1 Introductory notation

1.1 Markov and Lagrange values

Symbol $\mathcal{M}$ denotes double-infinite sequences from $\mathbb{N}^{\mathbb{Z}}$:

$$\mathcal{M} = ...a_{-2}a_{-1}a_0a_1a_2...$$

I will use $\lambda(\mathcal{M})$, $\mu(\mathcal{M})$ and $f(\mathcal{M})$ for Lagrange, Markov values and height function. Symbols γ and δ denote the lhs and rhs of sequence $\mathcal{M}$:

$$\begin{gathered}\gamma(\mathcal{M}) = [0; a_{-1}, a_{-2}, ...],\\ \delta(\mathcal{M}) = [0; a_1, a_2, ...],\\ f(\mathcal{M}) = a_0 + \gamma(\mathcal{M}) + \delta(\mathcal{M}).\end{gathered}$$

At last, symbols M and L denote the Markov and Lagrange spectra.

1.2 Centered sequence

Definition. A sequence $\mathcal{M}$ is called **centered**, if

$$\mu(\mathcal{M}) = f(\mathcal{M}). \tag{1}$$

Proposition. Markov spectrum can be defined with only centered sequences:

$$\{\mu(\mathcal{M}) \mid \mathcal{M} \in \mathbb{N}^{\mathbb{Z}}\} = M = \{\mu(\mathcal{M}) \mid \mathcal{M} \text{ is centered }\}.$$

1.3 Rectangle

Designation. Denote by

$$\{a_{i_1}, a_{i_1+1}, ..., a_{i_2}\} \ \ (i_1 \leqslant 0 \leqslant i_2)$$

the set of double-infinite sequences $\mathcal{M}$ with fixed terms $a_{i_1}, a_{i_1+1}, ..., a_{i_2}$ on the corresponding positions.

Terms a_s for $s < i_1$ and $s > i_2$ are arbitrary integers, chosen such that $\mathcal{M}$ is centered and, maybe, satisfies some conditions.

Segments Δ_1, Δ_2 and Δ are defined by the following equations:

$$\begin{array}{lclcl}\Delta_1 & = & [\,\Delta_1'\,;\,\Delta_1''\,] & = & [\,\min\gamma(\mathcal{M})\,;\,\max\gamma(\mathcal{M})\,]\,,\\ \Delta_2 & = & [\,\Delta_2'\,;\,\Delta_2''\,] & = & [\,\min\delta(\mathcal{M})\,;\,\max\delta(\mathcal{M})\,]\,,\\ \Delta & = & [\,\Delta'\,;\,\Delta''\,] & = & a_0 + \Delta_1 + \Delta_2,\end{array} \tag{2}$$

where $\mathcal{M}$ belongs to the set.

Note that we will use $'$ for the lower bound, and $''$ for the upper bound.

Definition. Rectangle is the segment Δ with the set of sequences, defining it.

1.4 Horizontal rectangle

Definition. Call a rectangle Δ **horizontal**, if

$$|\Delta_1| \geqslant |\Delta_2|. \tag{3}$$

In (3) we allow terms a_s for $s < i_1$ and $s > i_2$ to be integers $\{1, 2, 3\}$, regardless of the requirement that sequences $\mathcal{M} \in \Delta$ are centered.

In other words, Δ is horizontal, if and only if

$$\left|\left[0; a_{-1}, ..., a_{i_1}, \overline{3, 1}\right] - \left[0; a_{-1}, ..., a_{i_1}, \overline{1, 3}\right]\right| \geqslant \left|\left[0; a_1, ..., a_{i_2}, \overline{3, 1}\right] - \left[0; a_1, ..., a_{i_2}, \overline{1, 3}\right]\right|.$$

Clearly, we can always obtain a horizontal rectangle out of the vertical one, as we can reindex the sequence in the opposite direction.

1.5 Subrectangle

Consider a rectangle Δ, set by the sequence center

$$\{a_{i_1}, a_{i_1+1}, ..., a_{i_2}\}.$$

We will use a **shorter notation** for subrectangles, produced by setting integers a_i for $i < i_1$ or $i > i_2$:

$$\{b_\ell ... b_1,\ c_1 ... c_r\} \coloneqq \{b_\ell ... b_1 a_{i_1} ... a_{i_2} c_1 ... c_r\}.$$

For example:

$$\{213, 3\} \coloneqq \{213 a_{i_1} ... a_{i_2} 3\}, \tag{ex.1}$$

$$\{2, 0\} \coloneqq \{2 a_{i_1} ... a_{i_2}\}. \tag{ex.2}$$

We will also shorter the notation (2): lhs and rhs are $\Delta_1(312)$ and $\Delta_2(3)$ for subrectangle (ex.1) and $\Delta_1(2)$ and a_2 for (ex.2).

1.6 Geometrical interpretation

Consider the mapping

$$\widetilde{h} : \mathbb{N}^{\mathbb{Z}} \to \mathbb{R}^2,$$

$$\widetilde{h}(\mathcal{M}) = (\gamma(\mathcal{M}); \delta(\mathcal{M})).$$

In these terms, the Markov spectrum M is the projection of some subset $\mathcal{S} \subset C_4 \times C_4$ onto the diagonal.

Then **rectangle** Δ is indeed a rectangle $\Delta_1 \times \Delta_2$ and **subrectangles** are its subrectangles.

We will consider a family of rectangles whose projections cover the beginning of Hall's Ray.

Then we will present the algorithm to split rectangle into subrectangles so that their projections cover the projection of initial rectangle.

When we say that rectangles intersect, we, however, mean that their projections intersect.

The more «squarish» the rectangle, the easier the step.

That's why we will bound the aspect ratio of rectangles (see **good** rectangle).

Formulas to evaluate side lengths and aspect ratio are given in the section 2.

2 Calculations

2.1 Length of Δ_1 or Δ_2

Let's fix some terms of continued fraction $[0; q_1, q_2, q_3, ..., q_n]$.
We will often need to measure difference

$$[0; q_1, q_2, q_3, ..., q_n, \frac{1}{\Theta_R}] - [0; q_1, q_2, q_3, ..., q_n, \frac{1}{\Theta_L}],$$

where Θ's are some continuations of the continued fraction.
They generally look like $\Theta = \left[0; 12\overline{13}\right]$ or something[1]. We will set Θ's explicitly.
For the general proof, Θ's will be taken from Table 8.

Designation. For given continuation Θ_i, denote by ε_i the resulting continued fraction:

$$\varepsilon_i = [0; q_1, q_2, ..., q_n, \frac{1}{\Theta_i}]. \tag{4}$$

Then the following equality takes place:

$$|\varepsilon_i - \varepsilon_j| = \frac{|\Theta_i - \Theta_j|}{Q_n^2\,(1 + pQ_i)\,(1 + pQ_j)}, \tag{5}$$

where

$$p = \frac{Q_{n-1}}{Q_n}.$$

2.2 Rectangle aspect ratio

Consider some fixed center of rectangle $\{a_{i_1}...a_{i_2}\}$.
We will often extend it from the left (right) using continuations $\Theta_{\gamma_1}, \Theta_{\gamma_2}$ ($\Theta_{\delta_1}, \Theta_{\delta_2}$).
Rule (4) produces γ_1, γ_2 (δ_1, δ_2).
Finite continued fractions $\frac{P_{i_1}}{Q_{i_1}} = [0; a_{-1}, a_{-2}, ..., a_{i_1}]$ $\left(\frac{P_{i_2}}{Q_{i_2}} = [0; a_1, a_2, ..., a_{i_2}]\right)$ are their convergents.
Then

$$\left|\frac{\gamma_1 - \gamma_2}{\delta_1 - \delta_2}\right| = \frac{1}{q}\left|\frac{\Theta_{\gamma_1} - \Theta_{\gamma_2}}{\Theta_{\delta_1} - \Theta_{\delta_2}}\right| \frac{1 + p'\Theta_{\delta_1}}{1 + p\Theta_{\gamma_1}}\,\frac{1 + p'\Theta_{\delta_2}}{1 + p\Theta_{\gamma_2}} \approx \frac{1}{q}\left|\frac{\Theta_{\gamma_1} - \Theta_{\gamma_2}}{\Theta_{\delta_1} - \Theta_{\delta_2}}\right|, \tag{6}$$

where

$$p = \frac{Q_{i_1+1}}{Q_{i_1}}, \;\; p' = \frac{Q_{i_2-1}}{Q_{i_2}}, \;\; q = \frac{Q_{i_1}^2}{Q_{i_2}^2}.$$

[1] Here, as always in this book, $\overset{k}{\overline{abc}}$ means k-times repetition of abc, and $\overline{abc}$ means infinite repetition.

3 Rectangle boundaries

We will set boundaries for rectangles in a different way.

To distinguish rectangles, we will introduce the notion of left- or right-shortened rectangles.

It is given in subsections 3.1-3.3.

Subsections 3.4-3.5 provide the rules for boundaries Δ_1', Δ_1'', Δ_2', Δ_2''.

3.1 Resection

Definition. Call **resection** of a segment $A = [a; b]$ a process of removing subsegment $A_{12} = [a_1; b_1]$, leaving two segments $A_1 \sqcup A_2 = [a; a_1] \sqcup [b_1; b]$.

Definition. Call subsegment $A_{12} \subset A$ **normal**, if it is thicker than the two remaining subsegments:

$$|A_{12}| \leqslant min\,\{|A_1|,\ |A_2|\} \tag{7}$$

We call a resection **normal** if the resected subsegment is normal.

Proposition. For any normal resection, having

$$A + A = (A_1 \sqcup A_2) + (A_1 \sqcup A_2). \tag{8}$$

3.2 Shortened rectangle

Consider a horizontal rectangle Δ.

3.2.1 Case $i_1 \equiv i_2 \equiv 0 \pmod 2$

If both i_1 and i_2 are even, then left- and right-shortened rectangles are defined as follows.

Definition. Rectangle Δ is called left-shortened, if for subrectangle $\{3, 3\}$ the following condition takes place:

$$|\Delta_1(3)| \leqslant 1.4 \cdot |\Delta_2(3)|\,. \tag{9}$$

Here, as in (3), we allow terms a_s for $s < i_1$ and $s > i_2$ to be integers $\{1, 2, 3\}$, so (9) can be rewritten as

$$\left|\left[0; a_{-1}, ..., a_{i_1}, 3, \overline{3, 1}\right] - \left[0; a_{-1}, ..., a_{i_1}, \overline{3, 1}\right]\right| \leqslant \left|\left[0; a_1, ..., a_{i_2}, 3, \overline{3, 1}\right] - \left[0; a_1, ..., a_{i_2}, \overline{3, 1}\right]\right|.$$

For further convenience, we also introduce the opposite to (9) condition:

Definition. Rectangle Δ is called left-normal, if

$$|\Delta_1(3)| > 1.4 \cdot |\Delta_2(3)|\,. \tag{10}$$

Now we introduce the notion of right-shortened rectangle:

Definition. Rectangle Δ is called right-shortened, if for subrectangle $\{31, 13\}$ the following condition takes place:

$$|\Delta_1(13)| \leqslant 1.4 \cdot |\Delta_2(13)|\,. \tag{11}$$

As in 3.2.1, we allow terms a_s for $s < i_1$ and $s > i_2$ to be integers $\{1, 2, 3\}$, so (11) can be rewritten as

$$\left|[0; a_{-1}, ..., a_{i_1}, 1, 3, \overline{3, 1}] - [0; a_{-1}, ..., a_{i_1}, \overline{1, 3}]\right| \leqslant \left|[0; a_1, ..., a_{i_2}, 1, 3, \overline{3, 1}] - [0; a_1, ..., a_{i_2}, \overline{1, 3}]\right|.$$

We also introduce the opposite condition:

Definition. Rectangle Δ is called right-normal, if

$$|\Delta_1(13)| > 1.4 \cdot |\Delta_2(13)|. \tag{12}$$

3.2.2 Other cases

If $i_1 \equiv i_2 \pmod 2$ and i_1 is odd, then the conditions for left- and right-shortened rectangles are swapped.

In case $i_1 \not\equiv i_2 \pmod 2$, the notions of left- and right-shortened rectangles are inferred from subrectangles $\{1,\}$, $\{2,\}$, and $\{3,\}$.

We will refer to the relevant conditions from 3.5

3.3 Explanation of shortened

Notion left-shortened is used to set the lower bound Δ' for rectangle Δ.

Notion right-shortened is used to set the upper bound Δ''.

For example, when interested in Δ' when i_1 is even and i_2 is odd, one has to consider larger a_{i_1-1} and smaller a_{i_2+1}. As the rectangle is horizontal, we will add terms to the left first. So we will check the condition (11) for $\{2,\}$ or $\{3,\}$.

In other words, check whether $\{2,\}$ or $\{3,\}$ is left-shortened or left-normal (recall that $i_1 - 1$ and i_2 are odd, so left-shortenence is defined by (11)).

Which subrectangle ($\{2,\}$ or $\{3,\}$) to take, however, depends on terms a_{i_1+1} and a_{i_1} (see 3.4).

3.4 How we ensure centeredness

Consider integers $q_1, q_2, ..., q_n$. Let $\{\delta_n\}$ be the set of fractions $\delta_n = [0; q_1, q_2, ..., q_n, ...]$ with n fixed terms. We will suppose that n is even (for odd n the bounds are swapped).

At first, determine the smallest of fractions δ_n.

We will consider 2 cases: S (<u>S</u>hortened) and N (<u>N</u>ormal):

$$S.\ q_{n-1} = 3, q_n = 1. \tag{13a}$$
$$N.\ \text{Otherwise.} \tag{13b}$$

The lower bound δ'_n for segment, containing δ_n, is defined by:

$$\begin{array}{ll} S. & \delta'_n = [0; q_1, ..., q_n, 213\overline{12}]. \\ N. & \delta'_n = [0; q_1, ..., q_n, 3\overline{12}]. \end{array}$$

To set the upper bound δ''_n, largest of δ_n, consider 2 other cases:

$$S. \quad q_n = 3. \tag{14a}$$
$$N. \quad q_n \neq 3. \tag{14b}$$

Then

$$\begin{array}{ll} S. & \delta''_n = [0; q_1, ..., q_n, 121\overline{3\overline{12}}]. \\ N. & \delta''_n = [0; q_1, ..., q_n, 1\overline{3\overline{12}}]. \end{array}$$

These bounds will allow us to construct sequences $\mathcal{M}$, for which combination (31313) is forbidden and, therefore, the following condition takes place:

$$f_i(\mathcal{M}) \leqslant \lambda(\overline{31312}) \approx 4,5241, \ i \neq 0,$$

which will ensure (1).

3.5 Formal

Let's now turn to concrete definitions and bounds. Remind that we are looking at the horizontal rectangle Δ.

Suppose that $i_1 \equiv i_2 \pmod 2$, i_1 is even. (If i_1 is odd, then Δ' and Δ'' are swapped.)

3.5.1 Bounds for Δ'

I. Suppose both $\{\gamma_0(\mathcal{M})\}$ and $\{\delta_0(\mathcal{M})\}$ satisfy (13b) and the rectangle Δ is left-normal, that is, satisfies (10). We will denote such situation by $N - N - N$ (segment Δ_1 is left-normal, Δ_2 is left-normal, rectangle Δ is normal).

   In this case define Δ' by equation

$$\Delta' = f(\overline{21}3a_{i_1}...a_{i_2}3\overline{12}). \tag{15}$$

IIa. Sets $\{\gamma_0(\mathcal{M})\}$ and $\{\delta_0(\mathcal{M})\}$ meet condition (13b) and Δ is left-shortened, that is, meets condition (9). This is case $N - N - S$ (segments Δ_1 and Δ_2 are left-normal, rectangle Δ is left-shortened).

   Then

$$\Delta' = f(\overline{21}3a_{i_1}...a_{i_2}213\overline{12}). \tag{16}$$

IIb. Set $\{\gamma_0(\mathcal{M})\}$ meets (13b), $\{\delta_0(\mathcal{M})\}$ meets (13a). No matter which condition (9) of (10) is met. It is the case $N - S$ (segment Δ_1 is left-normal, segment Δ_2 left-shortened). Bound Δ' is defined by (16).

III. Set $\{\gamma_0(\mathcal{M})\}$ meets (13a), $\{\delta_0(\mathcal{M})\}$ meets (13b). In this $S - N$ case (segment Δ_1 is left-shortened, segment Δ_2 is left-normal) Δ' is defined as follows:

$$\Delta' = f(\overline{21}312a_{i_1}...a_{i_2}3\overline{12}). \tag{17}$$

IV. Both $\{\gamma_0(\mathcal{M})\}$ and $\{\delta_0(\mathcal{M})\}$ meet (13a). In this $S - S$ case (both segments Δ_1 and Δ_2 are left-shortened) Δ' is defined by

$$\Delta' = f(\overline{21}312a_{i_1}...a_{i_2}213\overline{12}). \tag{18}$$

Left side of the figure 1 illustrates these bounds.

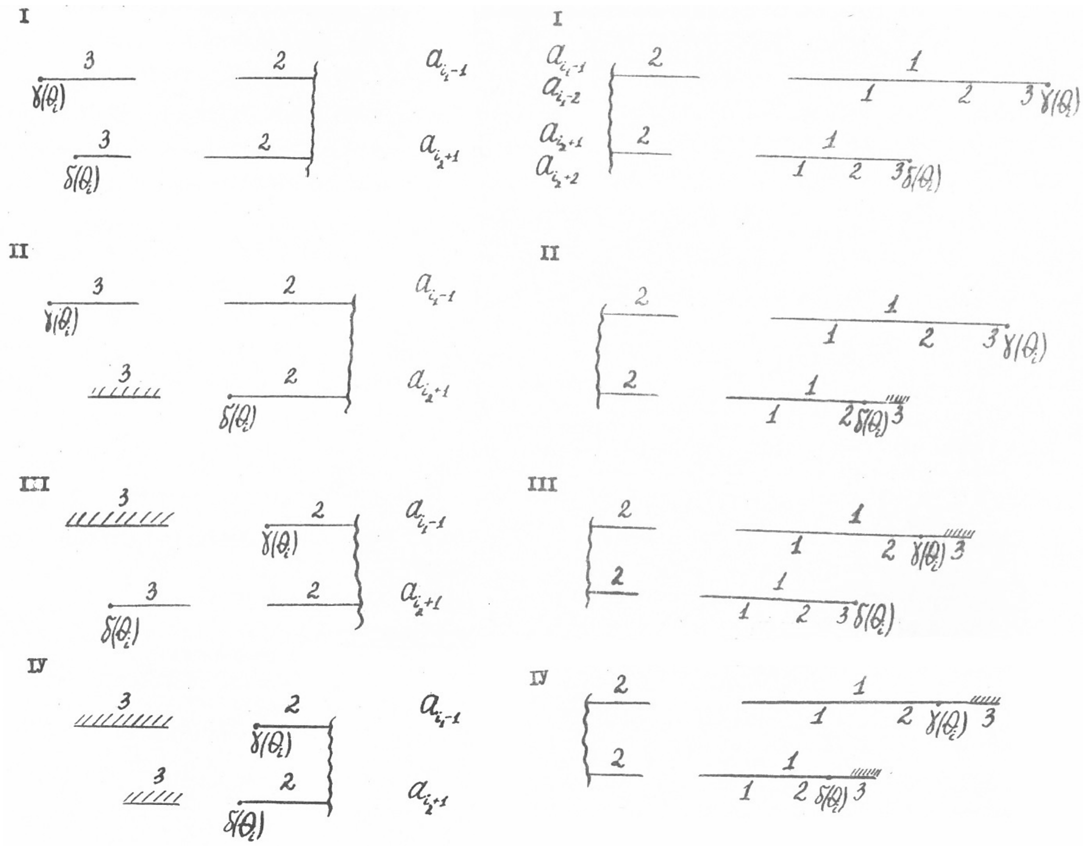


Figure 1: Bounds Δ' (left) and Δ'' (right).
Hatched areas correspond to values of a_{i_1-1} and a_{i_2+1} (on the left) or a_{i_1-2} and a_{i_2+2} (on the right), equal to 3, which can not appear in the concrete case.
$\Delta'_1 = \gamma(\Theta_i)$, $i = 3, 30$, $\Delta'_2 = \delta(\Theta_i)$, $i = 3, 30$, $\Delta' = \Delta'_1 + \Delta'_2$.
$\Delta''_1 = \gamma(\Theta_i)$, $i = 90, 94$, $\Delta''_2 = \delta(\Theta_i)$, $i = 90, 94$.

3.5.2 Bounds for Δ''

Now we will provide formulas for Δ'':

$$
\begin{array}{llr}
I & \Delta'' = f(\overline{21}31a_{i_1}...a_{i_2}13\overline{12}), & \text{(19a)} \\
II & \Delta'' = f(\overline{21}31a_{i_1}...a_{i_2}12131\overline{2}), & \text{(19b)} \\
III & \Delta'' = f(\overline{2}13121a_{i_1}...a_{i_2}13\overline{12}), & \text{(19c)} \\
IV & \Delta'' = f(\overline{2}13121a_{i_1}...a_{i_2}12131\overline{2}). & \text{(19d)}
\end{array}
$$

Figure (2) regulates the choice of the formulas.

Case	Δ''_1	Δ''_2	Rectangle	Δ''
I	(14b)	(14b)	(12) $N-N-N$	(19a)
IIa	(14b)	(14b)	(11) $N-N-S$	(19b)
IIb	(14b)	(14a)	$N-S$	(19b)
III	(14a)	(14b)	$S-N$	(19c)
IV	(14a)	(14a)	$S-S$	(19d)

Figure 2: Rules for choice of Δ'' in case $i_1 \equiv i_2 \pmod 2$.

Again, these bounds are illustrated on the right side of the figure 1.

3.5.3 Case $i_1 \not\equiv i_2 \pmod 2$

Now consider case $i_1 \not\equiv i_2 \pmod 2$, i_1 is even. We will use rules from figure (3) to choose one of 4 formulas for Δ'.

$$
\begin{array}{llr}
I & \Delta' = f(\overline{21}3a_{i_1}...a_{i_2}13\overline{12}), & \text{(20a)} \\
II & \Delta' = f(\overline{21}3a_{i_1}...a_{i_2}12131\overline{2}), & \text{(20b)} \\
III & \Delta' = f(\overline{21}312a_{i_1}...a_{i_2}13\overline{12}), & \text{(20c)} \\
IV & \Delta' = f(\overline{21}312a_{i_1}...a_{i_2}12131\overline{2}). & \text{(20d)}
\end{array}
$$

Case	Δ''_1	Δ''_2	Case name	Δ''
I	(13b)	(14b)	$N-N$	(20a)
II	(13b)	(14a)	$N-S$	(20b)
III	(13a)	(14b)	$S-N$	(19c)
IV	(13a)	(14a)	$S-S$	(19d)

Figure 3: Rules for choice of Δ' in case $i_1 \not\equiv i_2 \pmod 2$, i_1 is even.

To determine the bound Δ'', we will use subrectangle

$$\{1, 0\}.$$

Having $i_1 - 1 \equiv i_2 \pmod 2$, so we can use all the previous formulas to determine Δ''.

4 Good rectangle

4.1 Definition

Consider a horizontal rectangle $\Delta = \{a_{i_1} \ldots a_{i_2}\}$.

Definition. Rectangle Δ is called **good**, if subrectangles $\{2, 0\}$ and $\{1, 0\}$ intersect.

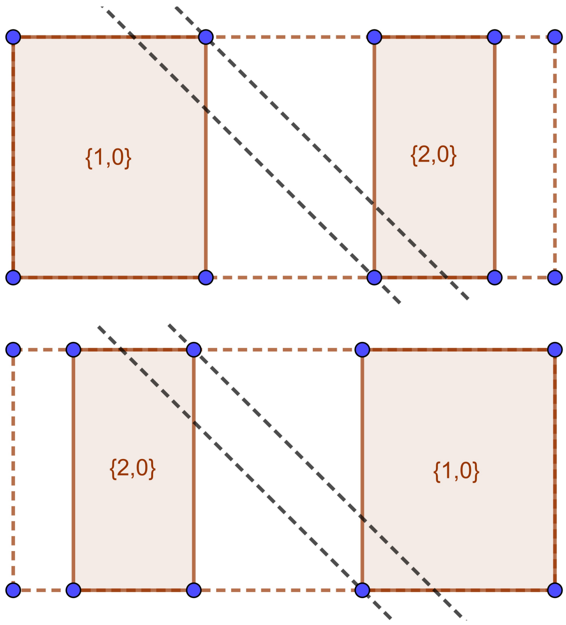


Figure 4: Good rectangles if i_1 is odd (upper) and even (lower).

For example, in case i_1 is even, goodness is equivalent to

$$\{2, 0\}'' \geqslant \{1, 0\}'. \tag{21}$$

Bounds of rectangles are determined by the rules from section 3.

If the rectangle Δ is not good (for example, (21) doesn't take place), then

$$(\{2, 0\}''; \{1, 0\}') \not\subset \Delta.$$

Clearly, if the rectangle is not good, then one can not split it into smaller subrectangles whose projections cover the projection of initial one. That's why we will only consider good rectangles during the proof.

4.2 Sufficient conditions of goodness

4.2.1 Results

A horizontal rectangle Δ is good, if

$$\frac{\Delta_1}{\Delta_2} < \begin{cases} 3.8, & i_1 \equiv i_2 \pmod 2, \quad \{\delta(\mathcal{M})\} \text{ meets (13b)}, \\ 3.43, & i_1 \not\equiv i_2 \pmod 2, \quad \{\delta(\mathcal{M})\} \text{ meets (13b)}, \\ 2.9, & \text{otherwise}. \end{cases} \tag{22}$$

4.2.2 Universal $2,9$ bound

We will introduce a sufficient conditions for a rectangle to be good.

Designation. For given continuations Θ_γ and Θ_δ, denote by γ and δ the resulting continued fractions:

$$\gamma = [0; a_{-1}, ..., a_{i_1}, \frac{1}{\Theta_\gamma}],$$
$$\delta = [0; a_1, ..., a_{i_2}, \frac{1}{\Theta_\delta}].$$

As stated in 1.1, γ's correspond to the lhs of $\mathcal{M}$, and δ's correspond to the rhs of $\mathcal{M}$. Variables Θ will be taken from the Figure 8 and will be specified in each case separately.

Suppose that i_1 is even.
Remind that $|\Delta_1| \geqslant |\Delta_2|$, which means that

$$\frac{|\gamma'' - \gamma'|}{|\delta'' - \delta'|} \geqslant 1,$$

where

$$\Theta_{\gamma'} = \Theta_{\delta'} = \Theta_1,\ \Theta_{\gamma''} = \Theta_{\delta''} = \Theta_{95}.$$

Using (6), rewrite it as

$$q\frac{1+p\Theta_1}{1+p'\Theta_1}\frac{1+p\Theta_{95}}{1+p'\Theta_{95}} \leqslant 1. \tag{23}$$

Goodness can be written as

$$\gamma' - \gamma'' < |\delta' - \delta''|, \tag{24}$$

where $\gamma' = \Delta_1(1)'$ and $\gamma'' = \Delta_1(2)''$.
For an arbitrary rectangle Δ, taking

$$\Theta_{\gamma'} = \Theta_{66},\ \Theta_{\gamma''} = \Theta_{63},\ \Theta_{\delta'} = \Theta_{90},\ \Theta_{\delta''} = \Theta_{30}.$$

Inequality (24) transforms into

$$0,313 < q\frac{1+p\Theta_{63}}{1+p'\Theta_{30}}\frac{1+p\Theta_{66}}{1+p'\Theta_{90}}. \tag{25}$$

Suppose that

$$\frac{\Delta_1}{\Delta_2} < 2,9, \tag{26}$$

which is equivalent to

$$\frac{1}{q}\frac{1+p'\Theta_1}{1+p\Theta_1}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} < 2,9. \tag{27}$$

Then (25) takes place. Indeed, it is so, if

$$\frac{1}{2,9}\frac{1+p'\Theta_1}{1+p\Theta_1}\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} > 0,313\frac{1+p'\Theta_{30}}{1+p\Theta_{63}}\frac{1+p'\Theta_{90}}{1+p\Theta_{66}}.$$

or, equivalent,

$$1,1 > \frac{1+p\Theta_1}{1+p\Theta_{63}} \frac{1+p\Theta_{95}}{1+p\Theta_{66}} \frac{1+p'\Theta_{30}}{1+p'\Theta_1} \frac{1+p'\Theta_{90}}{1+p'\Theta_{95}},$$

which is checked directly.

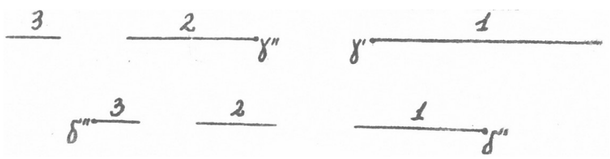
Condition (26) is sufficient for rectangle to be good, regardless of the parity of i_1 and i_2 and the left or right shortness or normalness of rectangle.

4.2.3 Case $i_1 \equiv i_2 \pmod 2$

Now consider case $i_1 \equiv i_2 \pmod 2$.

Suppose that the right hand side $\{\delta(\mathcal{M})\}$ is left-normal, that is, meets (13b).

Equivalent, the case $a_{i_2} = 1$, $a_{i_2-1} = 3$ doesn't take place.



This assumption allows us to substitute the folliwing γ's and δ's into (24):

$$\Theta_{\gamma'} = \Theta_{66},\ \Theta_{\gamma''} = \Theta_{63},\ \Theta_{\delta'} = \Theta_{90},\ \Theta_{\delta''} = \Theta_3.$$

The rhs left-normality leaves us hope that the bound Δ_2'' will be set to $\Delta_2(\Theta_3)$, so we are substituting $\Theta_{\delta''}$ with Θ_3 instead of Θ_{30}.

Now, instead of (25) we will get

$$0,253 < q\,\frac{1+p\Theta_{63}}{1+p'\Theta_3} \frac{1+p\Theta_{66}}{1+p'\Theta_{90}}. \tag{28}$$

Such choice of $\Theta_{\gamma'}$ and $\Theta_{\delta''}$ is fine, if the following inequality takes place:

$$\delta' - \delta'' > 1,4(\gamma' - \gamma''), \tag{29}$$

where

$$\Theta_{\gamma'} = \Theta_{68},\ \Theta_{\gamma''} = \Theta_{65},\ \Theta_{\delta'} = \Theta_{28},\ \Theta_{\delta''} = \Theta_1,$$

so we can rewrite (29) as

$$0,253 < q\,\frac{1+p\Theta_{65}}{1+p'\Theta_1} \frac{1+p\Theta_{68}}{1+p'\Theta_{28}}. \tag{30}$$

We can notice that (30) follows from (28). Indeed, that follows from the inequality

$$0,253\,\frac{1+p'\Theta_3}{1+p'\Theta_{63}} \frac{1+p\Theta_{90}}{1+p\Theta_{66}} > 0,269\,\frac{1+p'\Theta_1}{1+p'\Theta_{65}} \frac{1+p\Theta_{28}}{1+p\Theta_{68}},$$

or inequality

$$\frac{1+p\Theta_{65}}{1+p\Theta_{63}} \frac{1+p\Theta_{68}}{1+p\Theta_{66}} \frac{1+p'\Theta_3}{1+p'\Theta_1} \frac{1+p'\Theta_{90}}{1+p'\Theta_{28}} > 1,064.$$

Overall, we have proved that (28) is enough for rectangle to be good.

Suppose that

$$\frac{\Delta_1}{\Delta_2} < 3,8, \tag{31}$$

or

$$\frac{1}{q}\,\frac{1+p'\Theta_1}{1+p\Theta_1}\,\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} < 3,8.$$

Then (28) takes place. Indeed, it is so, if

$$\frac{1}{3,8}\,\frac{1+p'\Theta_1}{1+p\Theta_1}\,\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} > 0,253\,\frac{1+p'\Theta_3}{1+p'\Theta_{63}}\,\frac{1+p\Theta_{90}}{1+p\Theta_{66}}.$$

The last inequality is transformed into

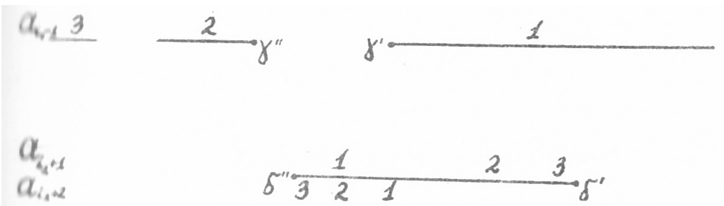
$$1,04 > \frac{1+p'\Theta_3}{1+p'\Theta_1}\,\frac{1+p'\Theta_{90}}{1+p'\Theta_{95}}\,\frac{1+p\Theta_1}{1+p\Theta_{63}}\,\frac{1+p\Theta_{95}}{1+p\Theta_{66}},$$

which is easily checked.

4.2.4 Case $i_1 \not\equiv i_2 \pmod 2$

Now consider case $i_1 \not\equiv i_2 \pmod 2$ and, again, the right hand side $\{\delta(\mathcal{M})\}$ is left-normal, in other words, case $a_{i_2} = 1$, $a_{i_2-1} = 3$ doesn't take place.

Suppose that i_1 is even.



Substituting

$$\Theta_{\gamma'} = \Theta_{66},\ \Theta_{\gamma''} = \Theta_{59},\ \Theta_{\delta'} = \Theta_3,\ \Theta_{\delta''} = \Theta_{90}$$

into (24), obtain

$$0,2885 < q\,\frac{1+p\Theta_{59}}{1+p'\Theta_3}\,\frac{1+p\Theta_{66}}{1+p'\Theta_{90}}. \tag{32}$$

Suppose that

$$\frac{\Delta_1}{\Delta_2} < 3,43 \tag{33}$$

or

$$\frac{1}{q}\,\frac{1+p'\Theta_1}{1+p\Theta_1}\,\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} < 3,43.$$

Then (32) takes place. Indeed, it is so, if

$$\frac{1}{3,43}\,\frac{1+p'\Theta_1}{1+p\Theta_1}\,\frac{1+p'\Theta_{95}}{1+p\Theta_{95}} > 0,2885\,\frac{1+p'\Theta_3}{1+p\Theta_{59}}\,\frac{1+p'\Theta_{90}}{1+p\Theta_{66}}.$$

or

$$1,0105 > \frac{1+p'\Theta_3}{1+p'\Theta_1}\,\frac{1+p'\Theta_{90}}{1+p'\Theta_{95}}\,\frac{1+p\Theta_1}{1+p\Theta_{59}}\,\frac{1+p\Theta_{95}}{1+p\Theta_{66}}.$$

Increasing the right part, obtain

$$1,0105 > 0,9896\,\frac{1+p\cdot 1,0551+p^2\cdot 0,20875}{1+p\cdot 0,983+p^2\cdot 0,237},$$

which is easily checked.

5 Initial set of rectangles

We present a set of rectangles whose projections cover the segment from the Freiman's constant to $\sqrt{21}$. All of them are good. Further sections will present the algorithm of splitting these rectangles into subrectangles correctly.

1) $\{\,3\underset{o}{4}3\,\}$

2) $\{\,313\underset{o}{4}312\,\}$

3) $\{\,123\underset{o}{4}431\,\}$

4) $\{\,313\underset{o}{4}313\,\}$

5) $\{\,21123\underset{o}{4}4322\,\}$

6) $\{\,31123\underset{o}{4}4323\,\}$

7) $\{\,31123\underset{o}{4}4322\,\}$

$8+2n,k)$ $\{\,\overset{k}{\overline{3}}13\overset{n}{\overline{121313}}1123\underset{o}{4}43222\overset{n}{\overline{313121}}\overset{k}{\overline{3}}\,\}$ (13.1)

$9+2n,k)$ $\{\,\overset{k-1}{\overline{3}}21313\overset{n}{\overline{121313}}1123\underset{o}{4}43222\overset{n}{\overline{313121}}3\overset{k}{\overline{13}}\,\}$ (13.2)

$9+2n,k,p)$ $\{\,\overset{p}{\overline{3}}\;\overset{k}{\overline{13}}\;21313\overset{n}{\overline{121313}}1123\underset{o}{4}43222\overset{n}{\overline{313121}}31\overset{k}{\overline{3}}2\overset{p}{\overline{13}}\,\}$ (13.3)

Figure 5: Initial set of rectangles.

6 Induction, $i_1 \not\equiv i_2 \pmod 2$

Let $\Delta = \{a_{i_1}, ..., a_{i_2}\}$ be a horizontal rectangle, satisfying conditions from sections 3 and 4.

This section deals with the case $i_1 \not\equiv i_2 \pmod 2$.

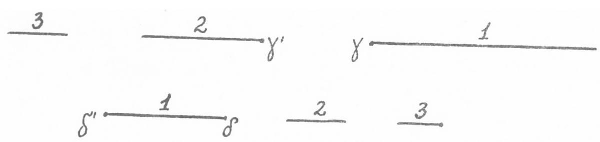
It is easier, because in this case both the lower-left and upper-right corners are filled with the "wide" rectangles $\{1,\}$ and $\{,1\}$ (see figure 6). In case $i_1 \equiv i_2 \pmod 2$ the lower-left corner, filled with $\{3,3\}$, will be a pain in the neck.

Without loss of generality, we may assume that i_1 is even and, therefore, i_2 is odd. As the reader will see, the agrument is repeated for the opposite parity, up to differences in illustrations.

6.1 Case $\{,1\}$ is good

At first, suppose that the subrectangle $\{,1\}$ is good.

It is so, if



$$\gamma - \gamma' < \delta - \delta', \tag{34}$$

where

$$\Theta_\gamma = \Theta_{66},\ \Theta_{\gamma'} = \Theta_{63},\ \Theta_\delta = \Theta_{70},\ \Theta_{\delta'} = \Theta_{90},$$

which can be rewritten with (6) as follows:

$$0,74 < q\,\frac{1 + p\Theta_{63}}{1 + p'\Theta_{66}}\,\frac{1 + p\Theta_{70}}{1 + p'\Theta_{90}}. \tag{35}$$

Constants Θ_γ and Θ_δ are chosen to ensure that

$$[\delta, \delta'] \subset \Delta_2,$$

regardless of the rules from the section 3 defining Δ_2' and Δ_2''.

From (35) it follows that the original rectangle Δ is contained within the union of the subrectangles $\{1,\}$ and $\{,1\}$.

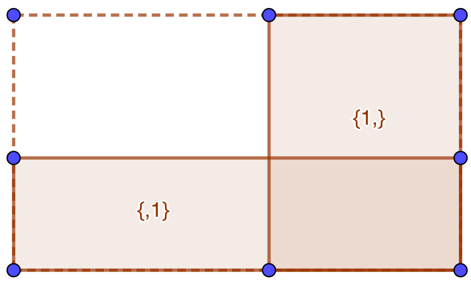


Figure 6: Subrectangles $\{1,\}$ and $\{,1\}$.

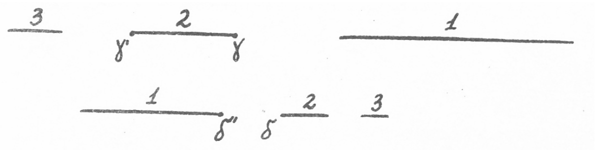
Now we will try contradiction and suppose that

$$0,74 \geqslant q\,\frac{1+p\Theta_{63}}{1+p'\Theta_{66}}\,\frac{1+p\Theta_{70}}{1+p'\Theta_{90}}\,. \tag{36}$$

We will use the inequality (36) until the end of the section.

6.2 Case Δ is left-shortened

Now suppose that the following inequality takes place:



$$\gamma-\gamma' > \delta-\delta',$$

where

$$\Theta_\gamma=\Theta_{63},\ \Theta_{\gamma'}=\Theta_{35},\ \Theta_\delta=\Theta_{63},\ \Theta_{\delta'}=\Theta_{70},$$

which can be rewritten as

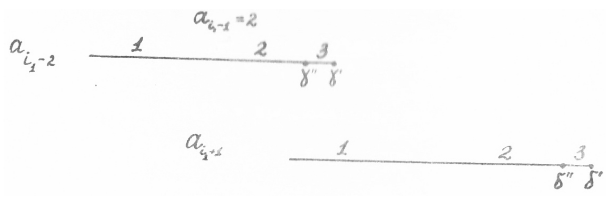
$$0,558 > q\,\frac{1+p\Theta_{35}}{1+p'\Theta_{63}}\,\frac{1+p\Theta_{63}}{1+p'\Theta_{70}}\,. \tag{37}$$

If the rectangle Δ is of type $S-S$ or $S-N$, that is, the lower bound Δ' is defined with (20c) or (20d), then the splitting is finished, as Δ is contained within the union of $\{1,\}$ and $\{2,\}$.

6.3 Case Δ is left-normal

Now turn to case if (37) takes place, and Δ is of type $N-S$ or $N-N$, so Δ' is set with one of the rules (20a) or (20b).

Let's see when the subrectangle $\{3,11\}$ is right-normal. The following condition should take place:



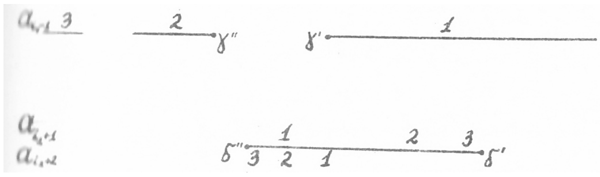
$$\gamma-\gamma' > 1,4(\delta-\delta'),$$

where

$$\Theta_\gamma=\Theta_{28},\ \Theta_{\gamma'}=\Theta_{22},\ \Theta_\delta=\Theta_{68},\ \Theta_{\delta'}=\Theta_{65},$$

which can be written as

$$0,345 > q\,\frac{1+p\Theta_{22}}{1+p'\Theta_{65}}\,\frac{1+p\Theta_{28}}{1+p'\Theta_{68}}\,. \tag{38}$$



If the condition (38) takes place, then the inequality

$$\gamma - \gamma' > \delta - \delta',$$

where

$$\Theta_\gamma = \Theta_{25},\ \Theta_{\gamma'} = \Theta_3,\ \Theta_\delta = \Theta_{66},\ \Theta_{\delta'} = \Theta_{63},$$

which can be written as

$$0,322 > q\,\frac{1+p\Theta_3}{1+p'\Theta_{63}}\,\frac{1+p\Theta_{25}}{1+p'\Theta_{66}}\,. \tag{39}$$

All in all, if rectangle Δ satisfies both conditions (38) and (39), then it can be split into $\{1,\}$, $\{2,\}$ and $\{3,\}$.

Now suppose that at least one of (38) and (39) doesn't take place. We will show that then we can split Δ into $\{1,\}$, $\{2,\}$ and $\{3,1\}$.

First, we need to check goodness of subcovering $\{3,1\}$. We will prove that it meets the sufficient condition (33). We need to show that

$$3,43(\gamma - \gamma') > |\delta - \delta'|\,, \tag{40}$$

where

$$\Theta_\gamma = \Theta_{28},\ \Theta_{\gamma'} = \Theta_1,\ \Theta_\delta = \Theta_{95},\ \Theta_{\delta'} = \Theta_{65}.$$

Inequality (40) is equivalent to

$$0,628 > q\,\frac{1+p\Theta_1}{1+p'\Theta_{65}}\,\frac{1+p\Theta_{28}}{1+p'\Theta_{95}}\,.$$

Recalling (37), we have to show that

$$0,558\,\frac{1+p'\Theta_{63}}{1+p\Theta_{35}}\,\frac{1+p'\Theta_{70}}{1+p\Theta_{63}} < 0,627\,\frac{1+p'\Theta_{65}}{1+p\Theta_1}\,\frac{1+p'\Theta_{95}}{1+p\Theta_{28}}\,,$$

and it is true.

Now show that the subrectangles $\{3,1\}$ and $\{2,1\}$ intersect.

First, let $a_{i_2} = 3$, so Δ_2' is set with the formula (20b).

The following inequality should take place:

$$\gamma - \gamma' < \delta - \delta', \tag{41}$$

where

$$\Theta_\gamma = \Theta_{30},\ \Theta_{\gamma'} = \Theta_{25},\ \Theta_\delta = \Theta_{70},\ \Theta_{\delta'} = \Theta_{90},$$

which is transformed into

$$0,328 < q\,\frac{1+p\Theta_{25}}{1+p'\Theta_{70}}\,\frac{1+p\Theta_{30}}{1+p'\Theta_{90}}\,. \tag{42}$$

If (38) doesn't take place, then we need the following inequality for (42):

$$0,345\,\frac{1+p'\Theta_{65}}{1+p\Theta_{22}}\,\frac{1+p'\Theta_{68}}{1+p\Theta_{28}} > 0,328\,\frac{1+p'\Theta_{70}}{1+p\Theta_{25}}\,\frac{1+p'\Theta_{90}}{1+p\Theta_{30}}$$
$$\Updownarrow$$
$$1,05 > \frac{1+p'\Theta_{70}}{1+p'\Theta_{65}}\,\frac{1+p'\Theta_{90}}{1+p'\Theta_{68}}\,\frac{1+p\Theta_{22}}{1+p\Theta_{25}}\,\frac{1+p\Theta_{28}}{1+p\Theta_{30}}.$$

From $a_{i_2}=3$ it follows that $p'\leqslant\frac{1}{3}$. Setting $p'=\frac{1}{3}$ and $p=\frac{1}{4}$ in the rhs, increasing it and obtaining the correct inequality. Thus, it is correct.

Now suppose that (38) takes place, while (39) doesn't. Setting $\Theta_\delta=\Theta_{66}$ (instead of Θ_{70}) in (41), get the following inequality:

$$0,288 < q\,\frac{1+p\Theta_{25}}{1+p'\Theta_{66}}\,\frac{1+p\Theta_{30}}{1+p'\Theta_{90}}. \tag{43}$$

Inequality (43) follows from denial of inequality (39), if

$$0,322\,\frac{1+p'\Theta_{63}}{1+p\Theta_{3}}\,\frac{1+p'\Theta_{66}}{1+p\Theta_{25}} > 0,298\,\frac{1+p'\Theta_{66}}{1+p\Theta_{25}}\,\frac{1+p'\Theta_{90}}{1+p\Theta_{30}}$$
$$\Updownarrow$$
$$1,08 > \frac{1+p\Theta_{3}}{1+p\Theta_{30}}\,\frac{1+p'\Theta_{90}}{1+p'\Theta_{63}}.$$

Setting $p=\frac{1}{4}$ and $p'=\frac{1}{3}$, easily check the last inequality.

6.4 Case we can consider $\{2,1\}$

Now consider the case, when $a_{i_2}\neq 3$ and Δ'_1 is defined with (20a).

For the subrectangle $\{2,1\}$ left-normality we need

$$1,4(\gamma-\gamma') < \|\delta-\delta'\|, \tag{44}$$

where

$$\Theta_\gamma=\Theta_{33},\ \Theta_{\gamma'}=\Theta_{29},\ \Theta_\delta=\Theta_{95},\ \Theta_{\delta'}=\Theta_{92},$$

so (44) can be rewritten as

$$0,182 < q\,\frac{1+p\Theta_{29}}{1+p'\Theta_{92}}\,\frac{1+p\Theta_{33}}{1+p'\Theta_{95}}. \tag{45}$$

If neither (38) nor (39) takes place, then it's easy to check (45).

In (41) we can set $\Theta_{\gamma'}=\Theta_{94}$ (instead of Θ_{90}), so (41) is transformed into

$$0,248 < q\,\frac{1+p\Theta_{25}}{1+p'\Theta_{70}}\,\frac{1+p\Theta_{30}}{1+p'\Theta_{94}}. \tag{46}$$

If (38) doesn't take place, then for (46) we will check the following:

$$0,345\,\frac{1+p'\Theta_{65}}{1+p\Theta_{22}}\,\frac{1+p'\Theta_{68}}{1+p\Theta_{28}} > 0,248\,\frac{1+p'\Theta_{70}}{1+p\Theta_{25}}\,\frac{1+p'\Theta_{94}}{1+p\Theta_{30}}$$
$$\Updownarrow$$
$$1,39 > \frac{1+p'\Theta_{70}}{1+p'\Theta_{65}}\,\frac{1+p'\Theta_{94}}{1+p'\Theta_{68}}\,\frac{1+p\Theta_{22}}{1+p\Theta_{25}}\,\frac{1+p\Theta_{28}}{1+p\Theta_{30}},$$

which is clear.

If (39) doesn't take place, then for (46) we will check

$$0,322\,\frac{1+p'\Theta_{63}}{1+p\Theta_{3}}\,\frac{1+p'\Theta_{66}}{1+p\Theta_{25}} > 0,248\,\frac{1+p'\Theta_{70}}{1+p\Theta_{25}}\,\frac{1+p'\Theta_{94}}{1+p\Theta_{30}}$$
$$\Updownarrow$$
$$1,29 > \frac{1+p\Theta_{3}}{1+p\Theta_{30}}\,\frac{1+p'\Theta_{70}}{1+p'\Theta_{63}}\,\frac{1+p'\Theta_{94}}{1+p'\Theta_{66}},$$

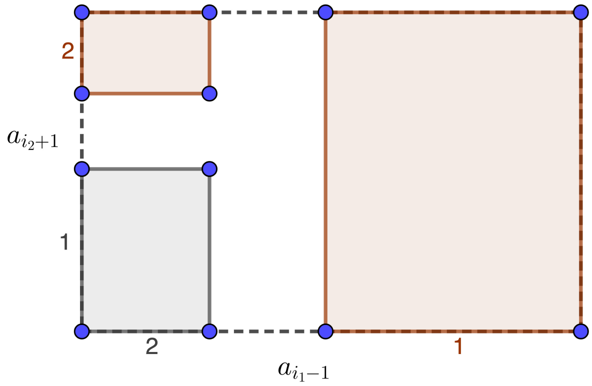
which is true.

6.5 Further plan

It remains for us to check the case if (36) takes place and (37) doesn't – instead, having the negation of (37):

$$0,558 \leqslant q\,\frac{1+p\Theta_{35}}{1+p'\Theta_{63}}\,\frac{1+p\Theta_{63}}{1+p'\Theta_{70}}. \tag{47}$$

We have proved that we can consider subrectangles $\{1,\}$ and $\{2,2\}$.



The picture illustrates the case when the boundaries Δ_1' and Δ_2'' of segments Δ_1 and Δ_2 are defined with constants $\Theta_{\Delta_1'} = \Theta_{\Delta_2''} = \Theta_{30} = \left[0; 213\overline{12}\right]$. In this case, having, in particular, $a_{i_1} = 1$, $a_{i_1+1} = 3$. We set segment Δ_1 to be left-shortened to forbid $a_{i_1-1} = 3$, because it can happen that we will face the forbidden combination (31313).

So, in the illustrated case, the only unused subrectangle is $\{2,1\}$. However, subrectangles $\{2,1\}$ and $\{2,2\}$ doesn't intersect, so we will be unable to construct a family of subrectangles, covering Δ.

How to overcome this difficulty? The idea being developed is to use $a_{i_1-1} = 3$ (and $a_{i_2+1} = 3$), but set a_{i_1-2}, a_{i_1-3} (and a_{i_2+2}, a_{i_2+3}), which produce the thinner subrectangles and doesn't occur in string (31313).

6.6 List of subrectangles

Let's turn to formal plan.

If Δ_1' and Δ_2'' were set with $\Theta_3 = \left[0; 3\overline{12}\right]$, then we would have been able to take $\{3,3\}$, which intersects with $\{2,2\}$.

Instead of $\{3,3\}$, we will take the following list of subrectangles:

$$\{23,32\}, \{23,33\} \text{ (or } \{223,33\} \text{ and } \{323,33\}), \{113,311\},$$
$$\{213,312\}, \{213,311\}, \{113,32\}, \{113,33\} \text{ (or } \{1113,33\}), \{213,33\}.$$

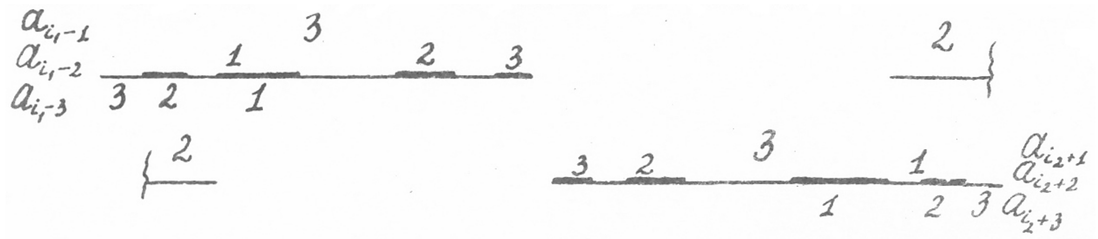


Figure 7: Illustration of the subrectangles replacing the subrectangle $\{3,3\}$.
I will make a 2d illustration in the future.

The next subrectangle to consider is $\{3,2\}$, but instead we will consider the following subrectangles:

$$\{33,213\},\{33,212\},\{23,213\},\{23,212\},\{23,211\}.$$

6.7 Correctness proof

We will show that the projections of adjacent subrectangles intersect and all of them are good.

Show that $\{23,32\}$ is good. Indeed, the condition

$$\gamma-\gamma' < 3,43(\delta-\delta'),$$

where

$$\Theta_\gamma=\Theta_{21},\ \Theta_{\gamma'}=\Theta_{15},\ \Theta_\delta=\Theta_{15},\ \Theta_{\delta'}=\Theta_{21},$$

is equivalent to

$$0,3<q\frac{1+p\Theta_{15}}{1+p'\Theta_{15}}\,\frac{1+p\Theta_{21}}{1+p'\Theta_{21}}$$

and follows from (47).

Subrectangle $\{23,32\}$ intersects with $\{2,2\}$. The following inequality is required:

$$\gamma-\gamma'<\delta-\delta',$$

where

$$\Theta_\gamma=\Theta_{30},\ \Theta_{\gamma'}=\Theta_{19},\ \Theta_\delta=\Theta_{16},\ \Theta_{\delta'}=\Theta_{59},$$

so obtain condition

$$0,4612<q\frac{1+p\Theta_{19}}{1+p'\Theta_{16}}\,\frac{1+p\Theta_{30}}{1+p'\Theta_{59}},$$

which follows from 15:

$$0,558\frac{1+p'\Theta_{63}}{1+p\Theta_{35}}\,\frac{1+p'\Theta_{70}}{1+p\Theta_{63}}>0,4612\frac{1+p'\Theta_{16}}{1+p\Theta_{19}}\,\frac{1+p'\Theta_{59}}{1+p\Theta_{30}}$$
$$\Updownarrow$$
$$1,2>\frac{1+p'\Theta_{16}}{1+p'\Theta_{63}}\,\frac{1+p'\Theta_{59}}{1+p'\Theta_{70}}\,\frac{1+p\Theta_{35}}{1+p\Theta_{19}}\,\frac{1+p\Theta_{63}}{1+p\Theta_{30}},$$

and the last line is obvious.

Checking the goodness of $\{23,33\}$. The condition

$$\gamma-\gamma'<3,43(\delta-\delta')$$

should take place for

$$\Theta_\gamma = \Theta_{21},\ \Theta_{\gamma'} = \Theta_{15},\ \Theta_\delta = \Theta_{22},\ \Theta_{\delta'} = \Theta_{28},$$

and it can be rewritten as follows:

$$0,533 < q\,\frac{1+p\Theta_{15}}{1+p'\Theta_{22}}\,\frac{1+p\Theta_{21}}{1+p'\Theta_{28}}\,. \tag{48}$$

If (48) takes place, then we take $\{23,33\}$. Checking that $\{23,33\}$ intersects with $\{23,32\}$ is easy. If (48) doesn't take place, then we take $\{223,33\}$ and $\{323,33\}$ instead.

At first, showing that $\{223,33\}$ and $\{23,32\}$ intersect.

Subrectangle $\{223,33\}$ is right-normal, since

$$1,4(\gamma-\gamma' < \delta-\delta'),$$

where

$$\begin{aligned}
\Theta_\gamma &= \big[0;3223\overline{1}\big] = 0,293294,\\
\Theta_{\gamma'} &= \big[0;32233\overline{1}\big] = 0,293176,\\
\Theta_\delta &= \Theta_{22},\\
\Theta_{\delta'} &= \big[0;3333\overline{1}\big] = 0,302806,
\end{aligned}$$

transforms into

$$0,457 < q\,\frac{1+p\Theta_\gamma}{1+p'\Theta_\delta}\,\frac{1+p\Theta_{\gamma'}}{1+p'\Theta_{'d'}}\,,$$

and it follows from (47), if

$$1,22 > \frac{1+p'\Theta_\delta}{1+p'\Theta_{63}}\,\frac{1+p'\Theta_{\delta'}}{1+p'\Theta_{70}}\,\frac{1+p\Theta_{35}}{1+p\Theta_\gamma}\,\frac{1+p\Theta_{63}}{1+p\Theta_{\gamma'}}\,,$$

which is clear.

Now the condition

$$\gamma-\gamma' > \delta-\delta',$$

where

$$\Theta_\gamma = \big[0;3223\overline{12}\big] = 0,293283,\ \Theta_{\gamma'} = \Theta_{16},\ \Theta_\delta = \Theta_{20},\ \Theta_{\delta'} = \Theta_{23},$$

transforms into

$$0,561 > q\,\frac{1+p\Theta_{16}}{1+p'\Theta_{20}}\,\frac{1+p\Theta_\gamma}{1+p'\Theta_{23}}\,,$$

and it follows from negation of (48).

Let's try to take next subrectangle $\{113,32\}$. Let's see if it intersects with $\{23,33\}$ (or $\{323,33\}$). The condition of intersection

$$\gamma-\gamma' < \delta-\delta',$$

where

$$\Theta_\gamma = \Theta_{16},\ \Theta_{\gamma'} = \Theta_{13},\ \Theta_\delta = \Theta_{18},\ \Theta_{\delta'} = \Theta_{25},$$

is equivalent to

$$0,7 < q\,\frac{1+p\Theta_{13}}{1+p'\Theta_{18}}\,\frac{1+p\Theta_{16}}{1+p'\Theta_{25}}\,. \tag{49}$$

Assume that (49) doesn't take place:

$$0,7 \geqslant q \frac{1+p\Theta_{13}}{1+p'\Theta_{18}} \frac{1+p\Theta_{16}}{1+p'\Theta_{25}}. \tag{50}$$

Consider subrectangle $\{113, 311\}$ and show that it intersects with $\{23, 33\}$. The following inequality should take place:

$$\gamma - \gamma' < \delta - \delta',$$

where

$$\Theta_\gamma = \Theta_{16},\ \Theta_{\gamma'} = \Theta_{13},\ \Theta_\delta = \Theta_{10},\ \Theta_{\delta'} = \Theta_{25},$$

Obtain

$$0,314 < q \frac{1+p\Theta_{13}}{1+p'\Theta_{10}} \frac{1+p\Theta_{16}}{1+p'\Theta_{25}},$$

which follows from (47).

Now let's show that $\{213, 312\}$ intersects with $\{113, 311\}$.

The condition

$$\gamma - \gamma' < \delta - \delta',$$

where

$$\Theta_\gamma = \Theta_8,\ \Theta_{\gamma'} = \Theta_5,\ \Theta_\delta = \Theta_3,\ \Theta_{\delta'} = \Theta_{11},$$

expands to

$$0,338 < q \frac{1+p\Theta_3}{1+p'\Theta_3} \frac{1+p\Theta_8}{1+p'\Theta_{11}},$$

and follows from (47).

Show that $\{213, 311\}$ intersects with $\{213, 312\}$. The condition

$$\gamma - \gamma' > \delta - \delta',$$

where

$$\Theta_\gamma = \Theta_5,\ \Theta_{\gamma'} = \Theta_3,\ \Theta_\delta = \Theta_5,\ \Theta_{\delta'} = \Theta_8,$$

transforms into

$$0,77 > q \frac{1+p\Theta_3}{1+p'\Theta_5} \frac{1+p\Theta_5}{1+p'\Theta_8}. \tag{51}$$

Inequality (51) follows from (50), because

$$0,7 \frac{1+p'\Theta_{18}}{1+p\Theta_{13}} \frac{1+p'\Theta_{25}}{1+p\Theta_{16}} < 0,77 \frac{1+p'\Theta_5}{1+p\Theta_3} \frac{1+p'\Theta_8}{1+p\Theta_5}$$
$$\Updownarrow$$
$$\frac{1+p'\Theta_{18}}{1+p'\Theta_5} \frac{1+p'\Theta_{25}}{1+p'\Theta_8} \frac{1+p\Theta_3}{1+p\Theta_{13}} \frac{1+p\Theta_5}{1+p\Theta_{16}} < 1,1.$$

Constant $\Theta_{\delta'}$ can be set to Θ_8, only if $\{213, 311\}$ is right-normal, for which the condition

$$\gamma - \gamma' > 1,4(\delta - \delta'),$$

where

$$\Theta_\gamma = \Theta_9,\ \Theta_{\gamma'} = \Theta_7,\ \Theta_\delta = \Theta_4,\ \Theta_{\delta'} = \Theta_6,$$

should take place. It is equivalent to

$$0,8 > q\,\frac{1+p\Theta_7}{1+p'\Theta_4}\,\frac{1+p\Theta_9}{1+p'\Theta_6}\,,$$

which is true in consideration of (50).

Finally, it is very easy to check that $\{113, 32\}$ intersects with $\{213, 311\}$.

Overall, regardless of whether condition 17 is satisfied or not, we come to subrectangle $\{113, 32\}$. Further we note that it intersects with $\{113, 33\}$. We will check here that the latter one is good. The following inequality is required:

$$\gamma - \gamma' < 3,8(\delta - \delta'),$$

where

$$\Theta_\gamma = \Theta_{14},\ \Theta_{\gamma'} = \Theta_7,\ \Theta_\delta = \Theta_{22},\ \Theta_{\delta'} = \Theta_{28},$$

and obtain

$$0,44 < q\,\frac{1+p\Theta_7}{1+p'\Theta_{22}}\,\frac{1+p\Theta_{14}}{1+p'\Theta_{28}}\,,$$

which follows from (47).

It is easily checked that $\{113, 33\}$ intersects with $\{213, 32\}$.

Let's find the condition for the subrectangle $\{2, 11\}$ to be right-normal.

$$\gamma - \gamma' > 1,4(\delta - \delta'),$$

where

$$\Theta_\gamma = \Theta_{64},\ \Theta_{\gamma'} = \Theta_{61},\ \Theta_\delta = \Theta_{65},\ \Theta_{\delta'} = \Theta_{68},$$

which is equivalent to

$$0,714 > q\,\frac{1+p\Theta_{61}}{1+p'\Theta_{65}}\,\frac{1+p\Theta_{64}}{1+p'\Theta_{68}}\,. \tag{52}$$

Let's find the condition for subrectangle $\{2, 11\}$ to intersect with $\{213, 32\}$:

$$\gamma - \gamma' > \delta - \delta',$$

where

$$\Theta_\gamma = \Theta_{63},\ \Theta_{\gamma'} = \Theta_3,\ \Theta_\delta = \Theta_{20},\ \Theta_{\delta'} = \Theta_{66},$$

which is equivalent to

$$0,662 > q\,\frac{1+p\Theta_3}{1+p'\Theta_{20}}\,\frac{1+p\Theta_{63}}{1+p'\Theta_{66}}\,. \tag{53}$$

The condition (52) follows from (53), because

$$0,662 > q\,\frac{1+p'\Theta_{30}}{1+p\Theta_3}\,\frac{1+p'\Theta_{66}}{1+p\Theta_{63}} < 0,714\,\frac{1+p'\Theta_{65}}{1+p\Theta_{61}}\,\frac{1+p'\Theta_{68}}{1+p\Theta_{64}}$$
$$\Updownarrow$$
$$\frac{1+p'\Theta_{20}}{1+p'\Theta_{65}}\,\frac{1+p'\Theta_{66}}{1+p'\Theta_{68}}\,\frac{1+p\Theta_{61}}{1+p\Theta_3}\,\frac{1+p\Theta_{64}}{1+p\Theta_{63}} < 1,078,$$

which is easily checked.

In consideration of (53), we come to $\{2, 11\}$.

Now suppose that (54), the negation of (53), takes place:

$$0,662 \leqslant q\,\frac{1+p\Theta_3}{1+p'\Theta_{20}}\,\frac{1+p\Theta_{63}}{1+p'\Theta_{66}}\,. \tag{54}$$

Return to condition (34). Let's see when the subrectangle $\{11,1\}$ is left-normal. The condition

$$1,4(\gamma-\gamma') < \delta-\delta',$$

where

$$\Theta_\gamma=\Theta_{68},\ \Theta_{\gamma'}=\Theta_{65},\ \Theta_\delta=\Theta_{92},\ \Theta_{\delta'}=\Theta_{95},$$

expands to

$$0,443 < q\,\frac{1+p\Theta_{65}}{1+p'\Theta_{92}}\,\frac{1+p\Theta_{68}}{1+p'\Theta_{95}}$$

and follows from (54). Thus in case $a_{i_2}\neq 3$ we can use the following Θ's in (34):

$$\Theta_\gamma=\Theta_{66},\ \Theta_{\gamma'}=\Theta_{63},\ \Theta_\delta=\Theta_{70},\ \Theta_{\delta'}=\Theta_{94},$$

so (34) expands to

$$0,556 < q\,\frac{1+p\Theta_{63}}{1+p'\Theta_{70}}\,\frac{1+p\Theta_{66}}{1+p'\Theta_{94}}\,. \tag{55}$$

Overall, if $a_{i_2}\neq 3$, then we can assume that

$$0,556 \geqslant q\,\frac{1+p\Theta_{63}}{1+p'\Theta_{70}}\,\frac{1+p\Theta_{66}}{1+p'\Theta_{94}}\,. \tag{56}$$

Consider subrectangle $\{33,213\}$. We will show that it intersects with $\{2,2\}$. The condition

$$\gamma-\gamma' < \delta-\delta',$$

where

$$\Theta_\gamma=\Theta_{30},\ \Theta_{\gamma'}=\Theta_{25},\ \Theta_\delta=\Theta_{30},\ \Theta_{\delta'}=\Theta_{63},$$

expands to

$$0,638 < q\,\frac{1+p\Theta_{25}}{1+p'\Theta_{30}}\,\frac{1+p\Theta_{30}}{1+p'\Theta_{63}}\,. \tag{57}$$

Condition (57) follows from (54), if

$$0,662\,\frac{1+p'\Theta_{20}}{1+p\Theta_3}\,\frac{1+p'\Theta_{66}}{1+p\Theta_{63}} > 0,638\,\frac{1+p'\Theta_{30}}{1+p\Theta_{25}}\,\frac{1+p'\Theta_{63}}{1+p\Theta_{30}}$$
$$\Updownarrow$$
$$1,037 > \frac{1+p'\Theta_{30}}{1+p'\Theta_{20}}\,\frac{1+p'\Theta_{63}}{1+p'\Theta_{66}}\,\frac{1+p\Theta_{3}}{1+p\Theta_{25}}\,\frac{1+p\Theta_{63}}{1+p\Theta_{30}}\,.$$

Transform this inequality, increasing the right part:

$$1,037 > \frac{1+0,8p'+0,15p'^2}{1+0,85p'+0,15p'^2}\,\frac{1+0,71p+0,13p^2}{1+0,66p+0,13p^2} =$$
$$= \left(1-\frac{0,05p'}{1+0,85p'+0,15p'^2}\right)\left(1+\frac{0,05p}{1+0,66p+0,13p^2}\right).$$

Checking the last inequality by substituting p' with $0,25$ and p with $0,8$.

Showing that $\{33, 213\}$ is left-normal. The following condition should take place:

$$\gamma - \gamma' > 1,4(\delta - \delta'),$$

where

$$\Theta_\gamma = \left[0; 3333\overline{31}\right] = 0,302806, \Theta_{\gamma'} = \Theta_{22}, \Theta_\delta = \Theta_{31}, \Theta_{\delta'} = \Theta_{33},$$

which expands to

$$0,85 > q\,\frac{1 + p\Theta_{22}}{1 + p'\Theta_{31}}\,\frac{1 + p\Theta_\gamma}{1 + p'\Theta_{33}}. \tag{58}$$

Now show that (58) follows from (56) in case $a_{i_2} \neq 3$. We need to check the inequality

$$0,556\,\frac{1 + p'\Theta_{70}}{1 + p\Theta_{63}}\,\frac{1 + p'\Theta_{94}}{1 + p\Theta_{66}} < 0,85\,\frac{1 + p'\Theta_{31}}{1 + p\Theta_{22}}\,\frac{1 + p'\Theta_{33}}{1 + p\Theta_\gamma}$$
$$\Updownarrow$$
$$\frac{1 + p'\Theta_{70}}{1 + p'\Theta_{31}}\,\frac{1 + p'\Theta_{94}}{1 + p'\Theta_{33}}\,\frac{1 + p\Theta_{22}}{1 + p\Theta_{63}}\,\frac{1 + p\Theta_\gamma}{1 + p\Theta_{66}} < 1,5.$$

If $a_{i_2} = 3$, then (58) follows from (36). Indeed, check the following inequality:

$$0,74\,\frac{1 + p'\Theta_{70}}{1 + p\Theta_{63}}\,\frac{1 + p'\Theta_{90}}{1 + p\Theta_{66}} < 0,85\,\frac{1 + p'\Theta_{31}}{1 + p\Theta_{21}}\,\frac{1 + p'\Theta_{33}}{1 + p\Theta_\gamma}$$
$$\Updownarrow$$
$$\frac{1 + p'\Theta_{70}}{1 + p'\Theta_{31}}\,\frac{1 + p'\Theta_{90}}{1 + p\Theta_{33}}\,\frac{1 + p\Theta_{22}}{1 + p\Theta_{63}}\,\frac{1 + p\Theta_\gamma}{1 + p\Theta_{66}} < 1,15.$$

Increasing the lhs, replacing p' with $0,33$ (since $a_{i_2} = 3$) and p with $0,25$ and checking the inequality.

The next step is to show that the subrectangles $\{33, 213\}$ and $\{33, 212\}$ intersect. We will check the inequality

$$\gamma - \gamma' > \delta - \delta'$$

for

$$\Theta_\gamma = \Theta_{25},\ \Theta_{\gamma'} = \Theta_{23},\ \Theta_\delta = \Theta_{32},\ \Theta_{\delta'} = \Theta_{35},$$

which is equivalent to

$$0,9 > q\,\frac{1 + p\Theta_{23}}{1 + p'\Theta_{32}}\,\frac{1 + p\Theta_{25}}{1 + p'\Theta_{35}}. \tag{59}$$

As with the inequality (58), the inequality (59) follows from (56) in case $a_{i_2} \neq 3$. If $a_{i_2} = 3$, then (59) follows from (36).

It is easy to check that $\{23, 213\}$ intersects with $\{33, 212\}$.

Now let's check that the subrectangle $\{23, 213\}$ is good. The following inequality should take place:

$$\gamma - \gamma' < 3,8(\delta - \delta'),$$

where

$$\Theta_\gamma = \Theta_{21},\ \Theta_{\gamma'} = \Theta_{15},\ \Theta_\delta = \Theta_{29},\ \Theta_{\delta'} = \Theta_{33},$$

which expands to

$$0,569 < q\,\frac{1 + p\Theta_{15}}{1 + p'\Theta_{29}}\,\frac{1 + p\Theta_{21}}{1 + p'\Theta_{33}}. \tag{60}$$

The condition (60) follows from (54), if

$$0,662\,\frac{1+p'\Theta_{20}}{1+p\Theta_3}\,\frac{1+p'\Theta_{66}}{1+p\Theta_{63}} > 0,569\,\frac{1+p'\Theta_{29}}{1+p\Theta_{15}}\,\frac{1+p'\Theta_{33}}{1+p\Theta_{21}}$$
$$\Updownarrow$$
$$\frac{1+p\Theta_{15}}{1+p\Theta_3}\,\frac{1+p\Theta_{21}}{1+p\Theta_{63}}\,\frac{1+p'\Theta_{20}}{1+p'\Theta_{29}}\,\frac{1+p'\Theta_{66}}{1+p'\Theta_{33}} > 0,85.$$

By replacing p' with $0,25$ and p with $0,8$, we will decrease the lhs. The resulting inequality is true, so (60) is checked.

Easily check the intersection of $\{23,213\}$ with the next subrectangle $\{23,212\}$.

Checking that $\{23,212\}$ intersects with $\{23,211\}$. The condition

$$\gamma-\gamma' > \delta-\delta'$$

for

$$\Theta_\gamma=\Theta_{19},\ \Theta_{\gamma'}=\Theta_{16},\ \Theta_\delta=\Theta_{38},\ \Theta_{\delta'}=\Theta_{41}$$

expands to

$$0,875 > q\,\frac{1+p\Theta_{16}}{1+p'\Theta_{38}}\,\frac{1+p\Theta_{19}}{1+p'\Theta_{41}}$$

and follows from (58).

The choice of $\Theta_{\gamma'}$ and Θ_δ is justified, if the subrectangle $\{23,212\}$ is left-normal. For this we need the following condition to take place:

$$\gamma-\gamma' > 1,4(\delta-\delta'),$$

where

$$\Theta_\gamma=\Theta_{17},\ \Theta_{\gamma'}=\Theta_{15},\ \Theta_\delta=\left[0;2123\overline{31}\right]=0,370705,\ \Theta_{\delta'}=\Theta_{39},$$

so obtain

$$0,91 > q\,\frac{1+p\Theta_{15}}{1+p'\Theta_\delta}\,\frac{1+p\Theta_{17}}{1+p'\Theta_{39}},$$

which follows from (58).

Now show that the subrectangle $\{23,211\}$ intersects with $\{2,1\}$. The following condition should take place:

$$\gamma-\gamma' > \delta-\delta',$$

where

$$\Theta_\gamma=\Theta_{63},\ \Theta_{\gamma'}=\Theta_{18},\ \Theta_\delta=\Theta_{54},\ \Theta_{\delta'}=\Theta_{70},$$

, which expands to

$$0,80 > q\,\frac{1+p\Theta_{18}}{1+p'\Theta_{54}}\,\frac{1+p\Theta_{63}}{1+p'\Theta_{70}}. \tag{61}$$

If $a_{i_2}\neq 3$, then (61) follows from (56): the inequality

$$0,556\,\frac{1+p'\Theta_{70}}{1+p\Theta_{63}}\,\frac{1+p'\Theta_{94}}{1+p\Theta_{66}} < 0,8\,\frac{1+p'\Theta_{54}}{1+p\Theta_{18}}\,\frac{1+p'\Theta_{70}}{1+p\Theta_{63}}$$
$$\Updownarrow$$
$$\frac{1+p'\Theta_{94}}{1+p'\Theta_{54}}\,\frac{1+p\Theta_{18}}{1+p\Theta_{66}} < 1,43$$

is checked by setting p' to $0,8$ and p to $0,25$.

If $a_{i_2} = 3$, then (61) follows from (36). Checking that

$$0,74\,\frac{1+p'\Theta_{70}}{1+p\Theta_{63}}\,\frac{1+p'\Theta_{90}}{1+p\Theta_{66}} < 0,8\,\frac{1+p'\Theta_{54}}{1+p\Theta_{18}}\,\frac{1+p'\Theta_{70}}{1+p\Theta_{63}}$$
$$\Updownarrow$$
$$\frac{1+p'\Theta_{94}}{1+p'\Theta_{54}}\,\frac{1+p\Theta_{18}}{1+p\Theta_{66}} < 1,08$$

by substituting p' with $\frac{1}{3}$ and p with $0,25$.

In case Δ' is defined with (20c) or (20d), then we are done. In cases (20a) and (20b) we also need to consider the subrectangle $\{3,1\}$.

At first show that it is good. To start with, show that $\{3,12\}$ is right-normal. Check that

$$\gamma - \gamma' > 1,4(\delta - \delta')$$

for

$$\Theta_\gamma = \Theta_{28},\ \Theta_{\gamma'} = \Theta_{22},\ \Theta_{\delta'} = \left[0;123\overline{31}\right] = 0,697556,\ \Theta_\delta = \left[0;12\overline{31}\right] = 0,693606,$$

which expands to

$$0,715 > q\,\frac{1+p\Theta_{22}}{1+p'\Theta_\delta}\,\frac{1+p\Theta_{28}}{1+p'\Theta_{\delta'}}. \tag{62}$$

Inequality (62) follows from (36), because

$$0,74\,\frac{1+p'\Theta_{70}}{1+p\Theta_{63}}\,\frac{1+p'\Theta_{90}}{1+p\Theta_{66}} < 0,715\,\frac{1+p'\Theta_\delta}{1+p\Theta_{22}}\,\frac{1+p'\Theta_{\delta'}}{1+p\Theta_{28}}$$
$$\Updownarrow$$
$$\frac{1+p\Theta_{22}}{1+p\Theta_{63}}\,\frac{1+p\Theta_{28}}{1+p\Theta_{66}}\,\frac{1+p'\Theta_{70}}{1+p'\Theta_\delta}\,\frac{1+p'\Theta_{90}}{1+p'\Theta_{\delta'}} < 0,965.$$

Checking the last inequality by setting both p and p' to $0,25$ and increasing the lhs.

Now check the condition

$$\delta - \delta' > \delta - \delta'$$

for

$$\Theta_\gamma = \Theta_{25},\ \Theta_{\gamma'} = \Theta_3,\ \Theta_\delta = \Theta_{81},\ \Theta_{\delta'} = \Theta_{83},$$

expanding to

$$0,693 > q\,\frac{1+p\Theta_3}{1+p'\Theta_{81}}\,\frac{1+p\Theta_{25}}{1+p'\Theta_{83}}. \tag{63}$$

The inequality (63) follows from (36), because

$$0,74\,\frac{1+p'\Theta_{70}}{1+p\Theta_{63}}\,\frac{1+p'\Theta_{90}}{1+p\Theta_{66}} < 0,693\,\frac{1+p'\Theta_{81}}{1+p\Theta_3}\,\frac{1+p'\Theta_{83}}{1+p\Theta_{25}}$$
$$\Updownarrow$$
$$\frac{1+p\Theta_3}{1+p\Theta_{63}}\,\frac{1+p\Theta_{25}}{1+p\Theta_{66}}\,\frac{1+p'\Theta_{70}}{1+p'\Theta_{81}}\,\frac{1+p'\Theta_{90}}{1+p'\Theta_{83}} < 0,935.$$

Again, the last inequality is checked by setting p and p' to $0,25$.

Now show that $\{3,1\}$ and $\{2,1\}$ intersect. The condition

$$\gamma - \gamma' < \delta - \delta',$$

where

$$\Theta_\gamma = \Theta_{30},\ \Theta_{\gamma'} = \Theta_{25},\ \Theta_\delta = \Theta_{71},\ \Theta_{\delta'} = \Theta_{89},$$

expands to

$$0,338 < q\,\frac{1+p\Theta_{25}}{1+p'\Theta_{71}}\,\frac{1+p\Theta_{30}}{1+p'\Theta_{89}}\,. \tag{64}$$

The inequality (64) follows from (47), because

$$\begin{gathered} 0,558\,\frac{1+p'\Theta_{63}}{1+p\Theta_{35}}\,\frac{1+p'\Theta_{70}}{1+p\Theta_{63}} > 0,338\,\frac{1+p'\Theta_{71}}{1+p\Theta_{25}}\,\frac{1+p'\Theta_{89}}{1+p\Theta_{30}} \\ \Updownarrow \\ 1,65 > \frac{1+p\Theta_{35}}{1+p\Theta_{25}}\,\frac{1+p\Theta_{63}}{1+p\Theta_{30}}\,\frac{1+p'\Theta_{71}}{1+p'\Theta_{63}}\,\frac{1+p'\Theta_{89}}{1+p'\Theta_{70}}\,, \end{gathered}$$

which is easily checked by substituting both p and p' with $0,8$.

7 Induction, $i_1 \equiv i_2 \pmod 2$

This section deals with the case $i_1 \equiv i_2 \pmod 2$. Again, let $\Delta = \{a_{i_1}, ..., a_{i_2}\}$ be a horizontal rectangle, satisfying conditions from sections 3 and 4.

As in section 6, we will assume that both i_1 and i_2 are even; for the different parity the same logic can be repeated, with bounds in the inequalities swapped and illustrations reflected.

7.1 Case $\{3, 2\}$ is right-normal

We start with the case

$$\gamma - \gamma' > \delta - \delta', \tag{65}$$

where

$$\Theta_\gamma = \Theta_{25},\ \Theta_{\gamma'} = \Theta_3,\ \Theta_\delta = \Theta_{66},\ \Theta_{\delta'} = \Theta_{63},$$

Freiman ceases to give explanatory illustrations to inequalities like (65) in this section and suggests the reader to make them on their own.

Inequality (65) expands to

$$0,31 > q\,\frac{1+p\Theta_3}{1+p'\Theta_{63}}\,\frac{1+p\Theta_{25}}{1+p'\Theta_{66}}. \tag{66}$$

The rounding is done to a higher precision so that (65) and (68) both follow from (66).

The constants Θ_{25} and Θ_3 are chosen such that the subrectangle $\{3, 2\}$ which will occur in the further splitting is right-normal. This condition allows us to choose the largest possible constant in the lhs of (66).

So, the following condition should take place:

$$\gamma - \gamma' > 1,4(\delta - \delta'), \tag{67}$$

where

$$\Theta_\gamma = \Theta_{28},\ \Theta_{\gamma'} = \Theta_{22},\ \Theta_\delta = \Theta_{64},\ \Theta_{\delta'} = \Theta_{61},$$

which expands to

$$0,345 > q\,\frac{1+p\Theta_{22}}{1+p'\Theta_{61}}\,\frac{1+p\Theta_{28}}{1+p'\Theta_{64}}. \tag{68}$$

Let's show that (68) follows from (66):

$$\begin{gathered} 0,344\,\frac{1+p'\Theta_{61}}{1+p\Theta_{22}} > 0,31\,\frac{1+p'\Theta_{66}}{1+p\Theta_3} \\ \Updownarrow \\ 1,1 > \frac{1+p\Theta_{22}}{1+p\Theta_3}\,\frac{1+p'\Theta_{66}}{1+p'\Theta_{61}}. \end{gathered} \tag{69}$$

Checking it by replacing p and p' with their upper bound 0.8 and increasing the rhs.

In consideration of (66), in case Δ' is defined with (15) and (16), we can take $\{1, \}$, $\{2, \}$, and $\{3, \}$, and in case (17) and (18) subrectangles $\{1, \}$ and $\{2, \}$ are enough.

In the rest section we will assume that (66) doesn't take place, that is,

$$0,31 \leqslant q\,\frac{1+p\Theta_3}{1+p'\Theta_{63}}\,\frac{1+p\Theta_{25}}{1+p'\Theta_{66}}. \tag{70}$$

7.2 Name

Consider condition

$$\gamma - \gamma' > \delta - \delta', \tag{71}$$

where

$$\Theta_\gamma = \Theta_{63},\ \Theta_{\gamma'} = \Theta_{36},\ \Theta_\delta = \Theta_{66},\ \Theta_{\delta'} = \Theta_{63},$$

which expands to

$$0,634 > q\,\frac{1 + p\Theta_{36}}{1 + p'\Theta_{63}}\,\frac{1 + p\Theta_{63}}{1 + p'\Theta_{66}}\,. \tag{72}$$

To justify the choice $\Theta_\gamma = \Theta_{\gamma'} = \Theta_{63}$, we need to check

$$\gamma - \gamma' > 1,4(\delta - \delta'), \tag{73}$$

where

$$\Theta_\gamma = \Theta_{64},\ \Theta_{\gamma'} = \Theta_{61},\ \Theta_\delta = \Theta_{64},\ \Theta_{\delta'} = \Theta_{61},$$

which expands to

$$0,714 > q\,\frac{1 + p\Theta_{61}}{1 + p'\Theta_{61}}\,\frac{1 + p\Theta_{64}}{1 + p'\Theta_{64}}\,. \tag{74}$$

The last inequality follows from (72), if

$$\frac{1 + p'\Theta_{63}}{1 + p'\Theta_{61}}\,\frac{1 + p'\Theta_{66}}{1 + p'\Theta_{64}}\,\frac{1 + p\Theta_{61}}{1 + p\Theta_{36}}\,\frac{1 + p\Theta_{64}}{1 + p\Theta_{63}} < 1,125, \tag{75}$$

which is easily checked by substituting $p = p' = 0.8$, as in (69).

If (71) takes place, then take subrectangles $\{1,\}$ and $\{2,\}$. Thus, in consideration of (72), cases (17) and (18) are over.

7.3 Name

Now turn to cases (15) and (16) (IIa). We will show that

$$\gamma - \gamma' < \delta - \delta', \tag{76}$$

where

$$\Theta_\gamma = \Theta_{30},\ \Theta_{\gamma'} = \Theta_{25},\ \Theta_\delta = \Theta_{59},\ \Theta_{\delta'} = \Theta_{3},$$

which can be rewritten as

$$0,3375 < q\,\frac{1 + p\Theta_{25}}{1 + p'\Theta_{3}}\,\frac{1 + p\Theta_{30}}{1 + p'\Theta_{59}}\,. \tag{77}$$

The last inequality follows from (70), if

$$1,09 < \frac{1 + p'\Theta_{63}}{1 + p'\Theta_{3}}\,\frac{1 + p'\Theta_{66}}{1 + p'\Theta_{59}}\,\frac{1 + p\Theta_{30}}{1 + p\Theta_{3}}\,,$$

which is checked by substituting $p = p' = 0,25$.

The inequality (76) shows that, in consideration of (72) and in case (15), the subrectangle $\{3,2\}$ intersects with $\{2,\}$, even if $\{3,2\}$ is right-shortened.

If Δ is left-shortened, then the splitting is over. If it is left-normal, then subrectangles $J_k = \{\overset{k}{3}, \overset{k}{3}\}$ are added.

Check that, for example, $J_1 = \{3,3\}$ is good.

We need

$$\gamma - \gamma' < 3,8(\delta - \delta'), \tag{78}$$

where

$$\Theta_\gamma = \Theta_{28},\ \Theta_{\gamma'} = \Theta_1,\ \Theta_\delta = \Theta_{28},\ \Theta_{\delta'} = \Theta_1,$$

which expands to

$$0,264 < q\,\frac{1+p\Theta_1}{1+p'\Theta_1}\,\frac{1+p\Theta_{28}}{1+p'\Theta_{28}} \tag{79}$$

and follows from (70), if

$$1,17 > \frac{1+p\Theta_3}{1+p\Theta_1}\,\frac{1+p\Theta_{25}}{1+p\Theta_{28}}\,\frac{1+p'\Theta_1}{1+p'\Theta_{63}}\,\frac{1+p'\Theta_{28}}{1+p'\Theta_{66}},$$

which is obvious.

Here Freiman references the $\sqrt{21}$ proof from §9. I will translate it soon.

7.4 Name

It remains to deal with (16) (IIb).

Suppose that the following inequality takes place:

$$\gamma - \gamma' < \delta - \delta' \tag{80}$$

with

$$\Theta_\gamma = \Theta_{30},\ \Theta_{\gamma'} = \Theta_{25},\ \Theta_\delta = \Theta_{63},\ \Theta_{\delta'} = \Theta_{36}.$$

Inequality (80) is rewritten as follows. Instead of 0,703 in the lhs take constant 0,73 to assure $\{3,2\}$ normality:

$$0,73 < q\,\frac{1+p\Theta_{25}}{1+p'\Theta_{36}}\,\frac{1+p\Theta_{30}}{1+p'\Theta_{63}}. \tag{81}$$

Subrectangle $\{3,2\}$ is normal, if

$$1,4(\gamma - \gamma') < \delta - \delta',$$

where

$$\Theta_\gamma = \Theta_{28},\ \Theta_{\gamma'} = \Theta_{22},\ \Theta_\delta = \Theta_{64},\ \Theta_{\delta'} = \Theta_{61},$$

which expands to

$$0,6773 < q\,\frac{1+p\Theta_{22}}{1+p'\Theta_{61}}\,\frac{1+p\Theta_{28}}{1+p'\Theta_{64}}$$

and follows from (81), if

$$1,076 > \frac{1+p'\Theta_{61}}{1+p'\Theta_{36}}\,\frac{1+p'\Theta_{64}}{1+p'\Theta_{63}}\,\frac{1+p\Theta_{30}}{1+p\Theta_{22}},$$

which is checked as (69).

In addition to subrectangles $\{1,\}$ and $\{2,\}$ we can now take $\{3,2\}$, and case (16) is done, if (81), (72), and (70) all take place.

Note that (81) does not depend on (72). We will need it below. If (81) doesn't take place, that is,

$$0,73 \geqslant q\,\frac{1+p\Theta_{25}}{1+p'\Theta_{36}}\,\frac{1+p\Theta_{30}}{1+p'\Theta_{63}}\,, \tag{82}$$

then in case (16) the subrectangles $\{2,\}$ and $\{3,2\}$ may not intersect when

$$\Delta_1'(2)+\Delta_2'(2) > \Delta_1''(3)+\Delta_2''(2).$$

The system of subrectangles covering the interval $(\Delta_1''(3)+\Delta_2''(2);\Delta_1'(2)+\Delta_2'(2))$ will be found further.

Suppose that (72) doesn't take place, that is,

$$0,634 \leqslant q\,\frac{1+p\Theta_{36}}{1+p'\Theta_{63}}\,\frac{1+p\Theta_{63}}{1+p'\Theta_{66}}\,. \tag{83}$$

At first, consider subrectangles $\{1,\}$ and $\{2,1\}$. Rectangle Δ is good, so they intersect.

Showing that $\{2,1\}$ is good. We need the following:

$$3,8(\gamma-\gamma') > \delta-\delta', \tag{84}$$

where

$$\Theta_\gamma=\Theta_{64},\ \Theta_{\gamma'}=\Theta_{29},\ \Theta_\delta=\Theta_{95},\ \Theta_{\delta'}=\Theta_{65},$$

which expands to

$$1,36 > q\,\frac{1+p\Theta_{29}}{1+p'\Theta_{65}}\,\frac{1+p\Theta_{64}}{1+p'\Theta_{95}}$$

and is easily checked.

Now consider subrectangle $\{3,1\}$, if it is good. The following is required:

$$3,8(\gamma-\gamma') > \delta-\delta',$$

where

$$\Theta_\gamma=\Theta_{28},\ \Theta_{\gamma'}=\Theta_{1},\ \Theta_\delta=\Theta_{95},\ \Theta_{\delta'}=\Theta_{65},$$

which expands to

$$0,696 > q\,\frac{1+p\Theta_{1}}{1+p'\Theta_{65}}\,\frac{1+p\Theta_{28}}{1+p'\Theta_{95}}\,. \tag{85}$$

Let the negation of (85), the inequality (86), take place:

$$0,696 \leqslant q\,\frac{1+p\Theta_{1}}{1+p'\Theta_{65}}\,\frac{1+p\Theta_{28}}{1+p'\Theta_{95}}\,. \tag{86}$$

Then considering the subrectangle $\{3,11\}$ and showing that it intersects with $\{2,1\}$. The following inequality is needed:

$$\gamma-\gamma' < \delta-\delta', \tag{87}$$

where

$$\Theta_\gamma=\Theta_{30},\ \Theta_{\gamma'}=\Theta_{25},\ \Theta_\delta=\Theta_{79},\ \Theta_{\delta'}=\Theta_{66}.$$

The choice of Θ's will be explained.

Inequality (87) is transformed into

$$0,702 < q \frac{1+p\Theta_{25}}{1+p'\Theta_{66}} \frac{1+p\Theta_{30}}{1+p'\Theta_{79}},$$

which instantly follows from (86).

We have to show that

$$1,4(\gamma - \gamma') < \delta - \delta', \tag{88}$$

where

$$\Theta_\gamma = \Theta_{33},\ \Theta_{\gamma'} = \Theta_{29},\ \Theta_\delta = \Theta_{68},\ \Theta_{\delta'} = \Theta_{65},$$

which expands to

$$0,574 < q \frac{1+p\Theta_{29}}{1+p'\Theta_{65}} \frac{1+p\Theta_{33}}{1+p'\Theta_{68}},$$

which follows from (86).

The next subrectangle is $\{2,2\}$. Let's show that it intersects with $\{3,11\}$ or $\{3,1\}$. The following is required:

$$\gamma - \gamma' > \delta - \delta', \tag{89}$$

where

$$\Theta_\gamma = \Theta_{63},\ \Theta_{\gamma'} = \Theta_{3},\ \Theta_\delta = \Theta_{65},\ \Theta_{\delta'} = \Theta_{58},$$

which expands to

$$1,275 > q \frac{1+p\Theta_{3}}{1+p'\Theta_{58}} \frac{1+p\Theta_{63}}{1+p'\Theta_{65}},$$

which takes place.

The condition (76) allows us to finish this case by taking subrectangles $\{2,3\}$, $\{3,2\}$, and $\{3,3\}$ or J_k (the latter one in case Δ is left-normal).

7.5 Name

Turn to case (16) (IIb) with (83) taking place.

In this case the system of subrectangles covering $[\Delta_1'(2) + \Delta_2'(2); \Delta_1'' + \Delta_2'']$ coincides with one for condition (15), and the system covering the interval to the left coincides with the one for (16) (IIb) with (72) taking place.

In cases (17) and (18) the subrectangles $\{3,1\}$ and $\{3,11\}$ can not be considered, because of the rule of setting the left end of Δ. The interval $(\Delta_1'(2) + \Delta_2'(1); \Delta_1''(2) + \Delta_2''(2))$ remains uncovered.

Let's move to the construction of the system of subrectangles covering

$$(\Delta_1''(3) + \Delta_2''(2), \Delta_1'(2) +' D_2'(2))\,. \tag{90}$$

Conditions (70) and (82) both take place.

At first consider the case (80) takes place with the following choice of Θ's:

$$\Theta_\gamma = \Theta_{57},\ \Theta_{\gamma'} = \Theta_{19},\ \Theta_\delta = \Theta_{10},\ \Theta_{\delta'} = \Theta_{63},$$

which expands to

$$0,677 < q \frac{1+p\Theta_{19}}{1+p'\Theta_{10}} \frac{1+p\Theta_{57}}{1+p'\Theta_{63}}. \tag{91}$$

Interval (90) is covered by the subrectangles $\{22,311\}$ and $\{22,32\}$.

We ensure that $\{3,2\}$ and $\{22,311\}$ intersect by demanding the condition (91).

Showing that $\{22,311\}$ is good. We need

$$\gamma - \gamma' < 3,43(\delta - \delta'), \tag{92}$$

where

$$\Theta_\gamma = \Theta_{60},\ \Theta_{\gamma'} = \Theta_{56},\ \Theta_\delta = \Theta_{14},\ \Theta_{\delta'} = \Theta_7,$$

which expands to

$$0,652 < q\,\frac{1+p\Theta_{56}}{1+p'\Theta_7}\,\frac{1+p\Theta_{60}}{1+p'\Theta_{14}}, \tag{93}$$

which follows from (91).

Showing that $\{22,32\}$ is also good. We need

$$\gamma - \gamma' < 3,43(\delta - \delta')$$

with

$$\Theta_\gamma = \Theta_{60},\ \Theta_{\gamma'} = \Theta_{56},\ \Theta_\delta = \Theta_{21},\ \Theta_{\delta'} = \Theta_{15},$$

expanding to

$$0,53 < q\,\frac{1+p\Theta_{56}}{1+p'\Theta_{15}}\,\frac{1+p\Theta_{60}}{1+p'\Theta_{21}}, \tag{94}$$

which follows from (91).

Subrectangles $\{22,311\}$ and $\{22,32\}$ intersect under condition

$$3,43(\gamma - \gamma') > \delta - \delta', \tag{95}$$

where

$$\Theta_\gamma = \Theta_{60},\ \Theta_{\gamma'} = \Theta_{56},\ \Theta_\delta = \Theta_{28},\ \Theta_{\delta'} = \Theta_1,$$

which transforms to

$$1,17 > q\,\frac{1+p\Theta_{56}}{1+p'\Theta_1}\,\frac{1+p\Theta_{60}}{1+p'\Theta_{28}}, \tag{96}$$

which follows from (82).

Finally, we have to show that

$$\gamma - \gamma' > \delta - \delta'$$

with

$$\Theta_\gamma = \Theta_{59},\ \Theta_{\gamma'} = \Theta_{30},\ \Theta_\delta = \Theta_{36},\ \Theta_{\delta'} = \Theta_{19},$$

expanding to

$$0,94 > q\,\frac{1+p\Theta_{30}}{1+p'\Theta_{19}}\,\frac{1+p\Theta_{59}}{1+p'\Theta_{36}}, \tag{97}$$

which follows from (82).

Further we can assume that the inequality (98) takes place:

$$0,677 \geqslant q\,\frac{1+p\Theta_{19}}{1+p'\Theta_{10}}\,\frac{1+p\Theta_{57}}{1+p'\Theta_{63}}. \tag{98}$$

Showing that $\{2,2\}$ is left-normal, that is,

$$\gamma - \gamma' > 1,4(\delta - \delta') \tag{99}$$

with

$$\Theta_\gamma = \Theta_{33},\ \Theta_{\gamma'} = \Theta_{29},\ \Theta_\delta = \Theta_{33},\ \Theta_{\delta'} = \Theta_{29},$$

expanding to

$$0,715 > q\,\frac{1+p\Theta_{29}}{1+p'\Theta_{29}}\,\frac{1+p\Theta_{33}}{1+p'\Theta_{33}}\,. \tag{100}$$

Inequality (100) follows from (98), if

$$A(p,p') = \frac{1+p'\Theta_{10}}{1+p\Theta_{19}}\,\frac{1+p'\Theta_{63}}{1+p\Theta_{57}}\,\frac{1+p\Theta_{29}}{1+p'\Theta_{29}}\,\frac{1+p\Theta_{33}}{1+p'\Theta_{33}} < 1,053. \tag{101}$$

Note that $F(p',p) \leqslant A(0.25, 0.8) = 0,997 < 1,053$.

Overall, if (98) takes place, the inequality (89) can be set with constants

$$\Theta_\gamma = \Theta_{54},\ \Theta_{\gamma'} = \Theta_{29},\ \Theta_\delta = \Theta_{29},\ \Theta_{\delta'} = \Theta_{25},$$

so it can be rewritten as

$$0,612 > q\,\frac{1+p\Theta_{29}}{1+p'\Theta_{25}}\,\frac{1+p\Theta_{54}}{1+p'\Theta_{29}}\,. \tag{102}$$

At first, let (102) not take place, that is,

$$0,612 \leqslant q\,\frac{1+p\Theta_{29}}{1+p'\Theta_{25}}\,\frac{1+p\Theta_{54}}{1+p'\Theta_{29}}\,. \tag{103}$$

Check the condition (80) with

$$\Theta_\gamma = \Theta_{41},\ \Theta_{\gamma'} = \Theta_{19},\ \Theta_\delta = \Theta_{63},\ \Theta_{\delta'} = \Theta_{10},$$

expanding to

$$0,492 < q\,\frac{1+p\Theta_{19}}{1+p'\Theta_{10}}\,\frac{1+p\Theta_{41}}{1+p'\Theta_{63}}\,. \tag{104}$$

It follows from (103).

So $\{112, 311\}$ intersects with $\{3, 2\}$.

Showing that $\{113, 311\}$ is good. Check the following inequality:

$$\gamma - \gamma' < 3,8(\delta - \delta') \tag{105}$$

with

$$\Theta_\gamma = \Theta_{55},\ \Theta_{\gamma'} = \Theta_{40},\ \Theta_\delta = \Theta_{14},\ \Theta_{\delta'} = \Theta_{7},$$

expanding to

$$0,49 < q\,\frac{1+p\Theta_{40}}{1+p'\Theta_{7}}\,\frac{1+p\Theta_{55}}{1+p'\Theta_{14}}\,, \tag{106}$$

and it follows from (103).

Now consider the subrectangle $\{112, 32\}$ and show that it intersects with $\{112, 311\}$. It is so, if $\{112, 3\}$ is good, and it is enough to check that

$$3,8(\gamma - \gamma') > \delta - \delta'$$

with

$$\Theta_\gamma = \Theta_{55},\ \Theta_{\gamma'} = \Theta_{40},\ \Theta_\delta = \Theta_{28},\ \Theta_{\delta'} = \Theta_{1},$$

which expands to

$$1,07 > q\,\frac{1+p\Theta_{40}}{1+p'\Theta_1}\,\frac{1+p\Theta_{55}}{1+p'\Theta_{28}},$$

which follows from (98).

Check that $\{112, 32\}$ is good. We need

$$\gamma - \gamma' < 3,43(\delta - \delta'),$$

where

$$\Theta_\gamma = \Theta_{55},\ \Theta_{\gamma'} = \Theta_{40},\ \Theta_\delta = \Theta_{21},\ \Theta_{\delta'} = \Theta_{15},$$

which can be rewritten as

$$0,486 < q\,\frac{1+p\Theta_{40}}{1+p'\Theta_{15}}\,\frac{1+p\Theta_{55}}{1+p'\Theta_{21}},$$

which follows from (103).

The next two subrectangles are $\{2112, 33\}$ and $\{3112, 33\}$.

Show that the first one intersects with $\{112, 32\}$. The inequality (89) should take place with the following choice of constants:

$$\Theta_\gamma = \Theta_{54},\ \Theta_{\gamma'} = \Theta_{47},\ \Theta_\delta = \Theta_{24},\ \Theta_{\delta'} = \Theta_{19}.$$

This inequality expands to

$$0,765 > q\,\frac{1+p\Theta_{47}}{1+p'\Theta_{19}}\,\frac{1+p\Theta_{54}}{1+p'\Theta_{24}},$$

which follows from (98).

It is very easy to check that both $\{3113, 33\}$ and $\{3112, 33\}$ are good.

Let's show that they intersect. It is sufficient to check that $\{112, 3\}$ is normal. For this we need the condition (67) with

$$\Theta_\gamma = \Theta_{55},\ \Theta_{\gamma'} = \Theta_{49},\ \Theta_\delta = \Theta_{28},\ \Theta_{\delta'} = \Theta_{22}.$$

Obtain the condition

$$0,442 < q\,\frac{1+p\Theta_{49}}{1+p'\Theta_{22}}\,\frac{1+p\Theta_{55}}{1+p'\Theta_{28}},$$

which follows from (103).

Showing that $\{22, 311\}$ intersects with $\{3112, 33\}$. We need the condition (80) with

$$\Theta_\gamma = \Theta_{57},\ \Theta_{\gamma'} = \Theta_{54},\ \Theta_\delta = \Theta_{25},\ \Theta_{\delta'} = \Theta_{10},$$

which expands to

$$0,612 < q\,\frac{1+p\Theta_{54}}{1+p'\Theta_{10}}\,\frac{1+p\Theta_{57}}{1+p'\Theta_{25}},$$

which follows from (103).

Now consider the subrectangles $\{22, 311\}$ and $\{22, 32\}$.

They are good, if the conditions (93) and (94) take place, respectively. In the investigated case we can use (91), but should rely on (103).

We will show that (103) follows from (93). We need

$$1,065 < \frac{1+p\Theta_{56}}{1+p\Theta_{29}}\,\frac{1+p\Theta_{60}}{1+p\Theta_{54}}\,\frac{1+p'\Theta_{25}}{1+p'\Theta_7}\,\frac{1+p'\Theta_{29}}{1+p'\Theta_{14}}.$$

This inequality is easily checked, taking into account that $a_{i_2} = 1$, $a_{i_1} = 3$ and $p' \geqslant 0,75$.

The condition (94) is easily checked.

The condition (96) guarantees the intersection of the last two subrectangles, while (97) ensures the intersection of $\{22, 32\}$ and $\{2, 2\}$.

7.6 Name

Now suppose that (102) takes place.

The first considered subrectangle is $\{312, 312\}$.

On page (**??**) the condition (76) i checked, and it shows that $\{312, 312\}$ intersects with $\{3, 2\}$.

The considered subrectangle is normal. Indeed,

$$\gamma - \gamma' > 1,4(\delta - \delta'), \tag{107}$$

where

$$\Theta_\gamma = \Theta_{33},\ \Theta_{\gamma'} = \Theta_{31},\ \Theta_\delta = \Theta_6,\ \Theta_{\delta'} = \Theta_4,$$

which expands to

$$0,745 > q\left(\frac{1 + p\Theta_{33}}{1 + p'\Theta_4}\right)^2, \tag{108}$$

and it follows from (102).

To show that $\{312, 312\}$ intersects with $\{312, 311\}$, proving the inequality (89) for the following constants:

$$\Theta_\gamma = \Theta_{49},\ \Theta_{\gamma'} = \Theta_{30},\ \Theta_\delta = \Theta_8,\ \Theta_{\delta'} = \Theta_5.$$

.

This is rewritten as

$$0,767 > q\left(\frac{1 + p\Theta_{49}}{1 + p'\Theta_5}\right)^2,$$

which trivially follows from (108).

The next subrectangle $\{212, 312\}$ is good, if

$$\gamma - \gamma' < 3,8(\delta - \delta') \tag{109}$$

with

$$\Theta_\gamma = \Theta_{39},\ \Theta_{\gamma'} = \Theta_{24},\ \Theta_\delta = \Theta_{69},\ \Theta_{\delta'} = \Theta_2,$$

which expands to

$$0,492 < q\left(\frac{1 + p\Theta_{34}}{1 + p'\Theta_6}\right)^2. \tag{110}$$

Suppose that (110) doesn't take place, that is,

$$0,492 \geqslant q\left(\frac{1 + p\Theta_{34}}{1 + p'\Theta_6}\right)^2. \tag{111}$$

In this case, consider the subrectangle $\{1212, 312\}$, which is good.

Show that $\{312, 311\}$ intersects either $\{212, 312\}$ or $\{1212, 312\}$. For this we need the subrectangle $\{2121, 31\}$ to be good:

$$3,8(\gamma - \gamma') > \delta - \delta', \tag{112}$$

where

$$\Theta_\gamma = \Theta_{37},\ \Theta_{\gamma'} = \Theta_{34},\ \Theta_\delta = \Theta_{14},\ \Theta_{\delta'} = \Theta_1,$$

which expands to

$$0,512 > q\,\frac{1+p\Theta_{34}}{1+p'\Theta_1}\,\frac{1+p\Theta_{37}}{1+p'\Theta_{14}}\,,$$

which follows from (111).

Checking that $\{212,311\}$ is good. We need the following:

$$3,8(\gamma-\gamma') < \delta-\delta',$$

where

$$\Theta_\gamma=\Theta_{39},\ \Theta_{\gamma'}=\Theta_{34},\ \Theta_\delta=\Theta_{14},\ \Theta_{\delta'}=\Theta_7.$$

Obtain the inequality

$$0,235 < q\left(\frac{1+p\Theta_{34}}{1+p'\Theta_{14}}\right)^2,$$

following from (70).

The next subrectangle is $\{312,32\}$. To show that it intersects with $\{212,311\}$, we need to check the condition (89) with

$$\Theta_\gamma=\Theta_{38},\ \Theta_{\gamma'}=\Theta_{30},\ \Theta_\delta=\Theta_{16},\ \Theta_{\delta'}=\Theta_{11},$$

which expands to

$$1,1 > q\,\frac{1+p\Theta_{30}}{1+p'\Theta_{11}}\,\frac{1+p\Theta_{38}}{1+p'\Theta_{16}}$$

and therefore follows from (97).

Showing that $\{312,32\}$ intersects with $\{312,33\}$. We need to check the condition (89) with the following constants:

$$\Theta_\gamma=\Theta_{32},\ \Theta_{\gamma'}=\Theta_{30},\ \Theta_\delta=\Theta_{23},\ \Theta_{\delta'}=\Theta_{19},$$

which transforms to

$$0,945 > q\left(\frac{1+p\Theta_{32}}{1+p'\Theta_{19}}\right)^2,$$

and follows from (97).

The next subrectangle is $\{212,32\}$.

Showing that it intersects with $\{312,32\}$. The condition (80) is needed with the following Θ's:

$$\Theta_\gamma=\Theta_{66},\ \Theta_{\gamma'}=\Theta_{32},\ \Theta_\delta=\Theta_{25},\ \Theta_{\delta'}=\Theta_{16}.$$

This in equivalent to

$$0,289 < q\,\frac{1+p\Theta_{32}}{1+p'\Theta_{16}}\,\frac{1+p\Theta_{66}}{1+p'\Theta_{25}}\,,$$

following from (70).

Showing that $\{212,32\}$ intersects with $\{1112,311\}$. We need the condition (80) with

$$\Theta_\gamma=\Theta_{41},\ \Theta_{\gamma'}=\Theta_{38},\ \Theta_\delta=\Theta_{20},\ \Theta_{\delta'}=\Theta_8,$$

expanding to

$$0,319 < q\,\frac{1+p\Theta_{38}}{1+p'\Theta_8}\,\frac{1+p\Theta_{41}}{1+p'\Theta_{20}}\,,$$

and following from (70).

Let's figure out when the subrectangle $\{11112, 311\}$ is right-normal. The condition (88) is needed with

$$\Theta_\gamma = \Theta_{46},\ \Theta_{\gamma'} = \Theta_{44},\ \Theta_\delta = \Theta_{12},\ \Theta_{\delta'} = \Theta_{14}.$$

This expands to

$$0,392 < q\left(\frac{1+p\Theta_{44}}{1+p'\Theta_{14}}\right)^2. \tag{113}$$

Now figuring out when $\{2122, 311\}$ is normal. Again, we require the condition (88) with

$$\Theta_\gamma = \Theta_{48},\ \Theta_{\gamma'} = \Theta_{49},\ \Theta_\delta = \Theta_{9},\ \Theta_{\delta'} = \Theta_{7},$$

expanding to

$$0,575 > q\left(\frac{1+p\Theta_{49}}{1+p'\Theta_{7}}\right)^2. \tag{114}$$

Now we will deduce (??) the condition when $\{1112, 311\}$ and $\{2112, 311\}$ overlap, under the assumption of (113) and (114).

The condition (80) is required with the following constants:

$$\Theta_\gamma = \Theta_{45},\ \Theta_{\gamma'} = \Theta_{45},\ \Theta_\delta = \Theta_{13},\ \Theta_{\delta'} = \Theta_{8},$$

expanding to

$$0,438 < q\frac{1+p\Theta_{45}}{1+p'\Theta_{8}}\,\frac{1+p\Theta_{47}}{1+p'\Theta_{13}}. \tag{115}$$

This follows from (113).

Now assume that (114) doesn't take place, that is,

$$0,575 \leqslant q\left(\frac{1+p\Theta_{49}}{1+p'\Theta_{7}}\right)^2. \tag{116}$$

In this case setting $\Theta_{\delta'}$ differently. Let $\Theta_{\delta'}$ be Θ_{10}. Now (80) transforms into

$$0,482 < q\frac{1+p\Theta_{45}}{1+p'\Theta_{10}}\,\frac{1+p\Theta_{47}}{1+p'\Theta_{13}},$$

which follows from (116).

Overall, the considered subrectangles intersect in case (116) takes place, and also in case both (113) and (114) take place.

Now assume that (115) doesn't take place, that is,

$$0,438 \geqslant q\frac{1+p\Theta_{45}}{1+p'\Theta_{8}}\,\frac{1+p\Theta_{47}}{1+p'\Theta_{13}}. \tag{117}$$

Showing that $\{1112, 311\}$ and $\{1112, 32\}$ overlap. We need the inequality (89) with the following constants:

$$\Theta_\gamma = \Theta_{45},\ \Theta_{\gamma'} = \Theta_{41},\ \Theta_\delta = \Theta_{16},\ \Theta_{\delta'} = \Theta_{13},$$

expanding to

$$0,484 > q\left(\frac{1+p\Theta_{45}}{1+p'\Theta_{13}}\right)^2,$$

which follows from (117).

To justify such choice of γ' and δ we need to show that $\{1112, 32\}$ is left-normal, that is,

$$\gamma - \gamma' > 1,4(\delta - \delta'),$$

where

$$\Theta_\gamma = \Theta_{42},\ \Theta_{\gamma'} = \Theta_{40},\ \Theta_\delta = \Theta_{17},\ \Theta_{\delta'} = \Theta_{15}.$$

Rewrite it as follows:

$$0,497 > q\left(\frac{1+p\Theta_{42}}{1+p'\Theta_{15}}\right)^2,$$

which is clear from (117).

We have to show that the subrectangles $\{2112, 311\}$ and $\{1112, 32\}$ overlap.

The condition (89) should take place, with

$$\Theta_\gamma = \Theta_{51},\ \Theta_{\gamma'} = \Theta_{43},\ \Theta_\delta = \Theta_{16},\ \Theta_{\delta'} = \Theta_{13},$$

which expands to

$$0,88 > q\,\frac{1+p\Theta_{43}}{1+p'\Theta_{13}}\,\frac{1+p\Theta_{51}}{1+p'\Theta_{16}},$$

which follows from (102).

Showing that $\{1112, 33\}$ is good. The following inequality should take place:

$$\gamma - \gamma' < 3,8(\delta - \delta')$$

with

$$\Theta_\gamma = \Theta_{40},\ \Theta_{\gamma'} = \Theta_{40},\ \Theta_\delta = \Theta_{28},\ \Theta_{\delta'} = \Theta_{22},$$

which follows from

$$0,315 < q\left(\frac{1+p\Theta_{40}}{1+p'\Theta_{28}}\right)^2,$$

following from (70).

It is easy to check that $\{1112, 33\}$ and $\{2112, 311\}$ overlap. Showing that $\{1112, 33\}$ overlaps with $\{2112, 32\}$. We need the condition (80) with

$$\Theta_\gamma = \Theta_{47},\ \Theta_{\gamma'} = \Theta_{45},\ \Theta_\delta = \Theta_{25},\ \Theta_{\delta'} = \Theta_{18}.$$

Obtain inequality

$$0,202 < q\,\frac{1+p\Theta_{45}}{1+p'\Theta_{18}}\,\frac{1+p\Theta_{47}}{1+p'\Theta_{25}},$$

following from (70).

Consider the subrectangle $\{3112, 32\}$. Proving its goodness. We need

$$3,8(\gamma - \gamma') > \delta - \delta'$$

with

$$\Theta_\gamma = \Theta_{35},\ \Theta_{\gamma'} = \Theta_{49},\ \Theta_\delta = \Theta_{17},\ \Theta_{\delta'} = \Theta_{15},$$

expanding to

$$0,69 < q\,\frac{1+p\Theta_{35}}{1+p'\Theta_{15}}\,\frac{1+p\Theta_{49}}{1+p'\Theta_{17}},$$

which follows from (102). Indeed,

$$1,12 > \frac{1+p\Theta_{35}}{1+p\Theta_{29}}\,\frac{1+p'\Theta_{49}}{1+p'\Theta_{54}}\,\frac{1+p'\Theta_{25}}{1+p'\Theta_{15}}\,\frac{1+p'\Theta_{29}}{1+p'\Theta_{17}}.$$

This is checked by substituting both p and p' with $0,8$, increasing the rhs.

The subrectangles $\{2112, 32\}$ and $\{3112, 32\}$ overlap. It is shown by the condition (80) with the following constants:

$$\Theta_\gamma = \Theta_{49},\ \Theta_{\gamma'} = \Theta_{51},\ \Theta_\delta = \Theta_{19},\ \Theta_{\delta'} = \Theta_{16}.$$

Obtain inequality

$$0,297 < q\,\frac{1+p\Theta_{49}}{1+p'\Theta_{16}}\,\frac{1+p\Theta_{51}}{1+p'\Theta_{19}},$$

following from (70).

Now showing that $\{3112, 32\}$ overlaps with $\{2112, 33\}$. We need (89) with the following constants:

$$\Theta_\gamma = \Theta_{54},\ \Theta_{\gamma'} = \Theta_{50},\ \Theta_\delta = \Theta_{23},\ \Theta_{\delta'} = \Theta_{20},$$

which follows from

$$0,9 > q\left(\frac{1+p\Theta_{54}}{1+p'\Theta_{20}}\right)^2,$$

which follows from (70).

7.7 Segment between $\{2, 1\}$ and $\{2, 2\}$

Now we will turn to constructing the system of subrectangles which covers the interval

$$\Delta_3 = \left(\Delta_1''(2) + \Delta_2''(2); \Delta_1'(2) + \Delta_2'(1)\right).$$

At first, we will figure out the condition that $\{2, 1\}$ and $\{2, 2\}$ overlap. We need the inequality (67) with

$$\Theta_\gamma = \Theta_{64},\ \Theta_{\gamma'} = \Theta_{61},\ \Theta_\delta = \Theta_{64},\ \Theta_{\delta'} = \Theta_{61},$$

which is equivalent to

$$0,714 > q\,\frac{1+p\Theta_{61}}{1+p'\Theta_{61}}\,\frac{1+p\Theta_{64}}{1+p'\Theta_{64}}. \tag{118}$$

If (118) takes place, then the inequality (89) also takes place, with

$$\Theta_\gamma = \Theta_{63},\ \Theta_{\gamma'} = \Theta_{30},\ \Theta_\delta = \Theta_{66},\ \Theta_{\delta'} = \Theta_{63},$$

obtaining the condition

$$0,696 > q\,\frac{1+p\Theta_{30}}{1+p'\Theta_{63}}\,\frac{1+p\Theta_{63}}{1+p'\Theta_{66}}. \tag{119}$$

How consider the case we're having the following constants in (80):

$$\Theta_\gamma = \Theta_{30},\ \Theta_{\gamma'} = \Theta_{25},\ \Theta_\delta = \Theta_{81},\ \Theta_{\delta'} = \Theta_{66},$$

expanding to

$$0,657 < q\,\frac{1+p\Theta_{35}}{1+p'\Theta_{66}}\,\frac{1+p\Theta_{30}}{1+p'\Theta_{81}}. \tag{120}$$

To justify the choice of $\Theta_{\gamma'}$ and Θ_δ, we need to check the condition (67) for constants

$$\Theta_\gamma = \Theta_{28},\ \Theta_{\gamma'} = \Theta_{22},\ \Theta_\delta = \Theta_{82},\ \Theta_{\delta'} = \Theta_{80},$$

which expands to

$$0,842 > q\,\frac{1+p\Theta_{22}}{1+p'\Theta_{80}}\,\frac{1+p\Theta_{28}}{1+p'\Theta_{82}}\,. \tag{121}$$

If (121) doesn't take place, then we can set $\Theta_{\gamma'} = \Theta_{20}$ and, instead of (120), we will obtain

$$0,761 < q\,\frac{1+p\Theta_{20}}{1+p'\Theta_{66}}\,\frac{1+p\Theta_{30}}{1+p'\Theta_{81}}\,, \tag{122}$$

which takes place, in the consideration of (121).

At first, assume that (121) doesn't take place, that is,

$$0,6569 \geqslant q\,\frac{1+p\Theta_{25}}{1+p'\Theta_{66}}\,\frac{1+p\Theta_{30}}{1+p'\Theta_{81}}\,. \tag{123}$$

Checking the condition (89) with the following constants:

$$\Theta_\gamma = \Theta_{25},\ \Theta_{\gamma'} = \Theta_{3},\ \Theta_\delta = \Theta_{84},\ \Theta_{\delta'} = \Theta_{81}.$$

which is equivalent to

$$0,6027 > q\,\frac{1+p\Theta_{3}}{1+p'\Theta_{81}}\,\frac{1+p\Theta_{25}}{1+p'\Theta_{84}}\,,$$

which follows from (123):

$$0,9175 > q\,\frac{1+p'\Theta_{66}}{1+p'\Theta_{84}}\,\frac{1+p\Theta_{3}}{1+p\Theta_{30}}\,,$$

which is easily checked by substituting p with $[0;1,3,4] \geqslant 0,764$ and p' with $0,25$.

7.8 Name

Now consider the subrectangles $\{213, 122\}$, $\{113, 123\}$, and $\{113, 122\}$.

It is easy to check that they overlap. We will check the inequality

$$\gamma - \gamma' < \delta - \delta', \tag{124}$$

where

$$\Theta_\gamma = \Theta_{30},\ \Theta_{\gamma'} = \Theta_{13},\ \Theta_\delta = \Theta_{86},\ \Theta_{\delta'} = \Theta_{65},$$

which expands to

$$0,522 < q\,\frac{1+p\Theta_{13}}{1+p'\Theta_{66}}\,\frac{1+p\Theta_{30}}{1+p'\Theta_{86}}\,.$$

Let this inequality doesn't hold, that is,

$$0,522 \geqslant q\,\frac{1+p\Theta_{13}}{1+p'\Theta_{66}}\,\frac{1+p\Theta_{30}}{1+p'\Theta_{86}}\,. \tag{125}$$

Showing that $\{113, 122\}$ and $\{113, 1211\}$ overlap. The condition (89) takes place with the following Θ's:

$$\Theta_\gamma = \Theta_{13},\ \Theta_{\gamma'} = \Theta_{10},\ \Theta_\delta = \Theta_{87},\ \Theta_{\delta'} = \Theta_{86},$$

obtaining

$$0,518 > q\left(\frac{1+p\Theta_{13}}{1+p'\Theta_{86}}\right)^2,$$

which follows from (125).

Now in (124) we can set $\Theta_\delta = \Theta_{88}$ and obtain

$$0,468 < q\frac{1+p\Theta_{13}}{1+p'\Theta_{66}}\frac{1+p\Theta_{30}}{1+p'\Theta_{88}}.$$

If this inequality doesn't take place, then the inequality (118) follows from

$$\frac{1+p\Theta_{61}}{1+p\Theta_{13}}\frac{1+p\Theta_{64}}{1+p\Theta_{30}}\frac{1+p'\Theta_{66}}{1+p'\Theta_{61}}\frac{1+p'\Theta_{68}}{1+p'\Theta_{64}} < 1,53,$$

while the condition (119) follows from

$$\frac{1+p\Theta_{63}}{1+p\Theta_{13}}\frac{1+p'\Theta_{88}}{1+p'\Theta_{63}} < 1,48.$$

The subrectangle $\{23, 112\}$ overlaps with $\{2, 2\}$, if the condition (89) with the following constants takes place:

$$\Theta_\gamma = \Theta_{63},\ \Theta_{\gamma'} = \Theta_{16},\ \Theta_\delta = \Theta_{70},\ \Theta_{\delta'} = \Theta_{59},$$

which can be rewritten as

$$0,988 > q\frac{1+p\Theta_{16}}{1+p'\Theta_{59}}\frac{1+p\Theta_{63}}{1+p'\Theta_{70}},$$

following from the fact that Δ is horizontal:

$$1 \geqslant q\frac{1+p\Theta_{1}}{1+p'\Theta_{1}}\frac{1+p\Theta_{95}}{1+p'\Theta_{95}}.$$

Indeed,

$$\frac{1+p\Theta_{16}}{1+p\Theta_{1}}\frac{1+p\Theta_{63}}{1+p\Theta_{95}}\frac{1+p'\Theta_{1}}{1+p'\Theta_{59}}\frac{1+p'\Theta_{95}}{1+p'\Theta_{70}} < 0,988.$$

Substituting $(1+p'\Theta_1)(1+p'\Theta_{95})$ with $(1+p'\Theta_{59})(1+p'\cdot 0,64)$, increasing the lhs.

Increasing p' to $0,8$, decreasing p to $0,75$ and checking the inequality.

7.9 Name

Consider the subrectangle $\{113, 1111\}$. We will show that it overlaps with $\{23, 112\}$. The condition (89) should take place, with the following constants:

$$\Theta_\gamma = \Theta_{19},\ \Theta_{\gamma'} = \Theta_{10},\ \Theta_\delta = \Theta_{67},\ \Theta_{\delta'} = \Theta_{62},$$

obtaining the inequality

$$1,15 > q\frac{1+p\Theta_{10}}{1+p'\Theta_{62}}\frac{1+p\Theta_{19}}{1+p'\Theta_{67}},$$

which follows from *broken link*. Indeed,

$$\frac{1+p\Theta_{10}}{1+p\Theta_{1}}\frac{1+p\Theta_{19}}{1+p\Theta_{95}}\frac{1+p'\Theta_{1}}{1+p'\Theta_{62}}\frac{1+p'\Theta_{95}}{1+p'\Theta_{67}} < 1,15,$$

and, increasing the lhs, obtain

$$\frac{1+p\Theta_{10}}{1+p\Theta_{10}}\,\frac{1+p\Theta_{19}}{1+p\Theta_{95}}\,\frac{1+p'\cdot 0,63}{1+p'\Theta_{67}} < 1,15,$$

which is obvious.

Let's find out when the subrectangles $\{113, 1111\}$ and $\{23, 1111\}$ overlap. The following condition is needed:

$$\gamma - \gamma' < \delta - \delta',$$

where

$$\Theta_\gamma = \Theta_{16},\ \Theta_{\gamma'} = \Theta_{13},\ \Theta_\delta = \Theta_{76},\ \Theta_{\delta'} = \Theta_{74},$$

which expands to

$$0,793 < q\,\frac{1+p\Theta_{13}}{1+p'\Theta_{74}}\,\frac{1+p\Theta_{16}}{1+p'\Theta_{76}}\,. \tag{126}$$

Now check the condition

$$1,4(\gamma - \gamma') < \delta - \delta',$$

where

$$\Theta_\delta = [0; 1111331] = 0,61035,\ \Theta_\gamma = \Theta_{17},\ \Theta_{\gamma'} = \Theta_{15},\ \Theta_{\delta'} = \Theta_{73},$$

which follows from

$$0,78 < q\,\frac{1+p\Theta_{25}}{1+p'\Theta_{73}}\,\frac{1+p\Theta_{17}}{1+p'\Theta_{96}}\,,$$

which yields from (126).

Checking the condition

$$\gamma - \gamma' > 1,7(\delta - \delta'),$$

where

$$\Theta_\gamma = \Theta_{14},\ \Theta_{\gamma'} = \Theta_{12},\ \Theta_\delta = \Theta_{77},\ \Theta_{\delta'} = \Theta_{75},$$

expanding to

$$q\,\frac{1+p\Theta_{12}}{1+p'\Theta_{75}}\,\frac{1+p\Theta_{14}}{1+p'\Theta_{77}} < 0,98,$$

which obviously follows from the horizontality of Δ, (23).

If (126) doesn't take place and

$$0,793 \geqslant q\,\frac{1+p\Theta_{13}}{1+p'\Theta_{74}}\,\frac{1+p\Theta_{16}}{1+p'\Theta_{76}}\,, \tag{127}$$

then checking that $\{113, 1111\}$ and $\{113, 1112\}$ overlap.

For this, we need the following inequality to take place:

$$\gamma - \gamma' > \delta - \delta',$$

where

$$\Theta_\gamma = \Theta_{13},\ \Theta_{\gamma'} = \Theta_{10},\ \Theta_\delta = \Theta_{78},\ \Theta_{\delta'} = \Theta_{76},$$

which follows from (127).

It is easy to check that $\{113, 1112\}$ overlaps with $\{23, 1111\}$.

Checking that

$$3,8(\gamma - \gamma') > \delta - \delta',$$

where

$$\Theta_\gamma = \Theta_{28},\ \Theta_{\gamma'} = \Theta_{22},\ \Theta_\delta = \Theta_{96},\ \Theta_{\delta'} = \Theta_{73},$$

obtaining

$$1,26 > q\,\frac{1 + p\Theta_{22}}{1 + p'\Theta_{73}}\,\frac{1 + p\Theta_{28}}{1 + p'\Theta_{96}}\,,$$

which follows from (23):

$$\frac{1 + p'\Theta_1}{1 + p'\Theta_{73}}\,\frac{1 + p'\Theta_{95}}{1 + p'\Theta_{96}}\,\frac{1 + p\Theta_{22}}{1 + p\Theta_1}\,\frac{1 + p\Theta_{28}}{1 + p\Theta_{95}} < 1,26,$$

which is true.

Finally, obtain the following sequence of subrectangles:

$$\{33, 1111\},\ \{23, 1112\},\ \{23, 1112\},\ \{23, 1113\},\ \{33, 1112\}.$$

Showing that $\{33, 1112\}$ overlap with $\{33, 1113\}$. We need

$$\gamma - \gamma' > \delta - \delta',$$

where

$$\Theta_\gamma = \Theta_{25},\ \Theta_{\gamma'} = \Theta_{23},\ \Theta_\delta = \Theta_{81},\ \Theta_{\delta'} = \Theta_{79},$$

equivalent to

$$1,12 > q\,\frac{1 + p\Theta_{23}}{1 + p'\Theta_{79}}\,\frac{1 + p\Theta_{25}}{1 + p'\Theta_{81}}\,,$$

which follows from (23).

Θ_{1}	= [0; $\overline{\text{3I}}$]	= 0,263762
Θ_{2}	= [0; 3I2$\overline{\text{I3}}$]	= 0,267649
Θ_{3}	= [0; 3$\overline{\text{I2}}$]	= 0,26794
Θ_{4}	= [0; 3I23$\overline{\text{3I}}$]	= 0,270448
Θ_{5}	= [0; 3I23$\overline{\text{I2}}$]	= 0,2707I0
Θ_{6}	= [0; 3I2$\overline{\text{3I}}$]	= 0,270738
Θ_{7}	= [0; 3II$\overline{\text{I3}}$]	= 0,27459
Θ_{8}	= [0; 3III3$\overline{\text{I2}}$]	= 0,27462
Θ_{9}	= [0; 3III3$\overline{\text{3I}}$]	= 0,274847
Θ_{10}	= [0; 3II$\overline{\text{I2}}$]	= 0,275I7
Θ_{11}	= [0; 3I$\overline{\text{I2}}$]	= 0,27954
Θ_{12}	= [0; 3II3$\overline{\text{3I}}$]	= 0,280392
Θ_{13}	= [0; 3II3$\overline{\text{I2}}$]	= 0,28097
Θ_{14}	= [0; 3I$\overline{\text{I3}}$]	= 0,28I05
Θ_{15}	= [0; 32$\overline{\text{3I}}$]	= 0,290550
Θ_{16}	= [0; 323$\overline{\text{I2}}$]	= 0,29062
Θ_{17}	= [0; 323$\overline{\text{3I}}$]	= 0;29I242
Θ_{18}	= [0; 32$\overline{\text{2I}}$]	= 0;292I6
Θ_{19}	= [0; 3$\overline{\text{2I}}$]	= 0;29709
Θ_{20}	= [0; 32I3$\overline{\text{I2}}$]	= 0;29774
Θ_{21}	= [0; 32$\overline{\text{I3}}$]	= 0;297773
Θ_{22}	= [0; 33$\overline{\text{3I}}$]	= 0;302444
Θ_{23}	= [0; 333$\overline{\text{I2}}$]	= 0;30025
Θ_{24}	= [0; 33$\overline{\text{2I}}$]	= 0;30330
Θ_{25}	= [0; 33$\overline{\text{I2}}$]	= 0;30600
Θ_{26}	= [0; 33I2$\overline{\text{I3}}$]	= 0;30603
Θ_{27}	= [0; 33I3$\overline{\text{I2}}$]	= 0;30638
Θ_{28}	= [0; 3$\overline{\text{3I}}$]	= 0;306394
Θ_{29}	= [0; 2$\overline{\text{I3}}$]	= 0;358258
Θ_{30}	= [0; 2I3$\overline{\text{I2}}$]	= 0;35859
Θ_{31}	= [0; 2I33$\overline{\text{3I}}$]	= 0;36I292
Θ_{32}	= [0; 2I33$\overline{\text{I2}}$]	= 0;36I58
Θ_{33}	= [0; 2I3$\overline{\text{3I}}$]	= 0;36I602
Θ_{34}	= [0; 2I2$\overline{\text{I3}}$]	= 0;365465
Θ_{35}	= [0; 2I2I3$\overline{\text{I2}}$]	= 0,36549
Θ_{36}	= [0; $\overline{\text{2I}}$]	= 0,36602
Θ_{37}	= [0; 2I2I$\overline{\text{I3}}$]	= 0,36779
Θ_{38}	= [0; 2I23$\overline{\text{I2}}$]	= 0,37II9
Θ_{39}	= [0; 2I2$\overline{\text{3I}}$]	= 0,37I249
Θ_{40}	= [0; 2II$\overline{\text{I3}}$]	= 0,378537
Θ_{41}	= [0; 2III3$\overline{\text{I2}}$]	= 0,37858
Θ_{42}	= [0; 2III3$\overline{\text{3I}}$]	= 0,3790I8
Θ_{43}	= [0; 2II$\overline{\text{I2}}$]	= 0,37963
Θ_{44}	= [0; 2IIII3$\overline{\text{3I}}$]	= 0,38309I
Θ_{45}	= [0; 2IIII3$\overline{\text{I2}}$]	= 0,38326
Θ_{46}	= [0; 2III$\overline{\text{I3}}$]	= 0,383272
Θ_{47}	= [0; 2II23$\overline{\text{I2}}$]	= 0,38605
Θ_{48}	= [0; 2II2$\overline{\text{3I}}$]	= 0,386033
Θ_{49}	= [0; 2II23$\overline{\text{3I}}$]	= 0,386237
Θ_{50}	= [0; 2II2$\overline{\text{2I}}$]	= 0,3865I
Θ_{51}	= [0; 2I$\overline{\text{I2}}$]	= 0,38800
Θ_{52}	= [0; 2II3$\overline{\text{3I}}$]	= 0,389648
Θ_{53}	= [0; 2II3$\overline{\text{2I}}$]	= 0,3899I6
Θ_{54}	= [0; 2II3$\overline{\text{I2}}$]	= 0,39076
Θ_{55}	= [0; 2I$\overline{\text{I3}}$]	= 0,39089I
Θ_{56}	= [0; 22$\overline{\text{3I}}$]	= 0,409544
Θ_{57}	= [0; 223$\overline{\text{I2}}$]	= 0,40968
Θ_{58}	= [0; 2$\overline{\text{2I}}$]	= 0,42265
Θ_{59}	= [0; 22I3$\overline{\text{I2}}$]	= 0,42398
Θ_{60}	= [0; 22$\overline{\text{I3}}$]	= 0,424042
Θ_{61}	= [0; 23$\overline{\text{3I}}$]	= 0,433577
Θ_{62}	= [0; 233$\overline{\text{I2}}$]	= 0,43365
Θ_{63}	= [0; 23$\overline{\text{I2}}$]	= 0,44093
Θ_{64}	= [0; 2$\overline{\text{3I}}$]	= 0,44I742
Θ_{65}	= [0; I$\overline{\text{I3}}$]	= 0,558256
Θ_{66}	= [0; II3$\overline{\text{I2}}$]	= 0,55905
Θ_{67}	= [0; II33$\overline{\text{I2}}$]	= 0,56635
Θ_{68}	= [0; II3$\overline{\text{3I}}$]	= 0,566423
Θ_{69}	= [0; II2$\overline{\text{I3}}$]	= 0,57600
Θ_{70}	= [0; II2I3$\overline{\text{I2}}$]	= 0,57602
Θ_{71}	= [0; I$\overline{\text{I2}}$]	= 0,57735
Θ_{72}	= [0; II23$\overline{\text{I2}}$]	= 0,59032
Θ_{73}	= [0; III$\overline{\text{I3}}$]	= 0,609I08
Θ_{74}	= [0; IIII3$\overline{\text{I2}}$]	= 0,60923
Θ_{75}	= [0; IIIII3$\overline{\text{3I}}$]	= 0,62098I
Θ_{76}	= [0; IIIII3$\overline{\text{I2}}$]	= 0,62I4I
Θ_{77}	= [0; IIII$\overline{\text{I3}}$]	= 0,62I460
Θ_{78}	= [0; III23$\overline{\text{I2}}$]	= 0,6288I
Θ_{79}	= [0; II$\overline{\text{I2}}$]	= 0, 63400
Θ_{80}	= [0; III3$\overline{\text{3I}}$]	= 0,63839
Θ_{81}	= [0; III3$\overline{\text{I2}}$]	= 0,639I0
Θ_{82}	= [0; II$\overline{\text{I3}}$]	= 0,64I742
Θ_{83}	= [0; I23$\overline{\text{I2}}$]	= 0,69399
Θ_{84}	= [0; I22I3$\overline{\text{I2}}$]	= 0,70225
Θ_{85}	= [0; I223$\overline{\text{I2}}$]	= 0,70938
Θ_{86}	= [0; I22$\overline{\text{2I}}$]	= 0,70783
Θ_{87}	= [0; I2II3$\overline{\text{I2}}$]	= 0,7I903
Θ_{88}	= [0; I2III3$\overline{\text{I2}}$]	= 0,72538I
Θ_{89}	= [0; $\overline{\text{I2}}$]	= 0,73206
Θ_{90}	= [0; I2I3$\overline{\text{I2}}$]	= 0,73620
Θ_{91}	= [0; I2$\overline{\text{I3}}$]	= 0,73624
Θ_{92}	= [0; I3$\overline{\text{3I}}$]	= 0,765465
Θ_{93}	= [0; I33$\overline{\text{I2}}$]	= 0,76569
Θ_{94}	= [0; I3$\overline{\text{I2}}$]	= 0,78868
Θ_{95}	= [0; $\overline{\text{I3}}$]	= 0,79I287

Figure 8: List of Θ's.

Contents