Week 5 Tutorial 5

(5.1) Suppose a company manufactures two products, A and B, using three inputs, labor, material R, and materials S. To make one unit of product A requires 6 pounds of R, 7.5 pounds of S, and 9 person-hours of labor; to make one unit of product B requires 12 pounds of R, 4.5 pounds of S, and 6 person-hours of labor. The demands for the products are such that the company can sell as much of each product as it can produce and earn a profit of 3 per unit of A and 4 per unit of B. However, only 900 pounds of R, 675 pounds of S, and 1200 person-hours of labor are available to the company each day.

1. Formulate the company's problem as a linear program to maximize profit.
2. Graph the feasible region for this problem.
3. Solve the problem graphically by finding the best extreme point.

Solution Let x_i, $i = 1, 2$ denote the units of the two products. The company wants to maximise the profit per day, which is

$$\max\ 3x_1 + 4x_2. \tag{0.1}$$

The constraints are

$$\begin{aligned} 9x_1 + 6x_2 &\leq 1200 && \text{constraint on the number of hours per day} \\ 6x_1 + 12x_2 &\leq 900 && \text{constraint on material R} \\ 7.5x_1 + 4.5x_2 &\leq 675 && \text{constraint on material A.} \end{aligned}$$

Furthermore we have to assume that $x_1 \geq 0$ and $x_2 > 0$. We see that the solution is

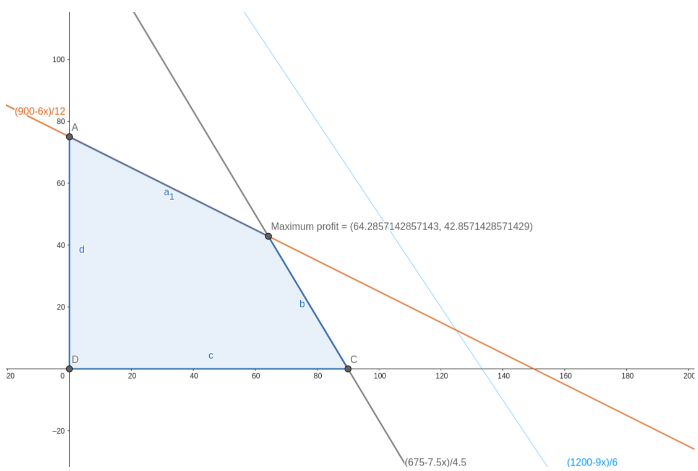

Figure 1: Graphical solution to Problem 1

$(64.286, 62.857)$.

(5.2) Consider the following linear program

$$\max_{(x_1,x_2)} \; x_1 + x_2 \tag{0.2a}$$
$$\text{s.t.} -3x_1 + 2x_2 \leq -1 \tag{0.2b}$$
$$x_1 - x_2 \leq 2 \tag{0.2c}$$
$$x_1, x_2 \geq 0 \tag{0.2d}$$

Show that

1. Show that the solution to it is unbounded.
2. State the dual problem and show that it is infeasible.
3. Explain how this relates to the duality results discussed in class.

Solution: The feasible set is unbounded, see Figure 2. Hence there is an optimum solution but it takes the value $+\infty$. The dual problem is

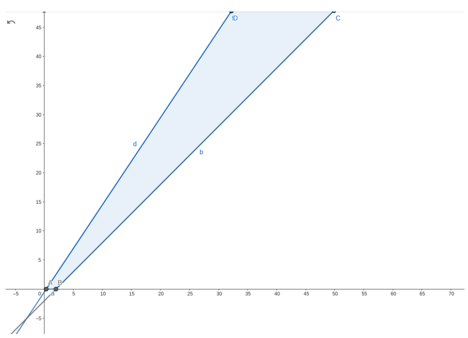

Figure 2: Unbouded feasible set of problem 5.2

$$
\begin{aligned}
\min_{(y_1,y_2)} \; & -y_1 + 2y_2 \\
\text{s.t.} \; & -3y_1 + y_2 \geq 1 \\
& 2y_1 - y_2 \geq 1 \\
& y_1, y_2 \geq 0.
\end{aligned}
$$

Adding the two inequality constraints we have that $y_1 \leq -2$, which contradicts the positivity constraint.
So we observe that if the primal problem is feasible, but has an unbounded solution then the dual problem is infeasible.

(5.3) Calculate the solution of the following equality constrained minimisation problem

$$\min 2x_1^2 + x_2^2 \text{ s. t. } x_1 + x_2 = 1.$$

Given a graphic interpretation of the solution, that is sketch the contour linear of the objective and the constraint.

Solution: The Lagrangian is given by

$$\mathcal{L} = 2x_1^2 + x_2^2 + \lambda(x_1 + x_2 - 1)$$

The gradient of $\mathcal{L}$ wrt x_1 and x_2 is given by

$$\frac{\partial \mathcal{L}}{\partial x_1} = 4x_1 + \lambda$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 2x_2 + \lambda$$

Setting the gradient of the Lagrangian to zero, gives $\lambda = -\frac{4}{3}$. Hence the optimal solution is $x_1 = \frac{1}{3}$ and $x_2 = \frac{2}{3}$.

(5.4) Calculate the solution to the following constrained optimisation problem

$$\max_{(x_1,x_2)} -(x_1-2)^2 - 2(x_2-1)^2$$
$$\text{s.t. } x_1 + 4x_2 \leq 3$$
$$x_1 \geq x_2$$

Solution: The Lagrangian is given by

$$\mathcal{L} = -(x_1-2)^2 - 2(x_2-1)^2 + \lambda_1(x_1 + 4x_2 - 3) + \lambda_2(x_2 - x_1).$$

The gradient of $\mathcal{L}$ wrt x_1 and x_2 is given by

$$\frac{\partial \mathcal{L}}{\partial x_1} = 2(x_1 - 2) + \lambda_1 - \lambda_2$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 4(x_2 - 1) + 4\lambda_1 + \lambda_2$$

We have that $\nabla_{x_i}\mathcal{L} = 0$ and that $\lambda_1(x_1 + 4x_2 - 3) = 0$ and $\lambda_2(x_2 - x_1) = 0$. To check the complemtary conditions we have to consider the following four cases

- $\lambda_1 = \lambda_2 = 0$ then $x_1 = 2$ and $x_2 = 1$.
- $\lambda_1 = 0$ and $x_2 = x_1$, then $x_1 = -\frac{4}{3}$ and $\lambda_2 = \frac{4}{3}$.
- $x_1 + 4x_2 - 3 = 0$, $\lambda_2 = 0$, then $x_1 = \frac{5}{3}, x_2 = \frac{1}{3}$, and $\lambda_1 = \frac{2}{3}$.
- $x_1 + 4x_2 - 3 = 0$ and $x_1 = x_2$, then $\lambda_1 = \frac{22}{25}$, $\lambda = -\frac{48}{25}$, $x_1 = \frac{3}{5}$ and $x_2 = \frac{3}{5}$.