10th Oct. 2023.

MA398 Week 2 Tutorial

1. (a)

1. (a) Given the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & -1 \\ 3 & -2 & -9 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ 2 \\ 9 \end{bmatrix}$$

Use the Gaussian elimination method described in the lecture notes to solve the system $Ax = b$. Show each step of your work.

Answer: Given system of equations:

$$\begin{aligned} x_1 - x_2 + x_3 &= 8, \\ 2x_1 + 3x_2 - x_3 &= 2, \\ 3x_1 - 2x_2 - 9x_3 &= 9. \end{aligned}$$

Starting with these equations, we can perform the following row operations: $-3 \times R_1 + R_2$ and $-2 \times R_1 + R_3$ to get:

$$\begin{aligned} x_1 - x_2 + x_3 &= 8, \\ 0x_1 + x_2 - 12x_3 &= -15, \\ 0x_1 + 5x_2 - 3x_3 &= -18. \end{aligned}$$

Then, to get rid of x_2 in the third equation, we can subtract 5 times the second row from the third row:

$$\begin{aligned} x_1 - x_2 + x_3 &= 8, \\ 0x_1 + x_2 - 12x_3 &= -15, \\ 0x_1 + 0x_2 + 57x_3 &= 57. \end{aligned}$$

From the third equation we get $x_3 = 1$. Substituting $x_3 = 1$ into the second equation gives $x_2 = -3$. Using these values in the first equation gives $x_1 = 4$.
So the solution to the system of equations is $x = (4, -3, 1)$.

1. (b) Jupyter Notebook

2. Forward Substitution

Given $Lx = b$

$$\Rightarrow \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

Expanding the above matrix yields

$$
\begin{aligned}
&a_{11}x_1 && && && = b_1\\
&a_{21}x_1 &+\ & a_{22}x_2 && && = b_2\\
&a_{31}x_1 &+\ & a_{32}x_2 &+\ & a_{33}x_3 && = b_3\\
&\vdots && \vdots && \vdots && \vdots\\
&a_{n1}x_1 &+\ & a_{n2}x_2 &+\ & a_{n3}x_3 + \cdots + a_{nn}x_n && = b_n
\end{aligned}
$$

Thus,

$$
\begin{aligned}
x_1 &= b_1 / a_{11}\\
x_2 &= (b_2 - a_{21}x_1)/a_{22}\\
x_3 &= (b_3 - a_{31}x_1 - a_{32}x_2)/a_{33}\\
&\vdots\\
x_n &= \left(b_n - \sum_{j=1}^{n-1} a_{nj}x_j\right)/a_{nn}
\end{aligned}
$$

In general,

$$x_1 = b_1/a_{11}$$

and

$$x_i = \left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j\right)/a_{ii}, \quad i = 2, 3, \ldots, n$$

Computational Cost.

In the summation sign, we have both $+$ and $\times$

$\times$: $1 + 2 + 3 + \cdots + n-1$

$+$: $0 + 1 + 2 + \cdots + n-2$ (Pigeon hole prin.)

Subtracting b_i is done $n-1$ times

Dividing by a_{ii} is done n times

$$\text{Flops} = \underset{[\times]}{\frac{n(n-1)}{2}} + \underset{[+]}{\frac{(n-2)(n-1)}{2}} + \underset{[b]}{n-1} + \underset{[a]}{n}$$

$$= \frac{n^2 - n}{2} + \frac{n^2 - 3n + 2}{2} + 2n - 1 = n^2 = O(n^2)$$

Algorithm FS (forward substitution)

Input: $L = (l_{ij})_{i,j=1}^{n} \in \mathbb{C}^{n\times n}$, $l_{ii} \neq 0 \ \forall i = 1, \dots, n$,

$b = (b_i)_{i=1}^{n} \in \mathbb{C}^n$

Output: $x \in \mathbb{C}^n$ Solution to $Lx = b$

1. $x_1 := b_1 / l_{11}$
2. for $i = 2$ to n do
3. $\quad h := 0$
4. $\quad$ for $j = 1$ to $i-1$ do
5. $\quad\quad h := h + l_{ij} x_j$
6. $\quad$ end for
7. $\quad x_i := (b_i - h)/l_{ii}$
8. end for

3. 3. (LU decomposition)

(a) Find the LU decomposition of the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ -6 & -1 & 2 \end{bmatrix},$$

and use it to solve $Ax = b$ with $b = (7, 8, -3)$.

Answer: The LU decomposition of a matrix is a process where we factorize the original matrix A into a product of a lower triangular matrix L and an upper triangular matrix U.

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ -6 & -1 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - \underline{2}R_1 \\ R_3 + \underline{3}R_1 \end{matrix} \rightarrow \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 0 & -4 & 11 \end{bmatrix} R_3 + \underline{}R_2$$

$$\downarrow$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 0 & 0 & 6 \end{bmatrix}$$

3. cont.

The LU factorization is given by:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 0 & 0 & 6 \end{bmatrix}.$$

Now that we have the LU decomposition of A, we can solve $Ax = b$ as follows:
First, we solve $Ly = b$ for y:

$$Ly = b$$

where $b = (7, 8, -3)$.
Then, we solve $Ux = y$ for x:

$$Ux = y$$

We can solve $Ly = b$ as follows:
Here,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix}, \quad b = \begin{pmatrix} 7 \\ 8 \\ -3 \end{pmatrix}.$$

Solving $Ly = b$ yields:

$$\left.\begin{aligned} y_1 &= 7, \\ 2y_1 + y_2 &= 8, \\ -3y_1 + y_2 + y_3 &= -3. \end{aligned}\right\}$$

using forward substitution

From the above system, we can find $y = (7, -6, -6)^T$.
Next, we use y to solve $Ux = y$:

$$U = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 0 & 0 & -1 \end{bmatrix}, \quad y = \begin{pmatrix} 7 \\ -6 \\ -6 \end{pmatrix}.$$

Solving $Ux = y$ yields:

$$\left.\begin{aligned} 2x_1 - x_2 + 3x_3 &= 7, \\ 4x_2 - 5x_3 &= -6, \\ -x_3 &= -6. \end{aligned}\right\}$$

using backward substitution

From the above system, we can find $x = (1, 1, 2)^T$, which is the solution to the original system $Ax = b$.

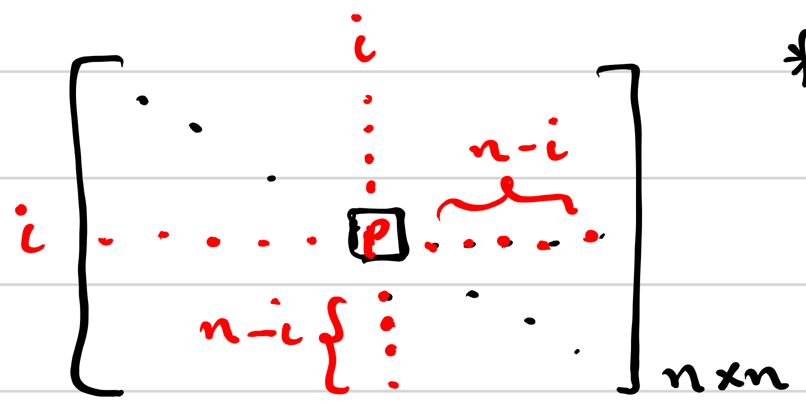
4.

Algorithm 2 LU

input: $A = (a_{ij})_{i,j=1}^n \in \mathbb{C}^{n \times n}$ with $\det(A_k) \neq 0,\ k = 1, \dots, n$.
output: $L \in \mathbb{C}^{n \times n}$ unit lower triangular, $U \in \mathbb{C}^{n \times n}$ upper triangular and regular with $A = LU$.

$U = A,\ L = I.$
for $k = 1$ to $n - 1$ **do**
  for $j = k + 1$ to n **do**
    $l_{j,k} := u_{j,k} / u_{k,k}$
    $u_{j,k} := 0$
    for $i = k + 1$ to n **do**
      $u_{j,i} := u_{j,i} - l_{j,k} u_{k,i}$
    end for
  end for
end for

One step in Gaussian



* multiply every element to the right of the pivot by a constant i.e $n-i$ multiplications.

* Then add it to the row underneath

\# We repeat this for the $n-i$ rows underneath

So we have $(n-i) \times (n-i)$ operations each for the addition and multiplication.

* We have to do this for $n-1$ different pivots.

So we have the total number of operations to be

$$\text{FLOPs} = 2\sum_{i=1}^{n-1} (n-i)^2 = 2\left[\sum_{i=1}^{n-1} n^2 - 2ni + i^2\right]$$

$$= 2n^2 \sum_{i=1}^{n-1} 1 - 4n\sum_{i=1}^{n-1} i + 2\sum_{i=1}^{n-1} i^2$$

$$= 2n^2(n-1) - 4n\left[\frac{n(n-1)}{2}\right]$$

$$+ 2\left[\frac{1}{6}\left\{(n-1)n\left(2[n-1]+1\right)\right\}\right]$$

$$= \cancel{2n^3} - \cancel{2n} - \cancel{2n^3} + \cancel{2n}$$

$$+ \frac{1}{3}\left[n\left[2(n-1)^2 + n - 1\right]\right] = \frac{1}{3}\left[n\left[2n^2 - 4n + 2 + n - 1\right]\right]$$

$$= \frac{1}{3}\left(2n^3 - 2n^2 + 1\right) = O(n^3)$$

⑤

- (Diagonal dominance) A matrix $A = (a_{ij})_{i,j=1}^n \in \mathbb{C}^{n\times n}$ is called strictly diagonal dominant if

$$|a_{ii}| > \sum_{j\neq i} |a_{ij}| \quad \text{for all } i \in \{1,\ldots,n\}.$$

Show that such a matrix is invertible and its LU factorisation exists. For this purpose, show that the remaining matrix $(u_{ij}^{(k)})_{i,j=k+1}^n$ after step k of the Gaussian elimination without pivoting still is strictly diagonal dominant. → note this

Answer: Without loss of generality, we show the assertion for the first step ($k = 1$) and for the second row. We will need that

From defn : → $$\sum_{i\neq 2} |u_{2i}^{(0)}| < |u_{22}^{(0)}| \quad \Rightarrow \quad \sum_{i=3}^n |u_{2i}^{(0)}| < |u_{22}^{(0)}| - |u_{21}^{(0)}|$$

came from there

and

From defn : → $$\sum_{i\neq 1} |u_{1i}^{(0)}| < |u_{11}^{(0)}| \quad \Rightarrow \quad \frac{1}{|u_{11}^{(0)}|}\sum_{i=3}^n |u_{1i}^{(0)}| < 1 - \frac{|u_{12}^{(0)}|}{|u_{11}^{(0)}|}.$$

also came from there, then divide.

Recall that

Xprsn for row operatns : → $$u_{2i}^{(1)} = u_{2i}^{(0)} - \frac{u_{21}^{(0)}}{u_{11}^{(0)}} u_{1i}^{(0)}, \quad i = 2,\ldots,n.$$

Xprsn for row operatns. Now we consider the Step after 1st Gaussian eliminatn.

Using this, we infer

we use ≤ sign becuz we now sum the terms together.

we plug out terms without i.

$$\begin{aligned}
\sum_{i=3}^n |u_{2i}^{(1)}| &= \sum_{i=3}^n \left|u_{2i}^{(0)} - \frac{u_{21}^{(0)}}{u_{11}^{(0)}} u_{1i}^{(0)}\right| \\
&\leq \sum_{i=3}^n |u_{2i}^{(0)}| + \sum_{i=3}^n \frac{|u_{21}^{(0)}|}{|u_{11}^{(0)}|} |u_{1i}^{(0)}| \\
&< |u_{22}^{(0)}| - |u_{21}^{(0)}| + |u_{21}^{(0)}| \frac{1}{|u_{11}^{(0)}|} \sum_{i=3}^n |u_{1i}^{(0)}| \\
&< |u_{22}^{(0)}| - |u_{21}^{(0)}| + |u_{21}^{(0)}| \left(1 - \frac{|u_{12}^{(0)}|}{|u_{11}^{(0)}|}\right) \\
&= |u_{22}^{(0)}| - \frac{|u_{21}^{(0)}|}{|u_{11}^{(0)}|} |u_{12}^{(0)}| \\
&\boxed{\leq \left|u_{22}^{(0)} - \frac{u_{21}^{(0)}}{u_{11}^{(0)}} u_{12}^{(0)}\right| = |u_{22}^{(1)}|}
\end{aligned}$$

Recall that

$$\boxed{\frac{1}{|u_{11}^{(0)}|}\sum_{i=3}^n |u_{1i}^{(0)}| < 1 - \frac{|u_{12}^{(0)}|}{|u_{11}^{(0)}|}.}$$

Simplifying cancels out $u_{21}^{(0)}$

triangle inequality →

which was to be shown.
Since the Gaussian elimination and the LU factorisation is possible all submatrices of A are regular, in particular $A = A_n$ itself is invertible.

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Note that $|u_{22}^{(0)}| > \sum_{i\neq 2} |u_{2i}^{(0)}|$ (Diagonal dominance)

$$|u_{22}^{(0)}| > |u_{21}^{(0)}| + \sum_{i=3}^n |u_{2i}^{(0)}|$$

$$\Rightarrow |u_{22}^{(0)}| - |u_{21}^{(0)}| > \sum_{i=3}^n |u_{2i}^{(0)}|$$

$$\Rightarrow \sum_{i=3}^n |u_{2i}^{(0)}| < |u_{22}^{(0)}| - |u_{21}^{(0)}|$$

(This is why the sign changed from ≤ to <)

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They are equal because of the expression used to compute row operations.

In the end, we have shown that the matrix after a Gaussian elimination process is still diagonally dominant.

Note that a matrix A is invertible if $\det(A) \neq 0$ (i.e not a singular matrix)

For the LU factorization of A to exist, then $\det(A) \neq 0$ i.e A must be invertible.

For the LU of A to be computable (without pivoting), we need $A_{ii} \neq 0$ (the diagonal entries) to be non-zero.

A diagonally dominant matrix ensures that the entries in the major diagonal are always non-zero.

6

6. (a) Let $A = (a_{ij})_{i,j=1}^{n} \in \mathbb{C}^{n\times n}$ be a matrix of bandwidth $w \in \{0, \ldots, n-1\}$, i.e.,

$$a_{ij} = 0 \quad \text{if } |j - i| > w.$$

Give an example of a 4×4 matrix of bandwidth $w = 2$ but not $w = 1$ which fulfils the strong row sum criterion (also known as strict diagonal dominance).

Answer: Example:

$$A = \begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}$$

6(b)

(b) Assume that the LU factorisation of a matrix $A \in \mathbb{C}^{n\times n}$ of bandwidth $w = 1$ can be computed with the algorithm LU (without pivoting!). Show that then the computed matrices L and U are of bandwidth $w = 1$, too.

Answer: By induction. Assume that $U^{(k-1)}$ and $L^{(k-1)}$ after step $k-1$ have bandwidth $w = 1$. Then $u_{ik}^{(k-1)} = 0$ if $i > k+1$ which yields that $l_{ik} = 0$ (if $i > k+1$). But this means that $L^{(k)}$ will have bandwidth $w = 1$. Moreover, only the row $i = k+1$ (if $i < n$) of $U^{(k-1)}$ may involve changes when updating to $U^{(k)}$.
From this row $i = k+1$ the multiple l_{ik} of row k is subtracted. The bandwidth assumption on $U^{(k-1)}$ implies that $u_{kj}^{(k-1)} = 0$ if $j > k+1$. Therefore, only the entries $u_{ij}^{(k-1)}$ with $j = k, \ldots, \min(k+1, n)$ may involve changes. But since $i = k+1$ we have for these entries that $|i-j| <= w = 1$. As a consequence, if $|i-j| > w = 1$ then $u_{ij}^{(k)} = u_{ij}^{(k-1)} = 0$ so that also $U^{(k)}$ will have bandwidth $w = 1$.

6(c)

(c) Formulate a specialised version of the algorithm LU for band matrices of bandwidth $w = 1$ where only the necessary operations are carried out. Ensure and check that the number of elementary executable operations is $O(n)$ as $n \to \infty$.

Answer: Cf. algorithm 1. Only the loops for i and j had to be adapted. In every step $k \in \{1, \ldots, n-1\}$ we have to perform at most one division to compute the $l_{k+1,k}$, and in order to update the $u_{k+1,j}$ we need at most one multiplication and one subtraction. Hence, the cost for step k is at most three operations. Altogether therefore

$$C_{LUB}(n) \le \sum_{k=1}^{n-1} 3 = 3(n-1) = O(n) \quad \text{as } n \to \infty.$$

Algorithm 1 LU for banded matrices

input: $A \in \mathbb{C}^{n\times n}$ of bandwidth w with $\det(A_j) \neq 0$ for $j = 0, \ldots, n$.
output: $L, U \in \mathbb{C}^{n\times n}$ where LU is the LU factorisation of A.

$L = I$, $U = A$
for $k = 1, \ldots, n-1$ **do**
  $l_{k+1,k} = u_{k+1,k}/u_{k,k}$
  $u_{k+1,k} = 0$
  $u_{k+1,k+1} = u_{k+1,k+1} - l_{k+1,k}u_{k,k+1}$
end for

Note that a matrix with bandwidth $w=1$ is a tridiagonal matrix

example:

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{32} & 0 & 0 & \cdots & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix}$$

Tridiagonal Matrix Example to Solve Math Problems

to find x value of math problem below :

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 6 & 3 & 9 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 21 \\ 69 \\ 34 \\ 22 \end{bmatrix}$$

U $\qquad\qquad\qquad$ L

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 6 & 3 & 9 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 4 & 3 \end{bmatrix} R_2 - 3R_1 \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\downarrow$

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & -6 & 9 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 4 & 3 \end{bmatrix} R_3 + \tfrac{1}{3}R_2 \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\downarrow$

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & -6 & 9 & 0 \\ 0 & 0 & 8 & 2 \\ 0 & 0 & 4 & 3 \end{bmatrix} R_4 - \tfrac{1}{2}R_3 \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & -1/3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\downarrow$

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & -6 & 9 & 0 \\ 0 & 0 & 8 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & -1/3 & 1 & 0 \\ 0 & 0 & 1/2 & 1 \end{bmatrix}$$