MA398 Matrix Analysis and Algorithms: Exercise Sheet 7

1. Consider the matrix

$$A = \begin{pmatrix} 1 & 5 \\ 2 & 3 \\ 4 & 3 \end{pmatrix},$$

perform a manual QR factorisation using the Gram-Schmidt orthonormalisation process outlined in the notes. Compute the matrices Q and R.

2. Using Python's numpy library, write a script to perform the following tasks:

   (a) Generate a random 3x2 matrix A.

   (b) Use numpy's built-in function to perform a QR factorisation of A.

   (c) Extract the matrices Q and R from the result of the QR factorisation.

   (d) Check the correctness of the factorization by verifying that $QR = A$ within some small tolerance.

   (e) Write a function to compute the Frobenius norm of a matrix (which is the square root of the sum of the absolute squares of its elements), and use this function to calculate the Frobenius norm of the difference between QR and A.

3. Consider the matrix

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 4 & 3 & 2 \end{pmatrix}$$

Perform a manual QR factorisation using the Householder reflections process. Compute the matrices Q and R. Verify the result by checking that $A = QR$.

4. Implement a Python function that performs the Jacobi method for a given system of linear equations, maximum number of iterations, and a specified tolerance for the norm of the residual vector. The function should return the solution vector and the number of iterations performed.

   Here is the function signature to get you started:

```
def jacobi_method(A, b, x0, max_iter, tol):
    # A is the input matrix
    # b is the right-hand side vector
    # x0 is the initial guess for the solution
    # max_iter is the maximum number of iterations
    # tol is the tolerance for the norm of the residual vector
    ...
```

   Test your function on a 3×3 system of equations. Experiment with different initial guesses, maximum number of iterations, and tolerance levels. Plot the norm of the residual vector as a function of the number of iterations. What do you observe about the convergence of the method?