Minimum Spanning Tree

August 23, 2025

Network Design Problem (Minimum Spanning Tree)

Input

- Connected undirected graph $G = (V, E)$;
- Cost $c(e) \geq 0$ for each $e \in E$

Output

A subset of edges $T \subseteq E$ so that the graph (V, T) is connected, and the total cost $c(T) = \sum_{e \in T} c(e)$ as small as possible.

Example

Example

Lemma

There is no circle in the optimal solution of the network design problem.

Greedy Algorithms?

Kruskal's Algorithm

1. Sort edges in increasing order of $c(e)$
2. Let $T = \emptyset$
3. For each edge $e \in E$ in order

   If ($T \cup \{e\}$ does not have a circle) Then add e to T
4. Output T

Minimum-Cost Edge

Lemma

Assume that all edge costs are distinct. Let edge $e = (v, w)$ be the minimum-cost edge in the graph. Then every minimum spanning tree contains the edge e.

Analysis of Kruskal's Algorithm

- Running time
- Correctness

Running time of Kruskal's Algorithm

Correctness of Kruskal's Algorithm

Another Greedy Algorithm?

Prim's Algorithm

1. Let $T = \emptyset$
2. Let $S = \{v\}$ for an arbitrary $v \in V$
3. While $S \neq V$

   Let $e = (u, w)$ be the minimum-cost edge cross S and $V \setminus S$

   Add w to S

   Add e to T.
4. Output T

Graph Cut

Cut Property

Lemma

Assume that all edge costs are distinct. Let S be any subsets of V, and let edge $e = (v, w)$ be the minimum-cost edge with one end in S and the other in $V \setminus S$. Then every minimum spanning tree contains the edge e

Proof of Cut Property

Analysis of Prim's Algorithm

- Running time
- Correctness

Uniqueness of MST

Theorem

Assume that all edge costs are distinct. There is a unique MST in G.

Thanks!