Solutions to Problem Sheet 3

(3.1) In this problem we see that strong duality does not hold if Slater's condition is violated.
The Lagrangian is given by **[1]**

$$\mathcal{L}(x_1, x_2, \lambda) = e^{-x_1} + \lambda \frac{x_1^2}{x_2}$$

and therefore the dual function is given by

$$g(\lambda) = \inf_{x_1, x_2 > 0} \mathcal{L}(x, \lambda)$$

is given by **[1]**

$$g(\lambda) = \begin{cases} 0 & \lambda \geq 0 \\ -\infty & \lambda < 0. \end{cases}$$

so the dual problem is

$$\max 0 \quad \text{s.t. } \lambda \geq 0.$$

This problem has the obvious optimal value $d^* = 0$, and the duality gap is $p^* - d^* = 1$. Hence strong duality does not hold. **[1]**
This is the case sind the inequality condition is not satisfied (since $x_1 = 0$ for any feasible pair (x_1, x_2)). **[1]**

(3.2) We note that the optimisation problem is convex, since the objective and all constraints are convex. Hence every local minimiser must be a global minimiser.
The Lagrangian is given by **[1]**

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = e^{x_1 - x_2} + \lambda_1(e^{x_1} + e^{x_2} - 20) - \lambda_2 x_1.$$

The KKT conditions are **[2]**

$$e^{x_1} + e^{x_2} - 20 \leq 0 \tag{3.1a}$$
$$-x_1 \leq 0 \tag{3.1b}$$
$$\lambda_1(e^{x_1} + e^{x_2} - 20) = 0 \tag{3.1c}$$
$$\lambda_2 x_1 = 0 \tag{3.1d}$$
$$e^{x_1 - x_2} + \lambda_1 e^{x_1} - \lambda_2 = 0 \tag{3.1e}$$
$$-e^{x_1 - x_2} + \lambda_1 e^{x_2} = 0. \tag{3.1f}$$

The three stated cases corresponds to (**one point of for each case**)

1. Both constraints are inactive, so $\lambda_1 = \lambda_2 = 0$. Then (3.1e) reduces to $e^{x_1 - x_2} = 0$, which has no solution

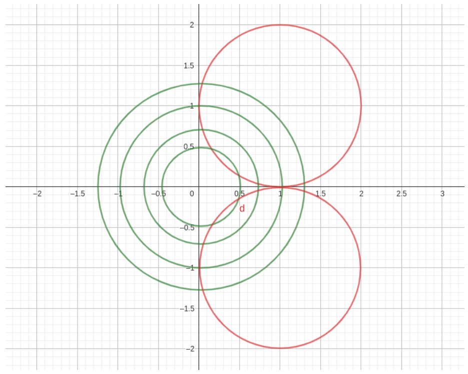

Figure 1: The feasible set of optimisation problem 3.3 corresponds to the intersection of the two red circles. The green lines are the contour lines of the objective

2. If the first constraint is inactive, so $\lambda_1 = 0$ and the second is active (implying that $\lambda_2 > 0$). The (3.1f) reduces to $e^{-x_2} = 0$ which again has no solution.
3. If the first constraint is active, so $\lambda_1 > 0$ and the first one inactive, the gradient of the Lagrangian reduces to

$$e^{x_1 - x_2} + \lambda_1 e^{x_1} = 0$$
$$-e^{x_1 - x_2} + \lambda_1 e^{x_2} = 0.$$

Adding the two equations gives $\lambda_1(e^{x_1} + e^{x_2}) = 20\lambda_1$. Therefore $\lambda_1 = 0$, but then $e^{x_1 - x_2} = 0$ which is not possible.

If both constraints are active we find that $x_1 = 0$ and $x_2 = \ln(19)$. The respective Lagrange multipliers are $(\lambda_1, \lambda_2) = \frac{1}{19^2}(1, 20)$. Hence $(0, \ln(19))$ is a KKT point. **[3]** Since we have a convex problem it is a global solution.**[1]**

(3.3) The sketch shows that the only feasible point is $(1, 0)$.
2 points for the sketch and 1 for any discussion why there is only one point in the feasible set.
The KKT conditions are**[2]**

$$\begin{aligned}
2x_1 + 2\lambda_1(x_1 - 1) + 2\lambda_2(x_1 - 1) &= 0 \\
2x_2 + 2\lambda_1(x_2 - 1) + 2\lambda_2(x_2 + 1) &= 0 \\
(x_1 - 1)^2 + (x_2 - 1)^2 &\leq 1 \\
(x_1 - 1)^2 + (x_2 + 1)^2 &\leq 1 \\
\lambda_1 &\geq 0 \\
\lambda_2 &\geq 0 \\
\lambda_1 \left((x_1 - 1)^2 + (x_2 - 1)^2 - 1\right) &= 0 \\
\lambda_2 \left((x_1 - 1)^2 + (x_2 + 1)^2 - 1)\right) &= 0.
\end{aligned}$$

In the point $(1, 0)$ the KKT conditions reduce to

$$\lambda_1 \geq 0, \quad \lambda_2 \geq 0, \, 2 = 0 \text{ and } -2\lambda_1 + 2\lambda_2 = 0.$$

which is not solvable. Therefore there is no solution. **[1]**

(3.4) If $\tilde{\mathbf{x}}$ minimises φ then **[2]**

$$\nabla f(\tilde{\mathbf{x}}) + 2\alpha \mathbf{A}^T(\mathbf{A}\tilde{\mathbf{x}} - \mathbf{b}) = \mathbf{0}$$

Therefore $\tilde{\mathbf{x}}$ is also a minimiser of

$$f(\mathbf{x}) + \mu^T(\mathbf{A}\mathbf{x} - \mathbf{b})$$

where $\mu = 2\alpha(\mathbf{A}\tilde{\mathbf{x}} - \mathbf{b})$. **[1]**
Therefore μ is the dual feasible with**[1]**

$$\begin{aligned} g(\mu) &= \inf_{\mathbf{x}}(f(\mathbf{x}) + \mu^T(\mathbf{A}\mathbf{x} - \mathbf{b}) \\ &= f(\tilde{\mathbf{x}}) + 2\alpha\|\mathbf{A}\tilde{\mathbf{x}} - \mathbf{b}\|^2. \end{aligned}$$

Therefore

$$f(\mathbf{x}) \geq f(\mathbf{x}) + 2\alpha\|\mathbf{A}\tilde{\mathbf{x}} - \mathbf{b}\|^2$$

for all $\mathbf{x}$ that satisfy $\mathbf{A}\mathbf{x} = \mathbf{b}$.**[1]**