Lecture 3

August 23, 2025

Stable Matching

- Given n professors and n students
- $P = \{p_1, \ldots, p_n\}$, $S = \{s_1, \ldots, s_n\}$
- Each student has a preference list of P
- Similarly, each professor has a preference list of S.

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Definition

For a perfect matching M, two pairs (p, s) and (p', s') are unstable if p prefers s' to s and s' prefers p to p'. A matching is stable if there is no unstable pair.

Stable Matching is Not Unique

Example

Let $n = 2$. There are two professors p_1, p_2 and two students s_1, s_2.

- p_1: $s_1 > s_2$
- p_2: $s_2 > s_1$
- s_1: $p_2 > p_1$
- s_2: $p_1 > p_2$

Gale-Shapley Algorithm

Invariant of Gale-Shapley Algorithm

Invariant of Gale-Shapley Algorithm

For a professor p, let

$$S(p) := \{s : \text{there is a stable matching that contains } (p, s)\}$$

Let $b(p)$ be the highest-ranked student in $S(p)$.

Theorem

Gale-Shapley Algorithm returns $b(p)$ for every professor p.

We prove this theorem by contradiction and case analysis.

Invariant of Gale-Shapley Algorithm

Invariant of Gale-Shapley Algorithm

For a student s, let

$$P(s) := \{p : \text{there is a stable matching that contains } (p, s)\}$$

Let $w(s)$ be the lowest-ranked professor in $P(s)$.

Theorem

Gale-Shapley Algorithm returns $w(s)$ for every student s.

Greedy Algorithms

- Makes local optimal choices
- Often the first approach that comes to mind
- Does not always output globally optimal solutions
- Try to find counterexamples
- Analyzing a greedy algorithm is usually non-trivial

Interval Scheduling

Input

- n tasks;
- a start time s_i for each task i;
- a finish time f_i for each task i;

Desired output

A set S of non-overlapping tasks maximizing $|S|$.

Examples

Approaches and Counter Examples

Thanks!