P and **NP**

August 23, 2025

Decision Problem

- The output of a decision problem Yes or No

Non-deterministic Polynomial Time

Definition

Let P be a computational problem and denote $L := \{x : P(x) = 1\}$. We say that P is in **NP**, if there is an algorithm $A(\cdot, \cdot)$ such that

- If $x \in L$, there is a string y, such that $A(x, y) = 1$;
- If $x \notin L$, then for every y, $A(x, y) = 0$;
- For every x and y, the running time of $A(x, y)$ is polynomial in $|x|$

Boolean Satisfiability Problem

SAT problem

For a given Boolean formula f, is there an assignment that satisfies f?

Boolean Satisfiability Problem

SAT problem

For a given Boolean formula f, is there an assignment that satisfies f?

Theorem

$SAT \in \mathbf{NP}$.

SAT $\in$ **NP**

$\mathbf{P}$ v.s. $\mathbf{NP}$

Theorem

$\mathbf{P} \subseteq \mathbf{NP}$.

Reduction

Karp-reduction

Definition

Let A, B be decision problems. A Karp-reduction from A to B is a polynomial time algorithm f such that, for every x,

- if $x \in L_A$, then $f(x) \in L_B$;
- if $x \notin L_A$, then $f(x) \notin L_B$

We write $A \leq_p B$ if there is Karp-reduction from A to B.

A Reduction from Largest Matching to Max-flow

NP-hardness and **NP**-completeness

Definition

- A problem A is **NP**-hard if for every problem $B \in \mathbf{NP}$, $B \leq_p A$;
- A problem A is **NP**-complete if $A \in \mathbf{NP}$ and A is **NP**-hard

Useful Facts

Lemma

Let A and B be problems such that $A \leq_p B$, then

- *if A is* **NP***-hard, then B is* **NP***-hard*
- *if $B \in \mathbf{P}$, then $A \in \mathbf{P}$*

Cook-Levin Theorem

Theorem

SAT is **NP***-hard.*

Thanks!