1 Introductory notation

1.1 Markov and Lagrange values

Symbol $\mathcal{M}$ denotes double-infinite sequences from $\mathbb{N}^{\mathbb{Z}}$:

$$\mathcal{M} = ...a_{-2}a_{-1}a_0a_1a_2...$$

I will use $\lambda(\mathcal{M})$, $\mu(\mathcal{M})$ and $f(\mathcal{M})$ for Lagrange, Markov values and height function. Symbols γ and δ denote the lhs and rhs of sequence $\mathcal{M}$:

$$\begin{gathered}\gamma(\mathcal{M}) = [0; a_{-1}, a_{-2}, ...],\\ \delta(\mathcal{M}) = [0; a_1, a_2, ...],\\ f(\mathcal{M}) = a_0 + \gamma(\mathcal{M}) + \delta(\mathcal{M}).\end{gathered}$$

At last, symbols M and L denote the Markov and Lagrange spectra.

1.2 Centered sequences

Definition. A sequence $\mathcal{M}$ is called **centered**, if

$$\mu(\mathcal{M}) = f(\mathcal{M}). \tag{1}$$

Proposition. *Markov spectrum can be defined with only centered sequences:*

$$\{\mu(\mathcal{M}) \mid \mathcal{M} \in \mathbb{N}^{\mathbb{Z}}\} = M = \{\mu(\mathcal{M}) \mid \mathcal{M} \textit{ is centered }\}.$$

1.3 Rectangle

Designation. Denote by

$$\{a_{i_1}, a_{i_1+1}, ..., a_{i_2}\} \ \ (i_1 \leqslant 0 \leqslant i_2)$$

the set of double-infinite sequences $\mathcal{M}$ with fixed terms $a_{i_1}, a_{i_1+1}, ..., a_{i_2}$ on the corresponding positions.

Terms a_s for $s < i_1$ and $s > i_2$ are arbitrary integers, chosen such that $\mathcal{M}$ is centered and, maybe, satisfies some conditions.

Segments Δ_1, Δ_2 and Δ are defined by the following equations:

$$\begin{array}{lclcl}\Delta_1 &=& [\,\Delta_1'\,;\,\Delta_1''\,] &=& [\,\min\gamma(\mathcal{M})\,;\,\max\gamma(\mathcal{M})\,]\,,\\ \Delta_2 &=& [\,\Delta_2'\,;\,\Delta_2''\,] &=& [\,\min\delta(\mathcal{M})\,;\,\max\delta(\mathcal{M})\,]\,,\\ \Delta &=& [\,\Delta'\,;\,\Delta''\,] &=& a_0+\Delta_1+\Delta_2,\end{array} \tag{2}$$

where $\mathcal{M}$ belongs to the set.

Definition. **Rectangle** is the segment Δ with the set of sequences, defining it.

1.4 Horizontal rectangle

Definition. Call a rectangle Δ **horizontal**, if

$$|\Delta_1| \geqslant |\Delta_2|. \tag{3}$$

In (3) we allow terms a_s for $s < i_1$ and $s > i_2$ to be integers $\{1, 2, 3\}$, regardless of the requirement that sequences $\mathcal{M} \in \Delta$ are centered.

In other words, Δ is horizontal, if and only if

$$\left|[0; a_{-1}, ..., a_{i_1}, \overline{3,1}] - [0; a_{-1}, ..., a_{i_1}, \overline{1,3}]\right| \geqslant \left|[0; a_1, ..., a_{i_2}, \overline{3,1}] - [0; a_1, ..., a_{i_2}, \overline{1,3}]\right|.$$

Clearly, we can always obtain a horizontal rectangle out of the vertical one, as we can reindex the sequence in the opposite direction.

1.5 Resection

Definition. Call **resection** of a segment $A = [a; b]$ a process of removing subsegment $A_{12} = [a_1; b_1]$, leaving two segments $A_1 \sqcup A_2 = [a; a_1] \sqcup [b_1; b]$.

Definition. Call subsegment $A_{12} \subset A$ **normal**, if it is thicker than the two remaining subsegments:

$$|A_{12}| \leqslant min\ \{|A_1|,\ |A_2|\} \tag{4}$$

We call a resection **normal** if the resected subsegment is normal.

Proposition. *For any normal resection, having*

$$A + A = (A_1 \sqcup A_2) + (A_1 \sqcup A_2). \tag{5}$$

1.6 Subrectangle

Consider a rectangle Δ, set by the sequence center

$$\{a_{i_1}, a_{i_1+1}, ..., a_{i_2}\}.$$

We will use a **shorter notation** for subrectangles, produced by setting integers a_i for $i < i_1$ or $i > i_2$:

$$\{b_\ell...b_1,\ c_1...c_r\} := \{b_\ell...b_1 a_{i_1}...a_{i_2} c_1...c_r\}.$$

For example:

$$\{213, 3\} := \{213 a_{i_1}...a_{i_2} 3\}, \tag{ex.1}$$

$$\{2, 0\} := \{2 a_{i_1}...a_{i_2}\}. \tag{ex.2}$$

We will also shorter the notation (2): lhs and rhs are $\Delta_1(312)$ and $\Delta_2(3)$ for subrectangle (ex.1) and $\Delta_1(2)$ and a_2 for (ex.2).

1.7 Geometrical interpretation

Consider the mapping

$$\widetilde{h} : \mathbb{N}^{\mathbb{Z}} \to \mathbb{R}^2,$$
$$\widetilde{h}(\mathcal{M}) = (\gamma(\mathcal{M}); \delta(\mathcal{M})).$$

In these terms, the Markov spectrum M is the projection of some subset $\mathcal{S} \subset C_4 \times C_4$ onto the diagonal.

Then **rectangle** Δ is indeed a rectangle $\Delta_1 \times \Delta_2$ and **subrectangles** are its subrectangles.

We will consider a family of rectangles whose projections cover the beginning of Hall's Ray.

Then we will present the algorithm to split rectangle into subrectangles so that their projections cover the projection of initial rectangle.

When we say that rectangles intersect, we, however, mean that their projections intersect.

The more «squarish» the rectangle, the easier the step.

That's why we will bound the aspect ratio of rectangles (see **good** rectangle).

Formulas to evaluate aspect side lengths and aspect ratio are given in the section 2.

2 Calculations

2.1 Length of Δ_1 or Δ_2

Let's fix some terms of continued fraction $[0; q_1, q_2, q_3, ..., q_n]$.
We will often need to measure difference

$$[0; q_1, q_2, q_3, ..., q_n, \frac{1}{\Theta_R}] - [0; q_1, q_2, q_3, ..., q_n, \frac{1}{\Theta_L}],$$

where Θ's are some continuations of the continued fraction.
They generally look like $\Theta = [0; 12\overline{13}]$ or something[1]. We will set Θ's explicitly.
For the general proof, Θ's will be taken from table 5.

Designation. For given continuation Θ_i denote by ε_i the resulting continued fraction:

$$\varepsilon_i = [0; q_1, q_2, ..., q_n, \frac{1}{\Theta_i}]. \tag{3.4}$$

Then the following equality takes place:

$$|\varepsilon_i - \varepsilon_j| = \frac{|\Theta_i - \Theta_j|}{Q_n^2\,(1 + pQ_i)\,(1 + pQ_j)}, \tag{3.5}$$

where

$$p = \frac{Q_{n-1}}{Q_n}.$$

2.2 Rectangle aspect ratio

Consider some fixed center of rectangle $\{a_{i_1}...a_{i_2}\}$.
We will often extend it from the left (right) using some continuations $\Theta_{\gamma_1}, \Theta_{\gamma_2}$ ($\Theta_{\delta_1}, \Theta_{\delta_2}$).
These values produce γ_1, γ_2 (δ_1, δ_2) using (3.4).
In other words, finite continued fractions $[0; a_{-1}, a_{-2}, ..., a_{i_1}] = \dfrac{P_{i_1}}{Q_{i_1}}$ $\left([0; a_1, a_2, ..., a_{i_2}] = \dfrac{P_{i_2}}{Q_{i_2}}\right)$ are convergents for γ_1, γ_2 (δ_1, δ_2).
Then

$$\left|\frac{\gamma_1 - \gamma_2}{\delta_1 - \delta_2}\right| = \frac{1}{q}\left|\frac{\Theta_{\gamma_1} - \Theta_{\gamma_2}}{\Theta_{\delta_1} - \Theta_{\delta_2}}\right| \frac{1 + p'\Theta_{\delta_1}}{1 + p\Theta_{\gamma_1}}\,\frac{1 + p'\Theta_{\delta_2}}{1 + p\Theta_{\gamma_2}} \approx \frac{1}{q}\left|\frac{\Theta_{\gamma_1} - \Theta_{\gamma_2}}{\Theta_{\delta_1} - \Theta_{\delta_2}}\right|, \tag{6}$$

where

$$p = \frac{Q_{i_1+1}}{Q_{i_1}}, \quad p' = \frac{Q_{i_2-1}}{Q_{i_2}}, \quad q = \frac{Q_{i_1}^2}{Q_{i_2}^2}.$$

[1]Here, as always in this book, $\overset{k}{\overline{abc}}$ means k-times repetition of abc, and $\overline{abc}$ means infinite repetition.

3 Rectangle boundaries

3.1 Nonformal

Consider integers $q_1, q_2, ..., q_n$. Let $\{\delta_n\}$ be the set of fractions $\delta_n = [0; q_1, q_2, ..., q_n, ...]$ with n fixed terms. We will suppose that n is even (for odd n the bounds are swapped).

At first, determine the smallest of fractions δ_n.

We will consider 2 cases: S (Shortened) and N (Normal):

$$\begin{array}{ll} S. & q_{n-1} = 3, q_n = 1. \\ N. & \text{Otherwise.} \end{array} \tag{7}$$

The lower bound δ'_n for segment, containing δ_n, is defined by:

$$\begin{array}{ll} S. & \delta'_n = [0; q_1, ..., q_n, 213\overline{12}]. \\ N. & \delta'_n = [0; q_1, ..., q_n, 3\overline{12}]. \end{array} \tag{8}$$

To set the upper bound δ''_n – biggest of δ_n, consider 2 other cases:

$$\begin{array}{ll} S. & q_n = 3. \\ N. & q_n \neq 3. \end{array} \tag{9}$$

Then

$$\begin{array}{ll} S. & \delta''_n = [0; q_1, ..., q_n, 1213\overline{12}]. \\ N. & \delta''_n = [0; q_1, ..., q_n, 13\overline{12}]. \end{array} \tag{10}$$

These bounds will allow us to construct sequences $\mathcal{M}$, for which combination (31313) is forbidden and, therefore, the following condition takes place:

$$f_i(\mathcal{M}) \leqslant \lambda(\overline{31312}) \approx 4,5241, \ i \neq 0, \tag{11}$$

which will ensure (1).

3.2 Formal

Let's now turn to concrete definitions and bounds. Remind that (**??**).

Suppose $i_1 = i_2 \pmod 2$, i_1 is even.

3.2.1 Bounds for Δ'

I. Suppose both $\{\gamma_0(\mathcal{M})\}$ and $\{\delta_0(\mathcal{M})\}$ don't meet condition (**??**) but meet (**??**). We will denote such situation by $H\!-\!H\!-\!H$ (segment Δ_1 is left-normal, Δ_2 is left-normal, rectangle Δ is normal).

In this case define Δ' by equation

$$\Delta' = \lambda(\overline{21}3a_{i_1}...a_{i_2}3\overline{12}). \tag{11.9}$$

IIa. Sets $\{\gamma_0(\mathcal{M})\}$ and $\{\delta_0(\mathcal{M})\}$ don't meet condition (**??**) but meet (**??**). This is case $H - H - Y$ (segments Δ_1 and Δ_2 are left-normal, rectangle Δ is left-shortened).

Then

$$\Delta' = \lambda(\overline{21}3a_{i_1}...a_{i_2}213\overline{12}). \tag{11.10}$$

IIb. Set $\{\gamma_0(\mathcal{M})\}$ doesn't meet (**??**), $\{\delta_0(\mathcal{M})\}$ does. No matter, what takes place, (**??**) of (**??**). It is case H_Y (segment Δ_1 is left-normal, segment Δ_2 left-shortened). Bound Δ_1 is defined by (11.10).

III. Set $\{\gamma_0(\mathcal{M})\}$ meets (**??**), $\{\delta_0(\mathcal{M})\}$ doesn't. In this $(Y-H)$ case Δ' is defined by the following:

$$\Delta' = \lambda(\overline{21}312a_{i_1}...a_{i_2}3\overline{12}). \tag{11.11}$$

IV. Both $\{\gamma_0(\mathcal{M})\}$ and $\{\delta_0(\mathcal{M})\}$ meet (**??**). In this $(Y-Y)$ case Δ' is defined by

$$\Delta' = \lambda(\overline{21}312a_{i_1}...a_{i_2}213\overline{12}). \tag{11.12}$$

Figure (1) illustrates the bounds. On the picture:
$\Delta'_1 = \gamma(\Theta_i)$, $i = 3, 30$, $\Delta'_2 = \delta(\Theta_i)$, $i = 3, 30$, $\Delta' = \Delta'_1 + \Delta'_2$.
Hatched areas correspond to values of a_{i_1-1} or a_{i_2+1} (equal 3), which can not appear in concrete case.

3.2.2 Bounds for Δ''

Now we will provide formulas for Δ'':

$$\Delta'' = \lambda(\overline{21}31a_{i_1}...a_{i_2}13\overline{12}) \tag{11.13}$$

$$\Delta'' = \lambda(\overline{21}31a_{i_1}...a_{i_2}1213\overline{12}) \tag{11.14}$$

$$\Delta'' = \lambda(\overline{21}3121a_{i_1}...a_{i_2}13\overline{12}) \tag{11.15}$$

$$\Delta'' = \lambda(\overline{21}3121a_{i_1}...a_{i_2}1213\overline{12}) \tag{11.16}$$

Figure (2) regulates the choise of the formulas.
On the figure (1): $\Delta''_1 = \gamma(\Theta_i)$, $i = 90, 94$, $\Delta''_2 = \delta(\Theta_i)$, $i = 90, 94$, $\Delta'' = \Delta''_1 + \Delta''_2$. Hatched areas correspond to restricted value 3 of variables a_{i_1-2} or a_{i_2+2}.

3.2.3 Case $i_1 \not\equiv i_2 \pmod 2$

Now take case $i_1 \not\equiv i_2 \pmod 2$, i_1 is even. We will use rules from figure (3) to choose one of 4 formulas for Δ'.

$$I \quad \Delta' = \lambda(\overline{21}3a_{i_1}...a_{i_2}13\overline{12}) \tag{11.17}$$

$$II \quad \Delta' = \lambda(\overline{21}3a_{i_1}...a_{i_2}1213\overline{12}) \tag{11.18}$$

$$III \quad \Delta' = \lambda(\overline{21}312a_{i_1}...a_{i_2}13\overline{12}) \tag{11.19}$$

$$IV \quad \Delta' = \lambda(\overline{21}312a_{i_1}...a_{i_2}1213\overline{12}) \tag{11.20}$$

To determine Δ'' we will use rectangle

$$\{1, 0\}.$$

For this rectangle have $i_1 - 1 \equiv i_2 \pmod 2$, so we can use all the previous formulas to determine Δ''.

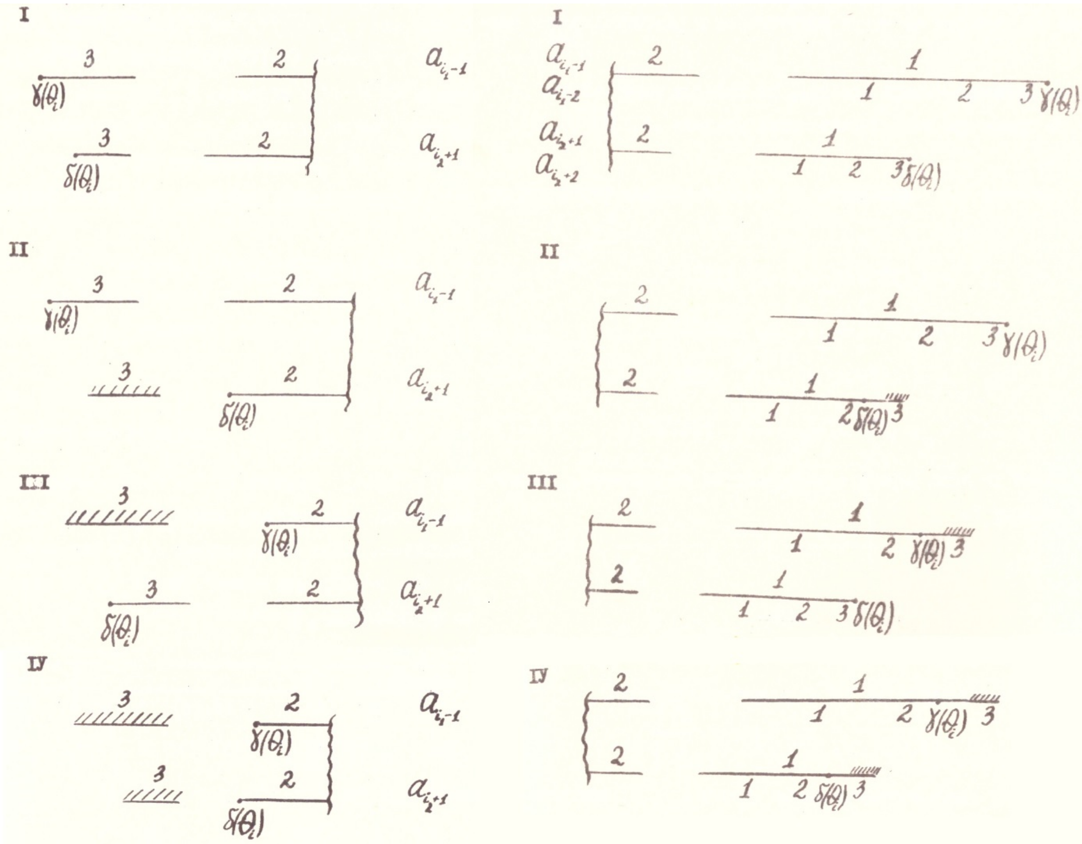


Figure 1: Bounds Δ' (left) and Δ'' (right).

N	Δ''_1	Δ''_2			Δ''
I	(5)	(5)	(5.I5)	Н – Н – Н	(I3)
IIa	(5)	(5)	(5.I3)	Н – Н – у	(I4)
IIб	(5)	(4)		Н – у	(I4)
III	(4)	(5)		у – Н	(I5)
IУ	(4)	(4)		у – у	(I6)

Figure 2: Rules for choose of Δ'' in case $i_1 = i_2 \pmod 2$.

N	Δ'_1	Δ'_2		Δ'
I	(I) не имеет места	(5)	Н – Н	(I7)
II	(I) не имеет места	(4)	Н – у	(I8)
III	(I)	(5)	у – Н	(I9)
IУ	(I)	(4)	у – у	(20)

Figure 3: Rules for choose of Δ' in case $i_1 \neq i_2 \pmod 2$, i_1 is even.

4 Good rectangle

4.1 Definition

Consider a horizontal rectangle $\Delta = \{a_{i_1} ... a_{i_2}\}$.

Definition. Rectangle Δ is called **good**, if subrectangles $\{2,0\}$ and $\{1,0\}$ intersect.

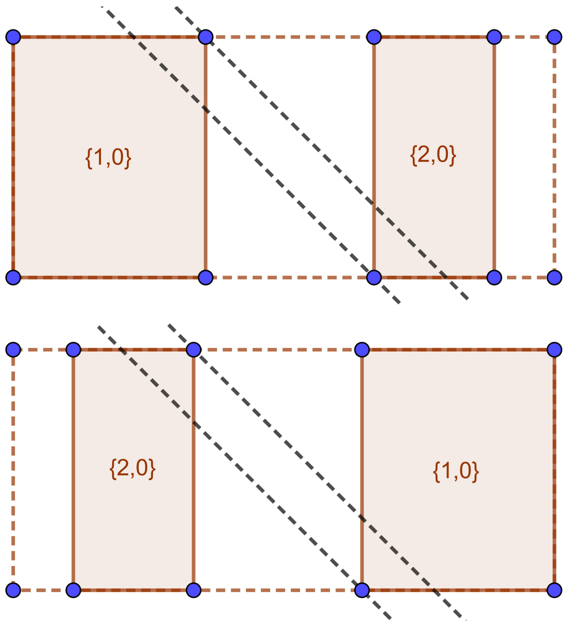


Figure 4: Good rectangles if i_1 is odd (upper) and even (lower).

For example, in case i_1 is even, goodness is equivalent to

$$\{2,0\}'' \geqslant \{1,0\}'. \tag{12}$$

Bounds of rectangles are determined by the rules from section 3.
If rectangle is not good (for example, (12) doesn't take place), then

$$(\{2,0\}''; \{1,0\}') \not\subset \Delta.$$

Clearly, if the rectangle is not good, then one can not split it into smaller rectangles. That's why we will only consider good rectangles during the proof.

4.2 Sufficient conditions of goodness

4.2.1 Results

A horizontal rectangle Δ is good, if

$$\frac{\Delta_1}{\Delta_2} < \begin{cases} 3.8, & i_1 \equiv i_2 \pmod 2, \\ 3.43, & i_1 \not\equiv i_2 \pmod 2. \end{cases} \tag{13}$$

4.2.2 Universal $2,9$ bound

Now let's introduce some sufficient conditions for rectangle to be good.

Remind that we suppose $|\Delta_1| \geqslant |\Delta_2|$, which means

$$q\frac{1+p\Theta_1}{1+p'\Theta_1}\frac{1+p\Theta_{95}}{1+p'\Theta_{95}} \leqslant 1.$$

Let's introduce the following designations:

$$\gamma = [0; a_{-1}, ..., a_{i_1}, \frac{1}{\Theta_\gamma}],$$

$$\delta = [0; a_1, ..., a_{i_2}, \frac{1}{\Theta_\delta}],$$

where Θ's are some Θ's from Figure (5), specified in each case separately.

If rectangle is good, then (12.1) should take place:

$$\gamma' - \gamma'' < |\delta' - \delta''|, \tag{12.1}$$

where $\Theta_{\gamma'} = \Theta_{66}$, $\Theta_{\gamma''} = \Theta_{63}$, $\Theta_{\delta'} = \Theta_{90}$, $\Theta_{\delta''} = \Theta_{30}$.

*As I understand, theese thetas are supposed to be $\gamma' = \gamma'_{\{1,0\}}$, $\gamma'' = \gamma''_{\{2,0\}}$, and (12.1) is just paraphrasing of (**??**) in the «worst» case. As usual, Freiman gives no comments.*

Inequality (12.1) transforms into

$$0,313 < q\frac{1+p\Theta_{63}}{1+p'\Theta_{30}}\frac{1+p\Theta_{66}}{1+p'\Theta_{90}}. \tag{12.2}$$

Suppose

$$\frac{\Delta_1}{\Delta_2} < 2,9, \tag{12.3}$$

which is equivalent to

$$\frac{1}{q}\frac{1+p'\Theta_1}{1+p\Theta_1}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} < 2,9. \tag{12.4}$$

Then (12.2) takes place. Indeed, it is so, if

$$\frac{1}{2,9}\frac{1+p'\Theta_1}{1+p\Theta_1}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} > 0,313\frac{1+p'\Theta_{30}}{1+p\Theta_{63}}\frac{1+p'\Theta_{90}}{1+p\Theta_{66}}.$$

or, equivalent,

$$1,1 > \frac{1+p\Theta_1}{1+p\Theta_{63}}\frac{1+p\Theta_{95}}{1+p\Theta_{66}}\frac{1+p'\Theta_{30}}{1+p'\Theta_1}\frac{1+p'\Theta_{90}}{1+p'\Theta_{95}},$$

Which is easily checked. *Ha-ha.*

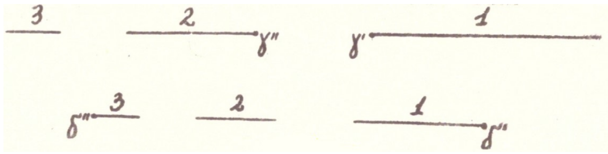
Condition (12.3) is sufficient for rectangle to be good, no matter, are i_1 and i_2 equivalent $\pmod 2$ or not, take conditions (**??**) and (**??**) place or not.

Θ_{1}	$=[0;\ \overline{31}]$	$= 0{,}263762$
Θ_{2}	$=[0;\ 312\overline{13}]$	$= 0{,}267649$
Θ_{3}	$=[0;\ 3\overline{12}]$	$= 0{,}26794$
Θ_{4}	$=[0;\ 3123\overline{31}]$	$= 0{,}270448$
Θ_{5}	$=[0;\ 3123\overline{12}]$	$= 0{,}270710$
Θ_{6}	$=[0;\ 312\overline{31}]$	$= 0{,}270738$
Θ_{7}	$=[0;\ 311\overline{13}]$	$= 0{,}27459$
Θ_{8}	$=[0;\ 31113\overline{12}]$	$= 0{,}27462$
Θ_{9}	$=[0;\ 31113\overline{31}]$	$= 0{,}274847$
Θ_{10}	$=[0;\ 311\overline{12}]$	$= 0{,}27517$
Θ_{11}	$=[0;\ 31\overline{12}]$	$= 0{,}27954$
Θ_{12}	$=[0;\ 3113\overline{31}]$	$= 0{,}280392$
Θ_{13}	$=[0;\ 3113\overline{12}]$	$= 0{,}28097$
Θ_{14}	$=[0;\ 31\overline{13}]$	$= 0{,}28105$
Θ_{15}	$=[0;\ 32\overline{31}]$	$= 0{,}290550$
Θ_{16}	$=[0;\ 323\overline{12}]$	$= 0{,}29062$
Θ_{17}	$=[0;\ 323\overline{31}]$	$= 0;291242$
Θ_{18}	$=[0;\ 32\overline{21}]$	$= 0;29216$
Θ_{19}	$=[0;\ 3\overline{21}]$	$= 0;29709$
Θ_{20}	$=[0;\ 3213\overline{12}]$	$= 0;29774$
Θ_{21}	$=[0;\ 32\overline{13}]$	$= 0;297773$
Θ_{22}	$=[0;\ 33\overline{31}]$	$= 0;302444$
Θ_{23}	$=[0;\ 333\overline{12}]$	$= 0;30025$
Θ_{24}	$=[0;\ 33\overline{21}]$	$= 0;30330$
Θ_{25}	$=[0;\ 33\overline{12}]$	$= 0;30600$
Θ_{26}	$=[0;\ 3312\overline{13}]$	$= 0;30603$
Θ_{27}	$=[0;\ 3313\overline{12}]$	$= 0;30638$
Θ_{28}	$=[0;\ 3\overline{31}]$	$= 0;306394$
Θ_{29}	$=[0;\ 2\overline{13}]$	$= 0;358258$
Θ_{30}	$=[0;\ 213\overline{12}]$	$= 0;35859$
Θ_{31}	$=[0;\ 2133\overline{31}]$	$= 0;361292$
Θ_{32}	$=[0;\ 2133\overline{12}]$	$= 0;36158$
Θ_{33}	$=[0;\ 213\overline{31}]$	$= 0;361602$
Θ_{34}	$=[0;\ 212\overline{13}]$	$= 0;365465$
Θ_{35}	$=[0;\ 21213\overline{12}]$	$= 0{,}36549$
Θ_{36}	$=[0;\ \overline{21}]$	$= 0{,}36602$
Θ_{37}	$=[0;\ 2121\overline{13}]$	$= 0{,}36779$
Θ_{38}	$=[0;\ 2123\overline{12}]$	$= 0{,}37119$
Θ_{39}	$=[0;\ 212\overline{31}]$	$= 0{,}371249$
Θ_{40}	$=[0;\ 211\overline{13}]$	$= 0{,}378537$
Θ_{41}	$=[0;\ 21113\overline{12}]$	$= 0{,}37858$
Θ_{42}	$=[0;\ 21113\overline{31}]$	$= 0{,}379018$
Θ_{43}	$=[0;\ 211\overline{12}]$	$= 0{,}37963$
Θ_{44}	$=[0;\ 211113\overline{31}]$	$= 0{,}383091$
Θ_{45}	$=[0;\ 211113\overline{12}]$	$= 0{,}38326$
Θ_{46}	$=[0;\ 2111\overline{13}]$	$= 0{,}383272$
Θ_{47}	$=[0;\ 21123\overline{12}]$	$= 0{,}38605$
Θ_{48}	$=[0;\ 2112\overline{31}]$	$= 0{,}386033$
Θ_{49}	$=[0;\ 21123\overline{31}]$	$= 0{,}386237$
Θ_{50}	$=[0;\ 2112\overline{21}]$	$= 0{,}38651$
Θ_{51}	$=[0;\ 21\overline{12}]$	$= 0{,}38800$
Θ_{52}	$=[0;\ 2113\overline{31}]$	$= 0{,}389648$
Θ_{53}	$=[0;\ 2113\overline{21}]$	$= 0{,}389916$
Θ_{54}	$=[0;\ 2113\overline{12}]$	$= 0{,}39076$
Θ_{55}	$=[0;\ 21\overline{13}]$	$= 0{,}390891$
Θ_{56}	$=[0;\ 22\overline{31}]$	$= 0{,}409544$
Θ_{57}	$=[0;\ 223\overline{12}]$	$= 0{,}40968$
Θ_{58}	$=[0;\ 2\overline{21}]$	$= 0{,}42265$
Θ_{59}	$=[0;\ 2213\overline{12}]$	$= 0{,}42398$
Θ_{60}	$=[0;\ 22\overline{13}]$	$= 0{,}424042$
Θ_{61}	$=[0;\ 23\overline{31}]$	$= 0{,}433577$
Θ_{62}	$=[0;\ 233\overline{12}]$	$= 0{,}43365$
Θ_{63}	$=[0;\ 23\overline{12}]$	$= 0{,}44093$
Θ_{64}	$=[0;\ 2\overline{31}]$	$= 0{,}441742$
Θ_{65}	$=[0;\ 1\overline{13}]$	$= 0{,}558256$
Θ_{66}	$=[0;\ 113\overline{12}]$	$= 0{,}55905$
Θ_{67}	$=[0;\ 1133\overline{12}]$	$= 0{,}56635$
Θ_{68}	$=[0;\ 113\overline{31}]$	$= 0{,}566423$
Θ_{69}	$=[0;\ 112\overline{13}]$	$= 0{,}57600$
Θ_{70}	$=[0;\ 11213\overline{12}]$	$= 0{,}57602$
Θ_{71}	$=[0;\ 1\overline{12}]$	$= 0{,}57735$
Θ_{72}	$=[0;\ 1123\overline{12}]$	$= 0{,}59032$
Θ_{73}	$=[0;\ 111\overline{13}]$	$= 0{,}609108$
Θ_{74}	$=[0;\ 11113\overline{12}]$	$= 0{,}60923$
Θ_{75}	$=[0;\ 111113\overline{31}]$	$= 0{,}620981$
Θ_{76}	$=[0;\ 111113\overline{12}]$	$= 0{,}62141$
Θ_{77}	$=[0;\ 1111\overline{13}]$	$= 0{,}621460$
Θ_{78}	$=[0;\ 11123\overline{12}]$	$= 0{,}62881$
Θ_{79}	$=[0;\ 11\overline{12}]$	$= 0{,}\ 63400$
Θ_{80}	$=[0;\ 1113\overline{31}]$	$= 0{,}63839$
Θ_{81}	$=[0;\ 1113\overline{12}]$	$= 0{,}63910$
Θ_{82}	$=[0;\ 11\overline{13}]$	$= 0{,}641742$
Θ_{83}	$=[0;\ 123\overline{12}]$	$= 0{,}69399$
Θ_{84}	$=[0;\ 12213\overline{12}]$	$= 0{,}70225$
Θ_{85}	$=[0;\ 1223\overline{12}]$	$= 0{,}70938$
Θ_{86}	$=[0;\ 122\overline{21}]$	$= 0{,}70783$
Θ_{87}	$=[0;\ 12113\overline{12}]$	$= 0{,}71903$
Θ_{88}	$=[0;\ 121113\overline{12}]$	$= 0{,}725381$
Θ_{89}	$=[0;\ \overline{12}]$	$= 0{,}73206$
Θ_{90}	$=[0;\ 1213\overline{12}]$	$= 0{,}73620$
Θ_{91}	$=[0;\ 12\overline{13}]$	$= 0{,}73624$
Θ_{92}	$=[0;\ 13\overline{31}]$	$= 0{,}765465$
Θ_{93}	$=[0;\ 133\overline{12}]$	$= 0{,}76569$
Θ_{94}	$=[0;\ 13\overline{12}]$	$= 0{,}78868$
Θ_{95}	$=[0;\ \overline{13}]$	$= 0{,}791287$

Figure 5: Table of Θ's.

4.2.3 Case $i_1 = i_2 \pmod 2$

Now consider case $i_1 = i_2 \pmod 2$ (see picture):



We will suppose case $a_{i_2} = 1$, $a_{i_2-1} = 3$ doesn't take place.
We can substitute the folliwing γ's and δ's into (12.1):

$$\Theta_{\gamma'} = \Theta_{66},\ \Theta_{\gamma''} = \Theta_{63},\ \Theta_{\delta'} = \Theta_{90},\ \Theta_{\delta''} = \Theta_{3},$$

and instead of (12.2) we will get

$$0,253 < q\frac{1+p\Theta_{63}}{1+p'\Theta_{3}}\frac{1+p\Theta_{66}}{1+p'\Theta_{90}}. \tag{12.5}$$

This choose of $\Theta_{\gamma'}$ and $\Theta_{\delta''}$ is fine, if the following inequality takes place:

$$\delta' - \delta'' > 1,4(\gamma' - \gamma''), \tag{12.6}$$

where

$$\Theta_{\gamma'} = \Theta_{68},\ \Theta_{\gamma''} = \Theta_{65},\ \Theta_{\delta'} = \Theta_{28},\ \Theta_{\delta''} = \Theta_{1},$$

so we can rewrite (12.6) as

$$0,253 < q\frac{1+p\Theta_{65}}{1+p'\Theta_{1}}\frac{1+p\Theta_{68}}{1+p'\Theta_{28}}. \tag{12.7}$$

We can notice that (12.7) follows from (12.5). Indeed, that follows from inequality

$$0,253\frac{1+p'\Theta_{3}}{1+p'\Theta_{63}}\frac{1+p\Theta_{90}}{1+p\Theta_{66}} > 0,269\frac{1+p'\Theta_{1}}{1+p'\Theta_{65}}\frac{1+p\Theta_{28}}{1+p\Theta_{68}}$$

or inequality

$$\frac{1+p\Theta_{65}}{1+p\Theta_{63}}\frac{1+p\Theta_{68}}{1+p\Theta_{66}}\frac{1+p'\Theta_{3}}{1+p'\Theta_{1}}\frac{1+p'\Theta_{90}}{1+p'\Theta_{28}} > 1,064.$$

Here Freiman even omits comment «which is checked directly».
Overall, we proved that (12.5) is enough for condition to be good.
Suppose

$$\frac{\Delta_1}{\Delta_2} < 3,8 \tag{12.8}$$

or

$$\frac{1}{q}\frac{1+p'\Theta_{1}}{1+p\Theta_{1}}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} < 3,8.$$

Then (12.5) takes place. Indeed, it is so, if

$$\frac{1}{3,8}\frac{1+p'\Theta_{1}}{1+p'\Theta_{1}}\frac{1+p\Theta_{95}}{1+p\Theta_{95}} > 0,253\frac{1+p'\Theta_{3}}{1+p'\Theta_{63}}\frac{1+p\Theta_{90}}{1+p\Theta_{66}}.$$

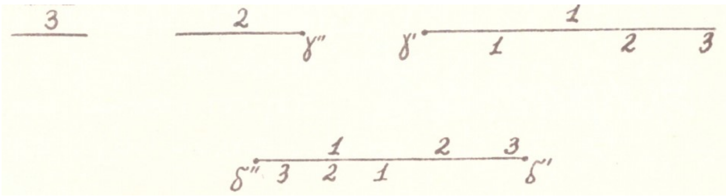
The last inequality is transformed into

$$1,04 > \frac{1+p'\Theta_3}{1+p'\Theta_1}\frac{1+p'\Theta_{90}}{1+p'\Theta_{95}}\frac{1+p\Theta_1}{1+p\Theta_{63}}\frac{1+p\Theta_{95}}{1+p\Theta_{66}},$$

which is easy to check.

4.2.4 Case $i_1 \neq i_2 \pmod 2$

Now consider case $i_1 \neq i_2 \pmod 2$ and, again, case $a_{i_2} = 1$, $a_{i_2-1} = 3$ doesn't take place. Suppose i_1 is even (see picture):



Taking

$$\Theta_{\gamma'} = \Theta_{66},\ \Theta_{\gamma''} = \Theta_{59},\ \Theta_{\delta'} = \Theta_3,\ \Theta_{\delta''} = \Theta_{90},$$

rewrite (12.1) as

$$0,2885 < q\frac{1+p\Theta_{59}}{1+p'\Theta_3}\frac{1+p\Theta_{66}}{1+p'\Theta_{90}}. \tag{12.9}$$

Suppose

$$\frac{\Delta_1}{\Delta_2} < 3,43 \tag{12.10}$$

or

$$\frac{1}{q}\frac{1+p'\Theta_1}{1+p\Theta_1}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} < 3,43.$$

Then (12.9) takes place. Indeed, it is so, if

$$\frac{1}{3,43}\frac{1+p'\Theta_1}{1+p\Theta_1}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} > 0,2885\frac{1+p'\Theta_3}{1+p\Theta_{59}}\frac{1+p'\Theta_{90}}{1+p\Theta_{66}}.$$

or

$$1,0105 > \frac{1+p'\Theta_3}{1+p'\Theta_1}\frac{1+p'\Theta_{90}}{1+p'\Theta_{95}}\frac{1+p\Theta_1}{1+p\Theta_{59}}\frac{1+p\Theta_{95}}{1+p\Theta_{66}},$$

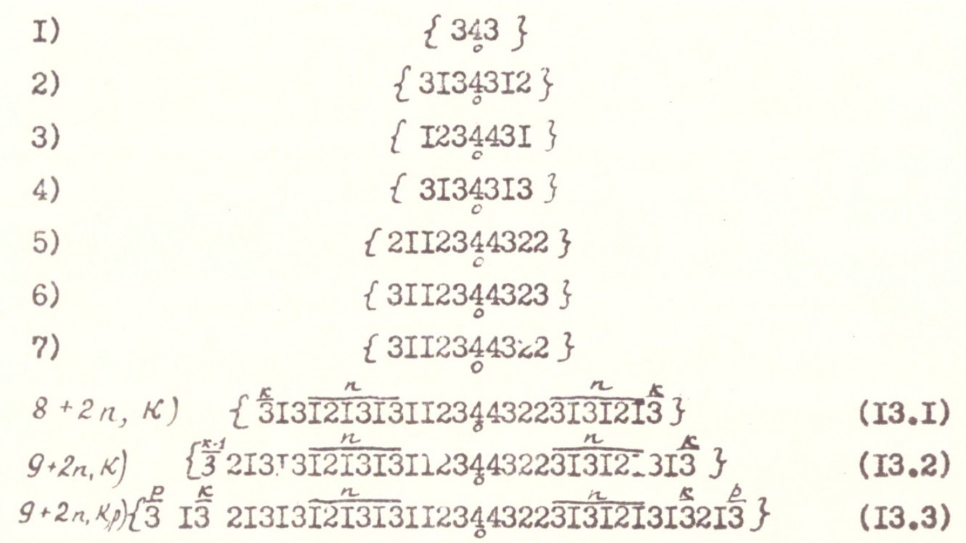
increasing the right part, obtain

$$1,0105 > 0,9896\frac{1+p\cdot 1,0551+p^2\cdot 0,20875}{1+p\cdot 0,983+p^2\cdot 0,237},$$

which is easily checked.

5 Freiman's constant. Initial set of rectangles

All these rectangles satisfy _*here will be a complete list of conditions.*_
At least all of them are good, in terms of section 4.


I) $\{3\underset{o}{4}3\}$

2) $\{3I3\underset{o}{4}3I2\}$

3) $\{I23\underset{o}{4}43I\}$

4) $\{3I3\underset{o}{4}3I3\}$

5) $\{2II23\underset{o}{4}4322\}$

6) $\{3II23\underset{o}{4}4323\}$

7) $\{3II23\underset{o}{4}4322\}$

$8+2n, k)$ $\{\overset{k}{3}I3\overline{I2I3I3}^{\,n}II23\underset{o}{4}4322\overline{2I3I2I}^{\,n}\overset{k}{3}\}$ (I3.I)

$9+2n, k)$ $\{\overset{k-1}{3}2I3I3\overline{I2I3I3}^{\,n}II23\underset{o}{4}4322\overline{3I3I2I}^{\,n}3I\overset{k}{3}\}$ (I3.2)

$9+2n, k, p)$ $\{\overset{p}{3}\ I\overset{k}{3}\ 2I3I3\overline{I2I3I3}^{\,n}II23\underset{o}{4}4322\overline{3I3I2I}^{\,n}3I\overset{k}{3}2I\overset{p}{3}\}$ (I3.3)

Figure 6: Freiman's constant. Initial set of rectangles.

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