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**Markov and Lagrange Spectra
for Continued Fractions**

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1 Foreword

The main subject of this paper is the Markov spectrum or, to be more precise, its final portion, called the Hall's Ray.

Its existence have been proven by Hall more than a century ago; however, evaluating the exact value of the start of the Hall's Ray took time.

In 1975, Freiman found and proved this constant.

This paper contains the translated and modernized approach from the original paper.

2 Introductory notation

2.1 Markov and Lagrange values

Symbol \mathcal{M} denotes double-infinite sequences from $\mathbb{N}^{\mathbb{Z}}$:

$$\mathcal{M} = \dots a_{-2}a_{-1}a_0a_1a_2\dots$$

I will use $\lambda(\mathcal{M})$, $\mu(\mathcal{M})$ and $f(\mathcal{M})$ for Lagrange, Markov values and height function. Symbols γ and δ denote the lhs and rhs of sequence \mathcal{M} :

$$\begin{aligned}\gamma(\mathcal{M}) &= [0; a_{-1}, a_{-2}, \dots], \\ \delta(\mathcal{M}) &= [0; a_1, a_2, \dots], \\ f(\mathcal{M}) &= a_0 + \gamma(\mathcal{M}) + \delta(\mathcal{M}).\end{aligned}$$

At last, symbols M and L denote the Markov and Lagrange spectra.

2.2 Centered sequence

Definition. A sequence \mathcal{M} is called **centered**, if

$$\mu(\mathcal{M}) = f(\mathcal{M}). \tag{1}$$

Proposition. *Markov spectrum can be defined with only centered sequences:*

$$\{\mu(\mathcal{M}) \mid \mathcal{M} \in \mathbb{N}^{\mathbb{Z}}\} = M = \{\mu(\mathcal{M}) \mid \mathcal{M} \text{ is centered}\}.$$

2.3 Rectangle

Designation. Denote by

$$\{a_{i_1}, a_{i_1+1}, \dots, a_{i_2}\} \quad (i_1 \leq 0 \leq i_2)$$

the set of double-infinite sequences \mathcal{M} with fixed terms $a_{i_1}, a_{i_1+1}, \dots, a_{i_2}$ on the corresponding positions.

Terms a_s for $s < i_1$ and $s > i_2$ are arbitrary integers, chosen such that \mathcal{M} is centered and, maybe, satisfies some conditions.

Segments Δ_1 , Δ_2 and Δ are defined by the following equations:

$$\begin{aligned}\Delta_1 &= [\Delta'_1; \Delta''_1] = [\min \gamma(\mathcal{M}); \max \gamma(\mathcal{M})], \\ \Delta_2 &= [\Delta'_2; \Delta''_2] = [\min \delta(\mathcal{M}); \max \delta(\mathcal{M})], \\ \Delta &= [\Delta'; \Delta''] = a_0 + \Delta_1 + \Delta_2,\end{aligned} \tag{2}$$

where \mathcal{M} belongs to the set.

Note that we will use ' for the lower bound, and '' for the upper bound.

Definition. **Rectangle** is the segment Δ with the set of sequences, defining it.

2.4 Horizontal rectangle

Definition. Call a rectangle Δ **horizontal**, if

$$|\Delta_1| \geq |\Delta_2|. \quad (3)$$

In (3) we allow terms a_s for $s < i_1$ and $s > i_2$ to be integers $\{1, 2, 3\}$, regardless of the requirement that sequences $\mathcal{M} \in \Delta$ are centered.

In other words, Δ is horizontal, if and only if

$$|[0; a_{-1}, \dots, a_{i_1}, \overline{3, 1}] - [0; a_{-1}, \dots, a_{i_1}, \overline{1, 3}]| \geq |[0; a_1, \dots, a_{i_2}, \overline{3, 1}] - [0; a_1, \dots, a_{i_2}, \overline{1, 3}]|.$$

Clearly, we can always obtain a horizontal rectangle out of the vertical one, as we can reindex the sequence in the opposite direction.

2.5 Subrectangle

Consider a rectangle Δ , set by the sequence center

$$\{a_{i_1}, a_{i_1+1}, \dots, a_{i_2}\}.$$

We will use a **shorter notation** for subrectangles, produced by setting integers a_i for $i < i_1$ or $i > i_2$:

$$\{b_\ell \dots b_1, c_1 \dots c_r\} := \{b_\ell \dots b_1 a_{i_1} \dots a_{i_2} c_1 \dots c_r\}.$$

For example:

$$\{213, 3\} := \{213 a_{i_1} \dots a_{i_2} 3\}, \quad (\text{ex.1})$$

$$\{2, 0\} := \{2 a_{i_1} \dots a_{i_2}\}. \quad (\text{ex.2})$$

We will also shorter the notation (2): lhs and rhs are $\Delta_1(312)$ and $\Delta_2(3)$ for subrectangle (ex.1) and $\Delta_1(2)$ and a_2 for (ex.2).

2.6 Geometrical interpretation

Consider the mapping

$$\begin{aligned} \tilde{h} : \mathbb{N}^{\mathbb{Z}} &\rightarrow \mathbb{R}^2, \\ \tilde{h}(\mathcal{M}) &= (\gamma(\mathcal{M}); \delta(\mathcal{M})). \end{aligned}$$

In these terms, the Markov spectrum M is the projection of some subset $\mathcal{S} \subset C_4 \times C_4$ onto the diagonal.

Then **rectangle** Δ is indeed a rectangle $\Delta_1 \times \Delta_2$ and **subrectangles** are its subrectangles.

We will consider a family of rectangles whose projections cover the beginning of Hall's Ray.

Then we will present the algorithm to split rectangle into subrectangles so that their projections cover the projection of initial rectangle.

When we say that rectangles intersect, we, however, mean that their projections intersect.

The more «squarish» the rectangle, the easier the step.

That's why we will bound the aspect ratio of rectangles (see **good** rectangle).

Formulas to evaluate side lengths and aspect ratio are given in the section 3.

3 Calculations

3.1 Length of Δ_1 or Δ_2

Let's fix some terms of continued fraction $[0; q_1, q_2, q_3, \dots, q_n]$.
We will often need to measure difference

$$[0; q_1, q_2, q_3, \dots, q_n, \frac{1}{\Theta_R}] - [0; q_1, q_2, q_3, \dots, q_n, \frac{1}{\Theta_L}],$$

where Θ 's are some continuations of the continued fraction.

They generally look like $\Theta = [0; 12\overline{13}]$ or something¹. We will set Θ 's explicitly.

For the general proof, Θ 's will be taken from Table 6.

Designation. For given continuation Θ_i , denote by ε_i the resulting continued fraction:

$$\varepsilon_i = [0; q_1, q_2, \dots, q_n, \frac{1}{\Theta_i}]. \quad (4)$$

Then the following equality takes place:

$$|\varepsilon_i - \varepsilon_j| = \frac{|\Theta_i - \Theta_j|}{Q_n^2 (1 + pQ_i) (1 + pQ_j)}, \quad (5)$$

where

$$p = \frac{Q_{n-1}}{Q_n}.$$

3.2 Rectangle aspect ratio

Consider some fixed center of rectangle $\{a_{i_1} \dots a_{i_2}\}$.

We will often extend it from the left (right) using continuations $\Theta_{\gamma_1}, \Theta_{\gamma_2}$ ($\Theta_{\delta_1}, \Theta_{\delta_2}$).

Rule (4) produces γ_1, γ_2 (δ_1, δ_2).

Finite continued fractions $\frac{P_{i_1}}{Q_{i_1}} = [0; a_{-1}, a_{-2}, \dots, a_{i_1}]$ ($\frac{P_{i_2}}{Q_{i_2}} = [0; a_1, a_2, \dots, a_{i_2}]$) are their convergents.

Then

$$\left| \frac{\gamma_1 - \gamma_2}{\delta_1 - \delta_2} \right| = \frac{1}{q} \left| \frac{\Theta_{\gamma_1} - \Theta_{\gamma_2}}{\Theta_{\delta_1} - \Theta_{\delta_2}} \right| \frac{1 + p'\Theta_{\delta_1}}{1 + p\Theta_{\gamma_1}} \frac{1 + p'\Theta_{\delta_2}}{1 + p\Theta_{\gamma_2}} \approx \frac{1}{q} \left| \frac{\Theta_{\gamma_1} - \Theta_{\gamma_2}}{\Theta_{\delta_1} - \Theta_{\delta_2}} \right|, \quad (6)$$

where

$$p = \frac{Q_{i_1+1}}{Q_{i_1}}, \quad p' = \frac{Q_{i_2-1}}{Q_{i_2}}, \quad q = \frac{Q_{i_1}^2}{Q_{i_2}^2}.$$

¹Here, as always in this book, \overline{abc}^k means k -times repetition of abc , and \overline{abc} means infinite repetition.

4 Rectangle boundaries

We will set boundaries for rectangles in a different way.

To distinguish rectangles, we will introduce the notion of left- or right-shortened rectangles.

It is given in subsections 4.1-4.3.

Subsections 4.4-4.5 provide the rules for boundaries $\Delta'_1, \Delta''_1, \Delta'_2, \Delta''_2$.

4.1 Resection

Definition. Call **resection** of a segment $A = [a; b]$ a process of removing subsegment $A_{12} = [a_1; b_1]$, leaving two segments $A_1 \sqcup A_2 = [a; a_1] \sqcup [b_1; b]$.

Definition. Call subsegment $A_{12} \subset A$ **normal**, if it is thicker than the two remaining subsegments:

$$|A_{12}| \leq \min \{|A_1|, |A_2|\} \quad (7)$$

We call a resection **normal** if the resected subsegment is normal.

Proposition. For any normal resection, having

$$A + A = (A_1 \sqcup A_2) + (A_1 \sqcup A_2). \quad (8)$$

4.2 Shortened rectangle

Consider a horizontal rectangle Δ .

4.2.1 Case $i_1 \equiv i_2 \equiv 0 \pmod{2}$

If both i_1 and i_2 are even, then left- and right-shortened rectangles are defined as follows.

Definition. Rectangle Δ is called left-shortened, if for subrectangle $\{3, 3\}$ the following condition takes place:

$$|\Delta_1(3)| \leq 1.4 \cdot |\Delta_2(3)|. \quad (9)$$

Here, as in (3), we allow terms a_s for $s < i_1$ and $s > i_2$ to be integers $\{1, 2, 3\}$, so (9) can be rewritten as

$$|[0; a_{-1}, \dots, a_{i_1}, 3, \overline{3, 1}] - [0; a_{-1}, \dots, a_{i_1}, \overline{3, 1}]| \leq |[0; a_1, \dots, a_{i_2}, 3, \overline{3, 1}] - [0; a_1, \dots, a_{i_2}, \overline{3, 1}]|.$$

For further convenience, we also introduce the opposite to (9) condition:

Definition. Rectangle Δ is called left-normal, if

$$|\Delta_1(3)| > 1.4 \cdot |\Delta_2(3)|. \quad (10)$$

Now we introduce the notion of right-shortened rectangle:

Definition. Rectangle Δ is called right-shortened, if for subrectangle $\{31, 13\}$ the following condition takes place:

$$|\Delta_1(13)| \leq 1.4 \cdot |\Delta_2(13)|. \quad (11)$$

As in 4.2.1, we allow terms a_s for $s < i_1$ and $s > i_2$ to be integers $\{1, 2, 3\}$, so (11) can be rewritten as

$$\left| [0; a_{-1}, \dots, a_{i_1}, 1, 3, \overline{3, 1}] - [0; a_{-1}, \dots, a_{i_1}, \overline{1, 3}] \right| \leq \left| [0; a_1, \dots, a_{i_2}, 1, 3, \overline{3, 1}] - [0; a_1, \dots, a_{i_2}, \overline{1, 3}] \right|.$$

We also introduce the opposite condition:

Definition. Rectangle Δ is called right-normal, if

$$|\Delta_1(13)| > 1.4 \cdot |\Delta_2(13)|. \quad (12)$$

4.2.2 Other cases

If $i_1 \equiv i_2 \pmod{2}$ and i_1 is odd, then the conditions for left- and right-shortened rectangles are swapped.

In case $i_1 \not\equiv i_2 \pmod{2}$, the notions of left- and right-shortened rectangles are inferred from subrectangles $\{1, \}$, $\{2, \}$, and $\{3, \}$.

We will refer to the relevant conditions from 4.5

4.3 Explanation of shortened

Notion left-shortened is used to set the lower bound Δ' for rectangle Δ .

Notion right-shortened is used to set the upper bound Δ'' .

For example, when interested in Δ' when i_1 is even and i_2 is odd, one has to consider larger a_{i_1-1} and smaller a_{i_2+1} . As the rectangle is horizontal, we will add terms to the left first. So we will check the condition (11) for $\{2, \}$ or $\{3, \}$.

In other words, check whether $\{2, \}$ or $\{3, \}$ is left-shortened or left-normal (recall that $i_1 - 1$ and i_2 are odd, so left-shortenence is defined by (11)).

Which subrectangle ($\{2, \}$ or $\{3, \}$) to take, however, depends on terms a_{i_1+1} and a_{i_1} (see 4.4).

4.4 How we ensure centeredness

Consider integers q_1, q_2, \dots, q_n . Let $\{\delta_n\}$ be the set of fractions $\delta_n = [0; q_1, q_2, \dots, q_n, \dots]$ with n fixed terms. We will suppose that n is even (for odd n the bounds are swapped).

At first, determine the smallest of fractions δ_n .

We will consider 2 cases: S (Shortened) and N (Normal):

$$S. \quad q_{n-1} = 3, q_n = 1. \quad (13a)$$

$$N. \quad \text{Otherwise.} \quad (13b)$$

The lower bound δ'_n for segment, containing δ_n , is defined by:

$$\begin{aligned} S. \quad & \delta'_n = [0; q_1, \dots, q_n, 213\overline{12}]. \\ N. \quad & \delta'_n = [0; q_1, \dots, q_n, 3\overline{12}]. \end{aligned}$$

To set the upper bound δ''_n , largest of δ_n , consider 2 other cases:

$$S. \quad q_n = 3. \quad (14a)$$

$$N. \quad q_n \neq 3. \quad (14b)$$

Then

$$\begin{aligned} S. \quad & \delta_n'' = [0; q_1, \dots, q_n, 1213\overline{12}]. \\ N. \quad & \delta_n'' = [0; q_1, \dots, q_n, 13\overline{12}]. \end{aligned}$$

These bounds will allow us to construct sequences \mathcal{M} , for which combination (31313) is forbidden and, therefore, the following condition takes place:

$$f_i(\mathcal{M}) \leq \lambda(\overline{31312}) \approx 4,5241, \quad i \neq 0,$$

which will ensure (1).

4.5 Formal

Let's now turn to concrete definitions and bounds. Remind that we are looking at the horizontal rectangle Δ .

Suppose that $i_1 \equiv i_2 \pmod{2}$, i_1 is even. (If i_1 is odd, then Δ' and Δ'' are swapped.)

4.5.1 Bounds for Δ'

- I. Suppose both $\{\gamma_0(\mathcal{M})\}$ and $\{\delta_0(\mathcal{M})\}$ satisfy (13b) and the rectangle Δ is left-normal, that is, satisfies (10). We will denote such situation by $N - N - N$ (segment Δ_1 is left-normal, Δ_2 is left-normal, rectangle Δ is normal).

In this case define Δ' by equation

$$\Delta' = f(\overline{21}3a_{i_1} \dots a_{i_2} 3\overline{12}). \quad (15)$$

- IIa. Sets $\{\gamma_0(\mathcal{M})\}$ and $\{\delta_0(\mathcal{M})\}$ meet condition (13b) and Δ is left-shortened, that is, meets condition (9). This is case $N - N - S$ (segments Δ_1 and Δ_2 are left-normal, rectangle Δ is left-shortened).

Then

$$\Delta' = f(\overline{21}3a_{i_1} \dots a_{i_2} 213\overline{12}). \quad (16)$$

- IIb. Set $\{\gamma_0(\mathcal{M})\}$ meets (13b), $\{\delta_0(\mathcal{M})\}$ meets (13a). No matter which condition (9) of (10) is met. It is the case $N - S$ (segment Δ_1 is left-normal, segment Δ_2 left-shortened). Bound Δ_1 is defined by (16).

- III. Set $\{\gamma_0(\mathcal{M})\}$ meets (13a), $\{\delta_0(\mathcal{M})\}$ meets (13b). In this $S - N$ case (segment Δ_1 is left-shortened, segment Δ_2 is left-normal) Δ' is defined as follows:

$$\Delta' = f(\overline{21}312a_{i_1} \dots a_{i_2} 3\overline{12}). \quad (17)$$

- IV. Both $\{\gamma_0(\mathcal{M})\}$ and $\{\delta_0(\mathcal{M})\}$ meet (13a). In this $S - S$ case (both segments Δ_1 and Δ_2 are left-shortened) Δ' is defined by

$$\Delta' = f(\overline{21}312a_{i_1} \dots a_{i_2} 213\overline{12}). \quad (18)$$

Left side of the figure 1 illustrates these bounds.

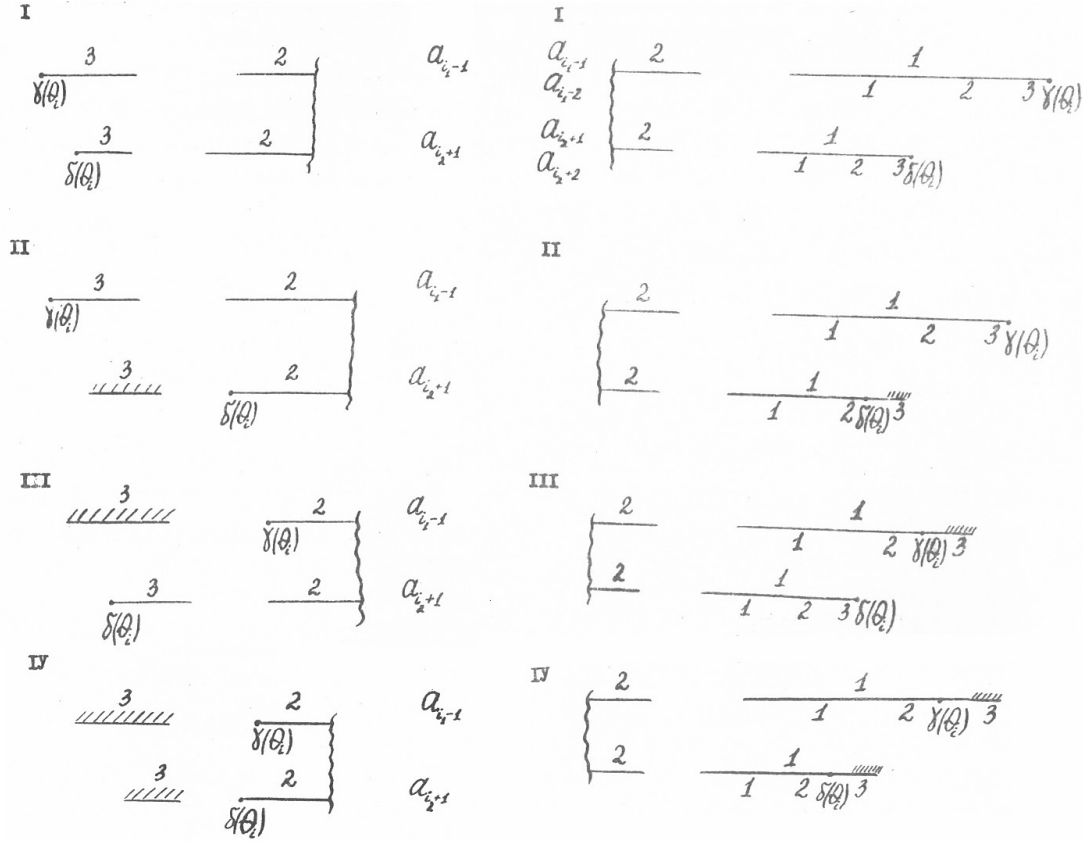


Figure 1: Bounds Δ' (left) and Δ'' (right).

Hatched areas correspond to values of a_{i_1-1} and a_{i_2+1} (on the left) or a_{i_1-2} and a_{i_2+2} (on the right), equal to 3, which can not appear in the concrete case.

$$\Delta'_1 = \gamma(\Theta_i), i = 3, 30, \Delta'_2 = \delta(\Theta_i), i = 3, 30, \Delta' = \Delta'_1 + \Delta'_2.$$

$$\Delta''_1 = \gamma(\Theta_i), i = 90, 94, \Delta''_2 = \delta(\Theta_i), i = 90, 94.$$

4.5.2 Bounds for Δ''

Now we will provide formulas for Δ'' :

$$I \quad \Delta'' = f(\overline{2131}a_{i_1}\dots a_{i_2}13\overline{12}), \quad (19a)$$

$$II \quad \Delta'' = f(\overline{2131}a_{i_1}\dots a_{i_2}1213\overline{12}), \quad (19b)$$

$$III \quad \Delta'' = f(\overline{213121}a_{i_1}\dots a_{i_2}13\overline{12}), \quad (19c)$$

$$IV \quad \Delta'' = f(\overline{213121}a_{i_1}\dots a_{i_2}1213\overline{12}). \quad (19d)$$

Figure (2) regulates the choice of the formulas.

Case	Δ''_1	Δ''_2	Rectangle	Δ''
I	(14b)	(14b)	(12) $N - N - N$	(19a)
IIa	(14b)	(14b)	(11) $N - N - S$	(19b)
IIb	(14b)	(14a)	$N - S$	(19b)
III	(14a)	(14b)	$S - N$	(19c)
IV	(14a)	(14a)	$S - S$	(19d)

Figure 2: Rules for choice of Δ'' in case $i_1 \equiv i_2 \pmod{2}$.

Again, these bounds are illustrated on the right side of the figure 1.

4.5.3 Case $i_1 \not\equiv i_2 \pmod{2}$

Now consider case $i_1 \not\equiv i_2 \pmod{2}$, i_1 is even. We will use rules from figure (3) to choose one of 4 formulas for Δ' .

$$I \quad \Delta' = f(\overline{213}a_{i_1}\dots a_{i_2}13\overline{12}), \quad (20a)$$

$$II \quad \Delta' = f(\overline{213}a_{i_1}\dots a_{i_2}1213\overline{12}), \quad (20b)$$

$$III \quad \Delta' = f(\overline{21312}a_{i_1}\dots a_{i_2}13\overline{12}), \quad (20c)$$

$$IV \quad \Delta' = f(\overline{21312}a_{i_1}\dots a_{i_2}1213\overline{12}). \quad (20d)$$

Case	Δ'_1	Δ'_2	Case name	Δ'
I	(13b)	(14b)	$N - N$	(20a)
II	(13b)	(14a)	$N - S$	(20b)
III	(13a)	(14b)	$S - N$	(19c)
IV	(13a)	(14a)	$S - S$	(19d)

Figure 3: Rules for choice of Δ' in case $i_1 \not\equiv i_2 \pmod{2}$, i_1 is even.

To determine the bound Δ'' , we will use subrectangle

$$\{1, 0\}.$$

Having $i_1 - 1 \equiv i_2 \pmod{2}$, so we can use all the previous formulas to determine Δ'' .

5 Good rectangle

5.1 Definition

Consider a horizontal rectangle $\Delta = \{a_{i_1} \dots a_{i_2}\}$.

Definition. Rectangle Δ is called **good**, if subrectangles $\{2, 0\}$ and $\{1, 0\}$ intersect.

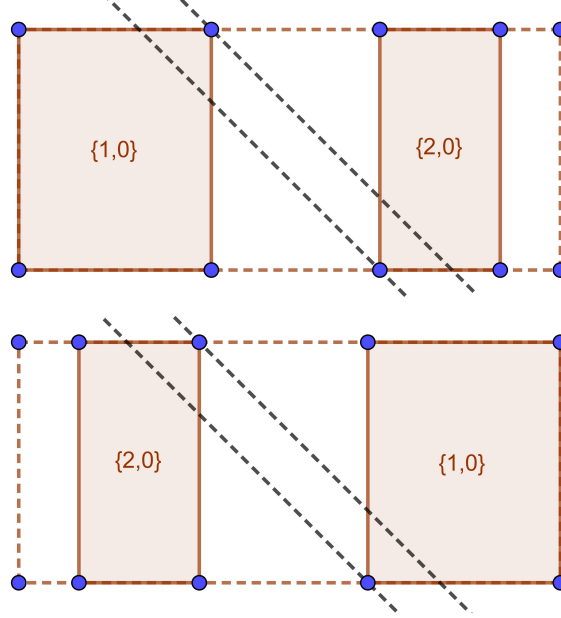


Figure 4: Good rectangles if i_1 is odd (upper) and even (lower).

For example, in case i_1 is even, goodness is equivalent to

$$\{2, 0\}'' \geq \{1, 0\}'. \quad (21)$$

Bounds of rectangles are determined by the rules from section 4.

If the rectangle Δ is not good (for example, (21) doesn't take place), then

$$(\{2, 0\}''; \{1, 0\}') \not\subset \Delta.$$

Clearly, if the rectangle is not good, then one can not split it into smaller subrectangles whose projections cover the projection of initial one. That's why we will only consider good rectangles during the proof.

5.2 Sufficient conditions of goodness

5.2.1 Results

A horizontal rectangle Δ is good, if

$$\frac{\Delta_1}{\Delta_2} < \begin{cases} 3.8, & i_1 \equiv i_2 \pmod{2}, \quad \{\delta(\mathcal{M})\} \text{ meets (13b),} \\ 3.43, & i_1 \not\equiv i_2 \pmod{2}, \quad \{\delta(\mathcal{M})\} \text{ meets (13b),} \\ 2.9, & \text{otherwise.} \end{cases} \quad (22)$$

5.2.2 Universal 2, 9 bound

We will introduce a sufficient conditions for a rectangle to be good.

Designation. For given continuations Θ_γ and Θ_δ , denote by γ and δ the resulting continued fractions:

$$\gamma = [0; a_{-1}, \dots, a_{i_1}, \frac{1}{\Theta_\gamma}],$$

$$\delta = [0; a_1, \dots, a_{i_2}, \frac{1}{\Theta_\delta}].$$

As stated in 2.1, γ 's correspond to the lhs of \mathcal{M} , and δ 's correspond to the rhs of \mathcal{M} . Variables Θ will be taken from the Figure 6 and will be specified in each case separately.

Suppose that i_1 is even.

Remind that $|\Delta_1| \geq |\Delta_2|$, which means that

$$\frac{|\gamma'' - \gamma'|}{|\delta'' - \delta'|} \geq 1,$$

where

$$\Theta_{\gamma'} = \Theta_{\delta'} = \Theta_1, \quad \Theta_{\gamma''} = \Theta_{\delta''} = \Theta_{95}.$$

Using (6), rewrite it as

$$q \frac{1 + p\Theta_1}{1 + p'\Theta_1} \frac{1 + p\Theta_{95}}{1 + p'\Theta_{95}} \leq 1.$$

Goodness can be written as

$$\gamma' - \gamma'' < |\delta' - \delta''|, \tag{23}$$

where $\gamma' = \Delta_1(1)'$ and $\gamma'' = \Delta_1(2)''$.

For an arbitrary rectangle Δ , taking

$$\Theta_{\gamma'} = \Theta_{66}, \quad \Theta_{\gamma''} = \Theta_{63}, \quad \Theta_{\delta'} = \Theta_{90}, \quad \Theta_{\delta''} = \Theta_{30}.$$

Inequality (23) transforms into

$$0, 313 < q \frac{1 + p\Theta_{63}}{1 + p'\Theta_{30}} \frac{1 + p\Theta_{66}}{1 + p'\Theta_{90}}. \tag{24}$$

Suppose that

$$\frac{\Delta_1}{\Delta_2} < 2, 9, \tag{25}$$

which is equivalent to

$$\frac{1}{q} \frac{1 + p'\Theta_1}{1 + p\Theta_1} \frac{1 + p'\Theta_{95}}{1 + p\Theta_{95}} < 2, 9. \tag{26}$$

Then (24) takes place. Indeed, it is so, if

$$\frac{1}{2, 9} \frac{1 + p'\Theta_1}{1 + p\Theta_1} \frac{1 + p'\Theta_{95}}{1 + p\Theta_{95}} > 0, 313 \frac{1 + p'\Theta_{30}}{1 + p\Theta_{63}} \frac{1 + p'\Theta_{90}}{1 + p\Theta_{66}}.$$

or, equivalent,

$$1, 1 > \frac{1+p\Theta_1}{1+p\Theta_{63}} \frac{1+p\Theta_{95}}{1+p\Theta_{66}} \frac{1+p'\Theta_{30}}{1+p'\Theta_1} \frac{1+p'\Theta_{90}}{1+p'\Theta_{95}},$$

which is checked directly.

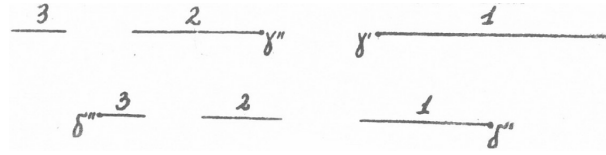
Condition (25) is sufficient for rectangle to be good, regardless of the parity of i_1 and i_2 and the left or right shortness or normalness of rectangle.

5.2.3 Case $i_1 \equiv i_2 \pmod{2}$

Now consider case $i_1 \equiv i_2 \pmod{2}$.

Suppose that the right hand side $\{\delta(\mathcal{M})\}$ is left-normal, that is, meets (13b).

Equivalent, the case $a_{i_2} = 1$, $a_{i_2-1} = 3$ doesn't take place.



This assumption allows us to substitute the following γ 's and δ 's into (23):

$$\Theta_{\gamma'} = \Theta_{66}, \Theta_{\gamma''} = \Theta_{63}, \Theta_{\delta'} = \Theta_{90}, \Theta_{\delta''} = \Theta_3.$$

The rhs left-normality leaves us hope that the bound Δ_2'' will be set to $\Delta_2(\Theta_3)$, so we are substituting $\Theta_{\delta''}$ with Θ_3 instead of Θ_{30} .

Now, instead of (24) we will get

$$0, 253 < q \frac{1+p\Theta_{63}}{1+p'\Theta_3} \frac{1+p\Theta_{66}}{1+p'\Theta_{90}}. \quad (27)$$

Such choice of $\Theta_{\gamma'}$ and $\Theta_{\delta''}$ is fine, if the following inequality takes place:

$$\delta' - \delta'' > 1, 4(\gamma' - \gamma''), \quad (28)$$

where

$$\Theta_{\gamma'} = \Theta_{68}, \Theta_{\gamma''} = \Theta_{65}, \Theta_{\delta'} = \Theta_{28}, \Theta_{\delta''} = \Theta_1,$$

so we can rewrite (28) as

$$0, 253 < q \frac{1+p\Theta_{65}}{1+p'\Theta_1} \frac{1+p\Theta_{68}}{1+p'\Theta_{28}}. \quad (29)$$

We can notice that (29) follows from (27). Indeed, that follows from the inequality

$$0, 253 \frac{1+p'\Theta_3}{1+p'\Theta_{63}} \frac{1+p\Theta_{90}}{1+p\Theta_{66}} > 0, 269 \frac{1+p'\Theta_1}{1+p'\Theta_{65}} \frac{1+p\Theta_{28}}{1+p\Theta_{68}},$$

or inequality

$$\frac{1+p\Theta_{65}}{1+p\Theta_{63}} \frac{1+p\Theta_{68}}{1+p\Theta_{66}} \frac{1+p'\Theta_3}{1+p'\Theta_1} \frac{1+p'\Theta_{90}}{1+p'\Theta_{28}} > 1, 064.$$

Overall, we have proved that (27) is enough for rectangle to be good.

Suppose that

$$\frac{\Delta_1}{\Delta_2} < 3, 8, \quad (30)$$

or

$$\frac{1}{q} \frac{1 + p'\Theta_1}{1 + p\Theta_1} \frac{1 + p'\Theta_{95}}{1 + p\Theta_{95}} < 3, 8.$$

Then (27) takes place. Indeed, it is so, if

$$\frac{1}{3, 8} \frac{1 + p'\Theta_1}{1 + p\Theta_1} \frac{1 + p'\Theta_{95}}{1 + p\Theta_{95}} > 0, 253 \frac{1 + p'\Theta_3}{1 + p'\Theta_{63}} \frac{1 + p\Theta_{90}}{1 + p\Theta_{66}}.$$

The last inequality is transformed into

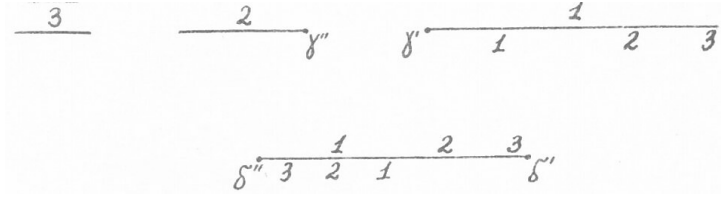
$$1, 04 > \frac{1 + p'\Theta_3}{1 + p'\Theta_1} \frac{1 + p'\Theta_{90}}{1 + p'\Theta_{95}} \frac{1 + p\Theta_1}{1 + p\Theta_{63}} \frac{1 + p\Theta_{95}}{1 + p\Theta_{66}},$$

which is easily checked.

5.2.4 Case $i_1 \not\equiv i_2 \pmod{2}$

Now consider case $i_1 \not\equiv i_2 \pmod{2}$ and, again, the right hand side $\{\delta(\mathcal{M})\}$ is left-normal, in other words, case $a_{i_2} = 1, a_{i_2-1} = 3$ doesn't take place.

Suppose that i_1 is even.



Substituting

$$\Theta_{\gamma'} = \Theta_{66}, \quad \Theta_{\gamma''} = \Theta_{59}, \quad \Theta_{\delta'} = \Theta_3, \quad \Theta_{\delta''} = \Theta_{90}$$

into (23), obtain

$$0, 2885 < q \frac{1 + p\Theta_{59}}{1 + p'\Theta_3} \frac{1 + p\Theta_{66}}{1 + p'\Theta_{90}}. \quad (31)$$

Suppose that

$$\frac{\Delta_1}{\Delta_2} < 3, 43 \quad (32)$$

or

$$\frac{1}{q} \frac{1 + p'\Theta_1}{1 + p\Theta_1} \frac{1 + p'\Theta_{95}}{1 + p\Theta_{95}} < 3, 43.$$

Then (31) takes place. Indeed, it is so, if

$$\frac{1}{3, 43} \frac{1 + p'\Theta_1}{1 + p\Theta_1} \frac{1 + p'\Theta_{95}}{1 + p\Theta_{95}} > 0, 2885 \frac{1 + p'\Theta_3}{1 + p\Theta_{59}} \frac{1 + p'\Theta_{90}}{1 + p\Theta_{66}}.$$

or

$$1,0105 > \frac{1+p'\Theta_3}{1+p'\Theta_1} \frac{1+p'\Theta_{90}}{1+p'\Theta_{95}} \frac{1+p\Theta_1}{1+p\Theta_{59}} \frac{1+p\Theta_{95}}{1+p\Theta_{66}}.$$

Increasing the right part, obtain

$$1,0105 > 0,9896 \frac{1+p \cdot 1,0551 + p^2 \cdot 0,20875}{1+p \cdot 0,983 + p^2 \cdot 0,237},$$

which is easily checked.

6 Initial set of rectangles

We present a set of rectangles whose projections cover the segment from the Freiman's constant to $\sqrt{21}$. All of them are good. Further sections will present the algorithm of splitting these rectangles into subrectangles correctly.

$$\begin{aligned}
 1) & \quad \{ 3\underset{\circ}{4}3 \} \\
 2) & \quad \{ 3\underset{\circ}{1}3\underset{\circ}{4}3\underset{\circ}{1}2 \} \\
 3) & \quad \{ 1\underset{\circ}{2}3\underset{\circ}{4}4\underset{\circ}{3}1 \} \\
 4) & \quad \{ 3\underset{\circ}{1}3\underset{\circ}{4}3\underset{\circ}{1}3 \} \\
 5) & \quad \{ 2\underset{\circ}{1}1\underset{\circ}{2}3\underset{\circ}{4}4\underset{\circ}{3}2\underset{\circ}{2} \} \\
 6) & \quad \{ 3\underset{\circ}{1}1\underset{\circ}{2}3\underset{\circ}{4}4\underset{\circ}{3}2\underset{\circ}{3} \} \\
 7) & \quad \{ 3\underset{\circ}{1}1\underset{\circ}{2}3\underset{\circ}{4}4\underset{\circ}{3}2\underset{\circ}{2} \} \\
 8+2n, \kappa) & \quad \{ \overset{\kappa}{3}\overset{\kappa}{1}\overset{\kappa}{3}\overset{\kappa}{1}\overset{\kappa}{2}\overset{\kappa}{1}\overset{\kappa}{3}\overset{\kappa}{1}\overset{\kappa}{3}\overset{\kappa}{1}\overset{\kappa}{1}\overset{\kappa}{2}3\underset{\circ}{4}4\underset{\circ}{3}2\underset{\circ}{2}3\overset{\kappa}{1}\overset{\kappa}{3}\overset{\kappa}{1}\overset{\kappa}{2}\overset{\kappa}{1}\overset{\kappa}{3} \} \quad (I3.1) \\
 9+2n, \kappa) & \quad \{ \overset{\kappa-1}{3}\overset{\kappa-1}{2}\overset{\kappa-1}{1}3\overset{\kappa-1}{1}\overset{\kappa-1}{3}\overset{\kappa-1}{1}\overset{\kappa-1}{2}\overset{\kappa-1}{3}\overset{\kappa-1}{1}\overset{\kappa-1}{1}\overset{\kappa-1}{2}3\underset{\circ}{4}4\underset{\circ}{3}2\underset{\circ}{2}3\overset{\kappa}{1}\overset{\kappa}{3}\overset{\kappa}{1}\overset{\kappa}{2}\overset{\kappa}{1}\overset{\kappa}{3} \} \quad (I3.2) \\
 9+2n, \kappa, \rho) & \quad \{ \overset{\rho}{3}\overset{\rho}{1}\overset{\rho}{3}\overset{\rho}{1}\overset{\rho}{2}\overset{\rho}{1}\overset{\rho}{3}\overset{\rho}{1}\overset{\rho}{3}\overset{\rho}{1}\overset{\rho}{1}\overset{\rho}{2}3\underset{\circ}{4}4\underset{\circ}{3}2\underset{\circ}{2}3\overset{\kappa}{1}\overset{\kappa}{3}\overset{\kappa}{1}\overset{\kappa}{2}\overset{\kappa}{1}\overset{\kappa}{3}\overset{\rho}{1}\overset{\rho}{2}\overset{\rho}{1}\overset{\rho}{3} \} \quad (I3.3)
 \end{aligned}$$

Figure 5: Initial set of rectangles.

$\Theta_1 = [0; 3\bar{1}] = 0,263762$	$\Theta_{35} = [0; 2\bar{1}2\bar{1}3\bar{1}\bar{2}] = 0,36549$	$\Theta_{67} = [0; 1\bar{1}3\bar{1}\bar{2}] = 0,56635$
$\Theta_2 = [0; 3\bar{1}2\bar{1}\bar{3}] = 0,267649$	$\Theta_{36} = [0; 2\bar{1}] = 0,36602$	$\Theta_{68} = [0; 1\bar{1}3\bar{1}\bar{1}] = 0,566423$
$\Theta_3 = [0; 3\bar{1}\bar{2}] = 0,26794$	$\Theta_{37} = [0; 2\bar{1}2\bar{1}\bar{1}\bar{3}] = 0,36779$	$\Theta_{69} = [0; 1\bar{1}2\bar{1}\bar{3}] = 0,57600$
$\Theta_4 = [0; 3\bar{1}23\bar{1}\bar{1}] = 0,270448$	$\Theta_{38} = [0; 2\bar{1}23\bar{1}\bar{2}] = 0,37119$	$\Theta_{70} = [0; 1\bar{1}2\bar{1}3\bar{1}\bar{2}] = 0,57602$
$\Theta_5 = [0; 3\bar{1}23\bar{1}\bar{2}] = 0,270710$	$\Theta_{39} = [0; 2\bar{1}2\bar{3}\bar{1}] = 0,371249$	$\Theta_{71} = [0; 1\bar{1}\bar{2}] = 0,57735$
$\Theta_6 = [0; 3\bar{1}23\bar{1}] = 0,270738$	$\Theta_{40} = [0; 2\bar{1}\bar{1}\bar{3}] = 0,378537$	$\Theta_{72} = [0; 1\bar{1}23\bar{1}\bar{2}] = 0,59032$
$\Theta_7 = [0; 3\bar{1}\bar{1}\bar{1}\bar{3}] = 0,27459$	$\Theta_{41} = [0; 2\bar{1}\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,37858$	$\Theta_{73} = [0; 1\bar{1}\bar{1}\bar{1}\bar{3}] = 0,609108$
$\Theta_8 = [0; 3\bar{1}\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,27462$	$\Theta_{42} = [0; 2\bar{1}\bar{1}\bar{1}3\bar{3}\bar{1}] = 0,379018$	$\Theta_{74} = [0; 1\bar{1}\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,60923$
$\Theta_9 = [0; 3\bar{1}\bar{1}\bar{1}3\bar{3}\bar{1}] = 0,274847$	$\Theta_{43} = [0; 2\bar{1}\bar{1}\bar{1}\bar{2}] = 0,37963$	$\Theta_{75} = [0; 1\bar{1}\bar{1}\bar{1}\bar{1}3\bar{3}\bar{1}] = 0,620981$
$\Theta_{10} = [0; 3\bar{1}\bar{1}\bar{1}\bar{2}] = 0,27517$	$\Theta_{44} = [0; 2\bar{1}\bar{1}\bar{1}\bar{1}3\bar{3}\bar{1}] = 0,383091$	$\Theta_{76} = [0; 1\bar{1}\bar{1}\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,62141$
$\Theta_{11} = [0; 3\bar{1}\bar{1}\bar{2}] = 0,27954$	$\Theta_{45} = [0; 2\bar{1}\bar{1}\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,38326$	$\Theta_{77} = [0; 1\bar{1}\bar{1}\bar{1}\bar{1}\bar{3}] = 0,621460$
$\Theta_{12} = [0; 3\bar{1}\bar{1}3\bar{3}\bar{1}] = 0,280392$	$\Theta_{46} = [0; 2\bar{1}\bar{1}\bar{1}\bar{1}\bar{3}] = 0,383272$	$\Theta_{78} = [0; 1\bar{1}\bar{1}23\bar{1}\bar{2}] = 0,62881$
$\Theta_{13} = [0; 3\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,28097$	$\Theta_{47} = [0; 2\bar{1}\bar{1}23\bar{1}\bar{2}] = 0,38605$	$\Theta_{79} = [0; 1\bar{1}\bar{1}\bar{2}] = 0,63400$
$\Theta_{14} = [0; 3\bar{1}\bar{1}\bar{3}] = 0,28105$	$\Theta_{48} = [0; 2\bar{1}\bar{1}23\bar{1}] = 0,386033$	$\Theta_{80} = [0; 1\bar{1}\bar{1}3\bar{3}\bar{1}] = 0,63839$
$\Theta_{15} = [0; 323\bar{1}] = 0,290550$	$\Theta_{49} = [0; 2\bar{1}\bar{1}23\bar{3}\bar{1}] = 0,386237$	$\Theta_{81} = [0; 1\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,63910$
$\Theta_{16} = [0; 323\bar{1}\bar{2}] = 0,29062$	$\Theta_{50} = [0; 2\bar{1}\bar{1}22\bar{1}] = 0,38651$	$\Theta_{82} = [0; 1\bar{1}\bar{1}\bar{3}] = 0,641742$
$\Theta_{17} = [0; 323\bar{3}\bar{1}] = 0,291242$	$\Theta_{51} = [0; 2\bar{1}\bar{1}\bar{2}] = 0,38800$	$\Theta_{83} = [0; 123\bar{1}\bar{2}] = 0,69399$
$\Theta_{18} = [0; 322\bar{1}] = 0,29216$	$\Theta_{52} = [0; 2\bar{1}\bar{1}3\bar{3}\bar{1}] = 0,389648$	$\Theta_{84} = [0; 122\bar{1}3\bar{1}\bar{2}] = 0,70225$
$\Theta_{19} = [0; 32\bar{1}] = 0,29709$	$\Theta_{53} = [0; 2\bar{1}\bar{1}32\bar{1}] = 0,389916$	$\Theta_{85} = [0; 1223\bar{1}\bar{2}] = 0,70938$
$\Theta_{20} = [0; 32\bar{1}3\bar{1}\bar{2}] = 0,29774$	$\Theta_{54} = [0; 2\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,39076$	$\Theta_{86} = [0; 1222\bar{1}] = 0,70783$
$\Theta_{21} = [0; 32\bar{1}\bar{3}] = 0,297773$	$\Theta_{55} = [0; 2\bar{1}\bar{1}\bar{3}] = 0,390891$	$\Theta_{87} = [0; 12\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,71903$
$\Theta_{22} = [0; 333\bar{1}] = 0,302444$	$\Theta_{56} = [0; 223\bar{1}] = 0,409544$	$\Theta_{88} = [0; 12\bar{1}\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,725381$
$\Theta_{23} = [0; 333\bar{1}\bar{2}] = 0,30025$	$\Theta_{57} = [0; 223\bar{1}\bar{2}] = 0,40968$	$\Theta_{89} = [0; \bar{1}\bar{2}] = 0,73206$
$\Theta_{24} = [0; 332\bar{1}] = 0,30330$	$\Theta_{58} = [0; 2\bar{2}\bar{1}] = 0,42265$	$\Theta_{90} = [0; 12\bar{1}3\bar{1}\bar{2}] = 0,73620$
$\Theta_{25} = [0; 33\bar{1}\bar{2}] = 0,30600$	$\Theta_{59} = [0; 22\bar{1}3\bar{1}\bar{2}] = 0,42398$	$\Theta_{91} = [0; 12\bar{1}\bar{3}] = 0,73624$
$\Theta_{26} = [0; 33\bar{1}2\bar{1}\bar{3}] = 0,30603$	$\Theta_{60} = [0; 22\bar{1}\bar{3}] = 0,424042$	$\Theta_{92} = [0; 133\bar{1}] = 0,765465$
$\Theta_{27} = [0; 33\bar{1}3\bar{1}\bar{2}] = 0,30638$	$\Theta_{61} = [0; 233\bar{1}] = 0,433577$	$\Theta_{93} = [0; 133\bar{1}\bar{2}] = 0,76569$
$\Theta_{28} = [0; 33\bar{1}\bar{3}] = 0,306394$	$\Theta_{62} = [0; 233\bar{1}\bar{2}] = 0,43365$	$\Theta_{94} = [0; 13\bar{1}\bar{2}] = 0,78868$
$\Theta_{29} = [0; 2\bar{1}\bar{3}] = 0,358258$	$\Theta_{63} = [0; 23\bar{1}\bar{2}] = 0,44093$	$\Theta_{95} = [0; \bar{1}\bar{3}] = 0,791287$
$\Theta_{30} = [0; 2\bar{1}3\bar{1}\bar{2}] = 0,35859$	$\Theta_{64} = [0; 23\bar{1}] = 0,441742$	
$\Theta_{31} = [0; 2\bar{1}33\bar{1}\bar{1}] = 0,361292$	$\Theta_{65} = [0; 1\bar{1}\bar{3}] = 0,558256$	
$\Theta_{32} = [0; 2\bar{1}33\bar{1}\bar{2}] = 0,36158$	$\Theta_{66} = [0; 1\bar{1}3\bar{1}\bar{2}] = 0,55905$	
$\Theta_{33} = [0; 2\bar{1}33\bar{1}] = 0,361602$		
$\Theta_{34} = [0; 2\bar{1}2\bar{1}\bar{3}] = 0,365455$		

Figure 6: List of Θ 's.

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