Additional Material (not covered in seminars)

(10.1) Let $f : [-\pi, \pi] \to \mathbb{R}$ be a 2π periodic function of the form

$$f(x) = \pi^2 - x^2 \text{ for } -\pi \leq x \leq \pi.$$

Show that its Fourier series representation is given by

$$f(x) = \frac{2\pi^2}{3} + 4(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x \ldots).$$

Recall that the Fourier series is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$$

with $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$ and $b_n = \int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$.

Solution: The parabola is even, so all $b_k = 0$. The Fourier coefficients a_k can be computed as

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (\pi^2 - x^2)dx = \frac{2}{\pi}\left(\pi^2 x - \frac{x^3}{3}\right) = \frac{4\pi^2}{3}$$

$$\begin{aligned}
a_k &= \frac{2}{\pi}\int_0^{\pi} (\pi^2 - x^2)\cos kx dx \\
&= \frac{2}{\pi}\left((\pi^2 - x^2)\frac{\sin kx}{k}|_0^{\pi} - \int_0^{\pi}(-2x\sin\frac{kx}{k})dx\right) \\
&= \frac{2}{\pi}\frac{2}{k}\int_0^{\pi} x\sin(kx)dx \\
&= \frac{4}{k\pi}\left(-x\frac{\cos kx}{k}|_0^{\pi} - \int_0^{\pi}(-\frac{\cos kx}{k}dx\right) \\
&= \frac{4}{k\pi^2}\left((-\pi\cos k\pi) + \frac{\sin kx}{k^2}|_0^{\pi}\right) \\
&= -\frac{4}{k^2}\cos k\pi \\
&= \begin{cases} -\frac{4}{k^2} \text{ if } k \text{ is even} \\ \frac{4}{k^2} \text{ if } k \text{ is odd} \end{cases}
\end{aligned}$$

(10.2) *Fourier series of even and odd functions:* The Fourier series contains a sum of terms while the integral formulae for the Fourier coefficients a_k and b_k contain products of the type $f(x)\cos nx$ and $f(x)\sin nx$. Let $q(x) = g(x)h(x)$, show that

Let $a \in \mathbb{R}$. Use that $\int_{-a}^{a} q(x)\,\mathrm{d}x = 0$ for odd functions and $\int_{-a}^{a} q(x)dx = 2\int_0^a q(x)\,\mathrm{d}x$ for even functions to show that

g		h		q
even	$\times$	even	$=$	even
even	$\times$	odd	$=$	odd
odd	$\times$	odd	$=$	even

- $a_k = 0$ if f is odd
- $b_k = 0$ if f is even.

Solution: We see that

$$q(-x) = g(-x)h(-x) = g(x)h(x) = q(x)$$
$$q(-x) = g(-x)h(-x) = -g(x)h(x) = -q(x)$$
$$q(-x) = g(-x)h(-x) = -g(x)(-h(x)) = q(x)$$

The sine function is odd, the cosine is even. So if

- f is odd, then $a_k = \int f(x)\cos kx dx = \int (odd) \times (even) = \int (odd) dx = 0$.
- f is even, then $b_k = \int f(x)\sin kx dx = \int (even) \times (odd) = \int (odd) dx = 0$.

(10.3) Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}$. Show that $\boldsymbol{A}$ has rank 2 and that its singular value decomposition is given by

$$\boldsymbol{U} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}, \boldsymbol{D} = \begin{pmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \text{ and } \boldsymbol{V} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Compute its rank 1 approximation.

Solution: The matrix $\boldsymbol{A}$ has rank, because the two rows are linear independent. We first calculate

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{pmatrix} 25 & 20 \\ 20 & 25 \end{pmatrix} \qquad \boldsymbol{A}\boldsymbol{A}^T = \begin{pmatrix} 9 & 12 \\ 12 & 41 \end{pmatrix}$$

Its characteristic polynomials are

$$(\lambda - 25)^2 - 400 = 0 \text{ and } (\mu - 9)(\mu - 41) - 144 = 0.$$

and we see that the eigenvalues $\sigma_1^2 = 45$ and $\sigma_2^2 = 5$. The eigenvectors of $\boldsymbol{A}^T\boldsymbol{A}$ are

$$\begin{pmatrix} 25 & 20 \\ 20 & 25 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 45 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 25 & 20 \\ 20 & 25 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 5 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

So the right singular vectors are $\boldsymbol{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

To calculate the left singular vectors we calculate $\boldsymbol{A}\boldsymbol{v}_1 = \sqrt{45} \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\boldsymbol{A}\boldsymbol{v}_2 = \sqrt{5} \frac{1}{\sqrt{10}} \begin{pmatrix} -3 \\ 1 \end{pmatrix}$