MA398 Exercise Sheet 6.

13th Nov., 2023.

1. Prove the uniqueness of the singular values in the Singular Value Decomposition (SVD) of a matrix. That is, show that for every matrix $A \in \mathbb{R}^{m\times n}$, the singular values in the SVD are uniquely determined.

#Note the shapes of all underlined entries.

The Singular Value Decomposition (SVD) of a matrix $A \in \mathbb{R}^{m\times n}$ is given by $A = U\Sigma V^T$, where:

$U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal matrices, $\Sigma \in \mathbb{R}^{m\times n}$ is a diagonal matrix whose elements σ_i are the singular values of A, and The rows and columns of U and V are the left and right singular vectors of A, respectively. First, observe that if $A = U\Sigma V^T$ is an SVD of A, then $A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma^T\Sigma V^T = V(\Sigma^2)V^T$ is an eigendecomposition of A^TA. Here, Σ^2 is a diagonal matrix whose entries are the squares of the singular values of A. The entries of V are the eigenvectors of A^TA.

$n \times n$ matrix

Now, the eigenvalues of a matrix are uniquely determined, up to order (as can be proved via the characteristic polynomial). This means that the singular values of A, being the square roots of the eigenvalues of A^TA, are also uniquely determined, up to order.

Note, however, that by convention, the singular values in an SVD are always arranged in descending order along the diagonal of Σ. Therefore, the ordering of the singular values is fixed in the SVD, and they are uniquely determined.

The Singular value decomposition (SVD) is a generalization of the eigen value decomposition. The eigen value decomposition is for square matrices, while SVD is for any matrix (square or rectangular).

Eigen Value Decomposition

Given a matrix $A \in \mathbb{C}^{n\times n}$ with rank n, then A has n eigen value-vector pair. Let the eigen values of A be $\sigma_1, \sigma_2, \dots, \sigma_n$ such that $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$. And let $x_1, x_2, \dots, x_n$ be the corresponding eigen vectors. We can form a matrix V whose columns are the eigen vectors of A i.e $V = [x_1\ x_2\ \cdots\ x_n]$

Then we can write the eigenvalue decomposition of A as

$$A = V \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} V^*$$

$$\therefore \quad A = V \Sigma V^* \quad , \quad \Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_n).$$

Note that V is a unitary matrix i.e, the columns of V ie $x_1, x_2, \dots, x_n$ are orthogonal to each other and $VV^* = I$ (identity).

Alternative proof. (Same way but shorter)

Answer: Let $A \in \mathbb{C}^{m \times n}$ and suppose that $A = U\Sigma V^* = \tilde{U}\tilde{\Sigma}\tilde{V}^*$ are two singular decompositions of A. Assume that $m \geq n$ (otherwise consider AA^* instead of A^*A in the following). We have that

$$A^*A = V\Sigma^*U^*U\Sigma V^* = V\Sigma^*\Sigma V^* \quad \Rightarrow \quad A^*AV = V\mathrm{diag}(\sigma_1^2, \dots, \sigma_n^2),$$

hence the eigenvalues of A^*A are $\sigma_1^2 \geq \dots \geq \sigma_n^2$.
Similarly, using the other singular values decomposition the values $\tilde{\sigma}_1^2 \geq \dots \geq \tilde{\sigma}_n^2$ are the eigenvalues of A^*A. We can conclude that $\sigma_i^2 = \tilde{\sigma}_i^2$ for all $i = 1, \dots, n$, and since the singular values are nonnegative $\sigma_i = \tilde{\sigma}_i$ $\forall i$.