23rd Oct, '23.

MA398 Exercise Sheet 2, Week 4

2. Intuitive explanation:

If the columns of Q are orthonormal: of length 1 and perpendicular to each other, then we know that

$$Q = [q_1 \; q_2 \; \ldots \; q_n] \;, \; q_i \in \mathbb{C}^n$$

$$\langle q_i, q_j \rangle = \begin{cases} 0 & , i \neq j \\ 1 & , i = j \end{cases}$$

$\Rightarrow QQ^*$ would be 1 on the major diagonal where $i=j$, and 0 everywhere else.

3. 3. Prove the following Theorem:

"If $A \in \mathbb{C}^{n\times n}$ is Hermitian then there is a unitary Q and a real $\Lambda \in \mathbb{R}^{n\times n}$ such that $A = Q\Lambda Q^*$."

$\hookrightarrow \Lambda$ is a diagonal matrix

Hint: Remember that a Hermitian matrix is one that is equal to its own conjugate transpose, i.e., $A = A^*$. You might want to utilize the Spectral Theorem for Hermitian matrices in your proof, which states that every Hermitian matrix can be diagonalized by a unitary matrix.

Answer: Firstly, we need to understand what Hermitian means in the context of complex matrices. A matrix $A \in \mathbb{C}^{n\times n}$ is Hermitian if it is equal to its own conjugate transpose, i.e., $A = A^*$.

Now, let us prove the theorem:

Proof:

The Spectral Theorem for Hermitian matrices states that a Hermitian matrix A can be diagonalized by a unitary matrix, i.e., there exists a unitary matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^*$.

We need to show that the diagonal elements of Λ are real. This can be deduced from the property of Hermitian matrices that their eigenvalues are real.

Consider the equation $Ax = \lambda x$, where x is an eigenvector corresponding to the eigenvalue λ.

Taking the complex conjugate transpose of both sides, we get $(Ax)^* = \lambda^* x^*$.

This becomes $x^* A^* = \lambda^* x^*$.

Since A is Hermitian, we can replace A^* with A. This gives us $x^* A = \lambda^* x^*$.

equate both of them.

But we know from the original eigenvalue equation that $x^* A = \lambda x^*$. Therefore, we can equate $\lambda = \lambda^*$, which implies that λ is real.

$\hookrightarrow$ left multiply eigenvector (transpose)

Therefore, all the diagonal elements of Λ, which are the eigenvalues of A, are real. Thus, we have shown that if A is Hermitian, then there exists a unitary matrix Q and a real diagonal matrix Λ such that $A = Q\Lambda Q^*$.

This concludes the proof.

(5.)

5. (Geometric series for matrices) Let $\|\cdot\|$ be a matrix norm on $\mathbb{C}^{n\times n}$. Assume that $\|X\| < 1$ for some $X \in \mathbb{C}^{n\times n}$. Show that,

(a) $I - X$ is invertible with $(I-X)^{-1} = \sum_{i=0}^{\infty} X^i$,

(b) $\|(I-X)^{-1}\| \le (1 - \|X\|)^{-1}$.

$\frac{a(1-r^n)}{1-r}$

Answer: For every $m \in \mathbb{N}$

$$\overbrace{\|\sum_{i=0}^{m} X^i\| \le \sum_i \|X^i\|}^{\text{triangle} \le} \le \overbrace{\sum_i \|X\|^i = \frac{1-\|X\|^{m+1}}{1-\|X\|}}^{\text{Sum of G.P}} \tag{1}$$

where the properties of a matrix norm were used. The right hand side converges as $m \to \infty$ since $\|X\| < 1$ (geometric series). An analogous argument shows that $n \mapsto \sum_{i=0}^{n} X^i$ is a Cauchy sequence. As a consequence, $\sum_{i=0}^{\infty} X^i$ exists and $X^i \to 0$ as $i \to \infty$ in $\mathbb{C}^{n\times n}$. We infer that

$$(I-X)\sum_{i=0}^{\infty} X^i = \lim_{m\to\infty} \overbrace{(I-X)\sum_{i=0}^{m} X^i}^{\text{Sum of GP \& cancelations.}} = \lim_{m\to\infty}(I - X^{m+1}) = I.$$ (question a)

Letting $m \to \infty$ in the right hand side of (1) we obtain the second claim:

$$\|(I-X)^{-1}\| = \lim_{m\to\infty} \|\sum_{i=0}^{m} X^i\| \le \lim_{m\to\infty} \frac{1-\|X\|^{m+1}}{1-\|X\|} = (1-\|X\|)^{-1}.$$ (question b)