

## MA398 Matrix Analysis and Algorithms: Exercise Sheet 6

1. Prove the uniqueness of the singular values in the Singular Value Decomposition (SVD) of a matrix. That is, show that for every matrix  $A \in \mathbb{R}^{m \times n}$ , the singular values in the SVD are uniquely determined.
2. Implement a function in Python that computes the Singular Value Decomposition (SVD) of a given matrix  $A$ . The function should take the matrix  $A$  as input and return the matrices  $U$ ,  $\Sigma$ , and  $V^T$ . You can make use of the NumPy library for matrix operations. You can also use `compute_svd(A)` of the NumPy library to compute the singular values.
3. Consider a 2-dimensional LSQ problem, with:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

- (a) Write a Python function to compute the pseudo-inverse of  $A$ , and use it to find the solution  $x = A^\dagger b$ .
- (b) Using the numpy library, verify that your computed pseudo-inverse is correct and your solution  $x$  matches the one obtained from numpy's built-in least squares solver `numpy.linalg.lstsq`.
- (c) Compute the condition number  $\kappa_2(A)$  of matrix  $A$  according to the provided definition, and confirm your computation with the function `numpy.linalg.cond`.
- (d) Generate a small perturbation  $\Delta b = [\delta \ \delta]$  for a range of  $\delta$  values and compute the corresponding  $\Delta x$  for each. Plot  $\frac{|\Delta x|_2}{|x|_2}$  versus  $\frac{\kappa_2(A)}{\eta \cos(\theta)} \frac{|\Delta b|_2}{|b|_2}$  to observe the bound provided in the theorem. For this part, you need to compute  $\eta$  and  $\cos(\theta)$  based on the given formulas. Note, you may need to be careful about the value of  $\delta$  to ensure that  $\Delta x$  and  $x$  are not zero vectors (which could lead to division by zero in your calculations).
- (e) Write a Python function implementing LSQ\_SVD algorithm, and verify it gives the same solution as before for the given  $A$  and  $b$ . Analyze the complexity of this function based on the operations it performs.

Hints:

- For the SVD of a matrix, you can use `numpy.linalg.svd`.
- For matrix multiplication, use the `@` operator or `numpy.dot`.
- For transposing a matrix, use `.T` attribute.
- For computing the 2-norm of a vector, use `numpy.linalg.norm`.