Lecture 3

LU factorisation

Learning Outcomes

- Gain knowledge of LU Factorization, understanding how to uniquely factorize a matrix A into a product of a lower triangular matrix L and an upper triangular matrix U, provided all principal sub-matrices of A are invertible.
- Develop an understanding of permutation matrices and learn how to use them to permute the rows of a matrix, ensuring the existence of LU factorization for regular matrices.
- Acquire the ability to apply an algorithm to find the LU factorization of a given matrix, recognizing the conditions under which the factorization is possible and illustrating the process through practical examples.

Introduction

The $\underline{j^{th}\text{ principal sub-matrix}}$ of $A \in \mathbb{C}^{n\times n}$ is given by $A_j = (a_{k,l})_{k,l=1}^{j}$.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$.*
(a) Assume that A_j is invertible for all $j = 1, \dots, n$. Then there is a unique factorisation $A = LU$ with L unit lower triangular and U regular and upper triangular.
(b) If A_j is singular for some j then there is no such factorisation.

Proof. (a) Proof is based on induction, similar proofs will follow → **Exercise.**
(b) Assume that $A = LU$ exists but A_j is singular. In block form:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix} = \begin{pmatrix} L_{11}U_{11} & \star \\ \star & \star \end{pmatrix}$$

with $A_{11} = A_j$, and since L_{11} is unit lower triangular and U_{11} is regular and upper triangular, $A_j = L_{11}U_{11}$ is the LU factorisation of A_j. But then

$$\det(A_j) = \det(L_{11}U_{11}) = \underbrace{\det(L_{11})}_{=1}\underbrace{\det(U_{11})}_{\neq 0} \neq 0$$

in contradiction to the singularity of A_j. $\square$

Algorithm 2 LU

input: $A = (a_{ij})_{i,j=1}^{n} \in \mathbb{C}^{n \times n}$ with $\det(A_k) \neq 0$, $k = 1, \ldots, n$.
output: $L \in \mathbb{C}^{n \times n}$ unit lower triangular, $U \in \mathbb{C}^{n \times n}$ upper triangular and regular with $A = LU$.

$U = A$, $L = I$.
for $k = 1$ to $n - 1$ **do**
    for $j = k + 1$ to n **do**
        $l_{j,k} := u_{j,k}/u_{k,k}$
        $u_{j,k} := 0$
        for $i = k + 1$ to n **do**
            $u_{j,i} := u_{j,i} - l_{j,k} u_{k,i}$
        end for
    end for
end for

As already discussed a matrix A has an LU factorisation if and only if it's leading principle sub-matrices are all non-singular. However, the matrix A could be perfectly good (has n-linearly independent columns), so LU factorisation would still be possible. To do it though we need to allow ourselves extra freedom so that if a zero shows in a pivot position, it needs to be moved away in such a way a proper pivot is obtained. This can be achieved by exchanging rows in the matrix we need to factorise. The vehicle for exchanging rows of a matrix is through the multiplication of that matrix by a permutation matrix.

What is a permutation matrix?

Definition 3.1. $P \in \mathbb{C}^{n \times n}$ *is a* <u>*permutation matrix*</u> *if every row and every column contains* $n - 1$ *zeros and* 1 *one.*

$$Pq = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

Further investigating this particular matrix, it can be seen that it can be written as,

$$P = \begin{pmatrix} e_3^T \\ e_1^T \\ e_2^T \end{pmatrix}$$

In addition, as can be seen multiplying this matrix of basis vectors by a vector results in picking out the entry index with 3, then 1, then 2 of that vector. Thus, standard basis vectors can be used to pick out entries.

How many possible re-orderings are there for an $n \times n$ matrix?

Remark:

$$P^{-1} = P^T \implies P^T P = I$$

Applying the same idea to a matrix will similarly permute the rows of the matrix.

Example:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

is regular since $\det(A) = 2$ but $A_1 = 0$ and A_2 both are singular.
Idea: Permutation of the rows of A, $\sigma : (1, 2, 3) \mapsto (2, 3, 1)$, which is equivalent to a multiplication with the permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad PA = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and PA has regular submatrices. In fact, as PA is upper triangular, the LU factorisation is obtained by setting $U = PA$ and choosing L to be the identity.
We may write $P = (e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(n)})^T$ where e_j is the j^{th} vector of the standard basis of $\mathbb{C}^n$. With $\pi = \sigma^{-1}$, and the following permutation vector,

$$\Pi = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_n \end{pmatrix}$$

where this is mainly the vector of permutation of the indices 0 to $n-1$.The permutation matrix can be constructed from Π as follows.

$$P = \begin{pmatrix} — & e_{\pi(1)} & — \\ & \vdots & \\ — & e_{\pi(n)} & — \end{pmatrix}.$$

This means that we don't really need to store the entire permutation matrix (which means storing all the zeros which are not necessary). It will be sufficient to just store the entries $\pi_1, \ldots, \pi_n$, or the indices that define the permutation. It also follows that, if we write,

$$A = \begin{pmatrix} — & a_1 & — \\ & \vdots & \\ — & a_n & — \end{pmatrix}$$

with the row vectors $a_i \in \mathbb{C}^n$ then

$$PA = \begin{pmatrix} — & a_{\pi(1)} & — \\ & \vdots & \\ — & a_{\pi(n)} & — \end{pmatrix}$$

hence, a multiplication with a permutation matrix from the *left* exchanges the *rows* according to the associated permutation. Similarly, a multiplication with a permutation matrix from the *right* exchanges the *columns*.

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}$ be regular. Then there is a permutation matrix $P \in \mathbb{C}^{n\times n}$, $L \in \mathbb{C}^{n\times n}$ unit lower triangular, and $U \in \mathbb{C}^{n\times n}$ upper triangular with $PA = LU$.*

Proof. By induction on n. The case $n = 1$ is trivial as one just has to choose $P = L = 1$ and $U = A$.
Let $n > 1$ and assume that the assertion is true for $n - 1$. Choose a permutation matrix P_1 such that $a := (P_1 A)_{1,1} \neq 0$. Such a matrix exists because A is regular, whence the first column of A will contain a nonzero entry. We write

$$P_1 A = \begin{pmatrix} a & u^* \\ l & B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{a} l & I \end{pmatrix} \begin{pmatrix} a & u^* \\ 0 & \tilde{A} \end{pmatrix}$$

where $l, u \in \mathbb{C}^{(n-1)\times 1}$, u^* is the adjoint of u, and $\tilde{A} = B - \frac{1}{a} l u^* \in \mathbb{C}^{(n-1)\times(n-1)}$. The matrix $\tilde{A}$ is regular since

$$0 \neq \det(P_1 A) = \underbrace{\det \begin{pmatrix} 1 & 0 \\ \frac{1}{a} l & I \end{pmatrix}}_{=1} \det \begin{pmatrix} a & u^* \\ 0 & \tilde{A} \end{pmatrix} = a \det(\tilde{A}).$$

By the induction hypothesis there are $\tilde{P}$ (permutation), $\tilde{L}$ (unit lower triangular) and $\tilde{U}$ (regular upper triangular) with $\tilde{P}\tilde{A} = \tilde{L}\tilde{U}$. Therefore

$$\begin{aligned} P_1 A &= \begin{pmatrix} 1 & 0 \\ \frac{1}{a} l & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (\tilde{P})^{-1}\tilde{L} \end{pmatrix} \underbrace{\begin{pmatrix} a & u \\ 0 & \tilde{U} \end{pmatrix}}_{=:U} \\ &= \begin{pmatrix} 1 & 0 \\ \frac{1}{a} l & (\tilde{P})^{-1}\tilde{L} \end{pmatrix} U \\ &= \begin{pmatrix} 1 & 0 \\ 0 & (\tilde{P})^{-1} \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ \frac{1}{a}\tilde{P} l & \tilde{L} \end{pmatrix}}_{=:L} U \end{aligned}$$

$$\Rightarrow \quad \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & (\tilde{P})^{-1} \end{pmatrix}^{-1}}_{=:P} P_1 A = LU$$

where P is a permutation matrix and L and U are of the desired structure, too. $\square$

Lecture 4

Gaussian Elimination with Pivoting

Learning Outcomes

- **Overcoming Pivot Issues in Gaussian Elimination:**
  - Develop strategies to address challenges encountered when the pivot, $u_{k,k}$, is zero or small, preventing stability issues. Understand the practicality and sufficiency of Gaussian Elimination with Partial Pivoting (GEPP) over Gaussian Elimination with Complete Pivoting (GECP) in resolving such issues.
- **Mastering the Implementation of Permutation Matrices:**
  - Learn to effectively use permutation matrices, P_k, for swapping rows to mitigate pivot issues, enabling the computation of $U^{(k)} = \tilde{L}_k P_k U^{(k-1)}$. Gain insights into achieving LU decomposition through appropriate permutation matrices, realizing the equation $LU = PA$.
- **Applying Algorithmic Solutions through Practical Examples:**
  - Acquire the ability to implement detailed algorithms like LUPP and GEPP for finding LU decomposition with partial pivoting and solving systems of linear equations. Apply this knowledge through practical examples to compute LU factorization and solve linear systems, focusing on the structure and properties of lower/upper triangular matrix operations.

Pivot Issues

Before step k:

$$U^{(k-1)} = \begin{pmatrix} \star & \cdots & \cdots & \star & \star & \cdot & \star \\ 0 & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \star & \star & \vdots & & \vdots \\ 0 & \cdots & 0 & u_{k,k} & \star & \cdots & \star \\ 0 & \cdots & 0 & \star & \star & \cdots & \star \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \star & \star & \cdots & \star \end{pmatrix}$$

and we have a **problem** if the pivot $u_{k,k}$ is zero.
In fact, a small $u_{k,k}$ is undesirable as it leads to stability problems $\to$ will see this later on.
There are basically two strategies to overcome this problem.

1. **GEPP**, Gaussian Elimination with Partial Pivoting: Swap rows to maximise $|u_{k,k}|$ among the entries $u_{l,k}$, $l = k, \ldots, n$.

2. **GECP**, Gaussian Elimination with Complete Pivoting: Swap rows and columns to maximise $|u_{k,k}|$ among the entries $u_{l,m}$, $l, m = k, \ldots, n$.

In the following we will only consider **GEPP** since, in practice, partial pivoting usually is sufficient and the gain in stability due to complete pivoting is negligible.
With an appropriate permutation matrix P_k that realises the swap of the rows we then compute $U^{(k)} = \tilde{L}_k P_k U^{(k-1)}$ so that in the end

$$U = \tilde{L}_{n-1} P_{n-1} \cdots \tilde{L}_1 P_1 A. \tag{4.1}$$

The permutation associated with P_l exchanges l with some number $i_l > l$ and leaves the other entries unchanged,

$$(1, \ldots, l-1, l, l+1, \ldots, i_l - 1, i_l, i_l + 1, \ldots, n) \mapsto (1, \ldots, l-1, i_l, l+1, \ldots, i_l - 1, l, i_l + 1, \ldots, n).$$

Example1:

$$A = \begin{pmatrix} -4 & 4 & -2 \\ 8 & -8 & 5 \\ 2 & 5 & 3 \end{pmatrix}$$

Exchange row 1 with 2 using P_{12} as follows,

$$P_{12} A = \begin{pmatrix} 8 & -8 & 5 \\ -4 & 4 & -2 \\ 2 & 5 & 3 \end{pmatrix}$$

Let,

$$\tilde{L}_1 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{4} & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{M_1} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{4} & 0 & 1 \end{pmatrix}}_{M_2}$$

Then,

$$\tilde{L}_1 P_{12} A = \begin{pmatrix} 8 & -8 & 5 \\ 0 & 0 & \frac{1}{2} \\ 0 & 7 & \frac{7}{4} \end{pmatrix}$$

Exchange row 2 with row 3 using P_{23},

$$P_{23} \tilde{L}_1 P_{12} A = \begin{pmatrix} 8 & -8 & 5 \\ 0 & 7 & \frac{7}{4} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = U$$

This completes the Gaussian elimination with pivoting step. The objective next is to find L by transforming $P_{23} \tilde{L}_1 P_{12} A = U$ to $PA = LU$. Thus,

$$\underbrace{P_{23} \tilde{L}_1 P_{23}^{-1}}_{\text{Lower Triangular}} \; P_{23} P_{12} A = U$$

Since $P_{23}\tilde{L}_1P_{23}^{-1}$ is lower triangular, it can be easily inverted from which the required lower triangular matrix can be obtained,

$$\begin{aligned} P_{23}P_{12}A &= (P_{23}\tilde{L}_1P_{23}^{-1})^{-1}U \\ &= LU \end{aligned}$$

Remark: Generally, for some $k < l$, the unit lower triangular matrix will have the following form,

$$M = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & & & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & & & & \vdots \\ \vdots & & 0 & 1 & \ddots & & & & & & \vdots \\ \vdots & & \vdots & m_{k+1,k} & \ddots & \ddots & & & & & \vdots \\ \vdots & & \vdots & \vdots & 0 & \ddots & \ddots & & & & \vdots \\ \vdots & & \vdots & m_{l,k} & \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \vdots & m_{i_l,k} & \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m_{n,k} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

Multiplying M by P_l from the left and P_l^{-1} from the right will result in swapping row l with row i_l followed by swapping column l by column i_l yielding another unit lower triangular matrix,

$$P_lMP_l^{-1} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & & & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & & & & \vdots \\ \vdots & & 0 & 1 & \ddots & & & & & & \vdots \\ \vdots & & \vdots & m_{k+1,k} & \ddots & \ddots & & & & & \vdots \\ \vdots & & \vdots & \vdots & 0 & \ddots & \ddots & & & & \vdots \\ \vdots & & \vdots & m_{i_l,k} & \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \vdots & m_{l,k} & \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m_{n,k} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

Example: If $k = 2$, $n = 6$, then

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.5 & 1 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0.1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Swap row 3 with row 5 using,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Permute M, gives,

$$PMP^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0.2 & 0 & 1 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 1 & 0 \\ 0 & 0.1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Multiplying M by P from the left, exchanged row 3 with row 5 while multiplying it by P from the right exchanged column 3 with column 5.
Consequently, the matrix,

$$L'_k := P_{n-1} \cdots P_{k+1} \tilde{L}_k P_{k+1}^{-1} \cdots P_{n-1}^{-1}$$

has the same structure as $\tilde{L}_k$ but just the entries in column k below the diagonal are permuted. Since

$$L'_k P_{n-1} \cdots P_{k+1} = P_{n-1} \cdots P_{k+1} \tilde{L}_k$$

we obtain from (4.1) that

$$U = \underbrace{L'_{n-1} \cdots L'_1}_{=:L^{-1}} \underbrace{P_{n-1} \cdot P_1}_{=:P} A \quad \Rightarrow LU = PA$$

which is a factorisation of the desired form.
Example: Let $n = 4$ then from equation (4.1) we get,

$$\begin{aligned} U &= \tilde{L}_3 P_3 \tilde{L}_2 P_2 \tilde{L}_1 P_1 A \\ &\equiv \tilde{L}_3 \underbrace{P_3 \tilde{L}_2 P_3^{-1}}_{L'_2} P_3 P_2 \tilde{L}_1 P_1 A \\ &= L'_3 L'_2 \underbrace{P_3 P_2 \tilde{L}_1 P_2^{-1} P_3^{-1}}_{L'_1} P_3 P_2 P_1 A \\ &= L'_3 L'_2 L'_1 P_3 P_2 P_1 A \end{aligned}$$

Note, the line before the last in the above equation is as a result of the following definition,

$$L'_k := P_{n-1} \cdots P_{k+1} \tilde{L}_k P_{k+1}^{-1} \cdots P_{n-1}^{-1}$$

which means that

$$L'_1 = P_3 P_2 \tilde{L}_1 P_2^{-1} P_3^{-1}$$

Algorithm 3 LUPP

input: $A = (a_{ij})_{i,j=1}^n \in \mathbb{C}^{n\times n}$ regular.
output: $L \in \mathbb{C}^{n\times n}$ unit lower triangular, $U \in \mathbb{C}^{n\times n}$ regular upper triangular, $P \in \mathbb{C}^{n\times n}$ permutation matrix with $PA = LU$.

```
U = A, L = I, P = I.
for k = 1 to n − 1 do
    choose i ∈ {k, ..., n} such that |u_{i,k}| is maximal
    exchange (u_{k,k}, ..., u_{k,n}) with (u_{i,k}, ..., u_{i,n})
    exchange (l_{k,1}, ..., l_{k,k−1}) with (l_{i,1}, ..., l_{i,k−1})
    exchange (p_{k,1}, ..., p_{k,n}) with (p_{i,1}, ..., p_{i,n})
    for j = k + 1 to n do
        l_{j,k} := u_{j,k}/u_{k,k}
        u_{j,k} := 0
        for i = k + 1 to n do
            u_{j,i} := u_{j,i} − l_{j,k} u_{k,i}
        end for
    end for
end for
```

A more elegant variant of the algorithms stores the permutation in a vector of length n initialised with the numbers $(1, \ldots, n)$ that finally contains the permutation π associated with P, i.e., the i^{th} entry of that vector contains $\pi(i)$. See example below.

Algorithm 4 GEPP (Gaussian elimination with partial pivoting)

input: $A \in \mathbb{C}^{n\times n}$ regular, $b \in \mathbb{C}^n$.
output: $x \in \mathbb{C}^n$ with $Ax = b$.

```
find PA = LU with algorithm LUPP
solve Ly = Pb with FS
solve Ux = y with BS
```

Example: Consider

$$A = \begin{pmatrix} -2 & 2 & 0 & 0 \\ 2 & -4 & 1 & 1 \\ 0 & 4 & -2 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 3 \end{pmatrix}$$

and execute **GEPP**. In what follows considering the structure and properties of sign operations when using lower/upper triangular matrix operations will become useful.
Step 1, computation of the LU factorisation with **LUPP**.
The last vector will contain the permutation π:

$$L^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad U^{(0)} = \begin{pmatrix} -2 & 2 & 0 & 0 \\ 2 & -4 & 1 & 1 \\ 0 & 4 & -2 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad \pi^{(0)} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

Apparently, $2 = |U_{1,1}^{(0)}| \geq \max_{j\geq 1} |U_{j,1}^{(0)}|$, hence we need no permutation. One elimination step leads to

$$L^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \end{pmatrix} \quad U^{(1)} = \begin{pmatrix} -2 & 2 & 0 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 4 & -2 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \quad \pi^{(1)} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

Now, $4 = |U^{(1)}_{3,2}| \geq \max_{j\geq 2} |U^{(1)}_{j,2}|$, hence we permute rows 2 and 3:

$$(L^{(1)})' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \end{pmatrix} \quad (U^{(1)})' = \begin{pmatrix} -2 & 2 & 0 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \quad \pi^{(2)} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$$

The next elimination step yields

$$L^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 1 \end{pmatrix} \quad U^{(2)} = \begin{pmatrix} -2 & 2 & 0 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \pi^{(2)} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$$

We have that $1 = |U^{(1)}_{4,3}| \geq \max_{j\geq 3} |U^{(1)}_{j,3}|$, hence we permute rows 3 and 4:

$$(L^{(2)})' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 1 & 0 \\ -1 & -\frac{1}{2} & 0 & 1 \end{pmatrix} \quad (U^{(2)})' = \begin{pmatrix} -2 & 2 & 0 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \pi^{(3)} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 2 \end{pmatrix}$$

No further elimination step is required as $(U^{(2)})'$ already is upper triangular. Hence

$$L^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 1 & 0 \\ -1 & -\frac{1}{2} & 0 & 1 \end{pmatrix} \quad U^{(3)} = \begin{pmatrix} -2 & 2 & 0 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \pi^{(3)} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 2 \end{pmatrix}$$

One may now check that $LU = PA$ with $U = U^{(3)}$, $L = L^{(3)}$, and

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

which is the permutation matrix associated with $\pi = \pi^{(3)}$.
Step2, solving $Ly = Pb$. We have that

$$Pb = \begin{pmatrix} e_{\pi(1)} \cdot b \\ \vdots \\ e_{\pi(n)} \cdot b \end{pmatrix} = \begin{pmatrix} b_{\pi(1)} \\ \vdots \\ b_{\pi(n)} \end{pmatrix}$$

which indicates how the action of the permutation matrix P on a vector can be computed given the associated permutation π. In our example, the solution is $y = (0, 2, 2, 1)^T$.
Step3, solving $Ux = y$.
One can check that $x = (1, 1, 1, 1)^T$ which indeed is the solution to $Ax = b$.

Lecture 5

Matrix Norms, Part I

Learning Outcomes

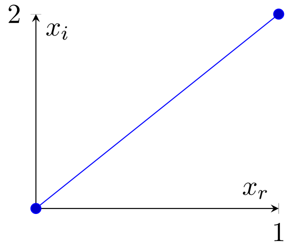
- **Understanding Vector and Matrix Norms:**
  - Gain a deep understanding of the properties and definitions of vector and matrix norms, and appreciate their crucial role in analyzing vectors and matrices in $\mathbb{C}^n$ and $\mathbb{C}^{m\times n}$.
- **Mastering Inner Products and Adjoint Operators:**
  - Develop proficiency in the concept of inner products on $\mathbb{C}^n$ and the properties of adjoint operators, applying this knowledge to effectively evaluate expressions involving vectors and matrices.
- **Exploring Unitary and Hermitian Matrices:**
  - Acquire insights into the properties of unitary and Hermitian matrices, understanding their significance in preserving orthonormality and symmetry in various mathematical computations and transformations.

5.1 Norms of complex vectors

The linear algebra course has mostly considered real valued vectors and real valued matrices. However, eigenvalues and eigenvectors are inherently complex valued. Thus, to address the general problem, complex valued vectors and matrices will be considered during this module. As you will see this will not complicate the problem much.
Considering complex numbers allows the introduction of the concept of vector norms in a conveniet way that you are familiar with.

$$x = x_r + jx_i$$
$$\bar{x} = x_r - jx_i$$

Question: How do we measure the magnitude of a complex number



The magnitude of that complex number is the distance from the number to the origin. i.e with Pythagoras theorem,

$$\begin{aligned}\underbrace{|x|}_{\text{magnitude}} &= \sqrt{x_r^2 + x_i^2} \\ &\equiv \sqrt{x\bar{x}}\end{aligned}$$

As can be noted the absolute value is a simple example of vector norms where $|.| : \mathbb{C} \to \mathbb{R}$. The absolute value has the following properties,

- $|x| > 0$ for $x \neq 0$.
- $|\alpha x| = |\alpha||x|$.
- $|x + y| \leq |x| + |y|$.

The notation of length (absolute value) can be extended to vectors.

In general, a vector norm on $\mathbb{C}^n$ is a map $\|\cdot\| : \mathbb{C}^n \to \mathbb{R}$ with

1. $\|x\| \geq 0$ for all $x \in \mathbb{C}^n$ and $\|x\| = 0$ if and only it $x = 0$,
2. $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{C}$, $x \in \mathbb{C}^n$,
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{C}^n$.

Another example on norms: is vector 2-norm (Euclidean length), $||x||_2$.
For $x \in \mathbb{C}^n$, i.e $x = (x_1, x_2, \dots, x_n)^T$

$$\begin{aligned}||x||_2 &= \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2} \\ &= \sqrt{\bar{x}_1 x_1 + \bar{x}_2 x_2 + \cdots + \bar{x}_n x_n} \\ &= \sqrt{x^* x}\end{aligned}$$

Theorem 5.2. *All norms on $\mathbb{C}^n$ are equivalent, i.e., given norms $\|\cdot\|_a$ and $\|\cdot\|_b$, there are constants $0 < c_1 \leq c_2 < \infty$ such that*

$$c_1\|x\|_a \leq \|x\|_b \leq c_2\|x\|_a \quad \forall x \in \mathbb{C}^n.$$

5.3 Inner product of complex vectors

For two real vectors, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ the dot product, $x.y : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.
Similarly, for two real vectors, $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^n$ the dot product, $\langle x, y\rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$.
As a motivational example, consider calculating the length of a complex vector, $x \in \mathbb{C}^n$ which is basically the Euclidean norm as has already been seen,

$$||x||_2^2 = |x|^2 = \bar{x}_1 x_1 + \bar{x}_2 x_2 + \cdots + \bar{x}_n x_n$$

which is a real number and is positive definite and it only vanishes if $x_1, x_2, \ldots, x_n$ vanish.
This suggests the following definition of inner product between two vectors, $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^n$,

$$\langle x, y\rangle = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n$$

This yields to the following Lemma,

Lemma 5.1. [1.6] *Given an inner product $\langle \cdot, \cdot \rangle$,*

$$|x| \mapsto \sqrt{\langle x, x\rangle}$$

is a norm on $\mathbb{C}^n$.

$$\begin{aligned}\langle x, x\rangle &= |x|^2 \\ &\equiv ||x||_2^2\end{aligned}$$

To re-emphasise, an <u>inner product</u> on $\mathbb{C}^n$ is a map $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ with

1. $\langle x, x\rangle \in \mathbb{R}^+$ for all $x \in \mathbb{C}^n$, and $\langle x, x\rangle = 0$ if and only if $x = 0$,
2. $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in \mathbb{C}^n$,
   Note: the change of the order in the inner product yields the complex conjugated number of the inner product. This axiom is clear from $\langle x, y\rangle = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n = \overline{\langle y, x\rangle}$.
3. $\langle x, \alpha y\rangle = \alpha\langle x, y\rangle$ for all $\alpha \in \mathbb{C}$ and $x, y \in \mathbb{C}^n$,
4. $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$ for all $x, y, z \in \mathbb{C}^n$.

More examples:

- p-norm: $\|x\|_p := (\sum_i |x_i|^p)^{1/p}$ for $p \in [1, \infty)$,
- maximum-norm: $\|x\|_\infty := \max_i |x_i|$,
- standard inner product: $\langle x, y\rangle := \sum_i \overline{x}_i y_i$.

An important property that holds is called the Cauchy-Schwarz, which is defined in the following Lemma.

Lemma 5.2. [1.5] *(Cauchy-Schwarz) We have that*

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle \quad \forall x, y \in \mathbb{C}^n$$

with equality if and only if $x = \alpha y$ or $y = \alpha x$ for some $\alpha \in \mathbb{C}$.

By taking the square root of both sides of the above inequality, the CauchySchwarz inequality can be written in its more familiar form,

$$|\langle x, y\rangle| \leq |x||y| \quad \forall x, y \in \mathbb{C}^n$$

Note that inner product of two vectors is a complex number, thus we take the absolute value of the complex number. Also note that $\langle x, x\rangle = |x|^2$ which is the norm of x. Similarly, $\langle y, y\rangle = |y|^2$ which is the norm of y. And this Schwarz inequality is saturated when $y = cx$, with $c \in \mathbb{C}$ i.e x is parallel to y

5.4 Adjoints

Given $A = (a_{k,l})_{k,l=1}^{m,n} \in \mathbb{C}^{m\times n}$, the <u>adjoint</u> $A^* \in \mathbb{C}^{n\times m}$ is the matrix with entries $(A^*)_{i,j} = \overline{a}_{j,i}$.

Example:

$$\begin{pmatrix} 1 & 3+j2 & 1+j0.5 \\ j7 & 4 & -2-j5 \\ j0.5 & 4+j1 & -j2 \end{pmatrix}^* = \begin{pmatrix} 1 & -j7 & -j0.5 \\ 3-j2 & 4 & 4-j1 \\ 1-j0.5 & -2+j5 & j2 \end{pmatrix}$$

A vector $x \in \mathbb{C}^n$ can be considered as an $n \times 1$ matrix allowing for the notation $x^* \in \mathbb{C}^{1\times n}$. Then

$$x^*y = \begin{pmatrix} \overline{x}_1 & \dots & \overline{x}_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \sum_{i=1}^{n} \overline{x}_i y_i = \langle x, y\rangle.$$

We also write

$$xy^* = (x_i\overline{y}_j)_{i,j=1}^{n,n} = x \otimes \overline{y} \in \mathbb{C}^{n\times n}.$$

moreover,

$$\langle Ax, y\rangle = (Ax)^*y = x^*A^*y = \langle x, A^*y\rangle$$

for all $A \in \mathbb{C}^{m\times n}$, $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$. This is actually where the definition of Adjoint came from.

Some further definitions:

1. $A \in \mathbb{C}^{n\times n}$ is <u>Hermitian</u> if $A^* = A$,
   $A \in \mathbb{R}^{n\times n}$ is <u>symmetric</u> if $A^T = A$.

$$\underbrace{\begin{pmatrix} 1 & 3+j2 & 1+j0.5 \\ 3-j2 & 4 & -2-j5 \\ 1-j0.5 & -2+j5 & 5 \end{pmatrix}}_{\text{Hemitian}}, \qquad \underbrace{\begin{pmatrix} 1 & 7 & 0.5 \\ 7 & 4 & 4 \\ 0.5 & 4 & 2 \end{pmatrix}}_{\text{Symmetric}}$$

   Note that the diagonal elements of a Hermitian matrix must be real numbers.

2. $Q \in \mathbb{C}^{m\times n}$ is <u>unitary</u> if $Q^*Q = I_n \in \mathbb{C}^{n\times n}$,
   $Q \in \mathbb{R}^{m\times n}$ is <u>orthogonal</u> if $Q^TQ = I_n \in \mathbb{R}^{n\times n}$.

   Q being unitary means that the columns are orthonormal. In the case $m = n$ we have that $Q^{-1} = Q^*$ and also $QQ^* = I$. The unitary matrix is the complex number version of orthogonality.

Example:

$$\begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}$$

Is this matrix Hermitian? Is it Unitary?
$A = A^*$, thus the matrix is Hermitian.
To verify unitarity,

$$AA^* \stackrel{?}{=} I$$
$$\begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example 2:

$$\begin{pmatrix} 0 & 1+j2 & j \\ \frac{5}{1+j2} & 0 & 0 \\ -j & 0 & 0 \end{pmatrix}$$

Is this matrix Hermitian? Is it Unitary?
Note that $\frac{5}{1+j2} = 1 - j2$, which means that $A^* = A$, thus the matrix is Hermitian. Now to check unitarity take,

$$\begin{aligned} AA^* &\stackrel{?}{=} I \\ &= \begin{pmatrix} 0 & 1+j2 & j \\ 1-j2 & 0 & 0 \\ -j & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1+j2 & j \\ 1-j2 & 0 & 0 \\ -j & 0 & 0 \end{pmatrix} \\ &\neq I \end{aligned}$$

Thus the matrix is not unitary.

3. A Hermitian matrix is <u>positive (semi-)definite</u> if

$$\langle x, Ax \rangle = x^* A x > 0 \quad (\geq 0) \qquad \forall x \in \mathbb{C}^n \backslash \{0\}.$$

Theorem 5.5. *For any unitary $Q \in \mathbb{C}^{m \times n}$*

$$\langle Qx, Qy \rangle = \langle x, y \rangle, \quad \|Qx\|_2 = \|x\|_2, \qquad \forall x, y \in \mathbb{C}^n.$$

Proof. **Exercise.** □

One can also show that if $A \in \mathbb{C}^{n \times n}$ is positive definite then

$$\langle x, y \rangle_A := \langle x, Ay \rangle \text{ is an inner product,} \tag{1.4}$$
$$\|x\|_A := \sqrt{\langle x, x \rangle_A} \text{ is a vector norm.} \tag{1.5}$$

5.6 Norms of matrices

As we now know, norm is just a measure of the length of the magnitude of the object to which it is applied to. In case of matrices, the input is a matrix $\in \mathbb{C}^{n\times m}$, while the output is a real number. The norm function has to obey the rules in the following definition.

Definition 5.1. [1.25] A matrix norm on $\mathbb{C}^{n\times n}$ is a mapping $\|\cdot\| : \mathbb{C}^{n\times n} \to \mathbb{R}$ with the properties

1. $\|A\| \geq 0$ for all $A \in \mathbb{C}^{n\times n}$, and $\|A\| = 0$ if and only if $A = 0$,
2. $\|\alpha A\| = |\alpha|\|A\|$ for all $\alpha \in \mathbb{C}, A \in \mathbb{C}^{n\times n}$,
3. $\|A + B\| \leq \|A\| + \|B\|$ for all $A, B \in \mathbb{C}^{n\times n}$,
4. $\|AB\| \leq \|A\|\|B\|$ for all $A, B \in \mathbb{C}^{n\times n}$.

The last condition makes the difference to a vector norm. It means that the norm is compatible with the matrix-matrix product.
Some norms measure the magnitude of the matrix which is a reflection of the magnitude of the underlying linear transformation. Some norms are easy to compute. Some norms are differentiable. None of the norms has all these properties. But equivalency of norms for matrix norms means that if a matrix happens to be large/small in that norm, it will be large/small in other norms.

Examples of entry-wise matrix norms: These norms consider the space $\mathbb{C}^{m\times n}$ of all $m \times n$ matrices as a vector of size mn, then apply all vector norms on $\mathbb{C}^{mn}$ as vector norms on the matrices $\mathbb{C}^{m\times n}$. Examples of vector norms on the space of matrices include,

$$\|A\|_{\max} := \max_{i,j} |a_{i,j}| \qquad \text{maximum norm,}$$

$$\|A\|_F := \left(\sum_{i,j} |a_{i,j}|^2\right)^{1/2} \qquad \text{Frobenius norm.}$$

$$= \begin{pmatrix} |a_{1,1}|^2 & + & \dots & + & |a_{1,n}|^2 & + \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ |a_{m,1}|^2 & + & \dots & + & |a_{m,n}|^2 & + \end{pmatrix}^{1/2}$$

Another way of assigning norms to a matrix is the operator norm.

Operator norms: Given norms $\|\cdot\|_{\hat{m}}$ on $\mathbb{C}^m$ and $\|\cdot\|_{\hat{n}}$ on $\mathbb{C}^n$ define

$$\|A\|_{(\hat{m},\hat{n})} := \max_{x\in\mathbb{C}^n\setminus\{0\}} \frac{\|Ax\|_{\hat{m}}}{\|x\|_{\hat{n}}} = \max_{\|x\|_{\hat{n}}=1} \|Ax\|_{\hat{m}}.$$

Definition 5.2. [1.26] *Given a vector norm $\|\cdot\|_v$ on $\mathbb{C}^n$, we define the induced (operator) norm $\|\cdot\|_m$ on $\mathbb{C}^{n\times n}$ by*

$$\|A\|_m := \max_{x\in\mathbb{C}^n\setminus\{0\}} \frac{\|Ax\|_v}{\|x\|_v} = \max_{\|x\|_v=1} \|Ax\|_v.$$

Theorem 5.7 (1.27)**.** *The induced norm $\|\cdot\|_m$ of a vector norm $\|\cdot\|_v$ is a matrix norm with $\|I_n\|_m = 1$ and*

$$\|Ax\|_v \leq \|A\|_m\|x\|_v \quad \forall A \in \mathbb{C}^{n\times n}, x \in \mathbb{C}.$$

Proof. The idea of the proof is to show that the matrix norm defined in this theorem above satisfies all the properties given in definition (5.1). Clearly $\|A\|_m \in \mathbb{R}$ and ≥ 0. We have

$$\|A\|_m = 0 \Leftrightarrow \frac{\|Ax\|_v}{\|x\|_v} = 0 \, \forall x \in \mathbb{C}^n \backslash \{0\} \Leftrightarrow \|Ax\|_v = 0 \, \forall x \in \mathbb{C}^n \Leftrightarrow A = 0$$

which shows the first point. For the second point we observe that

$$\|\alpha A\|_m = \max_{\|x\|_v=1} \|\alpha Ax\|_v = \max_{\|x\|_v=1} |\alpha| \|Ax\|_v = |\alpha| \max_{\|x\|_v=1} \|Ax\|_v = |\alpha| \|A\|_m,$$

and similarly the third property can be deduced from the corresponding property of the vector norm:

$$\begin{aligned} \|A+B\|_m = \max_{\|x\|_v=1} \|(A+B)x\|_v &\leq \max_{\|x\|_v=1} (\|Ax\|_v + \|Bx\|_v) \\ &\leq \max_{\|x\|_v=1} \|Ax\|_v + \max_{\|x\|_v=1} \|Bx\|_v = \|A\|_m + \|B\|_m. \end{aligned}$$

Clearly $\|I_n\|_m = \max_{\|x\|_v=1} \|I_n x\|_v = 1$, and for any $y \in \mathbb{C}^n \backslash \{0\}$

$$\|A\|_m = \max_{x \neq 0} \frac{\|Ax\|_v}{\|x\|_v} \geq \frac{\|Ay\|_v}{\|y\|_v} \quad \Rightarrow \quad \|Ay\|_v \leq \|A\|_m \|y\|_v.$$

Using this we can show the submultiplicativity, the fourth property of matrix norms:

$$\|AB\|_m = \max_{\|x\|_v=1} \|ABx\|_v \leq \max_{\|x\|_v=1} \|A\|_m \|Bx\|_v = \|A\|_m \max_{\|x\|_v=1} \|Bx\|_v = \|A\|_m \|B\|_m$$

□

Theorem 5.8. [1.28] *The matrix norm induced by the infinity norm is the* <u>*maximum row sum*</u>*,*

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{i,j}|, \quad A \in \mathbb{C}^{n \times n}$$

Example: compute the ∞-norm of the following matrix

$$\left\| \begin{pmatrix} -3 & 5 \\ 2 & -1 \end{pmatrix} \right\|_\infty$$

The 1-norm of the first row in that matrix is $|-3| + |5| = 8$.
The 1-norm of the second row in that matrix is $|2| + |-1| = 3$.
Thus the ∞-norm is 8.

Theorem 5.9. [1.29] *The matrix norm induced by the 1-norm is the* <u>*maximum column sum*</u>*,*

$$\|A\|_1 = \|A^*\|_\infty = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{i,j}|. \quad A \in \mathbb{C}^{n \times n}.$$

Example: compute the 1-norm of the following matrix

$$\left\| \begin{pmatrix} -3 & 5 \\ 2 & -1 \end{pmatrix} \right\|_1$$

The 1-norm of the first vector in that matrix is $|-3| + |2| = 5$.
The 1-norm of the second vector in that matrix is $|5| + |-1| = 6$.
Thus the 1-norm is 6.