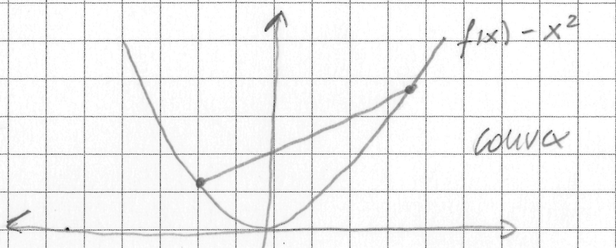
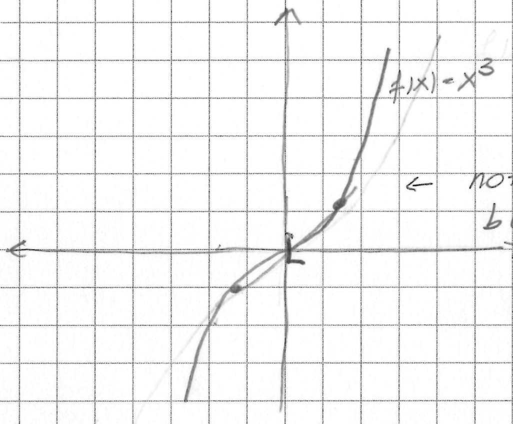
Convex functions

Def. $S \subseteq \mathbb{R}^n$; a function $f: S \to \mathbb{R}$ is called convex if S is convex and $\forall \vec{v}, \vec{w} \in S$ and $\forall \lambda \in [0,1]$

$<$ ← strictly convex

$$f(\lambda \vec{v} + (1-\lambda)\vec{w}) \leq \lambda f(\vec{v}) + (1-\lambda) f(\vec{w})$$





← not convex on $\mathbb{R}$ but convex on $\mathbb{R}_+$

⇐ Line segment between any two distinct points lies above the graph between the two points

Convex optimisation problem: set of constraints Ω and objective function F is convex

Thm: Let $f: \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then any local minimiser of f is a global minimiser.

Proof: Let $\vec{v}^*$ be a local minimiser & assume it's not global

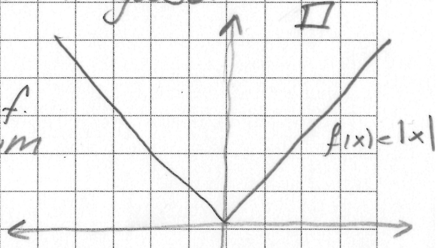
- $\exists \vec{w} \in \mathbb{R}^n$ s.t. $f(\vec{w}) < f(\vec{v}^*)$
- Since f is convex we have that for any $\lambda \in [0,1]$ and $\vec{v} = \lambda \vec{w} + (1-\lambda)\vec{v}^*$

holds for all line seg. connecting $\vec{w}$ and $\vec{v}^*$ $\Rightarrow$ $f(\vec{v}) \leq \lambda f(\vec{w}) + (1-\lambda) f(\vec{v}^*) < \lambda f(\vec{v}^*) + (1-\lambda) f(\vec{v}^*) = f(\vec{v}^*)$

- Every open neighborhood $\mathcal{U}$ of $\vec{v}^*$ contains a bit of this line segment
  $\Rightarrow$ every open neighborhood $\mathcal{U}$ of v^* contains an $\vec{v} \neq \vec{v}^*$ s.t. $f(\vec{v}) \leq f(\vec{v}^*)$
  $\Rightarrow$ ↯ to assumption that $\vec{v}^*$ is a local min $\Rightarrow$ $\vec{v}^*$ is global

$\square$

Thm makes no assumption about differentiability of f.
E.g. $f(x) = |x|$ is convex and $x = 0$ is global minimum



Examples of convex functions:

1) $f(\vec{x}) = \langle \vec{a}, \vec{x} \rangle + b \qquad \vec{a}, \vec{x} \in \mathbb{R}^n, b \in \mathbb{R}$

$$f(\lambda \vec{x} + (1-\lambda)\vec{y}) = \langle \vec{a}, \lambda(\vec{x} + (1-\lambda)\vec{y} \rangle + b$$
$$= \lambda \langle \vec{a}, \vec{x} \rangle + (1-\lambda) \langle a, y \rangle + b$$
$$= \lambda \langle \vec{a}, \vec{x} \rangle + \lambda b + (1-\lambda) \langle a, y \rangle + (1-\lambda) b$$
$$= \lambda f(x) + (1-\lambda) f(\vec{y})$$

ii) $f(x) = |x|$

$$f(\lambda x + (1-\lambda)y) = |\lambda x + (1-\lambda)y| \underset{\Delta \text{ inequality}}{\leq} \lambda|x| + (1-\lambda)|y|$$

iii) l_p-norms

$$\|\vec{x}\| = (|x_1|^2 + \dots + |x_n|^2)^{1/2} \quad \Leftarrow l_2 \text{ norm}$$

$$\|\lambda\vec{x} + (1-\lambda)\vec{y}\| \underset{\Delta\text{-inequ.}}{\leq} (\lambda\|\vec{x}\| + (1-\lambda)\|\vec{y}\|$$

iv) Operator norm

$$\|\vec{A}\|_2 := \max_{\vec{v}:\|\vec{v}\|=1} \frac{\|\vec{A}\vec{v}\|_2}{\|\vec{v}\|}$$

On can show that $\|\vec{A}\|_2 = \sigma_1(A) \Leftarrow$ largest singular value
see algebra 2

Characterise convexity using differentiability

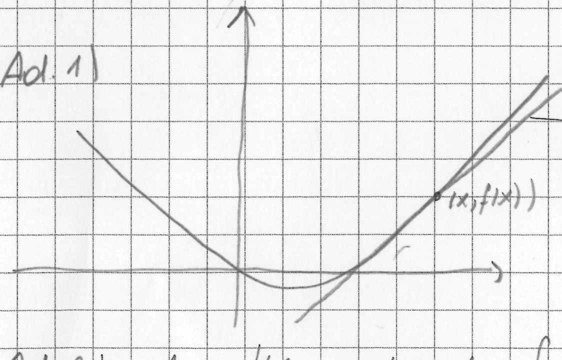
Thm 1) Let $f \in C^1(\mathbb{R}^n)$. Then f is convex if and only if $\forall \vec{v}, \vec{w} \in \mathbb{R}^d$

$$f(\vec{w}) \geq f(\vec{v}) + \nabla f(\vec{v})^T(\vec{w} - \vec{v})$$

2) Let $f \in C^2(\mathbb{R}^n)$. Then f is convex if and only if $\nabla^2 f(\vec{v})$ is positive semidefinite $\forall \vec{v} \in \mathbb{R}^n$

If $\nabla^2 f(\vec{v})$ is positive definite $\forall \vec{v} \in \mathbb{R}^n \Rightarrow f$ is strictly convex

Ad. 1)



$f(x) + \nabla f(x)^T(y-x)$ $\Leftarrow$ global underestimator of f

$\uparrow$ first order Taylor approximation

Ad. 2.) $f(x) = 1/x^2 \quad \text{dom } f = \{x \in \mathbb{R} : x \neq 0\}$

$f''(x) > 0$ for all $x \in \text{dom } f$ but f is not convex

Examples of convex fct.

v) $f(x) = e^x \quad f'' = e^x > 0 \Rightarrow$ convex

vi) $f(x) = -\log x \quad f'(x) = -1/x \quad f''(x) = 1/x^2 > 0$ for $x \neq 0$

Proof of 1) for $n=1$

We show that $f: \mathbb{R} \to \mathbb{R}$ is convex $\Leftrightarrow$ $f(y) \geq f(x) + f'(x)(y-x)$

"$\Rightarrow$" Assume f is convex therefore

$$f(x + t(y-x)) \leq (1-t) f(x) + t f(y) \quad | :t \qquad \text{for } t \in [0,1]$$

$$\frac{f(x+t(y-x)) - f(x)}{t} \leq -f(x) + f(y)$$

$$\text{or } f(y) \geq f(x) + \frac{f(x+t(y-x)) - f(x)}{t} \quad \Leftarrow \text{ take limit as } t \to 0$$

"$\Leftarrow$" Assume that $f(y) \geq f(x) + f'(x)(y-x)$ holds

Choose $x \neq y$ and $\lambda \in [0,1]$; $z = \lambda x + (1-\lambda) y$

$$f(x) \geq f(z) + f'(z)(x-z) \quad | \cdot \lambda$$

$$f(y) \geq f(z) + f'(z)(y-z) \quad | \cdot (1-\lambda)$$

$$\lambda f(x) + (1-\lambda) f(y) \geq \cancel{\lambda f(z)} + \cancel{(1-\lambda) f(z)} + \lambda f'(z)(x-z) + (1-\lambda) f'(z)(y-z)$$

$$\lambda f(x) + (1-\lambda) f(y) \geq f(z) + \lambda f'(z) x - \lambda \cancel{f'(z)} z + f'(z) y - f'(z) z - \lambda f'(z) y + \lambda \cancel{f'(z)} z$$

$$= \lambda x + (1-\lambda) y$$

$$\Rightarrow \quad \lambda f(x) + (1-\lambda) f(y) \geq f(z) \quad \Rightarrow f \text{ is convex}$$