Topological Data Analysis

Lecture 8

Hodge decomposition

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Motivation

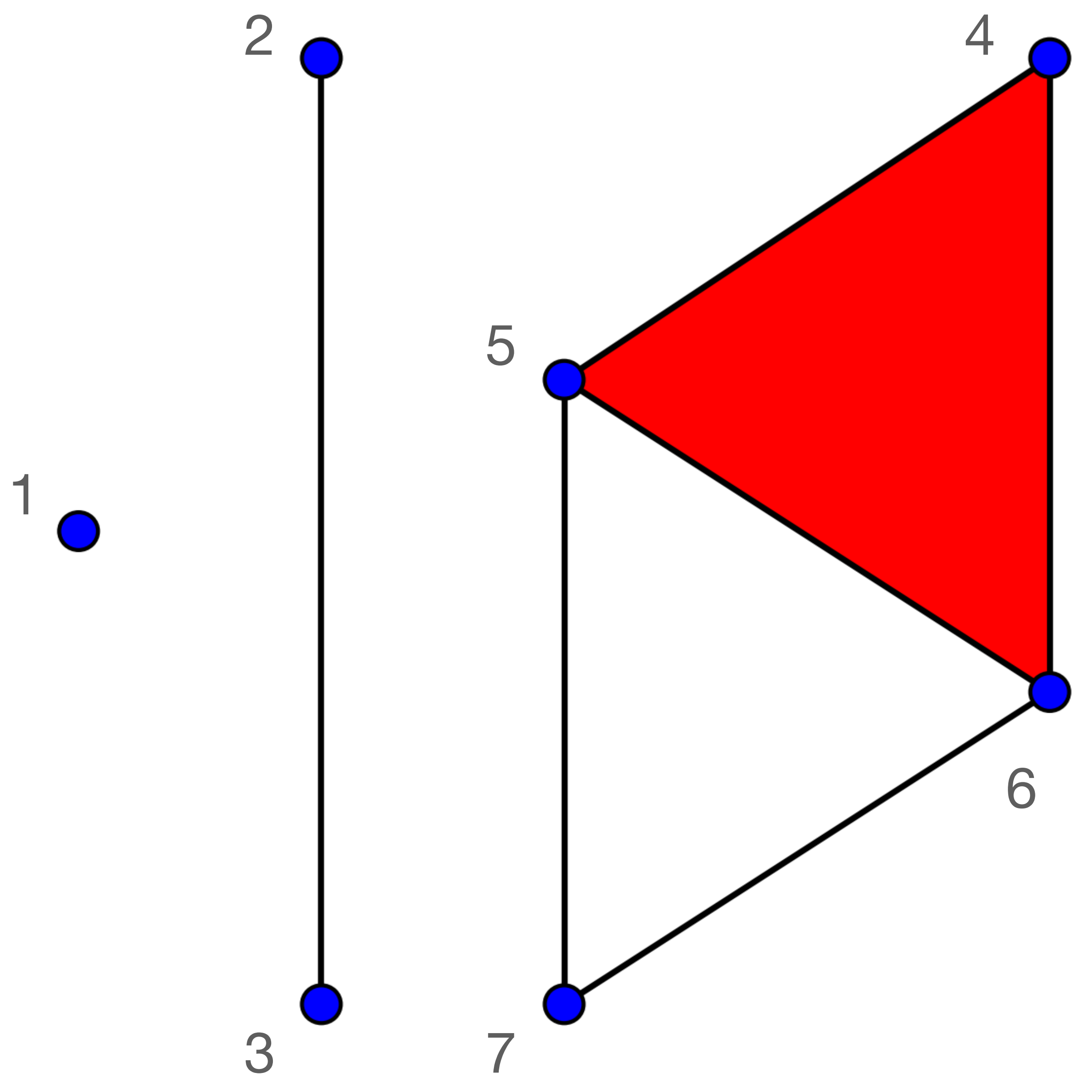
Higher-order Laplacian operator

$$C_2 \underset{\partial_2^*}{\overset{\partial_2}{\rightleftarrows}} C_1 \underset{\partial_1^*}{\overset{\partial_1}{\rightleftarrows}} C_0$$

$$\mathbf{L}_k = \mathbf{B}_k^T \mathbf{B}_k + \mathbf{B}_{k+1} \mathbf{B}_{k+1}^T$$

Betti numbers via higher-order Laplacian

$$\beta_k = \dim \ker(\mathbf{L}_k)$$



Simplicial complex *K*

$\ker(\mathbf{L}_k)$ recovers topological properties of a simplicial complex, but there is more!

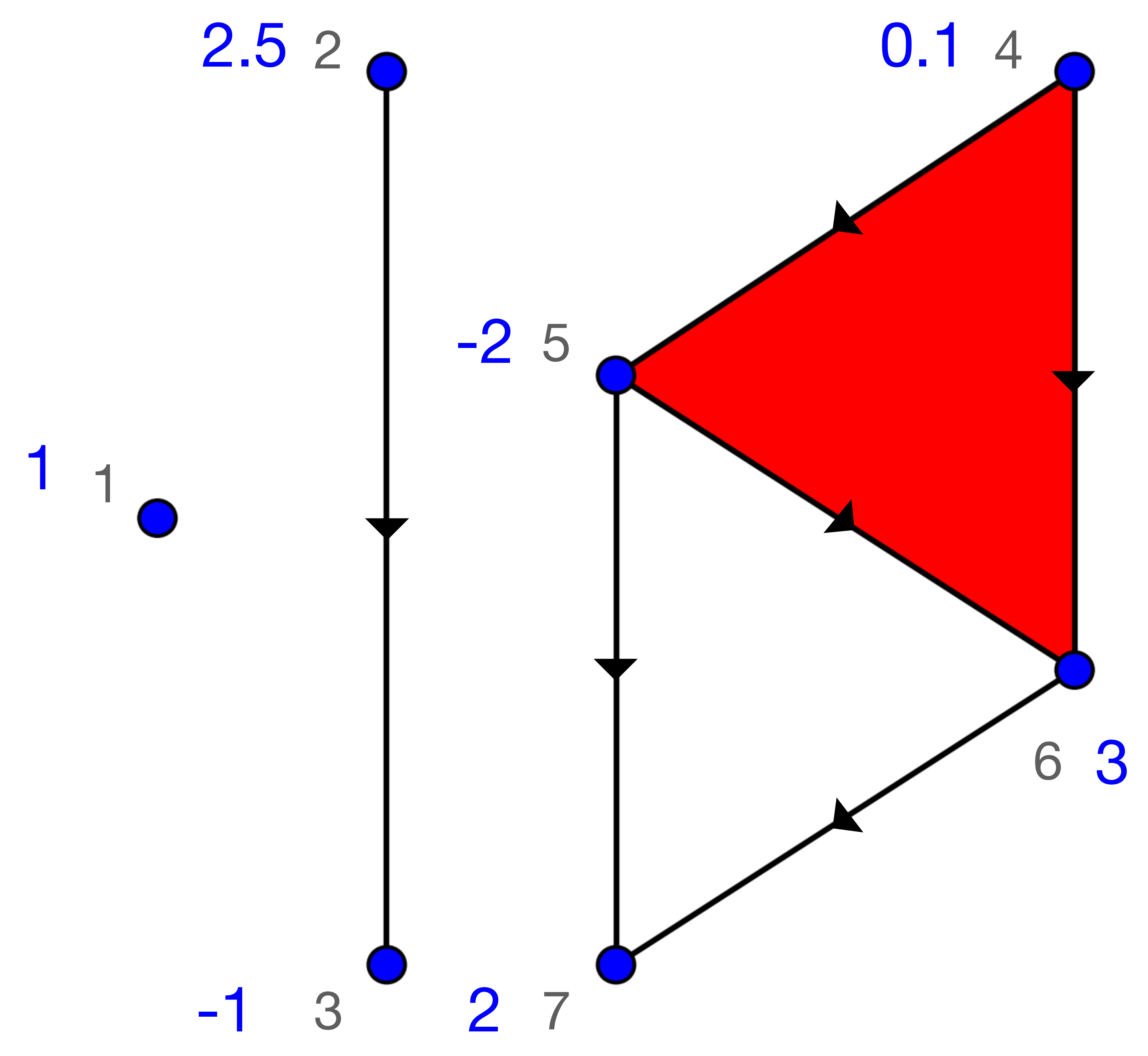
Functions on simplicial complex

Simplicial complex

$$K = \left(\Sigma_0, \Sigma_1, \ldots, \Sigma_{\dim(K)}\right)$$

Functions on a simplicial complex

$$c_k : \Sigma_k \to \mathbb{R}$$



Simplicial complex *K*

$$c_0 : \Sigma_0 \to \mathbb{R}$$

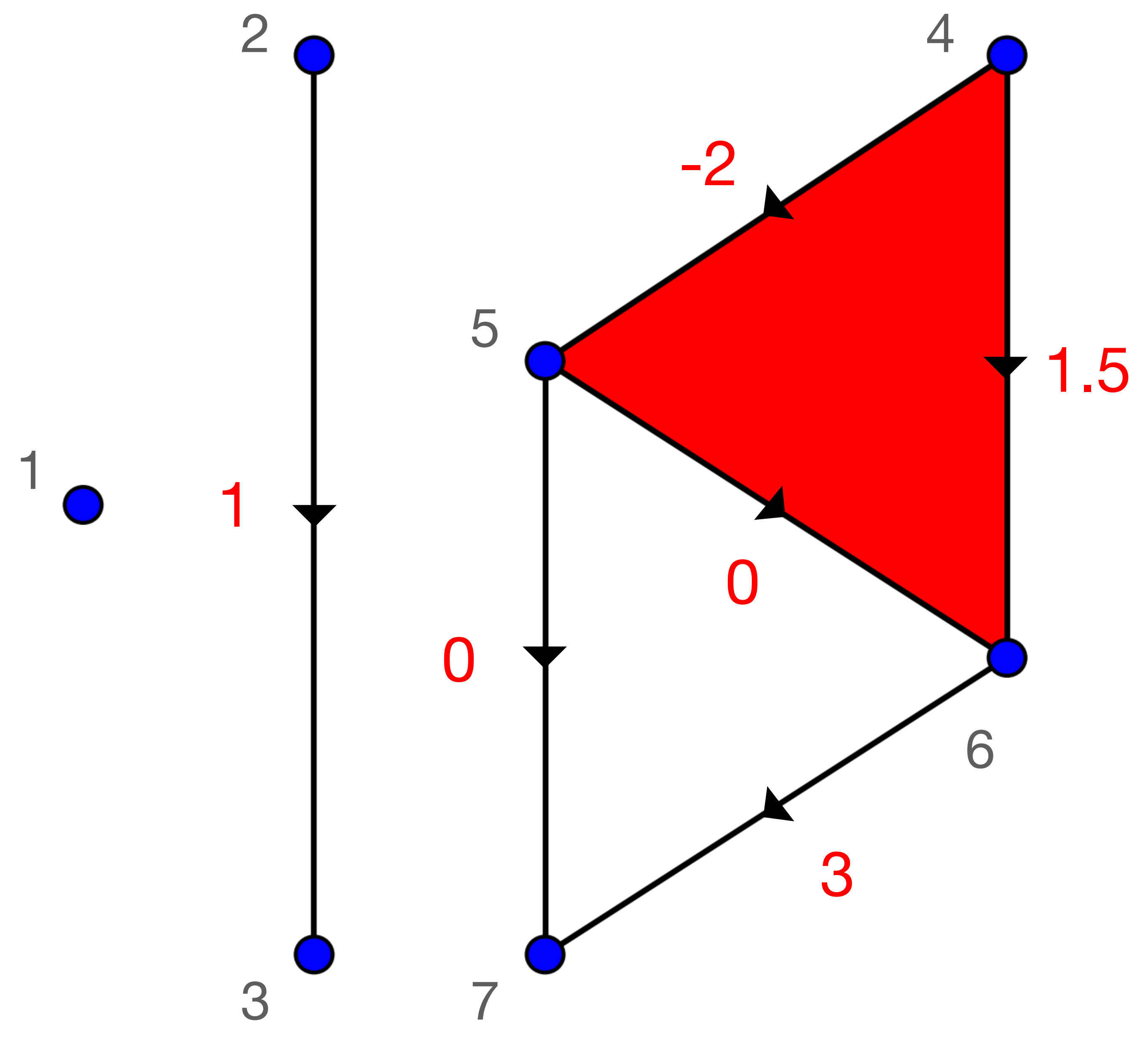
Functions on simplicial complex

Simplicial complex

$$K = \left(\Sigma_0, \Sigma_1, \ldots, \Sigma_{\dim(K)}\right)$$

Functions on a simplicial complex

$$c_k : \Sigma_k \to \mathbb{R}$$



Simplicial complex *K*

$$c_1 : \Sigma_1 \to \mathbb{R}$$

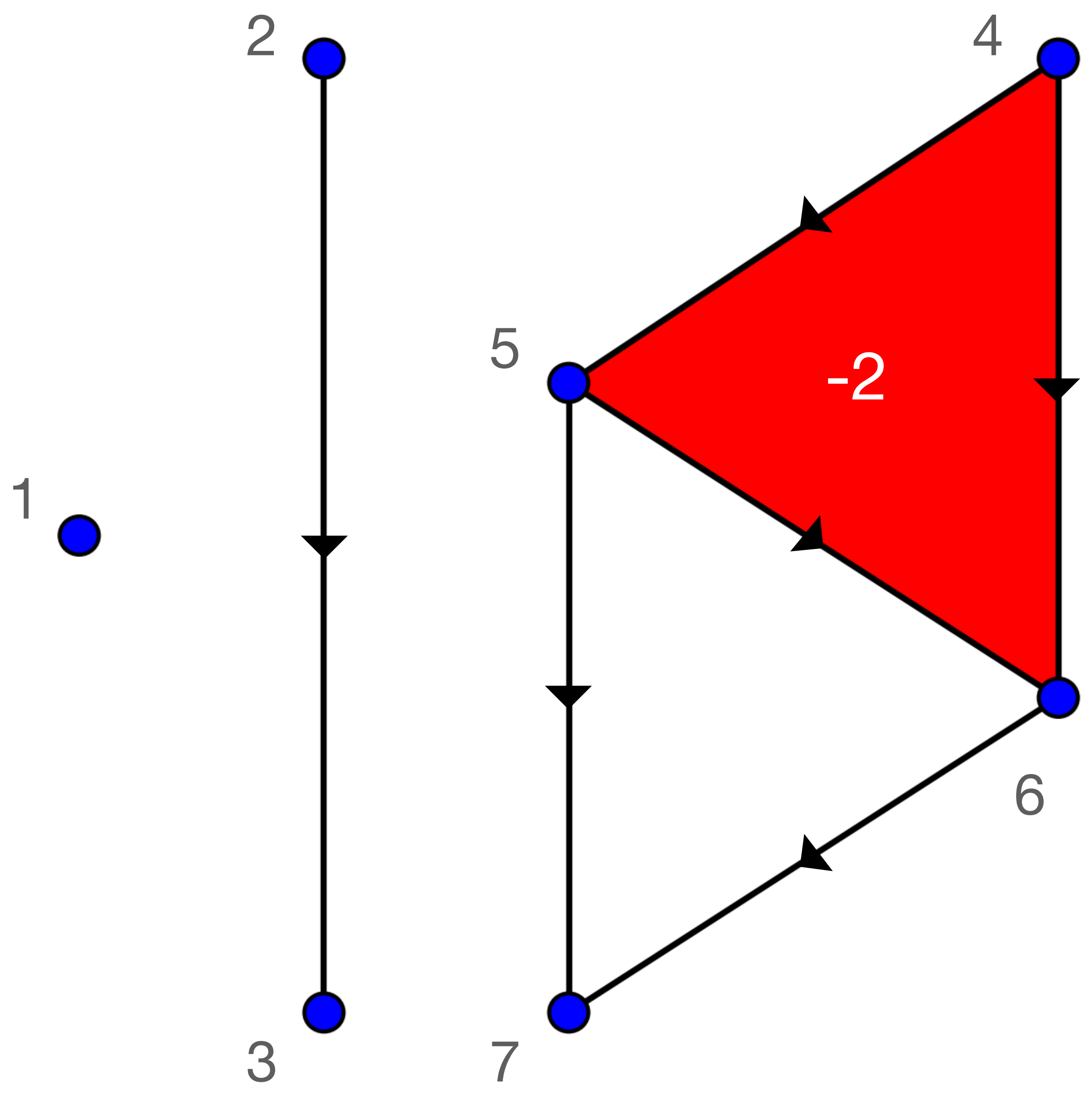
Functions on simplicial complex

Simplicial complex

$$K = \left(\Sigma_0, \Sigma_1, \ldots, \Sigma_{\dim(K)}\right)$$

Functions on a simplicial complex

$$c_k : \Sigma_k \to \mathbb{R}$$



Simplicial complex *K*

$$c_2 : \Sigma_2 \to \mathbb{R}$$

Functions on simplicial complex

Simplicial complex

$$K = \left(\Sigma_0, \Sigma_1, \ldots, \Sigma_{\dim(K)}\right)$$

Functions on a simplicial complex

$$c_k : \Sigma_k \to \mathbb{R}$$

Cochain complex

$$C^{k+1} \underset{\delta_k}{\overset{\delta_k^*}{\rightleftarrows}} C^k \underset{\delta_{k-1}}{\overset{\delta_{k-1}^*}{\rightleftarrows}} C^{k-1} \qquad \delta_k \delta_{k-1} = 0$$

Coboundary operator

$$\delta_k : C^{k+1} \to C^k$$

$$(\delta_k c)([v_0, \ldots, v_{k+1}]) = \sum_{i=0}^{k+1} (-1)^i c([v_0, \ldots, \hat{v}_i, \ldots, v_{k+1}])$$

Higher-order Laplacian operator

$$L_k = \delta_{k-1}\delta_{k-1}^* + \delta_k^* \delta_k$$

Functions on simplicial complex

Cochain complex

$$C^{k+1} \underset{\delta_k}{\overset{\delta_k^*}{\rightleftarrows}} C^k \underset{\delta_{k-1}}{\overset{\delta_{k-1}^*}{\rightleftarrows}} C^{k-1} \qquad \delta_k \delta_{k-1} = 0$$

$$H^k = \ker\, \delta_k / \ker\, \delta_{k-1}$$

Chain complex

$$C_{k+1} \underset{\partial_{k+1}^*}{\overset{\partial_{k+1}}{\rightleftarrows}} C_k \underset{\partial_k^*}{\overset{\partial_k}{\rightleftarrows}} C_{k-1} \qquad \partial_k \partial_{k+1} = 0$$

$$H_k = \ker\, \partial_k / \mathrm{im}\, \partial_{k+1}$$

Higher-order Laplacian operator

$$L_k = \delta_{k-1}\delta_{k-1}^* + \delta_k^* \delta_k$$

$$L_k = \partial_k^* \partial_k + \partial_{k+1} \partial_{k+1}^*$$

Higher-order Laplacian operator
Matrix notation

$$\mathbf{L}_k = \mathbf{B}_k^T \mathbf{B}_k + \mathbf{B}_{k+1} \mathbf{B}_{k+1}^T$$

Hodge decomposition

Chain complex

$$C_{k+1} \underset{\partial_{k+1}^*}{\overset{\partial_{k+1}}{\rightleftarrows}} C_k \underset{\partial_k^*}{\overset{\partial_k}{\rightleftarrows}} C_{k-1}$$

$$\partial_k \partial_{k+1} = 0 \iff \ker \partial_k \subseteq \mathrm{im} \partial_{k+1}$$

Higher-order Laplacian operator

$$\mathbf{L}_k = \mathbf{B}_k^T \mathbf{B}_k + \mathbf{B}_{k+1} \mathbf{B}_{k+1}^T$$

$$\mathbf{L}_k^{LOW} = \mathbf{B}_k^T \mathbf{B}_k$$

$$\mathbf{L}_k^{UP} = \mathbf{B}_{k+1} \mathbf{B}_{k+1}^T$$

Hodge decomposition

Every function on a simplicial complex (weighted oriented simplicial complex) can be decomposed into three orthogonal components

$$C_k = \mathrm{im}\ \partial_k^* \oplus \ker L_k \oplus \mathrm{im}\ \partial_{k+1}$$

$$C_k = \overbrace{\mathrm{im}\ \mathbf{B}_k^T \oplus \ker \mathbf{L}_k}^{\ker \mathbf{L}_k^{LOW}} \!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\underbrace{\phantom{\ker \mathbf{L}_k} \oplus \mathrm{im}\ \mathbf{B}_{k+1}}_{\ker \mathbf{L}_k^{UP}}$$

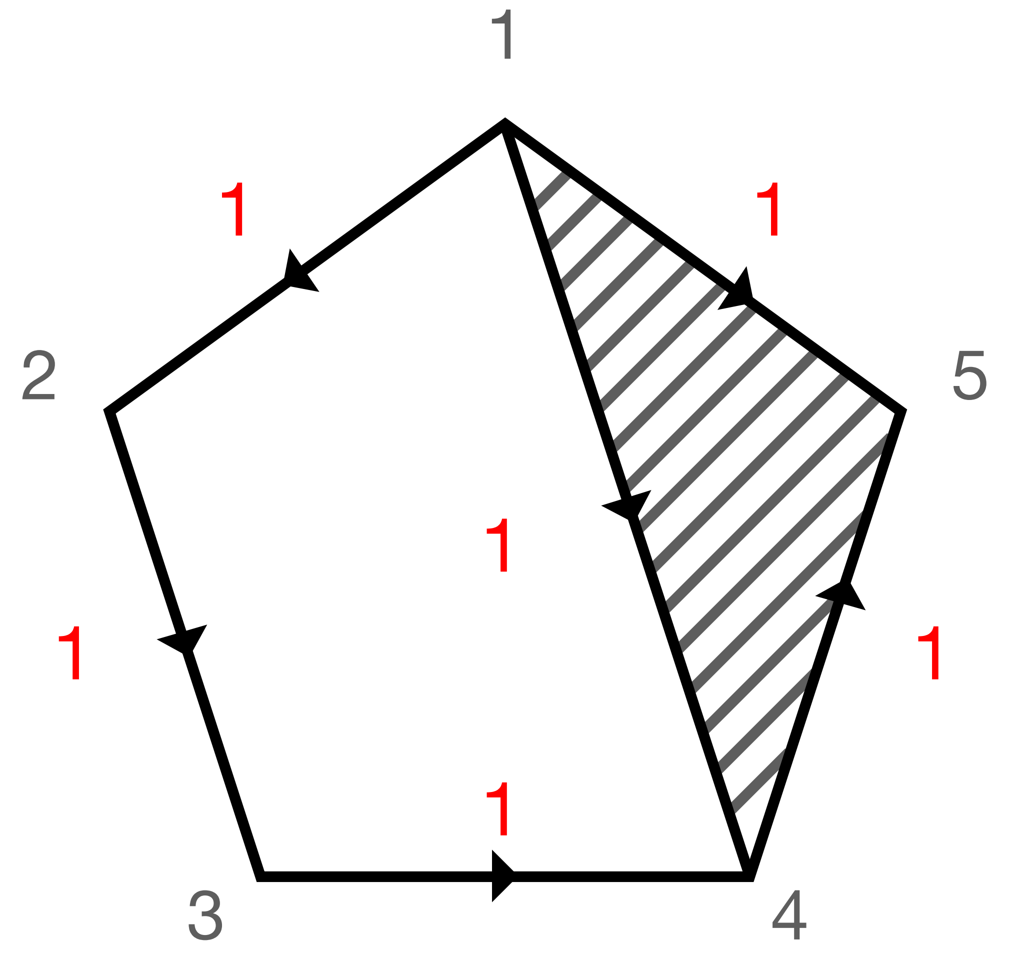
$$\mathbf{c}_k = \mathbf{B}_k^T \mathbf{c}_{k-1} + \mathbf{c}_k^H + \mathbf{B}_{k+1} \mathbf{c}_{k+1}$$

Harmonic component

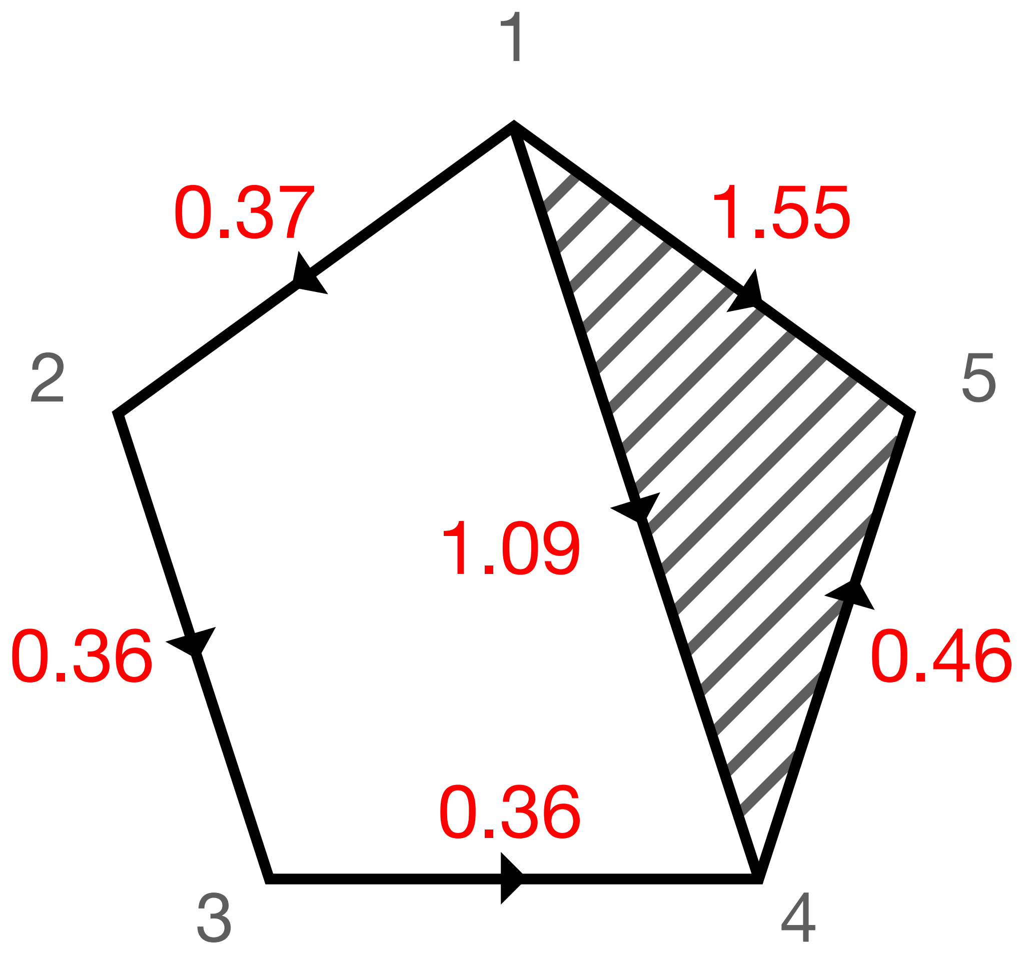
Solution of the discrete Laplace equation

$$\mathbf{L}_k \mathbf{c}_k^H = 0 \qquad \mathbf{c}_k^H \in \ker \mathbf{L}_k$$

Hodge decomposition

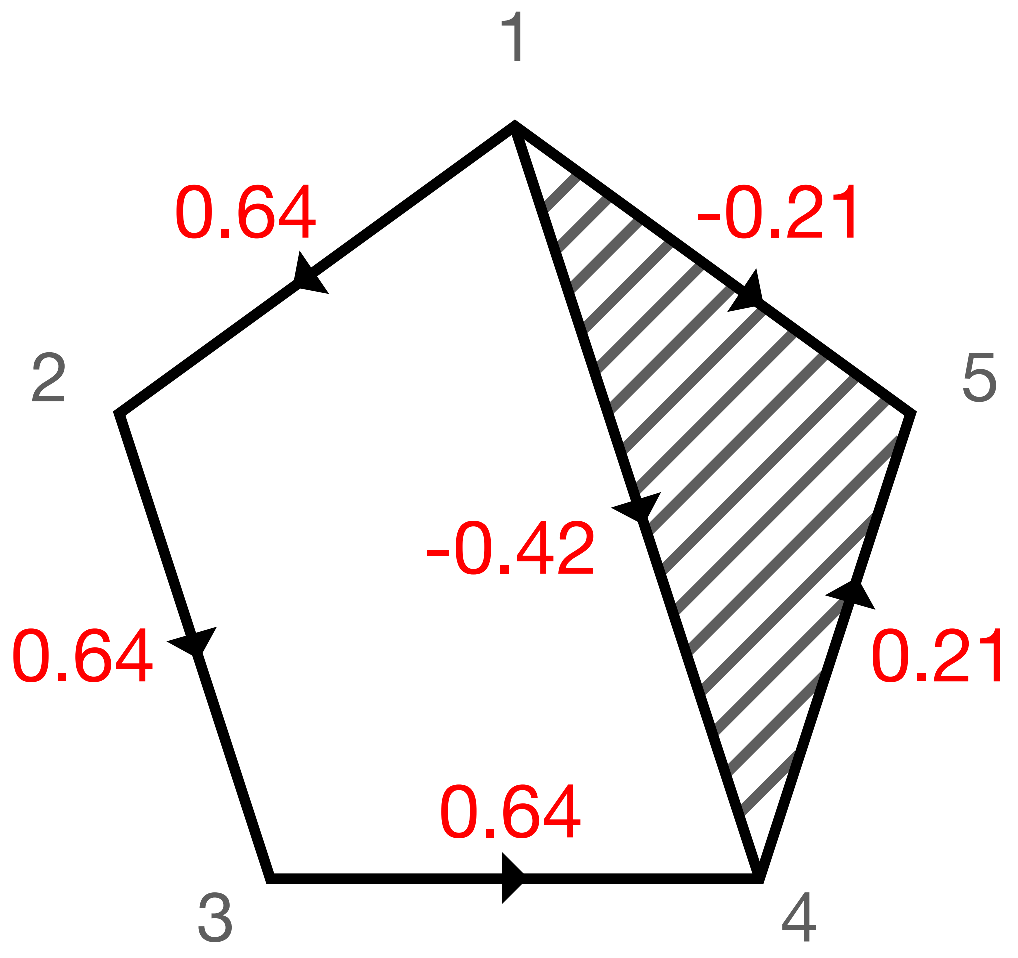


Edge signal $\mathbf{c}_1$

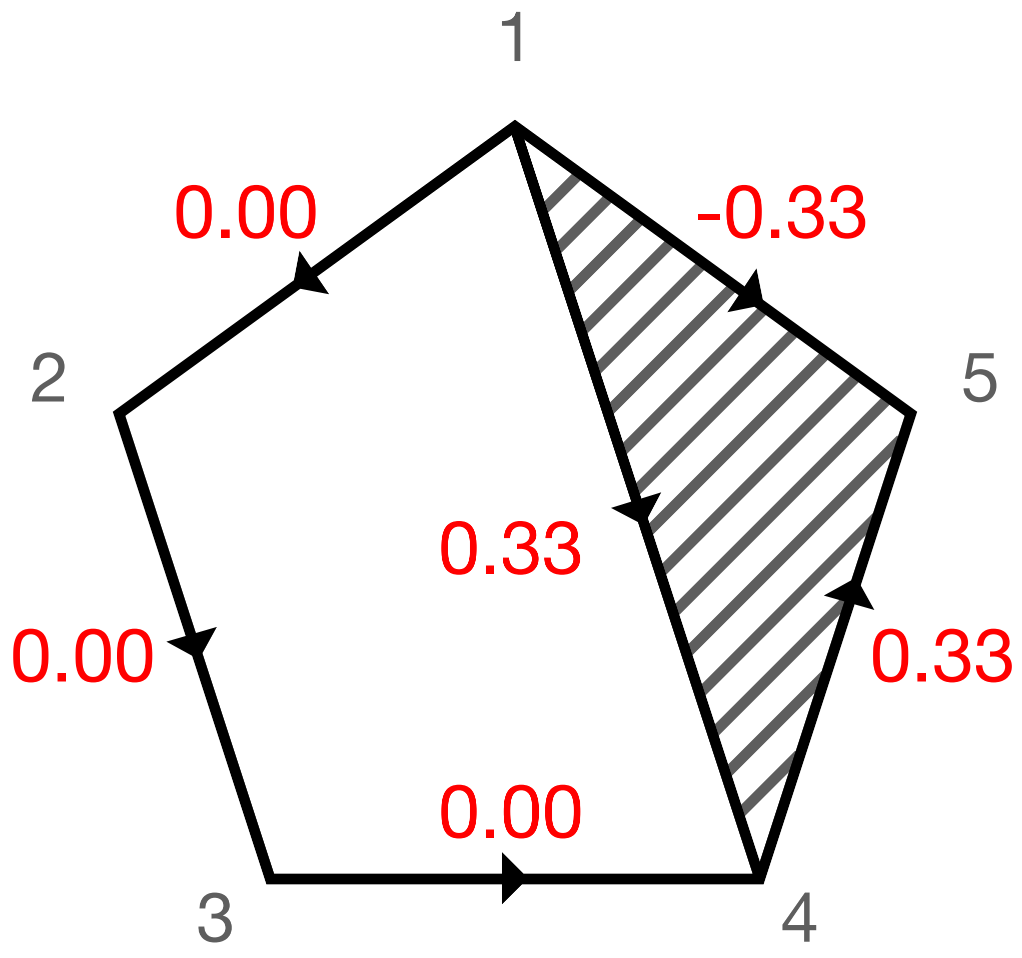


Gradient component $\mathbf{c}_1^G$

Zero curl



Harmonic component $\mathbf{c}_1^H$



Solenoidal component $\mathbf{c}_1^S$

Zero div

Hodge decomposition

Edge signals

Div

$$C_2 \underset{\partial_2^*}{\overset{\partial_2}{\rightleftarrows}} C_1 \underset{\partial_1^*}{\overset{\partial_1}{\rightleftarrows}} C_0$$

Curl **Grad**

Div

Netflow passing through a vertex

$\mathrm{div}(\mathbf{c}_1) = \mathbf{B}_1\mathbf{c}_1$

Curl

Flow around triangles edges

$\mathrm{curl}(\mathbf{c}_1) = \mathbf{B}_2^T\mathbf{c}_1$

Grad

$\mathrm{grad}(\mathbf{c}_0) = \mathbf{B}_1^T\mathbf{c}_0$

Zero curl $\mathrm{curl}(\mathbf{c}_1) = \mathbf{B}_2^T\mathbf{c}_1$

$\mathbf{B}_2\mathbf{B}_2^T\mathbf{c}_1 = 0$

$\ker \mathbf{L}_1^{LOW}$ **Zero curl**

$$C_1 = \mathrm{im}\ \mathbf{B}_1^T \oplus \ker \mathbf{L}_1 \oplus \mathrm{im}\ \mathbf{B}_2$$

Gradient **Harmonic** **Solenoidal**

$\ker \mathbf{L}_1^{UP}$ **Zero div**

$$\mathbf{c}_1 = \mathbf{B}_1^T\mathbf{c}_0 + \mathbf{c}_k^H + \mathbf{B}_2\mathbf{c}_2$$

$$\mathbf{c}_1 = \mathbf{c}_1^G + \mathbf{c}_1^H + \mathbf{c}_1^S$$

Zero div $\mathrm{div}(\mathbf{c}_1) = \mathbf{B}_1\mathbf{c}_1$

$\mathbf{B}_1\mathbf{B}_2\mathbf{c}_2 = 0$

Hodge decomposition

Given edge function $\mathbf{c}_1$

$$C_2 \underset{\partial_2^*}{\overset{\partial_2}{\rightleftarrows}} C_1 \underset{\partial_1^*}{\overset{\partial_1}{\rightleftarrows}} C_0$$

$$\mathbf{c}_1 = \mathbf{B}_1^T \mathbf{c}_0 + \mathbf{c}_k^H + \mathbf{B}_2 \mathbf{c}_2$$
$$\mathbf{c}_1 = \mathbf{c}_1^G + \mathbf{c}_1^H + \mathbf{c}_1^S$$

Div

$$\mathbf{B}_1 \mathbf{B}_1^T \mathbf{c}_0 = \mathbf{B}_1 \mathbf{c}_1$$

$$\mathrm{div}(\mathbf{c}_1) = \mathbf{B}_1 \mathbf{c}_1$$

Solve for $\mathbf{c}_0$

$$\mathbf{c}_0 = (\mathbf{B}_1 \mathbf{B}_1^T)^+ \mathbf{B}_1 \mathbf{c}_1$$

Curl

$$\mathbf{B}_2^T \mathbf{B}_2 \mathbf{c}_2 = \mathbf{B}_2^T \mathbf{c}_1$$

$$\mathrm{curl}(\mathbf{c}_1) = \mathbf{B}_2^T \mathbf{c}_1$$

Solve for $\mathbf{c}_2$

$$\mathbf{c}_2 = (\mathbf{B}_2^T \mathbf{B}_2)^+ \mathbf{B}_2^T \mathbf{c}_1$$

Gradient component $\mathbf{c}_1^G = \mathbf{B}_1^T \mathbf{c}_0$

$$\mathbf{c}_1^G = \mathbf{B}_1^T (\mathbf{B}_1 \mathbf{B}_1^T)^+ \mathbf{B}_1 \mathbf{c}_1$$

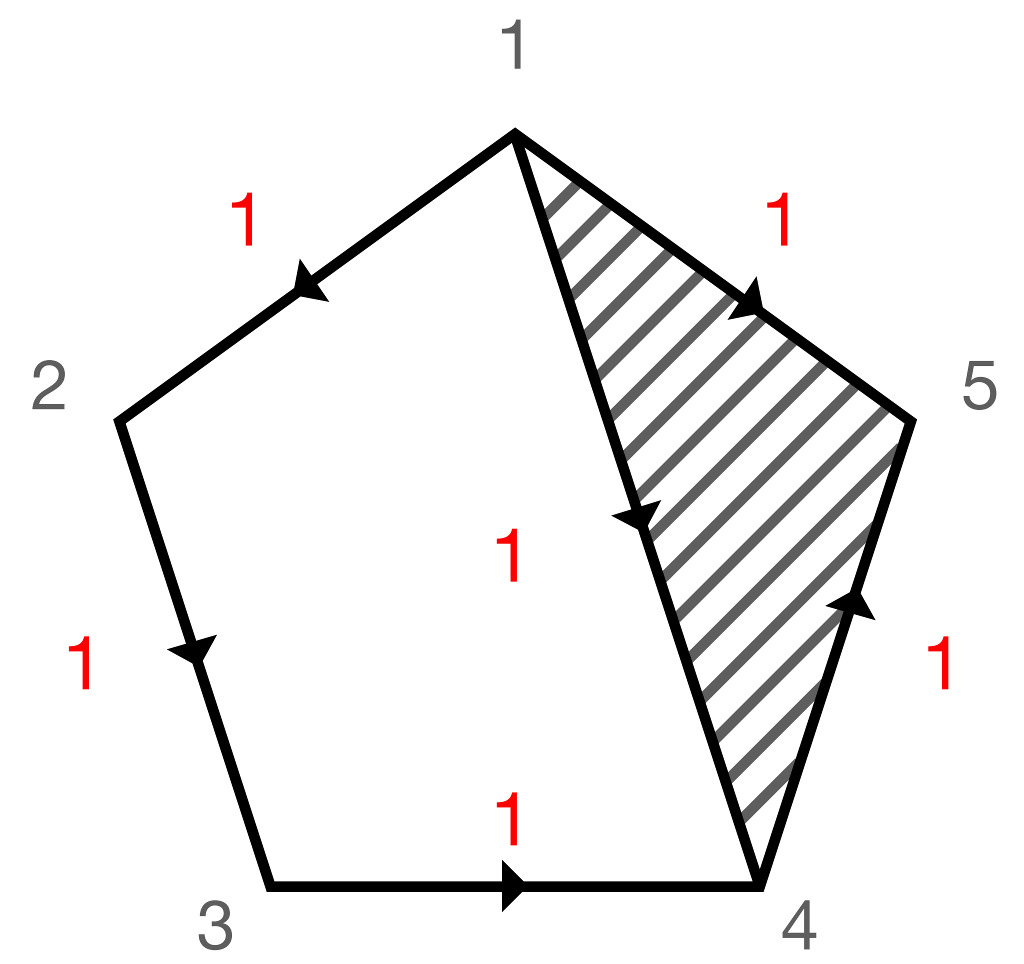
Solenoidal component $\mathbf{c}_1^S = \mathbf{B}_2 \mathbf{c}_2$

$$\mathbf{c}_1^S = \mathbf{B}_2 (\mathbf{B}_2^T \mathbf{B}_2)^+ \mathbf{B}_2^T \mathbf{c}_1$$

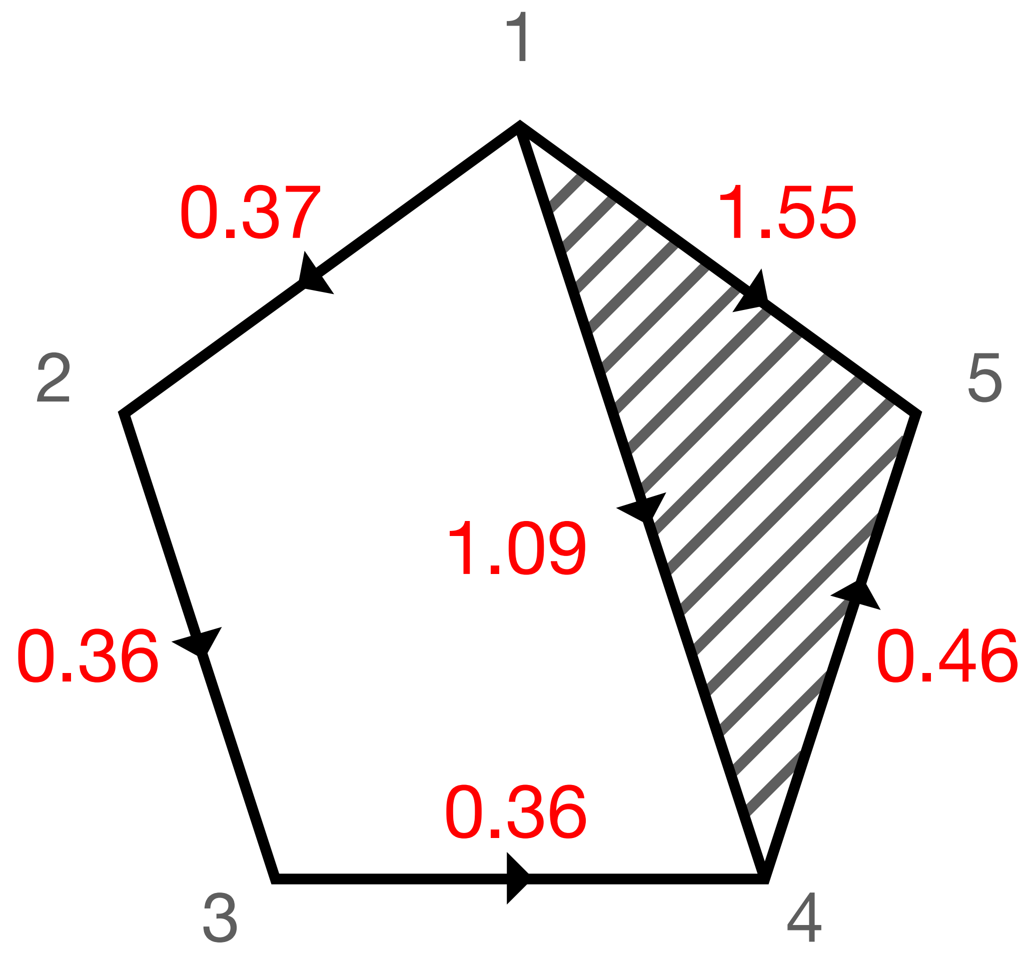
Harmonic component

$$\mathbf{c}_1^H = \mathbf{c}_1 - \mathbf{c}_1^G - \mathbf{c}_1^S$$

Hodge decomposition

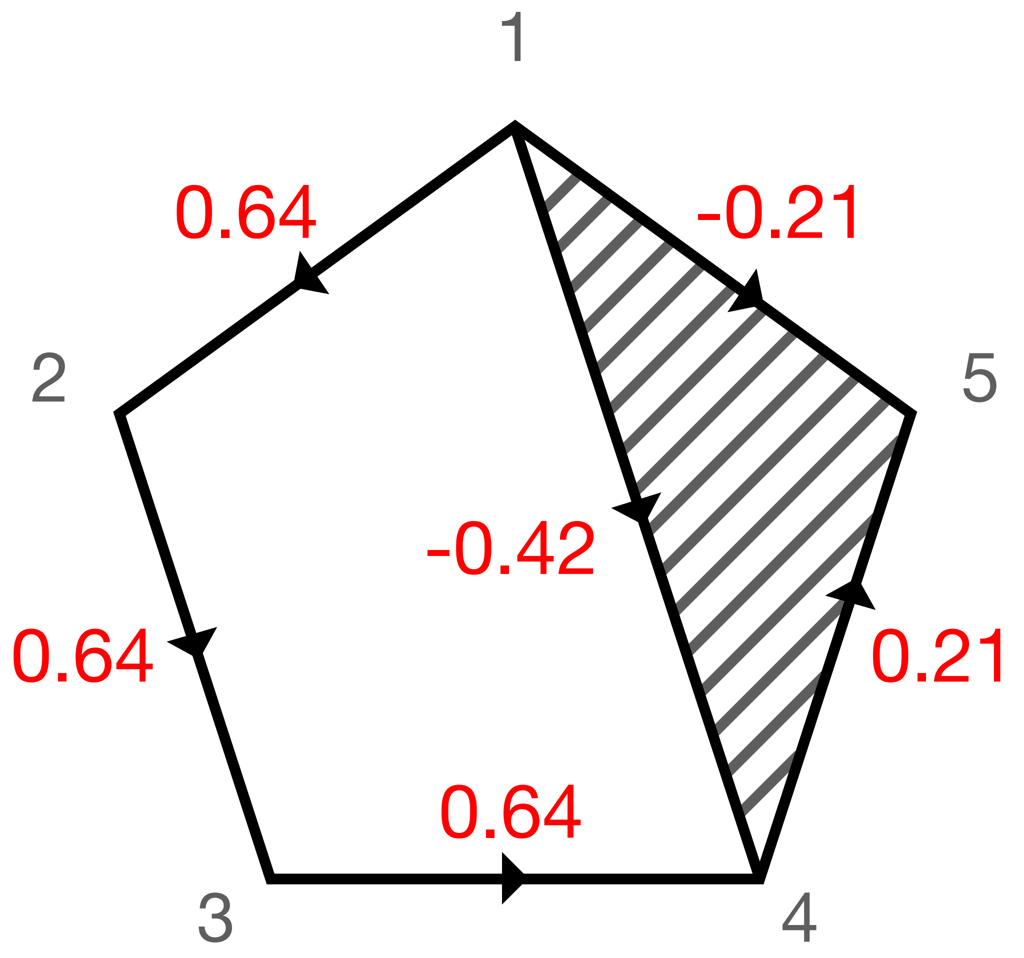


Edge signal $\mathbf{c}_1$

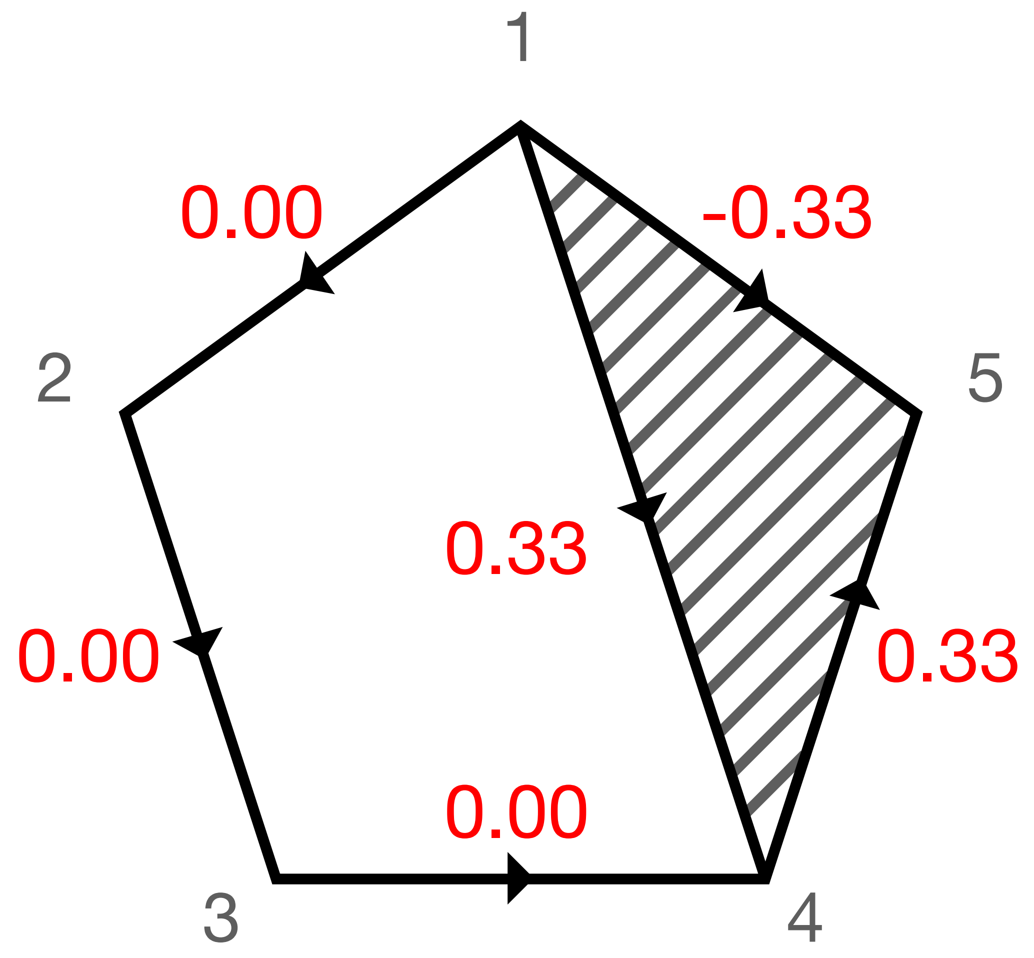


Gradient component $\mathbf{c}_1^G$

Zero curl



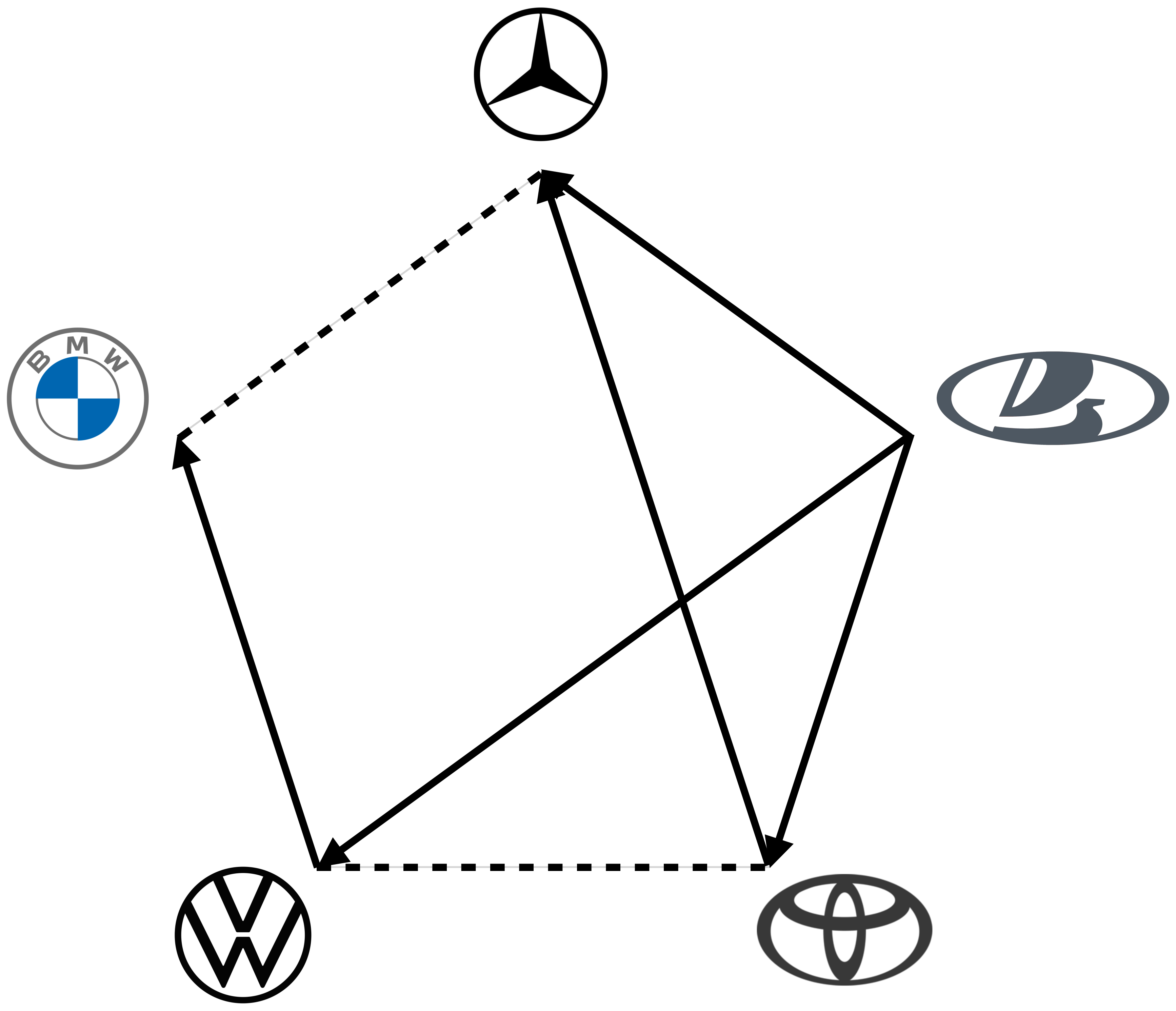
Harmonic component $\mathbf{c}_1^H$

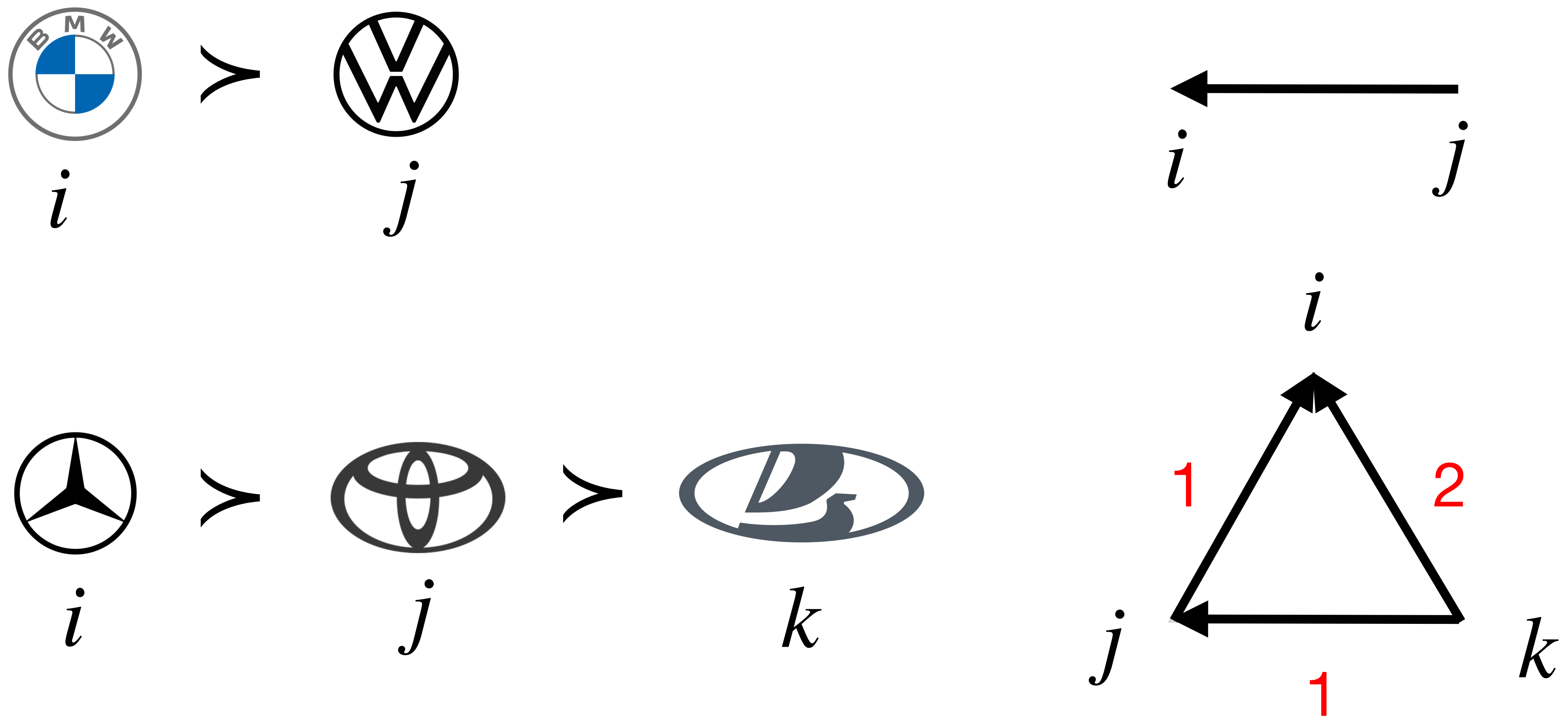


Solenoidal component $\mathbf{c}_1^S$

Zero div

Ranking problem



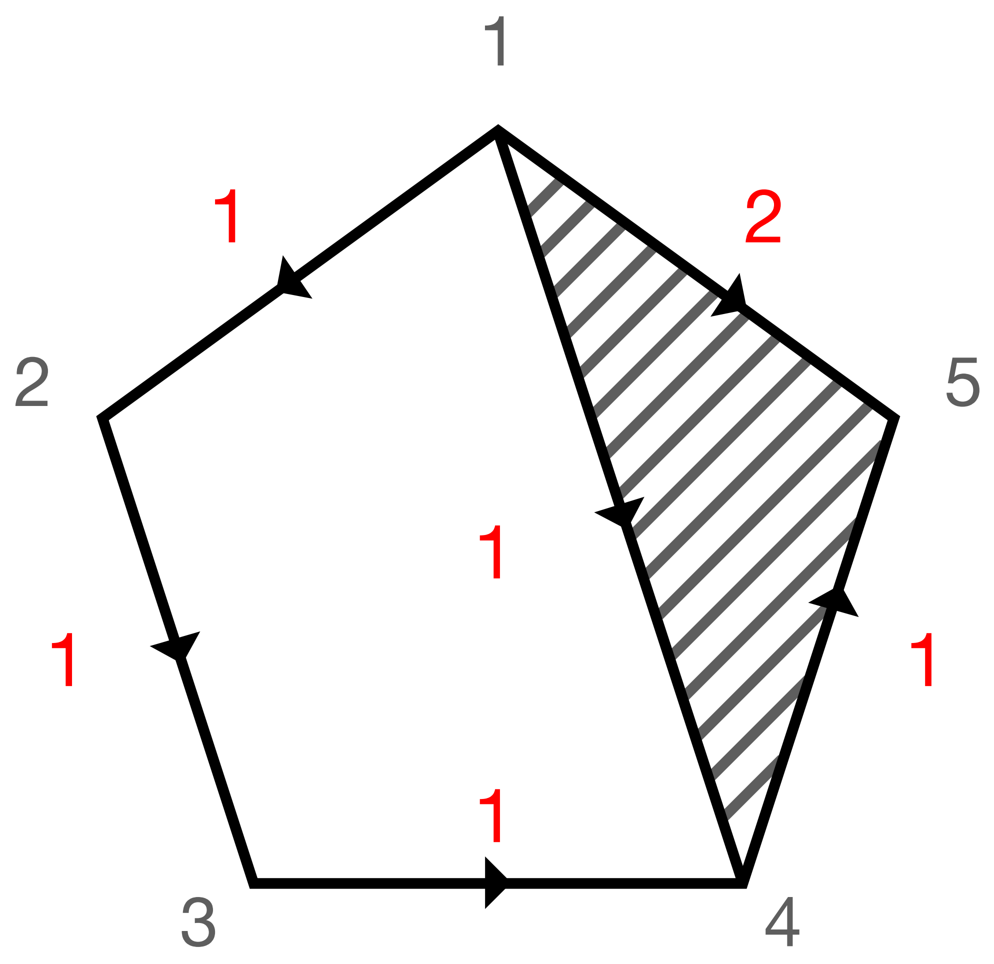


Cyclic rankings could occur!

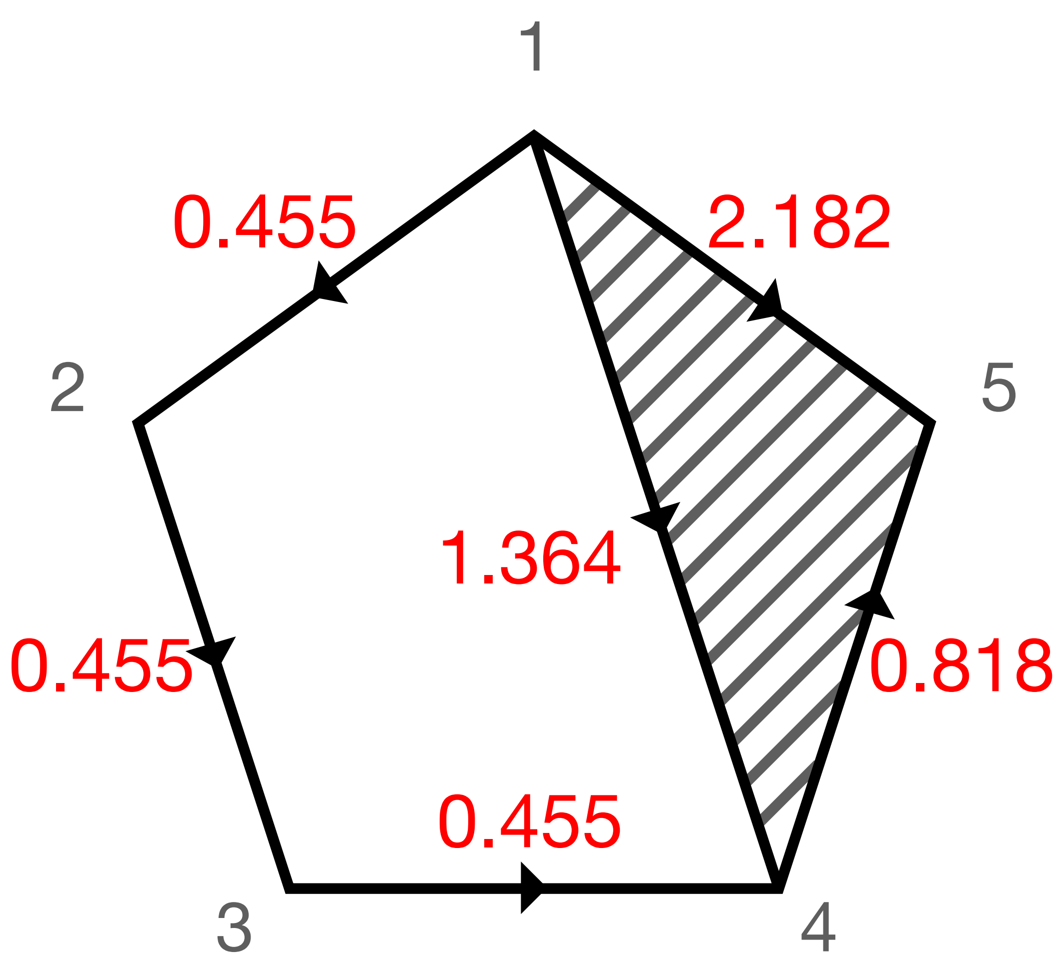
$$i \succ j \succ k \succ \ldots \succ i$$

Discrete Hodge Theory on Graphs
Johnson J., Goldring T. *Computing in Science & Engineering (2013)*

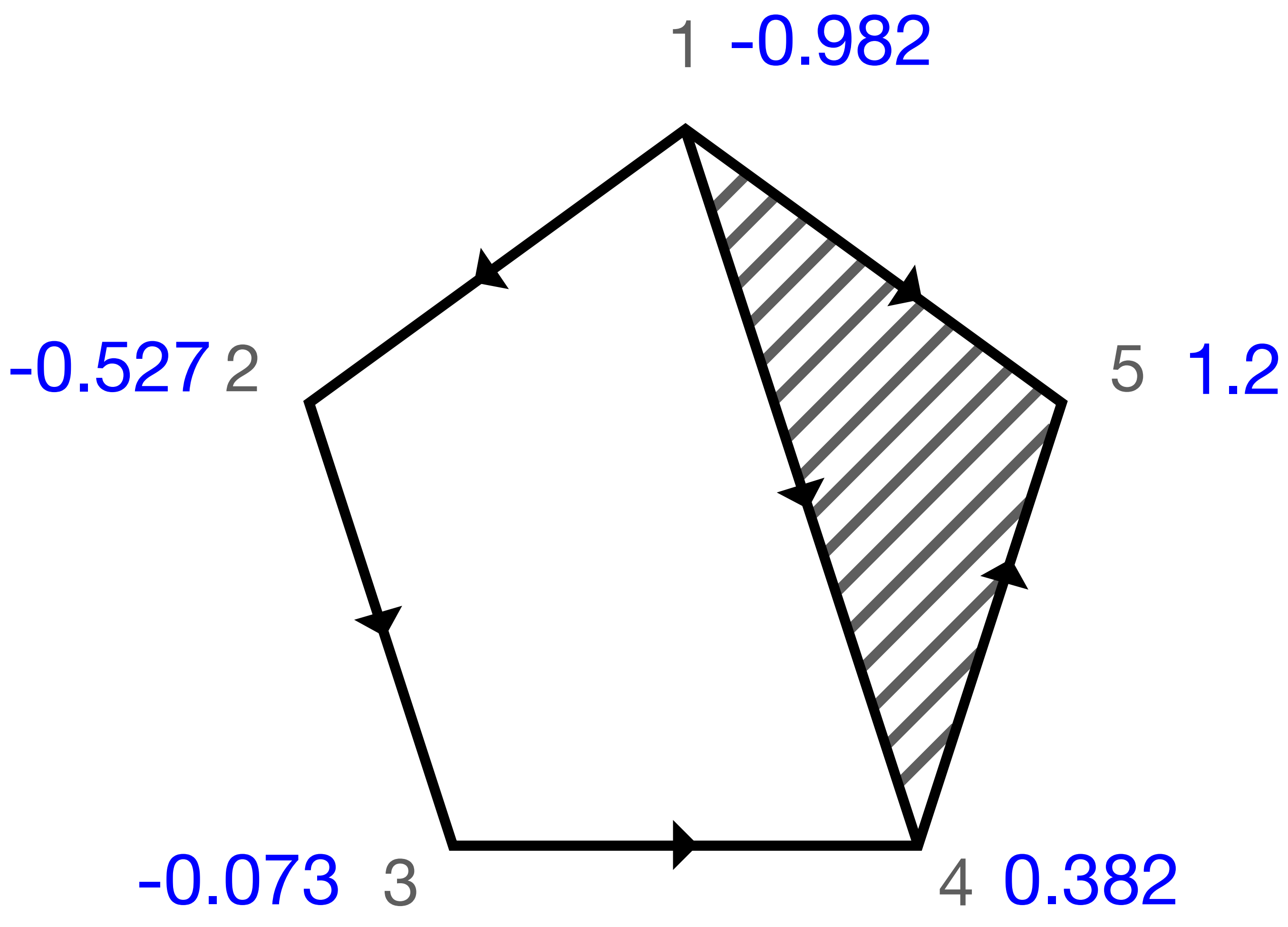
Ranking problem



Edge signal $\mathbf{c}_1$



Gradient component $\mathbf{c}_1^G$



Potential function $\mathbf{c}_0$

Gradient flow induces global ranking, given $\mathbf{c}_1$

Solve for $\mathbf{c}_0$ (potential function on vertices)

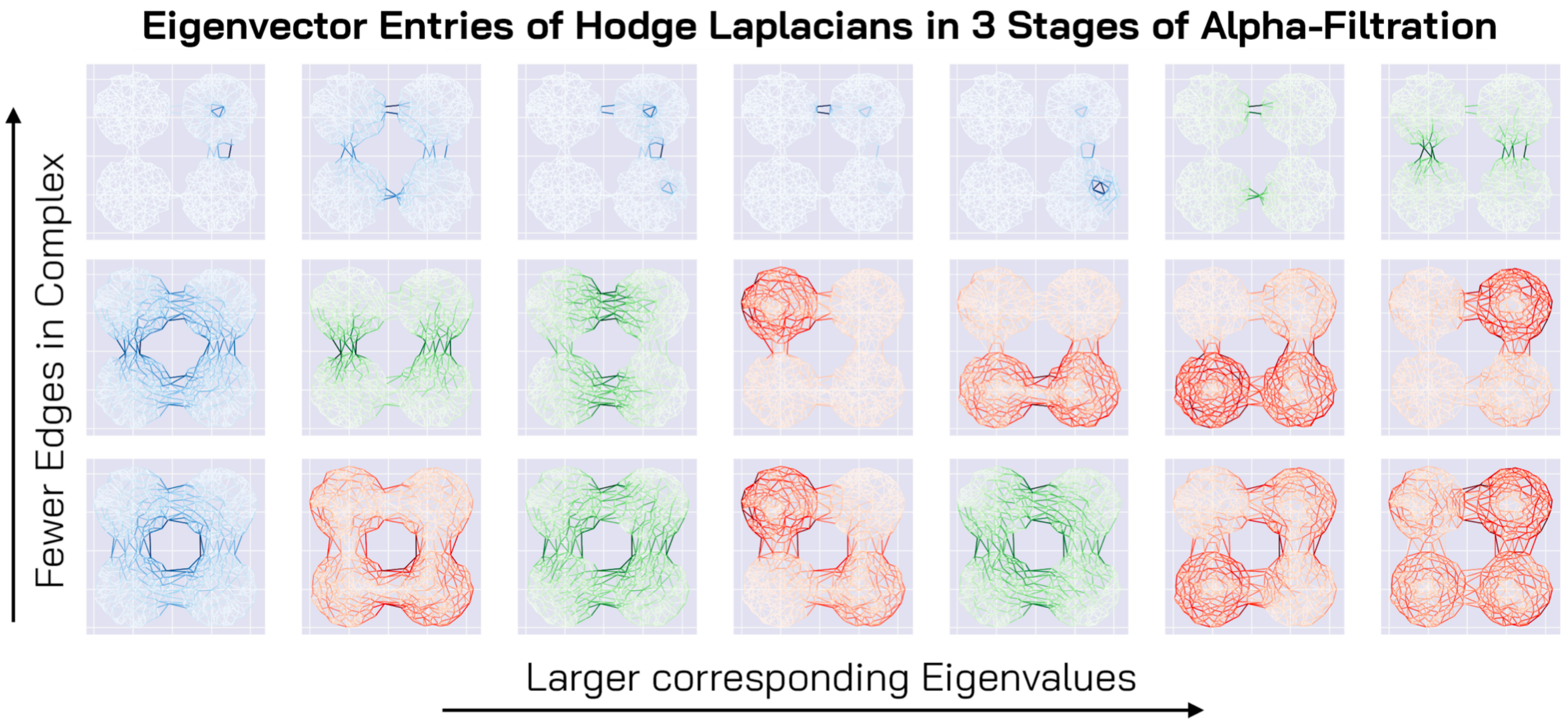
$$\mathbf{B}_1\mathbf{B}_1^T\mathbf{c}_0 = \mathbf{B}_1\mathbf{c}_1$$

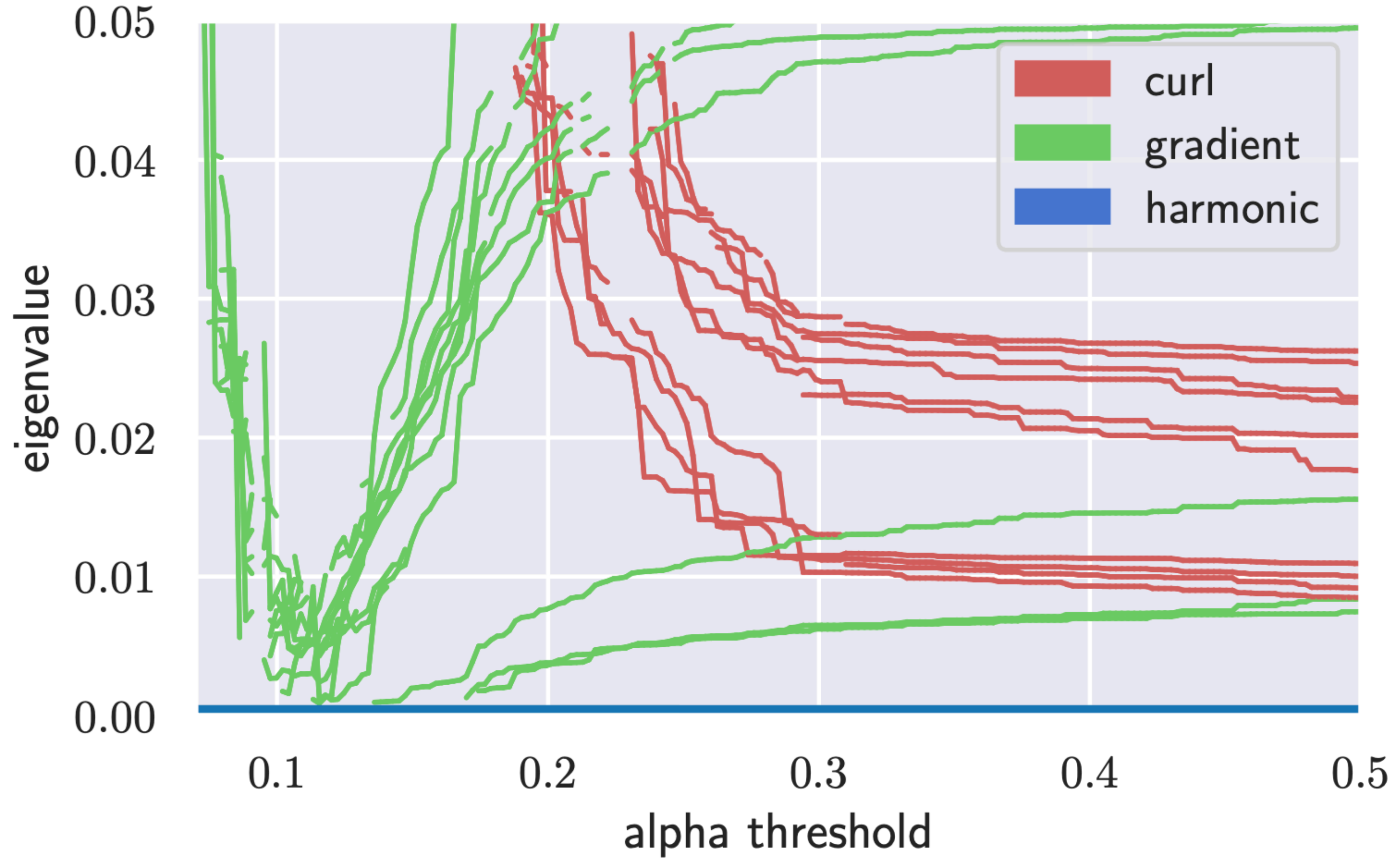
$$\mathbf{c}_0 = (\mathbf{B}_1\mathbf{B}_1^T)^+\mathbf{B}_1\mathbf{c}_1$$

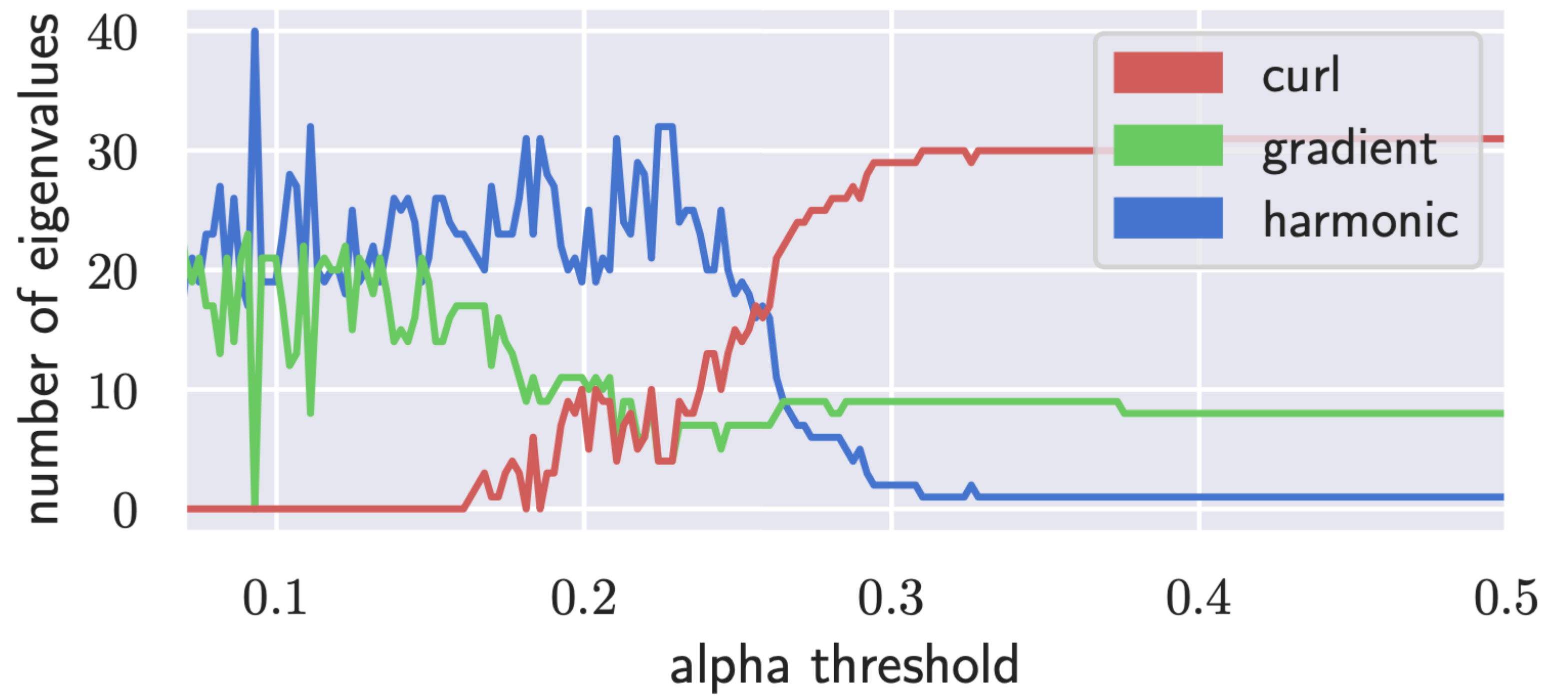
Order vertices according to the potential function

Discrete Hodge Theory on Graphs
Johnson J., Goldring T. *Computing in Science & Engineering (2013)*

Gradient, harmonic, solenoidal eigenvalues







Harmonic $\|\mathbf{L}_1^{UP}\mathbf{v}_i\| = 0,\ \|\mathbf{L}_1^{LOW}\mathbf{v}_i\| = 0$

Gradient $\|\mathbf{L}_1^{UP}\mathbf{v}_i\| = 0$

Curl $\|\mathbf{L}_1^{LOW}\mathbf{v}_i\| = 0$

Disentangling the Spectral Properties of the Hodge Laplacian: Not All Small Eigenvalues Are Equal
Grande V., Schaub M. *ICASSP (2024)*