1. (Jacobi) Consider the Jacobi method for solving

$$Ax = b, \quad \text{with } A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \; b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

and with start value $x^{(0)} = (0, 0, 0)^T$.

(a) State the iteration matrix $R = -D^{-1}(L + U)$, compute its spectral radius $\rho(R)$ and deduce that the Jacobi method converges.

Soln

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$ # which we can also write as $D = \operatorname{diag}(2, 2, 2)$

$$L = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

The iteration matrix $R = -D^{-1}(L + U)$ comes from

$$M x^{(k)} = b - N x^{(k-1)}$$

$$\Rightarrow x^{(k)} = M^{-1} b \boxed{- M^{-1} N} x^{(k-1)}$$

So, $M = D$ and $N = L + U$

Note: from Lemma 20.1, $e^{(k)} \to 0$ iff $\rho(R) < 1$.

$$L + U = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad -D^{-1} = \operatorname{diag}(-1/2, -1/2, -1/2)$$

$$R = \begin{pmatrix} -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$R = -\tfrac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$R = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial is given by

$$\det(R - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & 1/2 & 0 \\ 1/2 & -\lambda & 1/2 \\ 0 & 1/2 & -\lambda \end{vmatrix} = 0$$

$$-\lambda \begin{vmatrix} -\lambda & 1/2 \\ 1/2 & -\lambda \end{vmatrix} - 1/2 \begin{vmatrix} 1/2 & 1/2 \\ 0 & -\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 1/4\lambda + 1/4\lambda = 0$$

$$-\lambda(\lambda^2 - 1/2) = 0$$

$$\Rightarrow \lambda = 0, \quad \lambda = \pm\sqrt{1/2}$$

$$\Rightarrow \sigma = \{0, 1/\sqrt{2}, -1/\sqrt{2}\}$$

$$\rho(R) = 1/\sqrt{2} < 1$$

Hence the Jacobi method converges.

(c) Derive the estimate

$$\frac{\|e^{(k)}\|_2}{\|x\|_2} \leq \kappa_2(A)\frac{\|r^{(k)}\|_2}{\|b\|_2}$$

and give an upper bound for the number of steps required to get the relative error of the solution below 10^{-6}.

Proof: We have a SLE given as $Ax = b$ therefore,

$$\|Ax\|_2 = \|b\|_2$$

$$\|Ax\|_2 = \|b\|_2 .$$

but,

$$\|Ax\|_2 \le \|A\|_2 \|x\|_2$$

$$\Rightarrow \quad \|b\|_2 \le \|A_2\| \, \|x\|_2$$

$$\frac{1}{\|x\|_2} \le \frac{\|A\|_2}{\|b\|_2} \quad \text{——— (i)}$$

$$e^{(k)} := x - x^{(k)},$$

but x is not known. What we know is the residual

$$r^{(k)} := b - Ax^{(k)} \qquad (\text{but } Ax = b)$$

$$= Ax - Ax^{(k)}$$

$$= A(x - x^{(k)})$$

$$r^{(k)} = Ae^{(k)}$$

$$\Rightarrow \quad e^{(k)} = A^{-1} r^{(k)}$$

$$\therefore \|e^{(k)}\|_2 = \|A^{-1} r^{(k)}\|_2 \le \|A^{-1}\|_2 \|r^{(k)}\|_2$$

$$\|e^{(k)}\|_2 \le \|A^{-1}\|_2 \|r^{(k)}\|_2 \quad \text{——— (ii)}$$

Multiplying (i) and (ii) yields:

$$\frac{1}{\|x\|_2} \|e^{(k)}\|_2 \le \frac{\|A\|_2}{\|b\|_2} \|A^{-1}\|_2 \|r^{(k)}\|_2$$

Recall that $\kappa(A) = \|A^{-1}\| \, \|A\|$

$$\therefore \quad \frac{\|e^{(k)}\|_2}{\|x\|_2} \le \kappa(A) \frac{\|r^{(k)}\|_2}{\|b\|_2} \qquad \square$$

$$\frac{\|e^{(k)}\|_2}{\|x\|_2} \le K(A)\,\frac{\|r^{(k)}\|_2}{\|b\|_2}$$

Wk 4 lec notes Pg 54

Note that the relative error of the solution is

$$\frac{\|x - x^{(k)}\|}{\|x\|} = \frac{\|e^{(k)}\|}{\|x\|}$$

and that of the estimate is given by

$$\frac{\|Ax - Ax^{(k)}\|}{\|Ax\|} = \frac{\|r^{(k)}\|}{\|b\|}$$

Finding k to achieve $\frac{\|e^{(k)}\|_2}{\|x\|_2} \le \varepsilon$ if

$$\frac{\|r^{(k)}\|_2}{\|b\|_2} \le \frac{\varepsilon}{K(A)}.$$

Also, recall that $r^{(k)} = Ae^{(k)} = AR^k e_0$

$$\therefore \frac{\|e_k\|_2}{\|x\|_2} \le \frac{\|A\|_2\|R\|_2^k\|e_0\|_2}{\|b\|_2} \le \frac{\varepsilon}{K_2(A)}$$

$$\|R\|^k \le \frac{\varepsilon\|b\|_2}{K_2(A)\|A\|_2\|e_0\|_2}$$

$k \le k^*$, where k^* is given by

Pg 36, lec note week 2 # Example of normal matrices.	Theorem 7.1, lecture note week 3. & Theorem 7.2	Lemma 7.1, lecture note week 3

$$K^{\#} = \frac{\ln\left(\dfrac{\varepsilon \|b\|_2}{\kappa_2(A)\|A\|_2 \|e_0\|_2}\right)}{\ln(\|R\|)}$$

$$\|b\|_2 = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

Recall (from one of the lemmas discussed in the notes) that $\|A\|_2 = \rho(A)$, where A is Hermitian

$\lambda_A = \{2+\sqrt{2},\ 2,\ 2-\sqrt{2}\}$

$\Rightarrow \rho(A) = 2+\sqrt{2} = \|A\|_2$

Also,

$\lambda_{A^{-1}} = \left\{\frac{1}{2+\sqrt{2}}, \frac{1}{2}, \frac{1}{2-\sqrt{2}}\right\}$

$\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2$

$\|A^{-1}\|_2 = \rho(A^{-1}) = \rho(\lambda_{A^{-1}}) = 1/(2-\sqrt{2})$

$$\Rightarrow \kappa_2(A) = \frac{2+\sqrt{2}}{2-\sqrt{2}}$$

$\|R\|_2 = \rho(R) = 1/\sqrt{2}$

With a solution of $x = (1,1,1)^T$ and $x^{(0)} = (0,0,0)^T$, then

$\|e^0\|_2 = \|x - x^0\|_2 = \sqrt{3}$

$$K^{\#} \approx \underline{49}$$

(b) Recall the estimate

$$k \geq k^{\sharp} = \frac{\log(\|A\|_2\|e^{(0)}\|_2/\|b\|_2) - \log(\varepsilon_r)}{\log(\|R\|_2^{-1})}$$

for the number of steps in order to achieve that $\|r^{(k)}\|_2 \leq \varepsilon_r\|b\|_2$.
For the above specific data, give an upper bound for the number of steps required to get the relative error of the residual below 10^{-6}.

For this, we need $\dfrac{\|r^{(k)}\|_2}{\|b\|_2} \leq \varepsilon_r$

$$\frac{\|A\|_2 \|R\|_2^k \|e_0\|_2}{\|b\|_2} \leq \varepsilon_r$$

$$\|R\|_2^k \leq \frac{\varepsilon_r \|b\|_2}{\|A\|_2 \|e_0\|_2}$$

$k \leq k^{\#}$, where

$$k^{\#} = \frac{\ln(\varepsilon_r) + \ln(\|b\|_2) - \ln(\|A\|_2) - \ln(\|e_0\|_2)}{\ln(\|R\|_2)}$$

Plugging in the same values as before, we get,

$$k^{\#} \approx \underline{\underline{43.99}}$$

(d) State the definition of the graph $G(B)$ of a matrix $B \in \mathbb{C}^{n \times n}$.
Prove that $B \in \mathbb{C}^{n \times n}$ is irreducible if and only if its graph $G(B)$ is connected.

(d) The graph $G(B)$ of B is an oriented graph with vertices $1, \ldots, n$ and edges $i \to j$ if $a_{i,j} \neq 0$. We first show "$\Rightarrow$" by a contradiction argument. Assume that $G(B)$ is not connected. There is a vertex k to which not all vertices are connected by a chain of edges. Let $S \subsetneq \{1, \ldots, n\}$ denote the set of vertices connected to k. Pick any $j \in S$ and any $i \in \{1, \ldots, n\} \backslash S$. Then

$$b_{ij} = 0 \qquad (\star)$$

since otherwise i would be connected to $j \in S$, but since j is connected to k then also i would be connected to k in contradiction to $i \notin S$. After a suitable permutation ($B = P\tilde{B}P^T$) we may assume that $S = \{1, \ldots, p\}$ with $p < n$ and let $q = n - p$. By $(\star)$ the lower left block of size $q \times p$ in $\tilde{B}$ vanishes, hence B is not irreducible.
Now, we show "$\Leftarrow$". Assume that B is not irreducible. Up to renumbering of the vertices, the graphs of B and $\tilde{B}$ are the same. Therefore, it is sufficient to show that $G(\tilde{B})$ is not connected. Let $i > p$ and $j \leq p$ be two vertices of $G(\tilde{B})$ and suppose that there is a chain of edges

$$i = i_0 \to i_1 \to \cdots \to i_k = j$$

connecting them. Necessarily, there is an edge $i_l \to i_{l+1}$ with $i_l > p$ and $i_{l+1} \leq p$. But since $\tilde{a}_{i_l, i_{l+1}} = 0$ such an edge cannot exist. Hence, i cannot be connected to j so that $G(\tilde{B})$ is not connected.

$\square$

Que 2.

2. (SSOR) The symmetric successive over relaxation consists in performing the following iteration:

$$i = 1, \ldots, n: \qquad a_{ii} x_i^{(k+\frac{1}{2})} = \omega\Big(-\sum_{j=1}^{i-1} a_{ij} x_j^{(k+\frac{1}{2})} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} + b_i \Big) - (\omega - 1) a_{ii} x_i^{(k)},$$

$$i = n, \ldots, 1: \qquad a_{ii} x_i^{(k+1)} = \omega\Big(-\sum_{j=1}^{i-1} a_{ij} x_j^{(k+\frac{1}{2})} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k+1)} + b_i \Big) - (\omega - 1) a_{ii} x_i^{(k+\frac{1}{2})}.$$

Here, $x^{(k)}$ stands for the k^{th} iterate, and $x^{(k+\frac{1}{2})}$ is an intermediate value.
Show that SSOR is a linear iterative method with

$$M_{\text{SSOR}}^{-1} = \omega(2 - \omega)(D + \omega U)^{-1} D (D + \omega L)^{-1}.$$

Remark: Recalling that SOR uses $M_{\text{SOR}} = \frac{1}{\omega} D + L$ we see that SSOR essentially consists in performing an SOR step followed by a reverse SOR step with $\frac{1}{\omega} D + U$, which explains its name. A couple of SSOR steps sometimes are applied as a preconditioner in CG.

Answer: From the first part of the step we have that

$$a_{ii}x_i^{(k+\frac{1}{2})} + \sum_{j<i} \omega a_{ij} x_j^{(k+\frac{1}{2})} = \omega b_i - \sum_{j=i+1}^{n} \omega a_{ij} x_j^{(k)} - (\omega - 1) a_{ii} x_i^{(k)}$$

so that, after dividing by ω,

$$\left(\tfrac{1}{\omega}D + L\right)x^{(k+\frac{1}{2})} = b - \left(U + (1 - \tfrac{1}{\omega})D\right)x^{(k)}.$$

Similarly, the second part of the step gives

$$\left(\tfrac{1}{\omega}D + U\right)x^{(k+1)} = b - \left(L + (1 - \tfrac{1}{\omega})D\right)x^{(k+\frac{1}{2})}.$$

Observe that

$$\begin{aligned}
&\left(L + (1 - \tfrac{1}{\omega})D\right)\left(\tfrac{1}{\omega}D + L\right)^{-1} \\
=&\left(L + \tfrac{1}{\omega}D + (1 - \tfrac{2}{\omega})D\right)\left(\tfrac{1}{\omega}D + L\right)^{-1} \\
=&I + (1 - \tfrac{2}{\omega})D\left(\tfrac{1}{\omega}D + L\right)^{-1} \\
=&I + (\omega - 2)D\left(D + \omega L\right)^{-1}.
\end{aligned} \tag{1}$$

Inserting the formula for $x^{(k+\frac{1}{2})}$ into the one for $x^{(k)}$ therefore yields

$$\begin{aligned}
\left(\tfrac{1}{\omega}D + U\right)x^{(k+1)} &= b - \left(L + (1 - \tfrac{1}{\omega})D\right)\left(\tfrac{1}{\omega}D + L\right)^{-1}\left(b - \left(U + (1 - \tfrac{1}{\omega})D\right)x^{(k)}\right) \\
&= -(\omega - 2)D\left(D + \omega L\right)^{-1} b \\
&+ \left(U + (1 - \tfrac{1}{\omega})D\right)x^{(k)} \\
&+ (\omega - 2)D\left(D + \omega L\right)^{-1}\left(U + (1 - \tfrac{1}{\omega})D\right)x^{(k)}.
\end{aligned}$$

Similarly to (1)

$$\boxed{\left(\tfrac{1}{\omega}D + U\right)^{-1}\left(U + (1 - \tfrac{1}{\omega})D\right) = I + \left(D + \omega U\right)^{-1}(\omega - 2)D.}$$

We conclude that

$$\begin{aligned}
x^{(k+1)} &= \boxed{\left(\tfrac{1}{\omega}D + U\right)}^{-1}(2 - \omega)D\left(D + \omega L\right)^{-1} b \\
&+ \left(I + \left(D + \omega U\right)^{-1}(\omega - 2)D\right)x^{(k)} \\
&+ \left(\tfrac{1}{\omega}D + U\right)^{-1}(\omega - 2)D\left(D + \omega L\right)^{-1}\left(U + (1 - \tfrac{1}{\omega})D\right)x^{(k)} \\
&= M_{\mathrm{SSOR}}^{-1} b + x^{(k)} \\
&+ \left(D + \omega U\right)^{-1}(\omega - 2)D\,\omega\left(D + \omega L\right)^{-1}\tfrac{1}{\omega}\left(D + \omega L\right)x^{(k)} \\
&- M_{\mathrm{SSOR}}^{-1}\left(U + (1 - \tfrac{1}{\omega})D\right)x^{(k)} \\
&= M_{\mathrm{SSOR}}^{-1} b - M_{\mathrm{SSOR}}^{-1}\left(-M_{\mathrm{SSOR}}\right)x^{(k)} \\
&- M_{\mathrm{SSOR}}^{-1}\big(\underbrace{\tfrac{1}{\omega}D + L + U + (1 - \tfrac{1}{\omega})D}_{=D+L+U=A}\big)x^{(k)} \\
&= M_{\mathrm{SSOR}}^{-1} b - M_{\mathrm{SSOR}}^{-1}\big(\underbrace{A - M_{\mathrm{SSOR}}}_{=:N_{\mathrm{SSOR}}}\big)x^{(k)} \\
&= M_{\mathrm{SSOR}}^{-1}\left(b - N_{\mathrm{SSOR}}x^{(k)}\right).
\end{aligned}$$

$\square$

refactor this to match M_{SSOR}

related to each other

related to each other

artifically added

check next page about this

$$\begin{aligned}
&\left(\tfrac{1}{\omega}D + U\right)^{-1}\left(U + (1 - \tfrac{1}{\omega})D\right) \\
&= \left(\tfrac{1}{\omega}D + U\right)^{-1}\left(U + \tfrac{1}{\omega}D + (1 - \tfrac{2}{\omega})D\right) \\
&= I + \left(\tfrac{1}{\omega}D + U\right)^{-1}(1 - \tfrac{2}{\omega})D \\
&= I + \left[\tfrac{1}{\omega}(D + \omega U)\right]^{-1}\tfrac{1}{\omega}(\omega - 2)D \\
&= I + \omega(D + \omega U)\tfrac{1}{\omega}(\omega - 2)D \\
&= I + (D + \omega U)^{-1}(\omega - 2)D
\end{aligned}$$

$$-M_{SSOR}^{-1} \frac{1}{\omega}(D+\omega L)x^{(k)} \qquad - M_{SSOR}^{-1}\left(U+\left[1-\frac{1}{\omega}\right]D\right)x^{(k)}$$

$$= -M_{SSOR}^{-1}\left(\frac{1}{\omega}D + L + U + (1-1/\omega)D\right)x^{(k)}$$

$$= -M_{SSOR}^{-1}(D+L+U)x^{(k)}$$

$$= -M_{SSOR}^{-1} A x^{(k)}$$

clean version of the answer

Answer: From the first part of the step we have that

$$a_{ii}x_i^{(k+\frac{1}{2})} + \sum_{j<i} \omega a_{ij} x_j^{(k+\frac{1}{2})} = \omega b_i - \sum_{j=i+1}^{n} \omega a_{ij} x_j^{(k)} - (\omega - 1)a_{ii}x_i^{(k)}$$

so that, after dividing by ω,

$$(\tfrac{1}{\omega}D + L)x^{(k+\frac{1}{2})} = b - (U + (1 - \tfrac{1}{\omega})D)x^{(k)}.$$

Similarly, the second part of the step gives

$$(\tfrac{1}{\omega}D + U)x^{(k+1)} = b - (L + (1 - \tfrac{1}{\omega})D)x^{(k+\frac{1}{2})}.$$

Observe that

$$\begin{aligned}
&(L + (1 - \tfrac{1}{\omega})D)(\tfrac{1}{\omega}D + L)^{-1}\\
=&(L + \tfrac{1}{\omega}D + (1 - \tfrac{2}{\omega})D)(\tfrac{1}{\omega}D + L)^{-1}\\
=&I + (1 - \tfrac{2}{\omega})D(\tfrac{1}{\omega}D + L)^{-1}\\
=&I + (\omega - 2)D(D + \omega L)^{-1}.
\end{aligned} \tag{1}$$

Inserting the formula for $x^{(k+\frac{1}{2})}$ into the one for $x^{(k)}$ therefore yields

$$\begin{aligned}
(\tfrac{1}{\omega}D + U)x^{(k+1)} &= b - (L + (1 - \tfrac{1}{\omega})D)(\tfrac{1}{\omega}D + L)^{-1}(b - (U + (1 - \tfrac{1}{\omega})D)x^{(k)})\\
&= -(\omega - 2)D(D + \omega L)^{-1}b\\
&+ (U + (1 - \tfrac{1}{\omega})D)x^{(k)}\\
&+ (\omega - 2)D(D + \omega L)^{-1}(U + (1 - \tfrac{1}{\omega})D)x^{(k)}.
\end{aligned}$$

Similarly to (1)

$$(\tfrac{1}{\omega}D + U)^{-1}(U + (1 - \tfrac{1}{\omega})D) = I + (D + \omega U)^{-1}(\omega - 2)D.$$

We conclude that

$$\begin{aligned}
x^{(k+1)} &= (\tfrac{1}{\omega}D + U)^{-1}(2 - \omega)D(D + \omega L)^{-1}b\\
&+ (I + (D + \omega U)^{-1}(\omega - 2)D)x^{(k)}\\
&+ (\tfrac{1}{\omega}D + U)^{-1}(\omega - 2)D(D + \omega L)^{-1}(U + (1 - \tfrac{1}{\omega})D)x^{(k)}\\
&= M_{\mathrm{SSOR}}^{-1}b + x^{(k)}\\
&+ (D + \omega U)^{-1}(\omega - 2)D\omega(D + \omega L)^{-1}\tfrac{1}{\omega}(D + \omega L)x^{(k)}\\
&- M_{\mathrm{SSOR}}^{-1}(U + (1 - \tfrac{1}{\omega})D)x^{(k)}\\
&= M_{\mathrm{SSOR}}^{-1}b - M_{\mathrm{SSOR}}^{-1}(-M_{\mathrm{SSOR}})x^{(k)}\\
&- M_{\mathrm{SSOR}}^{-1}\underbrace{(\tfrac{1}{\omega}D + L + U + (1 - \tfrac{1}{\omega})D)}_{=D+L+U=A}x^{(k)}\\
&= M_{\mathrm{SSOR}}^{-1}b - M_{\mathrm{SSOR}}^{-1}\underbrace{(A - M_{\mathrm{SSOR}})}_{=:N_{\mathrm{SSOR}}}x^{(k)}\\
&= M_{\mathrm{SSOR}}^{-1}(b - N_{\mathrm{SSOR}}x^{(k)}).
\end{aligned}$$