The Rocket Problem

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Variable Mass Rocket Motion

If the mass in straight line motion problem is changing with time in addition to its position, then Newton's 2^{nd} Law no longer holds for our purposes and must be modified. This motion, of a body of variable mass(for example, a rocket burning propellant) moving in a straight line, is given by

$$m\frac{dv}{dt} = F + u\frac{dm}{dt}$$

For which

- m is the mass of the body at time t
- v is the velocity of the body at time t
- F is the algebraic sum of all the forces acting on the body at time t
- dm is the mass either joining or leaving the body in the time interval dt
- u is the velocity of dm(the mass joining or leaving), at the time it joins or leaves the body, relative to the chosen frame of reference.

Note: for our purposes, we will define a mass dm leaving a body in a time interval dt to be a negative quantity, and a mass dm joining a body in a time interval dt to be a positive one.

The Problem

A rocket which weighs 14515 grams and contains fuel weighing 14515 grams at the $t = 0$, is propelled straight up from the surface of the earth by burning 14515 grams of fuel per second and expelling it backwards at a constant velocity of A meters per second relative to the observer on the rocket. Assume that the only force acting on the rocket is that of gravity. Find the velocity of the rocket and the distance it travels as functions of time. Take the positive direction to be upward.

The Solution

From our initial definition, we know we can use the following equation:

$$m\frac{dv}{dt} = F + u\frac{dm}{dt}$$

From the question, we can obtain the following information:

- Getting a Figure for Mass
  - We have a constant component of our mass, which is 14515 grams. We will call this m_c.
  - We have an initial mass of propellant, which is also of mass 14515 grams. We will call this m_0.
  - We have a mass $\frac{dm}{dt}$ exiting our total mass per second t.
  - From the above, we can generalize our mass m at any given t to be represented by:

$$m = m_c + (m_0 - \frac{dm}{dt}t)$$

- Getting a Figure for F
  - We know the only force acting on the rocket is gravity g, or $-9.8\frac{m}{s^2}$.
  - Therefore to obtain F we can use our figure for mass above and obtain the following:

$$F = -mg$$

  or more precisely

$$F = -9.8(m_c + (m_0 - \frac{dm}{dt}t))$$

- Additionally, we know u to be $-A$ in the context of our problem.

Using this information, we come up with the following differential equation:

$$(m_c + (m_0 - \frac{dm}{dt}t))\frac{dv}{dt} = -9.8(m_c + (m_0 - \frac{dm}{dt}t)) - A\frac{dm}{dt}$$

And dividing out our mass from either side leaves us with the following:

$$\frac{dv}{dt} = -9.8 - \frac{A\frac{dm}{dt}}{(m_c + (m_0 - \frac{dm}{dt}t))}$$

Integrating both sides with respect to t gives us:

$$\int \frac{dv}{dt}dt = \int -9.8 - \frac{A\frac{dm}{dt}}{(m_c + (m_0 - \frac{dm}{dt}t))}dt$$

To simplify our problem to help us solve it, we will equate $\frac{dm}{dt}$ to be k and clean up our workspace a bit.

$$v = -9.8t - Ak\int \frac{1}{(m_c + m_0 - kt)}dt$$

Using U-substitution:

$$\int \frac{1}{(m_c + m_0 - kt)} dt = \frac{-1}{k} ln(m_c + m_0 - kt) + C$$

Which when substituted back into our main equation gives us:

$$v = -9.8t - Ak(\frac{-1}{k} ln(m_c + m_0 - kt) + C)$$

Which can be further simplified to the following:

$$v = -9.8t + Aln(m_c + m_0 - kt) + v_0$$

Now, to find our position with respect to time, we need to substitute v for $\frac{dy}{dt}$ and solve.

$$\int \frac{dy}{dt} dt = \int -9.8t + Aln(m_c + m_0 - kt) + v_0 dt$$

Which gives us:

$$y = \frac{-9.8}{2} t^2 + A \int ln(m_c + m_0 - kt) dt + v_0 t$$

Using integration by parts and U-substitution to evaluate the central integral, we get the following:

$$y = \frac{-9.8}{2} t^2 + \frac{A}{k} (m_c + m_0 - kt) ln(m_c + m_0 - kt) + At + v_0 t + y_0$$

References

Dover Books - ODEs