MA398 MATRIX ANALYSIS AND ALGORITHMS: ASSIGNMENT 2

Please submit your solutions to this assignment via Moodle by **noon on Monday, November 6$^{\text{th}}$**. Make sure that your submission is clearly marked with your name, university number, course and year of study. Note that some of the questions are practical questions that are concerned with real-world problems and require coding in Python.

- The written part of the solutions may preferably be typed in LaTeX, otherwise written on paper and subsequently scanned/photographed provided the images are clearly legible. You are required to deliver a single document entitled *MA398_Assignment2_FirstnameLastname.pdf*, outlining your solutions and explaining your interpretation and arguments to the questions .
- The Python code scripts relevant to each question should be submitted as *MA398_ Assignment2_ ExerciseN.ipynb* where you need to make sure that you document the environment in which I should be able to run your code.

To avoid losing marks unnecessarily, please make sure that,

- you show all intermediate steps of your calculations.
- you include all the coding that you have to do in order to obtain your answers and according to the instruction given above.
- include any plots you generate and label them appropriately.
- when you provide an answer to a question make sure that you justify your answer and provide details of any mathematical calculations that are required.

⚠Only in an emergency or if the Moodle submission is unavailable because of a general outage, the assignment should in the respective case be submitted by email to Randa.Herzallah@warwick.ac.uk, and olayinka.ajayi@warwick.ac.uk.

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1. (Matrix Conditioning and Diagonal Dominance:) Consider the matrix A given by:

$$A = \begin{bmatrix} 3 & 6.01 \\ 2 & 4 \end{bmatrix}$$

   (a) Compute the determinant of A and describe its significance with respect to the matrix being singular.

   Answer: det(A)=-0.0200

   (b) Using the provided norms:

   - Frobenius norm: $\|A\|_F = \sqrt{\sum_{i=1}^2 \sum_{j=1}^2 a_{ij}^2}$
   - L_1 norm (max column sum): $\|A\|_1 = \max_j \sum_{i=1}^2 |a_{ij}|$

   Compute

   - The norms for matrix A.
     Answer:

$$\|A\|_{\text{frob}} = 8.0697$$

$$\|A\|_{L_1} = 10.0100$$

- The norms for matrix A^{-1}.
  Answer:

$$\|A^{-1}\|_{\text{frob}} = 403.4851$$
$$\|A^{-1}\|_{L_1} = 450.5000$$

- The condition number of the matrix A.
  Answer:

$$\|A\|_{\text{frob}}\|A^{-1}\|_{\text{frob}} = 3.2560 * 10^{+03}$$
$$\|A\|_{L_1}\|A^{-1}\|_{L_1} = 4.5095 * 10^{+03}$$

(c) Using Python, calculate the condition number of A and interpret its value in the context of the matrix being well-conditioned or ill-conditioned.

Answer: Interpretation of the Condition Number of Matrix A:

The condition number of matrix A is computed to be:

$$\text{cond}(A) \approx 4.5095 \times 10^3$$

The condition number provides a measure of how sensitive the solution of a linear system $Ax = b$ is to perturbations in b. A condition number close to 1 indicates that the matrix is well-conditioned, implying that its solution will not be overly sensitive to small changes in b. However, as the condition number increases, the matrix becomes more ill-conditioned, suggesting that the solution can be very sensitive to tiny changes in b.

Given the high condition number of 4.5095×10^3 for matrix A, it is evident that A is ill-conditioned. This means that solutions to systems using this matrix might be unstable with respect to small perturbations in the input data.

(d) A matrix is said to be diagonally dominant if, for each row, the magnitude of the diagonal element is greater than the sum of the magnitudes of the other elements in that row. Check if matrix A is diagonally dominant. Based on this property, can you infer anything about the conditioning of matrix A?

Answer: 1. Diagonal Dominance: For A to be diagonally dominant, the absolute value of the diagonal element in each row should be greater than the sum of the absolute values of the other elements in that row. Matrix A does not satisfy this criterion for all rows and is therefore **not diagonally dominant**.

2. Condition Number: The computed condition number for A is:

$$\text{cond}(A) \approx 4.5095 \times 10^3$$

3. Relationship and Interpretation: Diagonal dominance is often considered a good heuristic for a matrix being well-conditioned, especially for certain iterative solvers. However, a matrix can still be ill-conditioned even if it is diagonally dominant. In the case of A, despite not being diagonally dominant, its high condition number more conclusively establishes it as **ill-conditioned**. Thus, solutions to systems using this matrix might be unstable with respect to small perturbations in the input data.

2. (Analysis of Linear System Stability and Perturbations:) Consider the linear system of equations represented by matrix A:

$$A = \begin{pmatrix} 1 & -0.55 \\ -2 & 1.06 \end{pmatrix}$$

and vector b:

$$b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Perform the following tasks using Python:

(a) **LU Factorization:**
Solve the linear system using LU factorization and obtain the solution vector.

(b) **Condition Number:**
Compute the condition number of the matrix A and comment on its significance.

(c) **Perturbed b Analysis:**
For b' defined as:

$$b' = \begin{pmatrix} 1+\epsilon \\ -1 \end{pmatrix}$$

where $0 < \epsilon < 1$:

- Consider several values of ϵ.
- Compute and plot the forward error and backward error for each value of ϵ.
- Comment on the observed results in relation to the variation in ϵ.

(d) **Perturbed Matrix Analysis:**
Analyze the solution for matrix A' defined as:

$$A' = \begin{pmatrix} 1+\delta & -0.55 \\ -2 & 1.06 \end{pmatrix}$$

where $0 < \delta < 1$. Discuss the changes, if any, in the solution due to the perturbation in the matrix.

Hints: You might find the following Python functions useful:

- For LU factorization: lu_factor(X)
- To compute the condition number: np.linalg.cond(X)
- To compute the vector or matrix norm: np.linalg.norm(z)

Remember to leverage appropriate libraries such as numpy and scipy for these operations.