Aggregation by Movement

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1 Introduction

Spatial pattern formation is a fascinating phenomenon that can be observed in a wide range of ecosystems, such as mussel beds and arid bushlands (Liu et al. 2016; Rietkerk et al. n.d.). There is evidence that the stability of these ecosystems is improved by spatial patterns, for example in mussel beds through improved nutrient availability while reducing the impact of disturbances due to water flow (Van De Koppel et al. 2008).

To explain the formation of the observed patterns, usually one of two mechanisms is put forward. The first mechanism is based on the activation-inhibition principle described by Alan Turing (Turing 1952). One of the ways this principle can be applied is by

The second mechanism is based on the lesser known phase-separation prinicipIe developed by Cahn and Hilliard. In ecological settings, this corresponds to density-dependent movement leading to pattern formation, as opposed to birth and death processes. A number of organisms whose spatial distribution can be explained by density-dependent movement is presented in (Liu et al. 2016).

In this report, the spatial aggregation of individuals based on density-dependent movement in continuous space will be explored. The simplest way to do this, is to update the position of all organisms at discrete time steps and update their position by drawing from a normal distribution with a density dependent variance. Mathematically, the change of position of individual i can be described by

$$x_i(t+dt) = x_i(t) + \sqrt{2D(\rho_R(\vec{r}_i,t))dt}u_{i,x} \tag{1}$$

$$y_i(t+dt) = y_i(t) + \sqrt{2D(\rho_R(\vec{r}_i,t))dt}u_{i,y} \tag{2}$$

where ...

2 Methods

2.1 Practice Calculation: A Single Organism

Before adding spatial interactions, a single organism diffusing in space will be analysed. This will act as a baseline and can be compared to the more complicated case.

To do this, we consider a simple environment with only a single organism and a constant diffusion term. Additionally, it will be assumed that the environment is infinite. The simulations will be run on a torus later, but that would make an analytical derivation more complicated.
The walker position will change according to the following equations:

$$x(t+dt) = x(t) + \sqrt{2Ddt}u_x \tag{3}$$

$$y(t+dt) = y(t) + \sqrt{2Ddt}u_y \tag{4}$$

where u_x, u_y are independent random numbers $\sim \mathcal{N}(0,1)$.
Let's consider the increments $dx(t+dt) = x(t+dt) - x(t)$. Then

$$x(t) = \sum_{\tau=0}^{t/dt} dx(\tau) + x(0),$$

that is, the position of x at time t can be described by adding up the increments. From Eq. 3, we can see that the increments are normally distributed with variance $2Ddt$. The distribution of the displacements $x(t) - x(0)$ is therefore a sum of normally distributed random variables, thus being again normally distributed with the variance being the sum of the variances

$$x(t) - x(0) \sim \mathcal{N}(0, \sum 2Ddt) = \mathcal{N}(0, 2Dt).$$

To make the following equations clearer, we will use $\tilde{x}(t) = x(t) - x(0)$ and $\tilde{y}(t) = y(t) - y(0)$. Using this, we calculate the mean squared displacement

$$MSD(t) = \langle \tilde{x}^2(t) + \tilde{y}^2(t) \rangle = \langle \tilde{x}^2(t) \rangle + \langle \tilde{y}^2(t) \rangle.$$

Again we focus on the term involving x.

$$\langle \tilde{x}^2(t) \rangle = \int_{-\infty}^{\infty} \tilde{x}^2 p(\tilde{x}, t) d\tilde{x} = \int_{-\infty}^{\infty} \tilde{x}^2 \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{\tilde{x}^2}{4Dt}} d\tilde{x}$$

Solving this integral requires a few steps, that will be only briefly mentioned. First, we do a variable transformation by setting $y = \frac{\tilde{x}}{\sqrt{4Dt}}$. The integral then becomes

$$\frac{4Dt}{\sqrt{\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2} dy.$$

This integral can be solved by partial integration and is equal to $\sqrt{\pi}/2$. Thus, the final result is

$$\langle \tilde{x}^2(t) \rangle = 2Dt.$$

For $\langle \tilde{y}^2(t) \rangle$, we get the exact same result, and the mean squared displacement is therefore

$$MSD(t) = \langle \tilde{x}^2(t) \rangle + \langle \tilde{y}^2(t) \rangle = 4Dt.$$

The average absolute distance of the organism from the initial position therefore grows with the square root of time.

3 Aggregation

Now, we will add

4 Results

5 Discussion

References

Liu, Quan-Xing et al. (Dec. 2016). "Phase Separation Driven by Density-Dependent Movement: A Novel Mechanism for Ecological Patterns". In: *Physics of Life Reviews* 19, pp. 107–121. ISSN: 15710645. DOI: `10.1016/j.plrev.2016.07.009`. (Visited on 04/04/2025).

Rietkerk, Max et al. (n.d.). "Self-Organization of Vegetation in Arid Ecosystems." In: ().

Turing, Alan (Aug. 1952). "The Chemical Basis of Morphogenesis". In: *Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences* 237.641, pp. 37–72. ISSN: 2054-0280. DOI: `10.1098/rstb.1952.0012`. (Visited on 04/12/2025).

Van De Koppel, Johan et al. (Oct. 2008). "Experimental Evidence for Spatial Self-Organization and Its Emergent Effects in Mussel Bed Ecosystems". In: *Science* 322.5902, pp. 739–742. ISSN: 0036-8075, 1095-9203. DOI: `10.1126/science.1163952`. (Visited on 04/12/2025).