TMA226 17/18 PROBLEM SET 2
ORTHOGONAL POLYNOMIALS

Introduction

We have seen in that given any finite-dimensional vector space V equipped with an inner product, we can find an orthogonal basis for V. The method we used to construct such an orthogonal basis is known as the Gram-Schmidt process.

The Gram-Schmidt process can be applied to any linearly independent collection of elements in an inner product space. In particular, we can use it to orthogonalise polynomials. The resulting orthogonal polynomials depend on the choice of inner product and the linearly independent set we started from. If we start with the power basis, we get a sequence of polynomials satisfying the following definition.

Definition 1. A *family of orthogonal polynomials* is a set $\{p_0, p_1, \ldots\}$ such that

$$\begin{aligned} p_n \in \mathcal{P}_n &\quad \text{(i.e. } p_n \text{ has degree } n) \\ (p_n, p_m) = 0 &\quad \text{if } n \neq m. \end{aligned}$$

Note that a family of orthogonal polynomials are unique up to some scaling factors. Often (but not always) the orthogonal polynomials are taken to be monomials, meaning that the coefficent in front of the highest order exponent is 1.

There are many possible inner products that we can use on the space of polynomials $\mathcal{P}$, and each choice of inner product results in a family of orthogonal polynomials specific to that inner product. This exercise set introduces the classical orthogonal polynomials: The Chebyshev, Hermite, Laguerre and Legendre polynomials. The focus of this exercise set will be on the orthogonality of these polynomials, in particular the applications of the Gram-Schmidt process and orthogonal projection. The reader should note that the these families of orthogonal polynomials have many unique properties that is not is not covered here.

Exercise 1. Let $\{p_0, p_1, \ldots\}$ be a family of orthogonal polynomials. Show that p_{n+1} is orthogonal to every polynomial having degree less than or equal to n.

Solution. Let p be a polynomial of degree up to n. Then, since $\{p_0, \ldots, p_n\}$ is a basis for $\mathcal{P}_n$, there are coefficients $c_0, \ldots, c_n$ such that $p = c_0 p_0 + \cdots + c_n p_n$. The orthogonality condition then gives

$$(p, p_{n+1}) = \left(\sum_{k=0}^{n} c_n p_k, p_{n+1} \right) \sum_{k=0}^{n} c_n \underbrace{(p_k, p_{n+1})}_{=0} = 0.$$

Exercise 2. Let $\{p_0, p_1, \ldots\}$ be a family of orthogonal polynomials, in monomial form. Show that the polynomials satisfy a recurrence relation

$$p_{n+1} = (x - A_n) p_n - B_n p_{n-1}, \quad p = 1, 2, \ldots \tag{1a}$$

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where

$$A_n = \frac{(p_n, xp_n)}{(p_n, p_n)} \tag{1b}$$

$$B_n = \frac{(p_n, xp_{n-1})}{(p_{n-1}, p_{n-1})} \tag{1c}$$

Solution. Since each p_n is a monomial, we have $p_{n+1} - xp_n \in \mathcal{P}_n$. Hence there are coefficients $c_0, \dots, c_n$ such that

$$p_{n+1} - xp_n = c_0 p_0 + \cdots + c_n p_n \tag{2}$$

Now take the inner product of both sides with p_k, for $k = 0, \dots, n$. Since $(p_j, p_k) = 0$ for $j \neq k$, we get

$$\begin{aligned}(p_{n+1}, p_k) - (xp_n, p_k) &= (c_0 p_0 + \cdots + c_n p_n, p_k) \\ &= c_0(p_0, p_k), + \cdots + c_n(p_n, p_k), \quad k = 0, \dots, n.\end{aligned}$$

Only one one the terms on the right side will be nonzer due to orthogonality. We arrange to get

$$c_k(p_k, p_k) = (p_n, xp_k), \quad k = 0, \dots, n.$$

Now $xp_k \in P_{k+1}$, and p_n is othgonal to every polynomial in $\mathcal{P}_n$, so we get $c_0 = \cdots c_{n-2} = 0$, so only c_{n-1} and c_n may be nonzero. We set $A_n = -c_n$ and $B_n = -c_{n-1}$,

$$A_n = -c_n = \frac{(p_n, xp_n)}{(p_n, p_n)} \qquad b_n = -c_{n-1} = \frac{(p_n, xp_{n-1})}{(p_{n-1}, p_{n-1})}$$

and inserting in (**??**) we get

$$p_{n+1} = xp_n + c_{n-1}p_{n-1} + c_n p_n = (x - A_n)p_n - B_n p_n.$$

Let p be a polynomial of degree up to n. Then, since $\{p_0, \dots, p_n\}$ is a basis for $\mathcal{P}_n$, there are coefficients $c_0, \dots, c_n$ such that $p = c_0 p_0 + \cdots + c_n p_n$. The orthogonality condition then gives

$$(p, p_{n+1}) = \left(\sum_{k=0}^{n} c_n p_k, p_{n+1} \right) \sum_{k=0}^{n} c_n \underbrace{(p_k, p_{n+1})}_{=0} = 0.$$

For the classical plynomials, there are explicit formulas for A_n and B_n that does not require evaluating integrals. We will later derive them for the Chebyshev polynomials.

Legendre polynomials and Gauss-Legendre quadrature

The Legendre polynomials are orthogonal in the inner product

$$(p, q) = \int_{-1}^{1} p(x)q(x)\,dx.$$

Like other classes orthogonal polynomials, Legendre polynomials play an important role in numerical analysis and physics. In particular, they used in the derivation of Gauss-Legendre quadrature rules. The Gauss-Legendre quadrature rules are optimal, in the sense that they achive the maximal degree of precision[1]. In physics, Legendre polynomials appear in the spherical harmonics, the functions that the describe the angular part of the solutions ot the eigenvalue problem for the Laplacian

[1] A quadrature rule has degree of precision q if it is exact for every polynomial of degree less than or equal to q. A quadrature rule with $n+1$ points has a degree of precision $n \leq q \leq 2n+1$.

in three dimensions. In particular, they appear in the eigenstates of the hydrogen atom[2].

Exercise 3. Determine first four Legendre polynomials P_0, P_1, P_2 and P_3 by uisng the Gram-Schmidt process to orthogonalise the power basis $\{1, x, x^2, x^3\}$.

Solution. Let $p_n(x) = x^n$ for $n = 1, 2, 3, 4$ denote the power basis. To compute the inner products it is useful to note that

$$\int_{-1}^{1} x^n \, dx = \begin{cases} \dfrac{2}{n+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

This gives us the inner products between basis functions:

$$(p_n, p_m) = (x^n, x^m) = \int_{-1}^{1} x^n x^m \, dx = \begin{cases} \dfrac{2}{n+m+1} & \text{if } n+m \text{ is even} \\ 0 & \text{if } n+m \text{ is odd.} \end{cases}$$

Now we carry out the Gram-Schmidt orthogonalisation process to determine the Legendre polynomials L_0, L_1, L_2 and L_3. We start by setting the first Legendre polynomial to be the constant polynomial p_0,

$$P_0(x) = p_0(x) = 1.$$

We compute $(P_0, p_1) = (1, x) = 0$, and get the next polynomial

$$P_1(x) = p_1(x) - \frac{(P_0, p_1)}{(P_0, P_0)} P_0(x) = p_1(x) - 0P_0(x) = x$$

We compute $(P_1, p_2) = (x, x^2) = 0$, $(P_0, p_2) = (1, x^2) = 2/3$ and $(P_0, P_0) = 2$, we get the third polynomial

$$\begin{aligned} P_2(x) &= p_2(x) - \frac{(P_1, p_2)}{(P_1, P_1)} P_1(x) - \frac{(P_0, p_2)}{(P_0, P_0)} P_0(x) \\ &= p_2(x) - 0P_1(x) - \frac{2/3}{2} P_0(x) = x^2 - \frac{1}{3} \end{aligned}$$

We compute

$$(P_2, p_3) = (x^2 - \frac{1}{3}, x^3) = (x^2, x^5) - \frac{1}{3}(1, x^3) = 0.$$

and $(P_1, p_3) = (x, x^3) = 2/5$, $(P_1, P_1) = (x, x) = 2/3$ and $(P_0, x^3) = (1, x^3) = 0$.

$$\begin{aligned} P_3(x) &= p_3(x) - \frac{(P_2, p_3)}{(P_2, P_2)} P_2(x) - \frac{(P_1, p_3)}{(P_1, P_1)} L_1(x) - \frac{(P_0, p_3)}{(P_0, P_0)} P_0(x) \\ &= p_2(x) - 0P_2(x) - \frac{2/5}{2/3} P_1(x) - 0P_0(x) = x^3 - \frac{3}{5} x. \end{aligned}$$

[2]The eigenstates of the hydrogen atom are the solutions $\psi(\boldsymbol{x})$ to the stationary Schrödinger equation, which can be written

$$-\left(\Delta + \frac{1}{|\boldsymbol{x}|}\right)\psi(\boldsymbol{x}) = E\psi(\boldsymbol{x}).$$

This is eigenvalue problem where the eigenvalues E are the (quantized) energies of the eigenstates. Here Δ is the Laplacian operator, which in spherical coordinates (r, ϕ, θ) takes the form

$$\Delta\psi = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \psi^2} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right).$$

Summarising, the four first Legendre polynomials are

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = x^2 - \frac{1}{3}, \qquad P_3(x) = x^3 - \frac{3}{5}x.$$

The Gauss-Legendre quadrature rules uses the roots of the Legendre polynomials as quadrature points. This would not be possible if not every Legendre polynomial P_n has n distinct roots in $(-1, 1)$.

Exercise 4. Show that the Legendre polynomial P_n has n distinct roots in the interval $(-1, 1)$. *Hint: try to find a polynomial that is orthogonal to P_n and has the same roots as P^n in $(-1, 1)$.*

Solution. Suppose that P_n has no more than m distinct roots at $x_1, x_2, \ldots, x_m$ for some $m < n$. First suppose all the roots are simple. Then we have

$$P_n(x) \underbrace{(x - x_1)(x - x_2) \cdots (x - x_m)}_{= p(x)} q(x), \tag{3}$$

where q is a polynomial of degree $n - m$ with no roots in $(-1, 1)$. Since $p(x)$ is polynomial of degree $m < n$, we have

$$0 = (p, P_n) = \int_{-1}^{1} p(x)^2 q(x)\, dx. \tag{4}$$

Now note that the sign of the integrand never changes in the interval $(-1, 1)$ since $p(x) \geq 0$ and $q(x)$ has no roots in $(-1, 1)$. Since q is not constant 0, we have to conclude that (**??**) cannot possibly hold, so we have found a contradiction!

Only a slight generalisation of the argument above is needed for to prove the claim Let r_i be the multiplicity of the root x_i. Then instead of (5), we have

$$P_n(x) \underbrace{(x - x_1)^{r_1}(x - x_2)^{r_2} \cdots (x - x_m)^{r_m}}_{= p(x)} q(x),$$

and q is a polynomial without roots in $(-1, 1)$. Now set

$$\widehat{p} = \prod_{\substack{i=1 \\ r_i \text{ is odd}}}^{m} (x - x_i),$$

i.e. p is the product of all the $(x - x_i)$ for which the the root x_i has odd multiplicity. Then p is a polynomial of degree at most $m < n$, so from orthogonality we have and we have

$$0 = (\widehat{p}, P_n) = \int_{-1}^{1} \widehat{p}(x) p(x) q(x)\, dx. \tag{5}$$

Now note that all the factors $(x - x_i)$ in $\widehat{p}p$ occur with some even powers, so $\widehat{p}(x)p(x) \geq 0$ for all $x \in (-1, 1)$. Then since q has constant sign on $(1, 1)$, we have to conclude that (**??**) cannot possibly hold, so we have found a contradiction! This means that P_n cannot have less than n distinct roots in $(-1, 1)$.

Exercise 5. Let $x_0, \ldots, x_n$ be the roots of the Legendre polynomial of degree $n+1$. Consider the quadrature rule

$$\int_{-1}^{1} f(x)\, dx \approx \sum_{n=0}^{n} f(x_i) w_i, \tag{6}$$

where the weights w_i are the integral of the of Lagrange polynomials,

$$w_i = \int_{-1}^{1} \ell_i(x)\,dx, \quad \ell_i(x) = \prod_{j\neq i}\left(\frac{x-x_j}{x_i-x_j}\right)$$

Show that this quadrature rule is exact for a polynomial p of degree up to $2n+1$. That is, show that

$$\int_{-1}^{1} p(x)\,dx = \sum_{n=0}^{n} f(x_i) w_i. \tag{7}$$

Hint: Use polynomial division to write $p(x) = q(x)P_{n+1}(x) + r(x)$*, where* p *and* q *are polynomials of degree less than or equal to* n*.*

Solution. It is clear that the quadrature rule is exact for a polynomial of degree n. We need to show that it is also exact for polynomials of degree $2n+1$.

Let p be a polynomial of degree $2n+1$. Then we can find (using polynomial division) polynomials q and r such that

$$p(x) = q(x)P_{n+1}(x) + r(x).$$

The degrees of q and r are at most n. Now note that

$$p(x_i) = q(x_i)\underbrace{P_{n+1}(x_i)}_{=0} + r(x_i) = r(x_i), \quad i = 0, \ldots, n \tag{$*$}$$

since $P_{n+1}(x) = (x-x_0)\cdots(x-x_n)$. The quadrature rule is exact for r, since r is a polynomial of degree at most n, so we get

$$\sum_{i=0}^{n} p(x_i)w_i = \sum_{i=0}^{n} r(x_i)w_i = \int_{-1}^{-1} r(x)\,dx.$$

Now using ($*$) and orthogonality we have

$$\begin{aligned}
\sum_{i=0}^{n} p(x_i)w_i &= \int_{-1}^{-1} r(x)\,dx \\
&= \int_{-1}^{-1} p(x_i) - q(x_i)P_{n+1}(x_i)\,dx \\
&= \int_{-1}^{-1} p(x_i)\,dx - \underbrace{(q, P_{n+1})}_{=0}
\end{aligned}$$

where the last term vanishes because P_{n+1} is orthogonal to every polynomial of degree less than $n+1$, and q has at most degree n.

Hermite polynomials

The Hermite polynomials are orthogonal in the inner product

$$(p, q) = \int_{-\infty}^{\infty} p(x)q(x)e^{-\frac{x^2}{2}}\,dx. \tag{8}$$

Hermite polynomials have application in probability[3], and they also appear as the solutions to the quantum harmonic oscillator[4].

Exercise 6. Use the recurrence relation (1) to compute the first four Hermite polynomials H_0, H_1, H_2 and H_3.
Hint: It is possible to show that

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}.$$

This identity and integration by parts gives us

$$\int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}}\,dx = \begin{cases} \sqrt{2\pi}(n-1)!! & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{9}$$

where $k!! = 1\cdot 3\cdot 5\cdots\cdots k$ is the semi-factorial of an odd number n. Use this to compute A_n and B_n.

Solution. From (9) we get

$$(x^n, x^n)\int_{-\infty}^{\infty} x^{n+m} e^{-\frac{x^2}{2}}\,dx = \begin{cases} \sqrt{2\pi}(n+m-1)!! & \text{if } n+m \text{ is even} \\ 0 & \text{if } n+m \text{ is odd,} \end{cases}$$

We set $H_0(x) = 1$, and we compute H_1 using Gram-Schmidt:

$$H_1(x) = x - \frac{(x,1)}{(1,1)} = x.$$

We can use the recurrence relation (1) to determine H_2 and H_3. First, we compute

$$A_1 = \frac{(H_1, xH_1)}{(H_1,H_1)} = \frac{\overbrace{(x,x^2)}^{=0}}{(x,x)} = 0$$

$$B_1 = \frac{(H_1, xH_0)}{(H_0,H_0)} = \frac{(x,x)}{(1,1)} = \frac{\sqrt{2\pi}}{\sqrt{2\pi}} = 1.$$

This gives

$$H_2(x) = xH^1(x) - H_0(x) = x^2 - 1.$$

Continuing, we compute

$$A_2 = \frac{(H_2, xH_2)}{(H_2,H_2)} = \frac{\overbrace{(x^2, x^3 - x)}^{=0}}{(x^2-1, x^2-1)} = 0$$

$$B_2 = \frac{(H_2, xH_1)}{(H_1,H_1)} = \frac{(x^2-1, x^2)}{(x,x)} = \frac{3\sqrt{2\pi} - \sqrt{2\pi}}{\sqrt{2\pi}} = 2,$$

[3] If X is a random variable with a standard normal distribution, then equation (8) is the covariance of $p(X)$ and $q(x)$. More precisely,

$$\mathrm{Cov}\,\big(p(X), q(X)\big) = \mathbb{E}\big[p(X)q(X)\big] = \int_{\mathbb{R}} p(x)q(x)e^{-\frac{x^2}{2}}\,dx.$$

[4] The Schrödinger equation for the stationary quantum harmonic oscillator is the eigenvalue problem

$$-(D^2 - x^2)\psi(x) = E\psi(x)$$

where D is the derivative operator.

and we get

$$H_3(x) = xH^2(x) - 2H_1(x) = x(x^2-1) - 2x = x^3 - 3x.$$

In summary, the first four Hermite polynomials are

$$H_0(x) = 1, \qquad H_1(x) = x, \qquad H_2(x) = x^2 - 1, \qquad H_3(x) = x^3 - 3x.$$

Exercise 7. Compute the orthogonal projection (with respect to the inner product (8)) of $q(x) = x^4$ onto $\mathcal{P}_3$.

Solution. In general, if $\{\phi_0, \dots, \phi_n\}$ are an orthogonal basis for V, then the orthogonal projection Pf of f on to V is

$$\begin{aligned} Pf &= \left(\frac{(\phi_0, f)}{(\phi_0, \phi_0)}\right)\phi_0 + \left(\frac{(\phi_1, f)}{(\phi_1, \phi_1)}\right)\phi_1 + \cdots + \left(\frac{(\phi_n, f)}{(\phi_n, \phi_n)}\right)\phi_n. \\ &= P_{\phi_0} f + P_{\phi_1} f + \cdots + P_{\phi_n} f, \end{aligned}$$

where $P_{\phi_i} f$ is orthogonal projection on f on span ϕ_i.

Since we have already computed the first four Hermite polygonals, we can compute the projection Pq directly.

$$\begin{aligned} P_{H_0} q(x) &= \left(\frac{(H_0, q)}{(H_0, H_0)}\right) H_0 = \left(\frac{(1, x^4)}{(1,1)}\right) = \frac{3\sqrt{2\pi}}{\sqrt{2\pi}} = 3 \\ P_{H_1} q(x) &= \left(\frac{(H_1, q)}{(H_1, H_1)}\right) H_1 = \left(\frac{(x, x^4)}{(x,x)}\right) x = 0 \\ P_{H_2} q(x) &= \left(\frac{(H_2, q)}{(H_2, H_2)}\right) H_3 = \left(\frac{(x^2-1, x^4)}{(x^2-1, x^2-1)}\right)(x^2-1) \\ &= \frac{15\sqrt{2\pi} - 3\sqrt{2\pi}}{3\sqrt{2\pi} - 2\sqrt{2\pi} + \sqrt{2\pi}}(x^2-1) = 6x^2 - 1) \\ P_{H_3} q(x) &= \left(\frac{(H_3, q)}{(H_3, H_3)}\right) H_1 = \left(\frac{(x^3-3x, x^4)}{(x^3-3x, x^3-3x)}\right) = 0 \end{aligned}$$

This gives us

$$Pq = P_{H_0} q + P_{H_1} q + \cdots + P_{H_n} q = 3 + 0 + (6x^2 - 6) + 0 = 6x^2 - 3.$$

Note that we could now compute $H_4 = x^4 - Pq = x^4 - 6x^2 + 3$.

Laguerre polynomials

The Laguerre polynomials are orthogonal in the inner product

$$(p, q) = \int_0^\infty p(x) q(x) e^{-x}\, dx. \tag{10}$$

Laguerre polynomials occurs in the radial part of the eigenstates of the hydrogen atom, among other uses.

Exercise 8. Compute the orthogonal (with respect to the inner product (11)) projection of $f(x) = x^3$ onto $\mathcal{P}_2$. Do this without computing (or looking up) the Hermite polynomials.

Hint: Compute the integrals $\int_0^\infty x^n e^{-x}\, dx$ using integration by parts.

Solution. First note that

$$\int_0^\infty x^n e^{-x}\, dx = n \int_0^\infty x^{n-1} e^{-x}\, dx = \cdots = n! \int_0^\infty e^{-x}\, dx = n!. \tag{11}$$

From which we obtain

$$(x^n, x^m) = \int_0^\infty x^n x^m e^{-x}\, dx = (n+m)!.$$

The orthogonal projection $p = Pf$ of f onto P_2 is the solution of

$$(p, q) = (f, q) \quad \forall q \in \mathcal{P}_2.$$

Since $\{p_0, p_1, p_2\}$, wth $p_k(x) = x^k$, is a basis for $\mathcal{P}_2$, this means

$$(p, p_i) = (f, p_i) \quad \text{for } i = 0, 1, 2.$$

Since $p \in \mathcal{P}_2$, we have $p = \xi_0 p_0 + \xi_1 p_1 + \xi_2 p_2$. Inserting this we obtain the linear system

$$\sum_{j=0}^{2} \underbrace{(p_j, p_i)}_{=A_{ij}} \xi_j = \underbrace{(f, p_i)}_{=b_i} \quad \text{for } i = 0, 1, 2,$$

which in matrix form we write $\mathbf{A}\boldsymbol{\xi} = \boldsymbol{b}$. Since $a_{ij} = (p_j, p_i) = (x^i, x^j) = (i+j)!$ (note that we start indexing from 0 here) and similarly $b_i = (3+i)!$, this system reads

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 6 \\ 2 & 6 & 24 \end{bmatrix} \begin{bmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \\ 120 \end{bmatrix}$$

The system can be solved with Gauss elimination.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 1 & 2 & 6 & 24 \\ 2 & 6 & 24 & 120 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 1 & 4 & 18 \\ 0 & 4 & 20 & 108 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 1 & 4 & 18 \\ 0 & 0 & 4 & 36 \end{array}\right]$$

The solution is $\xi_2 = 9$, $\xi_1 = 18 - 4\xi_2 = -18$ and $\xi_0 = 6 - \xi_1 - 2\xi_2 = 6$, hence the projection $p = Pf$ is

$$Pf(x) = \xi_0 p_0(x) + \xi_1 + p_1(x) + \xi_2 p_2(x) = 9x^2 - 18x + 6.$$

Exercise 9. Compute the first four Laguerre polynomials L_0, L_1, L_2 and L_3.

Solution. We can set $L_0(x) = 1$, and $L_n = x^n - P_{\mathcal{P}_{n-1}} x^n$ for $n = 1, 2, \ldots$, where $P_{\mathcal{P}_{n-1}} x^n$ is the orthogonal projection of x^n onto $\mathcal{P}_{n-1}$. In particular, from the previous exercise we get

$$L_3(x) = x^3 - P_{\mathcal{P}_2} x^3 x^3 - (9x^2 - 18x + 6) = x^3 - 9x^2 + 18x - 6.$$

We can sompute L_2 the same way. That is, we compute $P_{\mathcal{P}_1} x^2$ by solving a linear system as in the previous exercise:

$$\left[\begin{array}{cc|c} 1 & 1 & 6 \\ 1 & 2 & 24 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & 1 & 18 \end{array}\right]$$

so we get $P_{\mathcal{P}_1} x^2 = 18x - 12$, and

$$L_2(x) = x^2 - P_{\mathcal{P}_1} x^2 = x^2 - (18x - 12) = x^2 - 18x + 12.$$

The same way $L_1(x) = x - 6$. In summary, the first four Laguerre polynomials are

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= x - 6 \\ L_2(x) &= x^2 - 8x + 6 \\ L_3(x) &= x^3 - 9x^2 + 18x - 6. \end{aligned}$$

Chebyshev polynomials

The Chebyshev polynomials are orthogonal in the inner product

$$(p,q) = \int_{-1}^{1} \frac{p(x)q(x)}{\sqrt{1-x^2}}\,dx. \tag{12}$$

Chebyshev polynomials are important in numerical analysis. In particular, the chebyshev polynomials can be used to determine optimal[5] nodes for polynomial interpolation.

Exercise 10. Suppose that we have have polynomials $\{T_0, T_1, T_2, \ldots\}$ such that

$$T_n(\cos\theta) = \cos n\theta \quad \text{for } n = 0, 1, 2, \ldots. \tag{13}$$

Show that $(T_n, T_m) = 0$ when $n \neq m$.
Hint: Consider a change of variables $x = \cos\theta$, $dx/d\theta = \sqrt{1-x^2}$.

Solution. With the change of variables $x = \cos\theta$ (hence $dx/d\theta = \sqrt{1-x^2}$), we can write the inner product (12) as

$$(p,q) = \int_{-\pi}^{\pi} p(\cos\theta)q(\cos\theta)\,d\theta.$$

Assuming that $T_n(\cos\theta) = \cos n\theta$, we get

$$(T_n, T_m) = \int_{-\pi}^{\pi} T_n(\cos\theta)T_m(\cos\theta)\,d\theta. = \int_{-\pi}^{\pi} \cos(n\theta)\cos(m\theta)\,d\theta. \tag{$*$}$$

Recall the trigonometric identity $\cos(a+b) = \cos a \cos b - \sin a \sin b$. Using this identity, we get

$$\begin{aligned}\cos((n+m)\theta) &= \cos(n\theta)\cos(m\theta) - \sin(n\theta)\sin(m\theta)\\ \cos((n-m)\theta) &= \cos(n\theta)\cos(m\theta) + \sin(n\theta)\sin(m\theta).\end{aligned}$$

Adding the last two identities together, we have

$$\cos(n\theta)\cos(m\theta) = \frac{1}{2}\cos((n+m)\theta) + \frac{1}{2}\cos((n-m)\theta)$$

which we can insert into ($*$). Assuming that $n \neq m$ and $n, m > 0$, we then get

$$\begin{aligned}(T_n, T_m) &= \frac{1}{2}\int_{-\pi}^{\pi} \cos((n+m)\theta)\,d\theta + \frac{1}{2}\int_{-\pi}^{\pi} \cos((n-m)\theta)\,d\theta\\ &= \frac{1}{2(n+m)}\sin((n+m)\theta)\Big|_{-\pi}^{\pi} + \frac{1}{2(n-m)}\sin((n-m)\theta)\Big|_{-\pi}^{\pi} = 0\end{aligned}$$

since $\sin k\theta = 0$ for any integer k.

Exercise 11. Show the first two polynomials in $\{T_0, T_1, T_2, \ldots\}$ are

$$\begin{aligned}T_0(x) &= 1\\ T_1(x) &= x.\end{aligned}$$

Use trigonometric identities to show the recurrence formula

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad \text{for } n = 1, 2, \ldots. \tag{14}$$

[5] Optimal in the sense of minising the interpolation error in max-norm. The Chebyshev interpolation nodes also minimise the oscillations of the interpolant.

Solution. With T_0 and T_1 as above, we have

$$T_0(\cos\theta) = \cos(0\theta)$$
$$T_1(\cos\theta) = \cos(1\theta),$$

so $T_n(\cos\theta) = \cos n\theta$ for $n = 0$ and $n = 1$. The recurrence formula can be derived from the trigonometric identity $\cos(a + b) = \cos a \cos b - \sin a \sin b$. Specifically, we get

$$\cos((n+1)\theta) = \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta)$$
$$\cos((n-1)\theta) = \cos(n\theta)\cos(\theta) + \sin(n\theta)\sin(\theta),$$

and adding these two identies to cancel the sine terms gives us

$$\cos((n+1)\theta) = 2\cos(n\theta)\cos(\theta) - \cos((n-1)\theta).$$

Assuming T_k is a polynomial of degree k so that $\cos(k\theta) = T_k(\cos\theta)$ for $k = 0, 1, \ldots, n$, we can write this as

$$\cos((n+1)\theta) = 2T_1(\cos\theta)T_n(\cos\theta) - T_{n-1}(\cos\theta). \qquad (*)$$

Here, the right hand side is a polynomial of degree $n + 1$ in $\cos\theta$. Hence, we can define the polynomial T_{n+1} of degree $n+1$ using $(*)$. Since $T_1(x) = x$, we have

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Exercise 12. Use the recurrence relation (14) to determine the first four Chebyshev polynomials T_0, T_1, T_2 and T_3.

Solution. We already have $T_0(x) = 1$ and $T_1(x) = x$, and from (14) we get

$$T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1$$
$$T_3(x) = 2xT_2(x) - T_1(x) = 2x(2x^2 - 1) - x = 4x^3 - 3x.$$

Note that the Chebyshev polynomials T_n are actually normalised in max-norm over the interval $[-1, 1]$ rathen being written in monomial form.