MA398 MATRIX ANALYSIS AND ALGORITHMS: ASSIGNMENT 3

Please submit your solutions to this assignment via Moodle by **noon on Thursday, November 17th**. Make sure that your submission is clearly marked with your name, university number, course and year of study. Note that some of the questions are practical questions that are concerned with real-world problems and require coding in Python.

- The written part of the solutions may preferably be typed in LaTeX, otherwise written on paper and subsequently scanned/photographed provided the images are clearly legible. You are required to deliver a single document entitled *MA398_Assignment1_FirstnameLastname.pdf*, outlining your solutions and explaining your interpretation and arguments to the questions .
- The Python code scripts relevant to each question should be submitted as *MA398_ Assignment1_ ExerciseN.ipynb* where you need to make sure that you document the environment in which I should be able to run your code.

To avoid losing marks unnecessarily, please make sure that,

- you show all intermediate steps of your calculations.
- you include all the coding that you have to do in order to obtain your answers and according to the instruction given above.
- include any plots you generate and label them appropriately.
- when you provide an answer to a question make sure that you justify your answer and provide details of any mathematical calculations that are required.

⚠Only in an emergency or if the Moodle submission is unavailable because of a general outage, the assignment should in the respective case be submitted by email to Randa.Herzallah@warwick.ac.uk, and olayinka.ajayi@warwick.ac.uk.

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1. (Symmetric and positive definite matrices). Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric positive definite matrix and $B \in \mathbb{R}^{n\times n}$ be a real matrix such that $A - B - B^T$ is real symmetric positive definite. Show that the spectral radius of $C = -(A-B)^{-1}B$ is strictly less than one.

   Hint: Consider the eigenvalue problem of the matrix C, and use the property that a positive definite matrix is always nonsingular.

   Answer: Let λ be an eigenvalue of C, such that $Cx = \lambda x$ holds, where $x \neq 0$ is an eigenvector.

   Remark: Both λ and x may have nonzero imaginary parts when C is not symmetric, e.g. in the Gauss-Seidel method.

   Using the definition of C we get that,

$$-(A-B)^{-1}Bx = \lambda x \tag{1}$$
$$-Bx = \lambda(A-B)x \tag{2}$$

   where the value of $\lambda \neq 1$ because A is nonsingular. Thus (1) can be rewritten as,

$$-Bx = \lambda(A-B)x \tag{3}$$
$$-\bar{x}Bx = \bar{x}\lambda(A-B)x \tag{4}$$
$$-\bar{x}Bx + \lambda\bar{x}Bx = \bar{x}\lambda Ax \tag{5}$$
$$\bar{x}Bx = \frac{\lambda}{\lambda-1}\bar{x}Ax \tag{6}$$

where the bar means complex conjugation. In addition writing $x = y + jz$ where y and z represent the real and imaginary parts of x respectively, the following identity can be obtained,

$$\bar{x}Ax = y^T Ay + z^T Az$$

So positive definiteness implies $\bar{x}Ax > 0$ and $\bar{x}(A - B - B^T)x > 0$. From $\bar{x}(A - B - B^t)x > 0$ and (3) we obtain,

$$\begin{aligned}\bar{x}^T Ax - \bar{x}^T Bx - \bar{x}^T B^T x &= \left(1 - \frac{\lambda}{\lambda - 1} - \frac{\bar{\lambda}}{\bar{\lambda} - 1}\right)\bar{x}^T Ax \\ &= \frac{1 - |\lambda|^2}{|\lambda - 1|^2}\bar{x}^T Ax\end{aligned}$$

Since $\lambda \neq 1$ we have $|\lambda - 1|^2 > 0$. Hence recalling that $\bar{x}^T Ax > 0$, we conclude that $1 - |\lambda|^2 > 0$. Thus, $|\lambda| < 1$ occurs for every eigenvalue of C as required.

2. (Jacobi and SOR methods). Given a matrix $A \in \mathbb{R}^{n\times n}$, let the matrices L, D and U be such that $A = L + D + U$, with L strictly lower triangular, D diagonal and U strictly upper triangular. Then one step of the Jacobi method is given by

$$x_k = D^{-1}(b - (L + U)x_{k-1}), \tag{7}$$

and one step of the successive-over-relaxation (SOR) method with relaxation parameter $\omega \in \mathbb{R}$ is given by

$$x_k = (\omega L + D)^{-1}(\omega b - (\omega U + (\omega - 1)D)x_{k-1}). \tag{8}$$

The special case $\omega = 1$ corresponds to the Gauss-Seidel method.

(a) Using the result in question 1 show that if A is symmetric positive definite, then the Gauss-Seidel method converges.

Answer: For Gauss-Seidel method, B is the superdiagonal part of symmetric A, hence $A - B - B^T = D$, where D is the diagonal part of the matrix A, and if A is positive definite, then D is positive definite too.

(b) Using the result in question 1 show that if both A and $2D - A$ are symmetric positive definite, then the Jacobi method converges.

Answer: For the Jacobi method, we have $B = A - D$, and if A is symmetric, then $A - B - B^T = 2D - A$. (The latter matrix is the same as A except that the signs of the off-diagonal elements are reversed

(c) Derive an explicit formula for the iteration step given in equation (8). Your answer must be of the form

$$x_i^{(k+1)} = \textit{[your formula]} \qquad \text{for } i = 1, \ldots, n$$

with only elementary arithmetic operations $(+, -, \times, \div, \sqrt{\cdot}$, etc.) as well as the entries of $A, b, x^{(\ldots)}$ and ω appearing on the right hand side.

Answer:

$$x_i^{(k+1)} = (1-\omega)x_i^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j^{(k+1)} - \sum_{j>i} a_{ij}x_j^{(k)}\right), \quad i = 1, 2, \ldots, n.$$

(d) Consider the application of Jacobi and SOR methods to solving $Ax = b$ where A is the nonsymmetric 2×2 matrix,

$$A = \begin{pmatrix} 20 & -5 \\ 5 & 1 \end{pmatrix}$$

and where

$$b = \begin{pmatrix} 36 \\ 12 \end{pmatrix}$$

i. Find the spectral radius of the Jacobi iteration matrix.
Answer: The spectral radius of the Jacobi iteration method is

$$\|D^{-1}(L+U)\|_2 = 0.5$$

ii. Find the spectral radius of the SOR iteration matrix with $\omega = 1.1$ and compare that to the one found part i of this question. Comment on your results.
Answer: The spectral radius of the SOR iteration method is

$$\|(D+\omega L)^{-1}(\omega U + (\omega - 1)D)\|_2 = 0.221$$

The spectral radius is reduced by a factor of 2.2625

iii. Compute the residual error, $\|b - Ax^{(k)}\|$ of the first four iterates of the SOR method. Comment on your result.
Answer: The spectral radius of the SOR iteration method is

$$x^{(1)} = \begin{pmatrix} 1.98 \\ 2.211 \end{pmatrix}, \quad \|e^{(1)}\|_2 = 1.1498$$

$$x^{(2)} = \begin{pmatrix} 2.0252 \\ 1.965 \end{pmatrix}, \quad \|e^{(2)}\|_2 = 0.5811$$

$$x^{(3)} = \begin{pmatrix} 1.9936 \\ 2.007 \end{pmatrix}, \quad \|e^{(2)}\|_2 = 0.1422$$

$$x^{(4)} = \begin{pmatrix} 2.0014 \\ 1.9985 \end{pmatrix}, \quad \|e^{(2)}\|_2 = 0.0312$$

Convergence is almost achieved at the third iteration.

(e) Implement functions `jacobi(A, b, n)` and `sor(A, b, n, w)` in Python which compute the iterates for the Jacobi and SOR methods up to the $\mathtt{n}^{\text{th}}$ iteration and return a vector containing the ∞-norm of the residual at each iteration. Note that this need not be the only output and in general you should feel comfortable modifying and improving the base versions of the algorithms according to your particular solution strategy. You may want to check these on simple (compact) examples such as those discussed during the lectures.

3. (Steepest descent and weather prediction) Consider the weather data we analysed in Assignment 2. Assuming a second order polynomial write a Python script that computes 90 steps of the steepest descent (SD) to approximately predict the humidity based on pressure measurements and compare it to the previously studied linear methods (Jacobi and SOR with your choice of relaxation parameter ω). The data is provide in the file "WeatherData.csv". Use all data from day 1 to day 90 to identify the coefficients of your polynomial model and use everything beyond day 90 as test data. What are the best estimates for the last five days calculated by your estimated 2nd order polynomial. Briefly comment on your results.