MA265 Week 9, 5th Dec. 2023

Assignment 4.

① $f(x) = \sinh(x)$ on $[0,1]$

Alternating points are $\{0, d, 1\}$

We consider a straight line $P_1(x) = c_0 + c_1 x$

$$f(0) - P_1(0) = A$$
$$f(d) - P_1(d) = -A$$
$$f(1) - P_1(1) = A$$

where $d \in (0,1)$. Since the error has a turning point at $x = d$, we have

$$f'(d) - P_1'(d) = 0$$

$\therefore$

$$\sinh(0) - c_0 = A \Rightarrow -c_0 = A \quad \text{— (i)}$$
$$\sinh(d) - c_0 - c_1 d = -A \quad \text{— (ii)}$$
$$\sinh(1) - c_0 - c_1 = A \quad \text{— (iii)}$$
$$\cosh(d) - c_1 = 0 \quad \text{— (iv)}$$

(i) − (iii): $\sinh(1) - c_1 = 0 \Rightarrow c_1 = \sinh(1) = 1.1752$

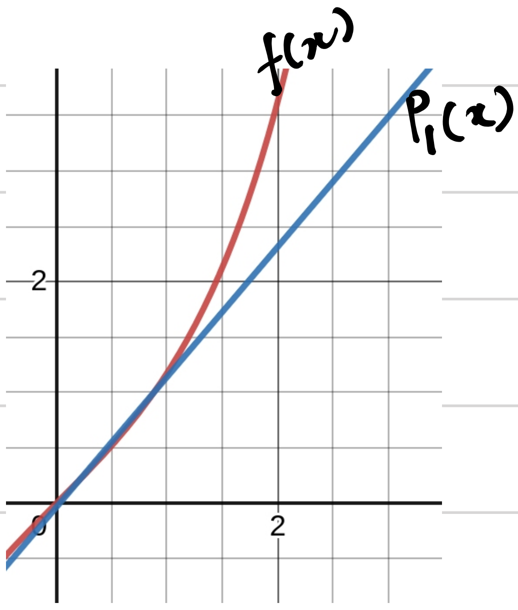
from (iv): $d = \cosh^{-1}(c_1) = 0.5836$

(ii) + (iii): $\sinh(d) - 2c_0 - c_1(d+1) + \sinh(1) = 0$

$$c_0 = \frac{\sinh(d) - (d+1)c_1 + \sinh(1)}{2} = -0.0343$$

$A = 0.0343$

plot →



③ $(1-x^2)y'' - xy' + n^2 y = 0$, $n = 0, 1, 2, \dots$

use subst $x = \cos\theta$

$\Rightarrow (1-\cos^2\theta)y'' - \cos\theta\, y' + n^2 y = 0$

$$\sin^2\theta \frac{d^2y}{dx^2} - \cos\theta \frac{dy}{dx} + n^2 y = 0 \quad \text{———} \ *$$

Now we proceed to substitute the derivatives wrt x with derivatives wrt θ.

Since $x = \cos\theta$

$$\frac{dx}{d\theta} = -\sin\theta \quad , \quad \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{dy}{dx} = \frac{-1}{\sin\theta} \frac{dy}{d\theta}$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx} = \frac{-1}{\sin\theta} \frac{d}{d\theta}\left(\frac{-1}{\sin\theta} \frac{dy}{d\theta}\right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{\sin\theta}\left(\frac{1}{\sin\theta} \frac{d^2y}{d\theta^2} - \frac{\cos\theta}{\sin^2\theta} \frac{dy}{d\theta}\right)$$

$$= \frac{1}{\sin^2\theta} \frac{d^2y}{d\theta^2} - \frac{\cos\theta}{\sin^3\theta} \frac{dy}{d\theta}$$

$$\therefore \sin^2\theta \frac{d^2y}{dx^2} = \frac{d^2y}{d\theta^2} - \frac{\cos\theta}{\sin\theta} \frac{dy}{d\theta}$$

$$\cos\theta \frac{dy}{dx} = -\frac{\cos\theta}{\sin\theta} \frac{dy}{d\theta}$$

$*$ then becomes:

$$\frac{d^2y}{d\theta^2} - \frac{\cos\theta}{\sin\theta} \frac{dy}{d\theta} + \frac{\cos\theta}{\sin\theta} \frac{dy}{d\theta} + n^2 y = 0$$

$$\frac{d^2y}{d\theta^2} + n^2 y = 0 \quad \text{———} \ **$$

The solution takes the form

$y(\theta) = e^{r\theta}$

$y' = r y \quad , \quad y'' = r^2 y$

** then becomes:

$r^2 y + n^2 y = 0 \quad , \quad y \neq 0$

$\Rightarrow r^2 + n^2 = 0 \quad \therefore \quad r = \pm n i$

The solutions are thus:

$y_1^*(\theta) = e^{in\theta} \quad , \quad y_2^*(\theta) = e^{-in\theta}$

General Solution is thus:

$y(\theta) = Ae^{in\theta} + Be^{-in\theta}$

$y(\theta) = A(\cos(n\theta) + i\sin(n\theta)) + B(\cos(n\theta) - i\sin(n\theta))$

$y(\theta) = (A+B)\cos(n\theta) + i(A-B)\sin(n\theta)$

$y(\theta) = c_1\cos(n\theta) + c_2\sin(n\theta)$

So,

$$\boxed{y_1(\theta) = \cos(n\theta), \quad y_2(\theta) = \sin(n\theta)}$$

4 $E(c_0, c_1) = \|f(x) - c_0 - c_1 x\|^2 = \int_a^b (f(x) - c_0 - c_1 x)^2 dx$

$= \int_a^b f(x)^2 - 2f(x)(c_0 + c_1 x) + (c_0 + c_1 x)^2 dx$

$= \int_a^b \left[f(x)^2 - 2c_0 f(x) - 2c_1 x f(x) + c_0^2 + 2c_0 c_1 x + c_1^2 x^2 \right] dx$

$= \int_a^b f(x)^2 dx - 2c_0 \int_a^b f(x) dx - 2c_1 \int_a^b x f(x) dx$

$+ \int_a^b c_0^2 + 2c_0 c_1 x + c_1^2 x^2 \, dx$

$$= \int_a^b f(x)^2\,dx - 2c_0\int_a^b f(x)\,dx - 2c_1\int_a^b x f(x)\,dx$$
$$+ \left[c_0^2 x + c_0 c_1 x^2 + \frac{c_1^2 x^3}{3}\right]_a^b$$

$$E(c_0, c_1) = \int_a^b f(x)^2\,dx - 2c_0\int_a^b f(x)\,dx - 2c_1\int_a^b x f(x)\,dx$$
$$+ c_0^2(b-a) + c_0 c_1 (b^2 - a^2) + \frac{1}{3}c_1^2(b^3 - a^3)$$

To find the optimal coefficients, we compute the gradient of E wrt c_0 & c_1 and set to 0.

$$\frac{\partial E}{\partial c_0} = \frac{\partial E}{\partial c_1} = 0$$

$$\frac{\partial E}{\partial c_0} = -2\int_a^b f(x)\,dx + 2c_0(b-a) + c_1(b^2 - a^2)$$

$$\frac{\partial E}{\partial c_1} = -2\int_a^b x f(x)\,dx + c_0(b^2 - a^2) + \frac{2}{3}c_1(b^3 - a^3)$$

Setting the derivatives to 0, we get:

$$2c_0(b-a) + c_1(b^2 - a^2) = 2\int_a^b f(x)\,dx$$
$$c_0(b^2 - a^2) + \frac{2}{3}c_1(b^3 - a^3) = 2\int_a^b x f(x)\,dx$$

$$\begin{pmatrix} 2(b-a) & (b^2 - a^2) \\ (b^2 - a^2) & \frac{2}{3}(b^3 - a^3) \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2\int_a^b f(x)\,dx \\ 2\int_a^b x f(x)\,dx \end{pmatrix}$$

Set $f(x) = e^x$ on the interval $[0, 1]$, we get

$$\begin{pmatrix} 2 & 1 \\ 1 & 2/3 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2\int_0^1 e^x\,dx \\ 2\int_0^1 x e^x\,dx \end{pmatrix} = \begin{pmatrix} 2e - 2 \\ 2 \end{pmatrix}$$

$$2c_0 + c_1 = 2e - 2 \quad \text{——— (i)}$$
$$c_0 + \tfrac{2}{3}c_1 = 2 \quad \text{——— (ii)}$$

(i) $- 2 \times$ (ii): $c_1 - \frac{4}{3}c_1 = 2e - 2 - 4$

$$-\tfrac{1}{3}c_1 = 2e - 6$$

$$\boxed{\Rightarrow c_1 = 18 - 6e}$$

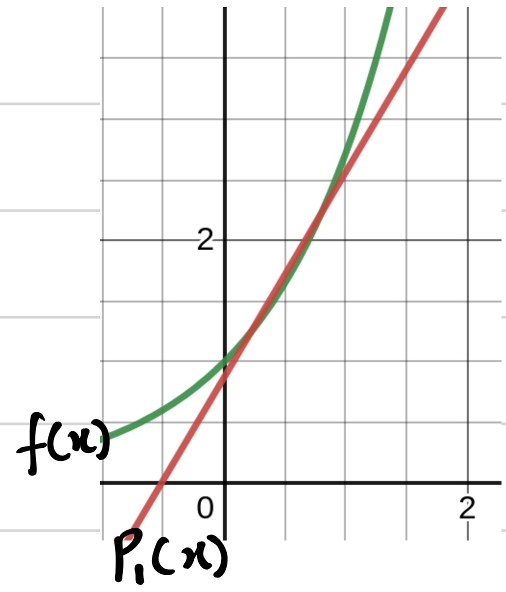
from (ii): $c_0 = 2 - \tfrac{2}{3}c_1$

$$c_0 = 2 - \frac{2}{3}(18 - 6e)$$

$$c_0 = 2 - 12 + 4e$$

$$\boxed{\Rightarrow c_0 = 4e - 10}$$

Plot $\longrightarrow$



Que 2.

(4.2) Since p_n^* is a minimax polynomial we have $(n+2)$ points x_i, $i = 0, \dots n+1$ at which

$$f(x_i) - p_n^*(x) = (-1)^i A \text{ for } i = 0, 1 \dots n+1$$

with $A = \|f - p_n^*\|_\infty$. Let $g(x) = f(-x)$, then

$$g(-x_i) - p_n^*(x_i) = (-1)^i A.$$

So $\{-x_i\}$ is an alternating set for $g(x), p_n^*(-x)$ and therefore $p_n^*(-x)$ is a minimax polynomial for g. But f is even, so $g = f$. And $p_n^*(-x)$ is also a minimax polynomial. Since the minimax polynomial is unique, p_n^* has to be even as well.
Since the minimax polynomial has to be even, it can't have any odd powers of x, which is why the coefficient of x^{n+1} has to be zero.
Since $f(x) = |x|$ is even we have that $p_1 = p_0$. By symmetry $p_0 = \frac{1}{2}$, therefore $p_1 = \frac{1}{2}$.