Shortest Path Problem

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Input

- A directed $G = (V, E)$;
- A cost C_e (positive or negative) for each edge $e \in E$
- A source $s \in V$ and a destination $t \in V$

Desired output

A path connects s and t with the smallest cost.

An Example

Paths in Graphs

Definition

A path in a graph $G = (V, E)$ is a sequence of nodes $P = (v_1, \ldots, v_k)$ with $v_i \in V$ and $(v_i, v_{i+1}) \in E$ for every i.

Definition

A simple path is a path which does not repeat any nodes.

Negative Cycles

Negative Cycles

Lemma

If G does not have any negative cycle, then its minimum-costed path from s to t is simple. In particular, it has length at most $n - 1$.

Recurrence Relation

Find a path from s to t of length at most $n - 1$ with the minimum cost.

Recurrence Relation

- $F[v, k]$ the minimum-costed path from s to v with length at most k
- $F[s, 0] = 0$ and $F[v, 0] = \infty$ for every $v \neq s$.
- $F[v, k] = \min\{F[v, k-1], \min_{(u,v)\in E}\{F[u, k-1] + C_{(u,v)}\}\}$

Bellman-Ford Algorithm

- $F[s, 0] = 0$
- For $v \in V \setminus \{s\}$
  $F[v, 0] = \infty$
- For $i = 1$ to $n - 1$
  For $v \in V$
  $F[v, k] = F[v, k-1]$
  For $u \in V$
  If $(u, v) \in E)$ and $F[u, k-1] + C_{(u,v)} < F[v, k]$
  Then $F[v, k] = F[u, k-1] + C_{(u,v)}$
- Output $F[n, t]$

Running Time Analysis

Thanks!