Network Flows and Graph Cuts

August 23, 2025

Maximum Flow Problems

Input

- A directed $G = (V, E)$;
- A capacity $C(e) \geq 0$ for each edge e;
- A source $s \in V$ and a sink $t \in V$

Goal

Find a way to send as much flow as possible from s to t

Path Decomposition

Lemma

For each flow f. There is a list of paths $P_1, \ldots, P_k$ with flow $a_1, \ldots, a_k$ such that $f = \sum_\ell a_\ell \cdot P_\ell$

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Maximum Bipartite Matching

Input

- A bipartite graph $G = (V, E)$, where $V = X \cup Y$

Goal

- A matching is a set of edges s.t each vertex is incident on at most one edge.
- Find a largest matching in G.

Example.

Classroom assignment.

Largest Matching v.s. Maximum Flow

Proof of Correctness

Minimum Cut Problem

Input

- A directed $G = (V, E)$;
- A cost $C(e) \geq 0$ for each edge e;
- A source $s \in V$ and a sink $t \in V$

Goal

Find a cut S such that $s \in S$ and $t \notin S$ minimizing the cost.

Max-Flow Min-Cut Theorem

Theorem

Maximum Flow is equal to Minimal Cut.

Upper Bound Flow Value by Cut Cost

Lemma

Let f be a s-t flow, and $(S, \overline{S})$ a s-t cut. Then $v(f) \leq C(S, \overline{S})$.

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Let f be a s-t flow, and $(S, \overline{S})$ a s-t cut. Then

$$v(f) = \sum_{e=(u,v):u \in S, v \notin S} f(e) - \sum_{e=(u,v):u \notin S, v \in S} f(e).$$

Proof

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Thanks!