Hall's Ray of the Markov and Lagrange Spectra

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Diophantine approximations

Consider $\alpha \in \mathbb{R} \backslash \mathbb{Q}$.
We want to approximate α with rational $\frac{p}{q}$, minimizing the distance

$$\left| \alpha - \frac{p}{q} \right| .$$

Some approximations are better than other:

$$\left| \pi - \frac{314}{100} \right| \approx 1.5 \cdot 10^{-3},$$
$$\left| \pi - \frac{355}{113} \right| \approx 2.6 \cdot 10^{-7}.$$

How do we find good diophantine approximations?

Continued fractions

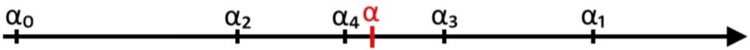
Representing α as a continued fraction:

$$\alpha = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}, \quad a_0 \in \mathbb{Z}, \; a_1, a_2, \ldots \in \mathbb{N}.$$

The continued fraction produces infinitely many rational convergents of α:

$$\alpha_n = \frac{P_n}{Q_n} = [a_0; a_1, a_2, \ldots, a_n].$$

The sequence $\{\alpha_n\}$ converges to α:



Convergents

The convergents of a continued fraction give the best diophantine approximations. For example:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}}},$$

$$\frac{355}{113} = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}.$$

Hurwitz theorem

The following theorem describes the best possible quality of approximation for an arbitrary α:

Theorem (Hurwitz)

For any irrational α, there exist infinitely many rationals $\frac{p}{q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

On the other hand, for any $c > \sqrt{5}$ the inequality

$$\left|\varphi - \frac{p}{q}\right| < \frac{1}{cq^2}. \tag{1}$$

has only finitely many solutions, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Markov value

Definition

Let the Markov value $M(\alpha)$ of an irrational α be the smallest constant c such that inequality (1) has infinitely many solutions.

Markov values allow us to introduce Lagrange spectrum.

Definition

Lagrange spectrum L is the set of the Markov values over all irrationals:

$$L := \{M(\alpha) \mid \alpha \in \mathbb{R} \backslash \mathbb{Q}\}.$$

Lagrange spectrum

Definition

$\mathcal{M}$ stands for a double-infinite sequence of positive integers:

$$\mathcal{M} = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots) \in \mathbb{N}^{\mathbb{Z}}.$$

Definition

The height function is defined as

$$f : \mathbb{N}^{\mathbb{Z}} \to \mathbb{R},$$
$$f(\mathcal{M}) = a_0 + [0; a_{-1}, a_{-2}, \ldots] + [0; a_1, a_2, \ldots].$$

Definition

The shift function is defined as

$$\sigma : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}},$$
$$\sigma(\ldots a_{-2} a_{-1} \underline{a_0} a_1 a_2 \ldots) = \ldots a_{-1} a_0 \underline{a_1} a_2 a_3 \ldots$$

Lagrange spectrum

We are now ready to give an alternative definition for the Lagrange spectrum.

Definition

Let the Lagrange value of a double-infinite sequence $\mathcal{M}$ be

$$\ell(\mathcal{M}) := \limsup_{k \to +\infty} f(\sigma^k \mathcal{M}).$$

Definition

Lagrange spectrum is the set of Lagrange values over all double-infinite sequences:

$$L := \{\ell(\mathcal{M}) \mid \mathcal{M} \in \mathbb{N}^{\mathbb{Z}}\}.$$

Lagrange spectrum geometry