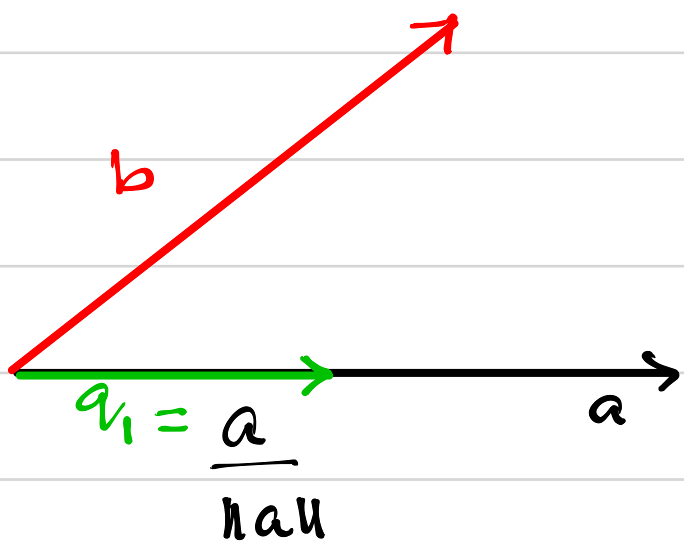
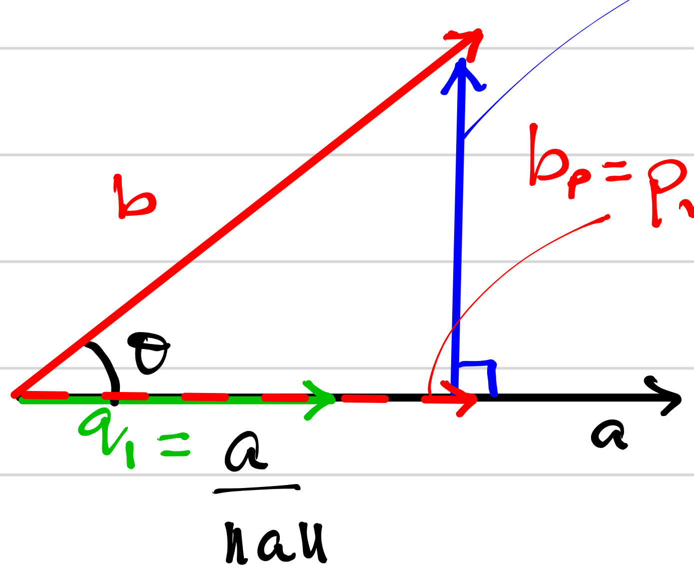
Gram Schmidt Process

20th Nov. 2023



The aim is to project the vector b on to the vector q_1.



$\hat{q}_2$ = perpendicular to the projection of b i.e b_p.

$b_p = \text{Proj}_{q_1}(b)$ = the projection of b on q_1

The projection means that we rotate b through the angle θ, until we land on a (or q_1).
So, how do we get $b_p = \text{Proj}_{q_1}(b)$

Observe the triangle formed by b_p, b and $\hat{q}_2$.
We note that by the trigonometry identity

$$\cos\theta = \frac{\|b_p\|}{\|b\|} \Rightarrow \|b_p\| = \|b\| \cos\theta$$

$$\|b_p\| = \|b\|\cos\theta \quad \text{——} \; *$$

But θ is the angle between b and q_1.

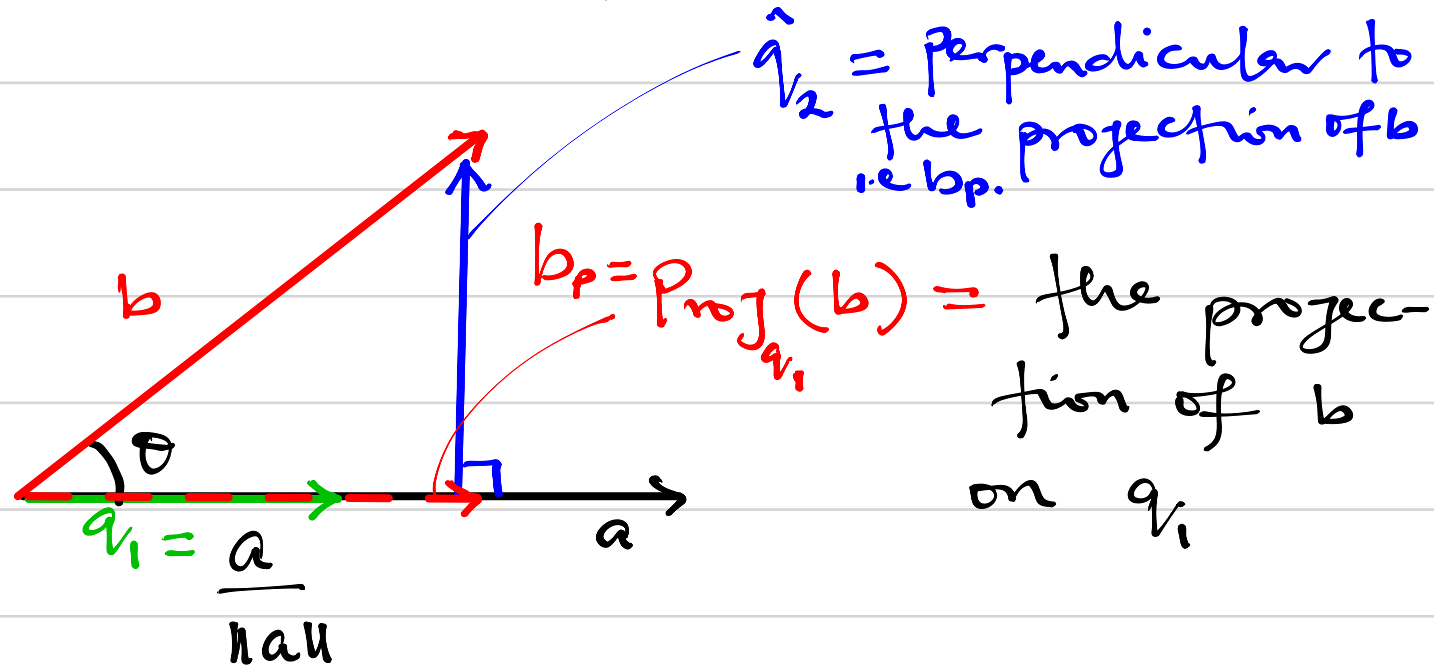
$$\Rightarrow \quad \cos\theta = \frac{\langle q_1, b\rangle}{\|q_1\|\,\|b\|}$$

So, plug into $*$:

$$\|b_p\| = \cancel{\|b\|}\,\frac{\langle q_1, b\rangle}{\|q_1\|\,\cancel{\|b\|}} = \frac{\langle q_1, b\rangle}{\|q_1\|} = \langle q_1, b\rangle \quad **$$

#Note that $\|q_1\| = 1$ (q_1 is a unit vector)

So, to get b_p, we observe that b_p is in the same direction as the unit vector q_1.



So, we can say that the projection of b on q_1 is given as:

$$b_p = \underbrace{\|b_p\|}_{\text{magnitude}}\,\overbrace{q_1}^{\text{direction}} = \langle q_1, b\rangle q_1$$

But what we want is the vector perpendicular to the projection of b, that is $\hat{q}_2$.

By vector addition: $b_p + b = \hat{q}_2$

$$\Rightarrow \quad \hat{q}_2 = b - b_p$$

$$\hat{q}_2 = b - b_p = b - \langle q_1, b\rangle q_1$$

To get $\hat{q}_2$ to be of unit length, we normalize to get:

$$q_2 = \frac{\hat{q}_2}{\|\hat{q}_2\|}$$

In general, given vectors $a_1, a_2, \dots, a_n$ their corresponding orthonormal vectors $q_1, q_2, \dots, q_n$ is given by

$$q_1 = \frac{a_1}{\|a_1\|}$$

$$q_2 = \frac{\hat{q}_2}{\|\hat{q}_2\|}, \quad \hat{q}_2 = a_2 - \langle q_1, a_2\rangle q_1$$

$$q_3 = \frac{\hat{q}_3}{\|\hat{q}_3\|}, \quad \hat{q}_3 = a_3 - \langle q_1, a_3\rangle q_1 - \langle q_2, a_3\rangle q_2$$

$$\vdots$$

and so on.

I hope this helps.

House Holder Reflection

20th Nov., 2023

We multiply A continuously (on the left) by some orthogonal (unitary) matrices, until it yields R, an upper triangular matrix.

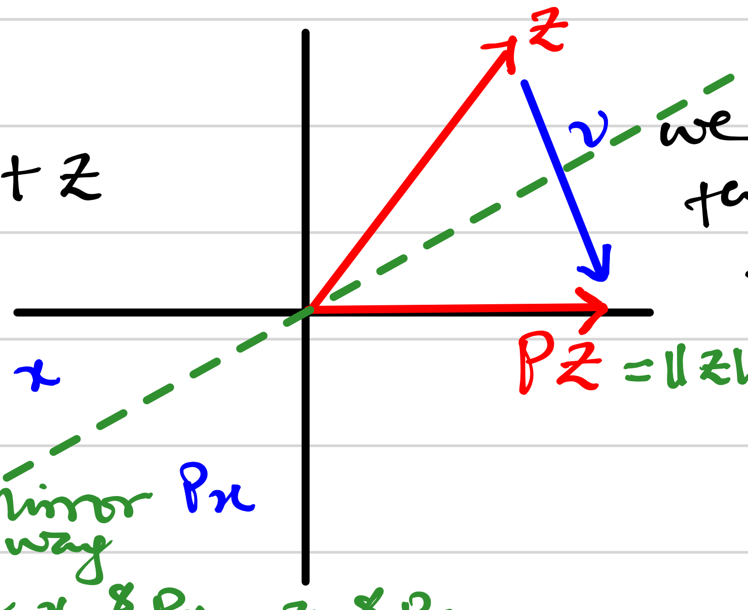
To start, we choose P such that

$$Pz = \|z\|_2 e_1 = z + v, \text{ where } v = \|z\|_2 e_1 - z$$

z is the target column, where we want that all the values below the diagonal index is zero.

P is orthogonal. So the norm of some vector multiplied by an orthogonal matrix is unchanged.

$\Rightarrow Pz = v + z$

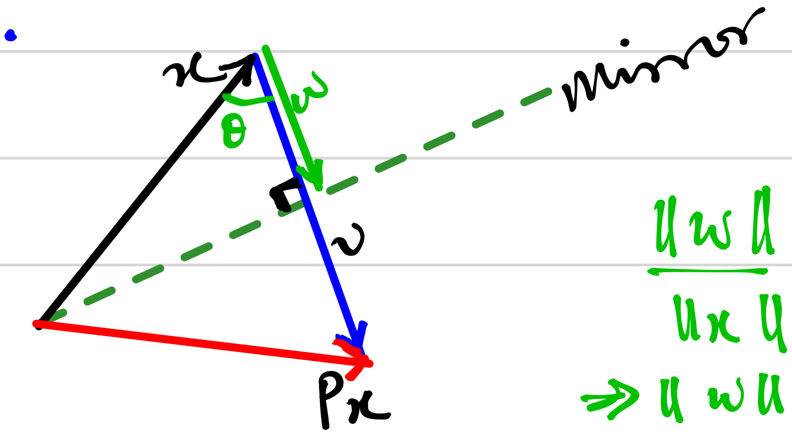


we want z to end up on the positive x-axis after multiplied by P.

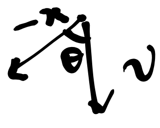
(assuming a 2×2 example).

Reflections don't change norms. Our reflection will give us an orthogonal matrix.

So the operation Pz means " reflect z through this mirror ". So for every x, Px is the image on the other side.



But θ is the angle between −x and v (note the −ve)



$$\frac{\|w\|}{\|x\|} = \cos\theta$$

$$\Rightarrow \|w\| = \|x\| \cos\theta$$

$$\Rightarrow \|w\| = \|x\|\cos\theta = \|x\| \frac{\langle -x, v\rangle}{\|x\|\,\|v\|}$$

$\hookrightarrow \|x\| = \|-x\|$

$$\therefore \|w\| = \frac{\langle -x, v\rangle}{\|v\|}$$

So the direction of w is a unit vector in the same direction as v.

$$\text{direction}(w) = \frac{v}{\|v\|}$$

$$\Rightarrow w = \underbrace{\frac{\langle -x, v\rangle}{\|v\|}}_{\text{(length) } \|w\|} \cdot \underbrace{\frac{v}{\|v\|}}_{\text{direction}} = -v \frac{\langle x, v\rangle}{\|v\|^2}$$

$x + w$ gets us to the mirror.
$x + 2w$ gets us to the reflection (i.e the point Px).

So, for any x

$$Px = x - 2v\frac{\langle x, v\rangle}{\|v\|^2}$$

$\|v\| = 1$ if v is already a unit vector.

It is possible to factor out x and obtain

$$P = I - 2\frac{vv^T}{\|v\|^2},$$

which is our projection matrix.

$\boxed{P^T P = I}$, P is orthogonal.

P is called a Householder reflector, and yields $Pz = \|z\| e_1$ (which was what we wanted).

Que 3. $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 4 & 3 & 2 \end{bmatrix}$

At Step 0 : $R^{(0)} = A, \quad Q^{(0)} = I$

First compute the vector $u^{(1)}$: (target vector)

$$u^{(1)} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \quad \text{(first column of A).}$$

Now we get $\hat{v} \longrightarrow$ takes us to basis-axis

basis vector

$$\hat{v}^{(1)} = u^{(1)} + \text{sign}(u_1^{(1)}) \|u^{(1)}\|_2 \cdot e_1$$

$$\|u^{(1)}\| = \sqrt{2^2 + 1^2 + 4^2} = \sqrt{21} = 4.5826$$

$$\hat{v}^{(1)} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + 1 \cdot \sqrt{21} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+\sqrt{21} \\ 1 \\ 4 \end{bmatrix}$$

For Convenience we normalize $\hat{v}$ in advance to get $v^{(1)}$:

$$v^{(1)} = \frac{\hat{v}^{(1)}}{\|\hat{v}^{(1)}\|_2} = \frac{1}{\sqrt{(2+\sqrt{21})^2 + 1^2 + 4^2}} \cdot \hat{v}^{(1)}$$

$$v^{(1)} = \frac{1}{\sqrt{42 + 4\sqrt{21}}} \begin{bmatrix} 2+\sqrt{21} \\ 1 \\ 4 \end{bmatrix}$$

Now we compute the first Householder (projection) matrix: $H_1 = I_3 - 2v^{(1)}(v^{(1)})^T$

$$2v^{(1)}(v^{(1)})^T = \frac{2}{42+4\sqrt{21}} \begin{bmatrix} 25+4\sqrt{21} & 2+\sqrt{21} & 8+4\sqrt{21} \\ 2+\sqrt{21} & 1 & 4 \\ 8+4\sqrt{21} & 4 & 16 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} -0.4364 & -0.2182 & -0.8729 \\ -0.2182 & 0.9668 & -0.1326 \\ -0.8729 & -0.1326 & 0.4696 \end{bmatrix}$$

So, we compute the next R matrix:

$$R^{(1)} = H_1 R^{(0)} = H_1 A$$

$$R^{(1)} = \begin{bmatrix} -4.5826 & -4.1461 & -3.7096 \\ 0 & -0.0856 & -0.1712 \\ 0 & -1.3425 & -2.685 \end{bmatrix}$$

Now, $u^{(2)} = \begin{bmatrix} 0 \\ -0.0856 \\ -1.3425 \end{bmatrix}$

Please follow the calculations in the Jupyter notebook.

$$\hat{v}^{(2)} = u^{(2)} + \text{Sign}(u_2^{(2)}) \cdot \|u^{(2)}\|_2 e_2$$

$$\hat{v}^{(2)} = \begin{bmatrix} -4.1461 \\ -4.4445 \\ -1.3425 \end{bmatrix} \Rightarrow v^{(2)} = \frac{\hat{v}^{(2)}}{\|\hat{v}^{(2)}\|}$$