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**Markov and Lagrange Spectra  
for Continued Fractions**

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# 1 Foreword

The main subject of this paper is the Markov spectrum or, to be more precise, its final portion, called the Hall's Ray.

Its existence have been proven by Hall more than a century ago; however, evaluating the exact value of the start of the Hall's Ray took time.

In 1975, Freiman found and proved this constant.

This paper contains the translated and modernized approach from the original paper.

## 2 Introductory notation

### 2.1 Markov and Lagrange values

Symbol  $\mathcal{M}$  denotes double-infinite sequences from  $\mathbb{N}^{\mathbb{Z}}$ :

$$\mathcal{M} = \dots a_{-2} a_{-1} a_0 a_1 a_2 \dots$$

I will use  $\lambda(\mathcal{M})$ ,  $\mu(\mathcal{M})$  and  $f(\mathcal{M})$  for Lagrange, Markov values and height function. Symbols  $\gamma$  and  $\delta$  denote the lhs and rhs of sequence  $\mathcal{M}$ :

$$\begin{aligned}\gamma(\mathcal{M}) &= [0; a_{-1}, a_{-2}, \dots], \\ \delta(\mathcal{M}) &= [0; a_1, a_2, \dots], \\ f(\mathcal{M}) &= a_0 + \gamma(\mathcal{M}) + \delta(\mathcal{M}).\end{aligned}$$

At last, symbols  $M$  and  $L$  denote the Markov and Lagrange spectra.

### 2.2 Centered sequence

**Definition.** A sequence  $\mathcal{M}$  is called **centered**, if

$$\mu(\mathcal{M}) = f(\mathcal{M}). \tag{1}$$

**Proposition.** *Markov spectrum can be defined with only centered sequences:*

$$\{\mu(\mathcal{M}) \mid \mathcal{M} \in \mathbb{N}^{\mathbb{Z}}\} = M = \{\mu(\mathcal{M}) \mid \mathcal{M} \text{ is centered}\}.$$

### 2.3 Rectangle

**Designation.** Denote by

$$\{a_{i_1}, a_{i_1+1}, \dots, a_{i_2}\} \quad (i_1 \leq 0 \leq i_2)$$

the set of double-infinite sequences  $\mathcal{M}$  with fixed terms  $a_{i_1}, a_{i_1+1}, \dots, a_{i_2}$  on the corresponding positions.

Terms  $a_s$  for  $s < i_1$  and  $s > i_2$  are arbitrary integers, chosen such that  $\mathcal{M}$  is centered and, maybe, satisfies some conditions.

Segments  $\Delta_1$ ,  $\Delta_2$  and  $\Delta$  are defined by the following equations:

$$\begin{aligned}\Delta_1 &= [\Delta'_1; \Delta''_1] = [\min \gamma(\mathcal{M}); \max \gamma(\mathcal{M})], \\ \Delta_2 &= [\Delta'_2; \Delta''_2] = [\min \delta(\mathcal{M}); \max \delta(\mathcal{M})], \\ \Delta &= [\Delta'; \Delta''] = a_0 + \Delta_1 + \Delta_2,\end{aligned} \tag{2}$$

where  $\mathcal{M}$  belongs to the set.

Note that we will use ' for the lower bound, and '' for the upper bound.

**Definition.** **Rectangle** is the segment  $\Delta$  with the set of sequences, defining it.

## 2.4 Horizontal rectangle

**Definition.** Call a rectangle  $\Delta$  **horizontal**, if

$$|\Delta_1| \geq |\Delta_2|. \quad (3)$$

In (3) we allow terms  $a_s$  for  $s < i_1$  and  $s > i_2$  to be integers  $\{1, 2, 3\}$ , regardless of the requirement that sequences  $\mathcal{M} \in \Delta$  are centered.

In other words,  $\Delta$  is horizontal, if and only if

$$|[0; a_{-1}, \dots, a_{i_1}, \overline{3}, \overline{1}] - [0; a_{-1}, \dots, a_{i_1}, \overline{1}, \overline{3}]| \geq |[0; a_1, \dots, a_{i_2}, \overline{3}, \overline{1}] - [0; a_1, \dots, a_{i_2}, \overline{1}, \overline{3}]|.$$

Clearly, we can always obtain a horizontal rectangle out of the vertical one, as we can reindex the sequence in the opposite direction.

## 2.5 Subrectangle

Consider a rectangle  $\Delta$ , set by the sequence center

$$\{a_{i_1}, a_{i_1+1}, \dots, a_{i_2}\}.$$

We will use a **shorter notation** for subrectangles, produced by setting integers  $a_i$  for  $i < i_1$  or  $i > i_2$ :

$$\{b_\ell \dots b_1, c_1 \dots c_r\} := \{b_\ell \dots b_1 a_{i_1} \dots a_{i_2} c_1 \dots c_r\}.$$

For example:

$$\{213, 3\} := \{213 a_{i_1} \dots a_{i_2} 3\}, \quad (\text{ex.1})$$

$$\{2, 0\} := \{2 a_{i_1} \dots a_{i_2}\}. \quad (\text{ex.2})$$

We will also shorter the notation (2): lhs and rhs are  $\Delta_1(312)$  and  $\Delta_2(3)$  for subrectangle (ex.1) and  $\Delta_1(2)$  and  $a_2$  for (ex.2).

## 2.6 Geometrical interpretation

Consider the mapping

$$\begin{aligned} \tilde{h} : \mathbb{N}^{\mathbb{Z}} &\rightarrow \mathbb{R}^2, \\ \tilde{h}(\mathcal{M}) &= (\gamma(\mathcal{M}); \delta(\mathcal{M})). \end{aligned}$$

In these terms, the Markov spectrum  $M$  is the projection of some subset  $\mathcal{S} \subset C_4 \times C_4$  onto the diagonal.

Then **rectangle**  $\Delta$  is indeed a rectangle  $\Delta_1 \times \Delta_2$  and **subrectangles** are its subrectangles.

We will consider a family of rectangles whose projections cover the beginning of Hall's Ray.

Then we will present the algorithm to split rectangle into subrectangles so that their projections cover the projection of initial rectangle.

When we say that rectangles intersect, we, however, mean that their projections intersect.

The more «squarish» the rectangle, the easier the step.

That's why we will bound the aspect ratio of rectangles (see **good** rectangle).

Formulas to evaluate side lengths and aspect ratio are given in the section 3.

### 3 Calculations

#### 3.1 Length of $\Delta_1$ or $\Delta_2$

Let's fix some terms of continued fraction  $[0; q_1, q_2, q_3, \dots, q_n]$ .  
We will often need to measure difference

$$[0; q_1, q_2, q_3, \dots, q_n, \frac{1}{\Theta_R}] - [0; q_1, q_2, q_3, \dots, q_n, \frac{1}{\Theta_L}],$$

where  $\Theta$ 's are some continuations of the continued fraction.

They generally look like  $\Theta = [0; 121\overline{3}]$  or something<sup>1</sup>. We will set  $\Theta$ 's explicitly.

For the general proof,  $\Theta$ 's will be taken from table 5.

**Designation.** For given continuation  $\Theta_i$  denote by  $\varepsilon_i$  the resulting continued fraction:

$$\varepsilon_i = [0; q_1, q_2, \dots, q_n, \frac{1}{\Theta_i}]. \quad (4)$$

Then the following equality takes place:

$$|\varepsilon_i - \varepsilon_j| = \frac{|\Theta_i - \Theta_j|}{Q_n^2 (1 + pQ_i) (1 + pQ_j)}, \quad (5)$$

where

$$p = \frac{Q_{n-1}}{Q_n}.$$

#### 3.2 Rectangle aspect ratio

Consider some fixed center of rectangle  $\{a_{i_1} \dots a_{i_2}\}$ .

We will often extend it from the left (right) using some continuations  $\Theta_{\gamma_1}, \Theta_{\gamma_2} (\Theta_{\delta_1}, \Theta_{\delta_2})$ .

These values produce  $\gamma_1, \gamma_2 (\delta_1, \delta_2)$  using (4).

In other words, finite continued fractions  $[0; a_{-1}, a_{-2}, \dots, a_{i_1}] = \frac{P_{i_1}}{Q_{i_1}} \left( [0; a_1, a_2, \dots, a_{i_2}] = \frac{P_{i_2}}{Q_{i_2}} \right)$

are convergents for  $\gamma_1, \gamma_2 (\delta_1, \delta_2)$ .

Then

$$\left| \frac{\gamma_1 - \gamma_2}{\delta_1 - \delta_2} \right| = \frac{1}{q} \left| \frac{\Theta_{\gamma_1} - \Theta_{\gamma_2}}{\Theta_{\delta_1} - \Theta_{\delta_2}} \right| \frac{1 + p'\Theta_{\delta_1}}{1 + p\Theta_{\gamma_1}} \frac{1 + p'\Theta_{\delta_2}}{1 + p\Theta_{\gamma_2}} \approx \frac{1}{q} \left| \frac{\Theta_{\gamma_1} - \Theta_{\gamma_2}}{\Theta_{\delta_1} - \Theta_{\delta_2}} \right|, \quad (6)$$

where

$$p = \frac{Q_{i_1+1}}{Q_{i_1}}, \quad p' = \frac{Q_{i_2-1}}{Q_{i_2}}, \quad q = \frac{Q_{i_1}^2}{Q_{i_2}^2}.$$

---

<sup>1</sup>Here, as always in this book,  $\overline{abc}^k$  means  $k$ -times repetition of  $abc$ , and  $\overline{abc}$  means infinite repetition.

## 4 Rectangle boundaries

We will set boundaries for rectangles in a different way.

To distinguish rectangles, we will introduce the notion of left- or right-shortened rectangles.

It is given in subsections 4.1-4.3.

Subsections 4.4-4.5 provide the rules for boundaries  $\Delta'_1, \Delta''_1, \Delta'_2, \Delta''_2$ .

### 4.1 Resection

**Definition.** Call **resection** of a segment  $A = [a; b]$  a process of removing subsegment  $A_{12} = [a_1; b_1]$ , leaving two segments  $A_1 \sqcup A_2 = [a; a_1] \sqcup [b_1; b]$ .

**Definition.** Call subsegment  $A_{12} \subset A$  **normal**, if it is thicker than the two remaining subsegments:

$$|A_{12}| \leq \min \{|A_1|, |A_2|\} \quad (7)$$

We call a resection **normal** if the resected subsegment is normal.

**Proposition.** For any normal resection, having

$$A + A = (A_1 \sqcup A_2) + (A_1 \sqcup A_2). \quad (8)$$

### 4.2 Shortened rectangle

Consider a horizontal rectangle  $\Delta$ .

#### 4.2.1 Case $i_1 = i_2 = 0 \pmod{2}$

If both  $i_1$  and  $i_2$  are even, then left- and right-shortened rectangles are defined as follows.

**Definition.** Rectangle  $\Delta$  is called left-shortened, if for subrectangle  $\{3, 3\}$  the following condition takes place:

$$|\Delta_1(3)| \leq 1.4 \cdot |\Delta_2(3)|. \quad (9)$$

Here, as in (3), we allow terms  $a_s$  for  $s < i_1$  and  $s > i_2$  to be integers  $\{1, 2, 3\}$ , so (9) can be rewritten as

$$|[0; a_{-1}, \dots, a_{i_1}, 3, \overline{3, 1}] - [0; a_{-1}, \dots, a_{i_1}, \overline{3, 1}]| \leq |[0; a_1, \dots, a_{i_2}, 3, \overline{3, 1}] - [0; a_1, \dots, a_{i_2}, \overline{3, 1}]|.$$

For further convenience, we also introduce the opposite to (9) condition:

**Definition.** Rectangle  $\Delta$  is called left-normal, if

$$|\Delta_1(3)| > 1.4 \cdot |\Delta_2(3)|. \quad (10)$$

Now we introduce the notion of right-shortened rectangle:

**Definition.** Rectangle  $\Delta$  is called right-shortened, if for subrectangle  $\{31, 13\}$  the following condition takes place:

$$|\Delta_1(13)| \leq 1.4 \cdot |\Delta_2(13)|. \quad (11)$$

As in 4.2.1, we allow terms  $a_s$  for  $s < i_1$  and  $s > i_2$  to be integers  $\{1, 2, 3\}$ , so (11) can be rewritten as

$$\left| [0; a_{-1}, \dots, a_{i_1}, 1, 3, \overline{3, 1}] - [0; a_{-1}, \dots, a_{i_1}, \overline{1, 3}] \right| \leq \left| [0; a_1, \dots, a_{i_2}, 1, 3, \overline{3, 1}] - [0; a_1, \dots, a_{i_2}, \overline{1, 3}] \right|.$$

We also introduce the opposite condition:

**Definition.** Rectangle  $\Delta$  is called right-normal, if

$$|\Delta_1(13)| > 1.4 \cdot |\Delta_2(13)|. \quad (12)$$

#### 4.2.2 Other cases

If  $i_1 \equiv i_2 \pmod{2}$  and  $i_1$  is odd, then the conditions for left- and right-shortened rectangles are swapped.

In case  $i_1 \not\equiv i_2 \pmod{2}$ , the notions of left- and right-shortened rectangles are inferred from subrectangles  $\{1, \}$ ,  $\{2, \}$ , and  $\{3, \}$ .

We will refer to the relevant conditions from 4.5

### 4.3 Explanation of shortened

Notion left-shortened is used to set the lower bound  $\Delta'$  for rectangle  $\Delta$ .

Notion right-shortened is used to set the upper bound  $\Delta''$ .

For example, when interested in  $\Delta'$  when  $i_1$  is even and  $i_2$  is odd, one has to consider larger  $a_{i_1-1}$  and smaller  $a_{i_2+1}$ . As the rectangle is horizontal, we will add terms to the left first. So we will check the condition (11) for  $\{2, \}$  or  $\{3, \}$ .

In other words, check whether  $\{2, \}$  or  $\{3, \}$  is left-shortened or left-normal (recall that  $i_1 - 1$  and  $i_2$  are odd, so left-shortenence is defined by (11)).

Which subrectangle ( $\{2, \}$  or  $\{3, \}$ ) to take, however, depends on terms  $a_{i_1+1}$  and  $a_{i_1}$  (see 4.4).

### 4.4 How we ensure centeredness

Consider integers  $q_1, q_2, \dots, q_n$ . Let  $\{\delta_n\}$  be the set of fractions  $\delta_n = [0; q_1, q_2, \dots, q_n, \dots]$  with  $n$  fixed terms. We will suppose that  $n$  is even (for odd  $n$  the bounds are swapped).

At first, determine the smallest of fractions  $\delta_n$ .

We will consider 2 cases:  $S$  (Shortened) and  $N$  (Normal):

$$S. \quad q_{n-1} = 3, q_n = 1. \quad (13a)$$

$$N. \quad \text{Otherwise.} \quad (13b)$$

The lower bound  $\delta'_n$  for segment, containing  $\delta_n$ , is defined by:

$$\begin{aligned} S. \quad & \delta'_n = [0; q_1, \dots, q_n, 213\overline{12}]. \\ N. \quad & \delta'_n = [0; q_1, \dots, q_n, 3\overline{12}]. \end{aligned}$$

To set the upper bound  $\delta''_n$ , largest of  $\delta_n$ , consider 2 other cases:

$$S. \quad q_n = 3. \quad (14a)$$

$$N. \quad q_n \neq 3. \quad (14b)$$



Then

$$\begin{aligned} S. \quad & \delta_n'' = [0; q_1, \dots, q_n, 1213\overline{12}]. \\ N. \quad & \delta_n'' = [0; q_1, \dots, q_n, 13\overline{12}]. \end{aligned}$$

These bounds will allow us to construct sequences  $\mathcal{M}$ , for which combination (31313) is forbidden and, therefore, the following condition takes place:

$$f_i(\mathcal{M}) \leq \lambda(\overline{31312}) \approx 4,5241, \quad i \neq 0,$$

which will ensure (1).

## 4.5 Formal

Let's now turn to concrete definitions and bounds. Remind that we are looking at the horizontal rectangle  $\Delta$ .

Suppose that  $i_1 = i_2 \pmod{2}$ ,  $i_1$  is even. (If  $i_1$  is odd, then  $\Delta'$  and  $\Delta''$  are swapped.)

### 4.5.1 Bounds for $\Delta'$

- I. Suppose both  $\{\gamma_0(\mathcal{M})\}$  and  $\{\delta_0(\mathcal{M})\}$  satisfy (13b) and the rectangle  $\Delta$  is left-normal, that is, satisfies (10). We will denote such situation by  $N - N - N$  (segment  $\Delta_1$  is left-normal,  $\Delta_2$  is left-normal, rectangle  $\Delta$  is normal).

In this case define  $\Delta'$  by equation

$$\Delta' = f(\overline{21}3a_{i_1} \dots a_{i_2} 3\overline{12}). \quad (15)$$

- IIa. Sets  $\{\gamma_0(\mathcal{M})\}$  and  $\{\delta_0(\mathcal{M})\}$  meet condition (13b) and  $\Delta$  is left-shortened, that is, meets condition (9). This is case  $N - N - S$  (segments  $\Delta_1$  and  $\Delta_2$  are left-normal, rectangle  $\Delta$  is left-shortened).

Then

$$\Delta' = f(\overline{21}3a_{i_1} \dots a_{i_2} 213\overline{12}). \quad (16)$$

- IIb. Set  $\{\gamma_0(\mathcal{M})\}$  meets (13b),  $\{\delta_0(\mathcal{M})\}$  meets (13a). No matter which condition (9) of (10) is met. It is the case  $N - S$  (segment  $\Delta_1$  is left-normal, segment  $\Delta_2$  left-shortened). Bound  $\Delta_1$  is defined by (16).

- III. Set  $\{\gamma_0(\mathcal{M})\}$  meets (13a),  $\{\delta_0(\mathcal{M})\}$  meets (13b). In this  $S - N$  case (segment  $\Delta_1$  is left-shortened, segment  $\Delta_2$  is left-normal)  $\Delta'$  is defined as follows:

$$\Delta' = f(\overline{21}312a_{i_1} \dots a_{i_2} 3\overline{12}). \quad (17)$$

- IV. Both  $\{\gamma_0(\mathcal{M})\}$  and  $\{\delta_0(\mathcal{M})\}$  meet (13a). In this  $S - S$  case (both segments  $\Delta_1$  and  $\Delta_2$  are left-shortened)  $\Delta'$  is defined by

$$\Delta' = f(\overline{21}312a_{i_1} \dots a_{i_2} 213\overline{12}). \quad (18)$$

Left side of the figure 1 illustrates these bounds.

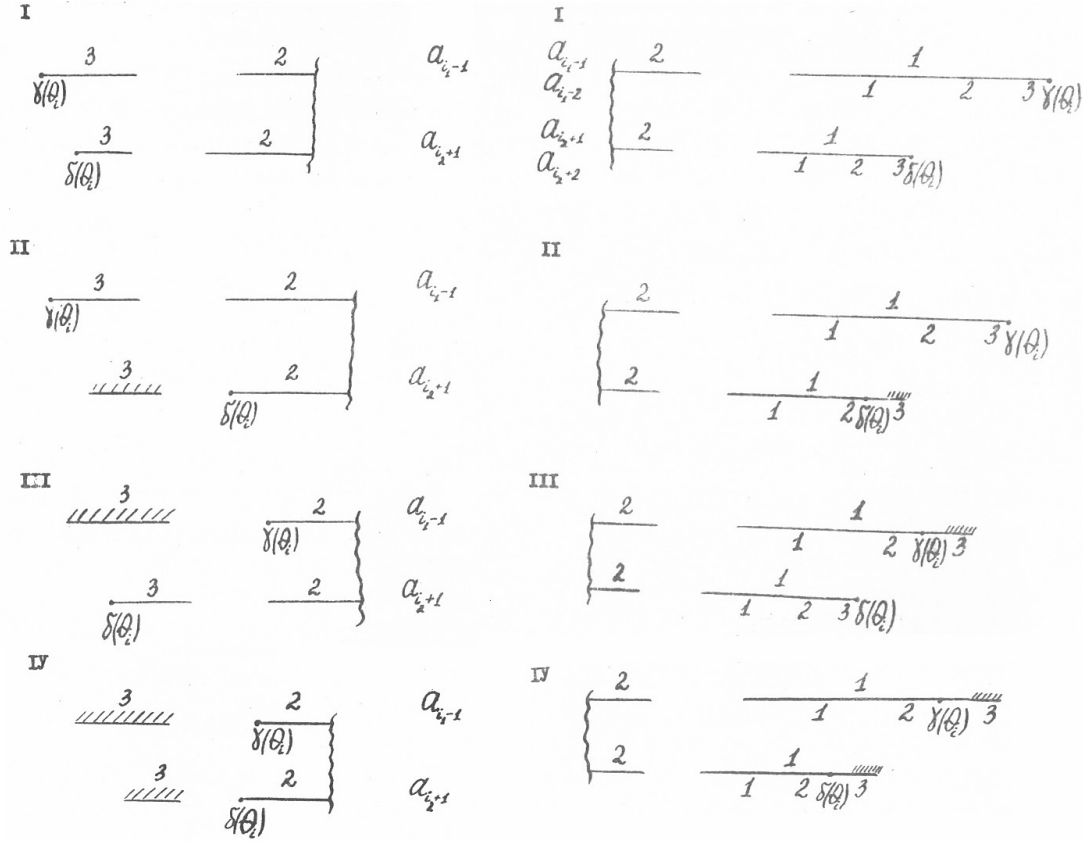


Figure 1: Bounds  $\Delta'$  (left) and  $\Delta''$  (right).

Hatched areas correspond to values of  $a_{i_1-1}$  and  $a_{i_2+1}$  (on the left) or  $a_{i_1-2}$  and  $a_{i_2+2}$  (on the right), equal to 3, which can not appear in the concrete case.

$$\Delta'_1 = \gamma(\Theta_i), i = 3, 30, \Delta'_2 = \delta(\Theta_i), i = 3, 30, \Delta' = \Delta'_1 + \Delta'_2.$$

$$\Delta''_1 = \gamma(\Theta_i), i = 90, 94, \Delta''_2 = \delta(\Theta_i), i = 90, 94.$$

### 4.5.2 Bounds for $\Delta''$

Now we will provide formulas for  $\Delta''$ :

$$I \quad \Delta'' = f(\overline{2131}a_{i_1}\dots a_{i_2}13\overline{12}), \quad (19a)$$

$$II \quad \Delta'' = f(\overline{2131}a_{i_1}\dots a_{i_2}1213\overline{12}), \quad (19b)$$

$$III \quad \Delta'' = f(\overline{213121}a_{i_1}\dots a_{i_2}13\overline{12}), \quad (19c)$$

$$IV \quad \Delta'' = f(\overline{213121}a_{i_1}\dots a_{i_2}1213\overline{12}). \quad (19d)$$

Figure (2) regulates the choise of the formulas.

Case	$\Delta''_1$	$\Delta''_2$	Rectangle	$\Delta''$
I	(14b)	(14b)	(12) $N - N - N$	(19a)
IIa	(14b)	(14b)	(11) $N - N - S$	(19b)
IIb	(14b)	(14a)	$N - S$	(19b)
III	(14a)	(14b)	$S - N$	(19c)
IV	(14a)	(14a)	$S - S$	(19d)

Figure 2: Rules for choise of  $\Delta''$  in case  $i_1 = i_2 \pmod{2}$ .

Again, these bounds are illustrated on the right side of the figure 1.

### 4.5.3 Case $i_1 \not\equiv i_2 \pmod{2}$

Now consider case  $i_1 \not\equiv i_2 \pmod{2}$ ,  $i_1$  is even. We will use rules from figure (3) to choose one of 4 formulas for  $\Delta'$ .

$$I \quad \Delta' = f(\overline{213}a_{i_1}\dots a_{i_2}13\overline{12}), \quad (20a)$$

$$II \quad \Delta' = f(\overline{213}a_{i_1}\dots a_{i_2}1213\overline{12}), \quad (20b)$$

$$III \quad \Delta' = f(\overline{21312}a_{i_1}\dots a_{i_2}13\overline{12}), \quad (20c)$$

$$IV \quad \Delta' = f(\overline{21312}a_{i_1}\dots a_{i_2}1213\overline{12}). \quad (20d)$$

Case	$\Delta''_1$	$\Delta''_2$	Case name	$\Delta''$
I	(13b)	(14b)	$N - N$	(20a)
II	(13b)	(14a)	$N - S$	(20b)
III	(13a)	(14b)	$S - N$	(19c)
IV	(13a)	(14a)	$S - S$	(19d)

Figure 3: Rules for choise of  $\Delta'$  in case  $i_1 \not\equiv i_2 \pmod{2}$ ,  $i_1$  is even.

To determine the bound  $\Delta''$ , we will use subrectangle

$$\{1, 0\}.$$

Having  $i_1 - 1 \equiv i_2 \pmod{2}$ , so we can use all the previous formulas to determine  $\Delta''$ .

## 5 Good rectangle

### 5.1 Definition

Consider a horizontal rectangle  $\Delta = \{a_{i_1} \dots a_{i_2}\}$ .

**Definition.** Rectangle  $\Delta$  is called **good**, if subrectangles  $\{2, 0\}$  and  $\{1, 0\}$  intersect.

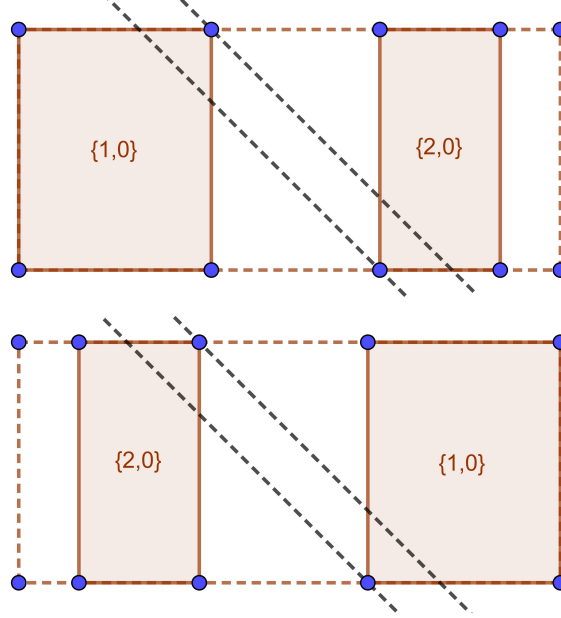


Figure 4: Good rectangles if  $i_1$  is odd (upper) and even (lower).

For example, in case  $i_1$  is even, goodness is equivalent to

$$\{2, 0\}'' \geq \{1, 0\}'. \quad (21)$$

Bounds of rectangles are determined by the rules from section 4.

If the rectangle  $\Delta$  is not good (for example, (21) doesn't take place), then

$$(\{2, 0\}''; \{1, 0\}') \not\subset \Delta.$$

Clearly, if the rectangle is not good, then one can not split it into smaller subrectangles whose projections cover the projection of initial one. That's why we will only consider good rectangles during the proof.

### 5.2 Sufficient conditions of goodness

#### 5.2.1 Results

A horizontal rectangle  $\Delta$  is good, if

$$\frac{\Delta_1}{\Delta_2} < \begin{cases} 3.8, & i_1 \equiv i_2 \pmod{2}, \\ 3.43, & i_1 \not\equiv i_2 \pmod{2}. \end{cases} \quad (22)$$

### 5.2.2 Universal 2, 9 bound

Now let's introduce some sufficient conditions for rectangle to be good.

Remind that we suppose  $|\Delta_1| \geq |\Delta_2|$ , which means that

$$q \frac{1 + p\Theta_1}{1 + p'\Theta_1} \frac{1 + p\Theta_{95}}{1 + p'\Theta_{95}} \leq 1.$$

Let's introduce the following designations:

$$\gamma = [0; a_{-1}, \dots, a_{i_1}, \frac{1}{\Theta_\gamma}],$$

$$\delta = [0; a_1, \dots, a_{i_2}, \frac{1}{\Theta_\delta}],$$

where  $\Theta$ 's are some  $\Theta$ 's from the Figure 5, specified in each case separately.

If rectangle is good, then (23) should take place:

$$\gamma' - \gamma'' < |\delta' - \delta''|, \quad (23)$$

where  $\Theta_{\gamma'} = \Theta_{66}$ ,  $\Theta_{\gamma''} = \Theta_{63}$ ,  $\Theta_{\delta'} = \Theta_{90}$ ,  $\Theta_{\delta''} = \Theta_{30}$ .

Inequality (23) transforms into

$$0,313 < q \frac{1 + p\Theta_{63}}{1 + p'\Theta_{30}} \frac{1 + p\Theta_{66}}{1 + p'\Theta_{90}}. \quad (24)$$

Suppose that

$$\frac{\Delta_1}{\Delta_2} < 2,9, \quad (25)$$

which is equivalent to

$$\frac{1}{q} \frac{1 + p'\Theta_1}{1 + p\Theta_1} \frac{1 + p'\Theta_{95}}{1 + p\Theta_{95}} < 2,9. \quad (26)$$

Then (24) takes place. Indeed, it is so, if

$$\frac{1}{2,9} \frac{1 + p'\Theta_1}{1 + p\Theta_1} \frac{1 + p'\Theta_{95}}{1 + p\Theta_{95}} > 0,313 \frac{1 + p'\Theta_{30}}{1 + p\Theta_{63}} \frac{1 + p'\Theta_{90}}{1 + p\Theta_{66}}.$$

or, equivalent,

$$1,1 > \frac{1 + p\Theta_1}{1 + p\Theta_{63}} \frac{1 + p\Theta_{95}}{1 + p\Theta_{66}} \frac{1 + p'\Theta_{30}}{1 + p'\Theta_1} \frac{1 + p'\Theta_{90}}{1 + p'\Theta_{95}},$$

which is checked directly.

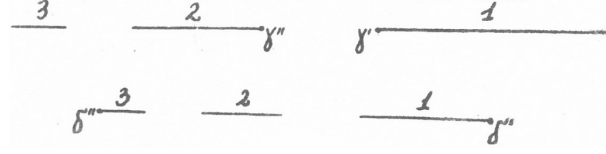
Condition (25) is sufficient for rectangle to be good, regardless of the parity of  $i_1$  and  $i_2$  and the left or right shortness or normalness of rectangle.

$\Theta_1 = [0; 3\bar{1}] = 0,263762$	$\Theta_{35} = [0; 2\bar{1}2\bar{1}3\bar{1}\bar{2}] = 0,36549$	$\Theta_{67} = [0; 1\bar{1}33\bar{1}\bar{2}] = 0,56635$
$\Theta_2 = [0; 3\bar{1}2\bar{1}\bar{3}] = 0,267649$	$\Theta_{36} = [0; 2\bar{1} ] = 0,36602$	$\Theta_{68} = [0; 1\bar{1}33\bar{1} ] = 0,566423$
$\Theta_3 = [0; 3\bar{1}\bar{2} ] = 0,26794$	$\Theta_{37} = [0; 2\bar{1}2\bar{1}\bar{3}] = 0,36779$	$\Theta_{69} = [0; 1\bar{1}2\bar{1}\bar{3} ] = 0,57600$
$\Theta_4 = [0; 3\bar{1}233\bar{1}] = 0,270448$	$\Theta_{38} = [0; 2\bar{1}23\bar{1}\bar{2}] = 0,37119$	$\Theta_{70} = [0; 1\bar{1}2\bar{1}3\bar{1}\bar{2}] = 0,57602$
$\Theta_5 = [0; 3\bar{1}23\bar{1}\bar{2}] = 0,270710$	$\Theta_{39} = [0; 2\bar{1}23\bar{1} ] = 0,371249$	$\Theta_{71} = [0; 1\bar{1}\bar{2} ] = 0,57735$
$\Theta_6 = [0; 3\bar{1}23\bar{1} ] = 0,270738$	$\Theta_{40} = [0; 2\bar{1}\bar{1}\bar{3} ] = 0,378537$	$\Theta_{72} = [0; 1\bar{1}23\bar{1}\bar{2}] = 0,59032$
$\Theta_7 = [0; 3\bar{1}\bar{1}\bar{1}\bar{3}] = 0,27459$	$\Theta_{41} = [0; 2\bar{1}\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,37858$	$\Theta_{73} = [0; 1\bar{1}\bar{1}\bar{1}\bar{3} ] = 0,609108$
$\Theta_8 = [0; 3\bar{1}\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,27462$	$\Theta_{42} = [0; 2\bar{1}\bar{1}\bar{1}33\bar{1} ] = 0,379018$	$\Theta_{74} = [0; 1\bar{1}\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,60923$
$\Theta_9 = [0; 3\bar{1}\bar{1}\bar{1}33\bar{1}] = 0,274847$	$\Theta_{43} = [0; 2\bar{1}\bar{1}\bar{2} ] = 0,37963$	$\Theta_{75} = [0; 1\bar{1}\bar{1}\bar{1}\bar{1}33\bar{1}] = 0,620981$
$\Theta_{10} = [0; 3\bar{1}\bar{1}\bar{2} ] = 0,27517$	$\Theta_{44} = [0; 2\bar{1}\bar{1}\bar{1}\bar{1}33\bar{1}] = 0,383091$	$\Theta_{76} = [0; 1\bar{1}\bar{1}\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,62141$
$\Theta_{11} = [0; 3\bar{1}\bar{1}\bar{2} ] = 0,27954$	$\Theta_{45} = [0; 2\bar{1}\bar{1}\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,38326$	$\Theta_{77} = [0; 1\bar{1}\bar{1}\bar{1}\bar{1}\bar{3} ] = 0,621460$
$\Theta_{12} = [0; 3\bar{1}\bar{1}33\bar{1} ] = 0,280392$	$\Theta_{46} = [0; 2\bar{1}\bar{1}\bar{1}\bar{3} ] = 0,383272$	$\Theta_{78} = [0; 1\bar{1}\bar{1}23\bar{1}\bar{2}] = 0,62881$
$\Theta_{13} = [0; 3\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,28097$	$\Theta_{47} = [0; 2\bar{1}\bar{1}23\bar{1}\bar{2}] = 0,38605$	$\Theta_{79} = [0; 1\bar{1}\bar{2} ] = 0,63400$
$\Theta_{14} = [0; 3\bar{1}\bar{1}\bar{3} ] = 0,28105$	$\Theta_{48} = [0; 2\bar{1}\bar{1}23\bar{1} ] = 0,386033$	$\Theta_{80} = [0; 1\bar{1}\bar{1}33\bar{1} ] = 0,63839$
$\Theta_{15} = [0; 323\bar{1} ] = 0,290550$	$\Theta_{49} = [0; 2\bar{1}\bar{1}233\bar{1} ] = 0,386237$	$\Theta_{81} = [0; 1\bar{1}\bar{1}3\bar{1}\bar{2} ] = 0,63910$
$\Theta_{16} = [0; 323\bar{1}\bar{2}] = 0,29062$	$\Theta_{50} = [0; 2\bar{1}\bar{1}22\bar{1} ] = 0,38651$	$\Theta_{82} = [0; 1\bar{1}\bar{1}\bar{3} ] = 0,641742$
$\Theta_{17} = [0; 3233\bar{1} ] = 0,291242$	$\Theta_{51} = [0; 2\bar{1}\bar{1}\bar{2} ] = 0,38800$	$\Theta_{83} = [0; 123\bar{1}\bar{2}] = 0,69399$
$\Theta_{18} = [0; 322\bar{1} ] = 0,29216$	$\Theta_{52} = [0; 2\bar{1}\bar{1}33\bar{1} ] = 0,389648$	$\Theta_{84} = [0; 122\bar{1}3\bar{1}\bar{2}] = 0,70225$
$\Theta_{19} = [0; 32\bar{1} ] = 0,29709$	$\Theta_{53} = [0; 2\bar{1}\bar{1}32\bar{1} ] = 0,389916$	$\Theta_{85} = [0; 1223\bar{1}\bar{2}] = 0,70938$
$\Theta_{20} = [0; 32\bar{1}3\bar{1}\bar{2}] = 0,29774$	$\Theta_{54} = [0; 2\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,39076$	$\Theta_{86} = [0; 1222\bar{1} ] = 0,70783$
$\Theta_{21} = [0; 32\bar{1}\bar{3} ] = 0,297773$	$\Theta_{55} = [0; 2\bar{1}\bar{1}\bar{3} ] = 0,390891$	$\Theta_{87} = [0; 12\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,71903$
$\Theta_{22} = [0; 333\bar{1} ] = 0,302444$	$\Theta_{56} = [0; 223\bar{1} ] = 0,409544$	$\Theta_{88} = [0; 12\bar{1}\bar{1}\bar{1}3\bar{1}\bar{2}] = 0,725381$
$\Theta_{23} = [0; 333\bar{1}\bar{2}] = 0,30025$	$\Theta_{57} = [0; 223\bar{1}\bar{2}] = 0,40968$	$\Theta_{89} = [0; \bar{1}\bar{2} ] = 0,73206$
$\Theta_{24} = [0; 332\bar{1} ] = 0,30330$	$\Theta_{58} = [0; 2\bar{2}\bar{1} ] = 0,42265$	$\Theta_{90} = [0; 12\bar{1}3\bar{1}\bar{2} ] = 0,73620$
$\Theta_{25} = [0; 33\bar{1}\bar{2} ] = 0,30600$	$\Theta_{59} = [0; 22\bar{1}3\bar{1}\bar{2}] = 0,42398$	$\Theta_{91} = [0; 12\bar{1}\bar{3} ] = 0,73624$
$\Theta_{26} = [0; 33\bar{1}2\bar{1}\bar{3}] = 0,30603$	$\Theta_{60} = [0; 22\bar{1}\bar{3} ] = 0,424042$	$\Theta_{92} = [0; 133\bar{1} ] = 0,765465$
$\Theta_{27} = [0; 33\bar{1}3\bar{1}\bar{2}] = 0,30638$	$\Theta_{61} = [0; 233\bar{1} ] = 0,433577$	$\Theta_{93} = [0; 133\bar{1}\bar{2}] = 0,76569$
$\Theta_{28} = [0; 33\bar{1} ] = 0,306394$	$\Theta_{62} = [0; 233\bar{1}\bar{2}] = 0,43365$	$\Theta_{94} = [0; 13\bar{1}\bar{2} ] = 0,78868$
$\Theta_{29} = [0; 2\bar{1}\bar{3} ] = 0,358258$	$\Theta_{63} = [0; 23\bar{1}\bar{2} ] = 0,44093$	$\Theta_{95} = [0; \bar{1}\bar{3} ] = 0,791287$
$\Theta_{30} = [0; 2\bar{1}3\bar{1}\bar{2}] = 0,35859$	$\Theta_{64} = [0; 23\bar{1} ] = 0,441742$	
$\Theta_{31} = [0; 2\bar{1}333\bar{1}] = 0,361292$	$\Theta_{65} = [0; 1\bar{1}\bar{3} ] = 0,558256$	
$\Theta_{32} = [0; 2\bar{1}33\bar{1}\bar{2}] = 0,36158$	$\Theta_{66} = [0; 1\bar{1}3\bar{1}\bar{2}] = 0,55905$	
$\Theta_{33} = [0; 2\bar{1}33\bar{1} ] = 0,361602$		
$\Theta_{34} = [0; 2\bar{1}2\bar{1}\bar{3}] = 0,365455$		

Figure 5: List of  $\Theta$ 's.

### 5.2.3 Case $i_1 = i_2 \pmod{2}$

Now consider case  $i_1 = i_2 \pmod{2}$ .



We will suppose case  $a_{i_2} = 1$ ,  $a_{i_2-1} = 3$  doesn't take place.

We can substitute the following  $\gamma$ 's and  $\delta$ 's into (23):

$$\Theta_{\gamma'} = \Theta_{66}, \Theta_{\gamma''} = \Theta_{63}, \Theta_{\delta'} = \Theta_{90}, \Theta_{\delta''} = \Theta_3,$$

and instead of (24) we will get

$$0,253 < q \frac{1+p\Theta_{63}}{1+p'\Theta_3} \frac{1+p\Theta_{66}}{1+p'\Theta_{90}}. \quad (27)$$

This choose of  $\Theta_{\gamma'}$  and  $\Theta_{\delta''}$  is fine, if the following inequality takes place:

$$\delta' - \delta'' > 1,4(\gamma' - \gamma''), \quad (28)$$

where

$$\Theta_{\gamma'} = \Theta_{68}, \Theta_{\gamma''} = \Theta_{65}, \Theta_{\delta'} = \Theta_{28}, \Theta_{\delta''} = \Theta_1,$$

so we can rewrite (28) as

$$0,253 < q \frac{1+p\Theta_{65}}{1+p'\Theta_1} \frac{1+p\Theta_{68}}{1+p'\Theta_{28}}. \quad (29)$$

We can notice that (29) follows from (27). Indeed, that follows from the inequality

$$0,253 \frac{1+p'\Theta_3}{1+p'\Theta_{63}} \frac{1+p\Theta_{90}}{1+p\Theta_{66}} > 0,269 \frac{1+p'\Theta_1}{1+p'\Theta_{65}} \frac{1+p\Theta_{28}}{1+p\Theta_{68}},$$

or inequality

$$\frac{1+p\Theta_{65}}{1+p\Theta_{63}} \frac{1+p\Theta_{68}}{1+p\Theta_{66}} \frac{1+p'\Theta_3}{1+p'\Theta_1} \frac{1+p'\Theta_{90}}{1+p'\Theta_{28}} > 1,064.$$

Overall, we proved that (27) is enough for rectangle to be good.

Suppose that

$$\frac{\Delta_1}{\Delta_2} < 3,8, \quad (30)$$

or

$$\frac{1}{q} \frac{1+p'\Theta_1}{1+p\Theta_1} \frac{1+p'\Theta_{95}}{1+p\Theta_{95}} < 3,8.$$

Then (27) takes place. Indeed, it is so, if

$$\frac{1}{3,8} \frac{1+p'\Theta_1}{1+p'\Theta_1} \frac{1+p\Theta_{95}}{1+p\Theta_{95}} > 0,253 \frac{1+p'\Theta_3}{1+p'\Theta_{63}} \frac{1+p\Theta_{90}}{1+p\Theta_{66}}.$$

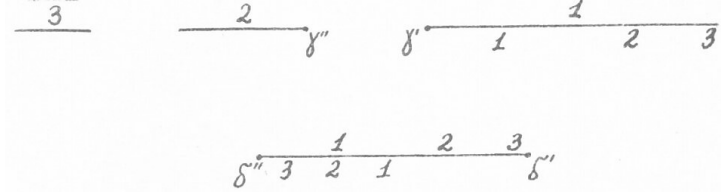
The last inequality is transformed into

$$1,04 > \frac{1+p'\Theta_3}{1+p'\Theta_1} \frac{1+p'\Theta_{90}}{1+p'\Theta_{95}} \frac{1+p\Theta_1}{1+p\Theta_{63}} \frac{1+p\Theta_{95}}{1+p\Theta_{66}},$$

which is easily checked.

#### 5.2.4 Case $i_1 \neq i_2 \pmod{2}$

Now consider case  $i_1 \neq i_2 \pmod{2}$  and, again, case  $a_{i_2} = 1, a_{i_2-1} = 3$  doesn't take place. Suppose that  $i_1$  is even.



Taking

$$\Theta_{\gamma'} = \Theta_{66}, \Theta_{\gamma''} = \Theta_{59}, \Theta_{\delta'} = \Theta_3, \Theta_{\delta''} = \Theta_{90},$$

rewrite (23) as

$$0,2885 < q \frac{1+p\Theta_{59}}{1+p'\Theta_3} \frac{1+p\Theta_{66}}{1+p'\Theta_{90}}. \quad (31)$$

Suppose

$$\frac{\Delta_1}{\Delta_2} < 3,43 \quad (32)$$

or

$$\frac{1}{q} \frac{1+p'\Theta_1}{1+p\Theta_1} \frac{1+p'\Theta_{95}}{1+p\Theta_{95}} < 3,43.$$

Then (31) takes place. Indeed, it is so, if

$$\frac{1}{3,43} \frac{1+p'\Theta_1}{1+p\Theta_1} \frac{1+p'\Theta_{95}}{1+p\Theta_{95}} > 0,2885 \frac{1+p'\Theta_3}{1+p\Theta_{59}} \frac{1+p'\Theta_{90}}{1+p\Theta_{66}}.$$

or

$$1,0105 > \frac{1+p'\Theta_3}{1+p'\Theta_1} \frac{1+p'\Theta_{90}}{1+p'\Theta_{95}} \frac{1+p\Theta_1}{1+p\Theta_{59}} \frac{1+p\Theta_{95}}{1+p\Theta_{66}},$$

increasing the right part, obtain

$$1,0105 > 0,9896 \frac{1+p \cdot 1,0551 + p^2 \cdot 0,20875}{1+p \cdot 0,983 + p^2 \cdot 0,237},$$

which is easily checked.



## 6 Initial set of rectangles

We present a set of rectangles whose projections cover the segment from the Freiman's constant to  $\sqrt{21}$ . All of them are good. Further sections will present the algorithm of splitting these rectangles into subrectangles correctly.

$$\begin{aligned}
 1) & \quad \{ 3 \underset{\circ}{4} 3 \} \\
 2) & \quad \{ 3 \underset{\circ}{1} 3 \underset{\circ}{4} 3 \underset{\circ}{1} 2 \} \\
 3) & \quad \{ 1 \underset{\circ}{2} 3 \underset{\circ}{4} 4 \underset{\circ}{3} 1 \} \\
 4) & \quad \{ 3 \underset{\circ}{1} 3 \underset{\circ}{4} 3 \underset{\circ}{1} 3 \} \\
 5) & \quad \{ 2 \underset{\circ}{1} 1 \underset{\circ}{2} 3 \underset{\circ}{4} 4 \underset{\circ}{3} 2 \underset{\circ}{2} \} \\
 6) & \quad \{ 3 \underset{\circ}{1} 1 \underset{\circ}{2} 3 \underset{\circ}{4} 4 \underset{\circ}{3} 2 \underset{\circ}{3} \} \\
 7) & \quad \{ 3 \underset{\circ}{1} 1 \underset{\circ}{2} 3 \underset{\circ}{4} 4 \underset{\circ}{3} 2 \underset{\circ}{2} \} \\
 8+2n, \kappa) & \quad \{ \overset{\kappa}{3} \overset{n}{1} \overset{n}{3} \overset{n}{1} \overset{n}{2} \overset{n}{1} \overset{n}{3} \overset{n}{1} \overset{n}{1} \overset{n}{2} 3 \underset{\circ}{4} 4 \underset{\circ}{3} 2 \underset{\circ}{2} 3 \overset{n}{1} \overset{n}{3} \overset{n}{1} \overset{n}{2} \overset{\kappa}{1} \overset{\kappa}{3} \} \quad (I3.1) \\
 9+2n, \kappa) & \quad \{ \overset{\kappa-1}{3} \overset{n}{2} \overset{n}{1} 3 \overset{n}{1} \overset{n}{3} \overset{n}{1} \overset{n}{2} \overset{n}{1} \overset{n}{3} \overset{n}{1} \overset{n}{1} \overset{n}{2} 3 \underset{\circ}{4} 4 \underset{\circ}{3} 2 \underset{\circ}{2} 3 \overset{n}{1} \overset{n}{3} \overset{n}{1} \overset{n}{2} \overset{\kappa}{1} \overset{\kappa}{3} \} \quad (I3.2) \\
 9+2n, \kappa, \rho) & \quad \{ \overset{\rho}{3} \overset{\kappa}{1} \overset{\kappa}{3} \overset{n}{2} \overset{n}{1} 3 \overset{n}{1} \overset{n}{3} \overset{n}{1} \overset{n}{2} 3 \underset{\circ}{4} 4 \underset{\circ}{3} 2 \underset{\circ}{2} 3 \overset{n}{1} \overset{n}{3} \overset{n}{1} \overset{n}{2} \overset{\kappa}{1} \overset{\kappa}{3} \overset{\rho}{2} \overset{\rho}{1} \overset{\rho}{3} \} \quad (I3.3)
 \end{aligned}$$

Figure 6: Initial set of rectangles.

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