Segment Tree

[J. L. Bentley; *Solutions to Klee's rectangle problem*, Technical Report, Carnegie-Mellon University, Pittsburgh, 1977]
[Section 10.3 in de Berg, Cheong, van Kreveld, Overmars; *Computational Geometry: Algorithms and Applications*, 3rd edition, 2008]
[Section 2.2 in de Langetepe Zachmann; *Geometric Data Structures for Computer Graphics*, 2006]

A *segment tree* is a static data structure for storing a set of intervals

$$I = \{[x_1, x_1'], [x_2, x_2'], \ldots, [x_n, x_n']\}$$

and can be used for solving problems e.g. concerning line segments.

Let $p_1, \ldots, p_m$, $m \leq 2n$, be the ordered list of distinct endpoints of the intervals in I. The ordered sequence of endpoints $p_1, \ldots, p_m$ partitions the real line into a set of *atomic intervals*:

$$(-\infty, p_1), [p_1, p_1], (p_1, p_2), [p_2, p_2], \ldots, (p_{n-1}, p_n), [p_n, p_n], (p_n, \infty)$$

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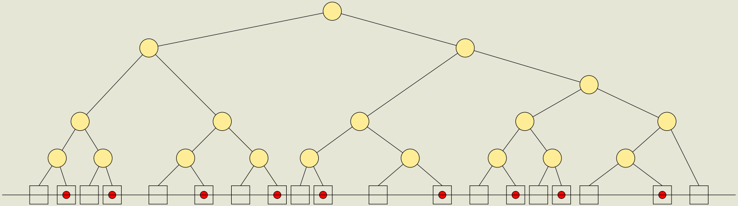

A segment tree is a leaf-oriented balanced binary tree on the atomic intervals according to left to right order.

An internal node v corresponds to the interval which is the union of the atomic intervals of the leaves of the subtree rooted at v.
Let $int(v)$ denote this interval.

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With each node v we store a set $I(v) \subseteq I$:
Interval $[x, x']$ is stored in $I(v)$ if and only if

$$int(v) \subseteq [x, x'] \quad \text{and} \quad int(parent(v)) \not\subseteq [x, x']$$

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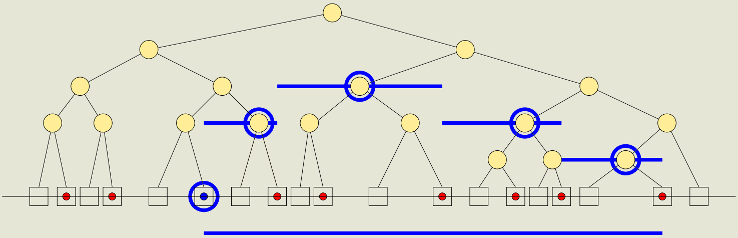

Lemma:

A segment tree on n intervals uses $O(n \log n)$ storage.

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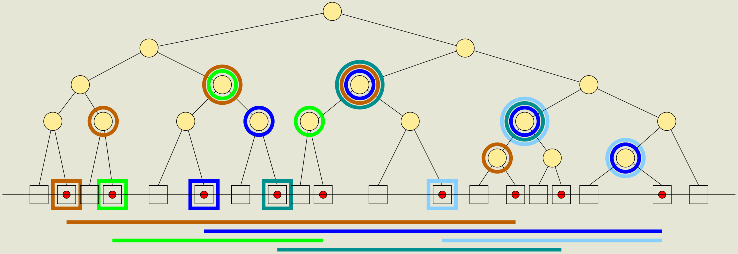
An interval is stored with at most two nodes at the same depth of the tree.



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A segment tree for a set of n intervals can be constructed in $O(n \log n)$ time.

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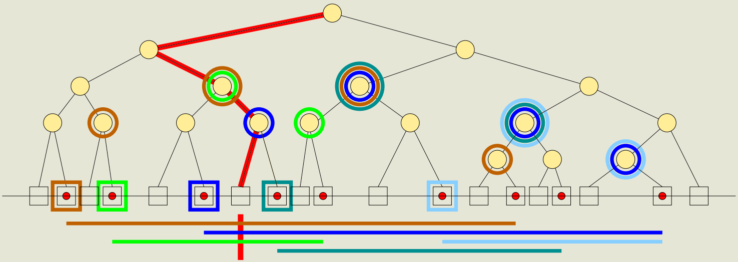
- build leaf-oriented balanced binary search tree on atomic intervals
- insert intervals one by one using InsertSegmentTree:

InsertSegmentTree($v, [x, x']$)

if $int(v) \subseteq [x, x']$
  then add $[x, x']$ to $I(v)$
  else **if** $int(lc(v)) \cap [x, x'] \neq \emptyset$
      then InsertSegmentTree($lc(v), [x, x']$)
    if $int(rc(v)) \cap [x, x'] \neq \emptyset$
      then InsertSegmentTree($rc(v), [x, x']$)

QUERY

QUERY(q_x) reports all segments containing query point q_x.



$\textsc{QuerySegmentTree}(v, q_x)$

Report all the intervals in $I(v)$
if v is not a leaf
  then if $q_x \in int(lc(v))$
    then $\textsc{QuerySegmentTree}(lc(v), q_x)$
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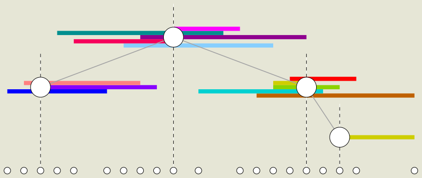
    else QUERYSEGMENTTREE$(rc(v), q_x)$

Lemma:
Using a segment tree, we can report all k intervals that contain a query point q_x, in time $O(k + \log n)$.

Interval Tree

[H. Edelsbrunner; *Dynamic Data Strcutures for Orthogonal Intersection Queries*, Tech. Report. TU Graz, 1980]
[E. M. McCreight; *Efficient Algorithms for Enumerating Intersection Intervals and Rectangles*, Tech. Report, Xerox Paolo Alto Reserach Center, 1980]
[Section 10.1 in de Berg, Cheong, van Kreveld, Overmars; *Computational Geometry: Algorithms and Applications*, 3rd edition, 2008]

An *interval tree* stores a set of intervals on a real line. Let $I = \{[x_1, x_1'], [x_2, x_2'], \ldots, [x_n, x_n']\}$ be a set of n closed intervals.



Let x_{mid} be the median of the interval endpoints of the intervals in I.

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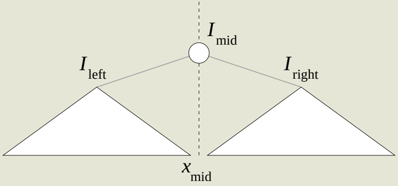

$$
\begin{aligned}
I_{\text{left}} &= \{[x_j, x'_j] \in I : x'_j < x_{\text{mid}}\} \\
I_{\text{mid}} &= \{[x_j, x'_j] \in I : x_j \leq x_{\text{mid}} \leq x'_j\} \\
I_{\text{right}} &= \{[x_j, x'_j] \in I : x_{\text{mid}} < x_j\}
\end{aligned}
$$

I_{mid} is stored at the root of the interval tree. The left subtree is an interval tree of I_{left} and the right subtree is an interval tree of I_{right}.

The interval tree of an empty set of intervals is an external node.

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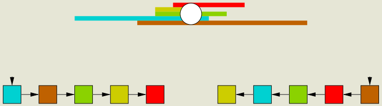
Associated data structures at internal node v:



- $L_{\text{left}}(v)$: list of the left endpoints of the intervals stored at v in increasing order.
- $L_{\text{right}}(v)$: list of right endpoints in decreasing order.

Theorem:
An interval tree for a set of n intervals uses $O(n)$ storage and has height $O(\log n)$. It can be built in $O(n \log n)$ time.

Stabbing Query

Stabbing Query: Given point x, report all intervals that contain x.

Query(v,x)

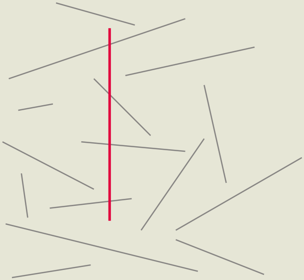
if v is not an external node
    then if $x < x_{\mathrm{mid}}(v)$
        then walk along $L_{\mathrm{left}}(v)$ until x is not contained anymore
            in the current interval; report containing intervals
            Query$(lc(v),x)$
        else walk along $L_{\mathrm{right}}(v)$ until x is not contained anymore
            in the current interval; report containing intervals
            Query$(rc(v),x)$

Theorem:
Using an interval tree, we can report all k intervals that contain a query point x, in time $O(k + \log n)$.

Vertical Stabbing Queries for Disjoint Line Segments

Let S be a set of pairwise disjoint line segments in the plane. We want to maintain S in a data structure that allows us to quickly find the segments intersected by a vertical query segment

$$q_x \times [q_y, q_y']$$

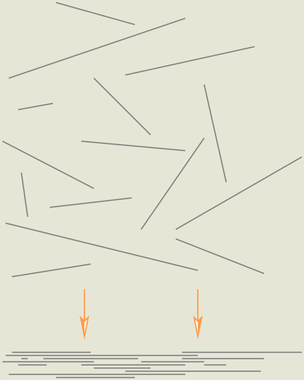


Augmented Segment Tree

for Vertical Stabbing Queries for Disjoint Line Segments

Underlying data structure:

Basically, the underlying data structure is a segment tree for the projection intervals of the segments in S onto the x-axis. However, we don't store the intervals in $I(v)$ at a node v explicitly.

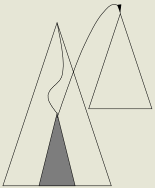


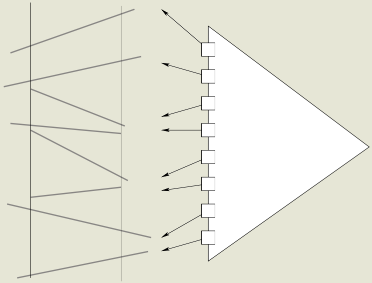
“Additional” information stored at each node v:

For a node v, let $S(v)$ be the set of segments corresponding to the intervals in $I(v)$.

"Additional" information stored at each node v:

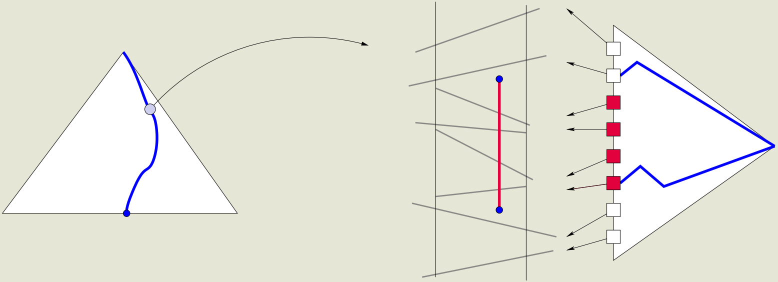
For a node v, let $S(v)$ be the set of segments corresponding to the intervals in $I(v)$. Store the segments in $S(v)$ in a leaf-oriented balanced binary search tree based on the order of the elements in the slab $int(v) \times (-\infty, \infty)$.





QUERY

$$\text{QUERY}(q_x \times [q_y, q'_y])$$



Search for q_x in the underlying segment tree. For each visited node v, search for q_y and q'_y in the balanced binary search tree for $S(v)$. Report segments which are between q_y and q'_y at q_x.

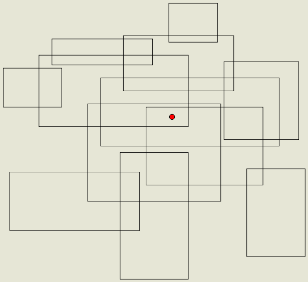
Lemma:

Let S be a set of n disjoint segments in the plane. S can be stored in a data structure such that the segments in S intersected by a vertical query segment can be reported in time $O(k+(\log n)^2)$, where k is the number of reported segments. The data structure uses $O(n \log n)$ storage space and can be built in $O(n(\log n)^2)$ time.

Construction time can be improved to $O(n \log n)$.

Point Stabbing Queries for Rectangles

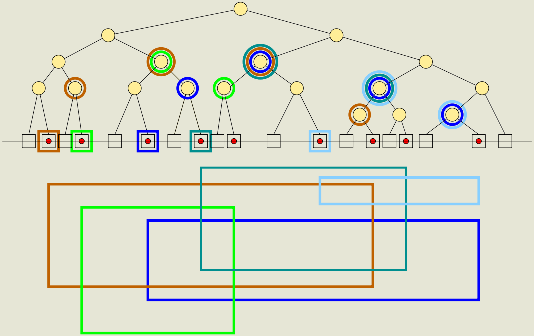
Segment trees can be augemented such that point enclosure problems for axis-aligned rectangles in 2D can be solved efficiently.



Point enclosure problems are also called “inverse range queries”.

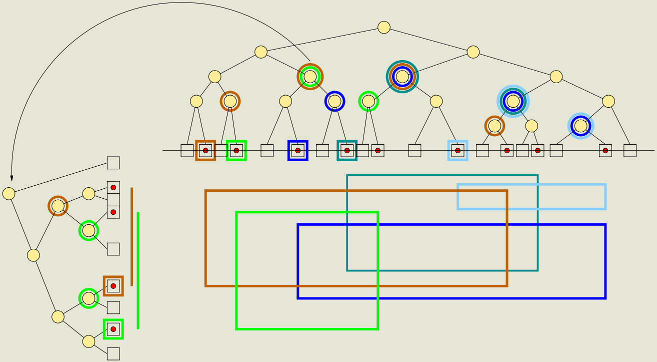
Multi-Level Segment Tree

The underlying data structure of a 2-*dimensional segment tree* is a segment tree for the projection intervals of the rectangles onto the x-axis.



"Additional" information stored at node v:

Let $R(v)$ be the set of rectangles whose x-interval are associated with v. The secondary data structure associated with v is a standard segment tree for the projection intervals of the rectangles in $R(v)$ onto the y-axis.



Theorem:
A 2-dimensional segment tree for solving point enclosure problems for n rectangles in the plane can be built in $O(n(\log n)^2)$ time and takes $O(n(\log n)^2)$ space.

QUERY

$$\text{QUERY}(q_x, q_y)$$

Search for q_x in the underlying segment tree. For each visited node v, search for q_y in the associated segment tree and report the rectangles whose y-intervals contain q_y.

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Search for q_x in the underlying segment tree. For each visited node v, search for q_y in the associated segment tree and report the rectangles whose y-intervals contain q_y.

Theorem:
Using a 2-dimensional segment tree, point enclosure queries for n rectangles in the plane can be answered in time $O(k + (\log n)^2)$ time, where k is the number of reported rectangles.

2-dimensional segment trees can be extended to higher dimensions for point stabbing queries for axis-aligned rectangular boxes.

As before the underlying data structure is a segment tree for the projection intervals with respect to the first coordinate.

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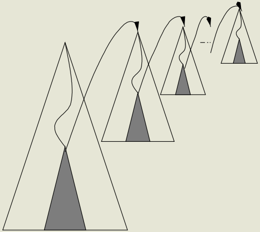
The secondary data structure associated with a node v is a $(d-1)$-dimensional segment tree according to the remaining coordinates for the boxes corresponding to the intervals stored at v. More precisely, it is a $(d-1)$-dimensional segment tree for the boxes formed by the remaining $(d-1)$ coordinates.



Theorem:
A d-dimensional segment tree for a set of n axis-aligned rectangular boxes in $\mathbb{R}^d$ can be built in $O(n(\log n)^d)$ time and takes $O(n(\log n)^d)$ space.

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