$$\frac{d\bar{v}}{d\tau} = \frac{3}{2}R\frac{D\bar{u}}{D\tau} + \left(1 - \frac{3}{2}R\right)\bar{g} + A\frac{U}{L}\left[\bar{u} - \bar{v}\right]$$

Proceeding to non dimensionalize

$$\underline{v} = \frac{\bar{v}}{\langle \bar{v} \rangle}, \quad \underline{u} = \frac{\bar{u}}{\langle \bar{u} \rangle}, \quad t = \frac{\tau}{\langle \tau \rangle}, \quad \underline{g} = \frac{\bar{g}}{\langle \bar{g} \rangle}, \quad \underline{x} = \frac{\bar{x}}{\langle \bar{x} \rangle}$$

$$\frac{d^2\bar{x}}{d\tau^2} = \frac{3}{2}R\frac{D\bar{u}}{D\tau} + \left(1 - \frac{3}{2}R\right)\bar{g} + A\frac{\langle \bar{u} \rangle}{\langle \bar{x} \rangle}\left[\bar{u} - \frac{d\bar{x}}{d\tau}\right]$$

$$\cancel{\frac{d}{d\tau}\left[\frac{d\underline{x}\langle \bar{x} \rangle}{d\tau}\right] = \frac{3}{2}R\frac{D\underline{u}\langle \bar{u} \rangle}{Dt\langle \tau \rangle} + \left(1 - \frac{3}{2}R\right)\underline{g}\langle \bar{g} \rangle + A\frac{\langle \bar{u} \rangle}{\langle \bar{x} \rangle}\left[\bar{u} - \bar{x}\right]}$$

$$\frac{d^2\underline{x}\langle \bar{x} \rangle}{dt^2\langle \tau \rangle^2} = \frac{3}{2}R\frac{D\underline{u}\langle \bar{u} \rangle}{Dt\langle \tau \rangle} + \left[1 - \frac{3}{2}R\right]\underline{g}\langle \bar{g} \rangle + A\frac{\langle \bar{u} \rangle}{\langle \bar{x} \rangle}\left[\underline{u}\langle \bar{u} \rangle - \frac{d\underline{x}\langle \bar{x} \rangle}{dt\langle \tau \rangle}\right]$$

$$\frac{d^2\underline{x}}{dt^2} = \frac{3}{2}R\frac{\langle \tau \rangle\langle \bar{u} \rangle}{\langle \bar{x} \rangle}\frac{D\underline{u}}{Dt} + \left[1 - \frac{3}{2}R\right]\frac{\underline{g}\langle \bar{g} \rangle\langle \tau \rangle^2}{\langle \bar{x} \rangle} + A\frac{\langle \bar{u} \rangle\langle \tau \rangle^2}{\langle \bar{x} \rangle^2}\left[\underline{u}\langle \bar{u} \rangle - \frac{\langle \bar{x} \rangle}{\langle \tau \rangle}\frac{d\underline{x}}{dt}\right]$$

Let $\dfrac{\langle \tau \rangle\langle \bar{u} \rangle}{\langle \bar{x} \rangle} = 1 \Rightarrow \langle \tau \rangle = \dfrac{\langle \bar{x} \rangle}{\langle \bar{u} \rangle} = \dfrac{L}{U}$

Then $\dfrac{\langle \bar{g} \rangle\langle \tau \rangle^2}{\langle \bar{x} \rangle} = 1 \Rightarrow \langle \bar{g} \rangle = \dfrac{\langle \bar{x} \rangle}{\langle \tau \rangle^2} = \dfrac{U^2}{L}$

then $\dfrac{\langle \bar{u} \rangle}{\langle \bar{x} \rangle^2}\langle \tau \rangle^2 = \dfrac{\langle \bar{u} \rangle}{\langle \bar{x} \rangle^2} \cdot \dfrac{\langle \bar{x} \rangle^2}{\langle \bar{u} \rangle^2} = \dfrac{1}{\langle \bar{u} \rangle}$

$$\frac{d^2\underline{x}}{dt^2} = \frac{3}{2}R\frac{D\underline{u}}{Dt} + \left[1 - \frac{3}{2}R\right]\underline{g} + A\left[\underline{u} - \frac{d\underline{x}}{dt}\right]$$

where,

$$\underline{u} = \frac{\bar{u}}{U}, \quad t = \frac{\tau U}{L}, \quad \underline{g} = \frac{\bar{g}L}{U^2}, \quad \underline{x} = \frac{\bar{x}}{L}$$

where U & L are the characteristic velocity and length respectively.

$$R = \frac{2\rho_f}{\rho_f + 2\rho_p}, \quad A = \frac{R}{St}, \quad St = \frac{1}{6\pi}\left(\frac{a}{L}\right)^2 Re$$

Let K denote the velocity magnitude of interest.
Then on the x-axis ($y=0$) around the source, # Note that K has been nondimensionalized.

$$\left(\frac{x}{2\pi x^2}\right)^2 = K^2 \Rightarrow \frac{1}{2\pi x} = K \Rightarrow x = \frac{1}{2\pi K}$$

Non-dimensionalizing yields $\bar{x} = (x, 0)$ to become

$$\underline{x} = \left(\frac{1}{2\pi K L}, 0\right)$$

Then,

$$x_0 = r\cos\theta, \quad y_0 = r\sin\theta \quad \text{where } r = \frac{1}{2\pi K L}$$

For an initial velocity same as the flow, we have

$$\underline{u} = \frac{\bar{u}}{u} = \left(\frac{u^*}{u}, \frac{v^*}{u}\right)$$

$$u^* = \frac{1}{u}\left[u_0 + \frac{1}{2\pi}\left\{\frac{x_0}{y_0^2 + x_0^2} - \frac{P(x_0 - l)}{y_0^2 + (x_0 - l)^2}\right\}\right]$$

$$v^* = \frac{1}{u}\left[\frac{1}{2\pi}\left\{\frac{y_0}{y_0^2 + x_0^2} - \frac{P y_0}{y_0^2 + (x_0 - l)^2}\right\}\right]$$

I have other concerns with this last result; since (x_0, y_0) is already nondimensionalized then $\underline{u}^*$ need not be divide by u, rather u_0 should be nondimensionalized as u_0/u and we should have

$$u^* = \frac{u_0}{u} + \frac{1}{2\pi}\left[\frac{x_0}{y_0^2 + x_0^2} - \frac{P(x_0 - l)}{y_0^2 + (x_0 - l)^2}\right] \text{ and } v^* = \frac{1}{2\pi}\left[\frac{y_0}{y_0^2 + x_0^2} - \frac{P y_0}{y_0^2 + (x_0 - l)^2}\right]$$

But I also have concerns for l, "should it also be nondimensionalized?"