1 Introductory notation

1.1 Markov and Lagrange values

Symbol $\mathcal{M}$ denotes double-infinite sequences from $\mathbb{N}^{\mathbb{Z}}$:

$$\mathcal{M} = ...a_{-2}a_{-1}a_0a_1a_2...$$

I will use $\lambda(\mathcal{M})$, $\mu(\mathcal{M})$ and $f(\mathcal{M})$ for Lagrange, Markov values and height function. Symbols γ and δ denote the lhs and rhs of sequence $\mathcal{M}$:

$$\gamma(\mathcal{M}) = [0; a_{-1}, a_{-2}, ...],$$
$$\delta(\mathcal{M}) = [0; a_1, a_2, ...],$$
$$f(\mathcal{M}) = a_0 + \gamma(\mathcal{M}) + \delta(\mathcal{M}).$$

At last, symbols M and L denote the Markov and Lagrange spectra.

1.2 Centered sequence

Definition. A sequence $\mathcal{M}$ is called **centered**, if

$$\mu(\mathcal{M}) = f(\mathcal{M}). \tag{1}$$

Proposition. Markov spectrum can be defined with only centered sequences:

$$\{\mu(\mathcal{M}) \mid \mathcal{M} \in \mathbb{N}^{\mathbb{Z}}\} = M = \{\mu(\mathcal{M}) \mid \mathcal{M} \text{ is centered }\}.$$

1.3 Rectangle

Designation. Denote by

$$\{a_{i_1}, a_{i_1+1}, ..., a_{i_2}\} \ (i_1 \leqslant 0 \leqslant i_2)$$

the set of double-infinite sequences $\mathcal{M}$ with fixed terms $a_{i_1}, a_{i_1+1}, ..., a_{i_2}$ on the corresponding positions.

Terms a_s for $s < i_1$ and $s > i_2$ are arbitrary integers, chosen such that $\mathcal{M}$ is centered and, maybe, satisfies some conditions.

Segments Δ_1, Δ_2 and Δ are defined by the following equations:

$$\begin{array}{lllll} \Delta_1 & = & [\,\Delta_1'\,;\,\Delta_1''\,] & = & [\,\min\gamma(\mathcal{M})\,;\,\max\gamma(\mathcal{M})\,]\,, \\ \Delta_2 & = & [\,\Delta_2'\,;\,\Delta_2''\,] & = & [\,\min\delta(\mathcal{M})\,;\,\max\delta(\mathcal{M})\,]\,, \\ \Delta & = & [\,\Delta'\,;\,\Delta''\,] & = & a_0 + \Delta_1 + \Delta_2, \end{array} \tag{2}$$

where $\mathcal{M}$ belongs to the set.

Note that we will use $'$ for the lower bound, and $''$ for the upper bound.

Definition. Rectangle is the segment Δ with the set of sequences, defining it.

1.4 Horizontal rectangle

Definition. Call a rectangle Δ **horizontal**, if

$$|\Delta_1| \geqslant |\Delta_2|. \tag{3}$$

In (3) we allow terms a_s for $s < i_1$ and $s > i_2$ to be integers $\{1, 2, 3\}$, regardless of the requirement that sequences $\mathcal{M} \in \Delta$ are centered.

In other words, Δ is horizontal, if and only if

$$\left|[0; a_{-1}, ..., a_{i_1}, \overline{3, 1}] - [0; a_{-1}, ..., a_{i_1}, \overline{1, 3}]\right| \geqslant \left|[0; a_1, ..., a_{i_2}, \overline{3, 1}] - [0; a_1, ..., a_{i_2}, \overline{1, 3}]\right|.$$

Clearly, we can always obtain a horizontal rectangle out of the vertical one, as we can reindex the sequence in the opposite direction.

1.5 Subrectangle

Consider a rectangle Δ, set by the sequence center

$$\{a_{i_1}, a_{i_1+1}, ..., a_{i_2}\}.$$

We will use a **shorter notation** for subrectangles, produced by setting integers a_i for $i < i_1$ or $i > i_2$:

$$\{b_\ell ... b_1,\ c_1 ... c_r\} \coloneqq \{b_\ell ... b_1 a_{i_1} ... a_{i_2} c_1 ... c_r\}.$$

For example:

$$\{213, 3\} \coloneqq \{213 a_{i_1} ... a_{i_2} 3\}, \tag{ex.1}$$

$$\{2, 0\} \coloneqq \{2 a_{i_1} ... a_{i_2}\}. \tag{ex.2}$$

We will also shorter the notation (2): lhs and rhs are $\Delta_1(312)$ and $\Delta_2(3)$ for subrectangle (ex.1) and $\Delta_1(2)$ and a_2 for (ex.2).

1.6 Geometrical interpretation

Consider the mapping

$$\widetilde{h} : \mathbb{N}^{\mathbb{Z}} \to \mathbb{R}^2,$$

$$\widetilde{h}(\mathcal{M}) = (\gamma(\mathcal{M}); \delta(\mathcal{M})).$$

In these terms, the Markov spectrum M is the projection of some subset $\mathcal{S} \subset C_4 \times C_4$ onto the diagonal.

Then **rectangle** Δ is indeed a rectangle $\Delta_1 \times \Delta_2$ and **subrectangles** are its subrectangles.

We will consider a family of rectangles whose projections cover the beginning of Hall's Ray.

Then we will present the algorithm to split rectangle into subrectangles so that their projections cover the projection of initial rectangle.

When we say that rectangles intersect, we, however, mean that their projections intersect.

The more «squarish» the rectangle, the easier the step.

That's why we will bound the aspect ratio of rectangles (see **good** rectangle).

Formulas to evaluate side lengths and aspect ratio are given in the section 2.

2 Calculations

2.1 Length of Δ_1 or Δ_2

Let's fix some terms of continued fraction $[0; q_1, q_2, q_3, ..., q_n]$.
We will often need to measure difference

$$[0; q_1, q_2, q_3, ..., q_n, \frac{1}{\Theta_R}] - [0; q_1, q_2, q_3, ..., q_n, \frac{1}{\Theta_L}],$$

where Θ's are some continuations of the continued fraction.
They generally look like $\Theta = \left[0; 12\overline{13}\right]$ or something[1]. We will set Θ's explicitly.
For the general proof, Θ's will be taken from Table 8.

Designation. For given continuation Θ_i, denote by ε_i the resulting continued fraction:

$$\varepsilon_i = [0; q_1, q_2, ..., q_n, \frac{1}{\Theta_i}]. \tag{4}$$

Then the following equality takes place:

$$|\varepsilon_i - \varepsilon_j| = \frac{|\Theta_i - \Theta_j|}{Q_n^2\,(1 + pQ_i)\,(1 + pQ_j)}, \tag{5}$$

where

$$p = \frac{Q_{n-1}}{Q_n}.$$

2.2 Rectangle aspect ratio

Consider some fixed center of rectangle $\{a_{i_1} ... a_{i_2}\}$.
We will often extend it from the left (right) using continuations $\Theta_{\gamma_1}, \Theta_{\gamma_2}$ $(\Theta_{\delta_1}, \Theta_{\delta_2})$.
Rule (4) produces γ_1, γ_2 (δ_1, δ_2).
Finite continued fractions $\frac{P_{i_1}}{Q_{i_1}} = [0; a_{-1}, a_{-2}, ..., a_{i_1}]$ $\left(\frac{P_{i_2}}{Q_{i_2}} = [0; a_1, a_2, ..., a_{i_2}]\right)$ are their convergents.
Then

$$\left|\frac{\gamma_1 - \gamma_2}{\delta_1 - \delta_2}\right| = \frac{1}{q}\left|\frac{\Theta_{\gamma_1} - \Theta_{\gamma_2}}{\Theta_{\delta_1} - \Theta_{\delta_2}}\right| \frac{1 + p'\Theta_{\delta_1}}{1 + p\Theta_{\gamma_1}} \frac{1 + p'\Theta_{\delta_2}}{1 + p\Theta_{\gamma_2}} \approx \frac{1}{q}\left|\frac{\Theta_{\gamma_1} - \Theta_{\gamma_2}}{\Theta_{\delta_1} - \Theta_{\delta_2}}\right|, \tag{6}$$

where

$$p = \frac{Q_{i_1+1}}{Q_{i_1}}, \quad p' = \frac{Q_{i_2-1}}{Q_{i_2}}, \quad q = \frac{Q_{i_1}^2}{Q_{i_2}^2}.$$

[1]Here, as always in this book, $\overset{k}{\overline{abc}}$ means k-times repetition of abc, and $\overline{abc}$ means infinite repetition.

3 Rectangle boundaries

We will set boundaries for rectangles in a different way.

To distinguish rectangles, we will introduce the notion of left- or right-shortened rectangles.

It is given in subsections 3.1-3.3.

Subsections 3.4-3.5 provide the rules for boundaries Δ_1', Δ_1'', Δ_2', Δ_2''.

3.1 Resection

Definition. Call **resection** of a segment $A = [a; b]$ a process of removing subsegment $A_{12} = [a_1; b_1]$, leaving two segments $A_1 \sqcup A_2 = [a; a_1] \sqcup [b_1; b]$.

Definition. Call subsegment $A_{12} \subset A$ **normal**, if it is thicker than the two remaining subsegments:

$$|A_{12}| \leqslant min \{|A_1|,\ |A_2|\} \tag{7}$$

We call a resection **normal** if the resected subsegment is normal.

Proposition. For any normal resection, having

$$A + A = (A_1 \sqcup A_2) + (A_1 \sqcup A_2). \tag{8}$$

3.2 Shortened rectangle

Consider a horizontal rectangle Δ.

3.2.1 Case $i_1 \equiv i_2 \equiv 0 \pmod 2$

If both i_1 and i_2 are even, then left- and right-shortened rectangles are defined as follows.

Definition. Rectangle Δ is called left-shortened, if for subrectangle $\{3, 3\}$ the following condition takes place:

$$|\Delta_1(3)| \leqslant 1.4 \cdot |\Delta_2(3)|. \tag{9}$$

Here, as in (3), we allow terms a_s for $s < i_1$ and $s > i_2$ to be integers $\{1, 2, 3\}$, so (9) can be rewritten as

$$\left|\left[0; a_{-1}, ..., a_{i_1}, 3, \overline{3,1}\right] - \left[0; a_{-1}, ..., a_{i_1}, \overline{3,1}\right]\right| \leqslant \left|\left[0; a_1, ..., a_{i_2}, 3, \overline{3,1}\right] - \left[0; a_1, ..., a_{i_2}, \overline{3,1}\right]\right|.$$

For further convenience, we also introduce the opposite to (9) condition:

Definition. Rectangle Δ is called left-normal, if

$$|\Delta_1(3)| > 1.4 \cdot |\Delta_2(3)|. \tag{10}$$

Now we introduce the notion of right-shortened rectangle:

Definition. Rectangle Δ is called right-shortened, if for subrectangle $\{31, 13\}$ the following condition takes place:

$$|\Delta_1(13)| \leqslant 1.4 \cdot |\Delta_2(13)|. \tag{11}$$

As in 3.2.1, we allow terms a_s for $s < i_1$ and $s > i_2$ to be integers $\{1, 2, 3\}$, so (11) can be rewritten as

$$\left|\left[0; a_{-1}, ..., a_{i_1}, 1, 3, \overline{3, 1}\right] - \left[0; a_{-1}, ..., a_{i_1}, \overline{1, 3}\right]\right| \leqslant \left|\left[0; a_1, ..., a_{i_2}, 1, 3, \overline{3, 1}\right] - \left[0; a_1, ..., a_{i_2}, \overline{1, 3}\right]\right|.$$

We also introduce the opposite condition:

Definition. Rectangle Δ is called right-normal, if

$$|\Delta_1(13)| > 1.4 \cdot |\Delta_2(13)|. \tag{12}$$

3.2.2 Other cases

If $i_1 \equiv i_2 \pmod 2$ and i_1 is odd, then the conditions for left- and right-shortened rectangles are swapped.

In case $i_1 \not\equiv i_2 \pmod 2$, the notions of left- and right-shortened rectangles are inferred from subrectangles $\{1, \}$, $\{2, \}$, and $\{3, \}$.

We will refer to the relevant conditions from 3.5

3.3 Explanation of shortened

Notion left-shortened is used to set the lower bound Δ' for rectangle Δ.

Notion right-shortened is used to set the upper bound Δ''.

For example, when interested in Δ' when i_1 is even and i_2 is odd, one has to consider larger a_{i_1-1} and smaller a_{i_2+1}. As the rectangle is horizontal, we will add terms to the left first. So we will check the condition (11) for $\{2, \}$ or $\{3, \}$.

In other words, check whether $\{2, \}$ or $\{3, \}$ is left-shortened or left-normal (recall that $i_1 - 1$ and i_2 are odd, so left-shortenence is defined by (11)).

Which subrectangle ($\{2, \}$ or $\{3, \}$) to take, however, depends on terms a_{i_1+1} and a_{i_1} (see 3.4).

3.4 How we ensure centeredness

Consider integers $q_1, q_2, ..., q_n$. Let $\{\delta_n\}$ be the set of fractions $\delta_n = [0; q_1, q_2, ..., q_n, ...]$ with n fixed terms. We will suppose that n is even (for odd n the bounds are swapped).

At first, determine the smallest of fractions δ_n.

We will consider 2 cases: S (<u>S</u>hortened) and N (<u>N</u>ormal):

$$S.\ q_{n-1} = 3, q_n = 1. \tag{13a}$$
$$N.\ \text{Otherwise.} \tag{13b}$$

The lower bound δ'_n for segment, containing δ_n, is defined by:

$$\begin{array}{ll} S. & \delta'_n = [0; q_1, ..., q_n, 213\overline{12}]. \\ N. & \delta'_n = [0; q_1, ..., q_n, 3\overline{12}]. \end{array}$$

To set the upper bound δ''_n, largest of δ_n, consider 2 other cases:

$$S. \quad q_n = 3. \tag{14a}$$
$$N. \quad q_n \neq 3. \tag{14b}$$

Then

$$\begin{array}{ll} S. & \delta''_n = [0; q_1, ..., q_n, 1213\overline{12}]. \\ N. & \delta''_n = [0; q_1, ..., q_n, 13\overline{12}]. \end{array}$$

These bounds will allow us to construct sequences $\mathcal{M}$, for which combination (31313) is forbidden and, therefore, the following condition takes place:

$$f_i(\mathcal{M}) \leqslant \lambda(\overline{31312}) \approx 4,5241, \ i \neq 0,$$

which will ensure (1).

3.5 Formal

Let's now turn to concrete definitions and bounds. Remind that we are looking at the horizontal rectangle Δ.

Suppose that $i_1 \equiv i_2 \pmod 2$, i_1 is even. (If i_1 is odd, then Δ' and Δ'' are swapped.)

3.5.1 Bounds for Δ'

I. Suppose both $\{\gamma_0(\mathcal{M})\}$ and $\{\delta_0(\mathcal{M})\}$ satisfy (13b) and the rectangle Δ is left-normal, that is, satisfies (10). We will denote such situation by $N - N - N$ (segment Δ_1 is left-normal, Δ_2 is left-normal, rectangle Δ is normal).

   In this case define Δ' by equation

$$\Delta' = f(\overline{21}3a_{i_1}...a_{i_2}3\overline{12}). \tag{15}$$

IIa. Sets $\{\gamma_0(\mathcal{M})\}$ and $\{\delta_0(\mathcal{M})\}$ meet condition (13b) and Δ is left-shortened, that is, meets condition (9). This is case $N - N - S$ (segments Δ_1 and Δ_2 are left-normal, rectangle Δ is left-shortened).

   Then

$$\Delta' = f(\overline{21}3a_{i_1}...a_{i_2}213\overline{12}). \tag{16}$$

IIb. Set $\{\gamma_0(\mathcal{M})\}$ meets (13b), $\{\delta_0(\mathcal{M})\}$ meets (13a). No matter which condition (9) of (10) is met. It is the case $N - S$ (segment Δ_1 is left-normal, segment Δ_2 left-shortened). Bound Δ_1 is defined by (16).

III. Set $\{\gamma_0(\mathcal{M})\}$ meets (13a), $\{\delta_0(\mathcal{M})\}$ meets (13b). In this $S - N$ case (segment Δ_1 is left-shortened, segment Δ_2 is left-normal) Δ' is defined as follows:

$$\Delta' = f(\overline{21}312a_{i_1}...a_{i_2}3\overline{12}). \tag{17}$$

IV. Both $\{\gamma_0(\mathcal{M})\}$ and $\{\delta_0(\mathcal{M})\}$ meet (13a). In this $S - S$ case (both segments Δ_1 and Δ_2 are left-shortened) Δ' is defined by

$$\Delta' = f(\overline{21}312a_{i_1}...a_{i_2}213\overline{12}). \tag{18}$$

Left side of the figure 1 illustrates these bounds.

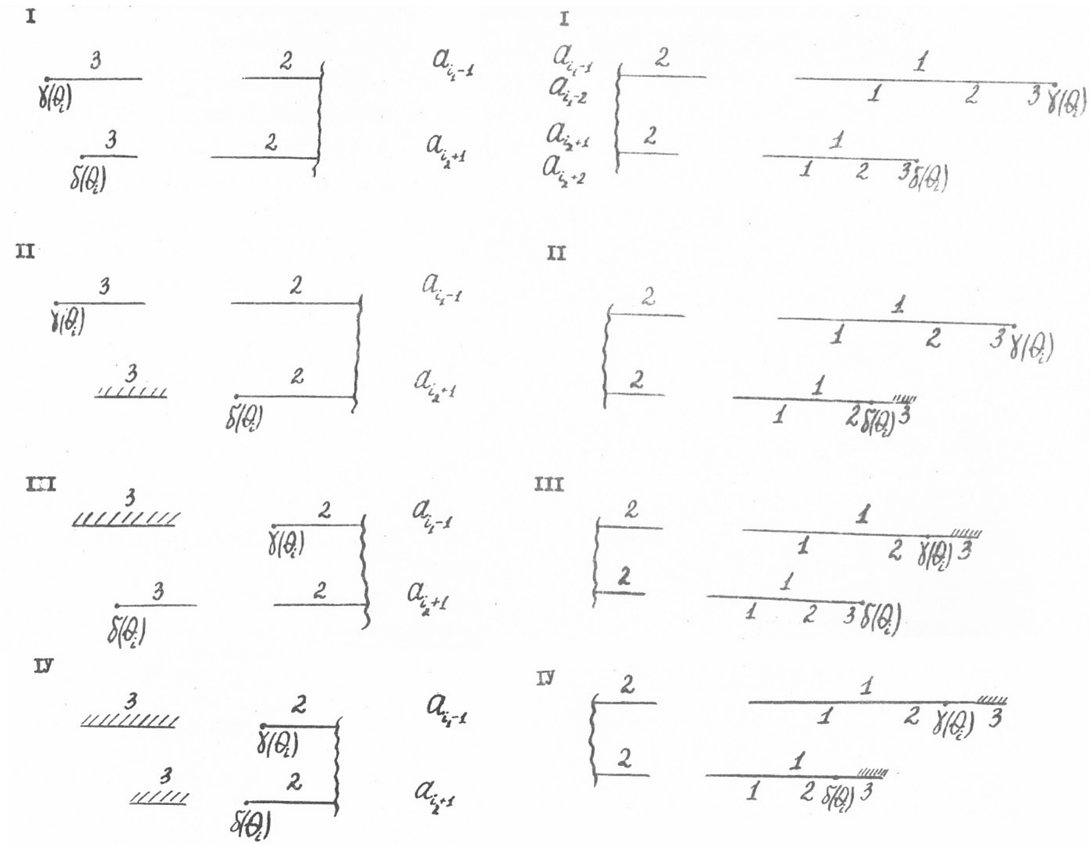


Figure 1: Bounds Δ' (left) and Δ'' (right).
Hatched areas correspond to values of a_{i_1-1} and a_{i_2+1} (on the left) or a_{i_1-2} and a_{i_2+2} (on the right), equal to 3, which can not appear in the concrete case.
$\Delta'_1 = \gamma(\Theta_i)$, $i = 3, 30$, $\Delta'_2 = \delta(\Theta_i)$, $i = 3, 30$, $\Delta' = \Delta'_1 + \Delta'_2$.
$\Delta''_1 = \gamma(\Theta_i)$, $i = 90, 94$, $\Delta''_2 = \delta(\Theta_i)$, $i = 90, 94$.

3.5.2 Bounds for Δ''

Now we will provide formulas for Δ'':

$$
\begin{aligned}
&I && \Delta'' = f(\overline{21}31a_{i_1}...a_{i_2}13\overline{12}), && (19a)\\
&II && \Delta'' = f(\overline{21}31a_{i_1}...a_{i_2}12131\overline{12}), && (19b)\\
&III && \Delta'' = f(\overline{21}3121a_{i_1}...a_{i_2}13\overline{12}), && (19c)\\
&IV && \Delta'' = f(\overline{21}3121a_{i_1}...a_{i_2}12131\overline{12}). && (19d)
\end{aligned}
$$

Figure (2) regulates the choise of the formulas.

Case	Δ''_1	Δ''_2	Rectangle	Δ''
I	(14b)	(14b)	(12) $N-N-N$	(19a)
IIa	(14b)	(14b)	(11) $N-N-S$	(19b)
IIb	(14b)	(14a)	$N-S$	(19b)
III	(14a)	(14b)	$S-N$	(19c)
IV	(14a)	(14a)	$S-S$	(19d)

Figure 2: Rules for choise of Δ'' in case $i_1 \equiv i_2 \pmod 2$.

Again, these bounds are illustrated on the right side of the figure 1.

3.5.3 Case $i_1 \not\equiv i_2 \pmod 2$

Now consider case $i_1 \not\equiv i_2 \pmod 2$, i_1 is even. We will use rules from figure (3) to choose one of 4 formulas for Δ'.

$$
\begin{aligned}
&I && \Delta' = f(\overline{21}3a_{i_1}...a_{i_2}13\overline{12}), && (20a)\\
&II && \Delta' = f(\overline{21}3a_{i_1}...a_{i_2}12131\overline{12}), && (20b)\\
&III && \Delta' = f(\overline{21}312a_{i_1}...a_{i_2}13\overline{12}), && (20c)\\
&IV && \Delta' = f(\overline{21}312a_{i_1}...a_{i_2}12131\overline{12}). && (20d)
\end{aligned}
$$

Case	Δ''_1	Δ''_2	Case name	Δ''
I	(13b)	(14b)	$N-N$	(20a)
II	(13b)	(14a)	$N-S$	(20b)
III	(13a)	(14b)	$S-N$	(19c)
IV	(13a)	(14a)	$S-S$	(19d)

Figure 3: Rules for choise of Δ' in case $i_1 \not\equiv i_2 \pmod 2$, i_1 is even.

To determine the bound Δ'', we will use subrectangle

$$\{1, 0\}.$$

Having $i_1 - 1 \equiv i_2 \pmod 2$, so we can use all the previous formulas to determine Δ''.

4 Good rectangle

4.1 Definition

Consider a horizontal rectangle $\Delta = \{a_{i_1}...a_{i_2}\}$.

Definition. Rectangle Δ is called **good**, if subrectangles $\{2,0\}$ and $\{1,0\}$ intersect.

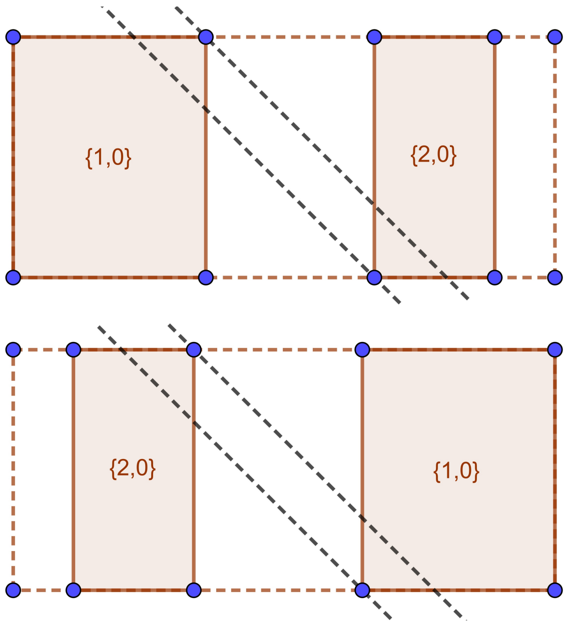


Figure 4: Good rectangles if i_1 is odd (upper) and even (lower).

For example, in case i_1 is even, goodness is equivalent to

$$\{2,0\}'' \geqslant \{1,0\}'. \tag{21}$$

Bounds of rectangles are determined by the rules from section 3.

If the rectangle Δ is not good (for example, (21) doesn't take place), then

$$(\{2,0\}''; \{1,0\}') \not\subset \Delta.$$

Clearly, if the rectangle is not good, then one can not split it into smaller subrectangles whose projections cover the projection of initial one. That's why we will only consider good rectangles during the proof.

4.2 Sufficient conditions of goodness

4.2.1 Results

A horizontal rectangle Δ is good, if

$$\frac{\Delta_1}{\Delta_2} < \begin{cases} 3.8, & i_1 \equiv i_2 \pmod 2, \quad \{\delta(\mathcal{M})\} \text{ meets (13b)}, \\ 3.43, & i_1 \not\equiv i_2 \pmod 2, \quad \{\delta(\mathcal{M})\} \text{ meets (13b)}, \\ 2.9, & \text{otherwise}. \end{cases} \tag{22}$$

4.2.2 Universal $2,9$ bound

We will introduce a sufficient conditions for a rectangle to be good.

Designation. For given continuations Θ_γ and Θ_δ, denote by γ and δ the resulting continued fractions:

$$\gamma = [0; a_{-1}, ..., a_{i_1}, \frac{1}{\Theta_\gamma}],$$
$$\delta = [0; a_1, ..., a_{i_2}, \frac{1}{\Theta_\delta}].$$

As stated in 1.1, γ's correspond to the lhs of $\mathcal{M}$, and δ's correspond to the rhs of $\mathcal{M}$. Variables Θ will be taken from the Figure 8 and will be specified in each case separately.

Suppose that i_1 is even.
Remind that $|\Delta_1| \geqslant |\Delta_2|$, which means that

$$\frac{|\gamma'' - \gamma'|}{|\delta'' - \delta'|} \geqslant 1,$$

where

$$\Theta_{\gamma'} = \Theta_{\delta'} = \Theta_1,\ \Theta_{\gamma''} = \Theta_{\delta''} = \Theta_{95}.$$

Using (6), rewrite it as

$$q\frac{1+p\Theta_1}{1+p'\Theta_1}\frac{1+p\Theta_{95}}{1+p'\Theta_{95}} \leqslant 1.$$

Goodness can be written as

$$\gamma' - \gamma'' < |\delta' - \delta''|, \tag{23}$$

where $\gamma' = \Delta_1(1)'$ and $\gamma'' = \Delta_1(2)''$.
For an arbitraty rectangle Δ, taking

$$\Theta_{\gamma'} = \Theta_{66},\ \Theta_{\gamma''} = \Theta_{63},\ \Theta_{\delta'} = \Theta_{90},\ \Theta_{\delta''} = \Theta_{30}.$$

Inequality (23) transforms into

$$0,313 < q\frac{1+p\Theta_{63}}{1+p'\Theta_{30}}\frac{1+p\Theta_{66}}{1+p'\Theta_{90}}. \tag{24}$$

Suppose that

$$\frac{\Delta_1}{\Delta_2} < 2,9, \tag{25}$$

which is equivalent to

$$\frac{1}{q}\frac{1+p'\Theta_1}{1+p\Theta_1}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} < 2,9. \tag{26}$$

Then (24) takes place. Indeed, it is so, if

$$\frac{1}{2,9}\frac{1+p'\Theta_1}{1+p\Theta_1}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} > 0,313\frac{1+p'\Theta_{30}}{1+p\Theta_{63}}\frac{1+p'\Theta_{90}}{1+p\Theta_{66}}.$$

or, equivalent,

$$1,1 > \frac{1+p\Theta_1}{1+p\Theta_{63}}\,\frac{1+p\Theta_{95}}{1+p\Theta_{66}}\,\frac{1+p'\Theta_{30}}{1+p'\Theta_1}\,\frac{1+p'\Theta_{90}}{1+p'\Theta_{95}},$$

which is checked directly.

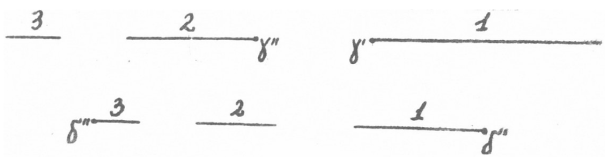
Condition (25) is sufficient for rectangle to be good, regardless of the parity of i_1 and i_2 and the left or right shortness or normalness of rectangle.

4.2.3 Case $i_1 \equiv i_2 \pmod 2$

Now consider case $i_1 \equiv i_2 \pmod 2$.

Suppose that the right hand side $\{\delta(\mathcal{M})\}$ is left-normal, that is, meets (13b).

Equivalent, the case $a_{i_2} = 1$, $a_{i_2-1} = 3$ doesn't take place.



This assumption allows us to substitute the folliwing γ's and δ's into (23):

$$\Theta_{\gamma'} = \Theta_{66},\ \Theta_{\gamma''} = \Theta_{63},\ \Theta_{\delta'} = \Theta_{90},\ \Theta_{\delta''} = \Theta_3.$$

The rhs left-normality leaves us hope that the bound Δ_2'' will be set to $\Delta_2(\Theta_3)$, so we are substituting $\Theta_{\delta''}$ with Θ_3 instead of Θ_{30}.

Now, instead of (24) we will get

$$0,253 < q\,\frac{1+p\Theta_{63}}{1+p'\Theta_3}\,\frac{1+p\Theta_{66}}{1+p'\Theta_{90}}. \tag{27}$$

Such choise of $\Theta_{\gamma'}$ and $\Theta_{\delta''}$ is fine, if the following inequality takes place:

$$\delta' - \delta'' > 1,4(\gamma' - \gamma''), \tag{28}$$

where

$$\Theta_{\gamma'} = \Theta_{68},\ \Theta_{\gamma''} = \Theta_{65},\ \Theta_{\delta'} = \Theta_{28},\ \Theta_{\delta''} = \Theta_1,$$

so we can rewrite (28) as

$$0,253 < q\,\frac{1+p\Theta_{65}}{1+p'\Theta_1}\,\frac{1+p\Theta_{68}}{1+p'\Theta_{28}}. \tag{29}$$

We can notice that (29) follows from (27). Indeed, that follows from the inequality

$$0,253\,\frac{1+p'\Theta_3}{1+p'\Theta_{63}}\,\frac{1+p\Theta_{90}}{1+p\Theta_{66}} > 0,269\,\frac{1+p'\Theta_1}{1+p'\Theta_{65}}\,\frac{1+p\Theta_{28}}{1+p\Theta_{68}},$$

or inequality

$$\frac{1+p\Theta_{65}}{1+p\Theta_{63}}\,\frac{1+p\Theta_{68}}{1+p\Theta_{66}}\,\frac{1+p'\Theta_3}{1+p'\Theta_1}\,\frac{1+p'\Theta_{90}}{1+p'\Theta_{28}} > 1,064.$$

Overall, we have proved that (27) is enough for rectangle to be good.

Suppose that

$$\frac{\Delta_1}{\Delta_2} < 3,8, \tag{30}$$

or

$$\frac{1}{q}\frac{1+p'\Theta_1}{1+p\Theta_1}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} < 3,8.$$

Then (27) takes place. Indeed, it is so, if

$$\frac{1}{3,8}\frac{1+p'\Theta_1}{1+p\Theta_1}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} > 0,253\frac{1+p'\Theta_3}{1+p'\Theta_{63}}\frac{1+p\Theta_{90}}{1+p\Theta_{66}}.$$

The last inequality is transformed into

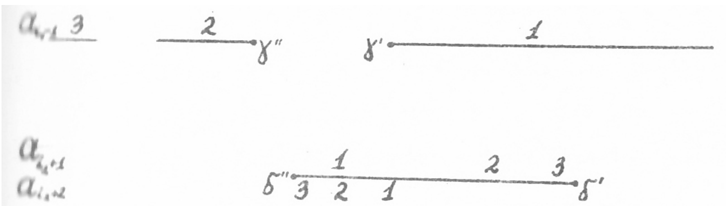
$$1,04 > \frac{1+p'\Theta_3}{1+p'\Theta_1}\frac{1+p'\Theta_{90}}{1+p'\Theta_{95}}\frac{1+p\Theta_1}{1+p\Theta_{63}}\frac{1+p\Theta_{95}}{1+p\Theta_{66}},$$

which is easily checked.

4.2.4 Case $i_1 \not\equiv i_2 \pmod 2$

Now consider case $i_1 \not\equiv i_2 \pmod 2$ and, again, the right hand side $\{\delta(\mathcal{M})\}$ is left-normal, in other words, case $a_{i_2} = 1$, $a_{i_2-1} = 3$ doesn't take place.

Suppose that i_1 is even.



Substituting

$$\Theta_{\gamma'} = \Theta_{66},\ \Theta_{\gamma''} = \Theta_{59},\ \Theta_{\delta'} = \Theta_3,\ \Theta_{\delta''} = \Theta_{90}$$

into (23), obtain

$$0,2885 < q\frac{1+p\Theta_{59}}{1+p'\Theta_3}\frac{1+p\Theta_{66}}{1+p'\Theta_{90}}. \tag{31}$$

Suppose that

$$\frac{\Delta_1}{\Delta_2} < 3,43 \tag{32}$$

or

$$\frac{1}{q}\frac{1+p'\Theta_1}{1+p\Theta_1}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} < 3,43.$$

Then (31) takes place. Indeed, it is so, if

$$\frac{1}{3,43}\frac{1+p'\Theta_1}{1+p\Theta_1}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} > 0,2885\frac{1+p'\Theta_3}{1+p\Theta_{59}}\frac{1+p'\Theta_{90}}{1+p\Theta_{66}}.$$

or

$$1,0105 > \frac{1+p'\Theta_3}{1+p'\Theta_1}\,\frac{1+p'\Theta_{90}}{1+p'\Theta_{95}}\,\frac{1+p\Theta_1}{1+p\Theta_{59}}\,\frac{1+p\Theta_{95}}{1+p\Theta_{66}}.$$

Increasing the right part, obtain

$$1,0105 > 0,9896\,\frac{1+p\cdot 1,0551+p^2\cdot 0,20875}{1+p\cdot 0,983+p^2\cdot 0,237},$$

which is easily checked.

5 Initial set of rectangles

We present a set of rectangles whose projections cover the segment from the Freiman's constant to $\sqrt{21}$. All of them are good. Further sections will present the algorithm of splitting these rectangles into subrectangles correctly.

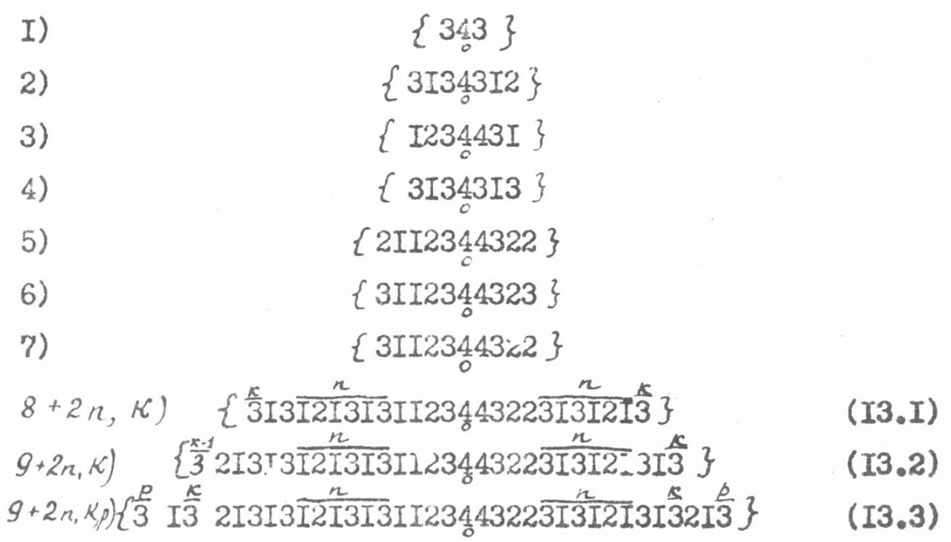


Figure 5: Initial set of rectangles.

6 Induction, $i_1 \not\equiv i_2 \pmod 2$

Let $\Delta = \{a_{i_1}, ..., a_{i_2}\}$ be a horizontal rectangle, satisfying conditions from sections 3 and 4.

This section deals with the case $i_1 \not\equiv i_2 \pmod 2$.

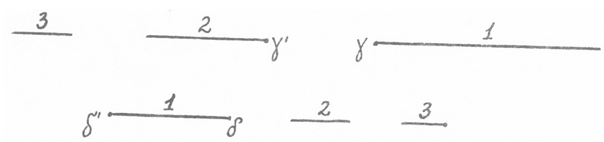
It is easier, because in this case both the lower-left and upper-right corners are filled with the "wide" rectangles $\{1,\}$ and $\{,1\}$ (see figure 6). In case $i_1 \equiv i_2 \pmod 2$ the lower-left corner, filled with $\{3,3\}$, will be a pain in the neck.

Without loss of generality, we may assume that i_1 is even and, therefore, i_2 is odd. As the reader will see, the agrument is repeated for the opposite parity, up to differences in illustrations.

6.1 Case $\{,1\}$ is good

At first, suppose that the subrectangle $\{,1\}$ is good.

It is so, if



$$\gamma - \gamma' < \delta - \delta',$$

where

$$\Theta_\gamma = \Theta_{66}, \; \Theta_{\gamma'} = \Theta_{63}, \; \Theta_\delta = \Theta_{70}, \; \Theta_{\delta'} = \Theta_{90},$$

which can be rewritten with (6) as follows:

$$0,74 < q\,\frac{1+p\Theta_{63}}{1+p'\Theta_{66}}\,\frac{1+p\Theta_{70}}{1+p'\Theta_{90}}. \tag{33}$$

Constants Θ_γ and Θ_δ are chosen to ensure that

$$[\delta, \delta'] \subset \Delta_2,$$

regardless of the rules from the section 3 defining Δ_2' and Δ_2''.

From (33) it follows that the original rectangle Δ is contained within the union of the subrectangles $\{1,\}$ and $\{,1\}$.

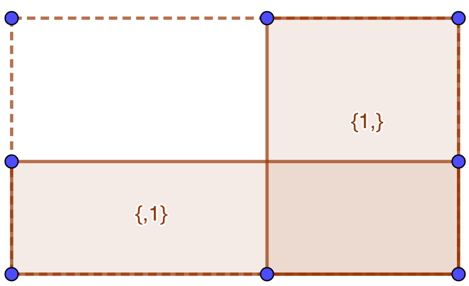


Figure 6: Subrectangles $\{1,\}$ and $\{,1\}$.

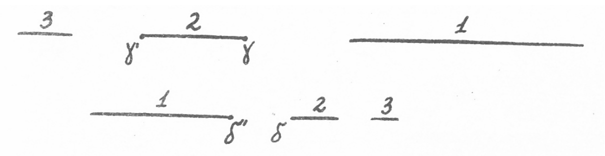
Now we will try contradiction and suppose that

$$0,74 \geqslant q\,\frac{1+p\Theta_{63}}{1+p'\Theta_{66}}\,\frac{1+p\Theta_{70}}{1+p'\Theta_{90}}\,. \tag{34}$$

We will use the inequality (34) until the end of the section.

6.2 Case Δ is left-shortened

Now suppose that the following inequality takes place:



$$\gamma-\gamma' > \delta-\delta',$$

where

$$\Theta_\gamma=\Theta_{63},\ \Theta_{\gamma'}=\Theta_{35},\ \Theta_\delta=\Theta_{63},\ \Theta_{\delta'}=\Theta_{70},$$

which can be rewritten as

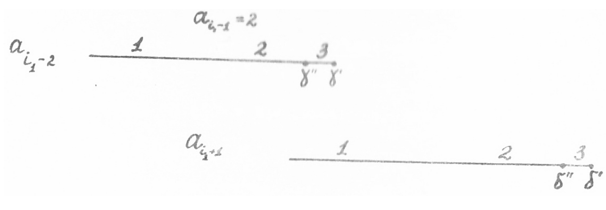
$$0,558 > q\,\frac{1+p\Theta_{35}}{1+p'\Theta_{63}}\,\frac{1+p\Theta_{63}}{1+p'\Theta_{70}}\,. \tag{35}$$

If the rectangle Δ is of type $S-S$ or $S-N$, that is, the lower bound Δ' is defined using rules (20c) or (20d), then the splitting is finished, as Δ is contained within the union of $\{1,\}$ and $\{2,\}$.

6.3 Case Δ is left-normal

Now turn to case if (35) takes place, and Δ is of type $N-S$ or $N-N$, so Δ' is set with one of the rules (20a) or (20b).

Let's see when the subrectangle $\{3,11\}$ is right-normal. The following condition should take place:



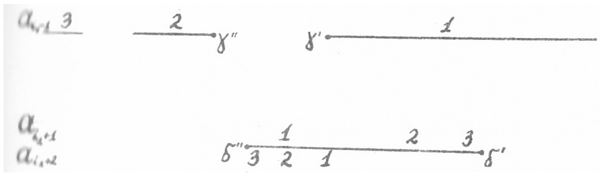
$$\gamma-\gamma' > 1,4(\delta-\delta'),$$

where

$$\Theta_\gamma=\Theta_{28},\ \Theta_{\gamma'}=\Theta_{22},\ \Theta_\delta=\Theta_{68},\ \Theta_{\delta'}=\Theta_{65},$$

which can be written as

$$0,345 > q\,\frac{1+p\Theta_{22}}{1+p'\Theta_{65}}\,\frac{1+p\Theta_{28}}{1+p'\Theta_{68}}\,. \tag{36}$$



If the condition (36) takes place, then the inequality

$$\gamma - \gamma' > \delta - \delta',$$

where

$$\Theta_\gamma = \Theta_{25},\ \Theta_{\gamma'} = \Theta_3,\ \Theta_\delta = \Theta_{66},\ \Theta_{\delta'} = \Theta_{63},$$

which can be written as

$$0,322 > q\,\frac{1 + p\Theta_3}{1 + p'\Theta_{63}}\,\frac{1 + p\Theta_{25}}{1 + p'\Theta_{66}}\,. \tag{37}$$

All in all, if rectangle Δ satisfies both conditions (36) and (37), then it can be split into $\{1,\}$, $\{2,\}$ and $\{3,\}$.

Now suppose that at least one of (36) and (37) doesn't take place. We will show that then we can split Δ into $\{1,\}$, $\{2,\}$ and $\{3,1\}$.

First, we need to check goodness of subcovering $\{3,1\}$. We will prove that it meets the sufficient condition (32). We need to show that

$$3,43(\gamma - \gamma') > |\delta - \delta'|\,, \tag{38}$$

where

$$\Theta_\gamma = \Theta_{28},\ \Theta_{\gamma'} = \Theta_1,\ \Theta_\delta = \Theta_{95},\ \Theta_{\delta'} = \Theta_{65}.$$

Inequality (38) is equivalent to

$$0,628 > q\,\frac{1 + p\Theta_1}{1 + p'\Theta_{65}}\,\frac{1 + p\Theta_{28}}{1 + p'\Theta_{95}}\,.$$

Recalling (35), we have to show that

$$0,558\,\frac{1 + p'\Theta_{63}}{1 + p\Theta_{35}}\,\frac{1 + p'\Theta_{70}}{1 + p\Theta_{63}} < 0,627\,\frac{1 + p'\Theta_{65}}{1 + p\Theta_1}\,\frac{1 + p'\Theta_{95}}{1 + p\Theta_{28}}\,,$$

and it is true.

Now show that the subrectangles $\{3,1\}$ and $\{2,1\}$ intersect.

First, let $a_{i_2} = 3$, so Δ_2' is set with the formula (20b).

The following inequality should take place:

$$\gamma - \gamma' < \delta - \delta', \tag{39}$$

where

$$\Theta_\gamma = \Theta_{30},\ \Theta_{\gamma'} = \Theta_{25},\ \Theta_\delta = \Theta_{70},\ \Theta_{\delta'} = \Theta_{90},$$

which is transformed into

$$0,328 < q\,\frac{1 + p\Theta_{25}}{1 + p'\Theta_{70}}\,\frac{1 + p\Theta_{30}}{1 + p'\Theta_{90}}\,. \tag{40}$$

If (36) doesn't take place, then we need the following inequality for (40):

$$0,345\,\frac{1+p'\Theta_{65}}{1+p\Theta_{22}}\,\frac{1+p'\Theta_{68}}{1+p\Theta_{28}} > 0,328\,\frac{1+p'\Theta_{70}}{1+p\Theta_{25}}\,\frac{1+p'\Theta_{90}}{1+p\Theta_{30}}$$
$$\Updownarrow$$
$$1,05 > \frac{1+p'\Theta_{70}}{1+p'\Theta_{65}}\,\frac{1+p'\Theta_{90}}{1+p'\Theta_{68}}\,\frac{1+p\Theta_{22}}{1+p\Theta_{25}}\,\frac{1+p\Theta_{28}}{1+p\Theta_{30}}.$$

From $a_{i_2} = 3$ it follows that $p' \leqslant \frac{1}{3}$. Setting $p' = \frac{1}{3}$ and $p = \frac{1}{4}$ in the rhs, increasing it and obtaining the correct inequality. Thus, it is correct.

Now suppose that (36) takes place, while (37) doesn't. Setting $\Theta_\delta = \Theta_{66}$ (instead of Θ_{70}) in (39), get the following inequality:

$$0,288 < q\,\frac{1+p\Theta_{25}}{1+p'\Theta_{66}}\,\frac{1+p\Theta_{30}}{1+p'\Theta_{90}}. \tag{41}$$

Inequality (41) follows from denial of inequality (37), if

$$0,322\,\frac{1+p'\Theta_{63}}{1+p\Theta_{3}}\,\frac{1+p'\Theta_{66}}{1+p\Theta_{25}} > 0,298\,\frac{1+p'\Theta_{66}}{1+p\Theta_{25}}\,\frac{1+p'\Theta_{90}}{1+p\Theta_{30}}$$
$$\Updownarrow$$
$$1,08 > \frac{1+p\Theta_{3}}{1+p\Theta_{30}}\,\frac{1+p'\Theta_{90}}{1+p'\Theta_{63}}.$$

Setting $p = \frac{1}{4}$ and $p' = \frac{1}{3}$, easily check the last inequality.

6.4 Case we can consider $\{2, 1\}$

Now consider the case, when $a_{i_2} \neq 3$ and Δ'_1 is defined with (20a).

For the subrectangle $\{2, 1\}$ left-normality we need

$$1,4(\gamma - \gamma') < \|\delta - \delta'\|, \tag{42}$$

where

$$\Theta_\gamma = \Theta_{33},\ \Theta_{\gamma'} = \Theta_{29},\ \Theta_\delta = \Theta_{95},\ \Theta_{\delta'} = \Theta_{92},$$

so (42) can be rewritten as

$$0,182 < q\,\frac{1+p\Theta_{29}}{1+p'\Theta_{92}}\,\frac{1+p\Theta_{33}}{1+p'\Theta_{95}}. \tag{43}$$

If neither (36) nor (37) takes place, then it's easy to check (43).

In (39) we can set $\Theta_{\gamma'} = \Theta_{94}$ (instead of Θ_{90}), so (39) is transformed into

$$0,248 < q\,\frac{1+p\Theta_{25}}{1+p'\Theta_{70}}\,\frac{1+p\Theta_{30}}{1+p'\Theta_{94}}. \tag{44}$$

If (36) doesn't take place, then for (44) we will check the following:

$$0,345\,\frac{1+p'\Theta_{65}}{1+p\Theta_{22}}\,\frac{1+p'\Theta_{68}}{1+p\Theta_{28}} > 0,248\,\frac{1+p'\Theta_{70}}{1+p\Theta_{25}}\,\frac{1+p'\Theta_{94}}{1+p\Theta_{30}}$$
$$\Updownarrow$$
$$1,39 > \frac{1+p'\Theta_{70}}{1+p'\Theta_{65}}\,\frac{1+p'\Theta_{94}}{1+p'\Theta_{68}}\,\frac{1+p\Theta_{22}}{1+p\Theta_{25}}\,\frac{1+p\Theta_{28}}{1+p\Theta_{30}},$$

which is clear.

If (37) doesn't take place, then for (44) we will check

$$0,322\,\frac{1+p'\Theta_{63}}{1+p\Theta_3}\,\frac{1+p'\Theta_{66}}{1+p\Theta_{25}} > 0,248\,\frac{1+p'\Theta_{70}}{1+p\Theta_{25}}\,\frac{1+p'\Theta_{94}}{1+p\Theta_{30}}$$
$$\Updownarrow$$
$$1,29 > \frac{1+p\Theta_3}{1+p\Theta_{30}}\,\frac{1+p'\Theta_{70}}{1+p'\Theta_{63}}\,\frac{1+p'\Theta_{94}}{1+p'\Theta_{66}},$$

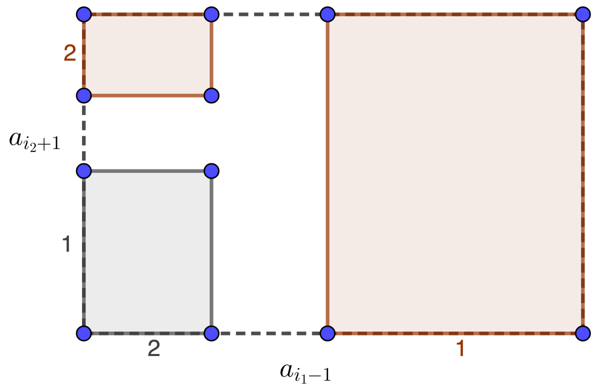
which is true.

6.5 Further plan

It remains for us to check the case if (34) takes place and (35) doesn't – instead, having the negation of (35):

$$0,558 \leqslant q\,\frac{1+p\Theta_{35}}{1+p'\Theta_{63}}\,\frac{1+p\Theta_{63}}{1+p'\Theta_{70}}. \tag{45}$$

We have proved that we can consider subrectangles $\{1,\}$ and $\{2,2\}$.



The picture illustrates the case when the boundaries Δ_1' and Δ_2'' of segments Δ_1 and Δ_2 are defined with constants $\Theta_{\Delta_1'} = \Theta_{\Delta_2''} = \Theta_{30} = \left[0; 213\overline{12}\right]$. In this case, having, in particular, $a_{i_1} = 1$, $a_{i_1+1} = 3$. We set segment Δ_1 to be left-shortened to forbid $a_{i_1-1} = 3$, because it can happen that we will face the forbidden combination (31313).

So, in the illustrated case, the only unused subsegment is $\{2,1\}$. However, subsegments $\{2,1\}$ and $\{2,2\}$ doesn't intersect, so we will be unable to construct a family of subsegments, covering Δ.

How to overcome this difficulty? The idea being developed is to use $a_{i_1-1} = 3$ (and $a_{i_2+1} = 3$), but set a_{i_1-2}, a_{i_1-3} (and a_{i_2+2}, a_{i_2+3}), which produce the thinner subrectangles and doesn't occur in string (31313).

6.6 List of subrectangles

Let's turn to formal plan.

If Δ_1' and Δ_2'' were set with $\Theta_3 = \left[0; 3\overline{12}\right]$, then we would have been able to take $\{3,3\}$, which intersects with $\{2,2\}$.

Instead of $\{3,3\}$, we will take the following list of subrectangles:

$$\{23,32\}, \{23,33\} \text{ (or } \{223,33\} \text{ and } \{323,33\}), \{113,311\},$$
$$\{213,312\}, \{213,311\}, \{113,32\}, \{113,33\} \text{ (or } \{1113,33\}), \{213,33\}.$$

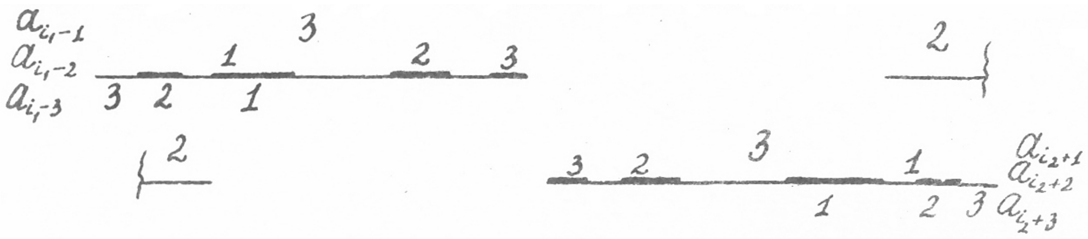


Figure 7: Illustration of the subrectangles replacing the subrectangle $\{3, 3\}$.
I will make a 2d illustration in the future.

The next subrectangle to consider is $\{3, 2\}$, but instead we will consider the following subrectangles:

$$\{33, 213\}, \{33, 212\}, \{23, 213\}, \{23, 212\}, \{23, 211\}.$$

Θ_{1}	$= [0;\ \overline{3I}]$	$= 0,263762$
Θ_{2}	$= [0;\ 3I2\overline{I3}]$	$= 0,267649$
Θ_{3}	$= [0;\ 3\overline{I2}]$	$= 0,26794$
Θ_{4}	$= [0;\ 3I23\overline{3I}]$	$= 0,270448$
Θ_{5}	$= [0;\ 3I23\overline{I2}]$	$= 0,2707I0$
Θ_{6}	$= [0;\ 3I2\overline{3I}]$	$= 0,270738$
Θ_{7}	$= [0;\ 3II\overline{I3}]$	$= 0,27459$
Θ_{8}	$= [0;\ 3III3\overline{I2}]$	$= 0,27462$
Θ_{9}	$= [0;\ 3III3\overline{3I}]$	$= 0,274847$
Θ_{10}	$= [0;\ 3II\overline{I2}]$	$= 0,275I7$
Θ_{11}	$= [0;\ 3I\overline{I2}]$	$= 0,27954$
Θ_{12}	$= [0;\ 3II3\overline{3I}]$	$= 0,280392$
Θ_{13}	$= [0;\ 3II3\overline{I2}]$	$= 0,28097$
Θ_{14}	$= [0;\ 3I\overline{I3}]$	$= 0,28I05$
Θ_{15}	$= [0;\ 32\overline{3I}]$	$= 0,290550$
Θ_{16}	$= [0;\ 323\overline{I2}]$	$= 0,29062$
Θ_{17}	$= [0;\ 323\overline{3I}]$	$= 0;29I242$
Θ_{18}	$= [0;\ 32\overline{2I}]$	$= 0;292I6$
Θ_{19}	$= [0;\ 3\overline{2I}]$	$= 0;29709$
Θ_{20}	$= [0;\ 32I3\overline{I2}]$	$= 0;29774$
Θ_{21}	$= [0;\ 32\overline{I3}]$	$= 0;297773$
Θ_{22}	$= [0;\ 33\overline{3I}]$	$= 0;302444$
Θ_{23}	$= [0;\ 333\overline{I2}]$	$= 0;30025$
Θ_{24}	$= [0;\ 33\overline{2I}]$	$= 0;30330$
Θ_{25}	$= [0;\ 33\overline{I2}]$	$= 0;30600$
Θ_{26}	$= [0;\ 33I2\overline{I3}]$	$= 0;30603$
Θ_{27}	$= [0;\ 33I3\overline{I2}]$	$= 0;30638$
Θ_{28}	$= [0;\ 3\overline{3I}]$	$= 0;306394$
Θ_{29}	$= [0;\ 2\overline{I3}]$	$= 0;358258$
Θ_{30}	$= [0;\ 2I3\overline{I2}]$	$= 0;35859$
Θ_{31}	$= [0;\ 2I33\overline{3I}]$	$= 0;36I292$
Θ_{32}	$= [0;\ 2I33\overline{I2}]$	$= 0;36I58$
Θ_{33}	$= [0;\ 2I3\overline{3I}]$	$= 0;36I602$
Θ_{34}	$= [0;\ 2I2\overline{I3}]$	$= 0;365465$
Θ_{35}	$= [0;\ 2I2I3\overline{I2}]$	$= 0,36549$
Θ_{36}	$= [0;\ \overline{2I}]$	$= 0,36602$
Θ_{37}	$= [0;\ 2I2I\overline{I3}]$	$= 0,36779$
Θ_{38}	$= [0;\ 2I23\overline{I2}]$	$= 0,37II9$
Θ_{39}	$= [0;\ 2I2\overline{3I}]$	$= 0,37I249$
Θ_{40}	$= [0;\ 2II\overline{I3}]$	$= 0,378537$
Θ_{41}	$= [0;\ 2III3\overline{I2}]$	$= 0,37858$
Θ_{42}	$= [0;\ 2III3\overline{3I}]$	$= 0,3790I8$
Θ_{43}	$= [0;\ 2II\overline{I2}]$	$= 0,37963$
Θ_{44}	$= [0;\ 2IIII3\overline{3I}]$	$= 0,38309I$
Θ_{45}	$= [0;\ 2IIII3\overline{I2}]$	$= 0,38326$
Θ_{46}	$= [0;\ 2III\overline{I3}]$	$= 0,383272$
Θ_{47}	$= [0;\ 2II23\overline{I2}]$	$= 0,38605$
Θ_{48}	$= [0;\ 2II2\overline{3I}]$	$= 0,386033$
Θ_{49}	$= [0;\ 2II23\overline{3I}]$	$= 0,386237$
Θ_{50}	$= [0;\ 2II2\overline{2I}]$	$= 0,3865I$
Θ_{51}	$= [0;\ 2I\overline{I2}]$	$= 0,38800$
Θ_{52}	$= [0;\ 2II3\overline{3I}]$	$= 0,389648$
Θ_{53}	$= [0;\ 2II3\overline{2I}]$	$= 0,3899I6$
Θ_{54}	$= [0;\ 2II3\overline{I2}]$	$= 0,39076$
Θ_{55}	$= [0;\ 2I\overline{I3}]$	$= 0,39089I$
Θ_{56}	$= [0;\ 22\overline{3I}]$	$= 0,409544$
Θ_{57}	$= [0;\ 223\overline{I2}]$	$= 0,40968$
Θ_{58}	$= [0;\ 2\overline{2I}]$	$= 0,42265$
Θ_{59}	$= [0;\ 22I3\overline{I2}]$	$= 0,42398$
Θ_{60}	$= [0;\ 22\overline{I3}]$	$= 0,424042$
Θ_{61}	$= [0;\ 23\overline{3I}]$	$= 0,433577$
Θ_{62}	$= [0;\ 233\overline{I2}]$	$= 0,43365$
Θ_{63}	$= [0;\ 23\overline{I2}]$	$= 0,44093$
Θ_{64}	$= [0;\ 2\overline{3I}]$	$= 0,44I742$
Θ_{65}	$= [0;\ I\overline{I3}]$	$= 0,558256$
Θ_{66}	$= [0;\ II3\overline{I2}]$	$= 0,55905$
Θ_{67}	$= [0;\ II33\overline{I2}]$	$= 0,56635$
Θ_{68}	$= [0;\ II3\overline{3I}]$	$= 0,566423$
Θ_{69}	$= [0;\ II2\overline{I3}]$	$= 0,57600$
Θ_{70}	$= [0;\ II2I3\overline{I2}]$	$= 0,57602$
Θ_{71}	$= [0;\ I\overline{I2}]$	$= 0,57735$
Θ_{72}	$= [0;\ II23\overline{I2}]$	$= 0,59032$
Θ_{73}	$= [0;\ III\overline{I3}]$	$= 0,609I08$
Θ_{74}	$= [0;\ IIII3\overline{I2}]$	$= 0,60923$
Θ_{75}	$= [0;\ IIIII3\overline{3I}]$	$= 0,62098I$
Θ_{76}	$= [0;\ IIIII3\overline{I2}]$	$= 0,62I4I$
Θ_{77}	$= [0;\ IIII\overline{I3}]$	$= 0,62I460$
Θ_{78}	$= [0;\ III23\overline{I2}]$	$= 0,6288I$
Θ_{79}	$= [0;\ II\overline{I2}]$	$= 0,\ 63400$
Θ_{80}	$= [0;\ III3\overline{3I}]$	$= 0,63839$
Θ_{81}	$= [0;\ III3\overline{I2}]$	$= 0,639I0$
Θ_{82}	$= [0;\ II\overline{I3}]$	$= 0,64I742$
Θ_{83}	$= [0;\ I23\overline{I2}]$	$= 0,69399$
Θ_{84}	$= [0;\ I22I3\overline{I2}]$	$= 0,70225$
Θ_{85}	$= [0;\ I223\overline{I2}]$	$= 0,70938$
Θ_{86}	$= [0;\ I22\overline{2I}]$	$= 0,70783$
Θ_{87}	$= [0;\ I2II3\overline{I2}]$	$= 0,7I903$
Θ_{88}	$= [0;\ I2III3\overline{I2}]$	$= 0,72538I$
Θ_{89}	$= [0;\ \overline{I2}]$	$= 0,73206$
Θ_{90}	$= [0;\ I2I3\overline{I2}]$	$= 0,73620$
Θ_{91}	$= [0;\ I2\overline{I3}]$	$= 0,73624$
Θ_{92}	$= [0;\ I3\overline{3I}]$	$= 0,765465$
Θ_{93}	$= [0;\ I33\overline{I2}]$	$= 0,76569$
Θ_{94}	$= [0;\ I3\overline{I2}]$	$= 0,78868$
Θ_{95}	$= [0;\ \overline{I3}]$	$= 0,79I287$

Figure 8: List of Θ's.

Contents