72532	7.157605
15	2.465971	7.334997
16	2.231303	7.495514
17	2.018966	7.640759
18	1.826836	7.772183
19	1.65299	7.8911
20	1.495687	7.998701
21	1.353354	8.096062
22	1.224565	8.184159
23	1.108032	8.263872
24	1.002589	8.335998
25	0.90718	8.401262
26	0.820851	8.460315
27	0.742737	8.513748
28	0.672056	8.562096
29	0.608101	8.605843
30	0.550232	8.645428
31	0.497871	8.681245

Source: Laboratory tutorial – Led by Paweł Gilewski, 2021

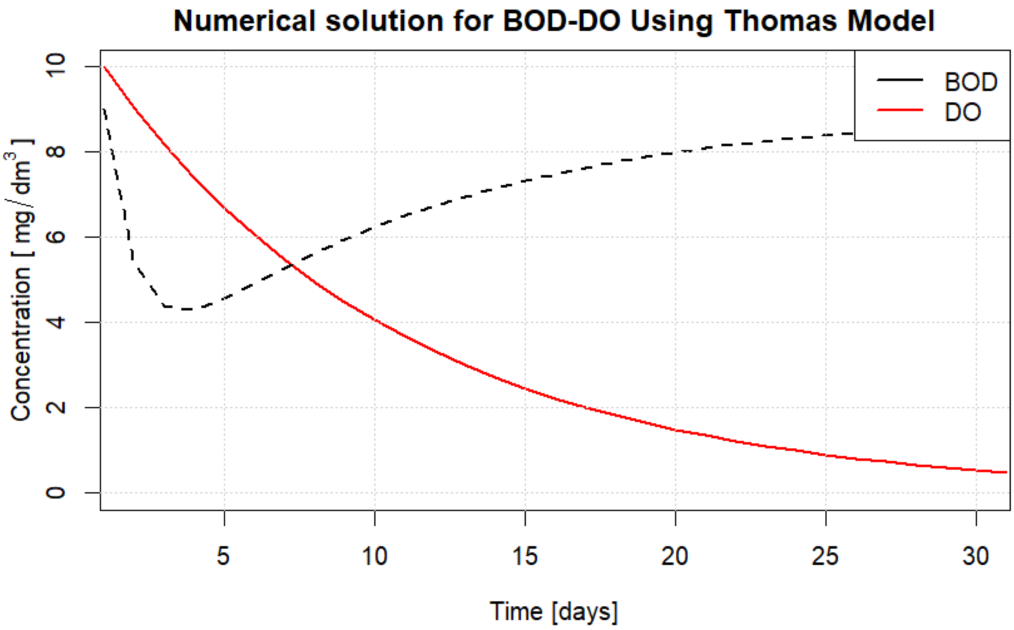

Figure 4: Showing BOD and DO Concentration over time (using Thomas Model) at BOD_0 = 10 mg/l

1.2 Materials and Methods - Lotka-Volterra Model

This section deals with the Lotka-Volterra Model, which is the second model considered during the practical class. The Lotka-Volterra Model is used to model population dynamics; biological, economic and social systems, and it can be applied to all forms of ecosystems beyond aquatic. The Lotka–Volterra model assumes that the prey consumption rate by a predator is directly proportional to the prey abundance, which implies that the predator feeding is limited only by the amount of prey in the environment. The model is built on the following assumptions;

- the prey population always finds enough food to grow. Its growth is only limited by constant growth rate,
- the food supply of the predator population depends entirely on the prey populations,
- the rate of change of population is proportional to its size and constant coefficients.
- The environment capacity is unlimited,
- any evolution changes in the environment are not considered.

As such, the description of the model and chronological order of how it was applied during the laboratory sessions is therefore discussed in the subsequent sections of this report.

Paweł, (2021) affirmed that Lotka-Volterra uses two state variables:

x(t) = prey population density

y(t) = predator population density.

And both state variables are time varying.

The prey equation:

$$\frac{dx}{dt}=\alpha \times x-\beta \times x \times y \qquad (1)$$

Where the term $\alpha \times x$ is responsible for exponential growth of prey population (assuming unlimited food supply). Prey population decay due to predator grazing is represented by

The predator equation:

$$\frac{dy}{dt}=\delta \times x \times y-\gamma \times y \qquad (2)$$

Where the term $\delta \times x \times y$ is responsible for predator growth, similar to the term $\beta \times x \times y$ in Eq. 1. And where, β is not necessarily equal to δ. Analogously $\gamma \times y$ describes predator population decay.

Parameters Used in the Model:

α - prey growth rate,
β - grazing efficiency coefficient,
δ - predator decay rate,
γ - predator growth rate per prey unit.

Equations 1 and 2 were implemented in R-studio 4.1.2 using appropriate syntax and semantics. First. the simulation was carried out using the parameters shown in Table 5. Thereafter, the value of the initial predator population density was changed to 0. The details of the results obtained for both the initial conditions and the modification is therefore presented in the next section.

Table 5: Parameters (with values) used for Lotka-Volterra Model

S/N	Values	Parameters
1	1	prey growth rate α
2	2	grazing efficiency coefficient β
3	1	predator decay rate δ
4	1	predator growth rate per unit γ
5	1	initial prey density
6	0.01 (Later changed to 0)	initial predator density
7	from 0 to 50 with interval of 1.	time span:

Source: Practical manual – by Paweł Gilewski, 2021

1.2.1 Discussion of Results - Lotka-Volterra Model

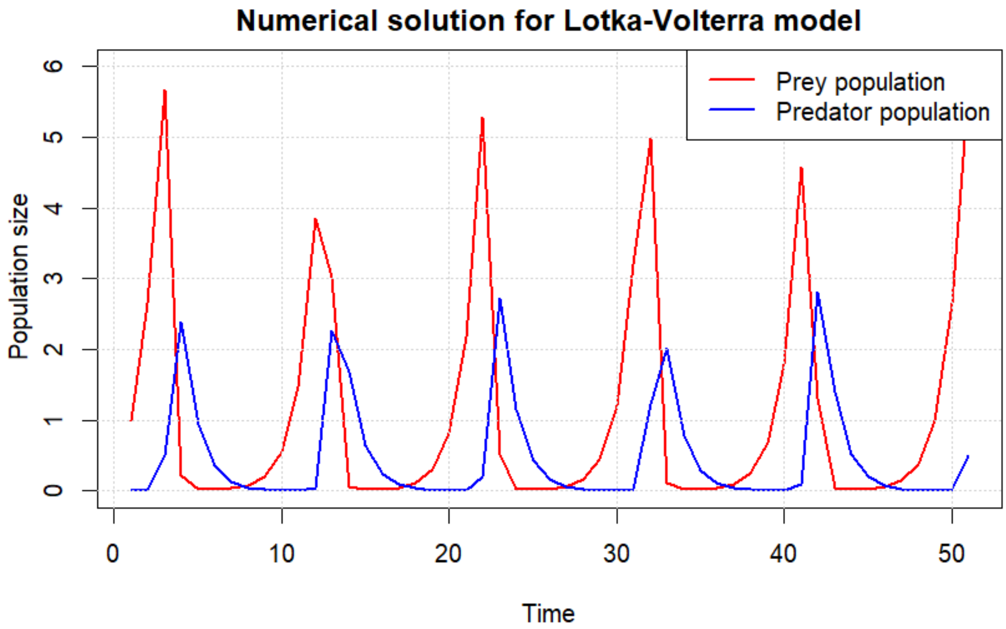

Figure 5: Showing Prey Predator Population over time (at initial predator population density = 0.01)

The result of the Lotka-Volterra Model shown in Figure 5 revealed that there is a direct relationship between the prey and the predator. That is, the population of the predator is influenced by the abundance of the prey population. When the prey population is low, predator population is also low, and when the prey population is high, predator population is also high. As such, the periodic relationship observed in the plot, reflects the highs and lows of population of both the prey and the predator. The result revealed by this simulation essentially validates Shim & Fishwick, (2008) affirmation that the Lotka–Volterra model assumes that the prey consumption rate by a predator is directly proportional to the prey abundance. Hence, the predator feeding is limited only by the amount of prey in the environment.

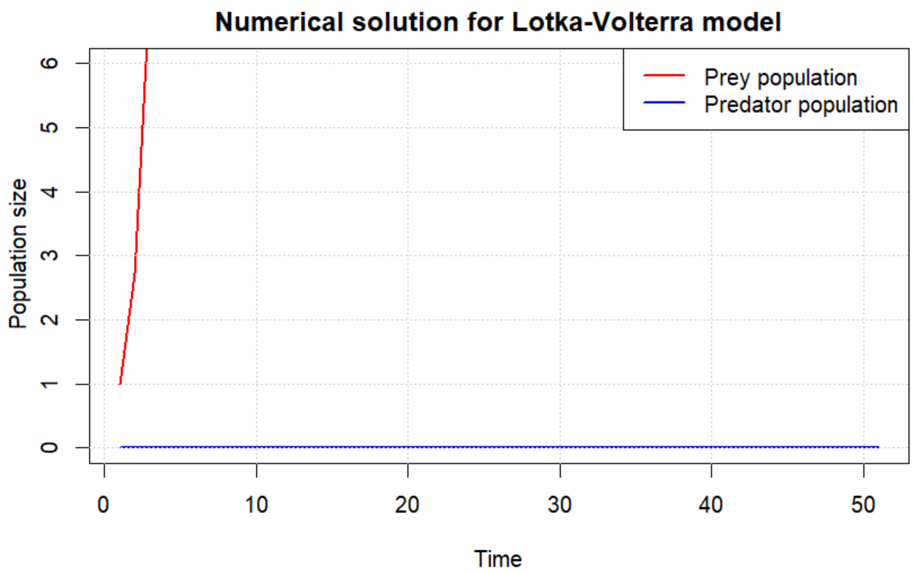

Figure 6: Showing Prey Predator Population over time (at initial predator population density = 0)

After attempting to observe the performance of the model for when initial predator density is 0.01, another attempt was made to observe the performance for when initial predator density is 0. The result depicted in Figure 6 revealed that there is a break down in the system – and that the prey population is unaffected by predator (since the predator density is 0), and also the prey continue to grow based on the prey growth rate α. The results obtained for these two scenarios depicted in Figure 5 and 6 revealed the population dynamics within a food chain, given the constraints and the assumption imposed by the model.

2.0 Conclusion

As we have seen from the analyses conducted, there is an inverse relationship between BOD and DO. That is, a decline in DO levels reflects a high level of BOD, while an increase in DO reflects a low level of BOD. This was evident from the simulation results obtained with the aid of the Streeter-Phelps BOD-DO, and the improved Thomas Model. As BOD decreases, DO on the other hand increases, with temperature and sediments playing important roles in oxygen solubility. Introduction of sedimentation coefficient into the model led to depletion of BOD. Conversely, as temperature increases, the aeration coefficient also increases. In addition, flow velocity greatly influence oxygen concentration in water. the higher the flow velocity, the more the dissolved oxygen concentration in water, and vice-versa.

Moreover, the Lotka-Volterra Model revealed that there is direct relationship between the prey and the predator within an ecosystem – the higher the prey, the more abundant the predator, and the lower the prey population, the less abundant the predator population. Although, all the models have considerable limitations, nevertheless, they are invaluable in helping to analyse, understand, and explain the complex relationships that exist between organisms, and as well as their interactions with their environment.

3.0 References

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