12. 已知 $A=\begin{bmatrix}1&1&3\\0&1&1\\-1&2&0\\0&1&1\end{bmatrix}$，分别求 $R(A)$ 及 $K(A)$ 的一组基.

$$A=\begin{pmatrix}1&1&3\\0&1&1\\-1&2&0\\0&1&1\end{pmatrix}\rightarrow\begin{pmatrix}1&0&2\\0&1&1\\0&3&3\\0&1&1\end{pmatrix}\rightarrow\begin{pmatrix}1&0&2\\0&1&1\\0&0&0\\0&0&0\end{pmatrix}$$

$\therefore$ $\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$ $\begin{pmatrix}1\\1\\2\\1\end{pmatrix}$ 为 $R(A)$ 的一组基

而对 $AX=A\cdot\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\theta$

通解为 $\begin{cases}x_1=-2x_3\\x_2=-x_3\end{cases}$

$\therefore$ $K(A)$ 一组基为 $\begin{pmatrix}-2\\-1\\1\end{pmatrix}$

13. 在 $F^{2\times2}$ 中定义线性变换 $f(X)=\begin{bmatrix}a & b\\ c & d\end{bmatrix}X, \forall X\in F^{2\times2}$，分别求 f 在基 $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ 与基 $\{E_{11}, E_{21}, E_{12}, E_{22}\}$ 下的矩阵.

分别记要求的两个矩阵为 A 和 B

$$\because\ [f(E_{11})\ f(E_{12})\ f(E_{21})\ f(E_{22})] = [E_{11}\ E_{12}\ E_{21}\ E_{22}]\cdot A$$

$$[f(E_{11})\ f(E_{21})\ f(E_{12})\ f(E_{22})] = [E_{11}\ E_{21}\ E_{12}\ E_{22}]\cdot B$$

$$\text{而 } f(E_{11}) = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}a & 0\\ c & 0\end{bmatrix} = [E_{11}\ E_{12}\ E_{21}\ E_{22}]\begin{bmatrix}a\\ 0\\ c\\ 0\end{bmatrix} = [E_{11}\ E_{21}\ E_{12}\ E_{22}]\begin{bmatrix}a\\ c\\ 0\\ 0\end{bmatrix}$$

$$f(E_{12}) = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} = \begin{bmatrix}0 & a\\ 0 & c\end{bmatrix} = [E_{11}\ E_{12}\ E_{21}\ E_{22}]\begin{bmatrix}0\\ a\\ 0\\ c\end{bmatrix} = [E_{11}\ E_{21}\ E_{12}\ E_{22}]\begin{bmatrix}0\\ 0\\ a\\ c\end{bmatrix}$$

$$f(E_{21}) = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} = \begin{bmatrix}b & 0\\ d & 0\end{bmatrix} = [E_{11}\ E_{12}\ E_{21}\ E_{22}]\begin{bmatrix}b\\ 0\\ d\\ 0\end{bmatrix} = [E_{11}\ E_{21}\ E_{12}\ E_{22}]\begin{bmatrix}b\\ d\\ 0\\ 0\end{bmatrix}$$

$$f(E_{22}) = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & b\\ 0 & d\end{bmatrix} = [E_{11}\ E_{12}\ E_{21}\ E_{22}]\begin{bmatrix}0\\ b\\ 0\\ d\end{bmatrix} = [E_{11}\ E_{21}\ E_{12}\ E_{22}]\begin{bmatrix}0\\ 0\\ b\\ d\end{bmatrix}$$

$$\therefore\ A = \begin{bmatrix}a & 0 & b & 0\\ 0 & a & 0 & b\\ c & 0 & d & 0\\ 0 & c & 0 & d\end{bmatrix} \qquad B = \begin{bmatrix}a & b & 0 & 0\\ c & d & 0 & 0\\ 0 & 0 & a & b\\ 0 & 0 & c & d\end{bmatrix}$$

14. 设线性变换 f 在基 $\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2,\boldsymbol{\varepsilon}_3$ 下矩阵为 $\boldsymbol{A}=(a_{ij})_{3\times3}$.

(1) 求 f 在基 $\boldsymbol{\varepsilon}_3,\boldsymbol{\varepsilon}_2,\boldsymbol{\varepsilon}_1$ 下矩阵;

(2) 求 f 在基 $\boldsymbol{\varepsilon}_1+k\boldsymbol{\varepsilon}_2,\boldsymbol{\varepsilon}_2,\boldsymbol{\varepsilon}_3$ 下矩阵.

(1) 由题得

$$[f(\varepsilon_1)\ f(\varepsilon_2)\ f(\varepsilon_3)] = [\varepsilon_1\ \varepsilon_2\ \varepsilon_3]\cdot\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

设 $[f(\varepsilon_3)\ f(\varepsilon_2)\ f(\varepsilon_1)] = [\varepsilon_3\ \varepsilon_2\ \varepsilon_1]\cdot B$

$$\because\ [\varepsilon_3\ \varepsilon_2\ \varepsilon_1] = [\varepsilon_1\ \varepsilon_2\ \varepsilon_3]\cdot\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\therefore\ B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1}\cdot\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\cdot\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\cdot\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_{33} & a_{32} & a_{31} \\ a_{23} & a_{22} & a_{21} \\ a_{13} & a_{12} & a_{11} \end{bmatrix}$$

14. 设线性变换 f 在基 $\varepsilon_1,\varepsilon_2,\varepsilon_3$ 下矩阵为 $A=(a_{ij})_{3\times3}$.

(1) 求 f 在基 $\varepsilon_3,\varepsilon_2,\varepsilon_1$ 下矩阵；

(2) 求 f 在基 $\varepsilon_1+k\varepsilon_2,\varepsilon_2,\varepsilon_3$ 下矩阵.

(2) 由题得

$$\begin{bmatrix} f(\varepsilon_1) & f(\varepsilon_2) & f(\varepsilon_3) \end{bmatrix} = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

设 $\begin{bmatrix} f(\varepsilon_1+k\varepsilon_2) & f(\varepsilon_2) & f(\varepsilon_3) \end{bmatrix} = \begin{bmatrix} \varepsilon_1+k\varepsilon_2, & \varepsilon_2, & \varepsilon_3 \end{bmatrix} \cdot C$

$\because \begin{bmatrix} \varepsilon_1+k\varepsilon_2, & \varepsilon_2, & \varepsilon_3 \end{bmatrix} = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore C = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ -ka_{11}+a_{21} & -ka_{12}+a_{22} & -ka_{13}+a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+ka_{12} & a_{12} & a_{13} \\ -ka_{11}+a_{21}-k^2a_{12}+ka_{22} & -ka_{12}+a_{22} & -ka_{13}+a_{23} \\ a_{31}+ka_{32} & a_{32} & a_{33} \end{bmatrix}$$

15. 证明下列映射是线性映射，并自选基偶，求线性映射的矩阵.

(1) $f(\boldsymbol{A})=\operatorname{tr}\boldsymbol{A}, \forall \boldsymbol{A}\in R^{n\times n}, f: R^{n\times n}\to R$； $1\times n^2$ 行向量

(2) $R[x]_3=\{a_0+a_1x+a_2x^2 \mid \forall a_i\in R\}, h(x,t)=x^2+tx$，且

$$f[p(x)]=\int_0^1 p(t)h(x,t)\,\mathrm{d}t \quad (\forall p(x)\in R[x]_3).$$

(1) ①
$$\begin{aligned} f(A+B) &= \operatorname{tr}(A+B) \\ &= \operatorname{tr}A+\operatorname{tr}B \\ &= f(A)+f(B) \end{aligned}$$

$$\begin{aligned} f(k\cdot A) &= \operatorname{tr}(k\cdot A) \\ &= k\cdot \operatorname{tr}A \\ &= k\cdot f(A) \end{aligned}$$

∴ 是线性映射

② 取基偶 $\{E_{11}, E_{12}\cdots E_{1n}, E_{21}, E_{22}\cdots, E_{2n}, \cdots\cdots E_{n1}, E_{n2}\cdots E_{nn}\}$ 和 $\{1\}$

∵ $[f(E_{11}), f(E_{12}) \cdots f(E_{1n}), f(E_{21}), f(E_{22})\cdots f(E_{2n})\cdots f(E_{n1}), f(E_{n2}), \cdots f(E_{nn})]$

$= [1]\cdot[\underbrace{1,0,\cdots,0}_{n}, \underbrace{0,1,0,\cdots,0}_{n} \cdots\cdots \underbrace{0,0,\cdots,1}_{n}]$

∴ 线性映射的矩阵为 $[\underbrace{1,0,\cdots,0}_{n}, \underbrace{0,1,0,\cdots,0}_{n} \cdots\cdots \underbrace{0,0,\cdots,1}_{n}]$

15. 证明下列映射是线性映射，并自选基偶，求线性映射的矩阵.

(1) $f(\boldsymbol{A})=\operatorname{tr}\boldsymbol{A}, \forall \boldsymbol{A}\in R^{n\times n}, f: R^{n\times n}\to R$； $1\times n^2$ 向量

(2) $R[x]_3=\{a_0+a_1x+a_2x^2 \mid \forall a_i\in R\}, h(x,t)=x^2+tx$，且

$$f[p(x)]=\int_0^1 p(t)h(x,t)\,\mathrm{d}t \quad (\forall p(x)\in R[x]_3).$$

(2) ①
$$f[p(x)+q(x)] = \int_0^1 [p(t)+q(t)]\,h(x,t)\,dt$$
$$= \int_0^1 p(t)\,h(x,t)\,dt + \int_0^1 q(t)\,h(x,t)\,dt$$
$$= f[p(x)] + f[q(t)]$$

$$f[k\cdot p(x)] = \int_0^1 k\cdot p(t)\cdot h(x,t)\,dt$$
$$= k\cdot\int_0^1 p(t)\cdot h(x,t)\,dt$$
$$= k\cdot f[p(x)]$$

∴ 是线性映射

② 取基 $\{1, x, x^2\}$

$$[f(1), f(x), f(x^2)] = [1, x, x^2]\cdot A$$

而 $f(1) = \int_0^1 x^2+tx\,dt = x^2t+\frac{1}{2}xt^2\Big|_0^1 = x^2+\frac{1}{2}x$

$f(x) = \int_0^1 x^2t+xt^2\,dt = \frac{1}{2}x^2t^2+\frac{1}{3}xt^3\Big|_0^1 = \frac{1}{2}x^2+\frac{1}{3}x$

$f(x^2) = \int_0^1 x^2t^2+xt^3\,dt = \frac{1}{3}x^2t^3+\frac{1}{4}xt^4\Big|_0^1 = \frac{1}{3}x^2+\frac{1}{4}x$

∴ 线性变换的矩阵为 $A = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$

16. 分别求第 15 题中 f 的值域及核的一组基.

<1> ① 求值域的一组基

矩阵 A 为 $[\underbrace{1,0,\cdots,0}_{n},\underbrace{0,1,0,\cdots,0}_{n}\ \cdots\cdots\ \underbrace{0,0,\cdots,1}_{n}]$

∴ A 的非零首元即为第一项

∴ 值域为 $L[f(E_{11})]=L[\operatorname{tr}E_{11}]$

$=L[1]$

基为 $\{1\}$ 维数为 1

② 求核的一组基

记 $x_{ij}'=x_{(i-1)\times n+j}$ (将 $1\times n^2$ 的向量以矩阵形式表示)

∵ $AX=\theta$

通解为 $x_{11}'=\sum\limits_{i=j\geq 2}(-1)\cdot x_{ij}'+\sum\limits_{i\neq j}0\cdot x_{ij}'$

∴ 一组基为 $\{-E_{11}+E_{ij}\}_{i=j\geq 2}\cup\{E_{ij}\}_{i\neq j}$

维数为 n^2-1

16. 分别求第 15 题中 f 的值域及核的一组基.

(2) ① 求值域的一组基

线性变换的矩阵为 $A = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$

$$A \to \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{6} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

∴ 值域为 $L[f(1), f(x)] = L[x^2+\frac{1}{2}x, \frac{1}{2}x^2+\frac{1}{3}x]$

一组基为 $\{x^2+\frac{1}{2}x, \frac{1}{2}x^2+\frac{1}{3}x\}$ 维数为 2

② 求核的一组基

对 $AX = \theta$

∴ 通解为 $\begin{cases} x_1 = \frac{1}{6}x_3 \\ x_2 = -x_3 \end{cases}$

∴ 基础解系为 $\begin{pmatrix} \frac{1}{6} \\ -1 \\ 1 \end{pmatrix}$

∴ 核的一组基为 $\frac{1}{6} - x + x^2$ 维数为 1