MA398 MATRIX ANALYSIS AND ALGORITHMS: ASSIGNMENT 3

Please submit your solutions to this assignment via Moodle by **noon on Monday, November 20$^{\text{th}}$**. Make sure that your submission is clearly marked with your name, university number, course and year of study. Note that some of the questions are practical questions that are concerned with real-world problems and require coding in Python.

- The written part of the solutions may preferably be typed in LaTeX, otherwise written on paper and subsequently scanned/photographed provided the images are clearly legible. You are required to deliver a single document entitled *MA398_Assignment3_FirstnameLastname.pdf*, outlining your solutions and explaining your interpretation and arguments to the questions .

- The Python code scripts relevant to each question should be submitted as *MA398_ Assignment3_ ExerciseN.ipynb* where you need to make sure that you document the environment in which I should be able to run your code.

To avoid losing marks unnecessarily, please make sure that,

- you show all intermediate steps of your calculations.

- you include all the coding that you have to do in order to obtain your answers and according to the instruction given above.

- include any plots you generate and label them appropriately.

- when you provide an answer to a question make sure that you justify your answer and provide details of any mathematical calculations that are required.

⚠️Only in an emergency or if the Moodle submission is unavailable because of a general outage, the assignment should in the respective case be submitted by email to Randa.Herzallah@warwick.ac.uk, and olayinka.ajayi@warwick.ac.uk.

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1. (Singular value decompositions).

   (a) Consider a non zero matrix $A \in \mathbb{R}^{m \times n}$ with rank r. The singular value decomposition of the matrix A is given by $A = U\Sigma V^T$ where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are orthogonal matrices and $\Sigma \in \mathbb{R}^{m \times n}$ is a diagonal matrix with entries $\sigma_1 > \sigma_2 > \cdots > \sigma_r$. Denoting by $u_i, i = 1, \ldots, m$ the column vectors of U and by $v_i, i = 1, \ldots, n$ the column vectors of v, show that,

$$Av_i = \left\{ \begin{array}{ll} \sigma_i u_i, & \text{for } i = 1, 2, \ldots, r \\ 0, & \text{for } i = r+1, r+2, \ldots, n \end{array} \right\}$$

$$A^T u_i = \left\{ \begin{array}{ll} \sigma_i v_i, & \text{for } i = 1, 2, \ldots, r \\ 0, & \text{for } i = r+1, r+2, \ldots, m \end{array} \right\}$$

   (b) Let $A \in \mathbb{R}^{m \times n}$, $m \geq n$, $\text{rank}(A) = n$ with singular value decomposition $A = U\Sigma V^T$.

   i. Determine the singular value decomposition of the matrix $(A^T A)^{-1}$.

   ii. Hence evaluate its spectral (the induced Euclidean matrix) norm, $\|(A^T A)^{-1}\|_2$.

2. (Polynomial (Over-)Fitting). Polynomial models can be used to provide predictions for a large number of real world problems. Here the main objective is to identify the coefficients $a_0, a_1, \ldots, a_d$ of a polynomial function of the form,

$$y = f(x, a) = \sum_{i=0}^{d} a_i x^i + \epsilon$$

where ϵ is a random component that represents the residual error, and d is the degree of the polynomial. Given n data points we can write a linear system of equations as follows,

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_{b} = \underbrace{\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^d \\ 1 & x_2 & x_2^2 & \cdots & x_2^d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^d \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_d \end{pmatrix}}_{a}$$

The matrix V is called a Vandermonde matrix. In first year linear algebra, you probably proved the qualitative result that V is invertible if and only if $n = d + 1$ and all the x_i are distinct.

(a) Numerically analyse the invertibility of the Vandermonde matrix through the following steps,
- Write a Python function Vandermonde(x,d) which assembles the above matrix, where `d = size(x)-1`, resulting in a square matrix.
- Generate equidistant points in $[-1, 1]$, and numerically calculate the condition number $\kappa(V)$ of V for several representative values $n \le 100$ in some matrix norm.

(b) Assuming $d = 3$, and $n = 5$, formulate the least square solution for the identification of the polynomial coefficients.

(c) This part of the question is concerned with predicting weather conditions using polynomial models. In particular given daily measurements of pressure and humidity provided in the file "WeatherData.csv" the objective is to identify a polynomial model that can predict the humidity based on pressure measurements. Use all data from day 1 to day 90 to identify the coefficients of your polynomial model and use everything beyond day 90 as test data.

  i. Plot the humidity measurements against their corresponding pressure measurements. (Think of axes labels, scale of the axes, fontsize, plotmarkers, etc.)

  ii. Starting from $d = 0$ to $d = 5$ identify the coefficients of the 6 polynomial models and state their values.

  iii. Compute the relative forward errors of the polynomial models you identified in part ii and then and plot them against their corresponding polynomial order.

  iv. Compute the relative forward errors of the polynomial models you identified in part ii on the test data and then and plot them against their corresponding polynomial order.

  v. What is the optimal order of polynomial that fits your weather data. Justify your answer.

  vi. What are the best estimates for the last five days calculated by your best polynomial model?