Divided and Conquer

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Master Theorem

Theorem

Let $a \geq 1, b > 1$, *and let* $T(n)$ *defined by* $T(1) = \Theta(1)$ *and* $T(n) = a \cdot T(n/b) + f(n)$*:*

- *If* $f(n) = \Theta(n^{\log_b a})$, *then* $T(n) = O(n^{\log_b a} \cdot \log n)$*;*
- *If there is an* $\epsilon > 0$ *with* $f(n) = O(n^{\log_b a - \epsilon})$, *then* $T(n) = O(n^{\log_b a})$.

Integer Multiplication

Input

Two integers x and y

Output

The multiplication $x \cdot y$

Running Time of Naive Algorithm

Divided and Conquer Approach

Divided and Conquer Approach

Multiplication(x, y)

- Let $r \leftarrow n/2$
- Let $z_1 \leftarrow (x_{n-1}, \ldots, x_{r+1})$ and $z_0 \leftarrow (x_r, \ldots, x_0)$
- Let $w_1 \leftarrow (y_{n-1}, \ldots, x_{r+1})$ and $w_0 \leftarrow (y_r, \ldots, x_0)$
- Let $p_0 \leftarrow$Multiplication(z_0, w_0);
- Let $p_1 \leftarrow$Multiplication(z_0, w_1);
- Let $p_2 \leftarrow$Multiplication(z_1, w_0);
- Let $p_3 \leftarrow$Multiplication(z_1, w_1);
- Output $2^{2(r+1)} \cdot p_3 + 2^{r+1} \cdot (p_2 + p_1) + p_0$

Analysis of Divided and Conquer

A New Approach of Divided and Conquer

An observation

$$z_0 \cdot w_1 + z_1 \cdot w_0 = (z_0 + z_1)(w_0 + w_1) - z_0 \cdot w_0 - z_1 \cdot w_1$$

Improved Approach

Multiplication(x, y)

- Let $r \leftarrow n/2$
- Let $z_1 \leftarrow (x_{n-1}, \ldots, x_{r+1})$ and $z_0 \leftarrow (x_r, \ldots, x_0)$
- Let $w_1 \leftarrow (y_{n-1}, \ldots, x_{r+1})$ and $w_0 \leftarrow (y_r, \ldots, x_0)$
- Let $p_0 \leftarrow$Multiplication(z_0, w_0);
- Let $p_1 \leftarrow$Multiplication(z_1, w_1);
- Let $p_2 \leftarrow$Multiplication$(z_0 + z_1, w_0 + w_1)$;
- Output $2^{2(r+1)} \cdot p_1 + 2^{r+1} \cdot (p_2 - p_1 - p_0) + p_0$

Running Time of the New Approach

Proof of Master Theorem

Theorem

Let $a \geq 1, b > 1$, and let $T(n)$ defined by $T(1) = \Theta(1)$ and $T(n) = a \cdot T(n/b) + f(n)$.

- *If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = O(n^{\log_b a} \cdot \log n)$;*

Proof of Master Theorem

Thanks!