Week 9 Tutorial 9

(9.1) *Inner products on vector spaces:* We recall that an inner product is a mapping from $V \times V$, where V is a vector space, into $\mathbb{R}$ satisfying

- Linearity in the first argument: $\langle \mathbf{x}_1 + \mathbf{x}_2, \mathbf{y} \rangle \leq \langle \mathbf{x}_1, \mathbf{y} \rangle + \langle \mathbf{x}_2, \mathbf{y} \rangle$
- Symmetry: $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.
- Positivity: $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ for all $\mathbf{x} \in V$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

Show that $\langle f, g \rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$ defines an inner product on the space of real-coefficient polynomial functions.

(9.2) *Gram Schmidt* Determine the first four Lagrange polynomials using Gram Schmidt to orthogonalise the power basis $\{1, x, x^2, x^3\}$.
We will use that

$$\int_{-1}^{1} x^n dx = \begin{cases} \frac{2}{n+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Hence the inner product between basis function is given by

$$(p_n, p_m) = \int_{-1}^{1} x^n x^m dx = \begin{cases} \frac{2}{n+m+1} & \text{if } n+m \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

- Set the first Legendre polynomial to $L_0 = p_0(x)$ and compute the next using that

$$L_1(x) = p_1(x) - \frac{\langle L_0, p_1 \rangle}{\langle L_0, L_0 \rangle} L_0(x)$$

- Continue up to order 4 to obtain the sequence

$$L_0(x) = 1, L_1(x) = x, L_2(x) = x^2 - \frac{1}{3}, L_3(x) = x^3 - \frac{3}{5}x.$$

- Let $x_0, \ldots x_n$ be the roots of the Legendre polynomial of degree $n+1$. Consider the quadrature rule

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=0}^{n} f(x_i) w_i(x)$$

  where the weights w_i are the integrals of the Lagrange polynomials

$$w_i = \int_{-1}^{1} L_i(x)dx \text{ with } L_i(x) = \Pi_{j \neq i} \left(\frac{x - x_j}{x_i - x_j} \right).$$

Show that this quadrature rule is exact for polynomials p up to order $2n+1$, that is show that

$$\int_{-1}^{1} p(x)dx = \sum_{i=0}^{n} f(x_i)w_i.$$

Hint: Use polynomial division to write $p(x) = q(x)P_{n+1}(x) + r(x)$ where p and q are polynomials of degree less than or equal to n.

(9.3) *Laguerre polynomials* The Laguerre polynomials are orthogonal on $(0, \infty)$ wrt the weight function $w(x) = e^{-x}$.

- Construct the first four Laguerre polynomials, starting with the lowest order $L_0(x) = 1$.
- Show that they satisfy the recurrence relation

$$L_{k+1}(x) = \frac{(2k+1-x)L_k(x) - kL_{k-1}(x)}{k+1}$$

for any $k \geq 0$.

(9.4) *Pade approximation* Show that $r_{2,1}$ for the exponential function is given by

$$r_{2,1}(x) = \frac{1 + \frac{2}{3}x + \frac{1}{6}x^2}{1 - \frac{1}{3}x}$$

Plot the exponential and the Pade approximation.