Additional Material (not covered in seminars)

(10.1) Let $f : [-\pi, \pi] \to \mathbb{R}$ be a 2π periodic function of the form

$$f(x) = \pi^2 - x^2 \text{ for } -\pi \le x \le \pi.$$

Show that its Fourier series representation is given by

$$f(x) = \frac{2\pi^2}{3} + 4(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x \ldots).$$

Recall that the Fourier series is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$$

with $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$, $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,\mathrm{d}x$ and $b_k = \int_{-\pi}^{\pi} f(x)\sin kx\,\mathrm{d}x$.

(10.2) *Fourier series of even and odd functions:* The Fourier series contains a sum of terms while the integral formulae for the Fourier coefficients a_k and b_k contain products of the type $f(x)\cos nx$ and $f(x)\sin nx$. Let $q(x) = g(x)h(x)$, show that

g		h		q
even	$\times$	even	$=$	even
even	$\times$	odd	$=$	odd
off	$\times$	odd	$=$	even

Let $a \in \mathbb{R}$. Use that $\int_{-a}^{a} q(x)\,\mathrm{d}x = 0$ for odd functions and $\int_{-a}^{a} q(x)dx = 2\int_0^a q(x)\,\mathrm{d}x$ for even functions to show that

- $a_k = 0$ if f is odd
- $b_k = 0$ if f is even.

(10.3) Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}$. Show that $\boldsymbol{A}$ has rank 2 and that its singular value decomposition is given by

$$\boldsymbol{U} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}, \boldsymbol{D} = \begin{pmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \text{ and } \boldsymbol{V} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Compute its rank 1 approximation.