(a) Frobenius norm $\|A\|_F = \sqrt{\text{trace}(A^*A)} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{i,j}|^2\right)^{1/2}$

Let $m=n$. For the Frobenius norm to be a matrix norm it must satisfy the following properties:

1. $\|A\|_F \geq 0 \ \forall A \in \mathbb{C}^{n\times n}$ and $\|A\|_F = 0$ iff $A=0$.
2. $\|\alpha A\|_F = |\alpha|\|A\|_F \ \forall \alpha \in \mathbb{C}, A \in \mathbb{C}^{n\times n}$
3. $\|A+B\|_F \leq \|A\|_F + \|B\|_F \ \forall A,B \in \mathbb{C}^{n\times n}$
4. $\|AB\|_F \leq \|A\|_F\|B\|_F \ \forall A,B \in \mathbb{C}^{n\times n}$

Property 1:

The frobenius norm takes the positive square root of a sum so $\|A\|_F \geq 0$.

Let $A=0$ then $\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2} = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} 0^2\right)^{1/2} = 0$

Let $\|A\|_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2} = 0$

$|a_{ij}|^2 \geq 0$ as the square of any number is greater than or equal to zero.

So for the sum of these values to be equal to zero $a_{ij} = 0 \ \forall i,j \in [1,n]$.

Hence $A=0$

Hence $\|A\|_F$ satisfies property 1.

Property 2:

Let $A \in \mathbb{C}^{n\times n}$, $\alpha \in \mathbb{C}$

$\|\alpha A\|_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|\alpha a_{i,j}|^2\right)^{1/2} = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|\alpha|^2|a_{i,j}|^2\right)^{1/2} = \left(|\alpha|^2\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{i,j}|^2\right)^{1/2} = |\alpha|\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{i,j}|^2\right)^{1/2} = |\alpha|\|A\|_F$

Hense $\|A\|_F$ satisfies the second property

Property 3:

Let $A,B \in \mathbb{C}^{n\times n}$

$\|A+B\|_F^2 = tr((A+B)^T(A+B)) = tr((A^T+B^T)(A+B)) = tr(A^TA + A^TB + B^TA + B^TB)$

$= tr(A^TA) + tr(A^TB) + tr(B^TA) + tr(B^TB)$

$= \|A\|_F^2 + tr(A^TB) + tr(B^TA) + \|B\|_F^2$

By Cauchy-Swartz applied to (a_{ij}) (b_{ij}) $|tr(A^TB)| \leq \sqrt{tr(A^TA)}\sqrt{tr(B^TB)} = \|A\|_F\|B\|_F$ and equivilantly

$tr(B^TA) = \sqrt{tr(B^TB)}\sqrt{tr(A^TA)} = \|B\|_F\|A\|_F = \|A\|_F\|B\|_F$

Hence $\|A+B\|_F^2 = \|A\|_F^2 + tr(A^TB) + tr(B^TA) + \|B\|_F^2 \leq \|A\|_F^2 + 2\|A\|_F\|B\|_F + \|B\|_F^2 = (\|A\|_F + \|B\|_F)^2$

So $\|A+B\|_F \leq \|A\|_F + \|B\|_F$

Hence $\|A\|_F$ satisfies property 3

Property 4:

$$\|AB\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij} b_{ij}|^2\right)^{1/2} \leq \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2 |b_{ij}|^2\right)^{1/2} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2 \sum_{i=1}^{m}\sum_{j=1}^{n} |b_{ij}|^2\right)^{1/2} = \|A\|_F \|B\|_F$$

ii Let U and V be unitary matrices

so $U^*U = I \quad V^*V = I$

For any unitary $U \in \mathbb{C}^{n \times n}$ $\quad \|Ux\|_2 = \|x\|_2$

$$\|UA\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |u_{ij} a_{ij}|^2\right)^{1/2} = \left(\sum_{j=1}^{n} |Ua_j|^2\right)^{1/2} = \left(\sum_{j=1}^{n} \|Ua_j\|_2^2\right)^{1/2} = \left(\sum_{j=1}^{n} \|a_j\|_2^2\right)^{1/2} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2} = \|A\|_F$$

$$\|AV\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij} v_{ij}|^2\right)^{1/2} = \left(\sum_{i=1}^{m} |a_i V|^2\right)^{1/2} = \left(\sum_{i=1}^{m} \|a_i V\|_2^2\right)^{1/2} = \left(\sum_{i=1}^{m} \|a_i\|_2^2\right)^{1/2} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2} = \|A\|_F$$

iii $\|A\|_F = \|U\Sigma V^*\|_F = \|\Sigma V^*\|_F = \|\Sigma\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |\Sigma_{ij}|^2\right)^{1/2}$

Since Σ is diagonal the only non-zero values are $\sigma_1, \dots, \sigma_r$

So $\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |\Sigma_{ij}|^2\right)^{1/2} = \sqrt{\sigma_1^2 + \dots + \sigma_r^2}$

b. $\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$

$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2}$

$$(\|x\|_1)^2 = (|x_1| + |x_2| + \dots + |x_n|)^2 = |x_1|^2 + |x_1||x_2| + \dots + |x_1||x_n| + |x_1||x_2| + |x_2|^2 + \dots + |x_n|^2$$

$$\geq |x_1|^2 + |x_2|^2 + \dots + |x_n|^2$$

So $\quad \|x\|_1 \geq \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2} = \|x\|_2$

c. $A = \begin{pmatrix} 4 & -2 & 0 \\ -3 & 6 & -2 \\ 0 & -3 & 5 \end{pmatrix}$

$\|A\|_\infty = \max_{ij} |a_{ij}| = 11$

$\|A\|_1 = \sum_{ij} |a_{ij}| = 11$

2. The economic model is $x = Ax + D$

$x = $ (electricity, water, Agriculture, manufacturing)

$$A = \begin{pmatrix} 0.2 & 0.25 & 0.1 & 0.1 \\ 0.15 & 0.1 & 0.005 & 0.005 \\ 0.3 & 0.6 & 0.1 & 0.3 \\ 0.6 & 0.5 & 0.1 & 0.1 \end{pmatrix} = \begin{pmatrix} 1/5 & 1/4 & 1/10 & 1/10 \\ 3/20 & 1/10 & 1/200 & 1/200 \\ 3/10 & 3/5 & 1/10 & 3/10 \\ 3/5 & 1/2 & 1/10 & 1/10 \end{pmatrix}$$

$$D = \begin{pmatrix} 100 \\ 150 \\ 120 \\ 70 \end{pmatrix}$$

a. $x = Ax + D \rightarrow (I - A)x = D$

$$\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1/5 & 1/4 & 1/10 & 1/10 \\ 3/20 & 1/10 & 1/200 & 1/200 \\ 3/10 & 3/5 & 1/10 & 3/10 \\ 3/5 & 1/2 & 1/10 & 1/10 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 100 \\ 150 \\ 120 \\ 70 \end{pmatrix}$$

$$\begin{pmatrix} 4/5 & -1/4 & -1/10 & -1/10 \\ -3/20 & 9/10 & -1/200 & -1/200 \\ -3/10 & -3/5 & 9/10 & -3/10 \\ -3/5 & -1/2 & -1/10 & 9/10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 100 \\ 150 \\ 120 \\ 70 \end{pmatrix}$$

$$\begin{pmatrix} 4/5 & -1/4 & -1/10 & -1/10 & 100 \\ -3/20 & 9/10 & -1/200 & -1/200 & 150 \\ -3/10 & -3/5 & 9/10 & -3/10 & 120 \\ -3/5 & -1/2 & -1/10 & 9/10 & 70 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{pmatrix} -3/5 & -1/2 & -1/10 & 9/10 & 70 \\ -3/20 & 9/10 & -1/200 & -1/200 & 150 \\ -3/10 & -3/5 & 9/10 & -3/10 & 120 \\ 4/5 & -1/4 & -1/10 & -1/10 & 100 \end{pmatrix} \xrightarrow{R_2 - \frac{1}{4}R_1} \begin{pmatrix} -3/5 & -1/2 & -1/10 & 9/10 & 70 \\ 0 & 41/40 & \frac{1}{50} & -\frac{23}{100} & \frac{530}{4} \\ -3/10 & -3/5 & 9/10 & -3/10 & 120 \\ 4/5 & -1/4 & -1/10 & -1/10 & 100 \end{pmatrix}$$

$$\xrightarrow{R_3 - \frac{1}{2}R_1} \begin{pmatrix} -3/5 & -1/2 & -1/10 & 9/10 & 70 \\ 0 & 41/40 & \frac{1}{50} & -\frac{23}{100} & \frac{530}{4} \\ 0 & -\frac{7}{20} & 19/20 & -3/4 & 85 \\ 4/5 & -1/4 & -1/10 & -1/10 & 100 \end{pmatrix} \xrightarrow{R_4 + \frac{4}{3}R_1} \begin{pmatrix} -3/5 & -1/2 & -1/10 & 9/10 & 70 \\ 0 & 41/40 & \frac{1}{50} & -\frac{23}{100} & \frac{530}{4} \\ 0 & -\frac{7}{20} & 19/20 & -3/4 & 85 \\ 0 & -\frac{11}{12} & -\frac{7}{30} & \frac{11}{10} & \frac{580}{3} \end{pmatrix}$$

$$\xrightarrow{R_3 + \frac{14}{41}R_2} \begin{pmatrix} -3/5 & -1/2 & -1/10 & 9/10 & 70 \\ 0 & 41/40 & \frac{1}{50} & -\frac{23}{100} & \frac{530}{4} \\ 0 & 0 & \frac{3923}{4100} & -\frac{3397}{4100} & \frac{5340}{41} \\ 0 & -\frac{11}{12} & -\frac{7}{30} & \frac{11}{10} & \frac{580}{3} \end{pmatrix}$$

$$\xrightarrow{R_4 + \frac{110}{123}R_2} \begin{pmatrix} -3/5 & -1/2 & -1/10 & 9/10 & 70 \\ 0 & 41/40 & \frac{1}{50} & -\frac{23}{100} & \frac{530}{4} \\ 0 & 0 & \frac{3923}{4100} & -\frac{3397}{4100} & \frac{5340}{41} \\ 0 & 0 & -\frac{53}{246} & \frac{110}{123} & \frac{12785}{41} \end{pmatrix}$$

$$\xrightarrow{R_4 + \frac{2650}{11769}R_3} \begin{pmatrix} -3/5 & -1/2 & -1/10 & 9/10 & 70 \\ 0 & 41/40 & \frac{1}{50} & -\frac{23}{100} & \frac{530}{4} \\ 0 & 0 & \frac{3923}{4100} & -\frac{3397}{4100} & \frac{5340}{41} \\ 0 & 0 & 0 & \frac{5553}{7846} & 341.1560031 \end{pmatrix}$$

So $\frac{5553}{7846} x_4 = 341.1560031$

$$x_4 = 482.0295336$$

$$\frac{3923}{4100} x_3 - \frac{3397}{4100} x_4 = \frac{5340}{41}$$

$$x_3 = 553.5188187$$

$$\frac{41}{40} x_2 + \frac{1}{50} x_3 - \frac{23}{100} x_4 = \frac{530}{4}$$

$$x_2 = 226.6306501$$

$-\frac{3}{5}x_1 - \frac{1}{2}x_2 - \frac{1}{10}x_3 + \frac{9}{10}x_4 = 70$

$x_1 = 325.2656223$

b. Let $M \in \mathbb{C}^{n \times n}$ st $M = LU$ and n is an integer power of 2.

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}$$ is the block decomposition of $M = LU$.

This gives the 4 equations:

$M_{11} = L_{11}U_{11}$ $\quad M_{12} = L_{11}U_{12}$ $\quad M_{21} = L_{21}U_{11}$ $\quad M_{22} = L_{21}U_{12} + L_{22}U_{22}$

The first equation is the LU factorisation of M_{11} as L_{11} is lower triangular and U_{11} is upper triangular. So we can recursivly call the function to solve $M_{11} = L_{11}U_{11}$. We will keep recursivly calling the function until the Matrices are 2x2. Suppose the Matrix we are finding the LU factorisation for is $A \in \mathbb{C}^{2 \times 2}$. We can then set $U_A = A$ and $L_A = I_2$. Then use the equations:

$(L_A)_{21} = (U_A)_{21}/(U_A)_{11}$, $(U_A)_{21} = 0$ & $(U_A)_{22} = (U_A)_{22} - (L_A)_{21}(U)_{12}$ to find L_A and U_A.

We can then calculate the inverse of L_{11} and U_{11} to find U_{12} and L_{21} from the second and third equation. The fourth equation then becomes $M_{22} - L_{21}U_{12} = L_{22}U_{22}$. Similar to the first equation this is the LU factorisation of $(M_{22} - L_{21}U_{12})$ so we can use the same equations to find L_{22} and U_{22}.

c. $A = \begin{pmatrix} 0.2 & 0.25 & 0.1 & 0.1 \\ 0.15 & 0.1 & 0.005 & 0.005 \\ 0.3 & 0.6 & 0.1 & 0.3 \\ 0.6 & 0.5 & 0.1 & 0.1 \end{pmatrix} = M \in \mathbb{C}^{4\times4}$ $\quad n=4>2$

$M_{11} = \begin{pmatrix} 0.2 & 0.25 \\ 0.15 & 0.1 \end{pmatrix}$ $\quad M_{12} = \begin{pmatrix} 0.1 & 0.1 \\ 0.005 & 0.005 \end{pmatrix}$ $\quad \underbrace{\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}}_{U}$

$M_{21} = \begin{pmatrix} 0.3 & 0.6 \\ 0.6 & 0.5 \end{pmatrix}$ $\quad M_{22} = \begin{pmatrix} 0.1 & 0.3 \\ 0.1 & 0.1 \end{pmatrix}$

With $M_{11} = L_{11}U_{11}$ $\quad M_{12} = L_{11}U_{12}$ $\quad M_{21} = L_{21}U_{11}$ $\quad M_{22} = L_{21}U_{12} + L_{22}U_{22}$

$M_{11} = L_{11}U_{11}$ is an LU Factorisation

Solve $M_{11} = L_{11}U_{11}$

Let $U_{11} = M_{11}$ $\quad L_{11} = I$

$M_{11} = \begin{pmatrix} 0.2 & 0.25 \\ 0.15 & 0.1 \end{pmatrix} \in \mathbb{C}^{2\times2}$

$n=2$ So

$u_{2,2} = u_{2,2} - l_{2,1}u_{1,2}$

$(M_{11})_{11} = 0.2$ $\quad (M_{11})_{12} = 0.25$ $\quad (M_{11})_{21} = 0.15$ $\quad (M_{11})_{22} = 0.1$

$(L_{11})_{21} = (U_{11})_{21} / (U_{11})_{11} = 0.15/0.2 = 0.75$

$(U_{11})_{21} = 0$

$(U_{11})_{22} = (U_{11})_{22} - (L_{11})_{21} \times (U_{11})_{12} = 0.1 - 0.75 \times 0.25 = -0.0875$

so $M_{11} = \begin{pmatrix} 0.2 & 0.25 \\ 0.15 & 0.1 \end{pmatrix} = L_{11}U_{11} = \begin{pmatrix} 1 & 0 \\ 0.75 & 1 \end{pmatrix} \begin{pmatrix} 0.2 & 0.25 \\ 0 & -0.0875 \end{pmatrix}$

Can now use L_{11} and U_{11} to find U_{12} and L_{21}

$M_{12} = L_{11}U_{12} \Rightarrow U_{12} = L_{11}^{-1}M_{12}$

$L_{11}^{-1} = \begin{pmatrix} 1 & 0 \\ -0.75 & 1 \end{pmatrix}$ $\quad$ So $U_{12} = \begin{pmatrix} 1 & 0 \\ -0.75 & 1 \end{pmatrix} \begin{pmatrix} 0.1 & 0.1 \\ 0.005 & 0.005 \end{pmatrix} = \begin{pmatrix} 0.1 & 0.1 \\ -0.07 & -0.07 \end{pmatrix}$

$M_{21} = L_{21}U_{11} \Rightarrow L_{21} = M_{21}U_{11}^{-1}$

$U_{11}^{-1} = \begin{pmatrix} -0.0875 & -0.25 \\ 0 & 0.2 \end{pmatrix}$ $\quad$ So $L_{21} = \begin{pmatrix} 0.3 & 0.6 \\ 0.6 & 0.5 \end{pmatrix} \begin{pmatrix} -0.0875 & -0.25 \\ 0 & 0.2 \end{pmatrix} = \begin{pmatrix} -0.02625 & 0.045 \\ -0.0525 & -0.05 \end{pmatrix}$

$$M_{22} = L_{21}U_{12} + L_{22}U_{22} \implies M_{22} - L_{21}U_{12} = L_{22}U_{22}$$

$$L_{22}U_{22} = M_{22} - L_{21}U_{12} = \begin{pmatrix} 0.1 & 0.3 \\ 0.1 & 0.1 \end{pmatrix} - \begin{pmatrix} -0.02625 & 0.045 \\ -0.0525 & -0.05 \end{pmatrix}\begin{pmatrix} 0.025 & 0.1 \\ 0.00125 & 0.005 \end{pmatrix}$$

$$= \begin{pmatrix} 0.1 & 0.3 \\ 0.1 & 0.1 \end{pmatrix} - \begin{pmatrix} -0.0006 & -0.0024 \\ -0.001375 & -0.0055 \end{pmatrix}$$

$$= \begin{pmatrix} 0.1006 & 0.3024 \\ 0.101375 & 0.1055 \end{pmatrix}$$

Solve $M_{22} - L_{21}U_{12} = L_{22}U_{22}$

Let $U_{22} = M_{22} - L_{21}U_{12}$ $\quad L_{22} = I$

$$M_{22} - L_{21}U_{12} = \begin{pmatrix} 0.1006 & 0.3024 \\ 0.101375 & 0.1055 \end{pmatrix} \in \mathbb{C}^{2\times 2} \quad n=2$$

$$(L_{22})_{21} = (U_{22})_{21} / (U_{22})_{11} = 0.101375 / 0.1006 = 1.007703777$$

$$(U_{22})_{21} = 0$$

$$(U_{22})_{22} = (U_{22})_{22} - (L_{22})_{21} \times (U_{22})_{12} = 0.1055 - 1.007703777 \times 0.3024 = -0.1992296223$$

$$L_{22} = \begin{pmatrix} 1 & 0 \\ 1.007703777 & 1 \end{pmatrix}$$

$$U_{22} = \begin{pmatrix} 0.1006 & 0.3024 \\ 0 & -0.1992296223 \end{pmatrix}$$

$$\text{So } L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.75 & 1 & 0 & 0 \\ -0.02625 & 0.045 & 1 & 0 \\ -0.0525 & -0.05 & 1.007703777 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 0.2 & 0.25 & 0.1 & 0.1 \\ 0 & -0.0875 & -0.07 & -0.07 \\ 0 & 0 & 0.1006 & 0.3024 \\ 0 & 0 & 0 & -0.1992296223 \end{pmatrix}$$