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Markov and Lagrange Spectra for Continued Fractions

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Contents

1 Foreword

The main subject of this paper is the Markov spectrum or, to be more precise, its final portion, called the Hall's Ray. Its existence has been proven by M. Hall in 1947; however, finding this constant took time.

In 1975, Freiman evaluated the exact value of the start of the Hall's Ray.

This paper contains the translated and modernized approach from the original paper.

It also pretends to be the verification of the Freiman's theorem.

I am grateful to my scientific advisor Carlos Matheus for valuable advise, remark about the thickness argument, and help in understanding the geometry of the Markov spectrum.

2 Introductory notation

2.1 Markov and Lagrange values

Symbol $\mathcal{M}$ denotes double-infinite sequences from $\mathbb{N}^{\mathbb{Z}}$:

$$\mathcal{M} = ...a_{-2}a_{-1}a_0a_1a_2...$$

I will use $\lambda(\mathcal{M})$, $\mu(\mathcal{M})$ and $f(\mathcal{M})$ for Lagrange, Markov values and height function. Symbols γ and δ denote the lhs and rhs of sequence $\mathcal{M}$:

$$\begin{gathered}\gamma(\mathcal{M}) = [0; a_{-1}, a_{-2}, ...],\\ \delta(\mathcal{M}) = [0; a_1, a_2, ...],\\ f(\mathcal{M}) = a_0 + \gamma(\mathcal{M}) + \delta(\mathcal{M}).\end{gathered}$$

At last, symbols M and L denote the Markov and Lagrange spectra.

2.2 Centered sequence

Definition. A sequence $\mathcal{M}$ is called **centered**, if

$$\mu(\mathcal{M}) = f(\mathcal{M}). \tag{1}$$

Proposition. *Markov spectrum can be defined with only centered sequences:*

$$\{\mu(\mathcal{M}) \mid \mathcal{M} \in \mathbb{N}^{\mathbb{Z}}\} = M = \{\mu(\mathcal{M}) \mid \mathcal{M} \textit{ is centered }\}.$$

2.3 Rectangle

Designation. Denote by

$$\{a_{i_1}, a_{i_1+1}, ..., a_{i_2}\} \ \ (i_1 \leqslant 0 \leqslant i_2)$$

the set of double-infinite sequences $\mathcal{M}$ with fixed terms $a_{i_1}, a_{i_1+1}, ..., a_{i_2}$ on the corresponding positions.

Terms a_s for $s < i_1$ and $s > i_2$ are arbitrary integers, chosen such that $\mathcal{M}$ is centered and, maybe, satisfies some conditions.

Segments Δ_1, Δ_2 and Δ are defined by the following equations:

$$\begin{array}{lllll}\Delta_1 & = & [\,\Delta_1'\,;\,\Delta_1''\,] & = & [\,\min\gamma(\mathcal{M})\,;\,\max\gamma(\mathcal{M})\,],\\ \Delta_2 & = & [\,\Delta_2'\,;\,\Delta_2''\,] & = & [\,\min\delta(\mathcal{M})\,;\,\max\delta(\mathcal{M})\,],\\ \Delta & = & [\,\Delta'\,;\,\Delta''\,] & = & a_0 + \Delta_1 + \Delta_2,\end{array} \tag{2}$$

where $\mathcal{M}$ belongs to the set.

Note that we will use $'$ for the lower bound, and $''$ for the upper bound.

Definition. Rectangle is the segment Δ with the set of sequences, defining it.

2.4 Horizontal rectangle

Definition. Call a rectangle Δ **horizontal**, if

$$|\Delta_1| \geqslant |\Delta_2|. \tag{3}$$

In (3) we allow terms a_s for $s < i_1$ and $s > i_2$ to be integers $\{1, 2, 3\}$, regardless of the requirement that sequences $\mathcal{M} \in \Delta$ are centered.

In other words, Δ is horizontal, if and only if

$$\left|[0; a_{-1}, ..., a_{i_1}, \overline{3, 1}] - [0; a_{-1}, ..., a_{i_1}, \overline{1, 3}]\right| \geqslant \left|[0; a_1, ..., a_{i_2}, \overline{3, 1}] - [0; a_1, ..., a_{i_2}, \overline{1, 3}]\right|.$$

Clearly, we can always obtain a horizontal rectangle out of the vertical one, as we can reindex the sequence in the opposite direction.

2.5 Subrectangle

Consider a rectangle Δ, set by the sequence center

$$\{a_{i_1}, a_{i_1+1}, ..., a_{i_2}\}.$$

We will use a **shorter notation** for subrectangles, produced by setting integers a_i for $i < i_1$ or $i > i_2$:

$$\{b_\ell ... b_1,\ c_1 ... c_r\} \coloneqq \{b_\ell ... b_1 a_{i_1} ... a_{i_2} c_1 ... c_r\}.$$

For example:

$$\{213, 3\} \coloneqq \{213 a_{i_1} ... a_{i_2} 3\}, \tag{ex.1}$$

$$\{2, 0\} \coloneqq \{2 a_{i_1} ... a_{i_2}\}. \tag{ex.2}$$

We will also shorter the notation (2): lhs and rhs are $\Delta_1(312)$ and $\Delta_2(3)$ for subrectangle (ex.1) and $\Delta_1(2)$ and a_2 for (ex.2).

2.6 Geometrical interpretation

Consider the mapping

$$\widetilde{h} : \mathbb{N}^{\mathbb{Z}} \to \mathbb{R}^2,$$

$$\widetilde{h}(\mathcal{M}) = (\gamma(\mathcal{M}); \delta(\mathcal{M})).$$

In these terms, the Markov spectrum M is the projection of some subset $\mathcal{S} \subset C_4 \times C_4$ onto the diagonal.

Then **rectangle** Δ is indeed a rectangle $\Delta_1 \times \Delta_2$ and **subrectangles** are its subrectangles.

We will consider a family of rectangles whose projections cover the beginning of Hall's Ray.

Then we will present the algorithm to split rectangle into subrectangles so that their projections cover the projection of initial rectangle.

When we say that rectangles intersect, we, however, mean that their projections intersect.

The more «squarish» the rectangle, the easier the step.

That's why we will bound the aspect ratio of rectangles (see **good** rectangle).

Formulas to evaluate side lengths and aspect ratio are given in the section 3.

3 Calculations

3.1 Length of Δ_1 or Δ_2

Let's fix some terms of continued fraction $[0; q_1, q_2, q_3, ..., q_n]$.
We will often need to measure difference

$$[0; q_1, q_2, q_3, ..., q_n, \frac{1}{\Theta_R}] - [0; q_1, q_2, q_3, ..., q_n, \frac{1}{\Theta_L}],$$

where Θ's are some continuations of the continued fraction.
They generally look like $\Theta = \left[0; 12\overline{13}\right]$ or something[1]. We will set Θ's explicitly.
For the general proof, Θ's will be taken from Table 6.

Designation. For given continuation Θ_i, denote by ε_i the resulting continued fraction:

$$\varepsilon_i = [0; q_1, q_2, ..., q_n, \frac{1}{\Theta_i}]. \tag{4}$$

Then the following equality takes place:

$$|\varepsilon_i - \varepsilon_j| = \frac{|\Theta_i - \Theta_j|}{Q_n^2 \left(1 + pQ_i\right) \left(1 + pQ_j\right)}, \tag{5}$$

where

$$p = \frac{Q_{n-1}}{Q_n}.$$

3.2 Rectangle aspect ratio

Consider some fixed center of rectangle $\{a_{i_1}...a_{i_2}\}$.
We will often extend it from the left (right) using continuations $\Theta_{\gamma_1}, \Theta_{\gamma_2}$ ($\Theta_{\delta_1}, \Theta_{\delta_2}$).
Rule (4) produces γ_1, γ_2 (δ_1, δ_2).
Finite continued fractions $\frac{P_{i_1}}{Q_{i_1}} = [0; a_{-1}, a_{-2}, ..., a_{i_1}]$ $\left(\frac{P_{i_2}}{Q_{i_2}} = [0; a_1, a_2, ..., a_{i_2}]\right)$ are their convergents.
Then

$$\left|\frac{\gamma_1 - \gamma_2}{\delta_1 - \delta_2}\right| = \frac{1}{q}\left|\frac{\Theta_{\gamma_1} - \Theta_{\gamma_2}}{\Theta_{\delta_1} - \Theta_{\delta_2}}\right| \frac{1 + p'\Theta_{\delta_1}}{1 + p\Theta_{\gamma_1}} \frac{1 + p'\Theta_{\delta_2}}{1 + p\Theta_{\gamma_2}} \approx \frac{1}{q}\left|\frac{\Theta_{\gamma_1} - \Theta_{\gamma_2}}{\Theta_{\delta_1} - \Theta_{\delta_2}}\right|, \tag{6}$$

where

$$p = \frac{Q_{i_1+1}}{Q_{i_1}}, \;\; p' = \frac{Q_{i_2-1}}{Q_{i_2}}, \;\; q = \frac{Q_{i_1}^2}{Q_{i_2}^2}.$$

[1]Here, as always in this book, $\overset{k}{\overline{abc}}$ means k-times repetition of abc, and $\overline{abc}$ means infinite repetition.

4 Rectangle boundaries

We will set boundaries for rectangles in a different way.

To distinguish rectangles, we will introduce the notion of left- or right-shortened rectangles.

It is given in subsections 4.1-4.3.

Subsections 4.4-4.5 provide the rules for boundaries Δ_1', Δ_1'', Δ_2', Δ_2''.

4.1 Resection

Definition. Call **resection** of a segment $A = [a; b]$ a process of removing subsegment $A_{12} = [a_1; b_1]$, leaving two segments $A_1 \sqcup A_2 = [a; a_1] \sqcup [b_1; b]$.

Definition. Call subsegment $A_{12} \subset A$ **normal**, if it is thicker than the two remaining subsegments:

$$|A_{12}| \leqslant min\,\{|A_1|,\ |A_2|\} \tag{7}$$

We call a resection **normal** if the resected subsegment is normal.

Proposition. *For any normal resection, having*

$$A + A = (A_1 \sqcup A_2) + (A_1 \sqcup A_2). \tag{8}$$

4.2 Shortened rectangle

Consider a horizontal rectangle Δ.

4.2.1 Case $i_1 \equiv i_2 \equiv 0 \pmod 2$

If both i_1 and i_2 are even, then left- and right-shortened rectangles are defined as follows.

Definition. Rectangle Δ is called left-shortened, if for subrectangle $\{3, 3\}$ the following condition takes place:

$$|\Delta_1(3)| \leqslant 1.4 \cdot |\Delta_2(3)|\,. \tag{9}$$

Here, as in (3), we allow terms a_s for $s < i_1$ and $s > i_2$ to be integers $\{1, 2, 3\}$, so (9) can be rewritten as

$$\left|\left[0; a_{-1}, ..., a_{i_1}, 3, \overline{3,1}\right] - \left[0; a_{-1}, ..., a_{i_1}, \overline{3,1}\right]\right| \leqslant \left|\left[0; a_1, ..., a_{i_2}, 3, \overline{3,1}\right] - \left[0; a_1, ..., a_{i_2}, \overline{3,1}\right]\right|.$$

For further convenience, we also introduce the opposite to (9) condition:

Definition. Rectangle Δ is called left-normal, if

$$|\Delta_1(3)| > 1.4 \cdot |\Delta_2(3)|\,. \tag{10}$$

Now we introduce the notion of right-shortened rectangle:

Definition. Rectangle Δ is called right-shortened, if for subrectangle $\{31, 13\}$ the following condition takes place:

$$|\Delta_1(13)| \leqslant 1.4 \cdot |\Delta_2(13)|\,. \tag{11}$$

As in 4.2.1, we allow terms a_s for $s < i_1$ and $s > i_2$ to be integers $\{1, 2, 3\}$, so (11) can be rewritten as

$$\left|\left[0; a_{-1}, ..., a_{i_1}, 1, 3, \overline{3, 1}\right] - \left[0; a_{-1}, ..., a_{i_1}, \overline{1, 3}\right]\right| \leqslant \left|\left[0; a_1, ..., a_{i_2}, 1, 3, \overline{3, 1}\right] - \left[0; a_1, ..., a_{i_2}, \overline{1, 3}\right]\right|.$$

We also introduce the opposite condition:

Definition. Rectangle Δ is called right-normal, if

$$|\Delta_1(13)| > 1.4 \cdot |\Delta_2(13)|. \tag{12}$$

4.2.2 Other cases

If $i_1 \equiv i_2 \pmod 2$ and i_1 is odd, then the conditions for left- and right-shortened rectangles are swapped.

In case $i_1 \not\equiv i_2 \pmod 2$, the notions of left- and right-shortened rectangles are inferred from subrectangles $\{1,\}$, $\{2,\}$, and $\{3,\}$.

We will refer to the relevant conditions from 4.5

4.3 Explanation of shortened

Notion left-shortened is used to set the lower bound Δ' for rectangle Δ.

Notion right-shortened is used to set the upper bound Δ''.

For example, when interested in Δ' when i_1 is even and i_2 is odd, one has to consider larger a_{i_1-1} and smaller a_{i_2+1}. As the rectangle is horizontal, we will add terms to the left first. So we will check the condition (11) for $\{2,\}$ or $\{3,\}$.

In other words, check whether $\{2,\}$ or $\{3,\}$ is left-shortened or left-normal (recall that $i_1 - 1$ and i_2 are odd, so left-shortenence is defined by (11)).

Which subrectangle ($\{2,\}$ or $\{3,\}$) to take, however, depends on terms a_{i_1+1} and a_{i_1} (see 4.4).

4.4 How we ensure centeredness

Consider integers $q_1, q_2, ..., q_n$. Let $\{\delta_n\}$ be the set of fractions $\delta_n = [0; q_1, q_2, ..., q_n, ...]$ with n fixed terms. We will suppose that n is even (for odd n the bounds are swapped).

At first, determine the smallest of fractions δ_n.

We will consider 2 cases: S (Shortened) and N (Normal):

$$S.\ q_{n-1} = 3, q_n = 1. \tag{13a}$$
$$N.\ \text{Otherwise.} \tag{13b}$$

The lower bound δ'_n for segment, containing δ_n, is defined by:

$$\begin{array}{ll} S. & \delta'_n = [0; q_1, ..., q_n, 213\overline{12}]. \\ N. & \delta'_n = [0; q_1, ..., q_n, 3\overline{12}]. \end{array}$$

To set the upper bound δ''_n, largest of δ_n, consider 2 other cases:

$$S. \quad q_n = 3. \tag{14a}$$
$$N. \quad q_n \neq 3. \tag{14b}$$

Then

$$\begin{array}{ll} S. & \delta''_n = [0; q_1, ..., q_n, 121 3\overline{12}]. \\ N. & \delta''_n = [0; q_1, ..., q_n, 13\overline{12}]. \end{array}$$

These bounds will allow us to construct sequences $\mathcal{M}$, for which combination (31313) is forbidden and, therefore, the following condition takes place:

$$f_i(\mathcal{M}) \leqslant \lambda(\overline{31312}) \approx 4,5241, \ i \neq 0,$$

which will ensure (1).

4.5 Formal

Let's now turn to concrete definitions and bounds. Remind that we are looking at the horizontal rectangle Δ.

Suppose that $i_1 \equiv i_2 \pmod 2$, i_1 is even. (If i_1 is odd, then Δ' and Δ'' are swapped.)

4.5.1 Bounds for Δ'

I. Suppose both $\{\gamma_0(\mathcal{M})\}$ and $\{\delta_0(\mathcal{M})\}$ satisfy (13b) and the rectangle Δ is left-normal, that is, satisfies (10). We will denote such situation by $N - N - N$ (segment Δ_1 is left-normal, Δ_2 is left-normal, rectangle Δ is normal).

   In this case define Δ' by equation

$$\Delta' = f(\overline{21}3a_{i_1}...a_{i_2}3\overline{12}). \tag{15}$$

IIa. Sets $\{\gamma_0(\mathcal{M})\}$ and $\{\delta_0(\mathcal{M})\}$ meet condition (13b) and Δ is left-shortened, that is, meets condition (9). This is case $N - N - S$ (segments Δ_1 and Δ_2 are left-normal, rectangle Δ is left-shortened).

   Then

$$\Delta' = f(\overline{21}3a_{i_1}...a_{i_2}213\overline{12}). \tag{16}$$

IIb. Set $\{\gamma_0(\mathcal{M})\}$ meets (13b), $\{\delta_0(\mathcal{M})\}$ meets (13a). No matter which condition (9) of (10) is met. It is the case $N - S$ (segment Δ_1 is left-normal, segment Δ_2 left-shortened). Bound Δ_1 is defined by (16).

III. Set $\{\gamma_0(\mathcal{M})\}$ meets (13a), $\{\delta_0(\mathcal{M})\}$ meets (13b). In this $S - N$ case (segment Δ_1 is left-shortened, segment Δ_2 is left-normal) Δ' is defined as follows:

$$\Delta' = f(\overline{21}312a_{i_1}...a_{i_2}3\overline{12}). \tag{17}$$

IV. Both $\{\gamma_0(\mathcal{M})\}$ and $\{\delta_0(\mathcal{M})\}$ meet (13a). In this $S - S$ case (both segments Δ_1 and Δ_2 are left-shortened) Δ' is defined by

$$\Delta' = f(\overline{21}312a_{i_1}...a_{i_2}213\overline{12}). \tag{18}$$

Left side of the figure 1 illustrates these bounds.

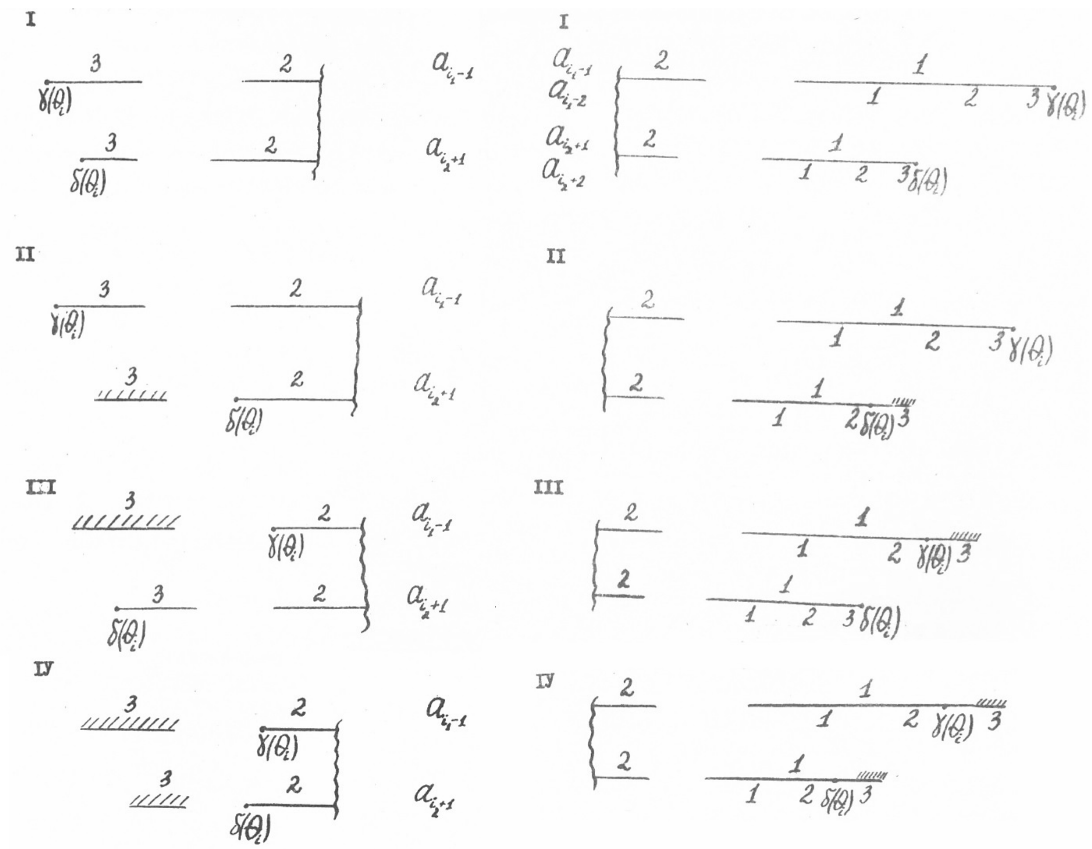


Figure 1: Bounds Δ' (left) and Δ'' (right).
Hatched areas correspond to values of a_{i_1-1} and a_{i_2+1} (on the left) or a_{i_1-2} and a_{i_2+2} (on the right), equal to 3, which can not appear in the concrete case.
$\Delta'_1 = \gamma(\Theta_i)$, $i = 3, 30$, $\Delta'_2 = \delta(\Theta_i)$, $i = 3, 30$, $\Delta' = \Delta'_1 + \Delta'_2$.
$\Delta''_1 = \gamma(\Theta_i)$, $i = 90, 94$, $\Delta''_2 = \delta(\Theta_i)$, $i = 90, 94$.

4.5.2 Bounds for Δ''

Now we will provide formulas for Δ'':

$$\begin{aligned}
&I && \Delta'' = f(\overline{21}31a_{i_1}...a_{i_2}13\overline{12}), && (19a)\\
&II && \Delta'' = f(\overline{21}31a_{i_1}...a_{i_2}12131\overline{2}), && (19b)\\
&III && \Delta'' = f(\overline{21}3121a_{i_1}...a_{i_2}13\overline{12}), && (19c)\\
&IV && \Delta'' = f(\overline{21}3121a_{i_1}...a_{i_2}1213\overline{12}). && (19d)
\end{aligned}$$

Figure (2) regulates the choise of the formulas.

Case	Δ''_1	Δ''_2	Rectangle	Δ''
I	(14b)	(14b)	(12) $N-N-N$	(19a)
IIa	(14b)	(14b)	(11) $N-N-S$	(19b)
IIb	(14b)	(14a)	$N-S$	(19b)
III	(14a)	(14b)	$S-N$	(19c)
IV	(14a)	(14a)	$S-S$	(19d)

Figure 2: Rules for choise of Δ'' in case $i_1 \equiv i_2 \pmod 2$.

Again, these bounds are illustrated on the right side of the figure 1.

4.5.3 Case $i_1 \not\equiv i_2 \pmod 2$

Now consider case $i_1 \not\equiv i_2 \pmod 2$, i_1 is even. We will use rules from figure (3) to choose one of 4 formulas for Δ'.

$$\begin{aligned}
&I && \Delta' = f(\overline{21}3a_{i_1}...a_{i_2}13\overline{12}), && (20a)\\
&II && \Delta' = f(\overline{21}3a_{i_1}...a_{i_2}1213\overline{12}), && (20b)\\
&III && \Delta' = f(\overline{21}312a_{i_1}...a_{i_2}13\overline{12}), && (20c)\\
&IV && \Delta' = f(\overline{21}312a_{i_1}...a_{i_2}1213\overline{12}). && (20d)
\end{aligned}$$

Case	Δ''_1	Δ''_2	Case name	Δ''
I	(13b)	(14b)	$N-N$	(20a)
II	(13b)	(14a)	$N-S$	(20b)
III	(13a)	(14b)	$S-N$	(19c)
IV	(13a)	(14a)	$S-S$	(19d)

Figure 3: Rules for choise of Δ' in case $i_1 \not\equiv i_2 \pmod 2$, i_1 is even.

To determine the bound Δ'', we will use subrectangle

$$\{1, 0\}.$$

Having $i_1 - 1 \equiv i_2 \pmod 2$, so we can use all the previous formulas to determine Δ''.

5 Good rectangle

5.1 Definition

Consider a horizontal rectangle $\Delta = \{a_{i_1} \dots a_{i_2}\}$.

Definition. Rectangle Δ is called **good**, if subrectangles $\{2, 0\}$ and $\{1, 0\}$ intersect.

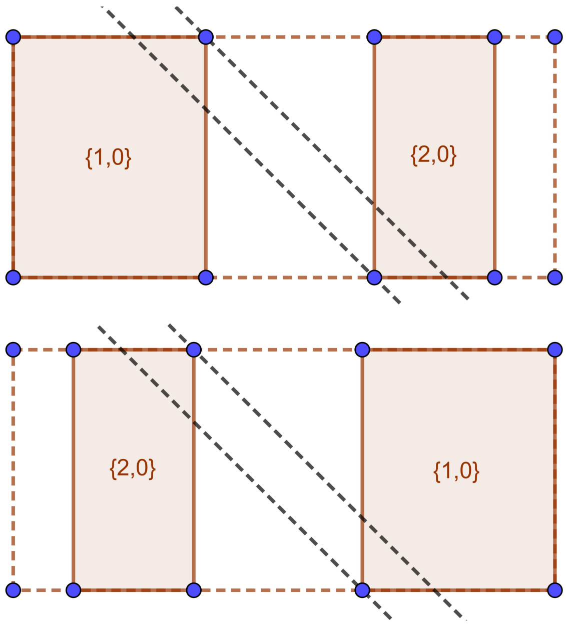


Figure 4: Good rectangles if i_1 is odd (upper) and even (lower).

For example, in case i_1 is even, goodness is equivalent to

$$\{2, 0\}'' \geqslant \{1, 0\}'. \tag{21}$$

Bounds of rectangles are determined by the rules from section 4.

If the rectangle Δ is not good (for example, (21) doesn't take place), then

$$(\{2, 0\}''; \{1, 0\}') \not\subset \Delta.$$

Clearly, if the rectangle is not good, then one can not split it into smaller subrectangles whose projections cover the projection of initial one. That's why we will only consider good rectangles during the proof.

5.2 Sufficient conditions of goodness

5.2.1 Results

A horizontal rectangle Δ is good, if

$$\frac{\Delta_1}{\Delta_2} < \begin{cases} 3.8, & i_1 \equiv i_2 \pmod 2, \quad \{\delta(\mathcal{M})\} \text{ meets (13b)}, \\ 3.43, & i_1 \not\equiv i_2 \pmod 2, \quad \{\delta(\mathcal{M})\} \text{ meets (13b)}, \\ 2.9, & \text{otherwise}. \end{cases} \tag{22}$$

5.2.2 Universal $2,9$ bound

We will introduce a sufficient conditions for a rectangle to be good.

Designation. For given continuations Θ_γ and Θ_δ, denote by γ and δ the resulting continued fractions:

$$\gamma = [0; a_{-1}, ..., a_{i_1}, \frac{1}{\Theta_\gamma}],$$
$$\delta = [0; a_1, ..., a_{i_2}, \frac{1}{\Theta_\delta}].$$

As stated in 2.1, γ's correspond to the lhs of $\mathcal{M}$, and δ's correspond to the rhs of $\mathcal{M}$. Variables Θ will be taken from the Figure 6 and will be specified in each case separately.

Suppose that i_1 is even.
Remind that $|\Delta_1| \geqslant |\Delta_2|$, which means that

$$\frac{|\gamma'' - \gamma'|}{|\delta'' - \delta'|} \geqslant 1,$$

where

$$\Theta_{\gamma'} = \Theta_{\delta'} = \Theta_1, \ \Theta_{\gamma''} = \Theta_{\delta''} = \Theta_{95}.$$

Using (6), rewrite it as

$$q\frac{1+p\Theta_1}{1+p'\Theta_1}\frac{1+p\Theta_{95}}{1+p'\Theta_{95}} \leqslant 1.$$

Goodness can be written as

$$\gamma' - \gamma'' < |\delta' - \delta''|, \tag{23}$$

where $\gamma' = \Delta_1(1)'$ and $\gamma'' = \Delta_1(2)''$.
For an arbitraty rectangle Δ, taking

$$\Theta_{\gamma'} = \Theta_{66}, \ \Theta_{\gamma''} = \Theta_{63}, \ \Theta_{\delta'} = \Theta_{90}, \ \Theta_{\delta''} = \Theta_{30}.$$

Inequality (23) transforms into

$$0,313 < q\frac{1+p\Theta_{63}}{1+p'\Theta_{30}}\frac{1+p\Theta_{66}}{1+p'\Theta_{90}}. \tag{24}$$

Suppose that

$$\frac{\Delta_1}{\Delta_2} < 2,9, \tag{25}$$

which is equivalent to

$$\frac{1}{q}\frac{1+p'\Theta_1}{1+p\Theta_1}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} < 2,9. \tag{26}$$

Then (24) takes place. Indeed, it is so, if

$$\frac{1}{2,9}\frac{1+p'\Theta_1}{1+p\Theta_1}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} > 0,313\frac{1+p'\Theta_{30}}{1+p\Theta_{63}}\frac{1+p'\Theta_{90}}{1+p\Theta_{66}}.$$

or, equivalent,

$$1,1 > \frac{1+p\Theta_1}{1+p\Theta_{63}}\,\frac{1+p\Theta_{95}}{1+p\Theta_{66}}\,\frac{1+p'\Theta_{30}}{1+p'\Theta_1}\,\frac{1+p'\Theta_{90}}{1+p'\Theta_{95}},$$

which is checked directly.

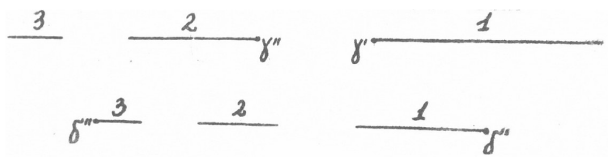
Condition (25) is sufficient for rectangle to be good, regardless of the parity of i_1 and i_2 and the left or right shortness or normalness of rectangle.

5.2.3 Case $i_1 \equiv i_2 \pmod 2$

Now consider case $i_1 \equiv i_2 \pmod 2$.

Suppose that the right hand side $\{\delta(\mathcal{M})\}$ is left-normal, that is, meets (13b).

Equivalent, the case $a_{i_2} = 1$, $a_{i_2-1} = 3$ doesn't take place.



This assumption allows us to substitute the folliwing γ's and δ's into (23):

$$\Theta_{\gamma'} = \Theta_{66},\ \Theta_{\gamma''} = \Theta_{63},\ \Theta_{\delta'} = \Theta_{90},\ \Theta_{\delta''} = \Theta_3.$$

The rhs left-normality leaves us hope that the bound Δ_2'' will be set to $\Delta_2(\Theta_3)$, so we are substituting $\Theta_{\delta''}$ with Θ_3 instead of Θ_{30}.

Now, instead of (24) we will get

$$0,253 < q\,\frac{1+p\Theta_{63}}{1+p'\Theta_3}\,\frac{1+p\Theta_{66}}{1+p'\Theta_{90}}. \tag{27}$$

Such choise of $\Theta_{\gamma'}$ and $\Theta_{\delta''}$ is fine, if the following inequality takes place:

$$\delta' - \delta'' > 1,4(\gamma' - \gamma''), \tag{28}$$

where

$$\Theta_{\gamma'} = \Theta_{68},\ \Theta_{\gamma''} = \Theta_{65},\ \Theta_{\delta'} = \Theta_{28},\ \Theta_{\delta''} = \Theta_1,$$

so we can rewrite (28) as

$$0,253 < q\,\frac{1+p\Theta_{65}}{1+p'\Theta_1}\,\frac{1+p\Theta_{68}}{1+p'\Theta_{28}}. \tag{29}$$

We can notice that (29) follows from (27). Indeed, that follows from the inequality

$$0,253\,\frac{1+p'\Theta_3}{1+p'\Theta_{63}}\,\frac{1+p\Theta_{90}}{1+p\Theta_{66}} > 0,269\,\frac{1+p'\Theta_1}{1+p'\Theta_{65}}\,\frac{1+p\Theta_{28}}{1+p\Theta_{68}},$$

or inequality

$$\frac{1+p\Theta_{65}}{1+p\Theta_{63}}\,\frac{1+p\Theta_{68}}{1+p\Theta_{66}}\,\frac{1+p'\Theta_3}{1+p'\Theta_1}\,\frac{1+p'\Theta_{90}}{1+p'\Theta_{28}} > 1,064.$$

Overall, we have proved that (27) is enough for rectangle to be good.

Suppose that

$$\frac{\Delta_1}{\Delta_2} < 3,8, \tag{30}$$

or

$$\frac{1}{q}\frac{1+p'\Theta_1}{1+p\Theta_1}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} < 3,8.$$

Then (27) takes place. Indeed, it is so, if

$$\frac{1}{3,8}\frac{1+p'\Theta_1}{1+p\Theta_1}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} > 0,253\frac{1+p'\Theta_3}{1+p'\Theta_{63}}\frac{1+p\Theta_{90}}{1+p\Theta_{66}}.$$

The last inequality is transformed into

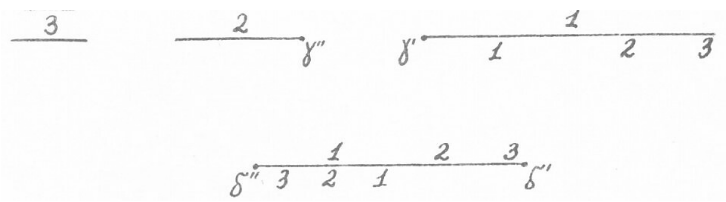
$$1,04 > \frac{1+p'\Theta_3}{1+p'\Theta_1}\frac{1+p'\Theta_{90}}{1+p'\Theta_{95}}\frac{1+p\Theta_1}{1+p\Theta_{63}}\frac{1+p\Theta_{95}}{1+p\Theta_{66}},$$

which is easily checked.

5.2.4 Case $i_1 \not\equiv i_2 \pmod 2$

Now consider case $i_1 \not\equiv i_2 \pmod 2$ and, again, the right hand side $\{\delta(\mathcal{M})\}$ is left-normal, in other words, case $a_{i_2} = 1$, $a_{i_2-1} = 3$ doesn't take place.

Suppose that i_1 is even.



Substituting

$$\Theta_{\gamma'} = \Theta_{66},\ \Theta_{\gamma''} = \Theta_{59},\ \Theta_{\delta'} = \Theta_3,\ \Theta_{\delta''} = \Theta_{90}$$

into (23), obtain

$$0,2885 < q\frac{1+p\Theta_{59}}{1+p'\Theta_3}\frac{1+p\Theta_{66}}{1+p'\Theta_{90}}. \tag{31}$$

Suppose that

$$\frac{\Delta_1}{\Delta_2} < 3,43 \tag{32}$$

or

$$\frac{1}{q}\frac{1+p'\Theta_1}{1+p\Theta_1}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} < 3,43.$$

Then (31) takes place. Indeed, it is so, if

$$\frac{1}{3,43}\frac{1+p'\Theta_1}{1+p\Theta_1}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} > 0,2885\frac{1+p'\Theta_3}{1+p\Theta_{59}}\frac{1+p'\Theta_{90}}{1+p\Theta_{66}}.$$

or

$$1,0105 > \frac{1+p'\Theta_3}{1+p'\Theta_1}\,\frac{1+p'\Theta_{90}}{1+p'\Theta_{95}}\,\frac{1+p\Theta_1}{1+p\Theta_{59}}\,\frac{1+p\Theta_{95}}{1+p\Theta_{66}}\,.$$

Increasing the right part, obtain

$$1,0105 > 0,9896\,\frac{1+p\cdot 1,0551+p^2\cdot 0,20875}{1+p\cdot 0,983+p^2\cdot 0,237},$$

which is easily checked.

6 Initial set of rectangles

Theorem 1 (Hall)**.** *The Markov and Lagrange spectra contain all the real numbers from some point, particularly, there exists a real μ such that*

$$[\mu; +\infty) \subset M \cap L.$$

Hall proved the constant μ to be smaller than 7. In 1975 Freiman evaluated it:

Theorem 2 (Freiman, 1975)**.** *For constants* $\mu = f\left(\overline{121313}2234\underset{0}{4}3211\overline{313121}\right)$ *and* $\nu = f\left(\overline{323444}31313\underset{0}{4}31312113\overline{3313121}\right)$, *having*

$$(\nu; \mu) \cap (\mathbb{R}\backslash L) = \emptyset, \quad [\mu; +\infty) \subset L \subset M.$$

We present a set of rectangles whose projections cover the segment from the Freiman's constant to $\sqrt{21}$. All of them are good. The reader is invited to find the the algorithm of splitting these rectangles into subrectangles correctly in [2].

1) $\{3\underset{0}{4}3\}$

2) $\{313\underset{0}{4}312\}$

3) $\{123\underset{0}{4}431\}$

4) $\{313\underset{0}{4}313\}$

5) $\{21123\underset{0}{4}4322\}$

6) $\{31123\underset{0}{4}4323\}$

7) $\{31123\underset{0}{4}4322\}$

$8+2n, k)$ $\{\overset{k}{3}13\overset{n}{\overline{121313}}1123\underset{0}{4}43223\overset{n}{\overline{131213}}\overset{k}{3}\}$ (13.1)

$9+2n, k)$ $\{\overset{k-1}{3}21313\overset{n}{\overline{121313}}1123\underset{0}{4}43223\overset{n}{\overline{13121}}31\overset{k}{3}\}$ (13.2)

$9+2n, k, p)$ $\{\overset{p}{3}\ 1\overset{k}{3}\ 21313\overset{n}{\overline{121313}}1123\underset{0}{4}43223\overset{n}{\overline{131213}}1\overset{k}{3}21\overset{p}{3}\}$ (13.3)

Figure 5: Initial set of rectangles.

Θ_{1}	$= [0; \overline{31}]$	= 0,263762
Θ_{2}	$= [0; 312\overline{13}]$	= 0,267649
Θ_{3}	$= [0; 3\overline{12}]$	= 0,26794
Θ_{4}	$= [0; 3123\overline{31}]$	= 0,270448
Θ_{5}	$= [0; 3123\overline{12}]$	= 0,270710
Θ_{6}	$= [0; 312\overline{31}]$	= 0,270738
Θ_{7}	$= [0; 311\overline{13}]$	= 0,27459
Θ_{8}	$= [0; 31113\overline{12}]$	= 0,27462
Θ_{9}	$= [0; 31113\overline{31}]$	= 0,274847
Θ_{10}	$= [0; 311\overline{12}]$	= 0,27517
Θ_{11}	$= [0; 31\overline{12}]$	= 0,27954
Θ_{12}	$= [0; 3113\overline{31}]$	= 0,280392
Θ_{13}	$= [0; 3113\overline{12}]$	= 0,28097
Θ_{14}	$= [0; 31\overline{13}]$	= 0,28105
Θ_{15}	$= [0; 32\overline{31}]$	= 0,290550
Θ_{16}	$= [0; 323\overline{12}]$	= 0,29062
Θ_{17}	$= [0; 323\overline{31}]$	= 0;291242
Θ_{18}	$= [0; 32\overline{21}]$	= 0;29216
Θ_{19}	$= [0; 3\overline{21}]$	= 0;29709
Θ_{20}	$= [0; 3213\overline{12}]$	= 0;29774
Θ_{21}	$= [0; 32\overline{13}]$	= 0;297773
Θ_{22}	$= [0; 33\overline{31}]$	= 0;302444
Θ_{23}	$= [0; 333\overline{12}]$	= 0;30025
Θ_{24}	$= [0; 33\overline{21}]$	= 0;30330
Θ_{25}	$= [0; 33\overline{12}]$	= 0;30600
Θ_{26}	$= [0; 3312\overline{13}]$	= 0;30603
Θ_{27}	$= [0; 3313\overline{12}]$	= 0;30638
Θ_{28}	$= [0; 3\overline{31}]$	= 0;306394
Θ_{29}	$= [0; 2\overline{13}]$	= 0;358258
Θ_{30}	$= [0; 213\overline{12}]$	= 0;35859
Θ_{31}	$= [0; 2133\overline{31}]$	= 0;361292
Θ_{32}	$= [0; 2133\overline{12}]$	= 0;36158
Θ_{33}	$= [0; 213\overline{31}]$	= 0;361602
Θ_{34}	$= [0; 212\overline{13}]$	= 0;365465
Θ_{35}	$= [0; 21213\overline{12}]$	= 0,36549
Θ_{36}	$= [0; \overline{21}]$	= 0,36602
Θ_{37}	$= [0; 2121\overline{13}]$	= 0,36779
Θ_{38}	$= [0; 2123\overline{12}]$	= 0,37119
Θ_{39}	$= [0; 212\overline{31}]$	= 0,371249
Θ_{40}	$= [0; 211\overline{13}]$	= 0,378537
Θ_{41}	$= [0; 21113\overline{12}]$	= 0,37858
Θ_{42}	$= [0; 21113\overline{31}]$	= 0,379018
Θ_{43}	$= [0; 211\overline{12}]$	= 0,37963
Θ_{44}	$= [0; 211113\overline{31}]$	= 0,383091
Θ_{45}	$= [0; 211113\overline{12}]$	= 0,38326
Θ_{46}	$= [0; 2111\overline{13}]$	= 0,383272
Θ_{47}	$= [0; 21123\overline{12}]$	= 0,38605
Θ_{48}	$= [0; 2112\overline{31}]$	= 0,386033
Θ_{49}	$= [0; 21123\overline{31}]$	= 0,386237
Θ_{50}	$= [0; 2112\overline{21}]$	= 0,38651
Θ_{51}	$= [0; 21\overline{12}]$	= 0,38800
Θ_{52}	$= [0; 2113\overline{31}]$	= 0,389648
Θ_{53}	$= [0; 2113\overline{21}]$	= 0,389916
Θ_{54}	$= [0; 2113\overline{12}]$	= 0,39076
Θ_{55}	$= [0; 21\overline{13}]$	= 0,390891
Θ_{56}	$= [0; 22\overline{31}]$	= 0,409544
Θ_{57}	$= [0; 223\overline{12}]$	= 0,40968
Θ_{58}	$= [0; 2\overline{21}]$	= 0,42265
Θ_{59}	$= [0; 2213\overline{12}]$	= 0,42398
Θ_{60}	$= [0; 22\overline{13}]$	= 0,424042
Θ_{61}	$= [0; 23\overline{31}]$	= 0,433577
Θ_{62}	$= [0; 233\overline{12}]$	= 0,43365
Θ_{63}	$= [0; 23\overline{12}]$	= 0,44093
Θ_{64}	$= [0; 2\overline{31}]$	= 0,441742
Θ_{65}	$= [0; 1\overline{13}]$	= 0,558256
Θ_{66}	$= [0; 113\overline{12}]$	= 0,55905
Θ_{67}	$= [0; 1133\overline{12}]$	= 0,56635
Θ_{68}	$= [0; 113\overline{31}]$	= 0,566423
Θ_{69}	$= [0; 112\overline{13}]$	= 0,57600
Θ_{70}	$= [0; 11213\overline{12}]$	= 0,57602
Θ_{71}	$= [0; 1\overline{12}]$	= 0,57735
Θ_{72}	$= [0; 1123\overline{12}]$	= 0,59032
Θ_{73}	$= [0; 111\overline{13}]$	= 0,609108
Θ_{74}	$= [0; 11113\overline{12}]$	= 0,60923
Θ_{75}	$= [0; 111113\overline{31}]$	= 0,620981
Θ_{76}	$= [0; 111113\overline{12}]$	= 0,62141
Θ_{77}	$= [0; 1111\overline{13}]$	= 0,621460
Θ_{78}	$= [0; 11123\overline{12}]$	= 0,62881
Θ_{79}	$= [0; 11\overline{12}]$	= 0, 63400
Θ_{80}	$= [0; 1113\overline{31}]$	= 0,63839
Θ_{81}	$= [0; 1113\overline{12}]$	= 0,63910
Θ_{82}	$= [0; 11\overline{13}]$	= 0,641742
Θ_{83}	$= [0; 123\overline{12}]$	= 0,69399
Θ_{84}	$= [0; 12213\overline{12}]$	= 0,70225
Θ_{85}	$= [0; 1223\overline{12}]$	= 0,70938
Θ_{86}	$= [0; 122\overline{21}]$	= 0,70783
Θ_{87}	$= [0; 12113\overline{12}]$	= 0,71903
Θ_{88}	$= [0; 121113\overline{12}]$	= 0,725381
Θ_{89}	$= [0; \overline{12}]$	= 0,73206
Θ_{90}	$= [0; 1213\overline{12}]$	= 0,73620
Θ_{91}	$= [0; 12\overline{13}]$	= 0,73624
Θ_{92}	$= [0; 13\overline{31}]$	= 0,765465
Θ_{93}	$= [0; 133\overline{12}]$	= 0,76569
Θ_{94}	$= [0; 13\overline{12}]$	= 0,78868
Θ_{95}	$= [0; \overline{13}]$	= 0,791287

Figure 6: List of Θ's.

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