MA398 Matrix Analysis and Algorithms: Exercise Sheet 8

1. (Jacobi) Consider the Jacobi method for solving

$$Ax = b, \quad \text{with } A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \; b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

and with start value $x^{(0)} = (0,0,0)^T$.

   (a) State the iteration matrix $R = -D^{-1}(L+U)$, compute its spectral radius $\rho(R)$ and deduce that the Jacobi method converges.

   (b) Recall the estimate

$$k \geq k^\sharp = \frac{\log(\|A\|_2 \|e^{(0)}\|_2 / \|b\|_2) - \log(\varepsilon_r)}{\log(\|R\|_2^{-1})}$$

   for the number of steps in order to achieve that $\|r^{(k)}\|_2 \leq \varepsilon_r \|b\|_2$.
   For the above specific data, give an upper bound for the number of steps required to get the relative error of the residual below 10^{-6}.

   (c) Derive the estimate

$$\frac{\|e^{(k)}\|_2}{\|x\|_2} \leq \kappa_2(A) \frac{\|r^{(k)}\|_2}{\|b\|_2}$$

   and give an upper bound for the number of steps required to get the relative error of the solution below 10^{-6}.

   (d) State the definition of the graph $G(B)$ of a matrix $B \in \mathbb{C}^{n\times n}$.
   Prove that $B \in \mathbb{C}^{n\times n}$ is irreducible if and only if its graph $G(B)$ is connected.

2. (SSOR) The <u>symmetric successive over relaxation</u> consists in performing the following iteration:

$$i = 1, \dots, n: \qquad a_{ii} x_i^{(k+\frac{1}{2})} = \omega\Big(-\sum_{j=1}^{i-1} a_{ij} x_j^{(k+\frac{1}{2})} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} + b_i \Big) - (\omega - 1) a_{ii} x_i^{(k)},$$

$$i = n, \dots, 1: \qquad a_{ii} x_i^{(k+1)} = \omega\Big(-\sum_{j=1}^{i-1} a_{ij} x_j^{(k+\frac{1}{2})} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k+1)} + b_i \Big) - (\omega - 1) a_{ii} x_i^{(k+\frac{1}{2})}.$$

Here, $x^{(k)}$ stands for the k^{th} iterate, and $x^{(k+\frac{1}{2})}$ is an intermediate value.
Show that SSOR is a linear iterative method with

$$M_{\text{SSOR}}^{-1} = \omega(2-\omega)(D+\omega U)^{-1} D (D + \omega L)^{-1}.$$

Remark: Recalling that SOR uses $M_{\text{SOR}} = \frac{1}{\omega} D + L$ we see that SSOR essentially consists in performing an SOR step followed by a reverse SOR step with $\frac{1}{\omega} D + U$, which explains its name. A couple of SSOR steps sometimes are applied as a preconditioner in CG.

3. Implement the Gauss-Seidel method, a variant of the Jacobi method, where $M := L + D$ and $N := U$. Write a Python function with the signature def gauss_seidel_method(A, b, x0, max_iter, tol, omega=1.0), where omega is the relaxation parameter.

   (a) Test your function on a system of linear equations with a diagonally dominant matrix A and different choices of omega. Plot the norm of the residual vector as a function of the number of iterations for different choices of omega.

(b) Use your implementation to investigate the impact of the relaxation parameter on the convergence of the method. For which values of omega does the method converge fastest?

(c) If possible, compare the performance (in terms of both accuracy and computational cost) of your Gauss-Seidel implementation with a basic Jacobi method implementation.