Week 7 Tutorial 7

(7.1) *Least square solution of a linear system:* We wish to solve a system of linear equations

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ with rk $\boldsymbol{A} = m$, $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{b} \in \mathbb{R}^m$. In many applications

- it is often not possible to calculate the inverse of $\boldsymbol{A}$,
- or we have more equations than unknowns, that is $m > n$, and therefore can not expect a unique solution.

In the later case one can calculate the least square solution, that is the vector $\boldsymbol{x}$ that minimises the squared Euclidean: distance

$$\min_{\boldsymbol{x}} \frac{1}{2} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|^2$$

(we multiply with $\frac{1}{2}$ to make the gradient nicer). Show that

1. the gradient of $f = \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|^2$ is given by
$$\nabla_{\boldsymbol{x}} f = \boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})$$
2. Calculate the critical point
$$\boldsymbol{x}^* = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{b}.$$
Is this point unique? Explain why $\boldsymbol{A}^T\boldsymbol{A}$ is invertible.

(7.2) *Norms:* Let $f : [0, 1] \to \mathbb{R}$ be continuous. Show that $\|f\|_\infty = \sup_x |f(x)|$ is indeed a norm.

(7.3) *Interpolation I:* Given $n + 1$ data points (x_i, y_i) with $x_i, y_i \in \mathbb{R}$ we wish to find a polynomial, which fits these points exactly.
One can for example consider a polynomial of degree n (then we have the same number of equations and unknowns):

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n.$$

Its coefficients are determined by setting $y_i = p(x_i)$ for $i = 1, \dots n + 1$.

1. Show that the coefficients $\mathbf{a} = (a_0, a_1, \dots a_n)$ can be found y solving the system $\boldsymbol{V}\boldsymbol{a} = \boldsymbol{y}$, where $\boldsymbol{V}$ is the so called Vandermonde matrix
$$\boldsymbol{V} = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & \dots & x_1^n \\ 1 & x_1 & x_2^2 & x_2^3 & \dots & x_2^n \\ \vdots & \vdots & \vdots & \vdots & & \\ 1 & x_n & x_n^2 & x_n^3 & \dots & x_n^n \end{pmatrix}$$
and $\boldsymbol{y} = (y_1, \dots y_n)$.

2. The determinant of $\boldsymbol{V}$ is given by $\det \boldsymbol{V} = \Pi_{0\leq i,j\leq n}(x_j - x_i)$. Why can this become problematic?

(7.4) *Interpolation II* Lagrangian interpolation is an attractive alternative to the Vandermonde interpolation, because it does not involve solving a system to find the interpolating polynomial. It is based on writing the polynomial in a different way, for example instead of writing $y = x^2 - 3x + 2$ we can also write $y = (x-2)(x-3)$.
Given a data set $(x_i, y_i) \in \mathbb{R}^2$, with $0 \leq i \leq n$. Then the Lagrange basis polynomials are given by

$$\ell_i(x) = \frac{\Pi_{k\neq i}(x - x_k)}{\Pi_{k\neq i}(x_i - x_k)} \quad i = 0, \dots n$$

- Show that

$$\ell_i(x_j) = \begin{cases} 1 & j = i \\ 0 & j \neq i, \end{cases}$$

  and that the Lagrange interpolating polynomial through those data points

$$p_n(x) = \sum_{k=0}^{n} y_k \ell_k(x)$$

  satisfies $p_n(x_i) = y_i$ for every $i = 0, \dots n$.

- Consider the function $f(x) = \frac{1}{x}$. Given the function values at $x_0 = 2$, $x_1 = 2.5$ and $x_2 = 4$, that is given the data pairs $(2, \frac{1}{2})$, $(\frac{5}{2}, \frac{2}{5})$ and $(4, \frac{1}{4})$, calculate the corresponding interpolating Lagrange basis polynomials $\{\ell_0, \ell_1, \ell_2\}$ and the interpolating Lagrange polynomial

$$p_2(x) = \sum_{k=0}^{2} y_k \ell_k(x).$$

  Sketch the solution and the function f.