MA398 Matrix Analysis and Algorithms: Exercise Sheet 2

1. Create a Python function **p_norm** that can calculate the p-norm and the inner product of two vectors. Hence, prove that the function **p_norm** correctly implements the p-norm definition and the function inner product correctly implements the standard inner product.

2. Implement a function that verifies whether a given square matrix is unitary. Hence, prove that the function **is_unitary** correctly implements the definition of a unitary matrix. Also, give an intuitive explanation of why this condition ensures the columns of Q are orthonormal.

3. Prove the following Theorem:

   "If $A \in \mathbb{C}^{n\times n}$ is Hermitian then there is a unitary Q and a real $\Lambda \in \mathbb{R}^{n\times n}$ such that $A = Q\Lambda Q^*$."

   Hint: Remember that a Hermitian matrix is one that is equal to its own conjugate transpose, i.e., $A = A^*$. You might want to utilize the Spectral Theorem for Hermitian matrices in your proof, which states that every Hermitian matrix can be diagonalized by a unitary matrix.

4. Implement a Python function **is_normal(A)** that checks whether a given square matrix A is normal. Recall from the lecture notes that a matrix A is normal if it satisfies $A^*A = AA^*$. The function should take as input a NumPy array A and return a Boolean value (True or False).

5. (Geometric series for matrices) Let $\|\cdot\|$ be a matrix norm on $\mathbb{C}^{n\times n}$. Assume that $\|X\| < 1$ for some $X \in \mathbb{C}^{n\times n}$. Show that,

   (a) $I - X$ is invertible with $(I - X)^{-1} = \sum_{i=0}^{\infty} X^i$,

   (b) $\|(I - X)^{-1}\| \leq (1 - \|X\|)^{-1}$.

6. (Cholesky factorisation) If $A \in \mathbb{C}^{n\times n}$ is Hermitian and positive definite then there exists a unique upper triangular matrix $R \in \mathbb{C}^{n\times n}$ with (real and) positive diagonal elements such that $A = R^*R$. (Hint: Induction on n.)

7. (Norms)

   (a) Is $\|A\|_{\max} = \max_{i,j} |a_{ij}|$ a matrix norm?

   (b) Show that $\|uv^*\|_2 = \|u\|_2\|v\|_2$ for all $u, v \in \mathbb{C}^n$. Does this also hold true if $\|\cdot\|_2$ is replaced by the Frobenius norm $\|\cdot\|_F$?

   (c) Let $p \in [1, \infty)$. Prove the following statement:

   $$\|x\|_\infty \leq \|x\|_p \leq \sqrt[p]{n}\|x\|_\infty \quad \forall x \in \mathbb{C}^n.$$