21/08 Nov '23

MA265 Assignment 3

Que 1:

$$\mathcal{L}(x_1, x_2, \lambda) = e^{-x_1} + \lambda \frac{x_1^2}{x_2}$$

$D = \{ (x_1, x_2) : x_2 > 0 \}$

The dual is given by $g(\lambda) = \inf_{x_1, x_2 > 0} \mathcal{L}(\bar{x}, \lambda)$

$$\mathcal{L}_{x_1} = -e^{-x_1} + \frac{2\lambda x_1}{x_2} = 0$$

$$\lambda = \frac{x_2 e^{-x_1}}{2x_1}$$

$$\therefore \mathcal{L} = e^{-x_1} + \frac{x_2 x_1^2 e^{-x_1}}{2 x_1 x_2} = \frac{(x_1+2)e^{-x_1}}{2}$$

$\lambda \geq 0 \implies x_1 > 0$, and the infimum of $\mathcal{L}$ for such x_1 is 0.

i.e $\inf_{x_1 \to \infty} \mathcal{L} = 0$

$\lambda < 0 \implies x_1 < 0$, and

$$\inf_{x_1 \to -\infty} \mathcal{L} = -\infty$$

$$\therefore g(\lambda) = \begin{cases} 0, & \lambda \geq 0 \\ -\infty, & \lambda < 0 \end{cases}$$

$d^* = \max g(\lambda) = 0$. The dual gap $p^* - d^* = 1 \neq 0$, Strong duality does not hold.

Que 2: For KKT conditions

$f_i(w) \leq 0$ [constrains]

$\lambda_i f_i(w^*) = 0$

$\nabla_w \mathcal{L}(w^*, \lambda^*, \mu^*) = 0$

b) (i) $\lambda_1 = \lambda_2 = 0 \Rightarrow \mathcal{L}_{x_1}$ reduces to

$e^{x_1 - x_2} = 0$

which has no solution

(ii) $\lambda_1 = 0, \lambda_2 > 0$, $\mathcal{L}_{x_2}$ reduces to

$-e^{x_1 - x_2} = 0$

which has no solution.

(iii) $\lambda_1 > 0$, $\lambda_2 = 0$, $\nabla \mathcal{L}$ reduces to

$$\left.\begin{aligned} e^{x_1 - x_2} + \lambda_1 e^{x_1} &= 0 \\ -e^{x_1 - x_2} + \lambda_1 e^{x_2} &= 0 \end{aligned}\right\} +$$

$$\lambda_1 (e^{x_1} + e^{x_2}) = 20 \lambda_1 = 0$$

$\Rightarrow \lambda_1 = 0$ (which is not possible).

KKT Condition
Pg 81 lec note.

$$\nabla_x \mathcal{L} = \begin{pmatrix} e^{x_1-x_2} + \lambda_1 e^{x_1} - \lambda_2 \\ -e^{x_1-x_2} + \lambda_1 e^{x_2} \end{pmatrix}$$

Part of Que 2

$$\nabla_{\lambda_i} \mathcal{L} = \begin{pmatrix} e^{x_1} + e^{x_2} - 20 \\ -x_1 \end{pmatrix}$$

$$e^{x_1-x_2} + \lambda_1 e^{x_1} - \lambda_2 = 0 \quad \text{——— (i)}$$
$$-e^{x_1-x_2} + \lambda_1 e^{x_2} = 0 \quad \text{——— (ii)}$$
$$e^{x_1} + e^{x_2} - 20 = 0 \quad \text{——— (iii)}$$
$$-x_1 = 0 \quad \text{——— (iv)}$$

(i) + (ii):

$$\lambda_1 (e^{x_2} + e^{x_1}) - \lambda_2 = 0 \quad \text{——— } *$$

from (iv): $x_1 = 0$ //

$\Rightarrow$ (iii) becomes: $1 + e^{x_2} - 20 = 0$

$$x_2 = \ln(19) \quad //$$

from (ii): $-\dfrac{e^{x_1}}{e^{x_2}} + \lambda_1 e^{x_2} = 0$

$$\therefore \frac{-1}{19} + \lambda_1 19 = 0$$

$$\lambda_1 = 1/19^2 \quad //$$

from $*$: $\dfrac{1}{19^2}(19+1) - \lambda_2 = 0$

$$\Rightarrow \lambda_2 = \frac{20}{19^2} \quad //$$

we have a Convex function. So the optimal point is the global solutn.

The only point in the feasible set is $(x_1, x_2) = (1,0)$

Part of Que 3

$$2(1) + 2\lambda_1(1-1) + 2\lambda_2(1-1) = 0$$
$$\Rightarrow \quad 2 = 0 \ ??$$

$$2(0) + 2\lambda_1(0-1) + 2\lambda_2(0+1) = 0$$
$$-2\lambda_1 + 2\lambda_2 = 0 \quad \text{———} \ *$$

$$(1-1)^2 + (0-1)^2 \leq 1$$
$$1 \leq 1$$

$$(1-1)^2 + (0+1)^2 \leq 1$$
$$1 \leq 1$$

$$\lambda_1, \lambda_2 \geq 0$$

the last 2 are somewhat captured above.

from $*$: $-2\lambda_1 + 2\lambda_2 = 0$
$$\lambda_1 = \lambda_2$$

This is not solvable (since we had $2=0$).

So no solution.

$\varphi(x) = f(x) + \alpha \|Ax - b\|^2$, $\alpha > 0$.

#Que 4

If $\tilde{x}$ is a minimizer, then

$$\nabla_x \varphi(x) = 0$$

$$\Rightarrow \quad \nabla f(\tilde{x}) + 2\alpha A^T(Ax - b) = 0 \quad \text{——} *$$

$$\nabla f(\tilde{x}) = -2\alpha A^T(Ax - b)$$

Therefore, $\tilde{x}$ is also a minimizer of

$$\mathcal{L} = f(x) + \mu^T(Ax - b)$$ —— #formulating as a lagrangian

where

$\mu = 2\alpha(Ax - b)$ # Note that we want to also obtain $*$ when we compute $\nabla \mathcal{L}$.

Therefore μ is the dual feasible with

$$g(\mu) = \inf_x \left(f(x) + \mu^T(Ax - b) \right)$$

Note that $\|x\|^2 = x^T x$

$$= f(\tilde{x}) + \alpha \|A\tilde{x} - b\|^2$$

Therefore,

$$f(x) \geq f(\tilde{x}) + \alpha \|A\tilde{x} - b\|^2$$

$\forall x$ s.t $Ax = b$.

We used that

$$\sup_{\lambda \geq 0, \mu} g(\lambda, \mu) \leq \inf_{w \in D} \{ f(w) \mid f_i(w) \leq 0, Aw = b \}$$

Page 77 lec note.

Que 4 (Another approach)

$$\varphi(x) = f(x) + \alpha \|Ax - b\|^2$$

At the minimum point

$$\nabla_x \varphi(x) = \nabla_x f(\tilde{x}) + 2\alpha A^T (A\tilde{x} - b) = 0$$

$$\nabla_x f(\tilde{x}) = -2\alpha A^T (A\tilde{x} - b)$$

The Lagrangian is given by

$$\mathcal{L}(x, \mu) = f(x) + \mu^T (Ax - b)$$

To find the dual problem

$$\nabla_x \mathcal{L}(x, \mu) = \nabla_x f(x) + \nabla_x (\mu^T (Ax - b)) = 0$$

Subst. $\tilde{x}$ for x:

$$\Rightarrow \nabla_x \mathcal{L}(\tilde{x}, \mu) = -2\alpha A^T (A\tilde{x} - b) + A^T \mu = 0$$

$$\mu = 2\alpha (A\tilde{x} - b)$$

Therefore μ is the dual feasible with

$$g(\mu) = \inf_x \left(f(x) + \mu^T (Ax - b) \right)$$

$$= f(\tilde{x}) + 2\alpha (A\tilde{x} - b)(A\tilde{x} - b)$$

$$= f(\tilde{x}) + 2\alpha \|A\tilde{x} - b\|^2$$

b) $g(\mu) \leq p^*$ if p^* is the optimal value for the primal problem.

$$\Rightarrow f(x) \geq f(\tilde{x}) + 2\alpha \|A\tilde{x} - b\|^2$$