Problem Sheet 1

Please turn in a solution by Monday on Week 3. The number of points attainable for each question is provided.

(1.1) Recall that a critical point of a differentiable function f is a point at which the gradient ∇f vanishes.

(a) Determine the critical points of the function

$$f(x, y) = x^3 - 12xy + 8y^3$$

and state whether they are maxima, minima or saddle points. **(6)**

(b) Determine the critical points of the function

$$f(x, y) = \sin(x)\cos(y) \quad \text{on } [0, 2\pi) \times [0, 2\pi),$$

and state whether they are maxima, minima or saddle points. **(6)**

(1.2) Find an example of:

(a) a function $f \in C^2(\mathbb{R})$ with a strict minimizer x such that $f''(x) = 0$ (that is, the second derivative is not positive definite). **(1)**

(b) A function $f\colon \mathbb{R} \to \mathbb{R}$ with a strict minimizer x^* that is not an isolated local minimizer. **Hint:** Consider a rapidly oscillating function that has minima that are arbitrary close together, but not equal. **(4)**

(1.3) A set $S \subseteq \mathbb{R}^n$ is called *convex*, if for any $\boldsymbol{x}, \boldsymbol{y} \in S$ and $\lambda \in [0, 1]$,

$$\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y} \in S.$$

In words, for any two points in S, the line segment joining them is also in S. Which of the following sets are convex?

(a) $S = \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\|_2 = 1\}$ (the unit sphere); **(2)**

(b) $S = \{\boldsymbol{x} \in \mathbb{R}^2 : 1 \leq x_1 - x_2 < 2\}$; **(2)**

(c) $S = \{\boldsymbol{x} \in \mathbb{R}^n : |x_1| + \cdots + |x_n| \leq 1\}$; **(2)**

(d) $S = \mathcal{S}^n_+ \subset \mathbb{R}^{n \times n}$, the set of symmetric, positive semidefinite matrices. **(2)**