MA398 Exercise Sheet 8 26th Nov. 2023.

Why does the iterative method work?

Consider the system of equation:

$$Ax = b \quad \text{———} \ *$$

Then let us define M and N such that

$$A = M + N$$

then we can write $*$ as

$$(M+N)x = b$$

$$Mx + Nx = b \quad \text{———} \ **$$

If we assume that there are two close solutions x_{k+1} and x_k (with reasonable error between their norms i.e $\| x_{k+1} - x_k \| < \varepsilon$).

Then we can write $**$ as

$$Mx_{k+1} + Nx_k = b$$

$$Mx_{k+1} = b - Nx_k$$

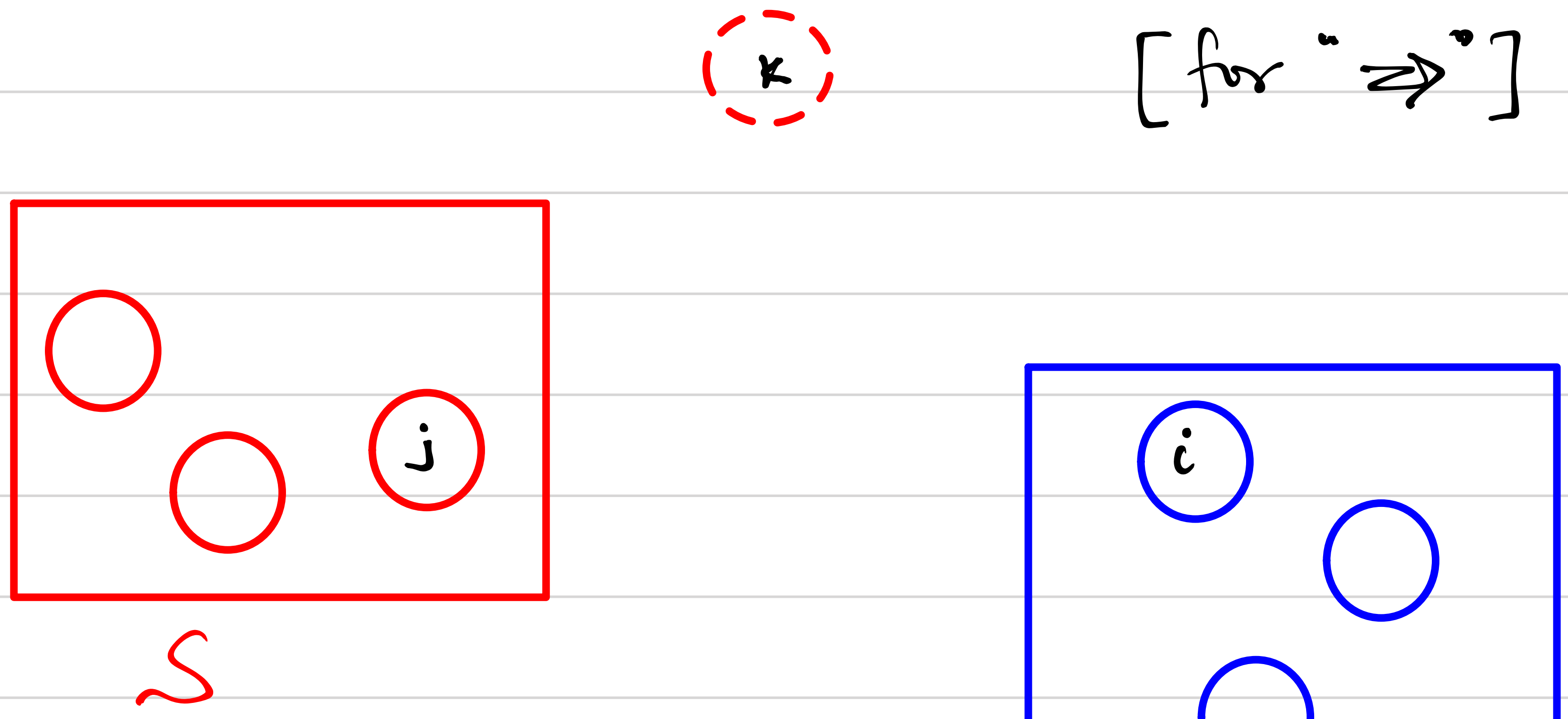
$$\boxed{x_{k+1} = M^{-1}(b - Nx_k)} \quad (++)$$

This forms the basis of the iterative approach to solve the system of linear equation.

The different methods: Jacobi, Gauss-Siedel etc. all define M and N differently (with some additional insight), but still applied to equation $(++)$.

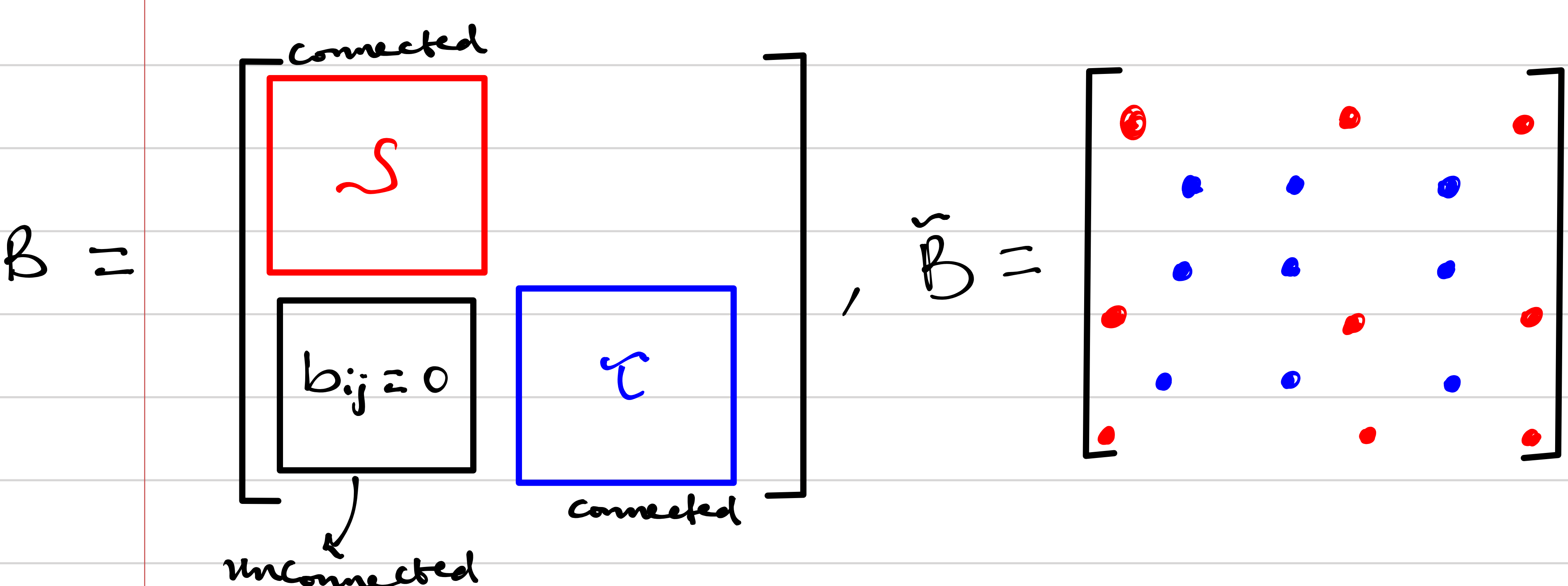
Useful diagram for Proof.

[for "⇒"]



$\tau = \{1, 2, \ldots, n\} \setminus \tilde{S}$

$\Rightarrow b_{ij} = 0$



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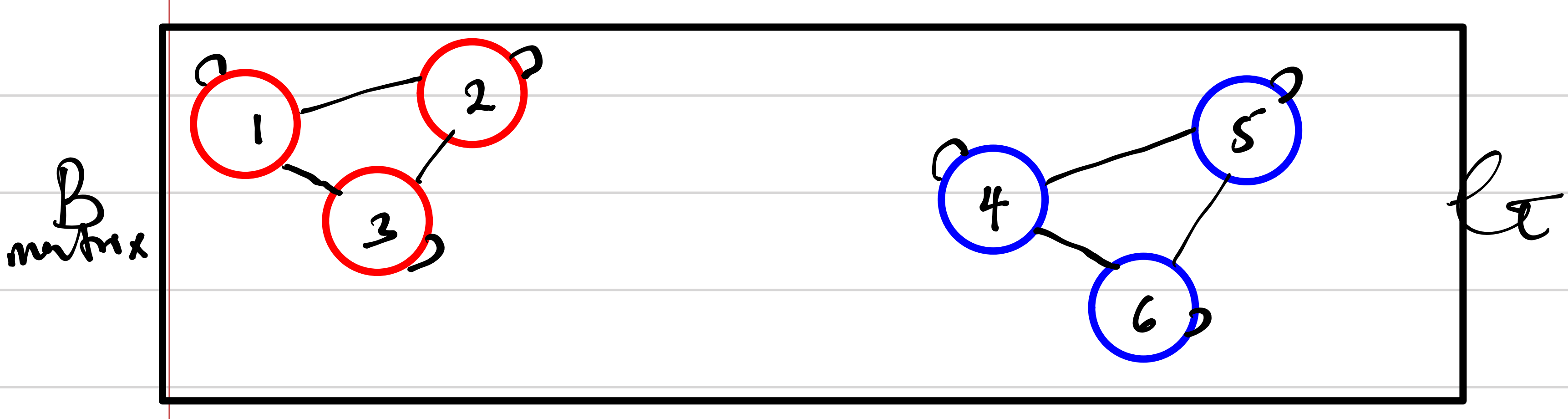

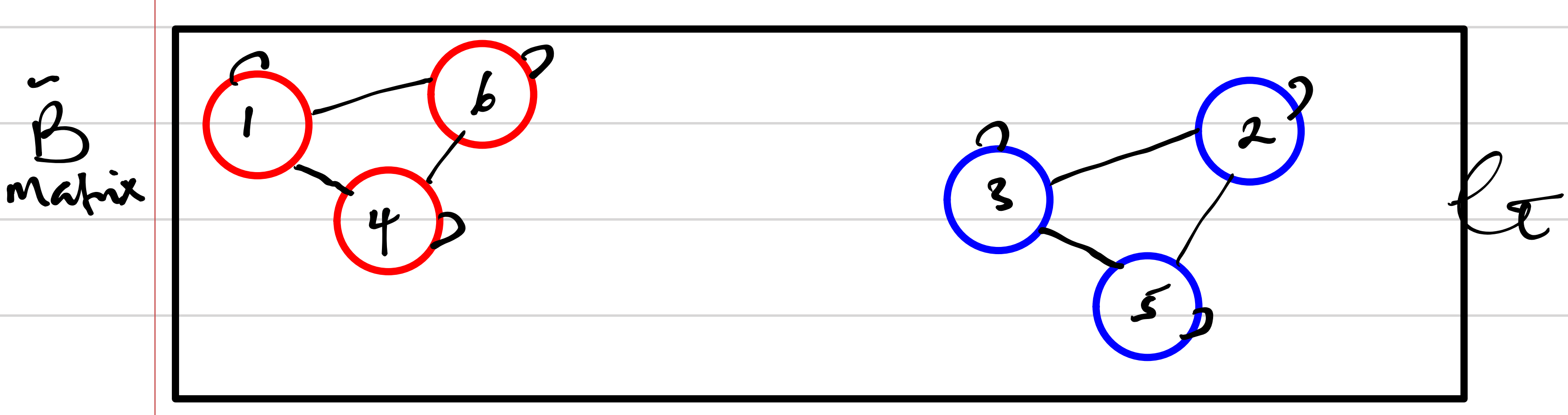

(some parts that need expanding)

Que 2:

Note: $(L + (1 - 1/w) D)(\frac{1}{w} D + L)^{-1}$

$= \underline{I} + (w-2) D (D + wL)^{-1}$

$\therefore (\frac{1}{w} D + U) x^{(k+1)} = b - (L + (1 - 1/w) D)(1/w D + L)^{-1} (b - (U + (1 - 1/w) D) x^{(k)})$

$= b - (\underline{I} + \underline{(w-2) D (D + wL)^{-1}})(b - (U + (1 - 1/w) D) x^{(k)})$

$= \cancel{b} - \cancel{b} + (U + (1 - 1/w) D) x^{(k)} - b (w-2) D (D + wL)^{-1}$
$+ (w-2) D (D + wL)^{-1} (U + (1 - 1/w) D) x^{(k)}$

$(\frac{1}{w} D + U) x^{(k+1)} =$
$- (w-2) D (D + wL)^{-1} b + (U + (1 - 1/w) D) x^{(k)}$
$+ (w-2) D (D + wL)^{-1} (U + (1 - 1/w) D) x^{(k)}$