

Hall's Ray of the Markov Spectrum

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Diophantine approximations

Consider $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

We want to approximate α with rational $\frac{p}{q}$, minimizing the distance

$$\left| \alpha - \frac{p}{q} \right|.$$

Some approximations are better than other:

$$\left| \pi - \frac{314}{100} \right| \approx 1.5 \cdot 10^{-3},$$
$$\left| \pi - \frac{355}{113} \right| \approx 2.6 \cdot 10^{-7}.$$

How do we find good diophantine approximations?

Continued fractions

Representing α as a continued fraction:

$$\alpha = [a_0; a_1, a_2, \dots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}, \quad a_0 \in \mathbb{Z}, \quad a_1, a_2, \dots \in \mathbb{N}.$$

The continued fraction produces infinitely many rational convergents of α :

$$\alpha_n = \frac{P_n}{Q_n} = [a_0; a_1, a_2, \dots, a_n].$$

The sequence $\{\alpha_n\}$ converges to α :

