# Week 9 Tutorial 9

**(9.1)** *Inner products on vector spaces:* We recall that an inner product is a mapping from $V \times V$, where $V$ is a vector space, into $\mathbb{R}$ satisfying

- Linearity in the first argument: $\langle \mathbf{x}_1 + \mathbf{x}_2, \mathbf{y} \rangle \leq \langle \mathbf{x}_1, \mathbf{y} \rangle + \langle \mathbf{x}_2, \mathbf{y} \rangle$
- Symmetry: $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.
- Positivity: $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ for all $\mathbf{x} \in V$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

Show that $\langle f, g \rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$ defines an inner product on the space of real-coefficient polynomial functions.

*Solution:* The first property follows from the linearity of integrals, the second one is obvious since $f(x)g(x) = g(x)f(x)$. For the last one we assume that $\langle f, f \rangle = 0$, where $f$ is a polynomial. Then $\int_{-1}^1 f(x)^2 dx = 0$. Since $f$ is a polynomial, $f$ is continuous. Therefore $f(x) \equiv 0$.

**(9.2)** *Gram Schmidt* Determine the first four Lagrange polynomials using Gram Schmidt to orthogonalise the power basis $\{1, x, x^2, x^3\}$.
We will use that

$$\int_{-1}^1 x^n dx = \begin{cases} \frac{2}{n+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Hence the inner product between basis function is given by

$$(p_n, p_m) = \int_{-1}^1 x^n x^m dx = \begin{cases} \frac{2}{n+m+1} & \text{if } n+m \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

- Set the first Legendre polynomial to $L_0 = p_0(x)$ and compute the next using that

$$L_1(x) = p_1(x) - \frac{\langle L_0, p_1 \rangle}{\langle L_0, L_0 \rangle} L_0(x)$$

- Continue up to order $4$ to obtain the sequence

$$L_0(x) = 1, L_1(x) = x, L_2(x) = x^2 - \frac{1}{3}, L_3(x) = x^3 - \frac{3}{5}x.$$

- Let $x_0, \ldots x_n$ be the roots of the Legendre polynomial of degree $n+1$. Consider the quadrature rule

$$\int_{-1}^1 f(x)dx \approx \sum_{i=0}^n f(x_i) w_i(x)$$

where the weights $w_i$ are the integrals of the Lagrange polynomials

$$w_i = \int_{-1}^{1} L_i(x)dx \text{ with } L_i(x) = \Pi_{j\neq i}\left(\frac{x-x_j}{x_i-x_j}\right).$$

Show that this quadrature rule is exact for polynomials $p$ up to order $2n+1$, that is show that

$$\int_{-1}^{1} p(x)dx = \sum_{i=0}^{n} f(x_i)w_i.$$

*Hint:* Use polynomial division to write $p(x) = q(x)P_{n+1}(x) + r(x)$ where $p$ and $q$ are polynomials of degree less than or equal to $n$.

*Solution:* Since $\langle L_0, p_1\rangle = \langle 1, x\rangle = 0$ we calculate

$$L_1(x) = p_1(x) - 0P_0(x) = x.$$

We continue and compute $\langle L_1, p_2\rangle = 0$, $\langle L_0, p_2\rangle = \langle 1, x\rangle = \frac{2}{3}$ and $\langle L_0, L_0\rangle = 2$ to get the third polynomial

$$\begin{aligned} L_2(x) &= p_2(x) - \frac{\langle L_1, p_2\rangle}{\langle L_1, L_1\rangle}L_1(x) - \frac{\langle L_0, p_2\rangle}{\langle L_0, L_0\rangle}L_0(x) \\ &= x^2 - \frac{1}{3}. \end{aligned}$$

The last set of coefficients is given by

$$\langle L_2, p_3\rangle = \langle x^2 - \frac{1}{3}, x^3\rangle = 0$$

and $\langle L_1, p_3\rangle = \langle x, x^3\rangle = \frac{2}{5}$, $\langle L_1, L_1\rangle = \langle x, x\rangle = \frac{2}{3}$ and $\langle L_0, x^3\rangle = \langle 1, x^3\rangle = 0$. Then

$$L_3(x) = p_3(x) - \frac{\langle L_2, p_3\rangle}{\langle L_2, L_2\rangle}L_2(x) - \frac{\langle L_1, p_3\rangle}{\langle L_1, L_1\rangle}L_1(x) - \frac{\langle L_0, p_3\rangle}{\langle L_0, L_0\rangle}L_0(x) = x^3 - \frac{3}{5}x.$$

**(9.3)** *Laguerre polynomials* The Laguerre polynomials are orthogonal on $(0, \infty)$ wrt the weight function $w(x) = e^{-x}$.

- Construct the first four Laguerre polynomials, starting with the lowest order $L_0(x) = 1$.
- Show that they satisfy the recurrance relation

$$L_{k+1}(x) = \frac{(2k+1-x)L_k(x) - kL_{k-1}(x)}{k+1}$$

for any $k \geq 0$.

*Solution:* We set $\varphi_1(x) = x - c_0\varphi_0$, such that $\langle e^{-x}\varphi_0, \varphi_1\rangle = \int_0^\infty xe^{-x}dx = -(x+1)e^{-x}|_{x=0}^\infty = 1$, $\langle e^{-x}\varphi_0, \varphi_0\rangle = \int_0^\infty e^{-x}dx = 1$. Hence

$$L_1(x) = x - \frac{\langle e^{-x}x, 1\rangle}{\langle e^{-x}1, 1\rangle} \cdot 1 = x - 1.$$

For the second we calculate

$$L_2(x) = x^2 - \frac{\langle e^{-x}x^2, 1\rangle}{\langle e^{-x}1, 1\rangle} \cdot 1 - \frac{\langle e^{-x}x^2, x-1\rangle}{\langle e^{-x}(x-1), x-1\rangle} \cdot (x-1)$$

We use that $\int (x-1)^2 e^{-x}dx = -e^{-x}(x^2+1)$ and $\int x^2(x-1)e^{-x}dx = -e^{-x}(x^3 + 2x^2 + 4x + 4)$ to obtain

$$L_2(x) = x^2 - 2 \cdot 1 - 4 \cdot (x-1) = x^2 - 4x - 2$$

Then the not scaled polynomials are given by

$$L_1(x) = x - 1,\ L_2(x) = x^2 - 4x + 2,\ L_3(x) = -x^3 + 9x^2 - 18x + 6.$$

If you scale them (divide by the absolute weighted norm) we obtain

$$L_1(x) = x - 1,\ L_2(x) = \frac{1}{2}(x^2 - 4x + 2),\ L_3(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6).$$

For the recurrance relation we check

**(9.4)** *Pade series* Show that $r_{2,1}$ for the exponential function is given by

$$r_{2,1}(x) = \frac{1 + \frac{2}{3}x + \frac{1}{6}x^2}{1 - \frac{1}{3}x}$$

Plot the exponential and the Pade approximation.

Solution: We have that

$$\{a_0, a_1, a_2, a_3\} = \{1, 1, \frac{1}{2}, \frac{1}{6}\}$$

The total order of the polynomial is $N = 3$, therefore $p_3 = 0$ and $q_1 = q_2 = 0$. We have to solve the following system

$$\begin{aligned} a_0 - p_0 &= 0 \\ a_0q_1 + a_1 - p_1 &= 0 \\ a_0q_2 + a_1q_1 + a_2 - p_2 &= 0 \\ a_0q_3 + a_1q_2 + a_2q_3 + a_3 - p_3 &= 0. \end{aligned}$$

The last equation reduces to $\frac{1}{2}q_1 + \frac{1}{6} = 0$, therefore $q_1 = -\frac{1}{3}$. Plugging this into the third equation implies that $p_2 = \frac{1}{6}$, and into the second one implies that $p_1 = \frac{2}{3}$.