Problem Sheet 3

Please turn in a solution by Monday on Week 7. The number of points attainable for each question is provided.

(3.4) Consider the convex problem

$$\min_{x_1, x_2 > 0} e^{-x_1}$$
$$\text{s.t. } \frac{x_1^2}{x_2} \leq 0$$

on the domain $D = \{(x_1, x_2)\colon x_2 > 0\}$. Since $x_1 = 0$ to satisfy the inequality constraint, we have that $p^* = \min_{x_1, x_2 > 0} e^{-x_1} = 1$.
Show that the dual function is given by **[2]**

$$g(\lambda) = \begin{cases} 0 & \lambda \geq 0 \\ -\infty & \lambda < 0. \end{cases}$$

Compute the optimal value of the dual problem and explain why strong duality does not hold. **[2]**

(3.5) Consider the constrained optimisation problem

$$\min_{x_1, x_2} e^{x_1 - x_2} \tag{3.1a}$$
$$\text{s.t. } e^{x_1} + e^{x_2} \leq 20 \tag{3.1b}$$
$$x_1 \geq 0 \tag{3.1c}$$

- State the Lagrange function and the KKT conditions. **[3]**
- Explain why no KKT points can exist in the following situations: **[3]**
  - Both constraints, that is (3.1b) and (3.1c), are inactive (and therefore both Lagrange multipliers are zero).
  - Constraint (3.1b) is active, while (3.1c) is inactive.
  - Constraint (3.1c) is active, while (3.1b) is inactive.

  Note that we say that an inequality constraint $f(x) \leq 0$ is active if $f(x) = 0$, and inactive if $f(x) < 0$ (and therefore forcing the respective Lagrange multiplier to be equal to zero).
- Consider the case when both constraints are active (hence both Langrange multipliers are strictly positive) and show that $(0, \ln(19))$ is a KKT point. **[3]**
- Assume that $(0, \ln(19))$ is a minimisier - is it a global or a local one? **[1]**.

(3.6) Consider the quadratic optimisation problem

$$\begin{aligned} &\min_{(x_1,x_2)} x_1^2 + x_2^2 \\ \text{s.t. } &(x_1-1)^2 + (x_2-1)^2 \leq 1 \\ &(x_1-1)^2 + (x_2+1)^2 \leq 1. \end{aligned}$$

- Sketch the contour lines of the objective function and the constrains and determine the feasible set graphically. **[3]**
- State the KKT conditions for the problem. Does the system have an optimal solution? **[3]**

(3.7) Consider the problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) \tag{3.2a}$$

$$\text{s.t.}\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{3.2b}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex and differentiable, $\mathbf{A} \in \mathbb{R}^{m\times n}$ with rank$\mathbf{A} = m$ and $\mathbf{b} \in \mathbb{R}^m$.
In the quadratic penalty method one considers the auxiliary function

$$\varphi(\mathbf{x}) = f(\mathbf{x}) + \alpha\|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2,$$

where $\alpha > 0$ is a parameter. This function consists of an objective plus the penalty term $\alpha\|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2$. The idea is that minimisers $\tilde{\mathbf{x}}$ to φ should be an approximate solution to the original problem.
Assume that $\tilde{\mathbf{x}}$ is a minimiser of φ.

- Show how to find, from $\tilde{\mathbf{x}}$ a dual feasible problem to (3.2). **[3]**
- Find the corresponding lower bound on the optimal value of (3.2). **[2]**