Solutions to Problem Sheet 4

(4.1) We look for a straight line $p_1(x) = c_0 + c_1 x$ such that

$$f(0) - p_1(0) = A$$
$$f(1) - p_1(1) = A$$
$$f(d) - p_1(d) = -A$$

where $d \in (0, 1)$. Since the error has a turning point at $x = d$, we have

$$f'(d) - p_1'(d) = 0.$$

This gives the following set of equations

$$\begin{aligned} -c_0 &= A \\ \sinh(1) - c_0 - c_1 &= A \\ \sinh(d) - c_0 - c_1 d &= -A \\ \cosh(d) - c_1 &= 0. \end{aligned}$$

Adding the first and the second equation gives $c_1 = \sinh(1) = 1.1752$. Therefore $\cosh(d) = \sinh(1)$. Plugging this into the third equation we have that $c_0 = \frac{1}{2}(\sinh(d) - \sinh(1)d) \approx -0.0343$.

(4.2) Since p_n^* is a minimax polynomial we have $(n + 2)$ points x_i, $i = 0, \ldots n + 1$ at which

$$f(x_i) - p_n^*(x) = (-1)^i A \text{ for } i = 0, 1 \ldots n + 1$$

with $A = \|f - p_n^*\|_\infty$. Let $g(x) = f(-x)$, then

$$g(-x_i) - p_n^*(x_i) = (-1)^i A.$$

So $\{-x_i\}$ is an alternating set for $g(x), p_n^*(-x)$ and therefore $p_n^*(-x)$ is a minimax polynomial for g. But f is even, so $g = f$. And $p_n^*(-x)$ is also a minimax polynomial. Since the minimax polynomial is unique, p_n^* has to be even as well.
Since the minimax polynomial has to be even, it can't have any odd powers of x, which is why the coefficient of x^{n+1} has to be zero.
Since $f(x) = |x|$ is even we have that $p_1 = p_0$. By symmetry $p_0 = \frac{1}{2}$, therefore $p_1 = \frac{1}{2}$.

(4.3) Using the substitution we obtain

$$\frac{dx}{d\theta} = -\sin\theta$$

and therefore

$$\frac{d^2y}{d\theta^2} + n^2 y = 0.$$

since $y' = \frac{dy}{dx} = -\frac{1}{\sin\theta}\frac{dy}{d\theta}$ and $y'' = \frac{1}{\sin\theta}\frac{d}{d\theta}(\frac{1}{\sin\theta}\frac{dy}{d\theta})$. The above ODE has two solutions

$$y_1 = \cos(n\theta) \text{ and } y_2 = \sin(n\theta).$$

The former gives the well-known Chebyshev polynomials if we transform back to the original variables.

(4.4) We calculate

$$\begin{aligned}
E(c_0, c_1) &= \int_a^b (f(x) - c_0 - c_1 x)^2 dx = \\
&= \int_a^b f(x)^2 - 2f(x)(c_0 + c_1 x) + (c_0^2 + 2c_0 c_1 x + c_1^2 x^2) dx \\
&= \int_a^b f(x)^2 dx - 2c_0 \int_a^b f(x) dx - 2c_1 \int_a^b x f(x) dx \\
&\quad + c_0^2(b-a) + c_0 c_1 (b^2 - a^2) + \frac{1}{3} c_1^2 (b^3 - a^3).
\end{aligned}$$

To find the optimal coefficients we calculate the gradient of E and set it to zero.

$$\frac{\partial E}{\partial c_0} = \frac{\partial E}{\partial c_1} = 0.$$

We compute

$$\begin{aligned}
\frac{\partial E}{\partial c_0} &= -2 \int_a^b f(x) dx + 2c_0(b-a) + c_1(b^2 - a^2) \\
\frac{\partial E}{\partial c_1} &= -2 \int_a^b x f(x) dx + c_0(b^2 - a^2) + c_1 \frac{2}{3}(b^3 - a^3).
\end{aligned}$$

Setting the partial derivatives to zero gives

$$\begin{aligned}
2(b-a)c_0 + c_1(b^2 - a^2) &= 2 \int_a^b f(x) dx \\
c_0(b^2 - a^2) + \frac{2}{3} c_1 (b^3 - a^3) &= 2 \int_a^b x f(x) dx.
\end{aligned}$$

This is exactly the linear system stated. This system has a unique solution for $b \neq a$. Consider $f(x) = e^x$ now, then

$$\int_0^1 e^x dx = e - 1 \quad \int_0^1 x e^x dx = [e^x(x-1)]|_{x=0}^1 = 1.$$

Therefore we have to solve the system

$$\begin{pmatrix} 2 & 1 \\ 1 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2e - 2 \\ 2 \end{pmatrix},$$

which has the solution

$$c_0 = 4e - 10 \quad c_1 = 18 - 6e.$$