

# Network Flows and Graph Cuts

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# Maximum Flow Problems

## Input

- A directed  $G = (V, E)$ ;
- A capacity  $C(e) \geq 0$  for each edge  $e$ ;
- A source  $s \in V$  and a sink  $t \in V$

## Goal

Find a way to send as much flow as possible from  $s$  to  $t$

# An Example

# Feasible Solutions

## Definition

An  $s$ - $t$  flow is a function  $f : E \rightarrow \mathbb{R}$  satisfying:

- Conservation:  $\sum_{e: \text{ into } v} f(e) = \sum_{e: \text{ out of } v} f(e)$  for all  $v \in V \setminus \{s, t\}$
- Capacity constraint:  $0 \leq f(e) \leq C(e)$  for all  $e \in E$

The value of this solution is  $\sum_{e: \text{ out of } s} f(e)$

# Path: A Basic Flow

# Path Decomposition

# Path Decomposition

## Lemma

*For each flow  $f$ . There is a list of paths  $P_1, \dots, P_k$  with flow  $a_1, \dots, a_k$  such that  $f = \sum_{\ell} a_{\ell} \cdot P_{\ell}$*

# Proof of Path Decomposition

# Minimum Cut Problem

## Input

- A directed  $G = (V, E)$ ;
- A cost  $C(e) \geq 0$  for each edge  $e$ ;
- A source  $s \in V$  and a sink  $t \in V$

## Goal

Find a cut  $S$  such that  $s \in S$  and  $t \notin S$  minimizing the cost.

# An Example

# Max-Flow Min-Cut Theorem

## Theorem

*Maximum Flow is equal to Minimal Cut.*

Thanks!