Problem Sheet 4

Please turn in a solution by Monday on Week 9. The number of points attainable for each question is provided.

(4.4) *Minimax polynomials:* Find the linear minimax approximation to $f(x) = \sinh(x)$ on $[0, 1]$. Sketch the solution. **[5]**

(4.5) *Minimax polynomial for even functions:* Let $f \in C[-1, 1]$ be even, that is $f(-x) = f(x)$ for all $x \in [-1, 1]$.

- Use the Oscillation Theorem to show that the minimax polynomial p_n^* is even for any $n \geq 0$. **[3]**
- Show that $p_{2n}^* = p_{2n+1}^*$ for any $n \geq 0$. **[1]**
- Find a linear minimax polynomial for $f(x) = |x|$ on $[-1, 1]$. **[4]**

(4.6) *Chebyshev polynomials of first kind:* Consider the Chebyshev differential equation of first kind

$$(1 - x^2)y'' - xy' + n^2 y = 0, \qquad n = 0, 1, 2, 3 \ldots.$$

Use the substitution $x = \cos\theta$ and show that the transformed ODE has solutions $y_1 = \cos(n\theta)$ and $y_2 = \sin(n\theta)$. The former is known as Chebyshev polynomials. **[4]**

(4.7) *Best approximation in the 2-norm:* We wish to find the best approximating linear function, that minimised the 2-norm. In particular, we want to minimise

$$\|f - p_1\|_2^2 = \int_a^b (f(x) - c_0 - c_1 x)^2 \mathrm{d}x$$

among all linear polynomials $p_1 = c_0 + c_1 x$, with $c_0, c_1 \in \mathbb{R}$.

- Show that the error function $E(c_0, c_1) = \|f(x) - c_0 - c_1 x\|^2$ can be written as **[1]**
$$E(c_0, c_1) = \int_a^b f(x)^2 \,\mathrm{d}x - 2c_0 \int_a^b f(x) \,\mathrm{d}x - 2c_1 \int_a^b x f(x) \,\mathrm{d}x \\ + c_0^2(b - a) + c_0 c_1 (b^2 - a^2) + \frac{1}{3} c_1^2 (b^3 - a^3).$$
- Show that the optimal polynomial can be found by solving the following linear system: **[3]**
$$\begin{pmatrix} 2(b-a) & b^2 - a^2 \\ b^2 - a^2 & \frac{2}{3}(b^3 - a^3) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2\int_a^b f(x)\mathrm{d}x \\ 2\int_a^b x f(x)\mathrm{d}x. \end{pmatrix}$$
- Find the best approximation wrt to the 2-norm for $f(x) = e^x$ for $x \in [0, 1]$. Plot the approximate solution $p_1(x)$ as well as $f(x)$. **[4]**