

Network Flows and Graph Cuts

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Maximum Flow Problems

Input

- A directed $G = (V, E)$;
- A capacity $C(e) \geq 0$ for each edge e ;
- A source $s \in V$ and a sink $t \in V$

Goal

Find a way to send as much flow as possible from s to t

An Example

Feasible Solutions

Definition

An s - t flow is a function $f : E \rightarrow \mathbb{R}$ satisfying:

- Conservation: $\sum_{e: \text{ into } v} f(e) = \sum_{e: \text{ out of } v} f(e)$ for all $v \in V \setminus \{s, t\}$
- Capacity constraint: $0 \leq f(e) \leq C(e)$ for all $e \in E$

The value of this solution is $\sum_{e: \text{ out of } s} f(e)$

Path: A Basic Flow

Path Decomposition

Path Decomposition

Lemma

For each flow f . There is a list of paths P_1, \dots, P_k with flow a_1, \dots, a_k such that $f = \sum_{\ell} a_{\ell} \cdot P_{\ell}$

Proof of Path Decomposition

Minimum Cut Problem

Input

- A directed $G = (V, E)$;
- A cost $C(e) \geq 0$ for each edge e ;
- A source $s \in V$ and a sink $t \in V$

Goal

Find a cut S such that $s \in S$ and $t \notin S$ minimizing the cost.

An Example

Max-Flow Min-Cut Theorem

Theorem

Maximum Flow is equal to Minimal Cut.

Thanks!