

# 1 Introductory notation

## 1.1 Markov and Lagrange values

Symbol  $\mathcal{M}$  denotes double-infinite sequences from  $\mathbb{N}^{\mathbb{Z}}$ :

$$\mathcal{M} = \dots a_{-2}a_{-1}a_0a_1a_2\dots$$

I will use  $\lambda(\mathcal{M})$ ,  $\mu(\mathcal{M})$  and  $f(\mathcal{M})$  for Lagrange, Markov values and height function. Symbols  $\gamma$  and  $\delta$  denote the lhs and rhs of sequence  $\mathcal{M}$ :

$$\begin{aligned}\gamma(\mathcal{M}) &= [0; a_{-1}, a_{-2}, \dots], \\ \delta(\mathcal{M}) &= [0; a_1, a_2, \dots], \\ f(\mathcal{M}) &= a_0 + \gamma(\mathcal{M}) + \delta(\mathcal{M}).\end{aligned}$$

At last, symbols  $M$  and  $L$  denote the Markov and Lagrange spectra.

## 1.2 Centered sequences

**Definition.** A sequence  $\mathcal{M}$  is called **centered**, if

$$\mu(\mathcal{M}) = f(\mathcal{M}). \quad (1)$$

**Proposition.** *Markov spectrum can be defined with only centered sequences:*

$$\{\mu(\mathcal{M}) \mid \mathcal{M} \in \mathbb{N}^{\mathbb{Z}}\} = M = \{\mu(\mathcal{M}) \mid \mathcal{M} \text{ is centered} \}.$$

## 1.3 Rectangle

**Designation.** Denote by

$$\{a_{i_1}, a_{i_1+1}, \dots, a_{i_2}\} \quad (i_1 \leq 0 \leq i_2)$$

the set of double-infinite sequences  $\mathcal{M}$  with fixed terms  $a_{i_1}, a_{i_1+1}, \dots, a_{i_2}$  on the corresponding positions.

Terms  $a_s$  for  $s < i_1$  and  $s > i_2$  are arbitrary integers, chosen such that  $\mathcal{M}$  is centered and, maybe, satisfies some conditions.

Segments  $\Delta_1$ ,  $\Delta_2$  and  $\Delta$  are defined by the following equations:

$$\begin{aligned}\Delta_1 &= [\Delta'_1; \Delta''_1] = [\min \gamma(\mathcal{M}); \max \gamma(\mathcal{M})], \\ \Delta_2 &= [\Delta'_2; \Delta''_2] = [\min \delta(\mathcal{M}); \max \delta(\mathcal{M})], \\ \Delta &= [\Delta'; \Delta''] = a_0 + \Delta_1 + \Delta_2,\end{aligned} \quad (2)$$

where  $\mathcal{M}$  belongs to the rectangle.

**Definition.** **Rectangle** is the segment  $\Delta$  with the set of sequences, defining it.

**Definition.** Call a rectangle  $\Delta$  **horizontal**, if

$$|\Delta_1| \geq |\Delta_2|. \quad (3)$$

Clearly, we can always obtain a horizontal rectangle out of the vertical one, as we can reindex the sequence in the opposite direction.

## 1.4 Resection

**Definition.** Call **resection** of a segment  $A = [a; b]$  a process of removing subsegment  $A_{12} = [a_1; b_1]$ , leaving two segments  $A_1 \sqcup A_2 = [a; a_1] \sqcup [b_1; b]$ .

**Definition.** Call subsegment  $A_{12} \subset A$  **normal**, if it is thicker than the two remaining subsegments:

$$|A_{12}| \leq \min \{|A_1|, |A_2|\} \quad (4)$$

We call a resection **normal** if the resected subsegment is normal.

**Proposition.** *For any normal resection, having*

$$A + A = (A_1 \sqcup A_2) + (A_1 \sqcup A_2). \quad (5)$$

## 1.5 Subrectangle

Consider a rectangle  $\Delta$ , set by the sequence center

$$\{a_{i_1}, a_{i_1+1}, \dots, a_{i_2}\}.$$

We will use a **shorter notation** for subrectangles, produced by setting integers  $a_i$  for  $i < i_1$  or  $i > i_2$ :

$$\{b_\ell \dots b_1, c_1 \dots c_r\} := \{b_\ell \dots b_1 a_{i_1} \dots a_{i_2} c_1 \dots c_r\}.$$

For example:

$$\{213, 3\} := \{213 a_{i_1} \dots a_{i_2} 3\}, \quad (\text{ex.1})$$

$$\{2, 0\} := \{2 a_{i_1} \dots a_{i_2}\}. \quad (\text{ex.2})$$

We will also shorter the notation (2): lhs and rhs are  $\Delta_1(312)$  and  $\Delta_2(3)$  for subrectangle (ex.1) and  $\Delta_1(2)$  and  $a_2$  for (ex.2).

## 1.6 Geometrical interpretation

Consider the mapping

$$\begin{aligned} \tilde{h} : \mathbb{N}^{\mathbb{Z}} &\rightarrow \mathbb{R}^2, \\ \tilde{h}(\mathcal{M}) &= (\gamma(\mathcal{M}); \delta(\mathcal{M})). \end{aligned}$$

In these terms, the Markov spectrum  $M$  is the projection of some subset  $\mathcal{S} \subset C_4 \times C_4$  onto the diagonal.

Then **rectangle**  $\Delta$  is indeed a rectangle  $\Delta_1 \times \Delta_2$  and **subrectangles** are its subrectangles.

We will consider a family of rectangles whose projections cover the beginning of Hall's Ray.

Then we will present the algorithm to split rectangle into subrectangles so that their projections cover the projections of initial rectangle.

When we say that rectangles intersect, we, however, mean that their projections intersect.

The more «squarish» the rectangle, the easier the step.

That's why we will bound the aspect ratio of rectangles (see **good** rectangle).

Formulas to evaluate aspect side lengths and aspect ratio are given in the section 2.

## 2 Calculations

### 2.1 Length of $\Delta_1$ or $\Delta_2$

Let's fix some terms of continued fraction  $[0; q_1, q_2, q_3, \dots, q_n]$ .  
We will often need to measure difference

$$[0; q_1, q_2, q_3, \dots, q_n, \frac{1}{\Theta_R}] - [0; q_1, q_2, q_3, \dots, q_n, \frac{1}{\Theta_L}],$$

where  $\Theta$ 's are some continuations of the continued fraction.

They generally look like  $\Theta = [0; 12\overline{13}]$  or something<sup>1</sup>. We will set  $\Theta$ 's explicitly.

For the general proof,  $\Theta$ 's will be taken from table 5.

**Designation.** For given continuation  $\Theta_i$  denote by  $\varepsilon_i$  the resulting continued fraction:

$$\varepsilon_i = [0; q_1, q_2, \dots, q_n, \frac{1}{\Theta_i}]. \quad (3.4)$$

Then the following equality takes place:

$$|\varepsilon_i - \varepsilon_j| = \frac{|\Theta_i - \Theta_j|}{Q_n^2 (1 + pQ_i) (1 + pQ_j)}, \quad (3.5)$$

where

$$p = \frac{Q_{n-1}}{Q_n}.$$

### 2.2 Rectangle aspect ratio

Consider some fixed center of rectangle  $\{a_{i_1} \dots a_{i_2}\}$ .

We will often extend it from the left (right) using some continuations  $\Theta_{\gamma_1}, \Theta_{\gamma_2} (\Theta_{\delta_1}, \Theta_{\delta_2})$ .

These values produce  $\gamma_1, \gamma_2 (\delta_1, \delta_2)$  using (3.4).

In other words, finite continued fractions  $[0; a_{-1}, a_{-2}, \dots, a_{i_1}] = \frac{P_{i_1}}{Q_{i_1}} \left( [0; a_1, a_2, \dots, a_{i_2}] = \frac{P_{i_2}}{Q_{i_2}} \right)$

are convergents for  $\gamma_1, \gamma_2 (\delta_1, \delta_2)$ .

Then

$$\left| \frac{\gamma_1 - \gamma_2}{\delta_1 - \delta_2} \right| = \frac{1}{q} \left| \frac{\Theta_{\gamma_1} - \Theta_{\gamma_2}}{\Theta_{\delta_1} - \Theta_{\delta_2}} \right| \frac{1 + p'\Theta_{\delta_1}}{1 + p\Theta_{\gamma_1}} \frac{1 + p'\Theta_{\delta_2}}{1 + p\Theta_{\gamma_2}} \approx \frac{1}{q} \left| \frac{\Theta_{\gamma_1} - \Theta_{\gamma_2}}{\Theta_{\delta_1} - \Theta_{\delta_2}} \right|, \quad (6)$$

where

$$p = \frac{Q_{i_1+1}}{Q_{i_1}}, \quad p' = \frac{Q_{i_2-1}}{Q_{i_2}}, \quad q = \frac{Q_{i_1}^2}{Q_{i_2}^2}.$$

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<sup>1</sup>Here, as always in this book,  $\overline{abc}^k$  means  $k$ -times repetition of  $abc$ , and  $\overline{abc}$  means infinite repetition.

### 3 Rectangle boundaries

#### 3.1 Nonformal

Let  $\{\delta_n\}$  be set of fractions  $\delta_n = [0; q_1, q_2, \dots, q_n, \dots]$  with  $n$  fixed terms. We will suppose that  $n$  is even (for odd  $n$  the bounds are swapped).

At first, determine the smallest of fractions  $\delta_n$ .

We will consider 2 cases:  $S$  (Shortened) and  $N$  (Normal):

$$\begin{aligned} S. \quad & q_{n-1} = 3, q_n = 1. \\ N. \quad & \text{Otherwise.} \end{aligned} \tag{7}$$

The lower bound  $\delta'_n$  for segment, containing  $\delta_n$ , is defined by:

$$\begin{aligned} S. \quad & \delta'_n = [0; q_1, \dots, q_n, 213\overline{12}]. \\ N. \quad & \delta'_n = [0; q_1, \dots, q_n, 3\overline{12}]. \end{aligned} \tag{8}$$

To set the upper bound  $\delta''_n$  – biggest of  $\delta_n$ , consider 2 other cases:

$$\begin{aligned} S. \quad & q_n = 3. \\ N. \quad & q_n \neq 3. \end{aligned} \tag{9}$$

Then

$$\begin{aligned} S. \quad & \delta''_n = [0; q_1, \dots, q_n, 1213\overline{12}]. \\ N. \quad & \delta''_n = [0; q_1, \dots, q_n, 13\overline{12}]. \end{aligned} \tag{10}$$

These bounds will allow us to construct sequences  $\mathcal{M}$ , for which combination (31313) is forbidden and, therefore, the following condition takes place:

$$f_i(\mathcal{M}) \leq \lambda(3\overline{1312}) \approx 4,5241, \quad i \neq 0, \tag{11}$$

which will ensure (1).

#### 3.2 Formal

Let's now turn to concrete definitions and bounds. Remind that (??).

Suppose  $i_1 = i_2 \pmod{2}$ ,  $i_1$  is even.

##### 3.2.1 Bounds for $\Delta'$

- I. Suppose both  $\{\gamma_0(\mathcal{M})\}$  and  $\{\delta_0(\mathcal{M})\}$  don't meet condition (??) but meet (??). We will denote such situation by  $H-H-H$  (segment  $\Delta_1$  is left-normal,  $\Delta_2$  is left-normal, rectangle  $\Delta$  is normal).

In this case define  $\Delta'$  by equation

$$\Delta' = \lambda(\overline{213}a_{i_1} \dots a_{i_2} 3\overline{12}). \tag{11.9}$$

- IIa. Sets  $\{\gamma_0(\mathcal{M})\}$  and  $\{\delta_0(\mathcal{M})\}$  don't meet condition (??) but meet (??). This is case  $H-H-Y$  (segments  $\Delta_1$  and  $\Delta_2$  are left-normal, rectangle  $\Delta$  is left-shortened).

Then

$$\Delta' = \lambda(\overline{213}a_{i_1} \dots a_{i_2} 213\overline{12}). \tag{11.10}$$

IIb. Set  $\{\gamma_0(\mathcal{M})\}$  doesn't meet (??),  $\{\delta_0(\mathcal{M})\}$  does. No matter, what takes place, (??) of (??). It is case  $H_Y$  (segment  $\Delta_1$  is left-normal, segment  $\Delta_2$  left-shortened). Bound  $\Delta_1$  is defined by (11.10).

III. Set  $\{\gamma_0(\mathcal{M})\}$  meets (??),  $\{\delta_0(\mathcal{M})\}$  doesn't. In this  $(Y - H)$  case  $\Delta'$  is defined by the following:

$$\Delta' = \lambda(\overline{21}312a_{i_1}...a_{i_2}3\overline{12}). \quad (11.11)$$

IV. Both  $\{\gamma_0(\mathcal{M})\}$  and  $\{\delta_0(\mathcal{M})\}$  meet (??). In this  $(Y - Y)$  case  $\Delta'$  is defined by

$$\Delta' = \lambda(\overline{21}312a_{i_1}...a_{i_2}213\overline{12}). \quad (11.12)$$

Figure (1) illustrates the bounds. On the picture:

$$\Delta'_1 = \gamma(\Theta_i), i = 3, 30, \Delta'_2 = \delta(\Theta_i), i = 3, 30, \Delta' = \Delta'_1 + \Delta'_2.$$

Hatched areas correspond to values of  $a_{i_1-1}$  or  $a_{i_2+1}$  (equal 3), which can not appear in concrete case.

### 3.2.2 Bounds for $\Delta''$

Now we will provide formulas for  $\Delta''$ :

$$\Delta'' = \lambda(\overline{21}31a_{i_1}...a_{i_2}13\overline{12}) \quad (11.13)$$

$$\Delta'' = \lambda(\overline{21}31a_{i_1}...a_{i_2}1213\overline{12}) \quad (11.14)$$

$$\Delta'' = \lambda(\overline{21}3121a_{i_1}...a_{i_2}13\overline{12}) \quad (11.15)$$

$$\Delta'' = \lambda(\overline{21}3121a_{i_1}...a_{i_2}1213\overline{12}) \quad (11.16)$$

Figure (2) regulates the choice of the formulas.

On the figure (1):  $\Delta''_1 = \gamma(\Theta_i)$ ,  $i = 90, 94$ ,  $\Delta''_2 = \delta(\Theta_i)$ ,  $i = 90, 94$ ,  $\Delta'' = \Delta''_1 + \Delta''_2$ . Hatched areas correspond to restricted value 3 of variables  $a_{i_1-2}$  or  $a_{i_2+2}$ .

### 3.2.3 Case $i_1 \not\equiv i_2 \pmod{2}$

Now take case  $i_1 \not\equiv i_2 \pmod{2}$ ,  $i_1$  is even. We will use rules from figure (3) to choose one of 4 formulas for  $\Delta'$ .

$$I \quad \Delta' = \lambda(\overline{21}3a_{i_1}...a_{i_2}13\overline{12}) \quad (11.17)$$

$$II \quad \Delta' = \lambda(\overline{21}3a_{i_1}...a_{i_2}1213\overline{12}) \quad (11.18)$$

$$III \quad \Delta' = \lambda(\overline{21}312a_{i_1}...a_{i_2}13\overline{12}) \quad (11.19)$$

$$IV \quad \Delta' = \lambda(\overline{21}312a_{i_1}...a_{i_2}1213\overline{12}) \quad (11.20)$$

To determine  $\Delta''$  we will use rectangle

$$\{1, 0\}.$$

For this rectangle have  $i_1 - 1 \equiv i_2 \pmod{2}$ , so we can use all the previous formulas to determine  $\Delta''$ .

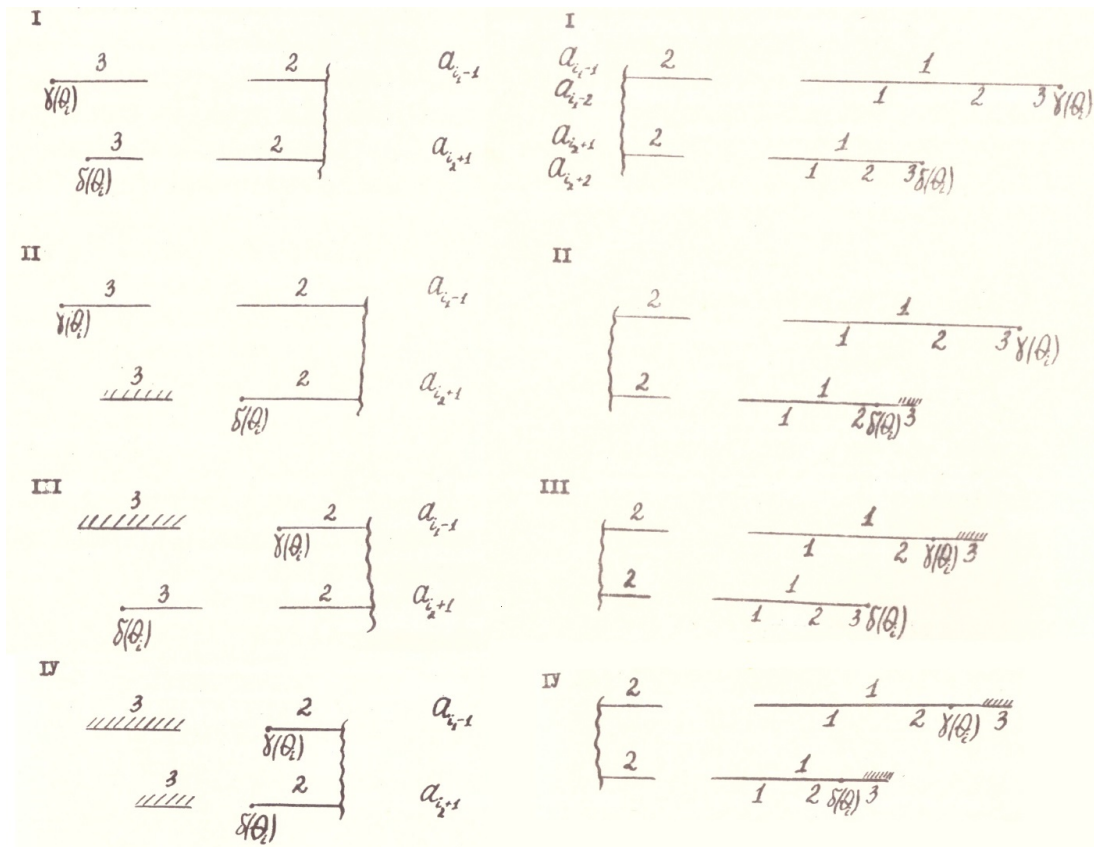


Figure 1: Bounds  $\Delta'$  (left) and  $\Delta''$  (right).

$N$	$\Delta'_i$	$\Delta''_i$	$\Delta''$
I	(5)	(5)	(5.I5) H-H-H (I3)
IIa	(5)	(5)	(5.I3) H-H-y (I4)
IIb	(5)	(4)	H-y (I4)
III	(4)	(5)	y-H (I5)
IV	(4)	(4)	y-y (I6)

Figure 2: Rules for choose of  $\Delta''$  in case  $i_1 = i_2 \pmod{2}$ .

$N$	$\Delta'_i$	$\Delta'_i$	$\Delta'$
I	(I) не имеет места	(5)	H-H (I7)
II	(I) не имеет места	(4)	H-y (I8)
III	(I)	(5)	y-H (I9)
IV	(I)	(4)	y-y (20)

Figure 3: Rules for choose of  $\Delta'$  in case  $i_1 \neq i_2 \pmod{2}$ ,  $i_1$  is even.

## 4 Good rectangle

### 4.1 Definition

Consider a horizontal rectangle  $\Delta = \{a_{i_1} \dots a_{i_2}\}$ .

**Definition.** Rectangle  $\Delta$  is called **good**, if subrectangles  $\{2, 0\}$  and  $\{1, 0\}$  intersect.

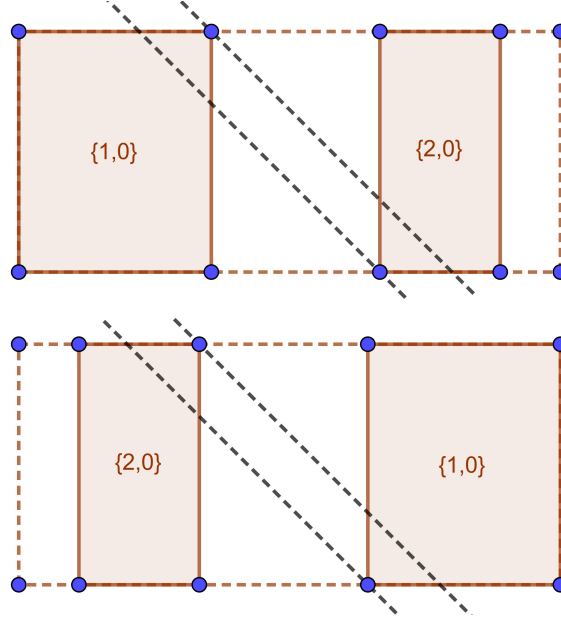


Figure 4: Good rectangles if  $i_1$  is odd (upper) and even (lower).

For example, in case  $i_1$  is even, goodness is equivalent to

$$\{2, 0\}'' \geq \{1, 0\}' \quad (12)$$

Bounds of rectangles are determined by the rules from section 3.

If rectangle is not good (for example, (12) doesn't take place), then

$$(\{2, 0\}''; \{1, 0\}') \not\subset \Delta.$$

Clearly, if the rectangle is not good, then one can not split it into smaller rectangles. That's why we will only consider good rectangles during the proof.

### 4.2 Sufficient conditions of goodness

#### 4.2.1 Results

A horizontal rectangle  $\Delta$  is good, if

$$\frac{\Delta_1}{\Delta_2} < \begin{cases} 3.8, & i_1 \equiv i_2 \pmod{2}, \\ 3.43, & i_1 \not\equiv i_2 \pmod{2}. \end{cases} \quad (13)$$

### 4.2.2 Universal 2, 9 bound

Now let's introduce some sufficient conditions for rectangle to be good.

Remind that we suppose  $|\Delta_1| \geq |\Delta_2|$ , which means

$$q \frac{1 + p\Theta_1}{1 + p'\Theta_1} \frac{1 + p\Theta_{95}}{1 + p'\Theta_{95}} \leq 1.$$

Let's introduce the following designations:

$$\gamma = [0; a_{-1}, \dots, a_{i_1}, \frac{1}{\Theta_\gamma}],$$

$$\delta = [0; a_1, \dots, a_{i_2}, \frac{1}{\Theta_\delta}],$$

where  $\Theta$ 's are some  $\Theta$ 's from Figure (5), specified in each case separately.

If rectangle is good, then (12.1) should take place:

$$\gamma' - \gamma'' < |\delta' - \delta''|, \quad (12.1)$$

where  $\Theta_{\gamma'} = \Theta_{66}$ ,  $\Theta_{\gamma''} = \Theta_{63}$ ,  $\Theta_{\delta'} = \Theta_{90}$ ,  $\Theta_{\delta''} = \Theta_{30}$ .

*As I understand, these thetas are supposed to be  $\gamma' = \gamma'_{\{1,0\}}$ ,  $\gamma'' = \gamma''_{\{2,0\}}$ , and (12.1) is just paraphrasing of (??) in the «worst» case. As usual, Freiman gives no comments.*

Inequality (12.1) transforms into

$$0,313 < q \frac{1 + p\Theta_{63}}{1 + p'\Theta_{30}} \frac{1 + p\Theta_{66}}{1 + p'\Theta_{90}}. \quad (12.2)$$

Suppose

$$\frac{\Delta_1}{\Delta_2} < 2,9, \quad (12.3)$$

which is equivalent to

$$\frac{1}{q} \frac{1 + p'\Theta_1}{1 + p\Theta_1} \frac{1 + p'\Theta_{95}}{1 + p\Theta_{95}} < 2,9. \quad (12.4)$$

Then (12.2) takes place. Indeed, it is so, if

$$\frac{1}{2,9} \frac{1 + p'\Theta_1}{1 + p\Theta_1} \frac{1 + p'\Theta_{95}}{1 + p\Theta_{95}} > 0,313 \frac{1 + p'\Theta_{30}}{1 + p\Theta_{63}} \frac{1 + p'\Theta_{90}}{1 + p\Theta_{66}}.$$

or, equivalent,

$$1,1 > \frac{1 + p\Theta_1}{1 + p\Theta_{63}} \frac{1 + p\Theta_{95}}{1 + p\Theta_{66}} \frac{1 + p'\Theta_{30}}{1 + p'\Theta_1} \frac{1 + p'\Theta_{90}}{1 + p'\Theta_{95}},$$

Which is easily checked. *Ha-ha.*

Condition (12.3) is sufficient for rectangle to be good, no matter, are  $i_1$  and  $i_2$  equivalent (mod 2) or not, take conditions (??) and (??) place or not.

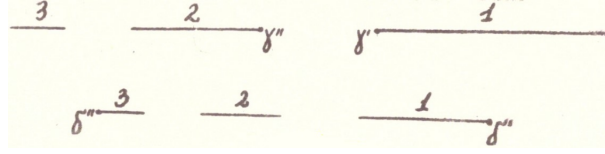


$\Theta_1 = [0; 3\bar{1}] = 0,263762$	$\Theta_{35} = [0; 2\bar{1}2\bar{1}3\bar{1}\bar{2}] = 0,36549$	$\Theta_{67} = [0; 1\bar{1}33\bar{1}\bar{2}] = 0,56635$
$\Theta_2 = [0; 3\bar{1}2\bar{1}\bar{3}] = 0,267649$	$\Theta_{36} = [0; 2\bar{1} ] = 0,36602$	$\Theta_{68} = [0; 1\bar{1}3\bar{3}\bar{1} ] = 0,566423$
$\Theta_3 = [0; 3\bar{1}\bar{2} ] = 0,26794$	$\Theta_{37} = [0; 2\bar{1}2\bar{1}\bar{1}\bar{3}] = 0,36779$	$\Theta_{69} = [0; 1\bar{1}2\bar{1}\bar{3} ] = 0,57600$
$\Theta_4 = [0; 3\bar{1}23\bar{3}\bar{1}] = 0,270448$	$\Theta_{38} = [0; 2\bar{1}23\bar{1}\bar{2} ] = 0,37119$	$\Theta_{70} = [0; 1\bar{1}2\bar{1}3\bar{1}\bar{2}] = 0,57602$
$\Theta_5 = [0; 3\bar{1}23\bar{1}\bar{2}] = 0,270710$	$\Theta_{39} = [0; 2\bar{1}2\bar{3}\bar{1} ] = 0,371249$	$\Theta_{71} = [0; 1\bar{1}\bar{2} ] = 0,57735$
$\Theta_6 = [0; 3\bar{1}23\bar{1} ] = 0,270738$	$\Theta_{40} = [0; 2\bar{1}\bar{1}\bar{1}\bar{3} ] = 0,378537$	$\Theta_{72} = [0; 1\bar{1}23\bar{1}\bar{2} ] = 0,59032$
$\Theta_7 = [0; 3\bar{1}\bar{1}\bar{1}\bar{3}] = 0,27459$	$\Theta_{41} = [0; 2\bar{1}\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,37858$	$\Theta_{73} = [0; 1\bar{1}\bar{1}\bar{1}\bar{3} ] = 0,609108$
$\Theta_8 = [0; 3\bar{1}\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,27462$	$\Theta_{42} = [0; 2\bar{1}\bar{1}\bar{1}3\bar{3}\bar{1}] = 0,379018$	$\Theta_{74} = [0; 1\bar{1}\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,60923$
$\Theta_9 = [0; 3\bar{1}\bar{1}\bar{1}3\bar{3}\bar{1}] = 0,274847$	$\Theta_{43} = [0; 2\bar{1}\bar{1}\bar{1}\bar{2} ] = 0,37963$	$\Theta_{75} = [0; 1\bar{1}\bar{1}\bar{1}\bar{1}3\bar{3}\bar{1}] = 0,620981$
$\Theta_{10} = [0; 3\bar{1}\bar{1}\bar{1}\bar{2} ] = 0,27517$	$\Theta_{44} = [0; 2\bar{1}\bar{1}\bar{1}\bar{1}3\bar{3}\bar{1}] = 0,383091$	$\Theta_{76} = [0; 1\bar{1}\bar{1}\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,62141$
$\Theta_{11} = [0; 3\bar{1}\bar{1}\bar{2} ] = 0,27954$	$\Theta_{45} = [0; 2\bar{1}\bar{1}\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,38326$	$\Theta_{77} = [0; 1\bar{1}\bar{1}\bar{1}\bar{1}\bar{3} ] = 0,621460$
$\Theta_{12} = [0; 3\bar{1}\bar{1}3\bar{3}\bar{1}] = 0,280392$	$\Theta_{46} = [0; 2\bar{1}\bar{1}\bar{1}\bar{1}\bar{3} ] = 0,383272$	$\Theta_{78} = [0; 1\bar{1}\bar{1}23\bar{1}\bar{2}] = 0,62881$
$\Theta_{13} = [0; 3\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,28097$	$\Theta_{47} = [0; 2\bar{1}\bar{1}23\bar{1}\bar{2}] = 0,38605$	$\Theta_{79} = [0; 1\bar{1}\bar{1}\bar{2} ] = 0,63400$
$\Theta_{14} = [0; 3\bar{1}\bar{1}\bar{3} ] = 0,28105$	$\Theta_{48} = [0; 2\bar{1}\bar{1}23\bar{1} ] = 0,386033$	$\Theta_{80} = [0; 1\bar{1}\bar{1}3\bar{3}\bar{1}] = 0,63839$
$\Theta_{15} = [0; 32\bar{3}\bar{1} ] = 0,290550$	$\Theta_{49} = [0; 2\bar{1}\bar{1}23\bar{3}\bar{1}] = 0,386237$	$\Theta_{81} = [0; 1\bar{1}\bar{1}3\bar{1}\bar{2} ] = 0,63910$
$\Theta_{16} = [0; 323\bar{1}\bar{2}] = 0,29062$	$\Theta_{50} = [0; 2\bar{1}\bar{1}22\bar{1} ] = 0,38651$	$\Theta_{82} = [0; 1\bar{1}\bar{1}\bar{3} ] = 0,641742$
$\Theta_{17} = [0; 323\bar{3}\bar{1}] = 0,291242$	$\Theta_{51} = [0; 2\bar{1}\bar{1}\bar{2} ] = 0,38800$	$\Theta_{83} = [0; 123\bar{1}\bar{2}] = 0,69399$
$\Theta_{18} = [0; 322\bar{1} ] = 0,29216$	$\Theta_{52} = [0; 2\bar{1}\bar{1}3\bar{3}\bar{1}] = 0,389648$	$\Theta_{84} = [0; 122\bar{1}3\bar{1}\bar{2}] = 0,70225$
$\Theta_{19} = [0; 32\bar{1} ] = 0,29709$	$\Theta_{53} = [0; 2\bar{1}\bar{1}32\bar{1} ] = 0,389916$	$\Theta_{85} = [0; 1223\bar{1}\bar{2} ] = 0,70938$
$\Theta_{20} = [0; 32\bar{1}3\bar{1}\bar{2}] = 0,29774$	$\Theta_{54} = [0; 2\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,39076$	$\Theta_{86} = [0; 1222\bar{1} ] = 0,70783$
$\Theta_{21} = [0; 32\bar{1}\bar{3} ] = 0,297773$	$\Theta_{55} = [0; 2\bar{1}\bar{1}\bar{3}] = 0,390891$	$\Theta_{87} = [0; 12\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,71903$
$\Theta_{22} = [0; 333\bar{1} ] = 0,302444$	$\Theta_{56} = [0; 223\bar{1}] = 0,409544$	$\Theta_{88} = [0; 12\bar{1}\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,725381$
$\Theta_{23} = [0; 333\bar{1}\bar{2}] = 0,30025$	$\Theta_{57} = [0; 223\bar{1}\bar{2}] = 0,40968$	$\Theta_{89} = [0; \bar{1}\bar{2} ] = 0,73206$
$\Theta_{24} = [0; 332\bar{1} ] = 0,30330$	$\Theta_{58} = [0; 2\bar{2}\bar{1} ] = 0,42265$	$\Theta_{90} = [0; 12\bar{1}3\bar{1}\bar{2} ] = 0,73620$
$\Theta_{25} = [0; 33\bar{1}\bar{2} ] = 0,30600$	$\Theta_{59} = [0; 22\bar{1}3\bar{1}\bar{2}] = 0,42398$	$\Theta_{91} = [0; 12\bar{1}\bar{3} ] = 0,73624$
$\Theta_{26} = [0; 33\bar{1}2\bar{1}\bar{3}] = 0,30603$	$\Theta_{60} = [0; 22\bar{1}\bar{3} ] = 0,424042$	$\Theta_{92} = [0; 133\bar{1} ] = 0,765465$
$\Theta_{27} = [0; 33\bar{1}3\bar{1}\bar{2}] = 0,30638$	$\Theta_{61} = [0; 233\bar{1} ] = 0,433577$	$\Theta_{93} = [0; 133\bar{1}\bar{2}] = 0,76569$
$\Theta_{28} = [0; 33\bar{1} ] = 0,306394$	$\Theta_{62} = [0; 233\bar{1}\bar{2}] = 0,43365$	$\Theta_{94} = [0; 13\bar{1}\bar{2} ] = 0,78868$
$\Theta_{29} = [0; 2\bar{1}\bar{3} ] = 0,358258$	$\Theta_{63} = [0; 23\bar{1}\bar{2} ] = 0,44093$	$\Theta_{95} = [0; \bar{1}\bar{3} ] = 0,791287$
$\Theta_{30} = [0; 2\bar{1}3\bar{1}\bar{2}] = 0,35859$	$\Theta_{64} = [0; 23\bar{1} ] = 0,441742$	
$\Theta_{31} = [0; 2\bar{1}33\bar{3}\bar{1}] = 0,361292$	$\Theta_{65} = [0; 1\bar{1}\bar{3} ] = 0,558256$	
$\Theta_{32} = [0; 2\bar{1}33\bar{1}\bar{2}] = 0,36158$	$\Theta_{66} = [0; 1\bar{1}3\bar{1}\bar{2}] = 0,55905$	
$\Theta_{33} = [0; 2\bar{1}33\bar{1} ] = 0,361602$		
$\Theta_{34} = [0; 2\bar{1}2\bar{1}\bar{3} ] = 0,365455$		

Figure 5: Table of  $\Theta$ 's.

### 4.2.3 Case $i_1 = i_2 \pmod{2}$

Now consider case  $i_1 = i_2 \pmod{2}$  (see picture):



We will suppose case  $a_{i_2} = 1$ ,  $a_{i_2-1} = 3$  doesn't take place.

We can substitute the following  $\gamma$ 's and  $\delta$ 's into (12.1):

$$\Theta_{\gamma'} = \Theta_{66}, \quad \Theta_{\gamma''} = \Theta_{63}, \quad \Theta_{\delta'} = \Theta_{90}, \quad \Theta_{\delta''} = \Theta_3,$$

and instead of (12.2) we will get

$$0,253 < q \frac{1+p\Theta_{63}}{1+p'\Theta_3} \frac{1+p\Theta_{66}}{1+p'\Theta_{90}}. \quad (12.5)$$

This choose of  $\Theta_{\gamma'}$  and  $\Theta_{\delta''}$  is fine, if the following inequality takes place:

$$\delta' - \delta'' > 1,4(\gamma' - \gamma''), \quad (12.6)$$

where

$$\Theta_{\gamma'} = \Theta_{68}, \quad \Theta_{\gamma''} = \Theta_{65}, \quad \Theta_{\delta'} = \Theta_{28}, \quad \Theta_{\delta''} = \Theta_1,$$

so we can rewrite (12.6) as

$$0,253 < q \frac{1+p\Theta_{65}}{1+p'\Theta_1} \frac{1+p\Theta_{68}}{1+p'\Theta_{28}}. \quad (12.7)$$

We can notice that (12.7) follows from (12.5). Indeed, that follows from inequality

$$0,253 \frac{1+p'\Theta_3}{1+p'\Theta_{63}} \frac{1+p\Theta_{90}}{1+p\Theta_{66}} > 0,269 \frac{1+p'\Theta_1}{1+p'\Theta_{65}} \frac{1+p\Theta_{28}}{1+p\Theta_{68}}$$

or inequality

$$\frac{1+p\Theta_{65}}{1+p\Theta_{63}} \frac{1+p\Theta_{68}}{1+p\Theta_{66}} \frac{1+p'\Theta_3}{1+p'\Theta_1} \frac{1+p'\Theta_{90}}{1+p'\Theta_{28}} > 1,064.$$

Here Freiman even omits comment «which is checked directly».

Overall, we proved that (12.5) is enough for condition to be good.

Suppose

$$\frac{\Delta_1}{\Delta_2} < 3,8 \quad (12.8)$$

or

$$\frac{1}{q} \frac{1+p'\Theta_1}{1+p\Theta_1} \frac{1+p'\Theta_{95}}{1+p\Theta_{95}} < 3,8.$$

Then (12.5) takes place. Indeed, it is so, if

$$\frac{1}{3,8} \frac{1+p'\Theta_1}{1+p'\Theta_1} \frac{1+p\Theta_{95}}{1+p\Theta_{95}} > 0,253 \frac{1+p'\Theta_3}{1+p'\Theta_{63}} \frac{1+p\Theta_{90}}{1+p\Theta_{66}}.$$

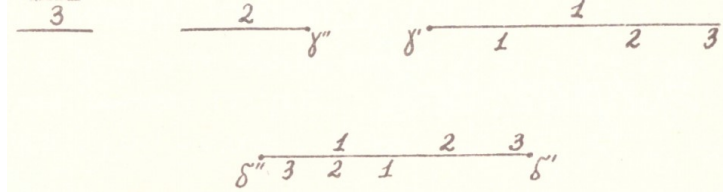
The last inequality is transformed into

$$1,04 > \frac{1+p'\Theta_3}{1+p'\Theta_1} \frac{1+p'\Theta_{90}}{1+p'\Theta_{95}} \frac{1+p\Theta_1}{1+p\Theta_{63}} \frac{1+p\Theta_{95}}{1+p\Theta_{66}},$$

which is easy to check.

#### 4.2.4 Case $i_1 \neq i_2 \pmod{2}$

Now consider case  $i_1 \neq i_2 \pmod{2}$  and, again, case  $a_{i_2} = 1, a_{i_2-1} = 3$  doesn't take place. Suppose  $i_1$  is even (see picture):



Taking

$$\Theta_{\gamma'} = \Theta_{66}, \Theta_{\gamma''} = \Theta_{59}, \Theta_{\delta'} = \Theta_3, \Theta_{\delta''} = \Theta_{90},$$

rewrite (12.1) as

$$0,2885 < q \frac{1+p\Theta_{59}}{1+p'\Theta_3} \frac{1+p\Theta_{66}}{1+p'\Theta_{90}}. \quad (12.9)$$

Suppose

$$\frac{\Delta_1}{\Delta_2} < 3,43 \quad (12.10)$$

or

$$\frac{1}{q} \frac{1+p'\Theta_1}{1+p\Theta_1} \frac{1+p'\Theta_{95}}{1+p\Theta_{95}} < 3,43.$$

Then (12.9) takes place. Indeed, it is so, if

$$\frac{1}{3,43} \frac{1+p'\Theta_1}{1+p\Theta_1} \frac{1+p'\Theta_{95}}{1+p\Theta_{95}} > 0,2885 \frac{1+p'\Theta_3}{1+p\Theta_{59}} \frac{1+p'\Theta_{90}}{1+p\Theta_{66}}.$$

or

$$1,0105 > \frac{1+p'\Theta_3}{1+p'\Theta_1} \frac{1+p'\Theta_{90}}{1+p'\Theta_{95}} \frac{1+p\Theta_1}{1+p\Theta_{59}} \frac{1+p\Theta_{95}}{1+p\Theta_{66}},$$

increasing the right part, obtain

$$1,0105 > 0,9896 \frac{1+p \cdot 1,0551 + p^2 \cdot 0,20875}{1+p \cdot 0,983 + p^2 \cdot 0,237},$$

which is easily checked.

## 5 Freiman's constant. Initial set of rectangles

All these rectangles satisfy *\_here will be a complete list of conditions.\_*  
 At least all of them are good, in terms of section 4.

$$\begin{aligned}
 1) & \quad \{ 3 \underset{\circ}{4} 3 \} \\
 2) & \quad \{ 3 \text{I} 3 \underset{\circ}{4} 3 \text{I} 2 \} \\
 3) & \quad \{ \text{I} 2 3 \underset{\circ}{4} 4 3 \text{I} \} \\
 4) & \quad \{ 3 \text{I} 3 \underset{\circ}{4} 3 \text{I} 3 \} \\
 5) & \quad \{ 2 \text{II} 2 3 \underset{\circ}{4} 4 3 2 2 \} \\
 6) & \quad \{ 3 \text{II} 2 3 \underset{\circ}{4} 4 3 2 3 \} \\
 7) & \quad \{ 3 \text{II} 2 3 \underset{\circ}{4} 4 3 \underset{\circ}{2} 2 \} \\
 8+2n, \kappa) & \quad \{ \overset{\kappa}{3} \text{I} \overline{\overset{n}{3} \text{I} 3 \text{I} 2 \text{I} 3 \text{I} 3 \text{I} 1} 2 3 \underset{\circ}{4} 4 3 2 2 \overline{\overset{n}{3} \text{I} 3 \text{I} 2 \text{I} 3} \overset{\kappa}{3} \} \quad (\text{I3.1}) \\
 9+2n, \kappa) & \quad \{ \overset{\kappa+1}{3} 2 \text{I} 3 \overline{\overset{n}{3} \text{I} 3 \text{I} 2 \text{I} 3 \text{I} 3 \text{I} 1} 2 3 \underset{\circ}{4} 4 3 2 2 \overline{\overset{n}{3} \text{I} 3 \text{I} 2 \text{I} 3} \overset{\kappa}{3} \} \quad (\text{I3.2}) \\
 9+2n, \kappa, \rho) & \quad \{ \overset{\rho}{3} \overset{\kappa}{\text{I} 3} 2 \text{I} 3 \overline{\overset{n}{3} \text{I} 3 \text{I} 2 \text{I} 3 \text{I} 3 \text{I} 1} 2 3 \underset{\circ}{4} 4 3 2 2 \overline{\overset{n}{3} \text{I} 3 \text{I} 2 \text{I} 3 \text{I} 3} \overset{\kappa}{3} \overset{\rho}{2} \text{I} 3 \} \quad (\text{I3.3})
 \end{aligned}$$

Figure 6: Freiman's constant. Initial set of rectangles.

# Contents

<b>1</b>	<b>Introductory notation</b>	<b>1</b>
1.1	Markov and Lagrange values . . . . .	1
1.2	Centered sequences . . . . .	1
1.3	Rectangle . . . . .	1
1.4	Resection . . . . .	2
1.5	Subrectangle . . . . .	2
1.6	Geometrical interpretation . . . . .	2
<b>2</b>	<b>Calculations</b>	<b>3</b>
2.1	Length of $\Delta_1$ or $\Delta_2$ . . . . .	3
2.2	Rectangle aspect ratio . . . . .	3
<b>3</b>	<b>Rectangle boundaries</b>	<b>4</b>
3.1	Nonformal . . . . .	4
3.2	Formal . . . . .	4
3.2.1	Bounds for $\Delta'$ . . . . .	4
3.2.2	Bounds for $\Delta''$ . . . . .	5
3.2.3	Case $i_1 \not\equiv i_2 \pmod{2}$ . . . . .	5
<b>4</b>	<b>Good rectangle</b>	<b>7</b>
4.1	Definition . . . . .	7
4.2	Sufficient conditions of goodness . . . . .	7
4.2.1	Results . . . . .	7
4.2.2	Universal 2, 9 bound . . . . .	8
4.2.3	Case $i_1 = i_2 \pmod{2}$ . . . . .	10
4.2.4	Case $i_1 \neq i_2 \pmod{2}$ . . . . .	11
<b>5</b>	<b>Freiman's constant. Initial set of rectangles</b>	<b>12</b>