6. 已知
$$\alpha_1=(1,2,1,0),\quad \alpha_2=(-1,1,1,1),$$
$$\beta_1=(2,-1,0,1),\quad \beta_2=(1,-1,3,7),$$
$$V_1=\mathrm{span}\{\alpha_1,\alpha_2\},\quad V_2=\mathrm{span}\{\beta_1,\beta_2\},$$
分别求 V_1+V_2 及 $V_1\cap V_2$ 的一组基.

6.

① $V_1+V_2 = L(\alpha_1^T, \alpha_2^T, \beta_1^T, \beta_2^T)$

而 $(\alpha_1^T, \alpha_2^T, \beta_1^T, \beta_2^T) = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & -1 \\ 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 7 \end{pmatrix}$

$$\xrightarrow{行} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & -3 \\ 0 & 2 & -2 & 2 \\ 0 & 1 & 1 & 7 \end{pmatrix}$$

$$\xrightarrow{行} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 3 & -5 & -3 \\ 0 & 1 & 1 & 7 \end{pmatrix}$$

$$\xrightarrow{行} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & -6 \\ 0 & 0 & 2 & 6 \end{pmatrix}$$

$$\xrightarrow{行} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{行} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ V_1+V_2 的基是 α_1 α_2 β_1

即 $(1,2,1,0)$ $(-1,1,1,1)$ $(2,-1,0,1)$

6. 已知

$$\boldsymbol{\alpha}_1=(1,2,1,0),\quad \boldsymbol{\alpha}_2=(-1,1,1,1),$$
$$\boldsymbol{\beta}_1=(2,-1,0,1),\quad \boldsymbol{\beta}_2=(1,-1,3,7),$$
$$V_1=\mathrm{span}\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2\},\quad V_2=\mathrm{span}\{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2\},$$

分别求 V_1+V_2 及 $V_1\cap V_2$ 的一组基.

② $V_1\cap V_2$ ： 任取 $y\in V_1\cap V_2$

$\therefore\ y\in V_1,\ y\in V_2$

设 $y = x_1\cdot\alpha_1 + x_2\cdot\alpha_2 = y_1\cdot\beta_1 + y_2\cdot\beta_2$

$$\therefore\ (\alpha_1^T, \alpha_2^T, -\beta_1^T, -\beta_2^T)\cdot\begin{pmatrix} x_1\\ x_2\\ y_1\\ y_2\end{pmatrix} = 0$$

$$\text{而}\ (\alpha_1^T, \alpha_2^T, -\beta_1^T, -\beta_2^T) \xrightarrow{\text{行}} \begin{pmatrix} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & -4\\ 0 & 0 & -1 & -3\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$\xrightarrow{\text{行}} \begin{pmatrix} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & -4\\ 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$\therefore\ \begin{cases} x_1 = -y_2\\ x_2 = 4y_2\\ y_1 = -3y_2\end{cases}$$

$$\therefore\ y = -y_2\alpha_1 + 4y_2\alpha_2 = y_2(-\alpha_1 + 4\alpha_2)$$

$\therefore\ V_1\cap V_2$ 的基是 $-\alpha_1+4\alpha_2$ 即 $(-5, 2, 3, 4)$

7. 已知 $C^{2\times 2}$ 的子空间：

$$V_1=\left\{\begin{bmatrix} x & y \\ y & x \end{bmatrix} \middle| \ \forall x, y \in C\right\},$$

$$V_2=\left\{\begin{bmatrix} x & y \\ x & y \end{bmatrix} \middle| \ \forall x, y \in C\right\},$$

分别求 $V_1 \cap V_2$ 及 V_1+V_2 的基.

7.

① $V_1 \cap V_2 = \left\{ \begin{bmatrix} x & x \\ x & x \end{bmatrix} \middle| \ \forall x \in C \right\}$

$\therefore$ 基为 $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

② $V_1 + V_2$：　V_1 的基为 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

V_2 的基为 $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$\therefore$ V_1+V_2 即由上述4者生成的空间

$\therefore$ 设 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$　$\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$　$\alpha_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$　$\alpha_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

$$(\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{行}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{行}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ α_1 α_2 α_3 是极大无关组

$\therefore$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ 是 V_1+V_2 的基

10. 设

$$V_1=\{(x_1,\cdots,x_n)\mid x_1+\cdots+x_n=0\},$$

$$V_2=\{(x_1,\cdots,x_n)\mid x_1=x_2=\cdots=x_n\},$$

试证：$C^n=V_1\oplus V_2$.

10.

① 证明 V_1, V_2 是直和

$\forall\ \eta = (x_1, \cdots, x_n) \in V_1 \cap V_2$

$\therefore\ \eta \in V_1 \Rightarrow x_1 + x_2 + \cdots + x_n = 0$

$\eta \in V_2 \Rightarrow x_1 = x_2 = \cdots = x_n$

解得 $x_1 = x_2 = \cdots = x_n = 0$

$\therefore\ \eta = \theta$ 即 $V_1 \cap V_2 = \{\theta\}$

$\therefore$ V_1, V_2 是直和

② 证明 $C^n = V_1 + V_2$

(a) $C^n \supseteq V_1 + V_2$ 显然成立

(b) 证明 $C^n \subseteq V_1 + V_2$

$\forall\ \eta = (t_1, t_2, \cdots, t_n) \in C^n$

令 $t = \dfrac{\sum_{i=1}^{n} t_i}{n}$

构 $\alpha_1 = (x_1 - t,\ x_2 - t,\ \cdots,\ x_{n-1} - t,\ (n-1)t - \sum_{i=1}^{n-1} x_i) \in V_1$

$\alpha_2 = (t, t, \cdots, t) \in V_2$

可验证 $\eta = \alpha_1 + \alpha_2 \Rightarrow C^n \subseteq V_1 + V_2$

综上：$C^n = V_1 \oplus V_2$

(接上) 如何构造 α_1, α_2

设 $\alpha_1 = (y_1, y_2 \text{ —— }, y_{n-1}, -y_1 - y_2 - \cdots - y_{n-1})$

$\alpha_2 = (z, z \text{ —— } z)$

$$\therefore \begin{cases} x_i = y_i + z & 1 \le i \le n-1 \quad ① \\ x_n = z - y_1 - y_2 - \cdots - y_{n-1} & ② \end{cases}$$

$\because \quad y_i = x_i - z$ 代入 ②

$$x_n = nz - \sum_{i=1}^{n-1} x_i$$

$$\therefore \quad z = \frac{\sum_{i=1}^{n} x_i}{n}$$

11. 设 $A,B\in F^{n\times n}$，且 $AB=O, B^2=B$. 又

$V_1=\{X|AX=0, X\in F^n\}$， $V_2=\{X|BX=0, X$

试证：(1) $F^n=V_1+V_2$；

(2) $F^n=V_1\oplus V_2 \Leftrightarrow r(A)+r(B)=n$.

11.

(1) ① $F^n \supseteq V_1+V_2$ 显然成立

② $F^n \subseteq V_1+V_2$

$\forall \eta \in F^n$

构造 $\eta = \underline{\eta - B\eta} + \underline{B\eta}$

$= \alpha_2 + \alpha_1$

其中 $\begin{cases} \alpha_1 = B\eta \\ \alpha_2 = \eta - B\eta \end{cases}$

$\because A\cdot\alpha_1 = A\cdot B\cdot\alpha_1 = O\cdot\alpha_1 = \theta$

$\therefore \alpha_1 \in V_1$

$\because B\cdot\alpha_2 = B(\eta - B\eta) = B\eta - B^2\eta = B\eta - B\eta = \theta$

$\therefore \alpha_2 \in V_2$

$\therefore \eta \in V_1+V_2$ 即 $F^n \subseteq V_1+V_2$

综上：$F^n = V_1+V_2$

11. 设 $\boldsymbol{A},\boldsymbol{B}\in F^{n\times n}$，且 $\boldsymbol{AB}=\boldsymbol{O},\boldsymbol{B}^2=\boldsymbol{B}$. 又

$$V_1=\{\boldsymbol{X}\mid \boldsymbol{AX}=\boldsymbol{0},\boldsymbol{X}\in F^n\},\quad V_2=\{\boldsymbol{X}\mid \boldsymbol{BX}=\boldsymbol{0},\boldsymbol{X}\in F^n\}.$$

试证：(1) $F^n=V_1+V_2$；

(2) $F^n=V_1\oplus V_2 \Leftrightarrow r(\boldsymbol{A})+r(\boldsymbol{B})=n$.

-2) A. 证明 $F^n = V_1 \oplus V_2 \Rightarrow r(A) + r(B) = n$

$\because$ 直和

$\therefore \dim(V_1+V_2) = \dim(V_1) + \dim(V_2)$

而 $\dim(V_1+V_2) = \dim(F^n) = n$

$\dim(V_1) = n - r(A)$

$\dim(V_2) = n - r(B)$

综上可得 $r(A) + r(B) = n$

B. 证明 $F^n = V_1 \oplus V_2 \Leftarrow r(A) + r(B) = n$

(见下页)

11. 设 $A,B\in F^{n\times n}$，且 $AB=O, B^2=B$. 又

$$V_1=\{X|AX=0,X\in F^n\},\quad V_2=\{X|BX=0,X\in F^n\}.$$

试证：(1) $F^n=V_1+V_2$；

(2) $F^n=V_1\oplus V_2\Leftrightarrow r(A)+r(B)=n$.

B. 证明 $F^n = V_1\oplus V_2 \Leftarrow r(A)+r(B)=n$

a. 先证明 $\forall \eta\in V_1$，$\exists \gamma\in F^n$，s.t $\eta=B\cdot\gamma$

记 $B=(\beta_1,\beta_2,\cdots,\beta_n)$

令 B的一组极大无关组为 $(\beta_{x_1},\beta_{x_2},\cdots,\beta_{x_{r(B)}})$

∵ $A\cdot B = A(\beta_1,\beta_2,\cdots,\beta_n)=O$

∴ $\beta_1,\beta_2,\cdots,\beta_n\in V_1$

而 $\beta_{x_1},\beta_{x_2}\cdots\beta_{x_{r(B)}}$ 可由 β_1 — β_n 表示

∴ $\beta_{x_1},\beta_{x_2}\cdots\beta_{x_{r(B)}}\in V_1$

又 V_1 的解空间的基的个数为 $n-r(A)=r(B)$

∴ $\beta_{x_1},\beta_{x_2}\cdots\beta_{x_{r(B)}}$ 即 V_1 的一组基

∴ $\forall\eta\in V_1$，η 可由 $\beta_{x_1},\beta_{x_2},\cdots,\beta_{x_{r(B)}}$ 表示

同样因为 $\beta_{x_1},\beta_{x_2}\cdots\beta_{x_{r(B)}}$ 可由 β_1 — β_n 表示

∴ η 可由 β_1 — β_n 表示

证得 $\exists\gamma\in F^n$，s.t $\eta=B\cdot\gamma$

b. 再证明 $F^n = V_1 \oplus V_2$

$\forall \eta \in V_1 \cap V_2$

则 $\eta \in V_1 \Rightarrow A\eta = \theta$

$\eta \in V_2 \Rightarrow B\eta = \theta$

而由 a 得 $\exists \gamma \in F^n$, s.t $\eta = B\gamma$

$$
\begin{aligned}
\therefore \eta &= B \cdot \gamma \\
&= B^2 \cdot \gamma \\
&= B \cdot (B\gamma) \\
&= B \cdot \eta \\
&= \theta
\end{aligned}
$$

$\therefore V_1 \cap V_2 = \{\theta\}$

结合 (1) 可得 $F^n = V_1 \oplus V_2$